Singular Bohr-Sommerfeld conditions for 1D Toeplitz operators: hyperbolic case

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Abstract

In this article, we state the Bohr-Sommerfeld conditions around a singular value of hyperbolic type of the principal symbol of a self-adjoint semiclassical Toeplitz operator on a compact connected Riemann surface. These conditions allow the description of the spectrum of the operator in a fixed size neighbourhood of the singularity. We provide numerical computations for three examples, each associated with a different topology.

1 Introduction

Let M be a compact, connected Riemann surface with area form ω. Assume that M is endowed with a prequantum bundle L, that is a Hermitian, holomorphic line bundle whose Chern connection has curvature $-i\omega$. Let K be another Hermitian holomorphic line bundle[1] and define the quantum Hilbert space $\mathcal{H}_k$ as the space of holomorphic sections of $L^{\otimes k} \otimes K$, for every positive integer k. We consider (Berezin-)Toeplitz operators (see for instance [7, 6, 8, 24] or the expository works [23, 28, 31]) acting on $\mathcal{H}_k$. The semiclassical limit corresponds to $k \to +\infty$.

The usual Bohr-Sommerfeld conditions, derived in [10], describe the intersection of the spectrum of a selfadjoint Toeplitz operator and a neighbourhood of any regular value of its principal symbol a_0, in terms of geometric quantities. More precisely, this intersection is the union of a finite number of families whose elements are, up to an error $O(k^{-2})$, the solutions of an equation of the form

$$c_0(\lambda) + k^{-1}(c_1(\lambda) + \epsilon\pi) \in 2\pi k^{-1}\mathbb{Z},$$

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[1] The reader must be warned that in this work, the letter K does not refer to the canonical bundle, unless explicitly stated otherwise.

where

- $c_0(\lambda)$ is the holonomy associated with the parallel transport in L along a connected component of the level set $a_0^{-1}(\lambda)$,
- $c_1(\lambda)$ contains the integral of a differential form involving the subprincipal symbol of the operator,
- $\epsilon \in \{0,1\}$ is an index associated with a half-forms structure.

Precise definitions of these quantities and a more explicit formulation of the Bohr-Sommerfeld rules can be found in section 4.6.

A natural question is whether one can write Bohr-Sommerfeld conditions near a singular value of the principal symbol. In the case of a non-degenerate singularity of elliptic type (a local extremum), it was answered positively in [22], and the result is quite simple: roughly speaking, the singular Bohr-Sommerfeld conditions are nothing but the limit of the regular Bohr-Sommerfeld conditions when the energy tends to the singular value. The hyperbolic case (presence of saddle points) is much more difficult, because of the complicated topology of a neighbourhood of the singular level. For instance, in the case of one hyperbolic point, the critical level looks like a figure eight, and crossing it has the effect of adding (or removing) one connected component from the regular level.

Let us mention that the case of Toeplitz operators is very close to the case of pseudodifferential operators. In this setting, the problem of describing the spectrum of a selfadjoint operator near a singular level of hyperbolic type was handled by Colin de Verdière and Parisse in a series of articles [14, 15, 16]. In this article, we use analogous techniques to write hyperbolic Bohr-Sommerfeld conditions in the context of Toeplitz operators. The novelty is that they can be applied in this context.

