Complete Calabi-Yau Metric on Noncompact Manifolds

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I have known Steve for about twenty years and has always been impressed by his great originality and his frank opinions. He also graduated from Berkeley and a grand student of Chern, that always made me feel close to him. I wish him happy birthday and recover fast from his illness.

Introduction

A compact Kähler manifold (X^n, ω) is called Calabi-Yau if $c_1(X) = 0 \in H^2(X, \mathbb{Z})$. Usually we assume $\pi_1(X)$ is finite.

- In 1954, 1957, Calabi conjectured that X carries Ricci-flat metric.
- Calabi reduced his conjecture to a complex Monge-Ampère equation.

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) = e^f \det\left(g_{i\bar{j}}\right)$$

with $g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi$ being positive definite.

Using the tools of nonlinear PDE, I proved the Calabi conjecture in 1976 and applied it to solve

- The uniqueness of Kähler complex structure on $\mathbb{C}P^n$
- Chern number inequality
- Severi conjecture — a complex surface that is homotopic to $\mathbb{C}P^2$ is biholomorphic to it.

This is the major starting point of the development of geometric analysis.

- This provides first known compact non-locally-homogeneous Einstein manifolds and the first known (nonflat) complete Ricci-flat Kähler manifolds.

- Soon afterwards, many noncompact complete Ricci-flat examples were found, which included the gravitational instantons, the Calabi constructions and the Eguchi-Hanson metrics.

- Chern number inequality $c_2 \cap [\omega]^n \geqslant 0$ and the equality holds if and only if the universal cover $\tilde{X} \cong \mathbb{C}^n$.
- The holonomy group is contained in $SU(n)$. This leads to the structure theorem of Calabi-Yau manifolds.
- There are many other applications in complex and algebraic geometry. For example, K3 is Kähler by Siu and Todorov.
- There is no similar theorem in G_2 manifolds. It is more difficult to construct examples of G_2 manifolds.

The Ricci-flat metric on Calabi-Yau manifolds are solutions of the Einstein field equation with no matter. The theory of motions of loops inside a Calabi-Yau manifold provide a model of a conformal field theory. So Calabi-Yau manifolds became very important in the study of superstring theory.

One of the most important topics contributed by both physicists and mathematicians is mirror symmetry.

Fano manifold

When the first Chern class is positive, it is called a Fano manifold.

Kollár, Mori and Miyaoka showed that smooth Fano varieties are rationally connected, and showed there is an effective bound for the degree of the Fano n-folds with respect to its anticanonical bundle.

Based on the work of Kollár and Matsusaka, it also implies that Fano n-folds form a bounded family.

Fano variety with singularities

For a long time, the most important question on Fano variety in algebraic geometry is:

Whether Fano varieties with mild singularities are bounded, i.e., they form finitely many families.

Birkar answered this question affirmatively in full generality in 2016. This breakthrough work has many important applications including Kähler-Einstein metric.

Uniqueness of extremal metric

- By Calabi-Yau Theorem, when $c_1(M) = 0$, there exists a unique KE metric in each Kähler class.
- Aubin and I independently proved that when $c_1(M) < 0$, there exists a unique KE metric.
- Bando-Mabuchi proved the uniqueness of KE metric (up to holomorphic isometry) on Fano manifolds.
- Donaldson and Chen-Tian proved that cscK metric in a fixed Kähler class, if it exists, is unique (up to holomorphic isometry).

For Fano manifolds, there are obstructions for the existence of Kähler-Einstein metric due to Matsushima and Futaki.

The conjecture that stability is linked to the existence of Kähler-Einstein metric started in 1977 when I tried to compare my proof of existence of Kähler-Einstein metric with the work of Bogomolov and Miyaoka.

Bogomolov used the stability of bundle to prove Chern number inequalities for algebraic surfaces, which were sharpened by Miyaoka and I to $3c_2 \geqslant c_1^2$. I believe these works must be linked.

