Transfer matrices, Riemann surfaces and topological phases of matter

(VD, V Chua, PRB 93 (2016) 134304)

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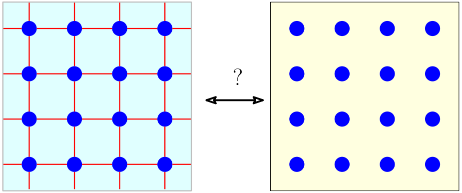
Gapped in bulk, but distinguished from trivial (atomic) insulator by "topology"



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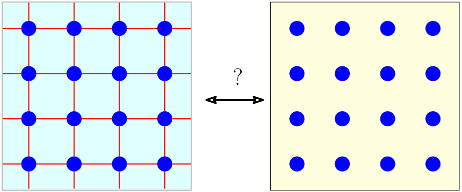
$\Updownarrow$

Cannot be continuously deformed into an atomic insulator without closing the bulk gap.



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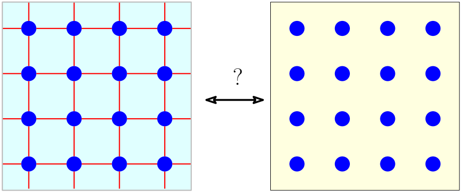
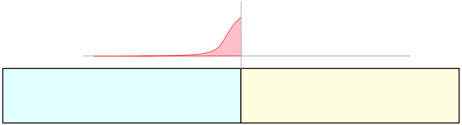


Exponentially localized modes at interface with atomic insulator, which cannot be removed by local deformations.



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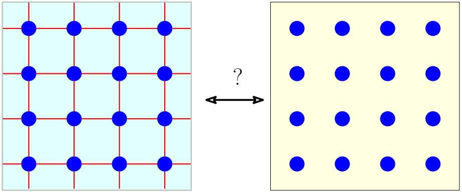

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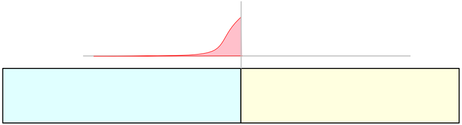
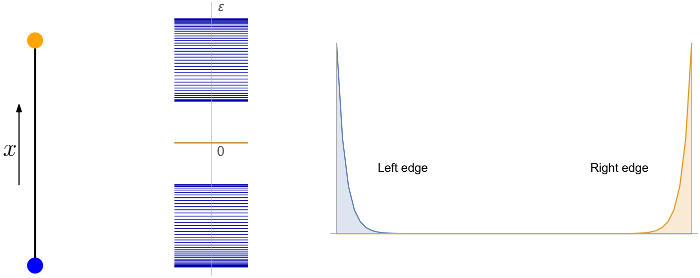


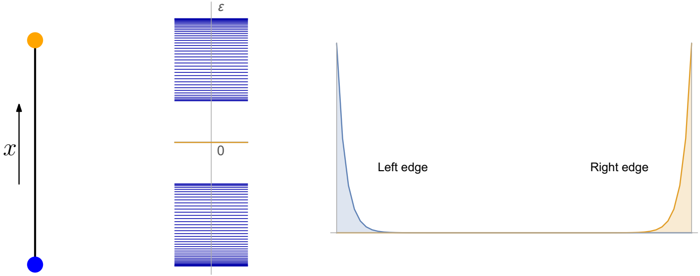
Described by single particle Hamiltonians without interactions:

$$\mathcal{H}|\psi\rangle = \varepsilon|\psi\rangle, \qquad |\psi\rangle = \sum_{\mathbf{n},\alpha} \psi_{\mathbf{n},\alpha} \, |\mathbf{n}, \alpha\rangle.$$

The Lore

Example: 1D Topological Insulator

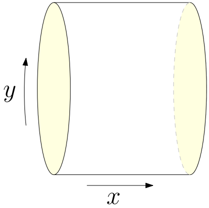


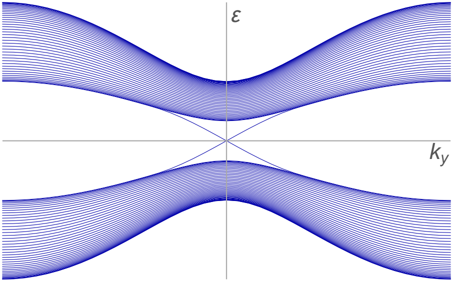


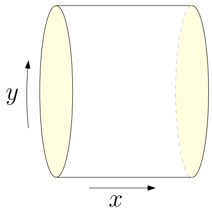
Topological invariant = Net Berry phase across the bulk Brillouin zone

