The Losev-Manin Shuffle Algebra

PAUL LAUBIE

Contents

1 Tools on shuffle algebras **3**
- 1.1 Shuffle algebra . . . 3
- 1.2 Twisted associative algebra . . . 7
- 1.3 Gröbner basis . . . 10
- 1.4 Anick's resolution . . . 13

2 Computation on the Losev-Manin shuffle algebra **15**
- 2.1 The Losev-Manin shuffle algebra . . . 15
- 2.2 Computing a Gröbner basis . . . 17
- 2.3 The Anick resolution . . . 22

Abstract

The goal is to define and compute properties of the Losev-Manin shuffle algebra. This algebra arises from geometry, namely from the Losev-Manin spaces, however we will not work on nor define those spaces. Those spaces are defined in [5] but their definition is not needed to understand what we are going to discuss. We will start from a definition of this shuffle algebra given by generators and relations.

On the one hand, we will define the structures and tools we are going to use, see [4, 2, 1]. We will be working on shuffle algebras, and adapt the notion of Gröbner basis on those algebras. Finally we will define the notion of Anick's resolution also adapted on shuffle algebras.

On the other hand, we will apply those tools on the Losev-Manin shuffle algebra, starting by defining it, and compute a Gröbner basis to construct the Anick's resolution.

1 Tools on shuffle algebras

Let's note $[n] = \{1, \cdots, n\}$.

1.1 Shuffle algebra

We start by defining the "nonsymmetric collections", the definition may seem useless or trivial, however it is linked to the definition of "symmetric collections" in 1.2, indeed it is a symmetric collection without the symmetry.

Definition 1.1.1. A nonsymmetric collection is a sequence $\mathcal{V} = \mathcal{V}(n)_{n\geq 0}$ of $\mathbb{F}$-vector spaces. A morphism between two nonsymmetric collections $\mathcal{V}$ and $\mathcal{W}$ is a collection of linear maps $\phi_n : \mathcal{V}(n) \mapsto \mathcal{W}(n)$, for $n \geq 0$. If each ϕ_n is an embedding of a subspace, we call the collection of their images a subcollection of $\mathcal{W}$, and write $\mathcal{V} \subset \mathcal{W}$.

Example 1.1.1. The nonsymmetric collection $\underline{\mathbb{F}}$ is defined as follow:

$$\underline{\mathbb{F}}(k) = \begin{cases} \mathbb{F} \text{ if } k = 0 \\ 0 \text{ else} \end{cases}$$

Definition 1.1.2. Let $\mathcal{V}$ and $\mathcal{W}$ be two nonsymmetric collections. The direct sum $\mathcal{V} \oplus \mathcal{W}$ is defined by the formula:

$$(\mathcal{V} \oplus \mathcal{W})(n) = \mathcal{V}(n) \oplus \mathcal{W}(n)$$

The shuffle tensor product $\mathcal{V} \boxtimes \mathcal{W}$ is defined by the formula:

$$(\mathcal{V} \boxtimes \mathcal{W})(n) = \bigoplus_{I \sqcup J = [n]} \mathcal{V}(|I|) \otimes \mathcal{W}(|J|)$$

Where the sum is taken over all partitions of $[n]$ into two disjoint subsets I and J.

Property 1.1.3. *The shuffle tensor product is associative.*

Proof. Let $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ be three nonsymmetric collections,

$$\begin{aligned} ((\mathcal{U} \boxtimes \mathcal{V}) \boxtimes \mathcal{W})(n) &= \bigoplus_{I \sqcup J \sqcup K = [n]} (\mathcal{U}(|I|) \otimes \mathcal{V}(|J|)) \otimes \mathcal{W}(|K|) \\ &\simeq \bigoplus_{I \sqcup J \sqcup K = [n]} \mathcal{U}(|I|) \otimes (\mathcal{V}(|J|) \otimes \mathcal{W}(|K|)) = (\mathcal{U} \boxtimes (\mathcal{V} \boxtimes \mathcal{W}))(n) \end{aligned}$$

$\square$

Property 1.1.4. *The nonsymmetric collection $\underline{\mathbb{F}}$ is the unit element of the shuffle tensor product.*

Proof. Let $\mathcal{V}$ be a nonsymmetric collection,

$$(\mathcal{V} \boxtimes \underline{\mathbb{F}})(n) = \bigoplus_{I \sqcup J = [n]} \mathcal{V}(|I|) \otimes \underline{\mathbb{F}}(|J|)$$
$$= \mathcal{V}(|[n]|) \otimes \mathbb{F} = \mathcal{V}(n)$$

Same for $\underline{\mathbb{F}} \boxtimes \mathcal{V}$. □

Definition 1.1.5. Hence the shuffle tensor power can be defined as follow: $\mathcal{V}^{\boxtimes n}$ is the shuffle tensor product of n copies of $\mathcal{V}$ (and $\mathcal{V}^{\boxtimes 0} = \underline{\mathbb{F}}$).

Definition 1.1.6. A shuffle algebra is a monoid in the category of nonsymmetric collections with respect to the shuffle tensor product.

More concretely, the structure of shuffle algebra on a nonsymmetric collection $\mathcal{A}$ is given by a collection of maps:

$$\mu_{I,J} : \mathcal{A}(|I|) \boxtimes \mathcal{A}(|J|) \to \mathcal{A}(n)$$

For each $I \sqcup J = [n]$, and unit element $e \in \mathcal{A}(0)$, following the properties:

- *associativity*
$$\mu_{I \sqcup J,K} \circ (\mu_{I,J} \otimes \mathrm{id}) = \mu_{I,J \sqcup K} \circ (\mathrm{id} \otimes \mu_{J,K})$$
For each I, J and K finite part of $\mathbb{N}^*$. We note this map $\mu I, J, K$.

- *unit*
$$\mu_{\emptyset,[n]} \circ (e \otimes \mathrm{id}) = \mu_{[n],\emptyset} \circ (\mathrm{id} \otimes e) = \mathrm{id}$$
For every $n \in N$.

Definition 1.1.7. A free shuffle algebra generated by a nonsymmetric collection $\mathcal{M}$ is:

$$\mathcal{T}_{\text{Ш}}(\mathcal{M}) = \bigoplus_{k \geq 0} \mathcal{M}^{\boxtimes k}$$

Let $\mathcal{X} = (\mathcal{X}(n))_{n \geq 0}$ be a sequence of finite sets, $\mathbb{F}\mathcal{X} = (\mathbb{F}\mathcal{X}(n))_{n \leq 0}$ is a nonsymmetric collection, let's define $\mathcal{T}_{\text{Ш}}(\mathcal{X}) = \mathcal{T}_{\text{Ш}}(\mathbb{F}\mathcal{X})$

Definition 1.1.8. Let $\mathcal{X} = (\mathcal{X}(n))_{n \geq 0}$ be a sequence of finite sets. A shuffle monomial of arity $n \in \mathbb{N}$ and length $k \in \mathbb{N}$ of $\mathcal{X}$ is a couple (π, ν), with:

- $\pi = (I_1, \cdots, I_k)$ is an ordered partition of $[n]$ of length k;
- $\nu = (m^{(1)}, \cdots, m^{(k)})$ such that $m^{(\lambda)} \in \mathcal{X}(|I_\lambda|)$.

Let's note it:

$$m^{(1)}_{I_1} \cdots m^{(k)}_{I_k}$$

Let $m = m^{(1)} \otimes \cdots \otimes m^{(k)}$ be the corresponding element in $\mathcal{T}_{\text{Ш}}(\mathcal{X})$ we identify it with the monomial.
Shuffle polynomials are linear combinations of monomials of the same arity.
The monomial for which $K = \emptyset$ and $k = 0$ is the trivial monomial.

