The Feynman propagator and self-adjointness

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Consider the wave equation on a Lorentzian spacetime (M, g): $\Box_g u = 0$, i.e. g is a symmetric 2-cotensor of signature $(1, n-1)$.

Typically the Cauchy problem is considered: one takes an (embedded) spacelike hypersurface S and specifies u and its normal derivative Vu at S. Then

- there is a unique local solution (near S), and
- if (M, g) is globally hyperbolic, i.e. there is a global time function t with S as a level set, there is a unique global solution.

The Cauchy problem is equivalent to a forcing (or inhomogeneous) problem: solve $\Box_g u = f$ where f is supported in $t \geq t_0$, by finding u which is supported in $t \geq t_0$, together with its analogue where $\geq$ is replaced by $\leq$. This way one does not need to choose a Cauchy surface S.

The solution operator $\Box_{g,R}^{-1} : f \mapsto u$ is the forward, or retarded solution operator. If one replaces $\geq$ by $\leq$, one obtains the backward, or advanced, solution operator, $\Box_{g,A}^{-1}$.

One question we address here is what the natural inverses of $\Box_g$ are. It turns out that in reasonable settings, there are two more natural inverses, the Feynman and anti-Feynman propagators (introduced by Feynman in the Minkowski setting!). These propagators are much more *global* than the advanced and retarded solution operator, i.e. even the local behavior of the solution depends on the global structure of the spacetime, though this dependence is via a smooth contribution.

The settings in which these global inverses exist are usually non-trapping, which says that the null-geodesic flow posesses a good structure. A concrete example is symbolic of order 0 Lorentzian metrics on $\mathbb{R}^n$, i.e. $|D^\alpha g_{ij}(x)| \leq C_\alpha (1+|x|)^{-|\alpha|}$, such as Minkowski metric or asymptotically Minkowski metrics, for which there are source/sink manifolds at spacetime infinity for the null-geodesic flow; in the asymptotically Minkowski case this is the standard light cone at infinity.

As we shall see, a nice way to encode the inverses is via the choice of appropriate function spaces (different spaces for the different inverses) on which $\Box_g$ is Fredholm.

Concretely, these will be variable order weighted Sobolev spaces $H^{s,r}$, where *an* order function is microlocal, i.e. is a function on the cotangent bundle, which is monotone along the Hamilton flow in T^*M, i.e. along the lifted null-geodesics, and satisfies certain threshold inequalities at the sources/sinks.

Precisely *which* order function this is depends on the underlying operator algebra. For $\Box_g$ in the symbolic metric setting this is the b-pseudodifferential operator algebra of Melrose, and the variable order for $H_b^{s,r}$ is the b-differential order s (the decay order is constant). Here the regularity (differential order) outside a compact set corresponds to homogeneous degree 0 vector fields, such as $x_i D_{x_j}$, while the weight is simply powers of $|x|$ there.

A closely related question is the behavior of the spectral family, $\Box_g - \lambda$, especially when $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under suitable global assumptions, $(\Box_g - \lambda)^{-1}$ then exists on either the Feynman ($\operatorname{Im} \lambda > 0$) or the anti-Feynman ($\operatorname{Im} \lambda < 0$) function spaces. Moreover, one can take limits as $\operatorname{Im} \lambda \to 0$, and the resulting inverses of $\Box_g - \lambda$, when $\lambda \in \mathbb{R}$ now, are exactly the Feynman and the anti-Feynman inverses.

In these cases the operator algebra is the scattering pseudodifferential operator algebra of Melrose, and the decay order r of $H^{s,r}_{\mathrm{sc}}$ is variable (the differential order is constant). There the regularity (differential order) corresponds to translation invariant vector fields, D_{x_j}, while the weight is still powers of $|x|$.

This is then closely related to the essential self-adjointness of $\Box_g$ on $C_c^\infty(M)$, which ends up being a regularity statement for $(\Box_g \pm \imath)^{-1}$, namely that they *both* map $C_c^\infty(M)$ to a suitable subspace of L^2. Concretely, this 'domain' can be taken to be

$$D = \{u \in H_{\mathrm{sc}}^{1,-1/2}(\overline{M}) \cap L^2_{\mathrm{sc}}(M) : \ \Box_g u \in L^2_{\mathrm{sc}}(M)\};$$

here H_{sc} is the standard weighted Sobolev space if $M = \mathbb{R}^n$; see V. '20. In fact, even Schwartz functions would work as D. In a sense this property is even more important than the actual self-adjointness.

