New critical and collective phenomena in the constrained random networks

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Outline of the talk

- Motivation

- .Numerical findings for constrained Erdos-Renyi network(CERN) and regular random graph (RRG)

- Spectral analysis. Eigenvalue tunneling and the ground state of regular random networks. New reincarnation of the eigenvalue instanton. Visualization of interplay «perturbative versus nonperturbative»

- Anderson one-particle localization on the CERN and RRG and many-body localization

- Conclusion

Motivation I

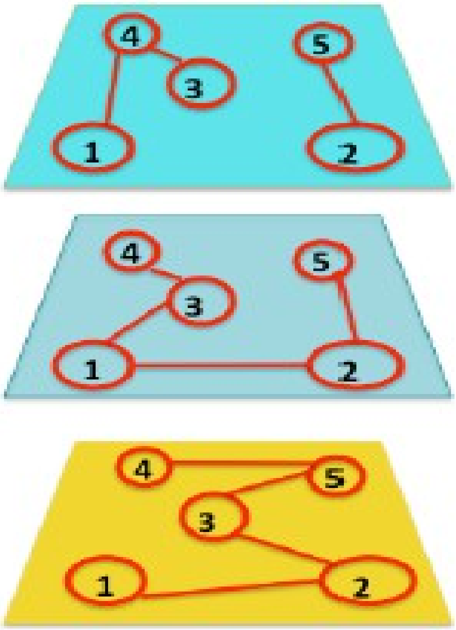
- The degree of the vertex in the «instanton liquid» is fixed by topology.Standard condition in chemistry and biology.
- Chiral condensate in QCD. How quasi-zero modes of Dirac operator in the instanton-antiinstanton ensemble get collectivized?
- The lattice QCD clearly demonstrates that restoration of the chiral symmetry at high temperature is the Anderson transition for the Dirac operator spectrum
- This localization phase transition in QCD involves the creation of the extended objects

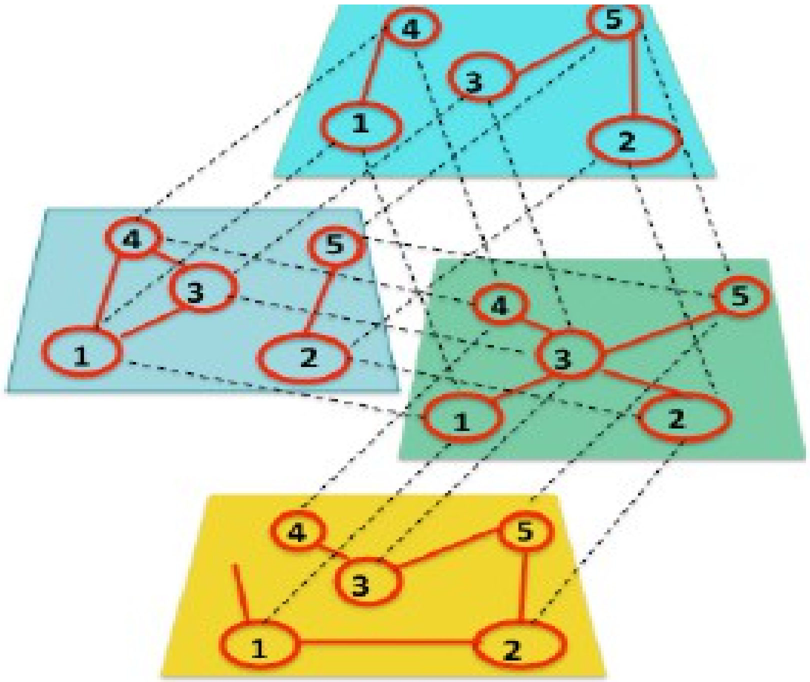
Motivation II

- Random matrix model — suitable playground for many nonperturbative and critical phenomena in gauge theory and 2d gravity
- Simplest nonperturbative phenomena- eigenvalue tunneling (selection of contour, Stokes lines etc) David 93, Marino-Shiappa-Vonk-Russo 2004-.... and many others
- Random matrix model- theory on unstable branes, tunneling - creation of stable brane, baby-universe (Verlinde-Mcgreevy, Klebanov-Maldacena-Seiberg, Gaiotto-Rastelli, Vafa et al)
- Can the random network be the laboratory for the investigation of nonperturbative effects?

Motivation III. Multilayer networks

- Multilayer or in more general case tensor networks — playground for discrete version of holographic duality
- How multilayear networks emerge?
- How interaction between the layers depends on the parameters of networks?
- How the external probe behaves on the multilayer networks
- Many real-world networks and all networks in brains (connectome) are multilayer





Examples of multilayer networks. Each layer can be network,of different type

For example   Erdos-Renyi and scale-free networks at different layers

Experimental data for one color

- The CERN network or RRG
- The degree of the network is conserved- constraint (standard in biology and chemistry, instanton charge in instanton liquid)
- Chemical potential for the triangles.
- Rich structure of the phase transitions. Cliques formation(=almost complete graph)
- For CERN network [1/p] cliques emerge

  (average number of links at vertex equals pN)

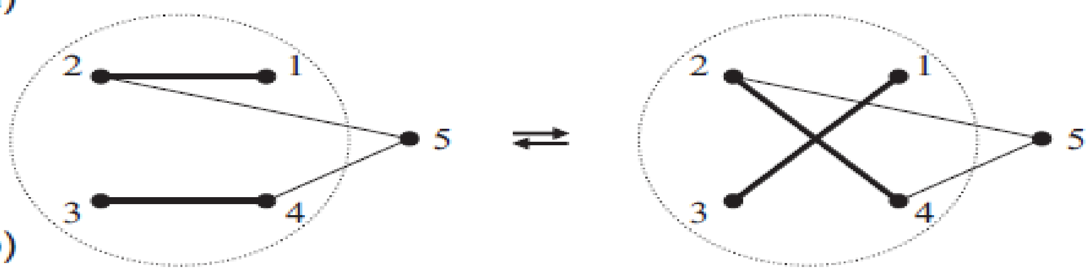
Model

$$H = -\mu N_\Delta, \qquad (\mu > 0)$$ (= Tr A^3, A -adjacency matrix)

- The possible moves in the network(= kinetic term)

undirected subgraphs-triads	[0]	[1]	[2]	[3]
concentration	c_0	c_1	c_2	c_3

(a)



(b)

Figure 1: a) Possible triads in a non-directed network; b) Single link permutation: links (12) and (34) are removed, and links (13) and (24) are created. Triad {135} goes from type [0] to type [1], triads {125, 345} – from type [2] to type [1], and triad {245} – from type [2] to type [3]: three new triads of type [1] and one triad of type [3] are created instead of three triads of type [2] and one of type [0], compare to Eq.(1).