Property 1.1.9. *The monomials of arity n form a basis of $\mathcal{T}_{\text{Ш}}(\mathcal{X})(n)$.*

Proof. We have:

$$\mathcal{T}_{\text{Ш}}(\mathcal{X})(n) = \mathbb{F}\mathcal{X}(n) \oplus \Big(\bigoplus_{I_1 \sqcup I_2 = [n]} \mathbb{F}\mathcal{X}(|I_1|) \otimes \mathbb{F}\mathcal{X}(|I_2|) \Big) \oplus \Big(\bigoplus_{I_1 \sqcup I_2 \sqcup I_3 = [n]} \mathbb{F}\mathcal{X}(|I_1|) \otimes \mathbb{F}\mathcal{X}(|I_2|) \otimes \mathbb{F}\mathcal{X}(|I_3|) \Big) \oplus \cdots$$

Hence a basis of $\mathcal{T}_{\text{Ш}}(\mathcal{X})(n)$ is :

$$\mathcal{X}(n) \uplus \Big(\biguplus_{I_1 \sqcup I_2 = [n]} \mathcal{X}(|I_1|) \times \mathcal{X}(|I_2|) \Big) \uplus \Big(\biguplus_{I_1 \sqcup I_2 \sqcup I_3 = [n]} \mathcal{X}(|I_1|) \times \mathcal{X}(|I_2|) \times \mathcal{X}(|I_3|) \Big) \uplus \cdots$$

Since $\biguplus_{I_1 \sqcup \cdots \sqcup I_k = [n]} \mathcal{X}(|I_1|) \times \cdots \times \mathcal{X}(|I_k|)$ is the set of monomials of length k and arity n, the result is proven.
(The two symbols $\uplus$ and $\sqcup$ are used differently, $A \sqcup B$ states that A and B are disjoint and builds the union, whereas $A \uplus B$ constructs a disjoint union even if A and B are not disjoints.) $\square$

Remark 1.1.2. This leads to a more explicit construction of a free shuffle algebra by taking the span of monomials of the same arity.

Definition 1.1.10. Let m be a monomial. A left (resp. right) factor m_{left} (resp. m_{right}) of m is a monomial such that we have a product map $\mu_{I,J}$ and a monomial r such that $m = \mu_{I,J}(m_{left}, r)$ (resp. $m = \mu_{I,J}(r, m_{right})$). Since the product map is uniquely determined by the factor, let's write it $m = m_{left}r$.
A divisor q of m is a monomial such that we have a product map $\mu_{I,J,K}$, a left and a right factor m_{left} and m_{right} such that $m = \mu_{I,J,K}(m_{left}, q, m_{right})$. Since the product map is uniquely determined by the choice of the factors, let's write it $m = m_{left}qm_{right}$
The monomial m is divisible by q.

Definition 1.1.11. A (double sided) ideal of a shuffle algebra is a nonsymmetric subcollection $\mathcal{I} \subset \mathcal{A}$ such that for $\mu : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ the structure map, we have:

$$\mu(\mathcal{I} \boxtimes \mathcal{A} + \mathcal{A} \boxtimes \mathcal{I}) \subset \mathcal{I}$$

Property 1.1.12. *Let $\mathcal{A}$ be a shuffle algebra and $\mathcal{I}$ an ideal of $\mathcal{A}$, the quotient $\mathcal{A}/\mathcal{I}$ is well defined and is a shuffle algebra.*

Proof. The nonsymmetric collection $\mathcal{A}/\mathcal{I}$ is defined by $(\mathcal{A}/\mathcal{I})(n) = \mathcal{A}(n)/\mathcal{I}(n)$.
The ideal condition over $\mathcal{I}$ ensures that the structure maps of $\mathcal{A}$ (the product maps $\mu_{I,J}$) go to the quotient.
Hence $\mathcal{A}/\mathcal{I}$ have a structure of shuffle algebra. $\square$

Definition 1.1.13. Let $\mathcal{A}$ be a shuffle algebra and $S = (S(n))_{n \geq 0}$ with $S(n) \subset \mathcal{A}(n)$. The Ideal $\mathcal{I}(S)$ generated by $\mathcal{S}$ is the smallest (for the inclusion) ideal of $\mathcal{A}$ containing S. It is well defined since an intersection of ideals is an ideal.

Definition 1.1.14. A presentation by generators and relations is a couple $(\mathcal{X}, R)$ such that $\mathcal{X} = (\mathcal{X}(n))_{n\geq 0}$ and $R = (R(n))_{n\geq 0}$, the $R(n)$ and the $\mathcal{X}(n)$ are finite, and $R(n) \subset \mathcal{T}_{Ш}(\mathcal{X})$.
The corresponding shuffle algebra is $\mathcal{A} = \mathcal{T}_{Ш}(\mathcal{X})/\mathcal{I}(R)$.

Definition 1.1.15 (Losev-Manin shuffle algebra)**.** Let $\mathcal{X} = (\emptyset, \{\alpha\}, \{\alpha\}, \cdots)$,
let $r_{n,a,b} = \sum_{\substack{A\ni a \\ B\ni b \\ A\sqcup B=[n]}} (\alpha_A\alpha_B - \alpha_B\alpha_A)$
and let $R = (\{r_{n,a,b} \mid a \neq b \leq n\})_{n\geq 0}$.
The Losev-Manin shuffle algebra is the shuffle algebra defined by generators and relations $(\mathcal{X}, R)$.

1.2 Twisted associative algebra

We now define the twisted associative algebras which are in fact the main objects we want to study. Twisted associative algebras are shuffle algebras with a symmetric structure on them, however, this symmetric structure makes it impossible to define a compatible order on monomials, and so impossible to define Gröbner bases.

Definition 1.2.1. A symmetric collection is a sequence $\mathcal{V}_{\mathfrak{S}} = \mathcal{V}_{\mathfrak{S}}(n)_{n\geq 0}$ of $\mathbb{F}$-vector spaces together with an action of $\mathfrak{S}_n$ on $\mathcal{V}_{\mathfrak{S}}(n)$. A morphism between two symmetric collections $\mathcal{V}_{\mathfrak{S}}$ and $\mathcal{W}_{\mathfrak{S}}$ is a collection of $\mathfrak{S}_n$-equivariant linear maps $\phi_n : \mathcal{V}(n) \mapsto \mathcal{W}(n)$, for $n \geq 0$. If each ϕ_n is an embedding of a subspace, we call the collection of their images a subcollection of $\mathcal{W}_{\mathfrak{S}}$, and write $\mathcal{V}_{\mathfrak{S}} \subset \mathcal{W}_{\mathfrak{S}}$.

Example 1.2.1. The nonsymmetric collection $\underline{\mathbb{F}}$ has a unique structure of symmetric collection (by trivial action of $\mathfrak{S}_n$). We note the symmetric collection $\underline{\mathbb{F}}_{\mathfrak{S}}$.

Definition 1.2.2. Let $\mathcal{V}_{\mathfrak{S}}$ and $\mathcal{W}_{\mathfrak{S}}$ be two symmetric collections. The direct sum $\mathcal{V}_{\mathfrak{S}} \oplus \mathcal{W}_{\mathfrak{S}}$ is defined by the formula:

$$(\mathcal{V}_{\mathfrak{S}} \oplus \mathcal{W}_{\mathfrak{S}})(n) = \mathcal{V}_{\mathfrak{S}}(n) \oplus \mathcal{W}_{\mathfrak{S}}(n)$$

The tensor product $\mathcal{V}_{\mathfrak{S}} \otimes \mathcal{W}_{\mathfrak{S}}$ is defined by the formula:

$$(\mathcal{V}_{\mathfrak{S}} \otimes \mathcal{W}_{\mathfrak{S}})(n) = \bigoplus_{k=0}^{n} \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \mathcal{V}_{\mathfrak{S}}(k) \otimes \mathcal{W}_{\mathfrak{S}}(n-k)$$

With $\mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \mathcal{V}_{\mathfrak{S}}(k) \otimes \mathcal{W}_{\mathfrak{S}}(n-k)$ the $\mathfrak{S}_n$-representation induced by the $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$-representation $\mathcal{V}_{\mathfrak{S}}(k) \otimes \mathcal{W}_{\mathfrak{S}}(n-k)$.

Property 1.2.3. *Let $\mathcal{V}_{\mathfrak{S}}$ and $\mathcal{W}_{\mathfrak{S}}$ be two symmetric collections and $\mathcal{V}$ and $\mathcal{W}$ the underlying nonsymmetric collections. We have that the underlying nonsymmetric collection of $\mathcal{V}_{\mathfrak{S}} \otimes \mathcal{W}_{\mathfrak{S}}$ is isomorphic to $\mathcal{V} \boxtimes \mathcal{W}$.*

Proof. It comes from the fact that for any $\mathfrak{S}_i$-representation E and $\mathfrak{S}_j$-representation F, we have:

$$\mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}} E \otimes F \simeq \bigoplus_{\substack{|I|=i \\ |J|=j \\ I \sqcup J = [i+j]}} E \otimes F$$

This follow from the definition of the induced representation of symmetric groups. $\square$

Remark 1.2.2. The shuffle tensor product of nonsymmetric collections is in fact the tensor product of symmetric collections on which we forget the symmetric structure.