Hence I concluded that the existence of Hermitian Yang-Mills connection (or Hermite-Einstein connection) on a holomorphic bundle should be linked to stability of the bundle.

This was finally proved by Donaldson in the case of algebraic surface. He uses properties of the restriction of the bundle to curves and the theory of secondary characteristic classes.

The proof of Uhlenbeck-Yau for general Kähler manifolds is to perturb the equation and to study the limit of the perturbed equation as the perturbation parameter goes to zero. (Donaldson later extended his proof for algebraic manifolds.)

With such success, I made the conjecture of existence of Kähler-Einstein metric on Fano manifolds in terms of stability of the manifold.

The existence of Kähler metrics with constant scalar curvature (cscK) is the same as the statement that the Ricci form is harmonic. Hence when the Kähler class is proportional to the first Chern class, it implies that cscK metric is Kähler-Einstein.

More precisely, I conjecture that for an ample line bundle L, the existence of cscK metric in $c_1(L)$ should be equivalent to the stability of L in the GIT sense.

When $L = K^{-1}$, Chen-Donaldson-Sun proved that if a Fano manifold is K-stable then it admits a Kähler-Einstein metric. The converse statement was proved by Berman.

There are many ways to define stability of manifolds including the concepts of Chow stability or Hilbert stability.

First of all, one had to make sure that algebraic stability, which is defined by embeddings of algebraic manifolds into complex projective space, can be linked to the existence of Kähler-Einstein metric.

I proposed to prove that any Hodge metric on an algebraic manifold can be approximated by normalized Fubini-Study metric induced on the manifold through embedding of the manifold into complex projective space by high powers of an ample line bundle.

I asked Tian to follow this line of argument to finish the first step of my conjecture on the equivalence of stability of Fano manifolds with the existence of Kähler-Einstein metrics.

I suggested Tian to use my method with Siu on the uniformization of Kähler manifolds to produce peak functions to achieve such a goal. (The purpose of my paper with Siu was also embedding of Kähler manifolds.)

The PhD thesis of Tian did not deal with high regularity approximation, this was settled by my other student Ruan in a latter thesis.

Y.-Tian-Zelditch expansion

The method can be said to be an understanding of the works of Kodaira in the analytic setting.

The C^∞ approximation of Tian and Ruan was strengthened by Catlin and Zelditch independently to the asymptotic expansion of the Bergman kernel $B_k(x)$ as $k \to \infty$,

$$B_k = a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots,$$

where $a_1 = \frac{1}{2}$ scalar curvature was proved by Lu.

Balanced embedding

However, there is an ambiguity due to the action of complex projective group.

When I studied first eigenvalue of the Laplacian with Bourguignon and Peter Li, we found a good position for the embedding upon action by the projective group, which we called the balanced condition:

$$\int \sigma(M) \frac{z_i \bar{z}_j}{|z_0|^2 + \cdots + |z_N|^2} \omega^n = \frac{\mathrm{vol}(M)}{N+1} \delta_{ij}$$

for some $\sigma \in SL(N+1, \mathbb{C})$.

My former student Luo in MIT observed that for a polarized manifold (M, L), if there exists a metric on L such that the Bergman function of L^k is constant for some k, then it is Gieseker-Mumford stable.

A theorem of Shouwu Zhang says that the existence of a unique balanced embedding is equivalent to the manifold being Chow-Mumford stable.

In view of the importance of the work of Futaki who found obstruction for Kähler-Einstein metric when there is group action, I suggest to Tian to understand the embedding into projective space and try to generalize Futaki invariant to the projective group action in ambient space.

Some functionals related to my proof of Calabi conjecture can be used. Some invariants such as alpha invariants were defined during my discussion with Tian.

Finally some concept of stability called K-stability was created. But Tian's definition is analytical and was not desirable. Donaldson then found the right algebro-geometric definition of K-stability.