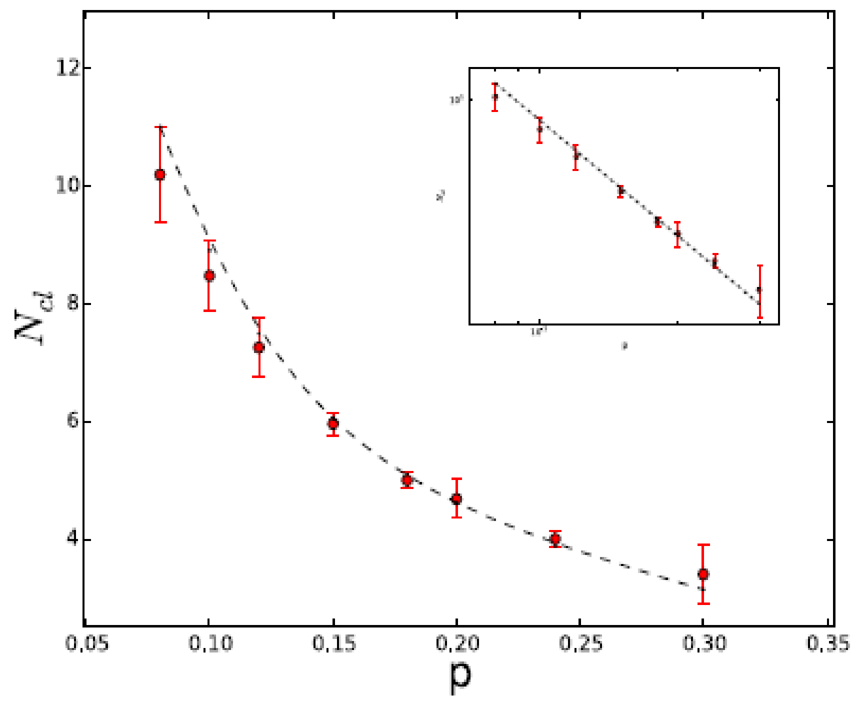


FIG. 1: The number of clusters N_{cl} as a function of the probability p in ER graph. The numerical data are obtained by averaging over 100 randomly generated graphs up to 512 vertices. Numerical values are fitted by the curve $p^{-0.95}$; the behavior in doubly logarithmic scale is shown in the insert.

The network is completely defragmented into the finite number of weakly coupled dense droplets above the critical point. Both for CERN and RRG

Model without degree conservation — Strauss model (1986), solved
In mean field approximation (Newman,Park -2004)

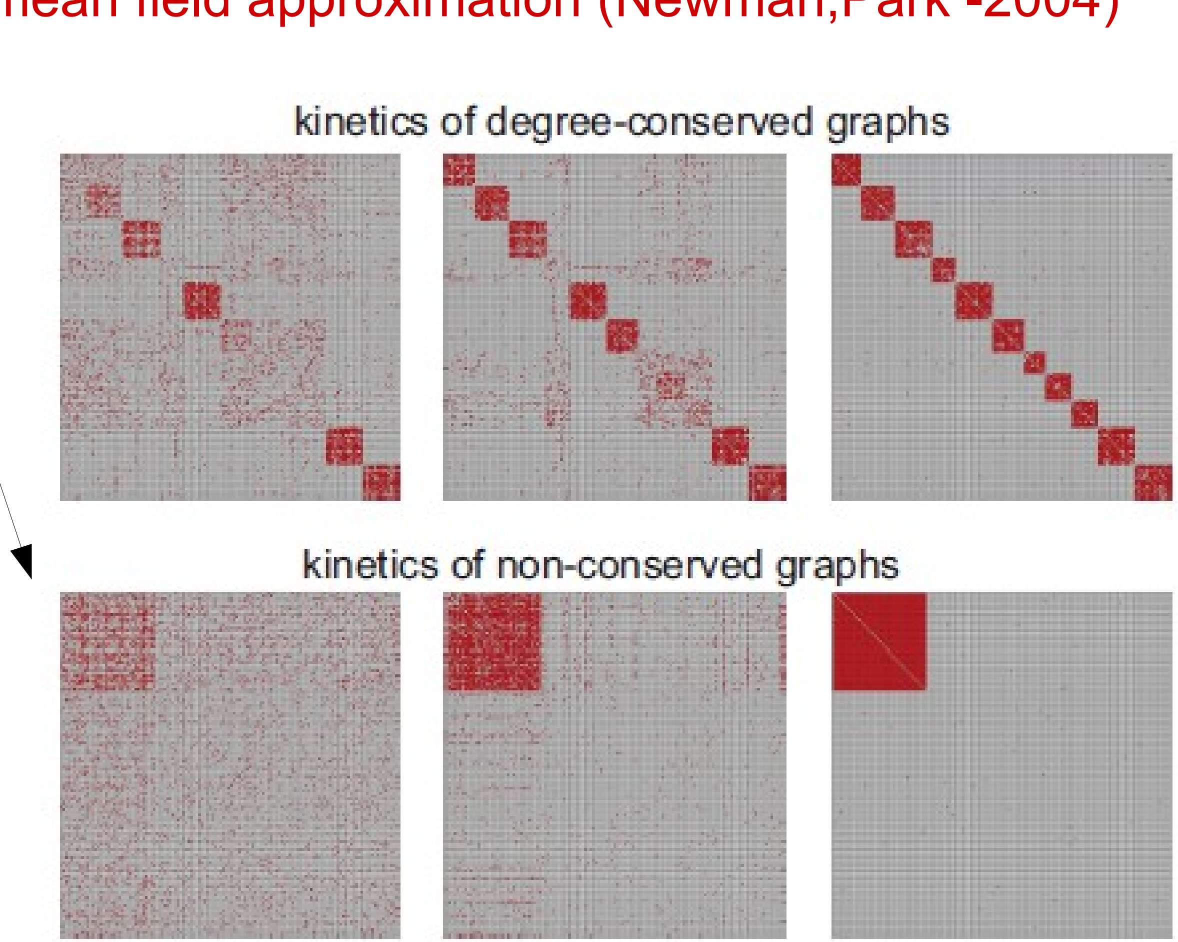
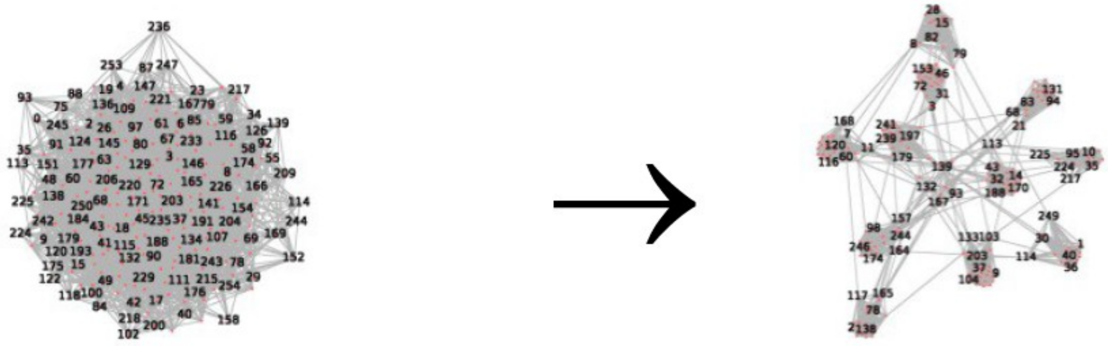


FIG. 5: Few typical samples of intermediate stages of the network evolution: upper panel – evolution with fixed vertex degree; lower panel – eolution with non-fixed vertex degree.

Intermediate «spin glass- like» pattern?

Typical evolution of the random initial network to the ground state



Number of the droplets in the ground state can be predicted!
All clusters are almost complete graphs.

Simplest model for the network defragmentation

$$F = -\mu N_3 s + M \ln M + (N - M) \ln(N - M)$$

Free energy as the function of the number of clusters s

$$F(s) = -\frac{\mu(k-1)^3}{6} s + \frac{N}{s} \ln \frac{N}{s} + N\left(1 - \frac{1}{s}\right) \ln\left(N\left(1 - \frac{1}{s}\right)\right)$$

Critical condition. Qulitatively correct behaviour however wrong value for the critical chemical potential for RRG

$$g s_{cr}^2 = \ln(s_{cr} - 1) \qquad g = \frac{\mu(d-1)^3}{6N}$$

Oversimplified model. To be corrected.