Property 1.2.4. *The tensor product of symmetric collections is associative.*

Proof. Let $\mathcal{U}_\mathfrak{S}, \mathcal{V}_\mathfrak{S}$ and $\mathcal{W}_\mathfrak{S}$ be three symmetric collections,

$$((\mathcal{U}_\mathfrak{S} \otimes \mathcal{V}_\mathfrak{S}) \otimes \mathcal{W}_\mathfrak{S})(n) = \bigoplus_{I \sqcup J \sqcup K = [n]} (\mathcal{U}_\mathfrak{S}(|I|) \otimes \mathcal{V}_\mathfrak{S}(|J|)) \otimes \mathcal{W}_\mathfrak{S}(|K|)$$
$$\simeq \bigoplus_{I \sqcup J \sqcup K = [n]} \mathcal{U}_\mathfrak{S}(|I|) \otimes (\mathcal{V}_\mathfrak{S}(|J|) \otimes \mathcal{W}_\mathfrak{S}(|K|)) = (\mathcal{U}_\mathfrak{S} \otimes (\mathcal{V}_\mathfrak{S} \otimes \mathcal{W}_\mathfrak{S}))(n)$$

□

Property 1.2.5. *The symmetric collection $\underline{\mathbb{F}}_\mathfrak{S}$ is the unit element of the tensor product of symmetric collections.*

Proof. Let $\mathcal{V}_\mathfrak{S}$ be a symmetric collection,

$$(\mathcal{V}_\mathfrak{S} \otimes \underline{\mathbb{F}}_\mathfrak{S})(n) = \bigoplus_{I \sqcup J = [n]} \mathcal{V}_\mathfrak{S}(|I|) \otimes \underline{\mathbb{F}}\mathfrak{S}(|J|)$$
$$= \mathcal{V}_\mathfrak{S}(|[n]|) \otimes \mathbb{F} = \mathcal{V}_\mathfrak{S}(n)$$

Same for $\underline{\mathbb{F}}_\mathfrak{S} \otimes \mathcal{V}_\mathfrak{S}$. □

Definition 1.2.6. Hence the tensor power of symmetric collections can be defined as follow: $\mathcal{V}_\mathfrak{S}^{\otimes n}$ is the tensor product of n copies of $\mathcal{V}_\mathfrak{S}$.

Definition 1.2.7. A twisted associative algebra is a monoid in the category of symmetric collections with respect to the tensor product.

Remark 1.2.3. One may find a more concrete way to define twisted associative algebras, the same way we gave a more concrete way to define a shuffle algebra.

Definition 1.2.8. A free twisted associative algebra generated by a symmetric collection $\mathcal{M}_\mathfrak{S}$ is:

$$\mathcal{T}_\Sigma(\mathcal{M}_\mathfrak{S}) = \bigoplus_{k \geq 0} \mathcal{M}_\mathfrak{S}^{\otimes k}$$

Let $\mathcal{X}_\mathfrak{S} = (\mathcal{X}_\mathfrak{S}(n))_{n \geq 0}$ be a sequence of finite sets together with an action of $\mathfrak{S}_n$ on each $\mathcal{X}_\mathfrak{S}(n)$, let $\mathbb{F}\mathcal{X}_\mathfrak{S}$ be the generated symmetric collection, let's define $\mathcal{T}_\Sigma(\mathcal{X}_\mathfrak{S}) = \mathcal{T}_\Sigma(\mathbb{F}\mathcal{X}_\mathfrak{S})$

Definition 1.2.9. A (double sided) ideal of a twisted associative algebra is a symmetric subcollection $\mathcal{I}_\mathfrak{S} \subset \mathcal{A}_\mathfrak{S}$ such that for $\mu : \mathcal{A}_\mathfrak{S} \otimes \mathcal{A}_\mathfrak{S} \to \mathcal{A}_\mathfrak{S}$ the structure map, we have:

$$\mu(\mathcal{I}_\mathfrak{S} \otimes \mathcal{A}_\mathfrak{S} + \mathcal{A}_\mathfrak{S} \otimes \mathcal{I}_\mathfrak{S}) \subset \mathcal{I}_\mathfrak{S}$$

Property 1.2.10. *Let $\mathcal{A}_\mathfrak{S}$ be a twisted associative algebra and $\mathcal{I}_\mathfrak{S}$ an ideal of $\mathcal{A}_\mathfrak{S}$, the quotient $\mathcal{A}_\mathfrak{S}/\mathcal{I}_\mathfrak{S}$ is well defined and is a twisted associative algebra.*

Proof. The symmetric collection $\mathcal{A}_\mathfrak{S}/\mathcal{I}_\mathfrak{S}$ is defined by $(\mathcal{A}_\mathfrak{S}/\mathcal{I}_\mathfrak{S})(n) = \mathcal{A}_\mathfrak{S}(n)/\mathcal{I}_\mathfrak{S}(n)$.
The ideal condition over $\mathcal{I}_\mathfrak{S}$ ensures that the structure maps of $\mathcal{A}_\mathfrak{S}$ go to the quotient.
Hence $\mathcal{A}_\mathfrak{S}/\mathcal{I}_\mathfrak{S}$ has a structure of twisted associative algebra. □

Definition 1.2.11. Let $\mathcal{A}_{\mathfrak{S}}$ be a twisted associative algebra and $S = (S(n))_{n\geq 0}$ with $S(n) \subset \mathcal{A}_{\mathfrak{S}}(n)$. The ideal $\mathcal{I}_{\mathfrak{S}}(S)$ generated by $\mathcal{S}$ is the smallest (for the inclusion) ideal of $\mathcal{A}_{\mathfrak{S}}$ containing S. It is well defined since an intersection of ideals is an ideal.

Property 1.2.12. *Let $\mathcal{A}_{\mathfrak{S}}$ be a twisted associative algebra, the underlying nonsymmetric collection is a shuffle algebra.*

Proof. One may construct the forgetful fonctor between the category of symmetric collections and the category of nonsymmetric collections, applying it to a twisted associative algebra proves the result. $\square$

Corollary 1.2.1. *The definition of shuffle monomials leads to a definition of monomials in the twisted associative algebras.*

Definition 1.2.13. A presentation by generators and relations is a couple $(\mathcal{X}_{\mathfrak{S}}, R)$ such that $\mathcal{X}_{\mathfrak{S}} = (\mathcal{X}_{\mathfrak{S}}(n))_{n\geq 0}$ and $R = (R(n))_{n\geq 0}$, the $R(n)$ and the $\mathcal{X}(n)$ are finite, and $R(n) \subset \mathcal{T}_{\Sigma}(\mathcal{X}_{\mathfrak{S}})$.
The corresponding twisted associative algebra is $\mathcal{A}_{\mathfrak{S}} = \mathcal{T}_{\Sigma}(\mathcal{X}_{\mathfrak{S}})/\mathcal{I}_{\mathfrak{S}}(R)$.

Theorem 1.2.1. *The Losev-Manin shuffle algebra has a unique structure of twisted associative algebra.*

Proof. Let $\mathcal{X} = (\emptyset, \{\alpha\}, \{\alpha\}, \cdots)$,
let $r_{n,a,b} = \sum_{\substack{A\ni a\\ B\ni b\\ A\sqcup B=[n]}} (\alpha_A\alpha_B - \alpha_B\alpha_A)$
and let $R = (\{r_{n,a,b} \mid a \neq b \leq n\})_{n\geq 0}$.
There is a unique structure of symmetric collection $\mathcal{X}_{\mathfrak{S}}$ on $\mathcal{X}$ (given by the trivial action of $\mathfrak{S}_n$).
Hence the twisted associative algebra $\mathcal{T}_{\Sigma}(\mathcal{X}_{\mathfrak{S}})/\mathcal{I}_{\mathfrak{S}}(R)$ is well defined.
Moreover, those actions give an action of $\mathfrak{S}_n$ on $R(n)$ by $\sigma.r_{n,a,b} = r_{n,\sigma(a),\sigma(b)}$. Hence the underlying nonsymmetric collection of $\mathcal{I}_{\mathfrak{S}}(R)$ is $\mathcal{I}(R)$.
Hence the underlying nonsymmetric collection of $\mathcal{T}_{\Sigma}(\mathcal{X}_{\mathfrak{S}})/\mathcal{I}_{\mathfrak{S}}(R)$ is the Losev-Manin shuffle algebra. $\square$

Remark 1.2.4. In fact, the algebraic object that arises from geometry is the Losev-Manin twisted associative algebra, but because of the next result (Property 1.3.1), we will mostly work with the shuffle algebra. The goal was to show that this algebra has many symmetries and very deep connections with the symmetric groups and their representations.

1.3 Gröbner basis

The goal is to get "canonical" elements (namely the normal forms) in the cosets of quotient algebras, in particular in the case of algebras defined by generators and relations. To do so, we need to define the notion of Gröbner basis. We will prove that it cannot be done over twisted associative algebras, thus we will forget their symmetry and define it over shuffle algebras.