The condition of K-stability is not easy to check in general. (The work of Abban-Zhuang eased the checking of K-stability for Fano surfaces and 3-folds). It would therefore be interesting to prove the existence of balanced condition for high power embeddings of a Fano manifold implies existence of Kähler-Einstein metrics.

It is highly desirable to clarify the condition of K-stability so that it can be checked effectively.

By applying Y.-Tian-Zelditch expansion, Donaldson proved that for a polarized manifold (with discrete automorphism group), the existence of cscK metric implies the asymptotic Chow stability.

It is a major open problem (first noted by Ross-Thomas) as to whether K-stability implies asymptotic Chow stability.

Ricci-flat metric with singularities

Let us now discuss another topic that I started in 1977, right after my settlement of the Calabi conjecture in 1976.

This is the program of construction of Ricci-flat Kähler manifolds with certain singularities dictated by the volume form. In the same paper of the Calabi conjecture, I started to prove the existence of such metrics with conical singularities.

Ricci-flat metric on noncompact manifolds

At the same time, I started to tackle the problem of classifying complete Kähler metrics with zero Ricci curvature on noncompact manifolds. I described this program in the Helsinki congress in 1978.

The results I knew at that time actually include those examples provided by Calabi and Eguchi-Hanson. But I was aiming at a general classification program.

Compactification

At that time, I had a program to study compactification of complete Kähler manifolds based on geometric analysis:

Given a Kähler manifold M, we would like to find a compact Kähler manifold M' such that M is holomorphically embedded into M' such that $M' - M$ is a divisor.

The program was to avoid the more algebraic approach due to Satake, Baily, Borel and Serre. And it would cover much more general manifolds.

I suggested to Siu and we worked out the case when the manifold admits a complete Kähler metric with curvature pinched between two negative constants with finite volume. We proved in 1978 that by adding one point to each end, M' can be formed as a complex variety.

When N. Mok came to Princeton university as a lecturer and J.-Q. Zhong came to Princeton to be my postdoctoral fellow, I suggested to them to relax the curvature condition that Siu and I used. It was quite successful.

Later S.-K. Yeung removed some hypotheses in the work of Mok-Zhong, proving my conjecture for complete Kähler manifold with finite volume and bounded curvature.

However, the argument did not show how the metric behaves in a neighborhood of $M' - M$. Furthermore, it works only when M has finite volume.

For complete noncompact Calabi-Yau, I knew that its volume is infinite.

Assuming that the topology of the manifold is reasonable, I expect that M is a Zariski open set of another compact Kähler manifold M'.

The volume form has zero Ricci curvature and is locally given by the wedge product of a holomorphic n-form with its conjugate.

If the fundamental group is trivial, the holomorphic n-form can be globally defined. It is the section of the canonical line bundle K. It goes to infinity near the divisor.

Hence the inverse of it defines a holomorphic section of K^{-1} which vanishes at the divisor supported on $M' - M$.

In my talk at the Helsinki congress, I asserted that the converse is true under a reasonable condition on the divisor.

In other words, suppose K^{-1} of a compact Kähler manifold has a holomorphic section s which vanishes on a divisor D, the complement $M' - D$ should admit a complete Ricci-flat Kähler metric if certain positivity condition holds for D.

I knew the key ingredient to handle this problem at least when D is nonsingular with possibility of high multiplicities, and reasonable positivity for the normal bundle of D, along the line of my paper with Cheng on Kähler-Einstein metric with negative scalar curvature.

The general case when D is singular is more tough. The only problem is to construct some ansatz of a model Kähler metric in a neighborhood of D.

If the singularity of D is very special, it can be done. But when D is assumed to be normal crossing, there is no good ansatz.

The major question is:

Question

Let $\bar{X}$ be a smooth n-dimensional Fano manifold, and $D = \sum D_i$ be an anticanonical divisor, which we assume to be simple normal crossing, but allow multiplicities for the components. Let $M = M' - D$, then M has a nowhere vanishing holomorphic volume form Ω. When does M admit a complete Calabi-Yau metric?

