Toda chain

Vincent PASQUIER

IPhT Saclay

Collaboration with Olivier Babelon, Simon Ruisjenaars

Former contributions

Former contributions:

- Gutzwiller
- Gaudin
- Sklyanin
- Pasquier Gaudin
- Karchev Lebedev
- Shatashvili, Nekrasov
- Kozlowski Techner
- $\cdots$

Schroedinger in periodic potential

Hamiltonian:

$$H = p^2 + 2\cos(2\pi x)$$

Schroedinger in periodic potential

Hamiltonian:

$$H = p^2 + 2\cos(2\pi x)$$

Schroedinger equation:

$$\omega^2 \psi(x) = -\psi''(x) + 2\cos(2\pi x)$$

Schroedinger in periodic potential

Hamiltonian:

$$H = p^2 + 2\cos(2\pi x)$$

Schroedinger equation:

$$\omega^2\psi(x) = -\psi''(x) + 2\cos(2\pi x)$$

- electrons moving in a periodic potential.
- Floquet theory $\psi(x) = e^{i\mu x}\sum_n a_n e^{2\pi i n x}$.

Hill determinant

Recursion:

$$a_{n+1} + a_{n-1} + V_n a_n = 0$$

with:

$$V_n = (n + \mu)^2 - \omega^2$$

Hill determinant

Recursion:

$$a_{n+1} + a_{n-1} + V_n a_n = 0$$

with:

$$V_n = (n + \mu)^2 - \omega^2$$

- Two decaying solutions at infinity need to be matched.
- Wronskian equal to zero is the infinite determinant of a tridiagonal matrix:

$$W = |1/V_n, 1, 1/V_n|$$

Hill determinant

Recursion:

$$a_{n+1} + a_{n-1} + V_n a_n = 0$$

with:

$$V_n = (n+\mu)^2 - \omega^2$$

- Two decaying solutions at infinity need to be matched.
- Wronskian equal to zero is the infinite determinant of a tridiagonal matrix:

$$W = |1/V_n, 1, 1/V_n|$$

Hill determinant-2

Periodicity and asymptotic:

$$W(\mu) = \frac{sin(\mu-\delta)\sin(\mu+\delta)}{\sin(\mu-\omega)\sin(\mu+\omega)}$$

Determines the spectrum as the zeros of W:

$$\mu = \delta(\omega)$$

Hofstadter

Hamiltonian:

$$H = X + X^{\dagger} + Y + Y^{\dagger}$$

Hofstadter

Hamiltonian:

$$H = X + X^\dagger + Y + Y^\dagger$$

$$X = e^{i\alpha n+\theta}, \;\; Yf(n) = f(n+1)$$

Hofstadter

Hamiltonian:

$$H = X + X^\dagger + Y + Y^\dagger$$

$$X = e^{i\alpha n+\theta},\ \ Yf(n) = f(n+1)$$

Schroedinger equation:

$$E\psi_n = \psi_{n+1} + \psi_{n-1} + 2\cos(\alpha n + \theta)\psi_n$$

Hofstadter

Hamiltonian:

$$H = X + X^\dagger + Y + Y^\dagger$$

$$X = e^{i\alpha n+\theta},\ \ Yf(n) = f(n+1)$$

Schroedinger equation:

$$E\psi_n = \psi_{n+1} + \psi_{n-1} + 2\cos(\alpha n + \theta)\psi_n$$

- Lattice electrons moving in a periodic potential α times the lattice period.
- 2D lattice electrons in a strong magnetic field with α fluxes per unit cell.

How to obtain spectrum:

Construct recursion relation for ψ:

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} E - V_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

$$V_k = 2\cos(2\pi\alpha k)$$

How to obtain spectrum:

Construct recursion relation for ψ:

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} E - V_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

$$V_k = 2\cos(2\pi\alpha k)$$

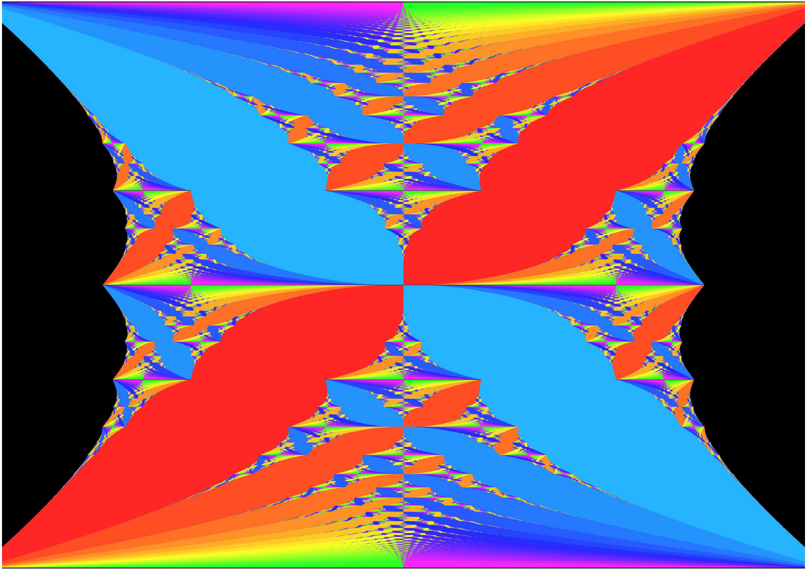
Spectrum for $|\psi_n|$ bounded when $|n \pm \infty$.
if $\alpha = p/q$,

$$-2 < tr(M^q) < 2$$

$$tr(M^q) = g(E) \pm 2\cos(q\theta)$$

$|g(E)|$ is a polynomial of degree q.
Typically q bands. Fractal spectrum.