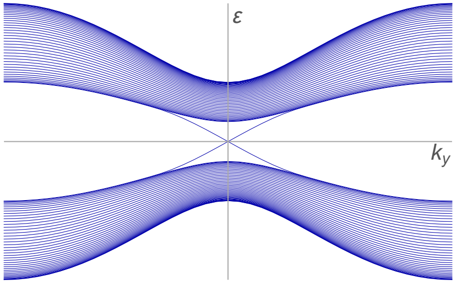
$$\mathfrak{C} = \frac{1}{2\pi}\oint_{S^1} \mathfrak{a} \in \mathbb{Z}, \qquad \mathfrak{a} \equiv -i\langle\widetilde{\psi}_k|d\widetilde{\psi}_k\rangle.$$

Winding number (Brouwer degree) of the map $BZ \cong S^1 \to S^1 : k \mapsto \arg(\widetilde{\psi}_k)$


y
x


ε
k_y



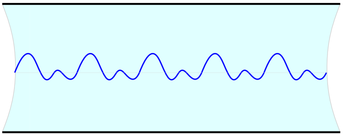


Topological invariant = Net Berry flux across the bulk Brillouin zone

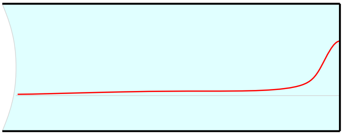
$$\mathfrak{C} = \frac{1}{2\pi}\oint_{\mathbb{T}^2} \mathfrak{F} \in \mathbb{Z}, \qquad \mathfrak{F} \equiv -i\langle d\widetilde{\psi}_{\mathbf{k}}| \wedge |d\widetilde{\psi}_{\mathbf{k}}\rangle.$$

First Chern number of a principal $\mathrm{U}(1)$ bundle over the Brillouin zone $BZ \cong \mathbb{T}^2$.

The bulk and the boundary

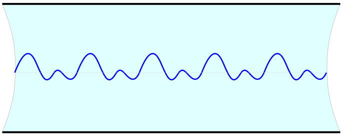


Bulk states: Momentum space
(Bloch's theorem)



Edge states: Position space
(Decaying ansatz solution)

The bulk and the boundary



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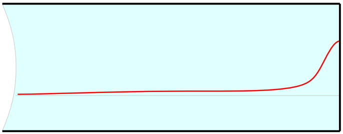
Bulk topological invariant



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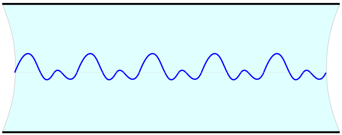

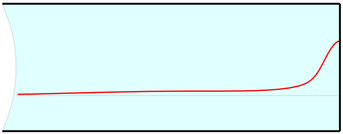

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Bulk topological invariant

Edge invariants

Bulk-boundary correspondence

e.g: Chern number = Number of "nontrivial" edge states on a given edge

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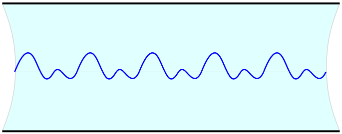

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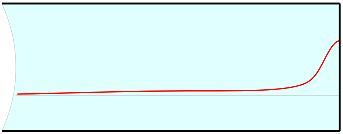

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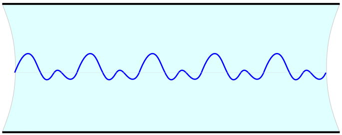
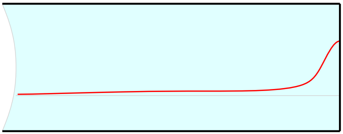
Treat edge and bulk on *an equal footing*??

The bulk and the boundary





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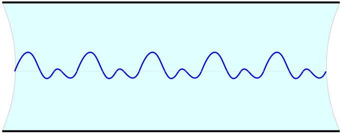
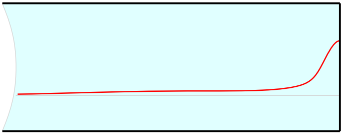
Treat edge and bulk on *an equal footing*??