1.1 Main result

Let A_k be a self-adjoint Toeplitz operator on M; its normalized symbol $a_0 + \hbar a_1 + \dots$ is real-valued. Assume that 0 is a critical value of the principal symbol a_0, that the level set $\Gamma_0 = a_0^{-1}(0)$ is connected and that every critical point contained in Γ_0 is non-degenerate and of hyperbolic type. Let $S = \{s_j\}_{1 \leq j \leq n}$ be the set of these critical points. Γ_0 is a compact graph embedded in M, and each of its vertices has degree 4 (this is a consequence of the usual Morse lemma, for instance). At each vertex s_j, we denote by e_m, $m = 1, 2, 3, 4$, the local edges, labeled with cyclic order $(1, 3, 2, 4)$ (with respect to the orientation of M near s_j) and such that e_1, e_2 (resp. e_3, e_4) correspond to the local unstable (resp. stable) manifolds. Cut $n+1$ edges of Γ_0, each one corresponding to a cycle γ_i in a basis $(\gamma_1, \dots, \gamma_{n+1})$ of $\mathrm{H}_1(\Gamma_0, \mathbb{Z})$, in such a way that the remaining graph is a tree $\mathcal{T}$; usually, $\mathcal{T}$ is called a spanning tree, and the basis $(\gamma_1, \ldots, \gamma_{n+1})$ is called a fundamental cycle basis (see for instance [3, pp. 25–26]). Our main result is the following:

Theorem (theorem 6.1, theorem 6.4)**.** *Zero is an eigenvalue of A_k up to $O(k^{-\infty})$ if and only if the following system of $3n+1$ linear equations with unknowns $(x_e \in \mathbb{C}_k)_{e \in \{\text{edges of } \mathcal{T}\}}$ (here $\mathbb{C}_k$ is the set of constant symbols, see section 2.1) has a non-trivial solution:*

1. *if the edges (e_1, e_2, e_3, e_4) connect at s_j (with the same convention as before for their labelling), then*

$$\begin{pmatrix} x_{e_3} \\ x_{e_4} \end{pmatrix} = T_j \begin{pmatrix} x_{e_1} \\ x_{e_2} \end{pmatrix},$$

2. *if the edges α and β are the extremities of a cut cycle γ_i, then*

$$x_\alpha = \exp\left(ik\theta(\gamma_i, k)\right) x_\beta,$$

   where the following orientation is assumed: γ_i can be represented as a closed path starting on the edge α and ending on the edge β.

Moreover, T_j is a matrix depending only on a semiclassical invariant $\varepsilon_j(k)$ of the system at the singular point s_j (see equation (8)), and $\theta(\gamma, k)$ admits an asymptotic expansion in non-positive powers of k. The first two terms of this expansion involve regularizations of the geometric invariants (actions and index) appearing in the usual Bohr-Sommerfeld conditions.

For spectral purposes, we use this theorem by replacing A_k by $A_k - E$ for E varying in a fixed size neighbourhood of the singular level. Away from the critical energy, we recover the regular Bohr-Sommerfeld conditions (see section 6.4).

This is very similar to the results of Colin de Verdière and Parisse [16], but the novelty lies in the framework that had to be set in order to extend their techniques to the Toeplitz setting (especially the sheaf theoretic approach to the spectral theory of Toeplitz operators), and also in the geometric invariants that are specific to this context.

1.2 Structure of the article

As said earlier, the case of Toeplitz operators is very close to the case of pseudodifferential operators; in mathematical terms, there is a microlocal equivalence between Toeplitz operators and pseudodifferential operators. When the phase space is the whole complex plane, this equivalence is realized by the Bargmann transform, and allows to use some of the results obtained in the pseudodifferential setting. This is why the article is organized as follows: first, we discuss microlocal properties of the Bargmann transform. Then, we introduce the sheaf of microlocal solutions of the equation $(A_k - E)u_k = 0$, explain its structure and recall the usual Bohr-Sommerfeld conditions. In section 5, we construct a microlocal normal form for A_k near each critical point $s_j, 1 \leq j \leq n$, on Bargmann spaces, and we use the properties of the Bargmann transform and the study of Colin de Verdière and Parisse [14] to describe the space of microlocal solutions of A_k near s_j. Finally, we adapt the reasoning of Colin de Verdière and Parisse [16] and Colin de Verdière and Vũ Ngọc [18] to obtain the singular Bohr-Sommerfeld conditions (in section 6). We give numerical evidence in the last section.