It is then not surprising that these propagator differences are positive (important for QFT) (V. '16):

$$\imath((\Box_g - \lambda + i0)^{-1} - (\Box_g - \lambda - i0)^{-1}) \geq 0 :$$

this is the positivity of the spectral measure!

This self-adjointness allows one to define, via the functional calculus, $(\Box_g - \imath\epsilon)^{-\alpha}$ when $\operatorname{Re}\alpha > \frac{n}{2}$. In a very nice recent paper Dang and Wrochna (arXiv '20) showed for n even the on-diagonal restriction of the Schwartz kernel has a meromorphic extension to $\mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1\}$, and at $\alpha = \frac{n}{2} - 1$ the limit, as $\epsilon \to 0+$, of the residue is, up to a constant factor, the scalar curvature of g.

They also computed the residue at other poles giving rise to a scalar Lorentzian version of the spectral action principle of Chamseddine-Connes, expressing the restriction to the diagonal of suitable functions of $\Box_g$ in terms of Hadamard coefficients.

In fact, our discussion is not really specific for the wave equation, rather it is a general non-elliptic phenomenon, so the basic setup we discuss is compact manifolds without boundary since the general setup is conceptually no more complicated.

Back in the setting of 2nd order PDE, another place where (anti-)Feynman propagators arise is ultrahyperbolic PDE such as $\sum_{j=1}^{k} D_{x_j}^2 - \sum_{j=k+1}^{n} D_{x_j}^2$, $k, n-k \geq 2$, or more generally the Laplacian associated to metrics of arbitrary non-Riemannian signature (with suitable global conditions). These are in fact *very much* like the wave equation *except* for the Cauchy problem — but our approach of constructing inverses works just as well!

There has been much work in mathematical quantum field theory on Feynman propagators. The closest works in terms of general outlook have been due to Bär, Dereziński, Gérard, Häfner, Siemssen, Strohmaier and Wrochna... and some of Zelditch's work also relates to this.

In particular, I became interested in this problem after some discussions with Ch. Gérard, and in the self-adjointness issues after discussions with Dereziński, who had partial results on this with Siemssen. The self-adjointness issues have also been addressed since then by Nakamura and Taira in other settings.

In this QFT setting one is interested in positivity properties of various propagator differences. A weaker version, up to smoothing (in a strong global sense) operators, follows from similar arguments, but not a strict positivity statement in general. However, the subject of this talk is not QFT, rather global non-elliptic analysis.

Before we discuss the full framework, let us discuss an aspect of it, propagation of singularities. This actually already had a major impact on quantum field theory due to the work of Duistermaat-Hörmander (early 70s) and Radzikowski (mid-90s).

Briefly recall that if M is a manifold (without boundary, not necessarily compact), e.g. $\mathbb{R}^n$, pseudodifferential operators, $P \in \Psi^m(M)$, are essentially quantizations of symbols $p \in S^m$ on T^*M: $P = \mathrm{Op}(p)$. In local coordinates $|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}$:

$$(Pu)(x) = (2\pi)^{-n} \int e^{\imath(x-y)\cdot\xi} p(x,\xi) u(y)\, d\xi\, dy.$$

The important behavior of p is $|\xi| \to \infty$; S^m corresponds to growth as $|\xi|^m$. Often one considers p with an asymptotic expansion in powers of $|\xi|$ (as $|\xi| \to \infty$), $p \sim \sum |\xi|^{m-j} p_j$; this holds for differential operators.

The behavior as $|\xi| \to \infty$, referred to as 'fiber infinity', is encoded by using dilations in the fibers (i.e. in ξ) (or a compactification), so the phase space can be considered as $T^*M \setminus o$ modulo dilations in the fibers, i.e. $S^*M = (T^*M \setminus o)/\mathbb{R}^+$.

Functions b on S^*M can be considered as homogeneous degree 0 functions on $T^*M \setminus o$, thus (modulo $\Psi^{-\infty}(M)$) $B = \mathrm{Op}(b) \in \Psi^0(M)$. Such $\mathrm{Op}(b)$ is a microlocalizer to $\operatorname{supp} b$, and is non-degenerate, namely *elliptic* on $\{b > 0\}$; b is the *principal symbol* of B, extending the notion for differential operators, while $\operatorname{supp}(b)$ is the operator wave front set $\mathrm{WF}'(B)$.