"Transfer matrices"

The bulk and the boundary





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Treat edge and bulk on *an equal footing*?? "Transfer matrices"

- ◦ Analytically tractable treatment in position space.
- ◦ Potential algebraic proof of bulk-boundary correspondence(s)?

PHYSICAL REVIEW B VOLUME 48, NUMBER 16 15 OCTOBER 1993-II

Edge states in the integer quantum Hall effect and the Riemann surface of the Bloch function

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(Received 27 May 1992; revised manuscript received 1 April 1993)

We study edge states in the integral quantum Hall effect on a square lattice in a rational magnetic field $\phi = p/q$. The system is periodic in the y direction but has two edges in the x direction. We have found that the energies of the edge states are given by the zero points of the Bloch function on some Riemann surface (RS) (complex energy surface) when the system size is commensurate with the flux. The genus of the RS, $g = q - 1$, is the number of the energy gaps. The energies of the edge states move around the holes of the RS as a function of the momentum in the y direction. The Hall conductance σ_{xy} is given by the winding number of the edge states around the holes, which gives the Thouless, Kohmoto, Nightingale, and den Nijs integers in the infinite system. This is a topological number on the RS. We can check that σ_{xy} given by this treatment is the same as that given by the Diophantine equation numerically. Effects of a random potential are also discussed.

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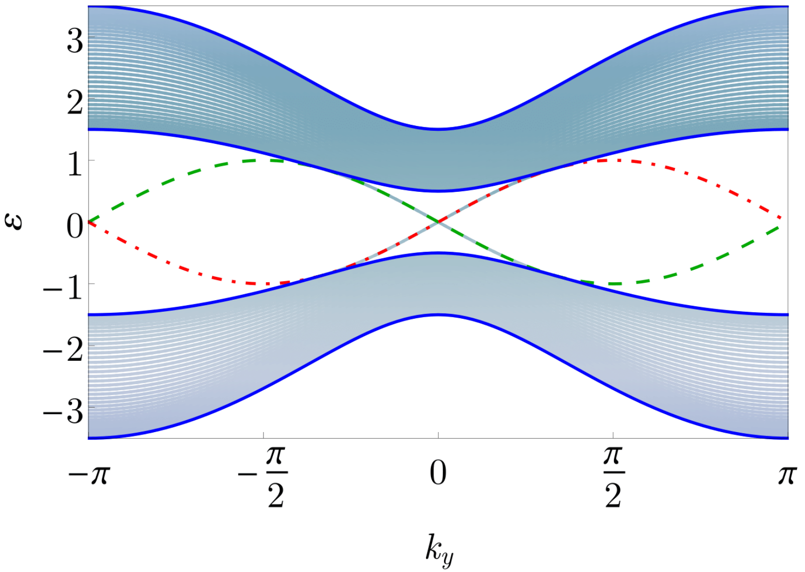

3
2
1
0
−1
−2
−3
ε
−π
−π/2
0
π/2
π
k_y

Transfer Matrices

30.05.2017

Formal setup:

Consider a family of vector spaces $\mathcal{V}_n \cong \mathbb{C}^{2r}$; $r \in \mathbb{Z}^+$, $n \in \mathcal{J} \subseteq \mathbb{Z}$

A transfer matrix is then a linear operator $T \in SL(2r, \mathbb{C})$. Given a vector $\phi_0 \in \mathcal{V}_0$, can use T to define $\phi_n \equiv T^n \phi_0 \in \mathcal{V}_n$. We are primarily interested in the asymptotics of $\|\phi_n\|$ as $n \to \pm\infty$. ($\mathcal{J} = \mathbb{Z}$)

But asymptotics of $\|\phi_n\| \sim$ spectrum of $T \equiv \sigma[T]$

↳ Let $T\phi_0 = \rho\phi_0$; $\|\phi_0\| = 1$. Then,

$$|\rho| = 1 \Rightarrow \|\phi_n\| = 1 \text{ as } n \to \pm\infty \quad : \text{"bulk"}$$

$$|\rho| \gtrless 1 \Rightarrow \|\phi_n\| \to 0 \text{ as } n \to \mp\infty \quad : \text{"edge"}$$

To explain what "bulk" and "edge" states mean, we consider an example:

- (Simple) Tight-binding model:

Hilbert space : $\mathcal{H} \equiv \text{span}\{|n\rangle\}_{n\in\mathcal{J}}$ $\equiv$ single particle states localized at site n.

dual: $\mathcal{H} \equiv \text{span}\{\langle n|\}_{n\in\mathcal{J}}$; $\langle m|n\rangle = \delta_{mn}$.

