Coulomb gas ensembles in 2D

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December 11, 2015

Coulomb gas in 2D

We consider n repelling particles in 2D confined by a potential $V : \mathbb{C} \to \mathbb{R}$. The interaction energy between the repelling particles is modelled by

$$\mathcal{E}_V^{\mathrm{int}} := \sum_{j,k:j\neq k} \log \frac{1}{|z_j - z_k|},$$

where z_j denotes the position of the j-th particle, and the potential energy is given by

$$\mathcal{E}_V^{\mathrm{pot}} := \sum_{j=1}^{n} V(z_j).$$

The total energy of a configuration $(z_1, \ldots, z_n) \in \mathbb{C}^n$ is then given by

$$\mathcal{E}_V := \mathcal{E}_V^{\mathrm{int}} + \mathcal{E}_V^{\mathrm{pot}}.$$

Coulomb gas. Gibbs model and inverse temperature

In any reasonable gas dynamics model, the low energy states should be more likely than the high energy states. Fix a positive constant β, and let Z_n be the constant ("partition function")

$$Z_n := \int_{\mathbb{C}^n} \mathrm{e}^{-\frac{\beta}{2}\mathcal{E}_V} \mathrm{dvol}_{2n},$$

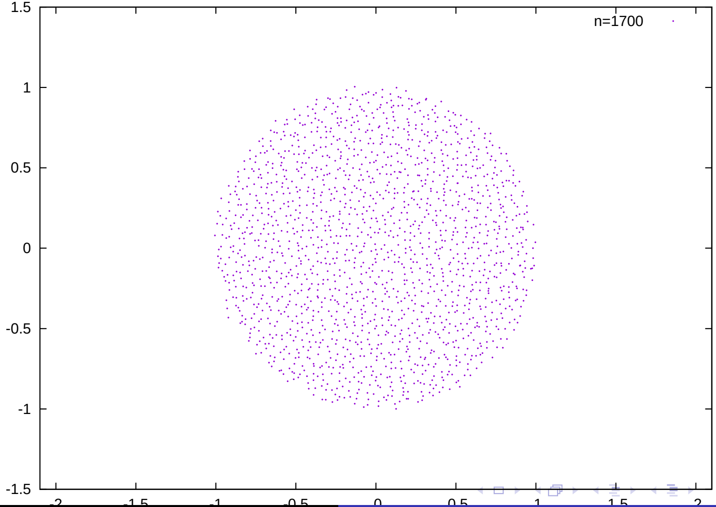
where vol_{2n} denotes standard volume measure in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Here, we need to assume that V grows at sufficiently at infinity to make the integral converge. The Gibbs model gives the joint density of states

$$\frac{1}{Z_n}\mathrm{e}^{-\frac{\beta}{2}\mathcal{E}_V},$$

which we use to define a probability point process $\Pi_n \in \mathrm{prob}(\mathbb{C}^n)$ by setting

$$\mathrm{d}\Pi_n := \frac{1}{Z_n}\mathrm{e}^{-\frac{\beta}{2}\mathcal{E}_V} \mathrm{dvol}_{2n}.$$

Simulation of the Ginibre ensemble $V(z) = m|z|^2$ (1700 pts)



Electron cloud interpretation. Marginal measures

The process Π_n models a cloud of electrons in a confining potential. Clearly, Π_n is random probabilty measure on $\mathbb{C}^n$. In order to study this process as $n \to +\infty$, it is advantageous to introduce the marginal probability measures $\Pi_n^{(k)}$ (for $0 \le k \le n$) given by

$$\Pi_n^{(k)}(e) := \Pi_n(e \times \mathbb{C}^{n-k}),$$

for Borel measurable subsets $e \subset \mathbb{C}^k$. In particular, $\Pi_n^{(n)} = \Pi_n$. The associated measures

$$\Gamma_n^{(k)} := \frac{n!}{(n-k)!}\Pi_n^{(k)}$$

are called **intensity (or correlation) measures**. To simplify the notation, we write $\Gamma_n := \Gamma_n^{(n)}$.

Aggregation of quantum droplets. Monotonicity

It is of interest to analyze what the addition of one more particle means for the process.