Property 1.3.1. *In general, it is not always possible to define a total ordering of basis elements of a twisted associative algebra which would lead to normal forms in quotient twisted associative algebras.*

Proof. The idea of the proof is to work on an example. Let $\mathcal{A}_{\mathfrak{S}} = \mathcal{T}_{\Sigma}(\mathcal{X}_{\mathfrak{S}})$ with $\mathcal{X}_{\mathfrak{S}} = (\emptyset, \{\alpha\}, \emptyset, \emptyset, \cdots)$ and $\mathcal{I}_{\mathfrak{S}} = \mathcal{I}_{\mathfrak{S}}(\alpha_1\alpha_2 - \alpha_2\alpha_1)$. One can remark, that $\mathcal{I}_{\mathfrak{S}}(\alpha_1\alpha_2) = \mathcal{I}_{\mathfrak{S}}(\alpha_2\alpha_1)$ let's note it $LT(\mathcal{I}_{\mathfrak{S}})$. Moreover, $\mathcal{I}_{\mathfrak{S}} \subsetneq LT(\mathcal{I}_{\mathfrak{S}})$, and thus the underlying vector spaces of $\mathcal{A}_{\mathfrak{S}}/\mathcal{I}_{\mathfrak{S}}$ and $\mathcal{A}_{\mathfrak{S}}/LT(\mathcal{I}_{\mathfrak{S}})$ have different dimensions.
A more complete proof is given in [2]. □

Remark 1.3.1. This result means that Gröbner basis cannot be adapted to twisted associative algebra. Hence we will define them over shuffle algebra.

Definition 1.3.2. Let $\mathcal{F}$ be a free shuffle algebra and let Ш_n be the set of its monomials of arity n. A monomials order Ξ on $\mathcal{F}$ is a sequence of orders Ξ_n on Ш_n such that:

- Each Ξ_n is a well-order,
- Each shuffle product maps (the $\mu_{I,J}$ in Definition 1.1.6) are strictly increasing in both variables.

Let $\mathcal{F} = \mathcal{T}_{\text{Ш}}(\mathcal{X})$ be a free shuffle algebra and Ξ a monomial order on it.

Property 1.3.3. *Let $\mathcal{I}$ be an ideal of $\mathcal{F}$, then $LT(\mathcal{I})$ the space of leading terms of $\mathcal{I}$ is also an ideal.*

Proof. By definition $LT(\mathcal{I})$ is the nonsymmetric subcollection of $\mathcal{F}$ generated by the leading terms of $\mathcal{I}$. Hence it suffices to prove that $LT(\mathcal{I})$ is stable by the product maps $\mu_{I,J}$. Let $P, Q \in \mathcal{I}$ and let p, q be their leading terms. Since $\mu_{I,J}$ is increasing, the leading terms of $\mu_{I,J}(P, Q)$ is $\mu_{I,J}(p, q)$.This ends the proof. □

Definition 1.3.4. Let $\mathcal{G} = (\mathcal{G}(n))_{n\geq 0}$, such that $\mathcal{G}(n) \subset \mathcal{I}(n)$. It is a Gröbner basis with respect to the monomial order Ξ as long as $LM(\mathcal{G})$ the set of leading monomials of $\mathcal{G}$ generates $LT(\mathcal{I})$ (as an ideal).

Property 1.3.5. *A Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ generates $\mathcal{I}$ (as an ideal).*

Proof. Let's assume that $\mathcal{I}(\mathcal{G}) \subsetneq \mathcal{I}$, let $f \in \mathcal{I} \setminus \mathcal{I}(\mathcal{G})$ with the least possible leading monomial. By definition of a Gröbner basis, we have $g \in \mathcal{G}$ such that $LM(g)$ is a divisor of $LM(f)$. Hence $LM(f) = m_{left}LM(g)m_{righ}$, let $r_g(f) = f - m_{left}gm_{right}$, we have $r_g(f) \in \mathcal{I}$ and $r_g(f) \notin \mathcal{I}(\mathcal{G})$, moreover, the leading term of $r_g(f)$ is smaller than the leading term of f. Hence the leading term of f is not minimal.

$\lightning$
Contradiction

□

Definition 1.3.6. Let $\mathcal{S}$ be a family of polynomials, a reduced monomial with respect to $\mathcal{S}$ is a monomial m such that m is not divisible by any leading monomial of elements of $\mathcal{S}$.
A reduced polynomial with respect to $\mathcal{S}$ is a polynomial P which is a linear combination of monomials that are reduced with respect to $\mathcal{S}$.
The family $\mathcal{S}$ is self-reduced as long as for any $P \in \mathcal{S}$, the polynomial P is reduced with respect to $\mathcal{S} \setminus \{P\}$.
A reduced form of a polynomial P with respect to $\mathcal{S}$ is a polynomial Q such that Q is reduced with respect to $\mathcal{P}$ and $P = Q + R$ with $R \in \mathcal{I}(\mathcal{S})$. In general, the reduced form is NOT unique.

Definition 1.3.7. Let's define the following algorithm for P a polynomial and $\mathcal{S}$ a family of polynomials:

```
Def Reduction(P,S):
  If P is reduced with respect to S:
    Return P;
  Else:
    Let m, m_left, m_right and s such that m = m_left LM(s) m_right
      with s in S
      and m the greater unreduced monomial of P;
    Let r_s(P) = P - m_left s m_right;
    Return Reduction(r_s(P),S);
```

Property 1.3.8. *The* Reduction *algorithm terminates and returns a reduced form of P with respect to $\mathcal{S}$ (hence any polynomial admit a reduced form).*

Proof. The algorithm terminates since the greater unreduced monomial of $r_s(P)$ is lesser than the greater unreduced monomial of P and since that Ξ_n is a well order. The polynomial returned is reduced and since at each step we subtract an element of $\mathcal{I}(\mathcal{S})$, it is a reduced form. □

Property 1.3.9. *Let $\mathcal{I}$ be an ideal of $\mathcal{F}$, then $\mathcal{G} \subset \mathcal{I}$ is a Gröbner basis if and only if the cosets of monomials that are reduced with respect to $\mathcal{G}$ form a basis of $\mathcal{F}/\mathcal{I}$ (of each component as a vector space).*

Proof. Let us note that the cosets of monomials that are reduced with respect to $\mathcal{G}$ form a basis of the quotient $\mathcal{F}/\mathcal{I}$ if and only if every coset contains a unique element that is reduced with respect to $\mathcal{G}$.

First of all, since each polynomial admits a reduced form, each coset contains a reduced element whether $\mathcal{G}$ is a Gröbner basis or not.
Suppose now that$\mathcal{G}$ is a Gröbner basis of $\mathcal{I}$. Suppose that the cosets of reduced monomials are linearly dependent, or, in other words, that the zero coset which is $\mathcal{I}$ contains a nonzero reduced element f. In that case, $LM(f) \in LT(\mathcal{I})$ is reduced with respect to $\mathcal{G}$, which is a contradiction.

$\lightning$
Contradiction

Suppose that $\mathcal{G}$ is not a Gröbner basis. This implies that there exists an element $f \in \mathcal{I} \setminus \{0\}$ for which $LM(f)$ is reduced with respect to $\mathcal{G}$. Hence the reduction of f (by the `Reduction` algorithm) is a nonzero reduced form of zero.

$\lightning$
Contradiction

$\square$

Corollary 1.3.1. *$\mathcal{G}$ is a Gröbner basis if and only if the reduced form with respect to $\mathcal{G}$ is unique for every polynomials.*

Corollary 1.3.2 (Diamond lemma)**.** *$\mathcal{G}$ is a Gröbner basis if and only if the result of the* `Reduction` *algorithm does not depend of the order of the reductions.*

Remark 1.3.2. This diamond lemma is a meaningful result if we interpret the Gröbner basis as a way to avoid patterns in monomials when seeing them as words over the alphabet of their generators.