Consider the Hamiltonian $\mathcal{H} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{H} = \sum_{n\in\mathcal{J}} \left[t|n+1\rangle\langle n| + \mu|n\rangle\langle n| + t|n\rangle\langle n+1| \right] \quad ; \quad t, \mu \in \mathbb{R}$$

(for $t \in \mathbb{C}$, absorb the phase in $|n\rangle$.)

Clearly, $\mathcal{H}^\dagger = \mathcal{H} \Rightarrow \sigma[\mathcal{H}] \subset \mathbb{R}$.

We seek to solve the (time-independent) Schrödinger equation

$$\mathcal{H}|\psi\rangle = \varepsilon|\psi\rangle \quad ; \quad |\psi\rangle \equiv \sum_{n\in\mathcal{J}} \psi_n |n\rangle \quad ; \quad \psi_n \in \mathbb{C}.$$

Substituting $\mathcal{H}$ and $|\psi\rangle$,

$$\sum_{m,n} \left[t|m+1\rangle\langle m| + \mu|m\rangle\langle m| + t|m\rangle\langle m+1| \right] \psi_n |n\rangle = \sum_n \varepsilon \psi_n |n\rangle$$

$$\Rightarrow \quad t\psi_{n+1} + \mu\psi_n + t\psi_{n-1} = \varepsilon\psi_n \quad \forall\, n \qquad \text{(not true at boundaries)}$$

$$\Rightarrow \quad \psi_{n+1} = \frac{\varepsilon-\mu}{t}\psi_n - \psi_{n-1}$$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix}}_{\phi_{n+1}} = \underbrace{\begin{pmatrix} \frac{\varepsilon-\mu}{t} & -1 \\ 1 & 0 \end{pmatrix}}_{T(\varepsilon)} \underbrace{\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}}_{\phi_n} \quad \Rightarrow \phi_{n+1} = T\phi_n \;;\; \phi_n \in \mathbb{C}^2 = \mathcal{V}_n \;\; (r=1)$$

$$\Rightarrow \phi_n = T^n \phi_0$$

Clearly, $T(\varepsilon) \in SL(2,\mathbb{C})\ \forall\, \varepsilon$. What can we compute with this?

Consider the eigenvalue problem $T\phi_0 = \rho\phi_0$; $\phi_0 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$

$$\therefore \begin{pmatrix} \frac{\varepsilon-\mu}{t} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \rho \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\varepsilon-\mu}{t}\beta - \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \rho\beta \\ \rho\alpha \end{pmatrix}$$

$$\therefore \beta = \rho\alpha \Rightarrow \phi_0 = \alpha \begin{pmatrix} \rho \\ 1 \end{pmatrix}$$

$$\therefore \phi_n = T^n\phi_0 = \rho^n\phi_0 = \alpha\begin{pmatrix} \rho^{n+1} \\ \rho^n \end{pmatrix}. \text{ But } \phi_n = \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \Rightarrow \psi_n = \alpha\rho^{n+1}$$

$$\therefore |\psi\rangle = \alpha\rho\sum_n \rho^n |n\rangle \Rightarrow \text{But } |\psi_n|^2 \propto \text{probability of particle being on site } n!$$

- for $|\rho| = 1$, define $\rho = e^{ik}$; $k \in (-\pi, \pi]$

$$\Rightarrow |\psi\rangle = \# \sum_{n\in\mathcal{J}} e^{ikn} |n\rangle \qquad \text{"Bloch wave"}$$

$\Rightarrow$ for $\mathcal{J} = \mathbb{Z}$ (infinite system), we only have these states (with a delta function normalization)

- For $|\rho| \gtrless 1$, the state is not normalizable on $\mathbb{Z}$, but is normalizable on half lines!