THEOREM 1. If $\beta = 2$, then

$$\forall k: \; \Gamma_n^{(k)} \le \Gamma_{n+1}^{(k)}.$$

This means that for the special inverse temperature $\beta = 2$, the addition of a new particle monotonically increases all the intensities.

REMARK 2. The assertion of Theorem 1 fails for $\beta > 2$. For $\beta < 2$, however, we conjecture that the assertion of Theorem 1 remains valid.

The determinantal nature of $\beta = 2$ case (1)

The proof of Theorem 1 is based on the fact that the point process Π_n is determinantal for $\beta = 2$. To explain what this means, we need the space Pol_n of all polynomials in z of degree $\leq n-1$. We equip Pol_n with the inner product structure of $L^2(\mathbb{C}, \mathrm{e}^{-V})$. Then under standard assumptions on V, point evaluations are bounded, and we obtain elements $K_w \in \mathrm{Pol}_n$ such that

$$p(w) = \langle p, K_w \rangle_{L^2(\mathbb{C}, \mathrm{e}^{-V})}.$$

The function $K(z,w) := K_w(z)$ may be written in the form

$$K(z,w) = \sum_{j=0}^{n-1} e_j(z) \bar{e}_j(w),$$

where the e_j form an ONB. It is called **the reproducing kernel**.

The determinantal nature of $\beta = 2$ case (2)

The determinantal structure of the process is easiest to see by considering intensities:

$$\mathrm{d}\Gamma_n^{(k)}(z) = \mathrm{e}^{-\sum_j V(z_j)} \det[K(z_i, z_j)]_{i,j=1}^k.$$

For instance, if we are interested in the intensity $\Gamma_n^{(1)}$, we should analyze $K(z,z)\mathrm{e}^{-V(z)}$. The expression

$$u_n(z) := \frac{1}{n} K(z,z)\mathrm{e}^{-V(z)}$$

is called the 1-**point function**. The determinantal case $\beta = 2$ models **Random Normal Matrices**.

Renormalization of the potential

To obtain a reasonable limit as $n \to +\infty$, we need to renormalize the potential. So we put $V := mQ$, where the parameter m is essentially proportional to n as n tends to infinity. Here, Q is a fixed confining potential.

N. B. Note that in the determinantal case, we just need to analyze the (polynomial) reproducing kernels $K(z, w)$ for the space of polynomials of degree $\leq n - 1$ with respect to the weight e^{-mQ} in the plane $\mathbb{C}$.

Approximation of the energy (1)

We recall that

$$\mathcal{E}_{mQ} = \mathcal{E}_{mQ}^{\mathrm{int}} + \mathcal{E}_{mQ}^{\mathrm{pot}} = \sum_{j,k:j\neq k} \log \frac{1}{|z_j - z_k|} + m \sum_{j=1}^{n} Q(z_j),$$

so that

$$\frac{\mathcal{E}_{mQ}}{n^2} = \frac{1}{n^2} \sum_{j,k:j\neq k} \log \frac{1}{|z_j - z_k|} + \frac{m}{n^2} \sum_{j=1}^{n} Q(z_j).$$

If $n/m = \tau$, then

$$\frac{\mathcal{E}_{mQ}}{n^2} = \frac{1}{n^2} \sum_{j,k:j\neq k} \log \frac{1}{|z_j - z_k|} + \frac{1}{n\tau} \sum_{j=1}^{n} Q(z_j).$$

Approximation of the energy (2)

If we put (for probability measures σ)

$$I_Q[\sigma] := \int_{\mathbb{C}}\int_{\mathbb{C}} \log\frac{1}{|\xi-\eta|}\mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\eta) + \int_{\mathbb{C}} Q\mathrm{d}\sigma,$$

then

$$\frac{\mathcal{E}_{mQ}}{n^2} \approx I_{Q/\tau}[\sigma],$$

where

$$\mathrm{d}\sigma = \frac{1}{n}\sum_{j=1}^{n}\mathrm{d}\delta_{z_j}.$$

Here, "$\approx$" means that we disregard the singularities which appear from diagonal terms in the integral. We write $I^{\sharp}_{Q/\tau}[\sigma]$ to indicate that we have removed the singular diagonal part from $I_{Q/\tau}[\sigma]$.