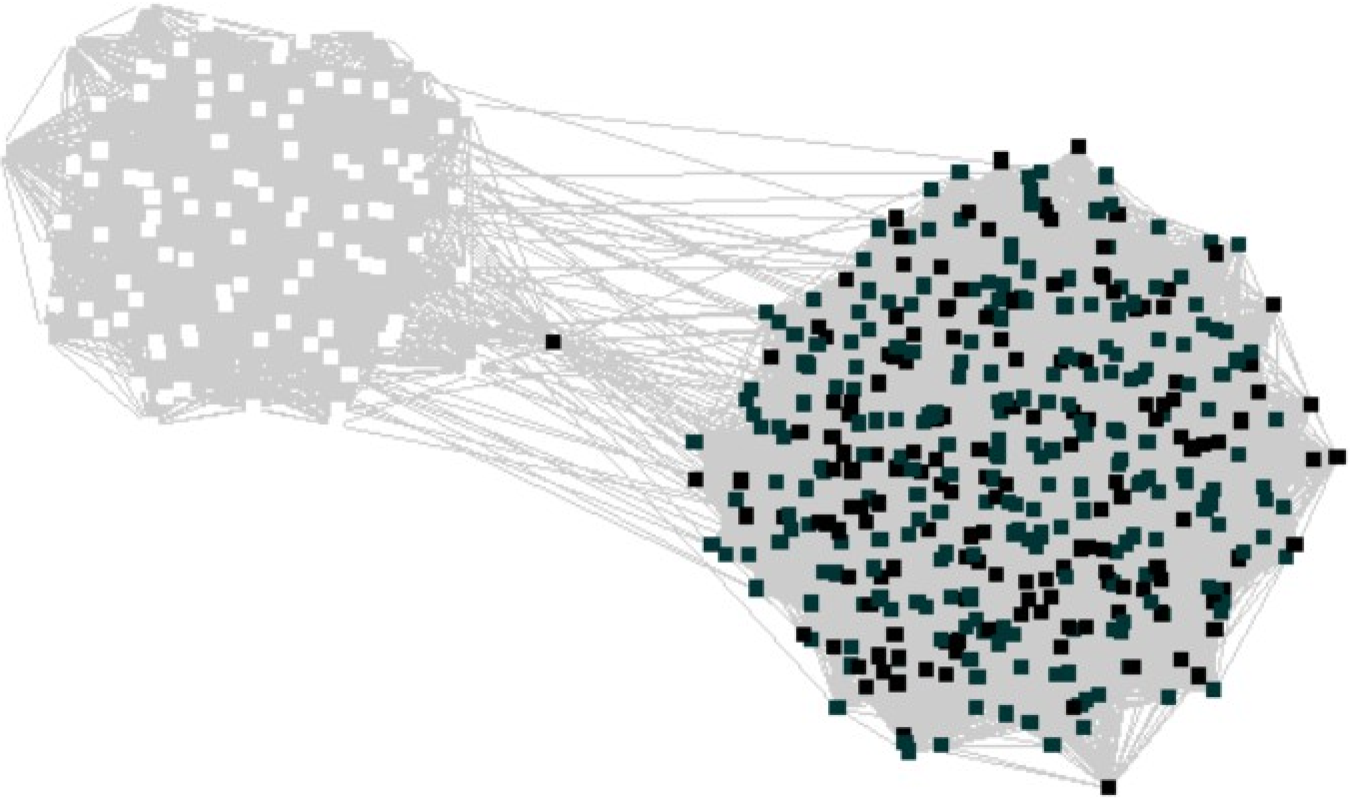
Experimental data for several colors

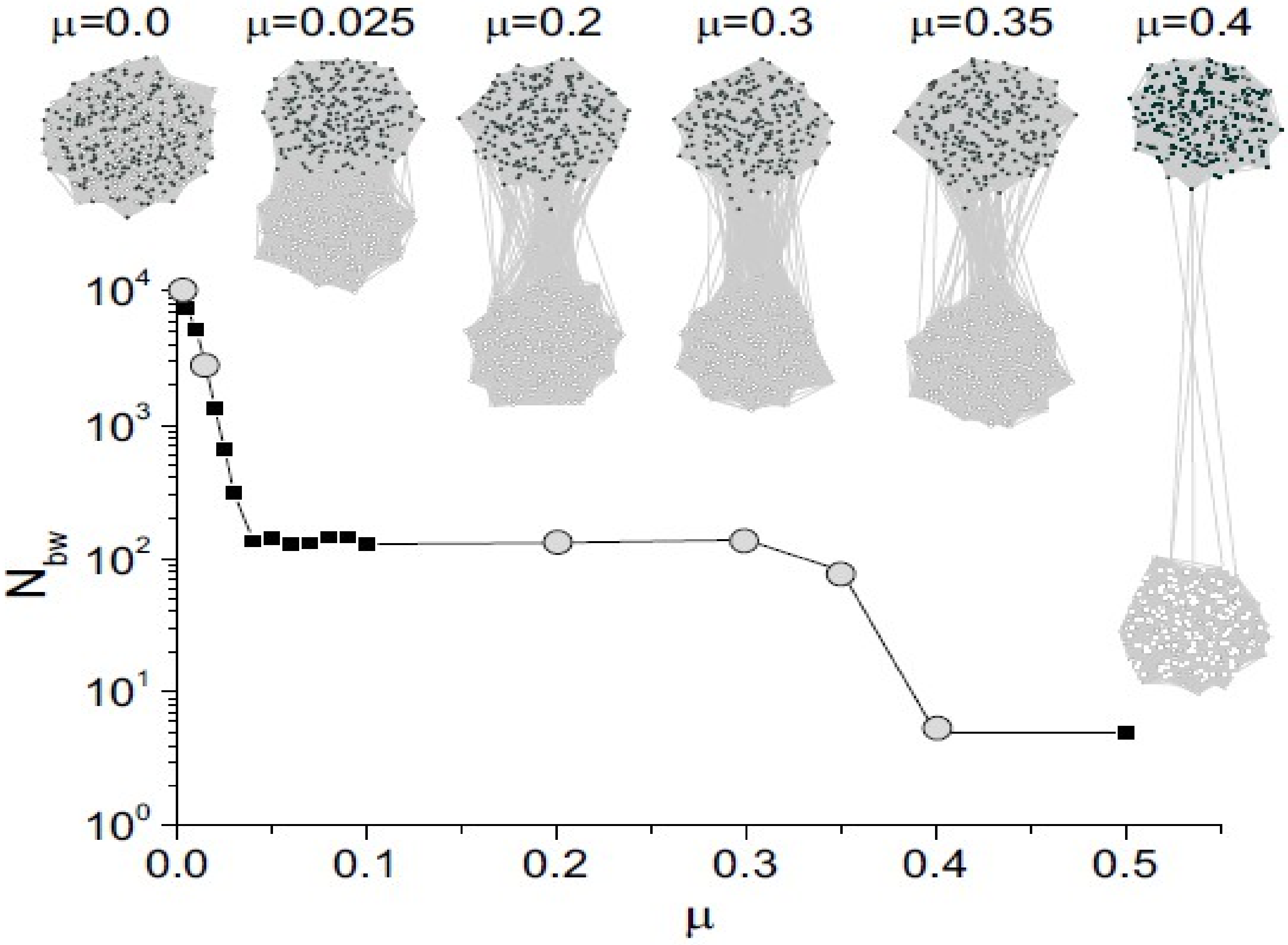
Chemical potentials for trimers are the same for all colors

- Two possibilities. Colored CERN network and colored RRG. The results for two cases are the same.
- There is critical behavior with the plateau formation for number of black-white links(two colors)

Immediate two color clusters formation from the homogeneous network even at small chemical potential

The unicolor trimers play the key role





Dependence of the number of white-black links on the chemical potential. Large entanglement of clusters.