2 Preliminaries and notations

2.1 Symbol classes

We introduce rather standard symbol classes. Let d be a positive integer. For u in $\mathbb{C}^d \simeq \mathbb{R}^{2d}$, let $m(u) = (1 + \|u\|^2)^{\frac{1}{2}}$. For every integer j, we define the symbol class $\mathcal{S}_j^d$ as the set of sequences of functions of $\mathcal{C}^\infty(\mathbb{C}^d)$ which admit an asymptotic expansion of the form $a(.,k) = \sum_{\ell \geq 0} k^{-\ell} a_\ell$ in the sense that

- $\forall \ell \in \mathbb{N} \quad \forall \alpha, \beta \in \mathbb{N}^{2d} \quad \exists\, C_{\ell,\alpha,\beta} > 0 \quad |\partial_z^\alpha \partial_{\bar{z}}^\beta a_\ell| \leq C_{\ell,\alpha,\beta} m^j$,
- $\forall L \in \mathbb{N}^* \quad \forall \alpha, \beta \in \mathbb{N}^{2d} \quad \exists\, C_{L,\alpha,\beta} > 0 \quad \left|\partial_z^\alpha \partial_{\bar{z}}^\beta \left(a - \sum_{\ell=0}^{L-1} k^{-\ell} a_\ell\right)\right| \leq C_{L,\alpha,\beta} k^{-L} m^j$.

We set $\mathcal{S}^d = \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j^d$. If, in the definition of $\mathcal{S}_0^1$, we only consider symbols independent of z, we obtain the class $\mathbb{C}_k$ of constant symbols.

2.2 Function spaces

Using standard notations, we denote by $\mathscr{S}(\mathbb{R})$ the Schwartz space of functions $f \in \mathcal{C}^\infty(\mathbb{R})$ such that for all $j, p \in \mathbb{N}$, $\sup_{t\in\mathbb{R}} |t^j f^{(p)}(t)| < +\infty$, by $\mathscr{D}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$, and by $\mathscr{S}'(\mathbb{R}) \subset \mathscr{D}'(\mathbb{R})$ the space of tempered distributions on $\mathbb{R}$ (the dual space of $\mathscr{S}(\mathbb{R})$). We recall that

$$\mathscr{S}(\mathbb{R}) = \bigcap_{j\in\mathbb{N}} \mathscr{S}_j(\mathbb{R}),$$

where $\mathscr{S}_j(\mathbb{R})$ is the space of functions f of $\mathcal{C}^j(\mathbb{R})$ with $\|f\|_{\mathscr{S}_j}$ finite, with

$$\|f\|_{\mathscr{S}_j} = \max_{0\leq p\leq j} \left(\sup_{t\in\mathbb{R}} \left| (1+t^2)^{(j-p)/2} f^{(p)}(t) \right| \right).$$

The topology of $\mathscr{S}(\mathbb{R})$ is defined by the countable family of semi-norms $\|.\|_{\mathscr{S}_j}, j \in \mathbb{N}$.

We recall the definition of Bargmann spaces [1, 2], which are spaces of square integrable functions with respect to a Gaussian weight: for $k \in \mathbb{N}^*$,

$$\mathcal{B}_k = \left\{ f\psi^k; f : \mathbb{C} \mapsto \mathbb{C} \text{ holomorphic}, \int_{\mathbb{R}^2} |f(z)|^2 \exp(-k|z|^2)\, d\lambda(z) < +\infty \right\}$$

with $\psi : \mathbb{C} \to \mathbb{C}, z \mapsto \exp\left(-\frac{1}{2}|z|^2\right)$, $\psi^k : \mathbb{C} \to \mathbb{C}^{\otimes k}$ its k-th tensor power, and λ the Lebesgue measure on $\mathbb{R}^2$. We denote by $\|\cdot\|_{\mathcal{B}_k}$ the naturally associated L^2-norm:

$$\|f\psi^k\|_{\mathcal{B}_k} = \left(\int_{\mathbb{R}^2} |f(z)|^2 \exp(-k|z|^2)\, d\lambda(z) \right)^{1/2}.$$