  → $|\rho| < 1 \Rightarrow |\psi_n| \to 0$ as $n \to \infty$

  ↳ Taking $\mathcal{J} = \mathbb{Z}^+$, $|\psi\rangle = \sum_n \rho^n |n\rangle$ is localized near $n=0$ "left edge"

  → $|\rho| > 1 \Rightarrow |\psi_n| \to 0$ as $n \to -\infty$: "right edge"

Aside: Symmetry

translation invariance ≡ "momentum is a good quantum number"

↳ symmetry group (of $\mathcal{H}$) = $\mathbb{Z}$ for $\mathcal{J} = \mathbb{Z}$

↳ abelian ⇒ all unitary irreps are 1D, labelled by characters $\chi = e^{ik} \in S^1$.

∴ Each eigenstate of $\mathcal{H}$ carries a 1D representation of $\mathbb{Z}$, and k labels the corresponding character.

- For $\mathcal{J} \subset \mathbb{Z}$, no symmetry group (but semigroup ?) and $\chi \notin S^1$ anymore: Representations of _ ?

Thus, to study this system, we should look at eigenvalues of T.
But $\det T = 1$, define $\Delta(\varepsilon) \equiv \mathrm{tr}\, T(\varepsilon)$ to get

$$\rho_\pm = \frac{+\Delta \pm \sqrt{\Delta^2 - 4}}{2} \quad : \quad |\rho| = 1 \text{ if } \Delta^2 - 4 \leq 0$$

The quantity $\Delta(\varepsilon)$ is the Floquet discriminant.

↳ Generically a polynomial in ε.

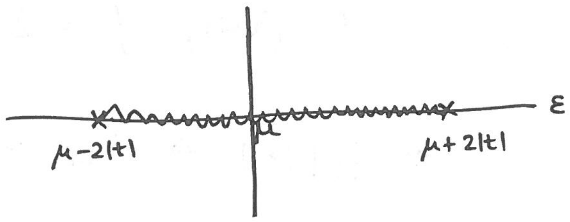
Allowing ε to be complex, it must live on a Riemann surface R with map $R \to \mathbb{C} : \varepsilon \mapsto \frac{+\Delta \pm \sqrt{\Delta^2 - 4}}{2}$.

More explicitly, consider the tight binding model, where

$$\Delta = \frac{\varepsilon - \mu}{t} \Rightarrow \Delta^2 - 4 = 0 \text{ for } \varepsilon = \mu \pm 2|t|$$

$\exists$ a square root branch cut

↳ Define it for $\varepsilon \in (\mu - 2|t|, \mu + 2|t|)$



$\therefore$ Bulk band $\Leftrightarrow$ Branch cut

(Riemann - Hurwitz)

Genus of $\mathcal{R}$ = (# of $\sqrt{\ }$ - branch cuts) − 1

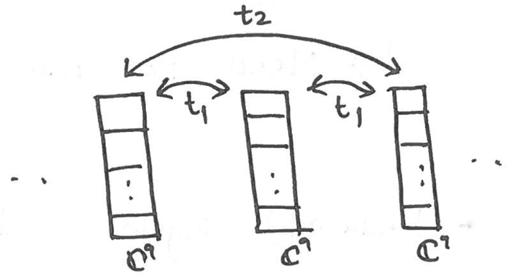
= (# of bulk bands) − 1 = 0 (for this case)

$\Rightarrow \mathcal{R} \cong S^2$, $\pi_1(S^2) = 0 \Rightarrow$ no noncontractible loops.

- More generally, the Floquet discriminant is a polynomial in ε of order $N \Rightarrow N$ bulk bands $\Rightarrow N$ branch cuts

$\therefore \exists$ an associated Riemann surface of genus $N-1$.

GENERALIZATION :



Consider a vector space $\mathbb{C}^q$; $q < \infty$ associated with each lattice site n

↳ physically spin/orbitals/ . . .