Theorem 1.3.1. *An ideal $\mathcal{I}$ admits a unique reduced Gröbner basis (up to multiplication by a scalar).*

Proof. Let us first prove uniqueness. If $\mathcal{G}$ is a Gröbner basis, then $LT(\mathcal{I}) = \mathcal{I}(LM(G))$; if $\mathcal{G}$ is reduced, then $LM(\mathcal{G}) \subset LM(\mathcal{I})$ must coincide with the set $\mathcal{M}$ of elements of $LM(\mathcal{I})$ that are not divisible by other elements of $LM(\mathcal{I})$. (In other words, $\mathcal{M}$ is the set of minimal elements of $LM(\mathcal{I})$ with respect to the partial order of divisibility). For each $m \in \mathcal{M}$ we have a unique $g \in \mathcal{G}$, moreover $m - g$ is reduces with respect to $\mathcal{I}$ since $\mathcal{G}$ is self-reduced, hence $m - g$ is the reduced form of m.
Let's construct $\mathcal{G}$ as we prove that it must look like. Let $g = m - h$ with $m \in \mathcal{M}$ and h its reduced form. The set M generates $LT(\mathcal{I})$ since if $\mathcal{M}$ does not generate $LT(\mathcal{I})$, the smallest element not divisible by any element of $\mathcal{M}$ must be in $\mathcal{M}$ by definition. Hence $\mathcal{G}$ is a Gröbner basis which is self reduced by construction. $\square$

1.4 Anick's resolution

Let $\mathcal{A}$ be a shuffle algebra defined by generators and relations $(\mathcal{X}, R)$ such that R is a Gröbner basis. Let $\widetilde{R}$ be the leading monomials of elements of R. and $\widetilde{\mathcal{A}}$ the shuffle algebra defined by $(\mathcal{X}, \widetilde{R})$. Let $\mathcal{F} = \mathcal{T}_{\text{Ш}}(\mathcal{X})$.

Definition 1.4.1. To build the Anick resolution, we need to define the k-chains and there tails. Let C_k be the set of k-chains, they are defined inductively.
The 0-chains are the generators and their tails are themselves.
The 1-chains are elements of $\widetilde{R}$, and let $m = m^{(1)}_{I_1} \cdots m^{(k)}_{I_k}$ be a 1-chain, its tail is $t(m) = m^{(2)}_{I_2} \cdots m^{(k)}_{I_k}$.
A monomial m is a n-chain as long as it satisfies those three conditions:

- *(1)*: $m = m' t(m)$ with m' a $(n-1)$-chain and $t(m)$ not divisible by any element of $\widetilde{R}$, we define $t(m)$ as its tail.
- *(2)*: $m' = m'' t(m')$ and $t(m')t(m)$ is divisible by an element of $\widetilde{R}$.
- *(3)*: No proper beginning of m satisfy both (1) and (2).

By convention, 1 is the only (-1)-chain and is its own tail.

Definition 1.4.2. Let $C_k(n)$ be the set of k-chains of arity n. Let $\mathcal{V}_k(n)$ be the span of $C_k(n)$, and let $\mathcal{V}_k$ be the associated nonsymmetric collection.

Definition / Property 1.4.3. Let $\widetilde{\mathcal{A}} = \mathcal{F}/\widetilde{R}$. We have an obvious resolution of $\mathbb{F}$ as a $\widetilde{\mathcal{A}}$-module:

$$\mathbb{F} \xleftarrow{\widetilde{\epsilon}} \mathbb{F} \boxtimes \widetilde{\mathcal{A}} \xleftarrow{\widetilde{d_0}} V_0 \boxtimes \widetilde{\mathcal{A}} \xleftarrow{\widetilde{d_1}} \cdots \xleftarrow{\widetilde{d_{k-1}}} V_{k-1} \boxtimes \widetilde{\mathcal{A}} \xleftarrow{\widetilde{d_k}} V_k \boxtimes \widetilde{\mathcal{A}} \xleftarrow{\widetilde{d_{k+1}}} \cdots$$

With $\widetilde{d_k} : m^{(1)}_{I_1} \cdots m^{(\lambda=1)}_{I_{\lambda+1}} \otimes 1 \mapsto m^{(1)}_{I_1} \cdots m^{(\lambda)}_{I_\lambda} \otimes m^{(\lambda+1)}_{I_{\lambda+1}}$.

The idea beyond the Anick resolution is to modify this resolution of $\widetilde{\mathcal{A}}$-module a bit to get a resolution of $\mathcal{A}$-module by adding smaller monomials to $\widetilde{d_k}$.

Definition / Property 1.4.4. The Anick resolution is a free resolution of $\mathbb{F}$ viewed as a (right) $\mathcal{A}$-module:

$$\mathbb{F} \xleftarrow{\epsilon} \mathbb{F} \boxtimes \mathcal{A} \xleftarrow{d_0} V_0 \boxtimes \mathcal{A} \xleftarrow{d_1} \cdots \xleftarrow{d_{k-1}} V_{k-1} \boxtimes \mathcal{A} \xleftarrow{d_k} V_k \boxtimes \mathcal{A} \xleftarrow{d_{k+1}} \cdots$$

The maps d_k are defined inductively together with the maps $\iota_k : \ker(d_k) \to V_k \otimes \mathcal{A}$. The maps d_k are right $\mathcal{A}$-linear and the maps ι_k are $\mathbb{F}$-linear hence, it suffices to define d_k on every k-chain:

- *Base case*: Let's define $d_{-1} = \epsilon$ and $d_0 : m \otimes 1 \mapsto 1 \otimes m$.
  And $\iota_{-1} : 1 \mapsto 1 \otimes 1$ and $\iota_0 : 1 \otimes m^{(1)}_{I_1} \cdots m^{(k)}_{I_k} \mapsto m^{(1)}_{I_1} \otimes m^{(2)}_{I_2} \cdots m^{(k)}_{i_k}$.

- *Inductive step*: Let gt be a $(k+1)$-chain such that t is its tail.

  $$d_{k+1} : gt \otimes 1 \mapsto g \otimes t - \iota_k(d_k(g \otimes t))$$

  Let $u \in \ker(d_{k+1})$, let $\lambda f \otimes s$ be the leading term of u, let $f = hr$ with r its tail, rs is not reduced, we have $s = xy$ such that fx is a $(k+1)$-chain.

  $$\iota_{k+1} = \lambda fx \otimes y + \iota_{k+1}(u - d_{k+1}(\lambda fx \otimes y))$$

  (The order used is $m_1 \otimes m_2 > m_3 \otimes m_4$ iff $m_1 m_2 > m_3 m_4$ in $\mathcal{F}$ with the elements of $\mathcal{A}$ in reduced form.)

Proof. One need to prove that this is indeed a resolution. It is proven in [6] for graded algebras, the proof can be adapted to shuffle algebras and it is done in [3]. $\square$

2 Computation on the Losev-Manin shuffle algebra

We are now going to use the tools we have defined to study the Losev-Manin shuffle algebra. This shuffle algebra arises from geometry namely from the Losev-Manin spaces. We will not directly study or define those spaces, we will rather prove some algebraic results on their shuffle algebra (which in fact can be translated into geometric properties). The goal is to compute its Gröbner basis and its Anick's resolution.

2.1 The Losev-Manin shuffle algebra

We will use A, B, C, I, J, K and L for finite parts of $\mathbb{N}^*$ and a, b, c, i, j, k, l and n for positive integers.

Let's note $[n] = \{1, \cdots, n\}$.

Let $\mathcal{X} = (\emptyset, \{\alpha\}, \{\alpha\}, \cdots)$ and $\mathcal{F} = \mathcal{T}_{\text{Ш}}(\mathcal{X})$ be the free shuffle algebra with one generator of each positive order named α.

Let's give an order on the monomials of $\mathcal{F}$. Let's use the lexicographic order on $2^{(\mathbb{N}^*)}$, let's use the lexicographic order on the monomials of $\mathcal{F}$ (seen as words over $2^{(\mathbb{N}^*)}$). This is a monomial order.

For $m = \alpha_{I_1} \cdots \alpha_{I_k}$ a non-trivial monomial of $\mathcal{F}$, let last$(m) = \max(I_k)$.

The components $\mathcal{F}(n)$ are vector spaces. Let $\mathcal{F}(n,k)$ be the sub-vector space spaned by the monomials of length exactly k.
We have $\mathcal{F}(n) = \bigoplus_{k=1}^{n} \mathcal{F}(n,k)$ and $\mathcal{F}(0) = \mathcal{F}(0,0)$.

Definition 2.1.1. For A and B disjoint finite parts of $\mathbb{N}^*$, let:

$$[\alpha_A, \alpha_B] = \alpha_A\alpha_B - \alpha_B\alpha_A$$

Remark 2.1.1. If $A \sqcup B \neq [n]$ for some $n \in \mathbb{N}$ then $[\alpha_A, \alpha_B]$ is not a shuffle polynomial, the notation is nonetheless useful in this case to more easily write down some shuffle polynomials.

Definition 2.1.2. Let $n \in \mathbb{N}$, for $a, b \in [n]$ and $a \neq b$, let $r_{n,a,b} = \sum_{\substack{A \ni a \\ B \ni b}} [\alpha_A, \alpha_B] \in \mathcal{F}(n)$.
Let $R = (\{r_{n,a,b} \mid a \neq b \leq n\})_{n \geq 0}$.
Let $\mathcal{R}$ be the ideal of $\mathcal{F}$ generated by the R.