Hofstadter



Toda chain

- Dynamics

$$H = \sum_{i=1}^{N} p_i^2 + e^{q_i - q_{i+1}}$$

Toda chain

- Dynamics

$$H = \sum_{i=1}^{N} p_i^2 + e^{q_i - q_{i+1}}$$

Two particles in the center of mass frame:

$$-\frac{d^2\psi}{dq^2} + 4\cosh(q)\psi = E\psi$$

- Hyperbolique Mathieu equation

Hill determinant

- Model is integrable in addition to the Hamiltonian N conserved quantities can be encoded in a transfer matrix $T(u)$
- The complete solution holds in a single Mathieu equation:

$$T(u)q(u) = i^N q(u + i\hbar) + i^{-N} q(u - i\hbar)$$

Hill determinant

- Model is integrable in addition to the Hamiltonian N conserved quantities can be encoded in a transfer matrix $T(u)$
- The complete solution holds in a single Mathieu equation:

$$T(u)q(u) = i^N q(u + i\hbar) + i^{-N} q(u - i\hbar)$$

- $q(u)$ is an entire function vanishing as $e^{-N\pi|u|/2}$ in the real axis.

Hill determinant

- Model is integrable in addition to the Hamiltonian N conserved quantities can be encoded in a transfer matrix $T(u)$
- The complete solution holds in a single Mathieu equation:

$$T(u)q(u) = i^N q(u + i\hbar) + i^{-N} q(u - i\hbar)$$

- $q(u)$ is an entire function vanishing as $e^{-N\pi|u|/2}$ in the real axis.
- Two solutions Q^+, Q^- with the incorrect asymptotic behavior can be constructed by solving the recursion.
- We can recover the correct asymptotic by dividing them with a product of $\sinh(u - u_k)$:

Bethe equations

$$\frac{Q_+(u) - \zeta Q_-(u)}{\prod_{k=1}^{N} \sinh(u - u_k)}$$

- Analyticity? how to eliminate the poles?
- Require the numerator to vanish, Wronskian:

$$W = |1/T(u+in), 1, 1/T(u+in)|$$

Bethe equations

$$\frac{Q_+(u) - \zeta Q_-(u)}{\prod_{k=1}^{N} \sinh(u - u_k)}$$

- Analyticity? how to eliminate the poles?
- Require the numerator to vanish, Wronskian:

$$W = |1/T(u + in), 1, 1/T(u + in)|$$

$$W = \frac{\prod_{k=1}^{N} \sinh(u - \delta_k)}{\prod_{k=1}^{N} \sinh(u - u_k)}$$

Bethe equations

$$\frac{Q_+(u) - \zeta Q_-(u)}{\prod_{k=1}^N \sinh(u-u_k)}$$

- Analyticity? how to eliminate the poles?
- Require the numerator to vanish, Wronskian:

$$W = |1/T(u+in), 1, 1/T(u+in)|$$

$$W = \frac{\prod_{k=1}^N \sinh(u-\delta_k)}{\prod_{k=1}^N \sinh(u-u_k)}$$

- Quantization conditions:

$$\zeta = \frac{Q_+(\delta_k)}{Q_-(\delta_k)}$$

Does not depend on k.

deformed Toda Lattice

- Is there some analogous of lattice Schroedinger for Toda?
- Ruisjenaars Hamiltonian:

$$H = \sum_1^N Y_k(1 - X_{k-1}/X_k)$$

deformed Toda Lattice

- Is there some analogous of lattice Schroedinger for Toda?
- Ruisjenaars Hamiltonian:

$$H = \sum_1^N Y_k(1 - X_{k-1}/X_k)$$

$$X = e^x, \;\; Yf(x) = f(x + i\alpha)$$

- α is the analogous of the flux per unit cell.
- i has migrated from X to Y:

$X^\dagger = X$, $Y^\dagger = Y$.

- Can easily be truncated on Harmonic oscillator basis.

Hill determinant

- Model is integrable in addition to the Hamiltonian N conserved quantities can be encoded in a transfer matrix $T(u)$
- The complete solution holds in a single Mathieu equation:

$$T(u)q(u) = i^N q(u + i\alpha) + i^{-N} q(u - i\alpha)$$

Hill determinant

- Model is integrable in addition to the Hamiltonian N conserved quantities can be encoded in a transfer matrix $T(u)$
- The complete solution holds in a single Mathieu equation:

$$T(u)q(u) = i^N q(u + i\alpha) + i^{-N} q(u - i\alpha)$$

- $T(u)$ is now a trigonometric polynomial: $T = \prod \sinh(u - u_k)$
  - $q(u)$ is an entire function.
  - Two solutions Q^+, Q^- with the incorrect asymptotic behavior cannot be constructed by solving the recursion.

The Hill determinant ??

$$W = |1/T(u + i\alpha n), 1, 1/T(u + i\alpha n)|$$

The Hill determinant ??

$$W = |1/T(u+i\alpha n), 1, 1/T(u+i\alpha n)|$$

- $T(u)$ is now a trigonometric polynomial, so, the situation is similar to Hofstadter, in particular periodicity for α rational.

The Hill determinant ??

$$W = |1/T(u+i\alpha n), 1, 1/T(u+i\alpha n)|$$

- $T(u)$ is now a trigonometric polynomial, so, the situation is similar to Hofstadter, in particular periodicity for α rational.

- Natural guess for the Hill determinant:

$$W = \frac{\prod_{k=1}^{N} \theta_1(u- \ \delta_k)}{\prod_{k=1}^{N} \theta_1(u-u_k)}$$

  - Period τ should be $i\alpha$ Absurd!

Conclusion

- Thank You Semyon for organizing such a nice conference!