I asked Tian to write up the detail of my idea when D is nonsingular. It turns out to be useful for many occasions.

When the anticanonical divisor at infinity is nonsingular with multiplicity one, we constructed complete Ricci-flat metric whose volume growth has the order $R^{2n/(n+1)}$, where n is the complex dimension of the manifold and R is the radius of the geodesic ball.

Conversely if a complete irreducible Ricci-flat manifold of complex dimension n has volume growth at most $R^{2n/(n+1)}$, would it be compactified to a compact Kähler manifold with nonsingular anticanonical divisor?

In order for $M' - D$ to admit a complete Ricci-flat Kähler metric, the pair (M', D) satisfies a stability condition. This condition should be put in as hypothesis for the existence.

So far this has not been carried out, except in the case of Sasakian-Einstein manifold where the cone over it is a Calabi-Yau cone. (This very interesting work is due to T. Collins and G. Szekelyhidi)

If X is an affine variety with an isolated singular point, one can ask whether X admits a Ricci flat Kähler cone metric. Fix the vector field ξ on X that gives the homothetic scaling on the cone.

Theorem (Collins-Szekelyhidi, 2017)

Let (X, ξ) be a normalized Fano cone singularity. Then (X, ξ) admits a Ricci flat Kähler cone metric if and only if it is K-stable.

Overall, for a long time, I expect the following picture to be true:

For classification of complete noncompact CY manifolds with finite topology, we should look at the category

$$C(n) = \{\text{stable pairs } (M, s)\},$$

where M is n-dimensional Kähler variety and s is a holomorphic section of the anticanonical line bundle K^{-1} with zero divisor D.

Two such pairs (M, s) and (M', s') are considered to be equivalent if there is another pair (N, t) which has morphisms from N to M and M' respectively so that the pullback of s and s' would be t.

s^{-1} is a meromorphic n-form and its residue gives rise to a holomorphic section s' on D. We require the pair (D, s') belongs to $C(n-1)$.

The stability condition for such pairs amounts to certain algebro-geometric condition for $M - D$ to admit a complete Kähler metric with bounded curvature and whose volume form is asymptotic to the square norm of s^{-1}.

There should be a one-to-one correspondence between the two categories. Namely, a complete noncompact Calabi-Yau M will map to the complement of D in some compact Kähler variety.

This should give a guiding principle to classify complete Kähler CY metrics on complex Euclidean space: that it corresponds to compactification of the Euclidean space in a suitable manner.

If the divisor $s = 0$ is nonsingular, I knew the way to handle the problem in my talk in Helsinki in 1978. The detail was written up with Tian. But even in that case, some more details need to be explored.

The work of Tosatti, Collins-Tosatti and Collins-Szekelyhidi on degeneration of CY metrics and Sasaki-Einstein metrics should be very much relevant. There are also works by Conlon, Hein and others on conical CY manifolds.

Theorem (Collins-Tosatti, 2015)

Let X^n be a Calabi-Yau manifold and $[\alpha] \in \partial\mathcal{K}$ be a nef class with $\int_X \alpha^n > 0$. Then there is a smooth incomplete Ricci-flat Kähler metric ω_0 on $X \backslash \mathrm{Null}(\alpha)$ such that given any path $[\alpha_t]$, $0 < t \leqslant 1$, of Kähler classes approaching $[\alpha]$ as $t \to 0$, the Ricci-flat Kähler metrics ω_t in the class $[\alpha_t]$ converge to ω_0 as $t \to 0$ in the $C^\infty_{\mathrm{loc}}(X \backslash \mathrm{Null}(\alpha))$ topology.

Moreover, (X, ω_t) converges as $t \to 0$ to the metric completion of $(X \backslash \mathrm{Null}(\alpha), \omega_0)$ in the Gromov-Hausdorff topology.