Definition 2.1.3. Let's define:

$$r_{K,a,b} = \sum_{\substack{A \ni a \\ B \ni b \\ A \sqcup B = K}} [\alpha_A, \alpha_B]$$

For K a finite subset of $\mathbb{N}^*$. Those are not shuffle polynomials (as in Remark 2.1.1) .

Definition 2.1.4. We recall that $\mathcal{A}$ defined by generators and relations by $(\mathcal{X}, R)$ is the Losev-Manin shuffle algebra. By definition, one may remark that the length is still well defined on this algebra (since for each $r_{n,a,b}$ all its monomials have the same length), moreover it has an augmentation map ϵ.

Property 2.1.5. *Let* $\mathcal{R}(n) = \mathcal{R} \cap \mathcal{F}(n)$ *and* $\mathcal{R}(n,k) = \mathcal{R} \cap \mathcal{F}(n,k)$. *One can remark that* $\mathcal{R}(n) = \bigoplus_{k=1}^{n} \mathcal{R}(n,k)$*, and* $\mathcal{R}(0) = \{0\}$.
Hence, $\mathcal{A}(n) = \bigoplus_{k=1}^{n} \mathcal{A}(n,k)$ *with* $\mathcal{A}(n,k) = \mathcal{F}(n,k)/\mathcal{R}(n,k)$*, and* $\mathcal{A}(0) = \mathcal{A}(0,0)$.

Remark 2.1.2. If we do the first computations, we get:
$\dim(\mathcal{A}(0,0)) = 1$.
$\dim(\mathcal{A}(1,1)) = 1$.
$\dim(\mathcal{A}(2,1)) = 1$ and $\dim(\mathcal{A}(2,2)) = 1$.
$\dim(\mathcal{A}(3,1)) = 1$, $\dim(\mathcal{A}(3,2)) = 4$ and $\dim(\mathcal{A}(3,3)) = 1$.
And with a bit more computations:
$\dim(\mathcal{A}(4,1)) = 1$, $\dim(\mathcal{A}(4,2)) = 11$, $\dim(\mathcal{A}(4,2)) = 11$ and $\dim(\mathcal{A}(4,4)) = 1$.
We recognize the eulerian numbers.

Definition 2.1.6. Let $\sigma \in \mathfrak{S}(n)$, a *descent* of σ is $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. The (n,k)-eulerian number is the number of elements of $\mathfrak{S}(n)$ with exactly k descents. Let's write it $a(n,k)$.

Definition 2.1.7. Let $m = \alpha_{I_1} \cdots \alpha_{I_k} \in \mathcal{F}(n,k)$ a monomial, an *ascent* of m is $i \in [n-1]$ such that $\max(I_i) > \min(I_{i+1})$. A *descents* of m is $i \in [n-1]$ such that $\max(I_i) < \min(I_{i+1})$.
Let $\mathcal{U}(n,k)$ be the sub-vector space of $\mathcal{F}(n,k)$ spaned by the monomials with at least one ascent.
Let $\mathcal{D}(n,k)$ be the sub-vector space of $\mathcal{F}(n,k)$ spaned by the ascent-free monomials.

2.2 Computing a Gröbner basis

The goal is to work on the relations defining $\mathcal{A}$ to find a Gröbner basis. Gröbner basis can be seen as a way to avoid the patterns of the leading monomials of the basis, and give a simple algorithm to compute the reduced form. However the algorithm is not unique and one need to check a diamond lemma in order to check if it is a Gröbner basis. The relations defining $\mathcal{A}$ aren't simple enough to easily check the diamond lemma hence we will instead prove that the reduced form is unique.

Property 2.2.1. *Let $n \in \mathbb{N}$ and $a, b, c \in [n]$ different, we have $r_{n,a,b} - r_{n,a,c} = r_{n,c,b}$.*

Proof. Let's compute this (we implicitly have $I \sqcup J = [n]$):

$$\begin{aligned} r_{n,a,b} &= \sum_{\substack{I \ni a \\ J \ni b}} [\alpha_I, \alpha_J] \\ &= \sum_{\substack{I \ni a,c \\ J \ni b}} [\alpha_I, \alpha_J] + \sum_{\substack{I \ni a \\ J \ni b,c}} [\alpha_I, \alpha_J] \end{aligned}$$

Hence:

$$\begin{aligned} r_{n,a,b} - r_{n,a,c} &= \sum_{\substack{I \ni a,c \\ J \ni b}} [\alpha_I, \alpha_J] + \sum_{\substack{I \ni a \\ J \ni b,c}} [\alpha_I, \alpha_J] - (\sum_{\substack{I \ni a,b \\ J \ni c}} [\alpha_I, \alpha_J] + \sum_{\substack{I \ni a \\ J \ni b,c}} [\alpha_I, \alpha_J]) \\ &= \sum_{\substack{I \ni a,c \\ J \ni b}} [\alpha_I, \alpha_J] - \sum_{\substack{I \ni a,b \\ J \ni c}} [\alpha_I, \alpha_J] \\ &= \sum_{\substack{I \ni a,c \\ J \ni b}} [\alpha_I, \alpha_J] + \sum_{\substack{J \ni a,b \\ I \ni c}} [\alpha_I, \alpha_J] = r_{K,c,b} \end{aligned}$$

□

Corollary 2.2.1. *Hence $(r_{n,i,i+1})_{i \in [n-1]}$ is a basis of $\mathcal{R}(n, 2)$, let's write them $r_{n,i}$. Moreover $(r_{n,i})_{i \in [n-1]}$ is in bijection with the basis of $\mathcal{U}(n, 2)$.*
Same as in Definition 2.1.3, it is useful to define the $r_{K,i}$ the same way.

Proof. By the last property, this is a generating family. Let's project this family in $\mathcal{U}(n, 2)$, we get the identity matrix, hence this family is free. □

Corollary 2.2.2. *For $k = 0$ or 1 or 2, we have $\dim(\mathcal{A}(n, k)) = a(n, k - 1)$.*

Proof. This is clear for $k = 0$ or 1. For $k = 2$, this is the last corollary. □

We now have a candidate for the Gröbner basis, which is $\{r_{n,i} \mid K \in \mathbb{N}, i \in [n - 1]\}$. We need to check the diamond lemma, since the relations are homogeneous of length 2, it is enough to check it on monomials of length 3. To do that, let's use the algorithm that reduces the leftmost descent at each step, and show that the set of relations that this algorithm uses is a basis of $\mathcal{R}(n, 3)$. To do so, we need a "relation between the relations" which is the following.

Definition 2.2.2. Let $n \in \mathbb{N}^*$, let's note (we implicitly have $J \sqcup K = [n]$):

$$[a, b], c = \sum_{J \ni c} r_{K,a,b} \alpha_J$$

and

$$c, [a, b] = \sum_{J \ni c} \alpha_J r_{K,a,b}$$

Let $[[a, b], c] = [a, b], c - c, [a, b]$.

Property 2.2.3. *We have the equality :* $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

Proof. (We implicitly have $J \sqcup K = [n]$ and $A \sqcup B \sqcup C = [n]$.)

$$\begin{aligned}
[[a, b], c] + [[b, c], a] + [[c, a], b] &= \sum_{J \ni c} r_{K,a,b} \alpha_J - (\sum_{J \ni c} \alpha_J r_{K,a,b}) \\
&\quad + \sum_{J \ni a} r_{K,b,c} \alpha_J - (\sum_{J \ni a} \alpha_J r_{K,a,c}) \\
&\quad + \sum_{J \ni b} r_{K,c,a} \alpha_J - (\sum_{J \ni b} \alpha_J r_{K,c,a}) \\
&= (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_A \alpha_B \alpha_C - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_B \alpha_A \alpha_C) \\
&\quad - (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_C \alpha_A \alpha_B - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_C \alpha_B \alpha_A) \\
&\quad + (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_B \alpha_C \alpha_A - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_C \alpha_B \alpha_A) \\
&\quad - (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_A \alpha_B \alpha_C - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_A \alpha_C \alpha_B) \\
&\quad + (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_C \alpha_A \alpha_B - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_A \alpha_C \alpha_B) \\
&\quad - (\sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_B \alpha_C \alpha_A - \sum_{\substack{A \ni a \\ B \ni b \\ C \ni c}} \alpha_B \alpha_A \alpha_C) \\
&= 0
\end{aligned}$$

□

Property 2.2.4. *The set*

$$E_{(n,3)} = \{r_{K,i}\alpha_J \mid K \sqcup J = [n]\} \cup \{\alpha_J r_{K,i} \mid K \sqcup J = [n], \max(J) > \min(K)\}$$

form a free family of $\mathcal{F}(n,3)$. Moreover this set is in bijection with the basis of $\mathcal{U}(n,3)$.