More generally, for $P = \mathrm{Op}(p)$, the *principal symbol*, $\sigma_m(P)$ is the equivalence class of p in S^m/S^{m-1}, capturing P modulo $\Psi^{m-1}(M)$; in the example above $|\xi|^m p_0$ can be taken as a representative.

For $P \in \mathrm{Diff}^m(M)$, $m \in \mathbb{N}$, $\sigma_m(P)$ captures the leading terms. If $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$, $\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

In general an operator P is called elliptic at a point $\alpha = (x,\xi) \in T^*M \setminus o$ if $p = \sigma_m(P)$ is invertible at α, i.e. non-zero in the scalar setting. The set of elliptic points is $\mathrm{Ell}(P)$, its complement of the characteristic set $\mathrm{Char}(P)$.

Near such points we have elliptic estimates:

$$\|B_1 u\|_{H^s} \leq C(\|B_3 P u\|_{H^{s-m}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$, provided $b_3 \neq 0$ on $\operatorname{supp} b_1$ (i.e. provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$) and $p \neq 0$ on $\operatorname{supp} b_1$.

The most basic non-elliptic phenomenon, when $P \in \Psi^m$ has real principal symbol p, is propagation of singularities along the bicharacteristics, i.e. integral curves of H_p, the *Hamilton vector field* given by the symplectic structure on T^*M, inside $\mathrm{Char}(p) = \{p = 0\}$; locally

$$H_p = \sum_{j=1}^{n} \frac{\partial p}{\partial \zeta_j}\frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j}\frac{\partial}{\partial \zeta_j},$$

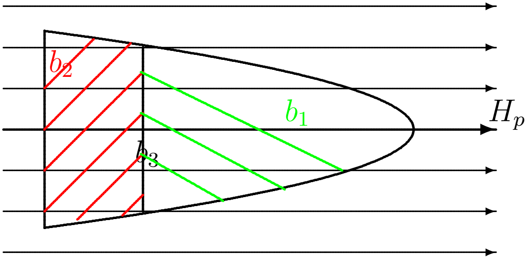
i.e. $H_p = (\partial_\zeta p, -\partial_z p)$, due to Hörmander '71.

This is only a meaningful statement at points at which H_p is not a multiple of the dilation vector field $\xi \cdot \partial_\xi$. A different way to say this is that H_p induces a vector field on S^*M (by making p homogeneous of degree 1), and then these *radial points* are stationary points of the induced vector field.

This is an estimate of the form

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$, provided $\operatorname{supp} b_1 \subset \{b_3 \neq 0\}$, and all bicharacteristics from points in $\operatorname{supp} b_1 \cap \mathrm{Char}(P)$ reach $\{b_2 \neq 0\}$ of while remaining in $\{b_3 \neq 0\}$. This is usually proved via a positive commutator estimate, which is a microlocal version of an energy estimate.



A complication for global non-elliptic theory, discussed later, is that H^s is often a *variable order* Sobolev space (see e.g. Unterberger '71, Duistermaat '72), i.e. s is a real-valued function on S^*M. This space is defined by:

- Let $s_0 = \inf s$, and let $A \in \Psi_\delta^s(M) \subset \Psi_\delta^{\sup s}(M)$, $\delta \in (0, 1/2)$, be elliptic,
- $$H^s = \{u \in H^{s_0} :\ Au \in L^2\},$$
- for instance if g is a Riemannian metric, one can take the principal symbol of A to be $|\xi|_g^s$.

Then $P \in \Psi^m$ maps $P : H^s \to H^{s-m}$ continuously still.

The microlocal elliptic and propagation estimates are valid if s is variable, provided, for the propagation estimate, one either restricts to *forward* bicharacteristics and requires $H_p s \geq 0$, or to *backward* bicharacteristics and requires $H_p s \leq 0$.

Duistermaat and Hörmander in 1972 constructed parametrices, i.e. approximate inverses, for operators with H_p non-radial; these are approximate inverses modulo smoothing operators.

- The work uses Fourier integral operators to reduce to a model case, D_{x_n}.
- This result in particular gave a proof of propagation of singularities.

They also showed that if $\mathrm{Char}(P)$ has k connected components, there are 2^k *distinguished parametrices*, namely in each component of the characteristic set one can specify whether one 'solves away' singularities forward or backward along the Hamilton flow.

This can be phrased in terms of the wave front set of the Schwartz kernel of the parametrix instead.

There are two extreme cases:

- in each component of the characteristic set we can propagate the estimates forwards, i.e. singularities backwards, or
- in each component we can propagate the estimates backwards.