Of course, this norm is still defined for elements of the form $f\psi^k$ satisfying the integrability condition with f not necessarily holomorphic; when this is the case, we denote it by $\|f\psi^k\|_{L^2,\exp}$. Furthermore, we introduce the subspace

$$\mathfrak{S}_k = \left\{ \varphi \in \mathcal{B}_k;\ \forall j \in \mathbb{N} \quad \sup_{z\in\mathbb{C}} \left(|\varphi(z)|(1+|z|^2)^{j/2} \right) < +\infty \right\} \tag{1}$$

of $\mathcal{B}_k$, with topology induced by the obvious associated family of semi-norms. It is the analogue of the Schwartz space on the Bargmann side; see section 3.1 for a more precise statement.

2.3 Weyl quantization and pseudodifferential operators

We briefly recall some standards notations and properties of the theory of pseudodifferential operators (for details, see *e.g.* [13, 19, 33]), replacing the usual small parameter $\hbar$ by k^{-1}, because this is all we need in the rest of the paper.

2.3.1 Pseudodifferential operators

A pseudodifferential operator in one degree of freedom is an operator (possibly unbounded) acting on $L^2(\mathbb{R})$ which is the Weyl quantization of a symbol $a(.,k) \in \mathcal{S}^1$, seen as a sequence of functions defined on the cotangent space $T^*\mathbb{R} \simeq \mathbb{R}^2$, more precisely:

$$\left(\mathrm{Op}_k^W(a)u\right)(x) = \frac{k}{2\pi} \int_{\mathbb{R}^2} \exp\left(ik(x-y)\xi\right) a\left(\frac{x+y}{2}, \xi, k\right) u(y)\, dy\, d\xi.$$

The leading term a_0 in the asymptotic expansion of $a(.,k)$ is the principal symbol of $A_k = \mathrm{Op}_k^W(a)$. A_k is said to be *elliptic* at $(x_0, \xi_0) \in T^*\mathbb{R}$ if $a_0(x_0, \xi_0) \neq 0$.

2.3.2 Wavefront set

Definition 2.1. A sequence u_k of elements of $\mathscr{D}'(\mathbb{R})$ is said to be *admissible* if for any pseudodifferential operator P_k whose symbol is compactly supported, there exists an integer $N \in \mathbb{Z}$ such that $\|P_k u_k\|_{L^2(\mathbb{R})} = O(k^N)$.

We recall the standard definition of the wavefront set $\mathrm{WF}(u_k)$ of an admissible sequence of distributions.

Definition 2.2. Let u_k be an admissible sequence of $\mathscr{D}'(\mathbb{R})$. A point (x_0, ξ_0) does not belong to $\mathrm{WF}(u_k)$ if and only if there exists a pseudodifferential operator P_k, elliptic at (x_0, ξ_0), such that $\|P_k u_k\|_{L^2(\mathbb{R})} = O(k^{-\infty})$.

One can refine these definitions in the case where u_k belong to $\mathscr{S}(\mathbb{R})$.

Definition 2.3. A sequence $(u_k)_{k\geq 1}$ of elements of $\mathscr{S}(\mathbb{R})$ is said to be

- $\mathscr{S}$*-admissible* if there exists N in $\mathbb{Z}$ such that every Schwartz semi-norm of u_k is $O(k^N)$,
- $\mathscr{S}$*-negligible* if every Schwartz semi-norm of u_k is $O(k^{-\infty})$. We write $u_k = O_{\mathscr{S}}(k^{-\infty})$.

Now, instead of using the L^2-norm in definition 2.2, one can actually consider the semi-norms $\|.\|_{\mathscr{S}_j}$.