Proof. Let's prove that this family is free:
The leading monomial of $r_{K,i}\alpha_J$ is $\alpha_A\alpha_B\alpha_J$ with $A = \{\lambda \in K \mid \lambda \leq i\}$ and $B = \{\lambda \in K \mid \lambda > i\}$. Same for the leading monomial of $\alpha_J r_{K,i}$ which is $\alpha_A\alpha_B\alpha_J$.
Assume that $\alpha_A\alpha_B\alpha_C$ is the leading monomial of two elements of the set $E_{(n,3)}$: then it is the leading monomial of $r_{A\sqcup B,\max(A)}\alpha_C$ and of $\alpha_A r_{B\sqcup C,\max(B)}$ but $\alpha_A r_{B\sqcup C,\max(B)} \notin E_{(n,3)}$ since:

$$\max(A) < \min(B) \leq \max(B) < \min(C)$$

$\lightning$
Contradiction

Let $\alpha_A\alpha_B\alpha_C$ be a monomial with at least one ascent, if A to B is an ascent then it is the leading monomial of $r_{A\sqcup B,\max(A)}\alpha_C \in E_{(n,3)}$. Else, B to C is an ascent so it is the leading monomial of $\alpha_A r_{B\sqcup C,\max(B)} \in E_{(n,3)}$ since A to B is not an ascent.
Hence taking the leading monomials gives us a bijection between $E_{(n,3)}$ and the basis of $\mathcal{U}(n,3)$. This gives us an ordering of $E_{(n,3)}$ such that if we project it on $\mathcal{U}(n,3)$, we get a triangular matrix with ones on the diagonal . Hence $E_{(n,3)}$ is free. $\square$

With this property, we've proven that $E_{(n,3)}$ is the set of relations that appear when we reduce the leftmost descent at each step for monomials of length 3.

Theorem 2.2.1. *The set*

$$E_{(n,3)} = \{r_{K,i}\alpha_J \mid K \sqcup J = [n]\} \cup \{\alpha_J r_{K,i} \mid K \sqcup J = [n], \max(J) > \min(K)\}$$

forms a basis of $\mathcal{R}(n,3)$.

Proof. The vector space $\mathcal{R}(n,3)$ is generated by

$$\{r_{K,i}\alpha_J \mid K \sqcup J = [n]\} \cup \{\alpha_J r_{K,i} \mid K \sqcup J = [n]\}$$

Let V be the span of $E_{(n,3)}$, since $E_{(n,3)}$ is free, it suffices to prove that

$$\{\alpha_J r_{K,i} \mid K \sqcup J = [n], \max(J) < \min(K)\} \subseteq V$$

Since $\max(J) < \min(K)$, let $a = \max(J)$, we have $J = [a]$, hence $r_{K,i} = r_{K,i,i+1}$ and if $J \neq [\max(J)]$ then $\alpha_J r_{K,i} \in E_{(n,3)}$.
Let's fix $b \in [n-1]$ and $c = b+1$. Let's inductively prove over k that $\alpha_{[b-k]}r_{K,b} \subseteq V$ with $K = [n] \setminus [b-k]$.

- *Base case*: Let $a = b - 1$. We have:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

  Since $a > b > c$, we have $[[a, b], c] \in V$, $[[c, a], b] \in V$ and $[b, c], a \in V$.
  Hence $a, [b, c] \in V$. Hence (we implicitly have $I \sqcup L = [n]$):

$$\sum_{I \ni a} \alpha_I r_{L,b,c} \in V$$

  Since if $I \neq [\max(I)]$ then $\alpha_I r_{L,b,c} \in E_{(n,3)}$, we have:

$$\sum_{I \in \{[a], \cdots, [b-1]\}} \alpha_I r_{L,b,c} \in V$$

  Hence $\alpha_{[b-1]} r_{K,b} \subseteq V$ with $K = [n] \setminus [b-1]$.

- *Inductive step*: Let $b = a + k$ and $c = b + 1$. We've shown that:

$$\sum_{I \in \{[a], \cdots, [b-1]\}} \alpha_I r_{L,b,c} \in V$$

  By inductive hypothesis we have $\alpha_I r_{L,b,c} \in V$ for any $I = [i]$ with $a < i < b$.
  Hence $\alpha_{[b-k]} r_{K,b} \subseteq V$ with $K = [n] \setminus [b-k]$.

Hence $E_{(n,3)}$ generate $\mathcal{R}(n, 3)$. □

Corollary 2.2.3. *We have* $\dim(\mathcal{A}(n, 3)) = a(n, 2)$.

Proof. It follows directly from the two last properties. □

Now that we've proven it for $k = 3$, it suffices to define the same object for $k > 3$ and do an induction to prove the result for any k.

Definition 2.2.5. Let $n \in \mathbb{N}^*$ and $k \leq n$.
Let $E_{(n,k,0)} = \{r_{K,i} m_{right}\} \subseteq \mathcal{F}(n, k)$ and let:

$E_{(n,k,j)} = \{m_{left} r_{K,i} m_{right} \mid l(m) = j$ and m_{left} is descent free and last$(m) > \min(K)\} \subseteq \mathcal{F}(n, k)$

with m_{left} and m_{right} some monomials.
Let $E_{(n,k)} = \bigcup_{j=0}^{k-2} E_{(n,k,j)}$.

Property 2.2.6. *The set* $E_{(n,k)}$ *forms a free family of* $\mathcal{F}(n, k)$*, moreover the leading monomials give a bijection between* $E_{(n,k)}$ *and a basis of* $\mathcal{U}(n, 3)$.

Proof. A leading monomial of an element of $E_{(n,k)}$ has at least one ascent. Let m be a monomial with a least one ascent, let $m = m_1 \alpha_I \alpha_J m_2$ such that $m_1 \alpha_I$ is ascent-free and there is an ascent between I and J, then m is the leading monomial of $m_1 r_{I \sqcup J, i} m_2$ for a unique i, moreover this is the only element of $E_{(n,k)}$ such that m is its leading monomial.
Hence $E_{(n,k)}$ is free and in bijection with a basis of $\mathcal{U}(n, k)$. □

Property 2.2.7. *Let $x \in \mathcal{F}(n,k)$, it admits a reduced form which is a $y \in \mathcal{D}(n,k)$ such $x = y + z$ with $z \in \mathcal{R}(n,k)$. Moreover if $E_{(n,k)}$ generates $\mathcal{R}(n,k)$ this reduced form is unique.*

Proof. It suffices to show it for a monomial m. Let's prove it inductively over the order of the monomials. If $m \in \mathcal{D}(n,k)$ it is already reduced. Else, we know that m is the leading monomial of a unique element e of $E_{(n,k)}$.

- *Base case*: If m is the smaller monomial with at least one ascent, $m - e \in \mathcal{D}(n,k)$ hence $m = e + (m - e)$ and $m - e$ is a reduced form.

- *Inductive step*: We know that monomials of $m - e$ are smaller than m hence we can reduce them by induction hypothesis, which gives us a reduced form for m.

Hence by additivity, we get a reduced form for every $x \in \mathcal{F}(n,k)$. In fact, we've constructed a bijection between $\mathcal{F}(n,k)/E_{(n,k)}$ and $\mathcal{D}(n,k)$, so if $E_{(n,k)}$ generates $\mathcal{R}(n,k)$, the reduced form is unique. ☐

Theorem 2.2.2. *The $E_{(n,k)}$ form a basis of $\mathcal{R}(n,k)$.*

Proof. We need to prove that $E_{(n,k)}$ generates $\mathcal{R}(n,k)$. Let V be the span of $E_{(n,k)}$, since the set $\{m_{left} r_{K,i} m_{right}\}$ with no condition on m_{left} generates $\mathcal{R}(n,k)$, it suffices to show that it is in V. Let's prove it inductively over the length of m_{left}.

- *Base case*: If $l(m_{left}) = 0$ then $r_{K,i} m_{right} \in E_{(n,k,0)}$ and there is nothing to prove.

- *Inductive step*: If $m_{left} \in \mathcal{F}(L)$ has at least one ascent, it admits a reduced form $m_{left} = y + z$ with y ascent-free and $z \in \mathcal{R}(L)$. By induction hypothesis, $z m'_{right} \in V$, hence $z r_{K,i} m_{right} \in V$. Hence if suffices to prove it for m_{left} ascent-free.
  Let $m_{left} r_{K,i} m_{right} = \sum_{\substack{B \ni b \\ C \ni c}} m'_{left} \alpha_A [\alpha_B, \alpha_C] m_{right}$. If $\max(A) > \min(B,C)$, this is an element of $E_{(n,k)}$. Else, by Theorem 2.2.1, it is in V.