The Gibbs model and energy heuristics

We recall the density of states from the Gibbs model

$$\mathrm{d}\Pi_n := \frac{1}{Z_n}\,\mathrm{e}^{-\frac{\beta}{2}\mathcal{E}(\lambda_1,\ldots,\lambda_n)}\mathrm{dvol}_{2n} = \frac{1}{Z_n}\,\mathrm{e}^{-n^2\frac{\beta}{2}I^{\sharp}_{Q/\tau}[\sigma]}\mathrm{dvol}_{2n}.$$

The factor n^2 in the exponent means that high energy states get severely punished and we expend generally convergence to the lowest energy state. To make this more precise, let $\hat{\sigma}_\tau \in \mathrm{prob}_c(\mathbb{C})$ minimize

$$\min_\sigma I_{Q/\tau}[\sigma].$$

The measure $\hat{\sigma}_\tau$ is called the *equilibrium measure*.

Johansson's marginal measure theorem

THEOREM 3. Under minimal growth and smoothness assumptions on Q, we have for fixed k that

$$\Pi_n^{(k)} \to \hat{\sigma}_\tau^{\otimes k} \quad \text{as } n \to +\infty, \text{ while } n = m\tau + \mathrm{o}(m),$$

in the weak-star sense of measures.

REMARK 4. In particular, the 1-point function converges to the equilibrium density. Theorem 3 was obtain by K. Johansson in the case of Coulomb gas on the real line [J1]. His techniques work also in the planar case, with some modifications [HM1].

Obstacle problem and the equilibrium measure

We consider the obstacle problem

$$\hat{Q}_\tau(z) := \sup\{q(z) : \ \ q \le Q \text{ on } \mathbb{C}, \ q \in \mathsf{Subh}_\tau(\mathbb{C})\},$$

where $\mathsf{Subh}_\tau(\mathbb{C})$ denotes the convex set of subharmonic functions $u : \mathbb{C} \to [-\infty, +\infty[$ with

$$u(z) \le 2\tau \log^+ |z| + \mathrm{O}(1).$$

For a measure σ, its logarithmic potential U^σ is

$$U^\sigma(\xi) := 2 \int_{\mathbb{C}} \log \frac{1}{|\xi - \eta|} \mathrm{d}\sigma(\eta).$$

THEOREM 5 (Frostman) For some constant c,

$$\hat{Q}_\tau = c - \tau U^{\hat{\sigma}_\tau}.$$

The support of the equilibrium measure

Let $S_\tau := \operatorname{supp} \hat{\sigma}_\tau$. This is called the **(spectral) droplet**.

THEOREM 6 (Kinderlehrer-Stampacchia theory) Under smoothness on Q, we have

$$\Delta \hat{Q}_\tau = 1_{S_\tau} \Delta Q,$$

so that

$$\mathrm{d}\hat{\sigma}_\tau = \frac{1_{S_\tau} \Delta Q}{4\pi\tau}.$$

REMARK 7 It follows that the study of the dynamics of the equilibrium measures $\hat{\sigma}_\tau$ reduces to the study of the supports S_τ. This is in contrast with the 1D theory.

Comparison with Hermitian ensebles

If we consider the degenerate case when $Q = +\infty$ on $\mathbb{C} \setminus \mathbb{R}$, we get the usual Hermitian ensebles (the eigenvalues are forced to be real). This can be thought of as a limit of smooth potentials

$$\widetilde{Q}(x + \mathrm{i}y) := Q(x) + ay^2,$$

where we let $a \to +\infty$. We expect that the droplets S_a tend to a compact subset of $\mathbb{R}$ as $a \to +\infty$, where the eigenvalues accumulate, and that the local vertical width of S_a corresponds to the local density of eigenvalues in the Hermitian ensemble. The relation

$$\tau \mathrm{d}\hat{\sigma}_\tau = \frac{1}{4\pi} \Delta \hat{Q}_\tau \mathrm{d}A$$

should survive also in the Hermitian case, although the right hand side must be understood in the sense of distribution theory. E.g., the Wigner semi-circle law comes from an obstacle problem with $Q(x) = x^2$ along the real line and $Q = +\infty$ elsewhere in $\mathbb{C}$.

