Zeros and Critical Points for Random Polynomials

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Theorem (Gauss-Lucas)

The critical points of a polynomial in one complex variable lie inside the convex hull of its zeros.

- **Q.** How are critical points distributed inside convex hull?
- **Q.** Are there long-range correlations between zeros and critical points?

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- So $\left\{\frac{d}{dw} p_N(w) = 0\right\} = \{\text{equilibria of E-field from } Div(p_N)\}$

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- IID zeros:

$$p_N(z) \overset{def}{=} \prod_{j=0}^{N-1} (z - \xi_j) \qquad \xi_j \sim \mu_{FS} \;\; i.i.d.$$

Pairing of Zeros and Crits for $SU(2)$ Polynomials

"Proof" of Pairing of Zeros and Crits

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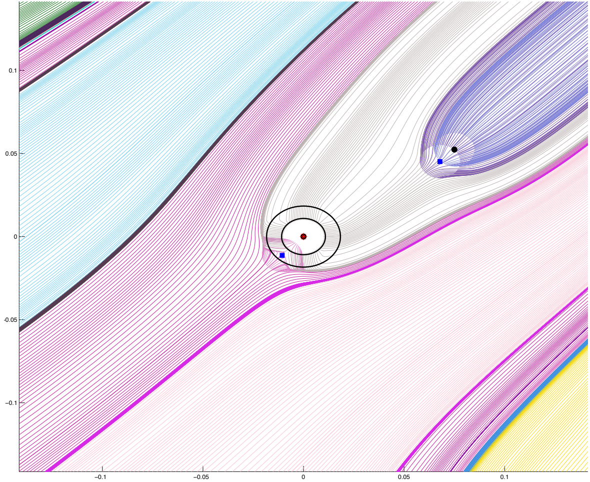

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Remark

Holds for general positive smooth hermitian metric h and can be extended to $N^{1-\eta}$ well-spaced zeros simultaneously.

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Holds for $\xi_j \sim \mu$ *a general smooth measure on* S^2.

Bargmann-Fock as Scaling Limit of $SU(2)$

- $SU(2)$ scaling limit:

$$p_N\left(z_0 + \frac{u}{\sqrt{N}}\right) \quad \longrightarrow \quad \sum_{j=0}^{\infty} a_j \, \frac{z^j}{\sqrt{j!}}$$

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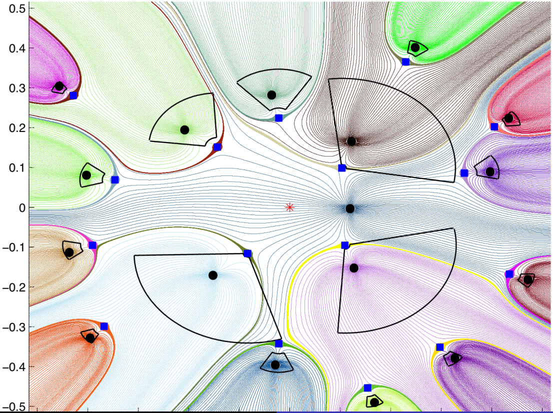

Zeros and Critical Points for Kac Polynomials

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