Hence $E_{(n,k)}$ generates $\mathcal{R}(n,k)$. ☐

Corollary 2.2.4. *Hence for all n and k, we have $\dim(\mathcal{A}(n,k)) = a(n, k-1)$, hence $\dim(\mathcal{A}(n)) = n!$. Moreover each element of $\mathcal{A}(n,k)$ has a unique reduced form.*

Corollary 2.2.5. *The $r_{n,i}$ are a Gröbner basis.*

2.3 The Anick resolution

Let's compute this resolution in the case of the shuffle algebra $\mathcal{A}$, with the Gröbner basis we have computed.

Property 2.3.1. *The k-chains are the monomials of length $k+1$ with exactly k ascent. Moreover their tails are of length* 1.

Proof. Let's prove it by induction:

- *Base case*: It is clear for C_0. For C_1, the leading monomial of $r_{n,i}$ is of length 2 and has 1 ascent, moreover each monomial of length 2 with 1 ascent is the leading monomial of some $r_{n,i}$.

- *Inductive step*: Let $m = m't(m)$ be a k-chain and $t(m)$ its tail. By induction hypothesis, m' is a monomial of length k with exactly $k-1$ ascent. Moreover, $m' = m''\alpha_{I_0}$ with α_{I_0} the tail of m'. Since $\alpha_{I_0}t(m)$ is divisible by an element of $\widetilde{R}$, it has at least one ascent.
  Let $t(m) = \alpha_{I_1}\cdots\alpha_{I_k}$, it doesn't have any ascent, otherwise it would be divisible by an element on $\widetilde{R}$. Hence the ascent is between α_{I_0} and α_{I_1}, hence $m'\alpha_{I_1}$ satisfies both (1) and (2). Hence m is of length $k+1$, it has k ascent and its tail is of length 1.

  Moreover each monomial of length $k+1$ with k ascent is a k-chain since it satisfies the conditions (1), (2) and (3) of Definition 1.4.1.

□

Definition 2.3.2. Let $n \in \mathbb{N}$, let $k \in \mathbb{N}^*$ and $i_1, \cdots, i_{k+1} \in K$. Let (with $I_1 \sqcup \cdots \sqcup I_{k+1} = [n]$):

$$c_n(i_1, \cdots, i_{k+1}) = \sum_{\substack{I_1 \ni i_1 \\ \vdots \\ I_{k+1} \ni i_{k+1} \\ \alpha_{I_1}\cdots\alpha_{I_{k+1}} \in C_k}} \alpha_{I_1}\cdots\alpha_{I_{k+1}} \in V_k$$

Let's call them symmetric k-chains.

Property 2.3.3. *The set $X_n = \{c_n(i_1, \cdots, i_k, n) \mid i_1 < \cdots < i_k < n\}$ is a basis of $V_k(n)$.*

Proof. Let's prove that taking the leading monomial gives us a bijection between X_n and $C_k(n)$.
The leading term of $c_n(i_1, \cdots, i_k, n)$ is $\alpha_{[i_1]}\alpha_{[i_2]\setminus[i_1]}\cdots\alpha_{[n]\setminus[i_k]}$.
Hence this gives us a bijection between X_n and $C_k(n)$. □

Definition 2.3.4. Let's define the following $\mathcal{A}$-module morphism from $V_k \boxtimes \mathcal{A}$ to $V_{k-1} \boxtimes \mathcal{A}$ (with $I \sqcup J = K$):

$$\varphi_k : c_n(i_1, \cdots, i_{k+1}) \otimes 1 \mapsto \sum_{\sigma \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) \sum_{J \ni i_{\sigma(k+1)}} c_I(i_{\sigma(1)}, \cdots, i_{\sigma(k)}) \otimes \alpha_J$$

Property 2.3.5. *We have $\varphi_k \circ \varphi_{k+1} = 0$.*

Proof. It suffices to prove it on a generating family:

$$\varphi_k \circ \varphi_{k+1}(c_n(i_1, \cdots, i_{k+2}) \otimes 1) = \sum_{\sigma \in \mathfrak{S}_{k+2}} \varepsilon(\sigma) \sum_{J \ni i_{\sigma(k+2)}} \varphi_{k+1}(c_I(i_{\sigma(1)}, \cdots, i_{\sigma(k+1)}) \otimes \alpha_J)$$

$$= \sum_{\sigma \in \mathfrak{S}_{k+2}} \varepsilon(\sigma) \sum_{J \ni i_{\sigma(k+2)}} \sum_{\sigma' \in \mathfrak{S}_{k+1}} \varepsilon(\sigma') \sum_{J' \ni i_{\sigma' \circ \sigma(k+1)}} c_{I'}(i_{\sigma' \circ \sigma(1)}, \cdots, i_{\sigma' \circ \sigma(k)}) \otimes \alpha_{J'} \alpha_J$$

$$= \sum_{\sigma \in \mathfrak{S}_{k+2}} \sum_{\sigma' \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) \varepsilon(\sigma') \sum_{J \ni i_{\sigma(k+2)}} \sum_{J' \ni i_{\sigma' \circ \sigma(k+1)}} c_{I'}(i_{\sigma' \circ \sigma(1)}, \cdots, i_{\sigma' \circ \sigma(k)}) \otimes \alpha_{J'} \alpha_J$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_{k+2} \\ \sigma' \in \mathfrak{S}_{k+1} \\ i_{\sigma' \circ \sigma(k+1)} > i_{\sigma(k+2)}}} \varepsilon(\sigma) \varepsilon(\sigma') \sum_{\substack{K \ni i_{\sigma(k+2)} \\ K \ni i_{\sigma' \circ \sigma(k+1)}}} c_{I'}(i_{\sigma' \circ \sigma(1)}, \cdots, i_{\sigma' \circ \sigma(k)}) \otimes r_{K, i_{\sigma' \circ \sigma(k+1)}, i_{\sigma(k+2)}}$$

$$= 0$$

□

Property 2.3.6. *Let $m = \alpha_{I_1} \cdots \alpha_{I_{k+1}} \in C_k$, the leading terms of $\varphi_k(m \otimes 1)$ and $d_k(m \otimes 1)$ are the same (with d_k the morphism of the Anick's resolution).*

Proof. We already know the leading term of $d_k(m \otimes 1)$, it is $t = \alpha_{I_1} \cdot \alpha_{I_k} \otimes \alpha_{I_{k+1}}$. Let's compute it for φ_k. Let c be a symmetric k-chain, let $m_c = \alpha_{J_1} \cdot \alpha_{J_{k+1}}$ be its leading term, we have that the leading term of $\varphi_k(c \otimes 1)$ is $\alpha_{J_1} \cdot \alpha_{J_k} \otimes \alpha_{J_{k+1}}$. Let $m = c_m + r$ with c_m a symmetric k-chain such that its leading term is m and r a linear combination of symmetric k-chains of lower leading terms. The leading term of $\varphi_k(m \otimes 1)$ is the leading term c_m hence it is t. □

Theorem 2.3.1. *We have $\varphi_k = d_k$ with d_k the morphism of the Anick's resolution.*
Hence φ_k gives a formula for the morphism of the Anick's resolution.

Proof. This result can probably be proven using spectral sequences over a filtration of the Anick's resolution, however, some work remains to complete the proof. □

Remark 2.3.1. Let's recall that the shuffle algebra we are studying comes from a twisted associative algebra and that we needed to "forget" its symmetry in order to be able to compute Gröbner basis and ultimately its Anick's resolution. It is remarkable that the symmetry comes back in the formula of the derivation map of the Anick's resolution.

References

[1] D. J. ANICK, *On the Homology of associative algebras*, American Mathematical Society, 1986.

[2] M. BREMNER, V. DOTSENKO, *Algebraic Operads: An Algorithmic Companion*.

[3] V. DOTSENKO, A. KHOROSHKIN, *Shuffle algebras, homology, and consecutive pattern avoidance*, MSP, 2013.

[4] J.-L. LODAY, B. VALLETTE, *Algebraic Operads*, Springer, 2012.

[5] A. LOSEV, Y. MANIN, *New Moduli Spaces of Pointed Curves and Pencils of Flat Connections*, 2000.

[6] V. A. UFNAROVSKIJ, *Combinatorial and Asymptotic Methods in Algebra*, Springer, 1990.