We now mention an application of Johansson's marginal measure theorem (Theorem 3) to linear statistics. For $f \in C_b(\mathbb{C})$, put

$$\mathrm{tr}_n f := f(z_1) + \cdots + f(z_n).$$

THEOREM 8 Under the assumptions of Theorem 3, we have the convergence

$$\frac{1}{n}\mathrm{tr}_n f \to \int_{\mathbb{C}} f \,\mathrm{d}\hat{\sigma}_\tau$$

in all moments as $m \to +\infty$ and $n = m\tau + \mathrm{o}(m)$.

REMARK 9 We may interpret this as the statement that when applied to a test function, the empirical measure converges to the equilibrium measure.

Fluctuations (1)

We now fix $\tau = 1$ and write $S = S_1$. In the context of Theorem 8, with smooth compactly test functions f, we would like to analyze first

$$\mathbb{E}\,\mathrm{tr}_n f - n\langle f, \hat{\sigma}\rangle.$$

THEOREM 10 Under smoothness of Q and simple-connectedness of S, and smoothness of ∂S,

$$\mathbb{E}\,\mathrm{tr}_n f - n\langle f, \hat{\sigma}\rangle \to \frac{1}{8\pi}\langle f, \Delta(1_S + L^S)\rangle,$$

where $L := \log \Delta Q$, and L^S is the harmonic extension to the outside of $L|_S$.

Fluctuations (2)

The next level to understand is **fluctuations**.

THEOREM 11 Under smoothness of Q and simple-connectedness of S, and smoothness of ∂S,

$$\mathrm{tr}_n f - \mathbb{E}\,\mathrm{tr}_n f \to N(0, s^2),$$

where

$$s^2 = \frac{1}{4\pi} \int_{\mathbb{C}} |\nabla f^S|^2 \mathrm{dvol}_2.$$

Techniques: correlation kernels

The specific choice we made of the inverse temperature gives us correlation kernel structure. That is, the whole process is determined by the *correlation kernel* $L(z, w)$, which depends on n, m, Q, which has the form

$$L(z, w) := K(z, w)\, \mathrm{e}^{-\frac{m}{2}(Q(z)+Q(w))},$$

where $K(z, w)$ is the *reproducing kernel* for the space of polynomials of degree $\leq n-1$ with inner product norm

$$\|f\|^2 = \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-mQ} \mathrm{d}A < +\infty.$$

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$$L(z, w) := K(z, w)\,\mathrm{e}^{-\frac{m}{2}(Q(z)+Q(w))},$$

where $K(z, w)$ is the *reproducing kernel* for the space of polynomials of degree $< n$ with inner product norm

$$\|f\|^2 = \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-mQ} \mathrm{d}A < +\infty.$$

Correlations

The determinant

$$\det\left(\left[L(z_i, z_j)\right]_{i,j=1}^{k}\right)$$

describes the intensity of finding a k-tuple of electrons at the points $z_1, \ldots, z_k$. E.g., $L(z, z)$ describes the density of electrons in position z.

Reproducing kernel expansion

Reproducing kernel expansions have a long history, rooted in the works of Hörmander, Fefferman, Boutet de Monvel, Sjöstrand, Berndtsson, etc. We use the recent version due to Berman, Berndtsson, and Sjöstrand to get the following.

THEOREM 12. We have, for $n \geq m-1$,

$$K(z,z)\mathrm{e}^{-mQ(z)} = m\Delta Q(z) + \frac{1}{2}\Delta \log \Delta Q(z) + O(m^{-1/2}),$$

on any compact subset Σ of the interior of S with $\Delta Q > 0$ on Σ.

There exists a polarized version of this diagonal approximation:

$$K_{m,n}(z,w)\mathrm{e}^{-mQ^*(z,w)} = m\Delta^* Q^*(z,w) + \frac{1}{2}\Delta^* \log \Delta^* Q^*(z,w) \\ + O\Big(m^{-1/2}\mathrm{e}^{(m/2)[Q(z)+Q(w)-2\mathrm{Re}Q^*(z,w)]}\Big).$$

Berezin quantization

The probablity measure

$$\mathrm{d}B^{\langle w\rangle}(z) = \frac{|K(z,w)|^2}{K(w,w)}\,\mathrm{e}^{-mQ(z)}\mathrm{d}A(z)$$

we call the Berezin measure. For $w \in S$ it converges to a point mass at w as $m, n \to +\infty$ while $n = m + O(1)$, while for $w \in \mathbb{C} \setminus S$ it converges to harmonic measure for w in the domain $\mathbb{C} \setminus S$. In case w is a bulk point (i.e., it is in the interior of S with $\Delta Q(w) > 0$), one can show that the Berezin measure – suitably blown up so that the scale $m^{-1/2}$ becomes 1 – tends to a radially symmetric Gaussian in the plane.