Lemma 2.4. *A point (x_0, ξ_0) does not belong to WF(u_k) if and only if there exists a pseudodifferential operator P_k, elliptic at (x_0, ξ_0), such that $P_k u_k = O_{\mathscr{S}}(k^{-\infty})$.*

Proof. The sufficient condition comes from the previous definition, so we only prove the necessary condition. We only adapt a standard argument used when one wants to deal with $\mathcal{C}^j$-norms (see [27, proposition IV−8]). Assume that (x_0, ξ_0) does not belong to $\mathrm{WF}(u_k)$; there exists a pseudodifferential operator P_k, elliptic at (x_0, ξ_0), such that $\|P_k u_k\|_{L^2(\mathbb{R})} = O(k^{-\infty})$. Consider a compactly supported smooth function χ equal to one in a neighbourhood of (x_0, ξ_0) and set $Q_k = \mathrm{Op}^{\mathrm{W}}(\chi) P_k$. For every $R \in \mathbb{R}[X]$ and every integer $j > 0$, $k^{-j} \frac{d^j}{dx^j} R\mathrm{Op}^{\mathrm{W}}(\chi)$ is a pseudodifferential operator of order 0, hence bounded $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by a constant $C > 0$ (by Calderon-Vaillancourt theorem, see [27, theorem II−36] or [19, theorem 7.11]). Thus, one has $\|k^{-j} \frac{d^j}{dx^j} R Q_k u_k\|_{L^2(\mathbb{R})} \leq C \|P_k u_k\|_{L^2(\mathbb{R})} = O(k^{-\infty})$. Hence, $\|R Q_k u_k\|_{H^s(\mathbb{R})} = O(k^{-\infty})$ for every integer $s > 0$, where we recall that the Sobolev space $H^s(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ whose elements have their s first derivatives in $L^2(\mathbb{R})$; Sobolev injections then yield that every $\mathcal{C}^j$-norm of $R Q_k u_k$ is $O(k^{-\infty})$. Since this holds for every polynomial R, we obtain the result. □

2.4 Geometric quantization and Toeplitz operators

We also recall the standard definitions and notations in the Toeplitz setting. Unless otherwise mentioned, "smooth" will always mean $\mathcal{C}^\infty$, and a section of a line bundle will always be assumed to be smooth. The space of sections of a bundle $E \to M$ will be denoted by $\mathcal{C}^\infty(M, E)$. Let M be a connected compact Kähler manifold, with fundamental 2-form $\omega \in \Omega^2(M, \mathbb{R})$. Assume M is endowed with a prequantum bundle $L \to M$, that is a Hermitian holomorphic line bundle whose Chern connection ∇ has curvature $-i\omega$. Let $K \to M$ be a Hermitian holomorphic line bundle. For every positive integer k, define the quantum space $\mathcal{H}_k$ as:

$$\mathcal{H}_k = H^0(M, L^k \otimes K) = \left\{\text{holomorphic sections of } L^k \otimes K\right\}.$$

The space $\mathcal{H}_k$ is a subspace of the space $L^2(M, L^k \otimes K)$ of sections of finite L^2-norm, where the scalar product is given by

$$\langle \varphi, \psi \rangle = \int_M h_k(\varphi, \psi) \mu_M$$

with h_k the Hermitian product on $L^k \otimes K$ induced by those of L and K, and μ_M the Liouville measure on M. Since M is compact, $\mathcal{H}_k$ is finite dimensional, and is thus given a Hilbert space structure with this scalar product.

2.4.1 Admissible and negligible sequences

Let $(s_k)_{k\geq 1}$ be a sequence such that for each k, s_k belongs to $\mathcal{C}^\infty(M, L^k \otimes K)$. We say that $(s_k)_{k\geq 1}$ is

- *admissible* if for every positive integer ℓ, for every vector fields $X_1, \ldots, X_\ell$ on M, and for every compact set $C \subset M$, there exist a constant $c > 0$ and an integer N (depending on $X_1, \ldots, X_\ell$ and C) such that

$$\forall p \in C \quad \|\nabla_{X_1} \ldots \nabla_{X_\ell} s_k(p)\| \leq c k^N,$$

- *negligible* if for every positive integers ℓ and N, for every vector fields $X_1, \ldots, X_\ell$ on M, and for every compact set $C \subset M$, there exists a constant $c > 0$ (depending on $X_1, \ldots, X_\ell$, C and N) such that

$$\forall p \in C \quad \|\nabla_{X_1} \ldots \nabla_{X_\ell} s_k(p)\| \leq c k^{-N}.$$