Three colors. Vizualization at plateau.

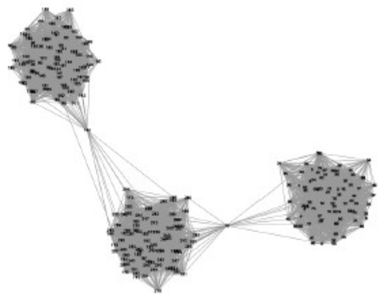


Рис. 3: Структура трехцветной сети с $\mu_3 = 0$

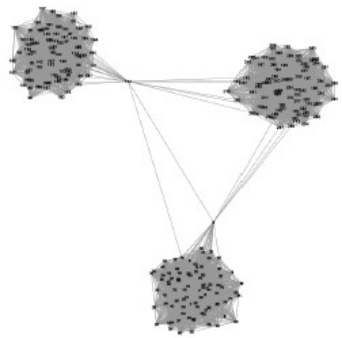
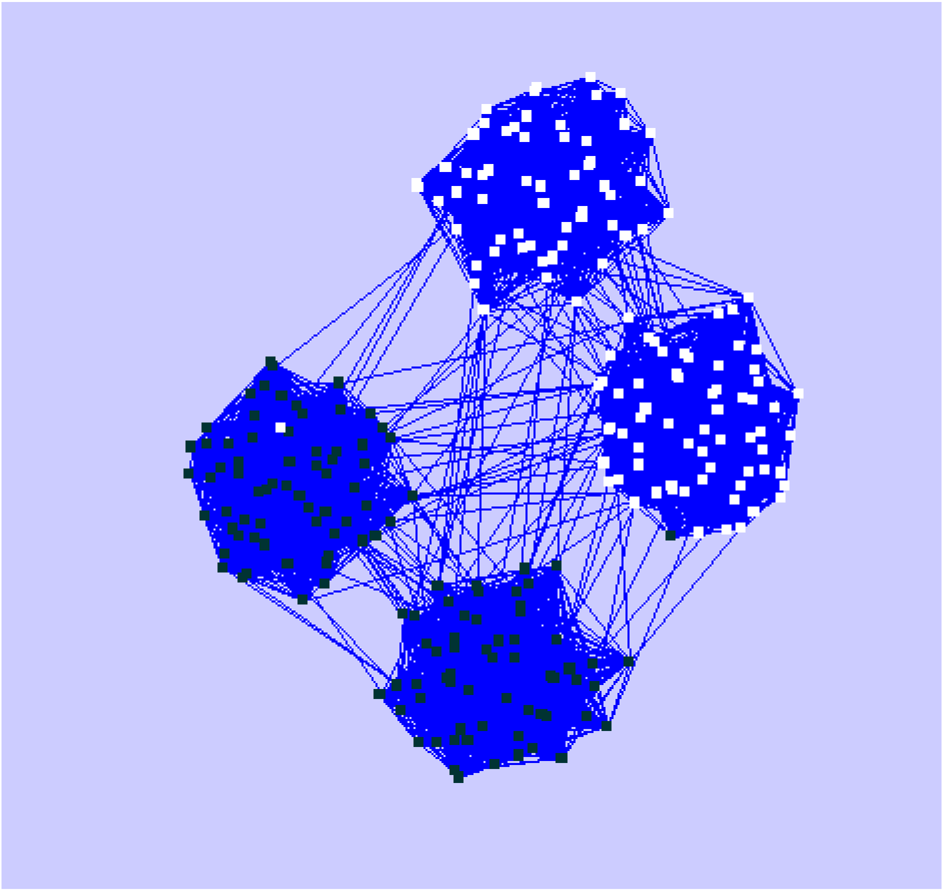


Рис. 4: Структура трехцветной сети с $\mu_3 \neq 0$

Ground state for network with chemical potential for closed triangles only .



Spectral anatomy of transitions

- Second zone formation from the separated eigenvalues moving from the central zone.
- Semicircle distribution before the phase transition. «Trianlge»-shape density in the central zone after the phase transition + second zone.
- Collision of the individual eigenvalues in two-color model at the point of plateau formation. Restoration of broken Z_2 symmetry! Strong dependence on the ratios of the chemical potentials of trimers in 3 color case

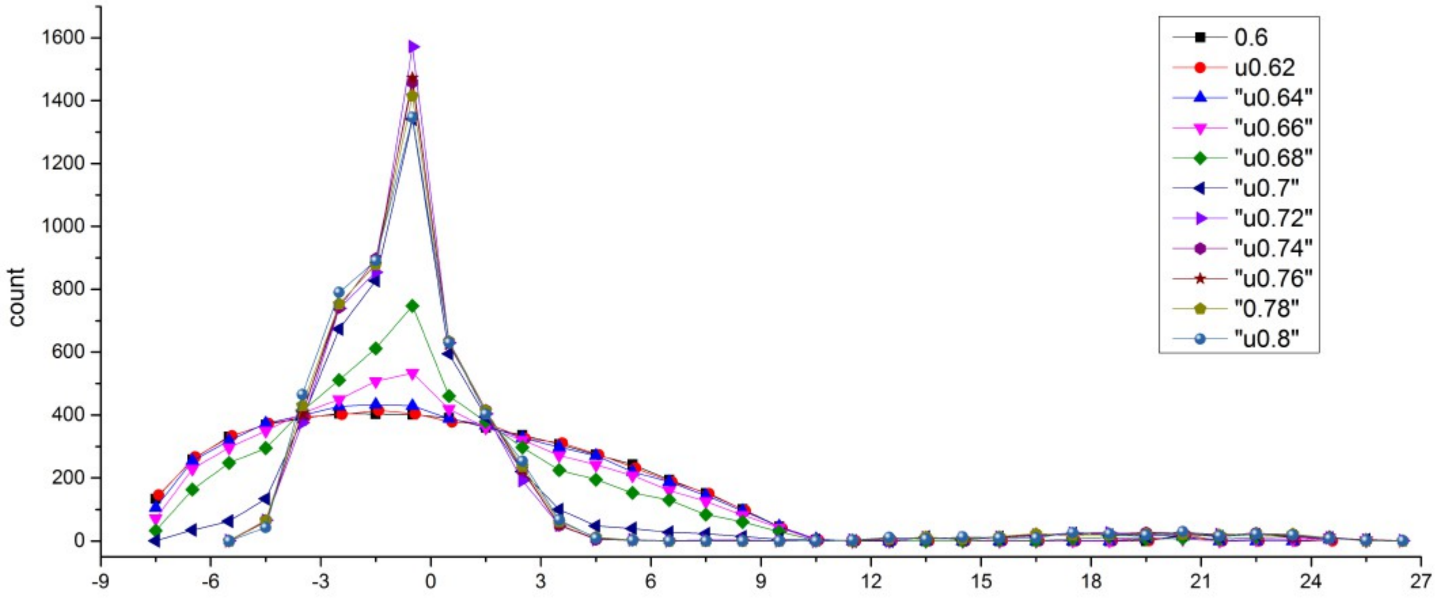
Spectral density in model without chemical potential

$$\rho(\lambda) = \frac{p}{2\pi(1-\lambda^2)}\sqrt{\frac{4(p-1)}{p^2} - \lambda^2}$$