Also consider finite range hopping, so that

$$\mathcal{H} = \sum_{n \in \mathcal{J}'} \sum_{\alpha\beta=1}^{q} \left[\sum_{l=1}^{R} \left(t_{l,\alpha\beta} |n+l, \alpha\rangle\langle n, \beta| + t^*_{l,\alpha\beta} |n,\alpha\rangle\langle n+l, \beta| \right) + \mu_{\alpha\beta} |n,\alpha\rangle\langle n,\beta| \right]$$

where $\mathcal{H} : \mathscr{H} \to \mathscr{H}$, $\mathscr{H} = \text{span}\{ |n,\alpha\rangle \}_{n \in \mathcal{J}, \alpha=1..q}$

Group these sites into supercells, so that $\exists$ only nearest neighbor hopping between the supercells to get

$$\mathcal{H} = \sum_{n \in \mathcal{J}} \sum_{ab=1}^{N} \left[J_{ab} |n+1, a\rangle\langle n, b| + M_{ab} |n,a\rangle\langle n,b| + J^*_{ab} |n,a\rangle\langle n+1,b| \right]$$

⇒ Can be thought of as a "basis transformation" on $\mathscr{H}$.

The Schrödinger equation becomes

$$J\psi_{n+1} + M\psi_n + J^\dagger \psi_{n-1} = \varepsilon \psi_n$$

If J is nonsingular, can write

$$\psi_{n+1} = J^{-1}(\varepsilon \mathbb{1} - M)\psi_n - J^{-1}J^\dagger \psi_{n-1}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix}}_{\phi_{n+1}} = \begin{pmatrix} J^{-1}(\varepsilon\mathbb{1} - M) & -J^{-1}J^\dagger \\ \mathbb{1} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}}_{\phi_n} \quad ; \phi_n \in \mathbb{C}^{2N} \; (r = N)$$

What happens when J is singular?

↳ Need to quotient out $\ker J$ and invert on $\mathbb{C}^N / \ker J$.

- (Reduced) Singular value decomposition:

$$J = V \Xi W^\dagger \quad ; \quad V = (v_1 \ldots v_r), \quad W = (w_1 \ldots w_r)$$

$\downarrow$

$r \equiv \text{rank } J. \qquad \Xi = \text{diag}\{\xi_1 \ldots \xi_r\} \; ; \; \xi_i > 0 \; \forall i$

$$\therefore \quad J = \sum_{i=1}^{r} \xi_i v_i w_i^\dagger \quad ; \quad \ker J \oplus \text{span}\{w\} = \mathbb{C}^N \; ?$$

Would like to construct an orthonormal basis of $\mathbb{C}^N$ using these vectors (v's, w's) : Need an extra condition

$$J^2 = 0 \iff \langle v_i, w_j \rangle = 0 \quad \forall i, j$$

Can always do this by enlarging the supercell. Thus, $r \leq \frac{N}{2}$, and form the basis

$$\{ \underbrace{v_1 \ldots v_r}_{V}, \underbrace{w_1 \ldots w_r}_{W}, \underbrace{x_1 \ldots x_{N-2r}}_{X} \}$$

so that

$$V^\dagger V = W^\dagger W = \mathbb{1}_r, \quad X^\dagger X = \mathbb{1}_{N-2r}, \quad V^\dagger W = V^\dagger X = W^\dagger X = 0.$$

Given $\psi \in \mathbb{C}^N$, expand in this basis as

$$\psi = \sum_{i=1}^{r} v_i \underbrace{\langle v_i, \psi \rangle}_{\alpha_i} + \sum_{i=1}^{r} w_i \underbrace{\langle w_i, \psi \rangle}_{\beta_i} + \sum_{i=1}^{N-2r} x_i \underbrace{\langle x_i, \psi \rangle}_{\gamma_i}$$

$$= V\alpha + W\beta + X\gamma \quad ; \quad \alpha, \beta \in \mathbb{C}^r, \; \gamma \in \mathbb{C}^{N-2r}$$

so that $\alpha = V^\dagger \psi$, $\beta = W^\dagger \psi$ etc.