There were some more progress due to construction of different ansatz. A first nonflat complete Calabi-Yau metric on complex Euclidean space was due to an observation of LeBrun that the Euclidean version of Taub-NUT metric gives such examples.

It will be interesting to find a suitable example of compactification associated to each Taub-NUT metric.

LeBrun showed that the Taub-NUT metric on $\mathbb{C}^2$ has the the same volume form as the standard metric, yet it only has cubic volume growth. Yang Li constructed a complete Ricci-Flat metric on $\mathbb{C}^3$ with maximal volume growth.

One can ask about the classification problem of Calabi-Yau metrics on $\mathbb{C}^n$ with maximal volume growth, at least when the volume form is fixed.

If my above program is correct, the compactification of $\mathbb{C}^n$ would give many different type of complete Ricci-flat metrics that are different from each other.

There are examples of complete noncompact CY manifolds due to various authors (such as H.-J. Hein, Y. Li, Conlon-Rochon). But these constructions are rather special, either on surface or $\mathbb{C}^n$.

For the more general situations, the recent works of Yang Li and T. Collins are significant.

Theorem (Li-Collins)

Let $\bar{X}$ be a smooth n-dimensional Fano manifold with $n \geqslant 3$, whose anticanonical bundle is $(d_1 + d_2)L_0$ for some positive line bundle L_0, and d_1, d_2 are positive integers. Let D_1, D_2 be two transversally intersecting smooth divisors in the linear system associated to $d_1 L_0$ and $d_2 L_0$ respectively. Then $X = \bar{X} \setminus D_1 \cup D_2$ admits a complete Calabi-Yau metric.

Their strategy to construct the Calabi-Yau metric on noncompact manifolds is

1. first construct an approximate metric by solving a non-archimedean Monge-Ampère equation, which is reminiscent of the semiflat metric in SYZ conjecture;
2. suitably improve the decay rate of the volume form error;
3. run a noncompact version of the proof of the Calabi-Yau Theorem to find a completion of the approximate metric.

As mentioned above, I conjectured that under some mild conditions, a complete noncompact Calabi-Yau manifolds should admit a (quasi-)projective compactification.

For noncompact Calabi-Yau, we have the explicit Ricci-flat metrics of Eguchi-Hanson and Taub-NUT, but at present there are no closed form expressions for the metric on a nontrivial compact Calabi-Yau.

Analytic or numerical expressions of Calabi-Yau and KE metrics are useful in string theory.

Numerical study of Calabi-Yau metric

Numerical approximations to Calabi-Yau metric was first obtained by Headrick-Wiseman on K3 surface.

Donaldson had a nice numerical approach to Calabi-Yau metric by applying the Y.-Tian-Zelditch expansion to show that the balanced metric converges to the Ricci-flat metric.

Doran et. al. obtained numerical Kähler-Einstein metric with positive scalar curvature on the third del Pezzo surface.

Toric variety

Toric variety is an important playground to study Kähler-Einstein metrics since the work of Mabuchi and Donaldson. In fact, K-stability was originally introduced and studied by Donaldson for toric varieties.

In the toric setting, K-stability, test configuration and Futaki invariant can be described in terms of convex functions on moment polytopes. The alpha invariants of toric variety can be computed relatively easily.

In my first paper on the Calabi conjecture, I have already started to discuss Kähler-Einstein metric with degeneracy along a divisor D.

In general, there should be residue of the Kähler-Einstein metric on D. We could prescribe degeneracy condition of the metric on D and prove the existence of Kähler-Einstein metric, which should be unique.

In that same paper, I consider those Kähler metric blowing up weakly along the divisor.

This would correspond to s being holomorphic sections of multi-anticanonical line bundle nondegenerate along the zero locus.

I proved the existence of Ricci-flat Kähler metric in a rather general situation at that time. I think one can push it to the optimal case with the modern development of techniques.

However, the detailed information of the metric along the divisor is not well understood. Once we understand them, there should be a great deal of applications to algebraic geometry.

Thank you!