These are the Feynman and anti-Feynman parametrices.

Answering a question of Wightman they also showed that one can *choose* the distinguished parametrices so that $\imath$ times their difference from the Feynman parametrix is positive.

This positivity is important for quantum field theory purposes: two-point functions, which are essentially such differences, must be positive for basic constructions in QFT.

Radzikowski ('96) used the Duistermaat-Hörmander work to explain microlocally the Hadamard condition traditionally used in QFT.

Note that for the wave operator in a connected globally hyperbolic spacetime, the characteristic set has 2 connected components, hence there are 4 distinguished parametrices, up to smoothing:

- retarded (forward in time),
- advanced (backward in time),
- Feynman (forward along H_p), and
- anti-Feynman (backward along H_p).

All this is up to smoothing, so there is quite a bit of freedom. As for limiting it, Duistermaat and Hörmander also stated that they 'do not see how to fix the indetermination'.

In order to do so, we need to work globally, i.e. we need to have an operator we can actually invert.

Theorem

(V. '18, Gell-Redman, Haber, V. '16) In the non-elliptic settings considered here, if the characteristic set of P has k connected components, then there are 2^k natural Fredholm problems for P, specified by the direction of propagation in each connected component. The difference of the extreme (generalized) inverses is ι^{-1} times positive.

Recall that Fredholm is invertibility up to a finite dimensional obstruction:

- $P : X \to Y$ continuously,
- Ran P closed
- Ker P, $Y/$ Ran P are finite dimensional.
- The settings include compact spaces as well as generalizations of Minkowski-like spacetimes.
- The spaces X, Y are essentially (weighted) Sobolev spaces.
- The (generalized) inverses correspond to the D-H parametrices, but there are (essentially) no choices.

The Fredholm properties are guaranteed by the Fredholm estimates

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_{Z_1})$$

and

$$\|v\|_{Y^*} \leq C(\|P^*v\|_{X^*} + \|u\|_{Z_2}),$$

where the inclusion maps $X \to Z_1$ and $Y^* \to Z_2$ are compact.

One often wants actual invertibility; another interesting question is the computation of the index of P.

Due to the lack of time, the boundary setting (Melrose's b-analysis) will be underemphasized; most phenomena are present already in the compact boundaryless setting (though of course the actual wave operator then is *not* an example).

The simplest example is elliptic (pseudo)differential operators on compact manifolds without boundary M, basic geometric examples being the Laplacian on differential forms, and Dirac operators.

- $P \in \Psi^m(M)$ elliptic (at least principally classical), i.e. $\sigma_m(P)$ invertible,
- $X = H^s = H^s(M)$, $Y = H^{s-m}(M)$, $s \in \mathbb{R}$,
- so $X^* = H^{-s}(M)$, $Y^* = H^{-s+m}(M)$,
- $Z_1 = H^{-N}(M)$, $Z_2 = H^{-N}(M)$, N large.

The Fredholm property follows from the elliptic estimate

$$\|\phi\|_{H^r} \leq C(\|L\phi\|_{H^{r-m}} + \|\phi\|_{H^{-N}}),$$

with $L = P$, $r = s$, resp. $L = P^*$, $r = -s + m$. Note that the choice of s is irrelevant here (elliptic regularity).

The non-elliptic problems we consider are problems in which the elliptic estimate is replaced by estimates of the form

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

i.e. with a loss of one derivative relative to the elliptic setting, and

$$\|v\|_{H^{s'}} \leq C(\|P^*v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$ being the case of interest.

Such estimates imply that $P : X \to Y$ is Fredholm if

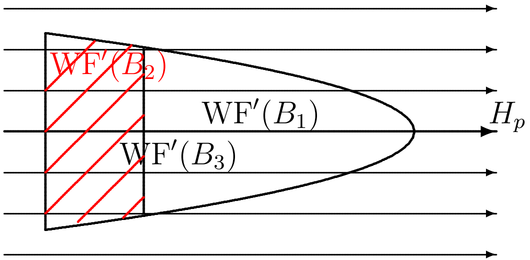
$$X = \{u \in H^s : \; Pu \in H^{s-m+1}\}, \; Y = H^{s-m+1}.$$

Here X is a first order coisotropic space associated to the characteristic set of P.

Recall the propagation of singularities estimate

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

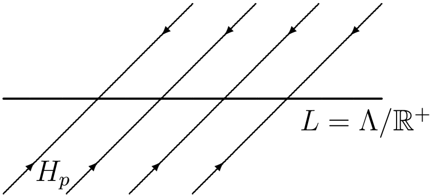
$B_j \in \Psi^0$, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.