The recursion relation becomes

$$J\psi_{n+1} + J^\dagger \psi_{n-1} = (\varepsilon \mathbb{1} - M)\psi_n$$

$\Rightarrow$ Invert $\varepsilon\mathbb{1} - M$ and define $\mathcal{G} \equiv (\varepsilon\mathbb{1} - M)^{-1}$: resolvant (singular on $\sigma[M]$)

to get

$$\psi_n = \mathcal{G} J \psi_{n+1} + \mathcal{G} J^\dagger \psi_{n-1}$$

But

$$J\psi_{n+1} = V \Xi W^\dagger (V\alpha_{n+1} + W\beta_{n+1} + X\gamma_{n+1}) = V\Xi\beta_{n+1}$$

$$J^\dagger\psi_{n-1} = W\Xi V^\dagger (V\alpha_{n-1} + W\beta_{n-1} + X\gamma_{n-1}) = W\Xi\alpha_{n-1}$$

so that

$$\psi_n = \mathcal{G}V.\Xi\beta_{n+1} + \mathcal{G}W.\Xi\alpha_{n-1}$$

Take projections along V, W to get

$$\alpha_n = V^\dagger\psi_n = \mathcal{G}_{VV}\cdot\Xi\beta_{n+1} + \mathcal{G}_{WV}\cdot\Xi\alpha_{n-1}$$

$$\beta_n = W^\dagger\psi_n = \mathcal{G}_{VW}\cdot\Xi\beta_{n+1} + \mathcal{G}_{WW}\cdot\Xi\alpha_{n-1}$$

Rearrange to get

$$\underbrace{\begin{pmatrix}\beta_{n+1}\\ \alpha_n\end{pmatrix}}_{\phi_{n+1}} = T\underbrace{\begin{pmatrix}\beta_n\\ \alpha_{n-1}\end{pmatrix}}_{\phi_{n-1}} \; ; \; T = \begin{pmatrix}\Xi^{-1}\mathcal{G}_{VW}^{-1} & -\Xi^{-1}\mathcal{G}_{VW}^{-1}\mathcal{G}_{WW}\Xi \\ \mathcal{G}_{VV}\cdot\mathcal{G}_{VW}^{-1} & (\mathcal{G}_{WV} - \mathcal{G}_{VV}\mathcal{G}_{VW}^{-1}\mathcal{G}_{WW}).\Xi\end{pmatrix}$$

where by construction, $\mathcal{G}_{VV}^\dagger = \mathcal{G}_{VV}$, $\mathcal{G}_{WW}^\dagger = \mathcal{G}_{WW}$, $\mathcal{G}_{VW}^\dagger = \mathcal{G}_{WV}$.

Also, $\phi_n \in \mathbb{C}^{2r}$ (instead of $\mathbb{C}^{2N}$), where $r = \text{rank}\, J$

↳ Use rank(J) to identify the relevant (propagating) degrees of freedom.

Real symplectic : $T \in Sp(2r, \mathbb{R}) \Rightarrow [\mathcal{G}_{ab}, \Xi] = 0$; $a, b \in \{v, w\}$

If that is the case, then eigenvalues come in reciprocal pairs $\{\rho, \frac{1}{\rho}\}$. Let the characteristic polynomial be

$$P(\rho) = \sum_{n=0}^{2r} a_n \rho^n = 0 = P\left(\frac{1}{\rho}\right) = \sum_{n=0}^{2r} \frac{a_n}{\rho^n}$$

$\Rightarrow P(x)$ is palindromic, i.e. $a_i = a_{2r-i}$, so that

$$P(x) = \sum_{n=0}^{r-1} a_n \rho^n + a_r \rho^r + \sum_{n=r+1}^{2r} a_{2r-n} \rho^n$$

$$= \sum_{n=0}^{r-1} a_n (\rho^n + \rho^{2r-n}) + a_r \rho^r$$

$$= \rho^r \left[a_r + \sum_{n=0}^{r-1} a_n \left(\rho^{r-n} + \rho^{-(r-n)} \right) \right]$$

Define $\rho + \rho^{-1} = \Delta \Rightarrow \rho^n + \rho^{-n} = 2T_n\left(\frac{\Delta}{2}\right)$: Chebyshev polynomial

$$\Rightarrow P(x) . \rho^{-r} = a_r + \sum_{n=0}^{r-1} 2a_n T_{r-n}\left(\frac{\Delta}{2}\right) = 0$$: Solve for Δ

$\Rightarrow r$ solutions $\Delta_1 .. \Delta_r$

$$\therefore \quad \rho_{n,\pm} = \frac{1}{2}\left[\Delta_n \pm \sqrt{\Delta_n^2 - 4} \right] \quad ; n = 1 .. r$$

$\Rightarrow$ Decompose the system into r chains.