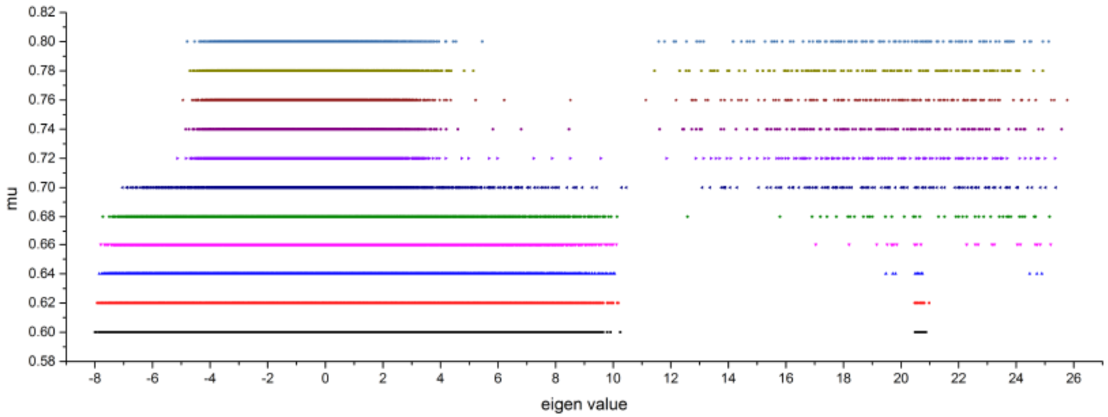
RRG
(Kesten-Mackey)

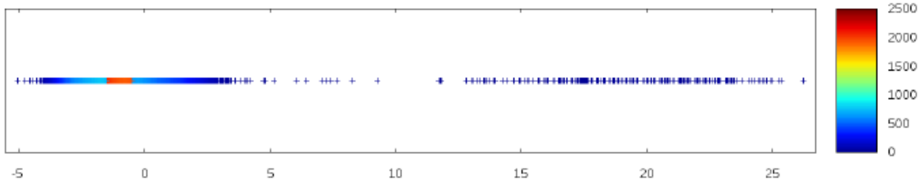
Wigner semicircle for the Erdos-Renyi network

Spectral density of adjacency matrix before and after phase transition



Second zone formation from clusters. Number of the Isolated eigenvalues of the adjacency matrix equals to the number of clusters

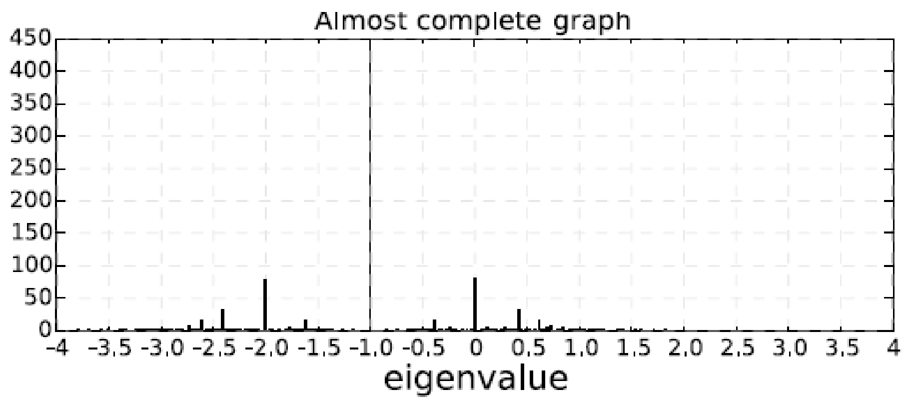




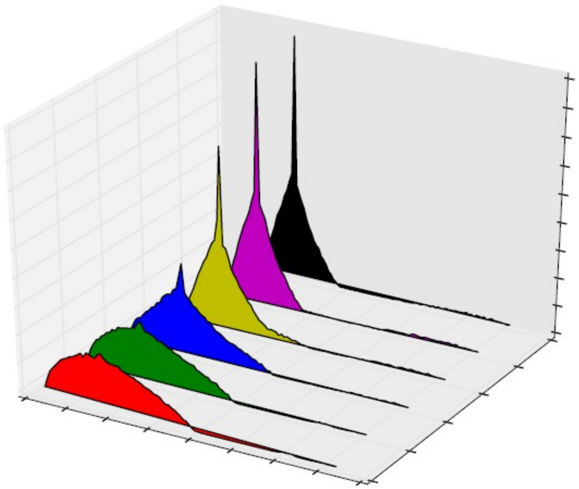
The second zone in the spectrum of adjecency matrix corresponds
To the soft modes in spectrum of graph Laplacian

Summary on spectral density

- Perturbative modes only - Wigner semicircle
- Perturbative modes around single «instanton»