The basic problem with this estimate is the term $\|B_2 u\|_{H^s}$ on the right hand side – how does one control this?

One option is *complex absorption*. The point is then that bicharacteristics reach the elliptic set of an operator Q with real principal symbol, and one works with $P - iQ$. In this case one can propagate estimates either forward or backward along the H_p-flow depending on the sign of the principal symbol of Q. This relates to the Feynman propagators discussed earlier.

A more natural option is to have some structure of the bicharacteristic flow: we need that there are submanifolds L of S^*M which act as sources/sinks in the normal direction.



- The most natural place these arise is *radial sets*, i.e. points in T^*M where H_p is tangent to the dilation orbits. Note that Hörmander's theorem provides no information here.
- In non-degenerate settings, i.e. when H_p is non-zero, the biggest possible dimension of a radial set is that of M, in which case it is a conic Lagrangian submanifold of T^*M.
- In this case, they act as source or sink within $\mathrm{Char}(P)$; in the source case H_p flows to the zero section within Λ, in the sink case from the zero section.
- This also arises in scattering theory, where it was studied by Melrose '94.

Let $\tilde{p}$ be the principal symbol of $\frac{1}{2i}(P - P^*) \in \Psi^{m-1}$, and define $\tilde{\beta}$ by

$$\tilde{p}|_\Lambda = \tilde{\beta}\frac{H_p\rho}{\rho},$$

where ρ is an elliptic homogeneous degree 1 function, which is independent of choices (even that of the metric defining the adjoint!).

In this case there is an analogue of the propagation of singularity theorem, but there is a threshold, $(m-1)/2 - \tilde{\beta}$:

- if the Sobolev order is higher than this, one can propagate estimates from $L = \Lambda/\mathbb{R}^+$, without needing a priori control like $B_2 u$,
- if the Sobolev order is below this, one can propagate estimates to L, needing control in a punctured neighborhood of L.

- If $s \geq s_0 > (m-1)/2 - \tilde{\beta}$, then

$$\|B_1 u\|_{H^s} \leq C(\|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{s_0}}),$$

  $B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ tend to L while remaining in $\mathrm{Ell}(B_3)$.
- If $s < (m-1)/2 - \tilde{\beta}$ then

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

  $B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $(\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)) \setminus L$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.

Replacing P by P^* changes the sign of $\tilde{\beta}$, and it naturally leads to estimates on the required dual spaces.

As a consequence, if there are radial sets L_1, L_2 such that all bicharacteristics in $\mathrm{Char}(P) \setminus (L_1 \cup L_2)$ escape to L_1 in one of the directions along the bicharacteristics and to L_2 in the other, one has the required Fredholm estimate provided one can arrange the Sobolev spaces so that

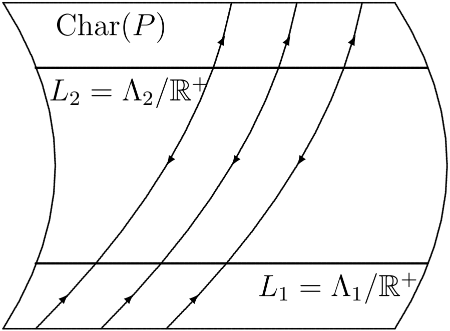
- at L_1 the Sobolev order is above the threshold for P,
- at L_2 the Sobolev order is above the threshold for P^* (i.e. below threshold for P),
- the Sobolev order is monotone decreasing from L_1 to L_2.
- One can actually do this independently in every one of the k connected components of $\mathrm{Char}(P)$, hence the 2^k Fredholm problems.

Namely,

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$$\|v\|_{H^{s'}} \leq C(\|P^*v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$.


Char(P)
$L_2 = \Lambda_2/\mathbb{R}^+$
$L_1 = \Lambda_1/\mathbb{R}^+$

The simplest example is multiplication by a function with non-degenerate zeros; locally this is $P = x_1$. (We do need a global problem though!)

- The characteristic set is $T^*_{x_1=0}\mathbb{R}^n \setminus o$.
- The radial sets are the two halves of $N^*_{x_1=0}\mathbb{R}^n \setminus o$.
- If the zero set is connected, there are 2 Fredholm problems.