Ginibre interpretation

The observation that the Berezin measure – rescaled – tends to the Gaussian at interior points with $\Delta Q > 0$, corresponds to the blown-up process converging to $\mathrm{Gin}(\infty)$, with correlation kernel

$$L_\infty(z, w) = \mathrm{e}^{z\bar{w}} \mathrm{e}^{-\frac{1}{2}(|z|^2 + |w|^2)}.$$

This corresponds to the reproducing kernel for the Bargmann-Fock space. The stochastic process is translation invariant with infinitely many points equidistributed in the entire plane.

Boundary point blow-up (1)

In case of the usual Ginibre ensemble, with reproducing kernel

$$K(z, w) = m \sum_{j=0}^{n-1} \frac{(mz\bar{w})^j}{j!},$$

we can make explicit calculations. The droplet S is the closed unit disk, so the boundary is the unit circle. If we blow up at a boundary point, the reproducing kernel tends to the reproducing kernel for a naturally defined subspace of the Bargmann-Fock space. The concrete expression involves the error function. This is most likely universal for smooth boundary points of S, for other (real-analytic) weights Q.

The analysis of the Ginibre ensemble suggested that for interior points and for boundary points, the limit of the blow-ups of the correlation kernel is determined by the reproducing kernel of a Hilbert space of entire functions. Probably this is universal. In fact, for GUE we have the sine kernel at bulk points, which is the reproducing kernel for the Paley-Wiener space. And at the boundary we have the Airy process, with a different local scaling of $m^{-2/3}$. The Airy kernel is also associated with a space of entire functions. Moreover, the different typical distance $m^{-2/3}$ comes from the fact that the Wigner semi-circle law has zero density at the boundary point, with a square-root type approach.

Boundary point blow-up (3)

To obtain a more satisfactory analysis of the polynomial kernel $K(z, w)$ near the boundary of the droplet $S = S_1$, we really need an asymptotic expression for the orthogonal polynomials. This would then also help in the analysis of the free energy $\log Z_n$.

The orthogonal polynomials and the kernel

Let $p_0, p_1, p_2, \ldots$ denote the normalized (holomorphic) orthogonal polynomials in $L^2(\mathbb{C}, \mathrm{e}^{-mQ})$, such that p_j has degree j. Then

$$K(z, w) = \sum_{j=0}^{n-1} p_j(z) \bar{p}_j(w),$$

is the reproducing kernel for the polynomial subspace (degree $\leq n - 1$). We consider asymptotics as $n = m\tau + o(1)$. The kernel expansion technique of [AHM1], [Ameur1] (which goes back to [BBS]) works well in the bulk of the droplet S_τ, and with effort within distance $m^{-1/2} \log m$ from the boundary ∂S_τ. But to go further and analyze in depth the behavior of $K(z, w)$ near ∂S_τ, we need to understand the individual orthogonal polynomials.

Orthogonal polynomial expansion

It was observed in [AHM2], [AHM3] that the orthogonal polynomials have the following limit:

$$|p_n(z)|^2 \mathrm{e}^{-mQ(z)} \mathrm{d}A(z) \to \mathrm{d}\omega_\tau(z), \qquad n = m\tau + o(1),$$

where the right-hand side expresses the *harmonic measure from* ∞ *in* $\mathbb{C} \setminus S_\tau$. In other words, the (first) hitting probability from Brownian starting at infinity and ending at ∂S_τ. With some further effort involving Euler-Maclaurin summation, a second correction term could be guessed from [AHM3]. Note that the left-hand side expresses a probability measure, which is analogous to how the mod-squared of the wave function is a probability distribution. This suggests that it might be possible to analyze p_n near ∂S_τ and in particular give a more detailed understanding of "the wave function probability".