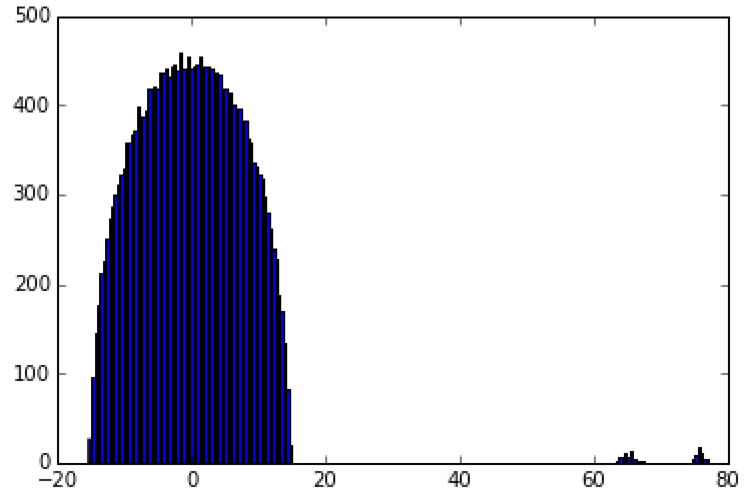


Hidden p-adic structure

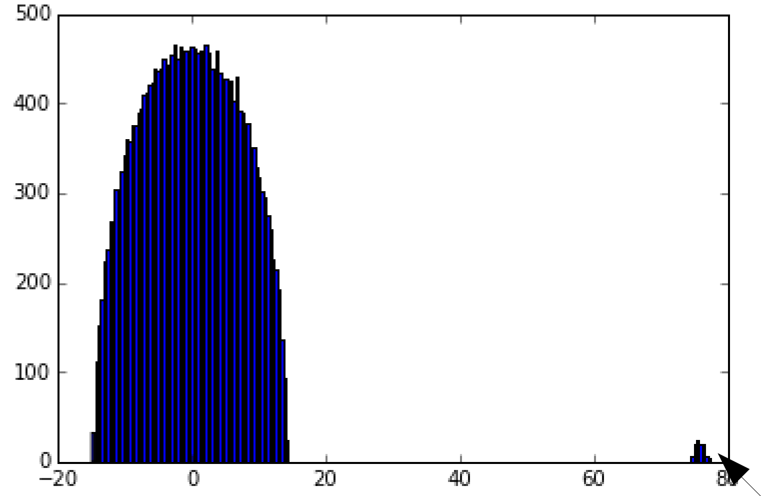


Account of «instanton» interactions via perturbative modes

Two-color CERN. Spectrum before plateau formation

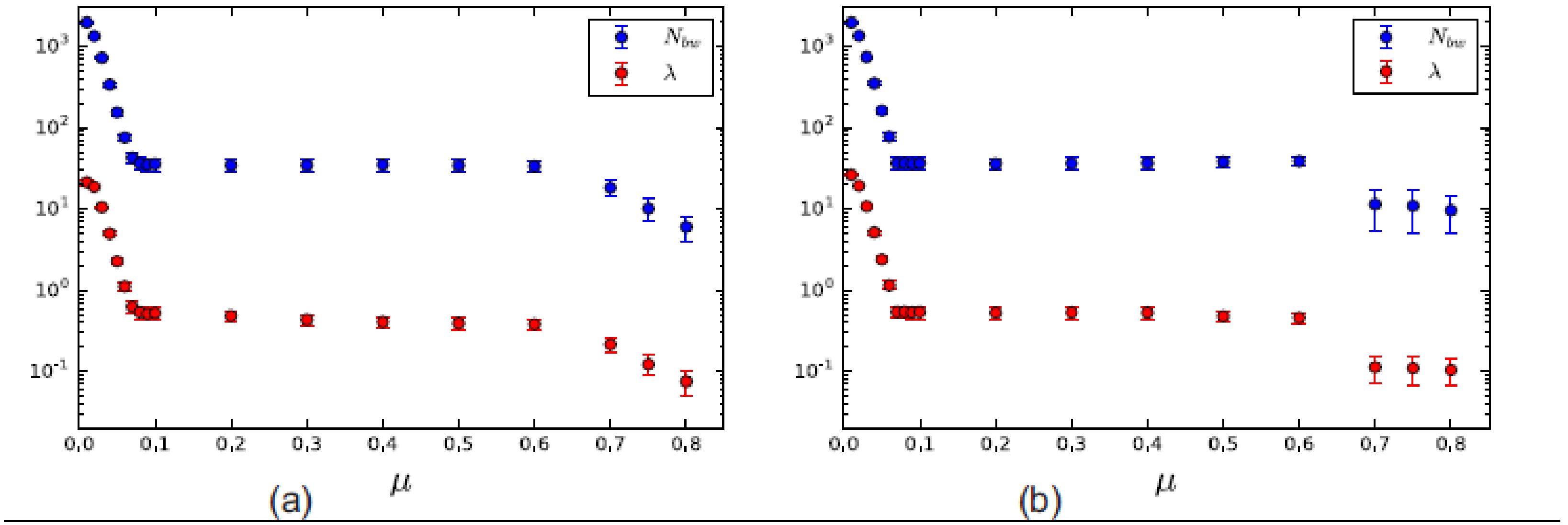


The first nonvanishing eigenvalue of the Laplacian — algebraic connectivity. The multiplicity of the zero eigenvalue — number of disconnected components of the network



Corresponds to the zero eigenvalue of Laplacian

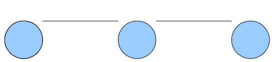
Spectrum of adjacency matrix at the plateau exit



CERN

RRG

The dependence of the first nonvanishing eigenvalue of Laplacian matrix
(red) and dependence of the number of interlayer links (blue) on the chemical potential for trimers



Trimer

$$\lambda_2(\gamma) = cN_{bw}(\gamma)$$

c- some constant

The problem of evolution of λ_2.

The key phenomena — intersection and rearrangements of the spectrum of first modes In the block matrices

$$det\begin{pmatrix} (A-\lambda) & C \\ C^T & (D-\lambda) \end{pmatrix}, = det(A-\lambda)det(D-C^T(A-\lambda)^{-1}C) = 0$$

Chemical potential for trimers

Condition for the plateau entrance

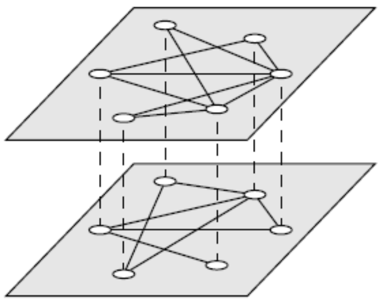
$$\lambda_2^A(\gamma) = \lambda_3(L)$$

Eigenvalue of the One layer Laplacian

Eigenvalue of the Laplacian of the whole network

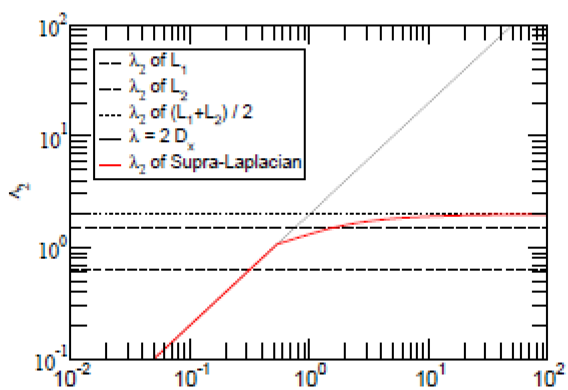
Two layers are absolutely correlated at the plateau!

There was example of the semiinfinite plateau in more simple two-layer network- phenomena of superdiffusion (Gomez,Arenas....2013-14)



$$\mathcal{L} = \left(\begin{array}{c|c} D_1 L_1 + D_x I & -D_x I \\ \hline -D_x I & D_2 L_2 + D_x I \end{array} \right)$$

Laplacian of the whole network



Analogue of our plateau in the model with One-one interaction between layers

However there are many differences with our case.

The layers are absolutely synchronized at plateau. Phenomena of Superdiffusion due to synchronization

Matrix model description

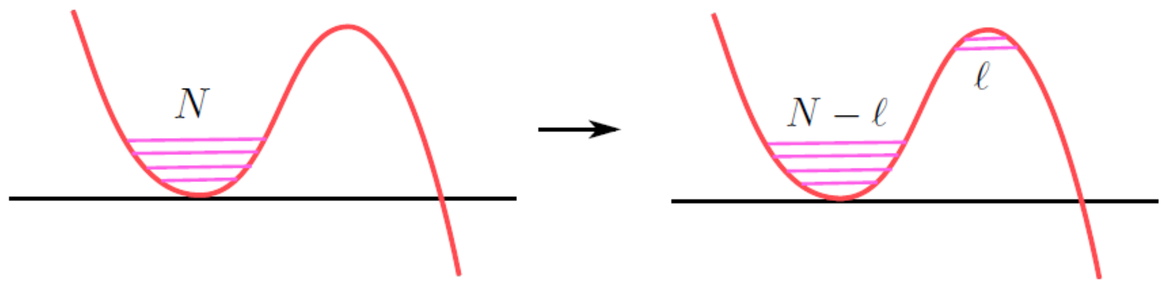
Adjacency matrix is symmetric random matrix involving 1 and 0 only

$$Z = \int dM exp(aTrM^2 - \mu TrM^3)$$

Additional constraint: sum of elements in each row and each column is fixed

a- parameter of the network . Chemical potential for the number of triangles yields «interaction cubic term».