A non-compact setting in which this arises is scattering theory:

- The Fourier conjugate of $\Delta - \lambda$, $\lambda > 0$, is $|\xi|^2 - \lambda$,
- While $\mathbb{R}^n$ is non-compact, this problem is fully elliptic at infinity, including decay, in the *scattering algebra*

$$|D_x^\alpha D_\xi^\beta a| \leq C_{\alpha\beta}\langle x\rangle^{l-|\alpha|}\langle\xi\rangle^{m-|\beta|},$$

- Fredholm theory is applicable in the scattering setting directly, giving two inverses, which are the *incoming/outgoing resolvents* $R(\lambda \pm i0)$.
- The positivity of $\imath$ times the difference of these is exactly the positivity of the spectral measure.

One can get more interesting problems if one has more than one connected components of the characteristic set.

For instance: asymptotically Euclidean spaces with more than one end:

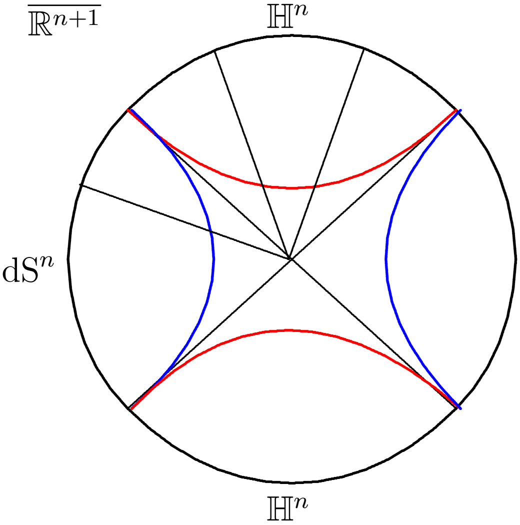
- The characteristic set is at infinity,
- In each component one sets the direction of propagation independently,
- Always forward/backward along the flow gives $R(\lambda \pm i0)$.

A perhaps more 'natural' example is the Klein-Gordon equation on asymptotically Minkowski spaces with positive mass $P = \Box_g - m^2$. The characteristic set has two connected components, and the sources/sinks are at infinity. This agains fits into the scattering algebra.

Back to the wave equation on Minkowski-like spaces $(\tilde{M}, \tilde{g})$:

- There is an analytic setup (Melrose's b-algebra) for the Fredholm setup.
- There is a principal symbol in this algebra, which is completely analogous to the compact/scattering settings, and all the analysis applies.
- The principal symbol does not capture operators modulo relatively compact operators unlike in the compact/scattering settings.
- An additional family of operators, the Mellin transformed normal operators are needed for this.
- One obtains Fredholm theory on weighted Sobolev spaces as long as the weights avoid a discrete set of reals, with the same amount of choice as in the compact/scattering settings (principal symbols).

- Let $\tilde{M} = \mathbb{R}^{n+1}$ with the Minkowski metric and $\Box$ be the wave operator.
- Let ρ be a homogeneous degree 1 positive function, e.g. a Euclidean distance from the origin.
- The conjugate of $\rho^2\Box$ by the Mellin transform along the dilation orbits gives a family of operators P_σ, σ the Mellin dual parameter, on $\mathbb{S}^n$ (smooth transversal to the dilation orbits).
- P_σ is elliptic inside the light cone, but Lorentzian outside the light cone.
- The conormal bundle of the light cone consists of radial points.
- The characteristic set has two components, and there are four components of the radial set: a future and a past component within each component of the characteristic set.



$\overline{\mathbb{R}^{n+1}}$
$\mathbb{H}^n$
dS^n
$\mathbb{H}^n$

In one component Σ_+ of the characteristic set, the bicharacteristics go from the past component of the radial set L_{+-} to the future one L_{++}; in the other component Σ_- they go from the future component of the radial set L_{-+} to the past one L_{--}.

In this case the interior of the light cone is naturally identified with hyperbolic space, while the exterior with de Sitter space.

Reasonable choices of Fredholm problems:

- Make the Sobolev spaces high regularity at the past radial sets and low at the future radial sets: this is the *forward propagator*.
- Make the Sobolev spaces low regularity at the past radial sets and high at the future radial sets: this is the *backward propagator*.
- Make the Sobolev spaces high regularity at the sources L_{+-} and L_{-+} and low regularity at the sinks, or vice versa. These are the Feynman propagators, and they propagate estimates for P_σ in the direction of the Hamilton flow in the first case, and against the Hamilton flow in the second.
- Note that the adjoint of these inverses always propagates estimates in the *opposite* direction!

Happy Birthday, Steve!