Aside : Homology ?

$J^2 = 0 \Rightarrow \quad \ldots C_{n-1} \xrightarrow{J} C_n \xrightarrow{J} C_{n+1} \longrightarrow \ldots$ is a chain complex, with $C_n = \mathbb{C}^N \; \forall n$. Clearly, $\text{im } J \cong \mathbb{C}^r$, $\ker J \cong \mathbb{C}^{2N-2r}$

$$\Rightarrow H_n(C_*) = \frac{\ker J}{\text{im } J} \cong \mathbb{C}^{N-2r}$$

By SVD, we are essentially quotienting out this homology ?

Chern Insulator:

Easiest to state as the Bloch Hamiltonian

$$\mathcal{H}_{Bloch} = \sin k_x \, \sigma^x + \sin k_y \, \sigma^y + (2 - m - \cos k_x - \cos k_y)\sigma^z$$

where σ's are Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The bulk chern number is

$$C = \begin{cases} +1 & ; \; 0 \leq m \leq 2 \\ -1 & ; \; 2 < m < 4 \\ 0 & ; \; \text{otherwise} \end{cases} \quad \Big\} \text{ Gap closes for } m = 0, 2, 4$$

For finite strip along x, inverse Fourier transform and compute the Schrödinger equation as

$$J \psi_{n+1} + M \psi_n + J^\dagger \psi_{n-1} = \varepsilon \psi_n$$

with

$$J = \frac{1}{2i}(\sigma^x - i\sigma^z) = \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} \quad ; \quad \text{rank } J = 1$$

$$M = \sin k_y \, \sigma^y + \Lambda(k_y)\,\sigma^z \quad ; \quad \Lambda(k_y) \equiv 2 - m - \cos k_y$$

The transfer matrix becomes

$$T = \frac{1}{|\Lambda|}\begin{pmatrix} -\varepsilon^2 + \Lambda^2 + \sin^2 k_y & \varepsilon - \sin k_y \\ -(\varepsilon + \sin k_y) & 1 \end{pmatrix}$$

Floquet discriminant

$$\Delta(\varepsilon, k_y) = \frac{1}{|\Lambda|}(1 - \varepsilon^2 + \Lambda^2 + \sin^2 k_y) \quad ; \quad \text{band edges: } \Delta^2 = 4$$

For edge states, impose Dirichlet boundary condition

Left: $\phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T\phi_0 = \rho\phi_0 \Rightarrow \begin{pmatrix} \# \\ -(\varepsilon + \sin k_y) \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$

$\Rightarrow \varepsilon_L = -\sin k_y$

Right: $\phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\phi_0 = \rho\phi_0 \Rightarrow \begin{pmatrix} \varepsilon - \sin k_y \\ \# \end{pmatrix} = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

$\Rightarrow \varepsilon_R = \sin k_y$

Consider the "left" one : $\rho_L = \left(-\varepsilon^2 + \Lambda^2 + \sin^2 k_y\right)\Big|_{\varepsilon = -\sin k_y} = \Lambda^2$

$= (2 - m - \cos k_y)^2$

$\exists$ a Riemann surface $\forall\, k_y$, but they all have genus 1

$\Rightarrow$ Map to a single Riemann surface (by rescaling)

$|\rho_L| = 1$ for $2 - m - \cos k_y = \pm 1$

$\Rightarrow \cos k_y = (2 \mp 1) - m$

Take $m \in (0, 2) \Rightarrow k_y = \pm \cos^{-1}(1 - m) \equiv \pm\theta$

$\therefore$ for $-\theta < k_y < \theta$, ε on top sheet

↳ cross over at $k_y = \pm\theta$.

Top sheet : $|\rho| > 1$

Bottom sheet : $|\rho| < 1$

$\therefore$ $\varepsilon_L(k_y)$ is noncontractible on R (with winding number $+1$)

$\varepsilon_R(k_y)$ winds the other way round.

Finally : $k_y \mapsto T(\varepsilon_L(k_y), k_y)$ is a curve on $Sp(2, \mathbb{R})$ and

$\pi_1(Sp(2, \mathbb{R})) \cong \mathbb{Z} \Rightarrow$ Associated winding # $\equiv$ Maslov index