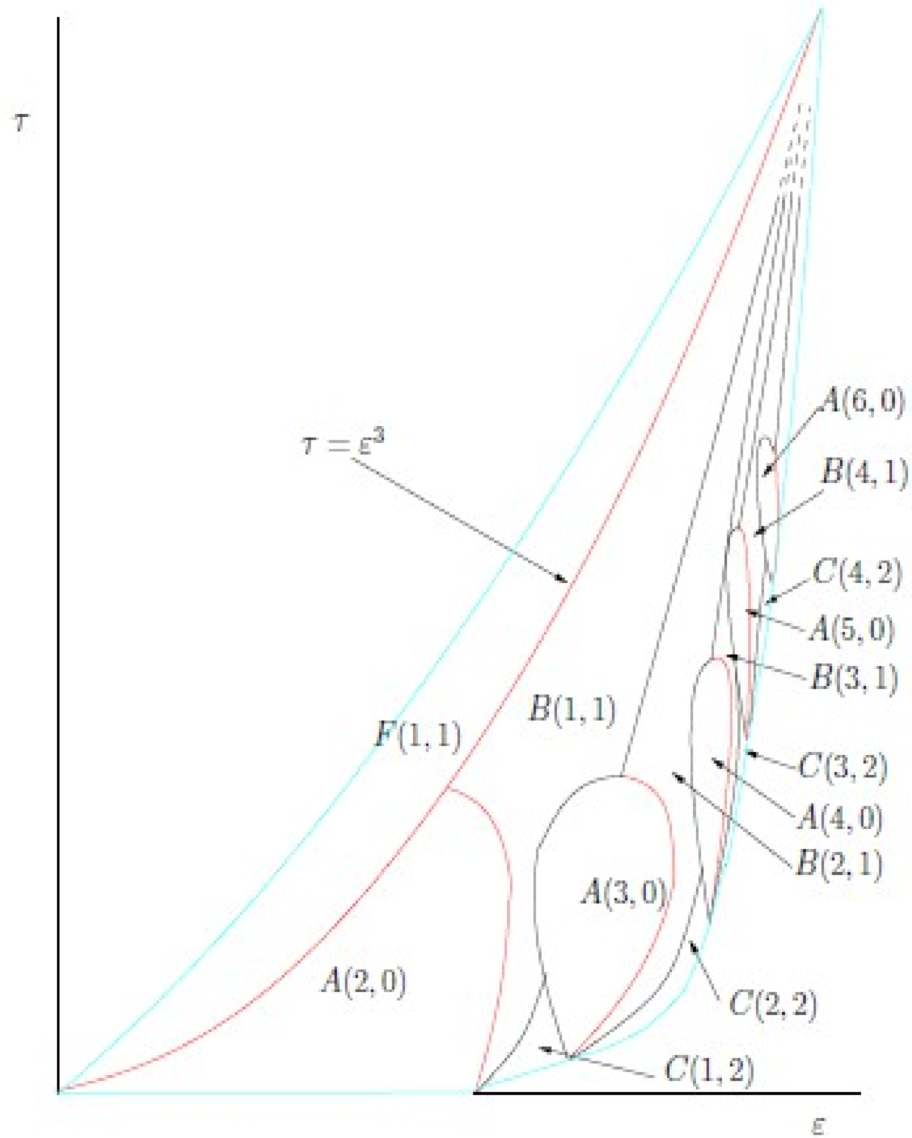
The matrix model counterpart of the cluster formation.
Eigenvalue tunneling — instanton, nonperturbative phenomena in many physical situations. Formation of stable D-branes in the string theory from unstable D0,s.
Baby-universes in 2d gravity, domain walls in SUSY YM



Unusual questions for matrix models

- Degree conservation constraint — non-singlet(can not be expressed in terms Tr). Not only eigenvalues matter!
- Chemical potential for trimers — nonsinglet Hamiltonian
- The chemical potentials for numbers of links and triangles (Strauss)- grand canonical, only number of links fixed(our case) — mixed , numbers of links and trianges fixed-microcanonical ensemble(Kenyon et al, 2017)

Critical lines in (number of links — number of triangles) plane , Numerics(Kenyon 2017)



Some Physics

- Random network — example of topological gravity without metric dependence. Similar potential in the matrix model
- Chemical potential for triangles= 2d cosmological constant
- Cluster creation= stable brane creation in the topological gravity. It is described as the eigenvalue tunneling
- Two color model. Top gravity+Ising model

Plateau formation

- Plateau for intercolor links= plateau for second eigenvalues of the Laplacian matrix
- New zero mode in the spectrum of Laplacian matrix. For string — new massless field
- Formation of the string between the extended objects. The modes inside clusters get collectivized!

Many-body localization

- The question concerns the transport in the interacting many-body system at some temperature. No transport below some critical temperature(Altshuler,Aleiner,Basko)
- The problem gets mapped into the localization of the single degree of freedom in the Hilbert space (Fock space). The «Fock space» is represented by the Bethe lattice and more generally by the RRG, There are results for the Andersen localization on RRG with on-site disorder.(Tikhonov, Skvortsov,Mirlin; Altshuler, Ioffe,Kravtsov)

Standard criteria for localization

- Level spacing distribution (between the neigbohr levels)

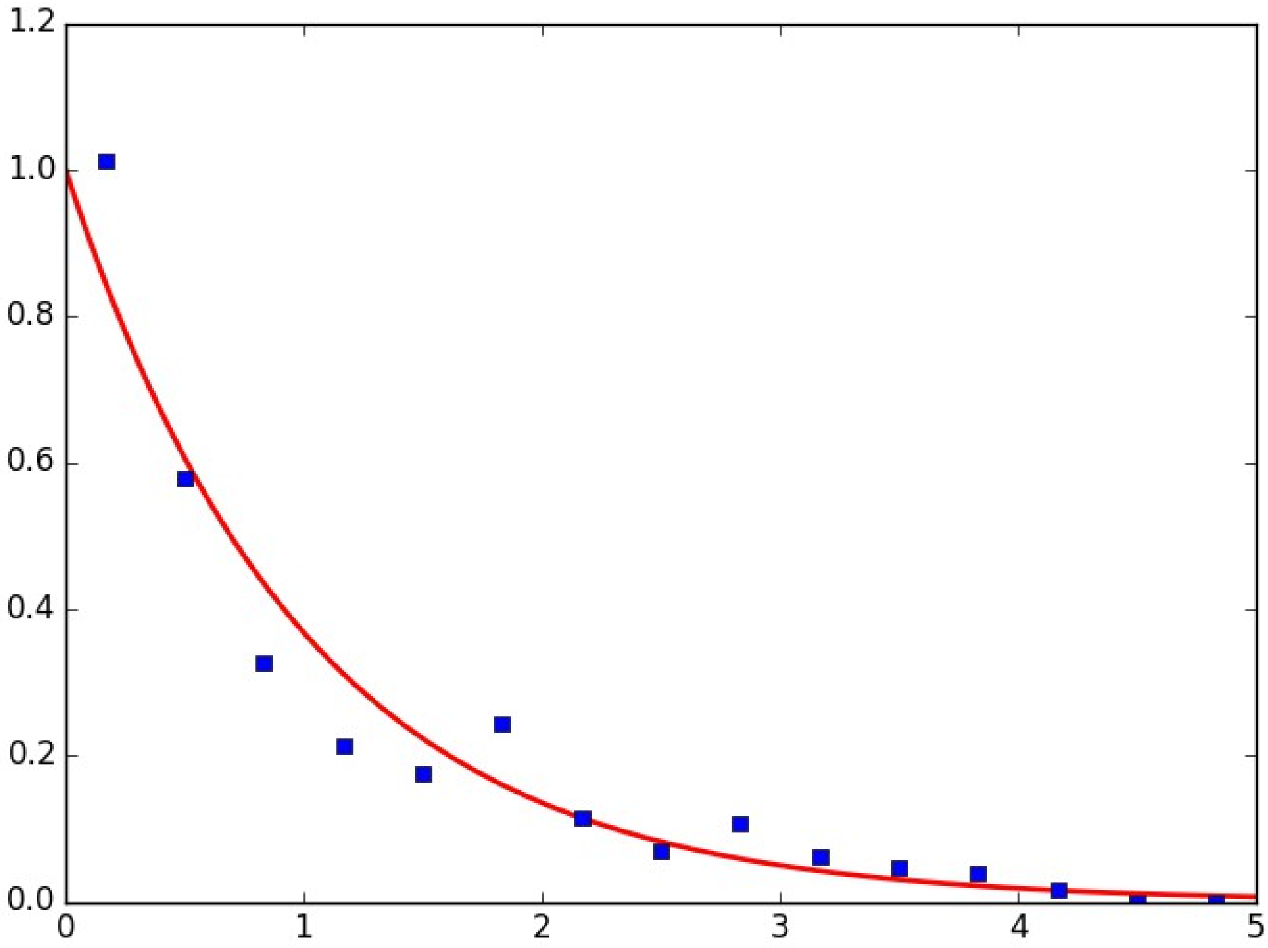
$$\begin{cases} P_{deloc}(s) = s\, e^{-as^2} \\ P_{loc}(s) = e^{-s} \end{cases}$$