Orthogonal polynomial expansion (2)

We should think of the bulk of S_τ as the domain of "diffusion", where information travels only approximately the distance $O(m^{-1/2})$. The exterior $\mathbb{C} \setminus S_\tau$ however is "rigid", and information travels instantaneously. Based on such thinking, we look for p_n of the form

$$p_n(z) \sim C_{m,n}\, \phi(z)^n \phi'(z)\, \mathrm{e}^{\frac{1}{2} m \mathcal{Q}(z)} (\mathcal{B}^0(z) + m^{-1}\mathcal{B}^1(z) + \dots),$$

where $C_{m,n} = O(m^{1/4})$ is a normalizing constant, ϕ is the conformal mapping $\mathbb{C} \setminus S_\tau \to \mathbb{D}_e := \{z : |z| > 1\}$ which fixes the point at infinity, $\mathcal{Q}(z)$ is a bounded holomorphic function in $\mathbb{C} \setminus S_\tau$ whose *real part equals Q on ∂S_τ*, and the functions $\mathcal{B}^j$ are to be found.

Orthogonal polynomial expansion (3)

The functions $\mathcal{B}^j$ are obtained algorithmically. For instance,

$$\mathcal{B}^0(z) = \mathrm{e}^{\mathcal{H}(z)},$$

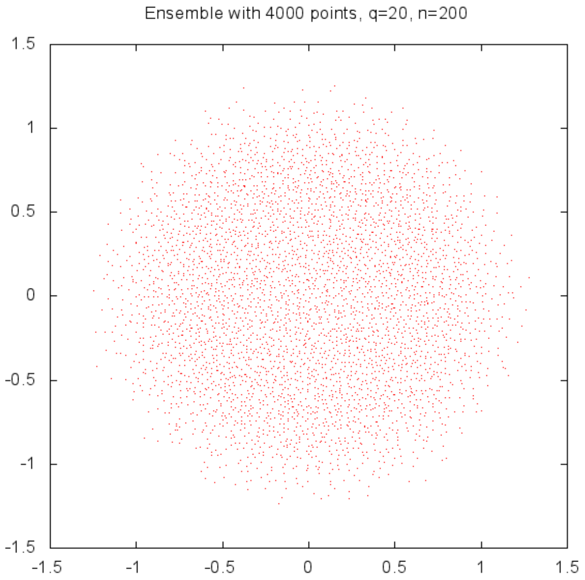
where $\mathcal{H}(z)$ is the bounded holomorphic function in $\mathbb{C} \setminus S_\tau$ whose real part equals

$$\operatorname{Re} \mathcal{H}(z) = \frac{1}{4} \log \frac{\Delta Q(z)}{|\phi'(z)|^2}, \qquad z \in \partial S_\tau.$$

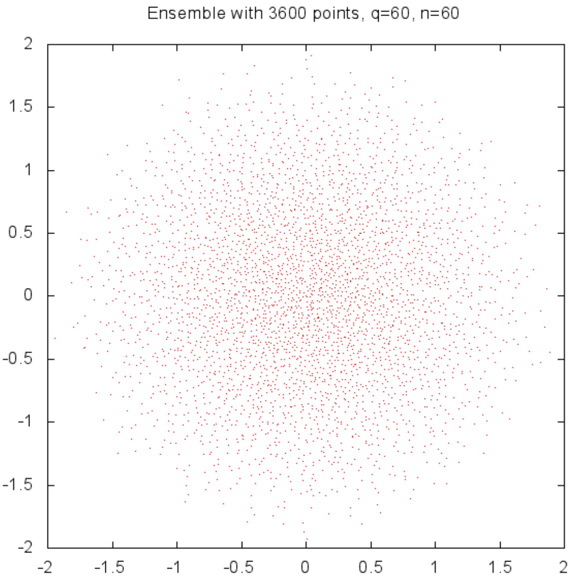
Polyanalytic ensembles

We might consider polynomials in z and $\bar{z}$, with the degree in $\bar{z}$ at most $q-1$, and the degree in z at most $n-1$. This was studied in [HH1] in the Ginibre case $Q(z)=|z|^2$, and in the general case in [HH2], [H1].

Simulation of the polyanalytic Ginibre (200X20 pts)



Simulation of the polyanalytic Ginibre (60X60 pts)



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