- Participation ratio or inverse participation ratio

- Area or volume law for the entanglement

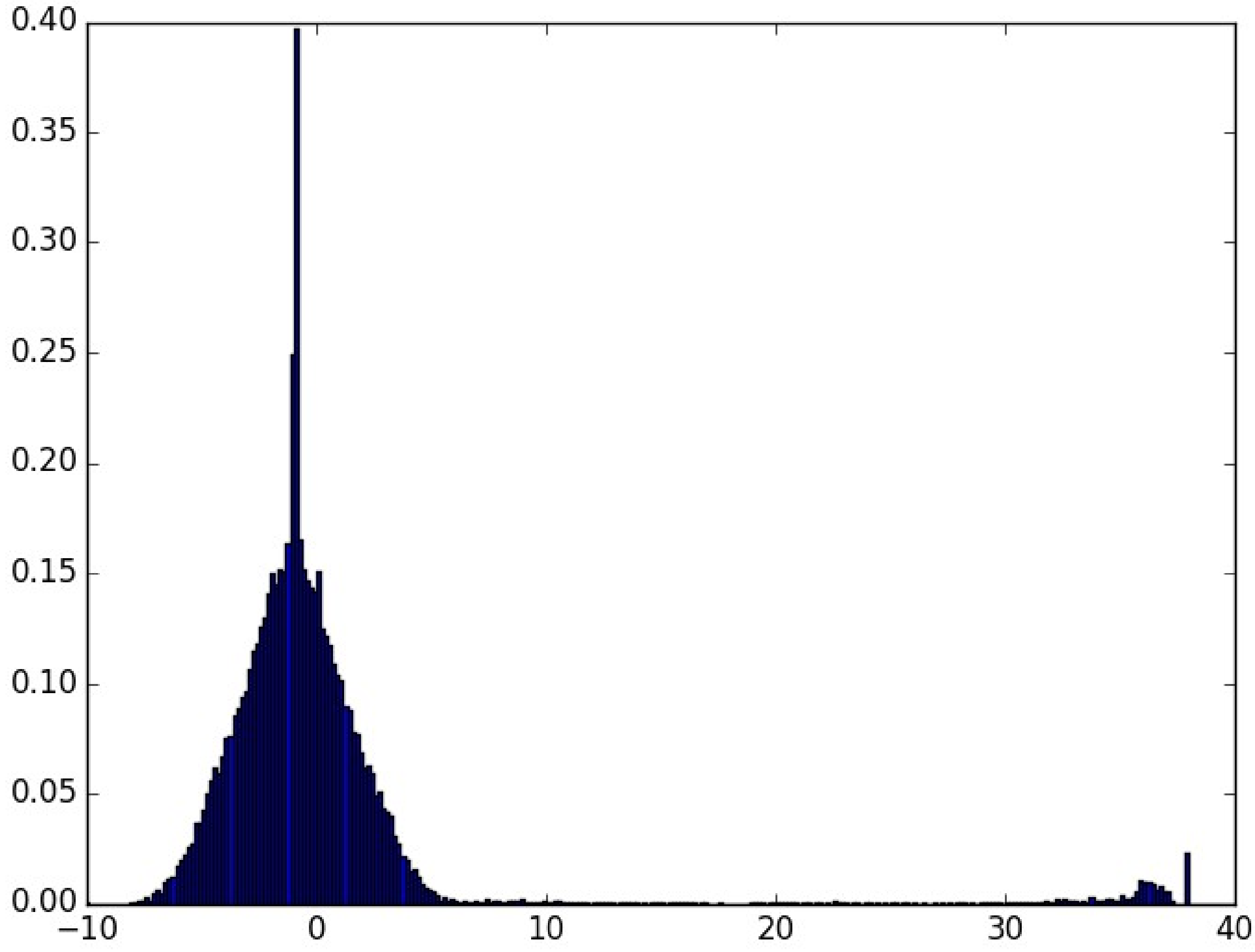
Let us treat our RRG and CERN as the «Fock space» for some interacting many-body system. Could we say smth about this many-body systems using our new critical phenomena?

First check- level spacing distribution



Level spacing distribution in the «nonperturbative» zone.
Poisson distribution — insulator zone.

The spectral density of adjacency matrix above the phase transition



The delocalization in «perturbative» and localiation in «nonperturbative» bands

Immediate questions

- If we treat our critical behavior as one in the Hilbert space what is the initial interacting many-body system?

- What is the meaning of the chemical potential for the short 3-cycle in the Hilbert space

- In which phase: ergodic, non-ergodic or mixed state are the eigenmodes in the perturbative delocalized zone?

Question 1;

Controversial issue of non-ergodic phase in the delocalized regime.
(Biroli et,al, De Luca et al, Altshuler et al, Kravtsov et al, Mirlin et.al......)

We have considered the artificial network of clusters with the same parameters as our initial random network. Its spectral Density is different from the network prepared from the initial random network. Hence in the delocalized regime there is dependence on the initial condition! Mark of non-ergodicity.

Recent result of Altshuler, Ioffe,Kravtsov — non-ergodicity corresponds to the one-step replica symmetry breaking. A lot In common with block-decomposition of the adjacency matrix In our case

Question 2

– According to the idea of the Fock space localization(AGKL) In the localization regime the degree of freedom on the network is identified with the state in the initial basis(system).

--In our case in the localization zone the eigenvectors are Identified with clusters hence clusters provide the initial basis of states

– Perturbative delocalized modes in the main triangle-shape region are treated as superposition of the large number of states in the cluster basis

Conclusion I.

- Phase transition in the colorless network — nonperturbative formation of the ground state with multiple stable objects from unstable network. Phase transition in the multi-color network- nonperturbative restoration of the broken Z_N symmetry in the ground state
- Nonperturbative formation of zone of soft or zero modes in the spectrum of network Laplacian
- Localization-delocalization transition in CERN and RGG with chemical potential for triads. Non-ergodic delocalized phase in CERN and RRG?

Conclusion II

- The phenomena seem to be universal for many real life networks(social, transport etc). Only two key requirements — degree conservation and wish to increase the number of short cycles or chains (triangles , trimers etc)
- The critical phenomena in the multicolor

  (large N !) networks are relevant for the brain network- connectom (in progress)
- A lot of interesting open questions