We say that $(s_k)_{k\geq 1}$ is *negligible* over an open set $U \subset M$ if the previous estimates hold for every compact subset of U. The *microsupport* $\mathrm{MS}(s_k)$ of an admissible sequence $(s_k)_{k\geq 1}$ is the complement of the set of points of M which admit a neighbourhood where $(s_k)_{k\geq 1}$ is negligible. Finally, we say that two admissible sequences $(t_k)_{k\geq 1}$ and $(s_k)_{k\geq 1}$ are *microlocally equal* on an open set U if $\mathrm{MS}(t_k - s_k) \cap U = \emptyset$; unless explicitly stated otherwise, the symbol $\sim$ will indicate microlocal equivalence.

2.4.2 Toeplitz operators

Let Π_k be the orthogonal projector of $L^2(M, L^k \otimes K)$ onto $\mathcal{H}_k$. A *Toeplitz operator* is any sequence $(T_k : \mathcal{H}_k \to \mathcal{H}_k)_{k\geq 1}$ of operators of the form

$$T_k = \Pi_k M_{f(.,k)} + R_k$$

where $f(.,k)$ is a sequence of $\mathcal{C}^\infty(M)$ with an asymptotic expansion $f(.,k) = \sum_{\ell\geq 0} k^{-\ell} f_\ell$ for the $\mathcal{C}^\infty$ topology, $M_{f(.,k)}$ is the operator of multiplication by $f(.,k)$ and R_k is an operator acting on $\mathcal{H}_k$ with $\|R_k\| = O(k^{-\infty})$. Let $\mathcal{T}$ be the set of Toeplitz operators, and define the contravariant symbol map

$$\sigma_{\text{cont}} : \mathcal{T} \to \mathcal{C}^\infty(M)[[\hbar]]$$

sending T_k into the formal series $\sum_{\ell\geq 0} \hbar^\ell f_\ell$. We will mainly work with the *normalized symbol*

$$\sigma_{\text{norm}} = \left(\text{Id} + \frac{\hbar}{2}\Delta\right)\sigma_{\text{cont}}$$

where $\Delta = \partial^*\partial$ is the holomorphic Laplacian acting on $\mathcal{C}^\infty(M)$; unless otherwise mentioned, when we talk about a subprincipal symbol, this refers to the normalized symbol.

We can define the notions of admissibility, negligibility, microsupport and microlocal equivalence for Toeplitz operators, using the fact that their Schwartz kernels are sequences of sections of some line bundle (see [10, equation (4.1)] for a more precise statement).

2.4.3 The case of the complex plane

Let us briefly recall how to adapt the previous constructions to the case of the whole complex plane. We consider the Kähler manifold $\mathbb{C} \simeq \mathbb{R}^2$ with coordinates (x, ξ), standard complex structure and symplectic form $\omega_0 = d\xi \wedge dx$. Let $L_0 = \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$ be the trivial fiber bundle with standard Hermitian metric h_0 and connection ∇^0 with 1-form $\frac{1}{i}\alpha$, where $\alpha_u(v) = \frac{1}{2}\omega_0(u, v)$; endow L_0 with the unique holomorphic structure compatible with h_0 and ∇^0. For every positive integer k, the quantum space at order k is

$$\mathcal{H}_k^0 = H^0(\mathbb{R}^2, L_0^k) \cap L^2(\mathbb{R}^2, L_0^k),$$

and it turns out that $\mathcal{H}_k^0 = \mathcal{B}_k$ (see section 2.2 for the definition of $\mathcal{B}_k$); indeed, if we choose the holomorphic coordinate $z = \frac{x - i\xi}{\sqrt{2}}$, then a section φ of $L_0^k \to \mathbb{R}^2$ is holomorphic if and only if

$$\partial_{\bar{z}}\varphi + \frac{kz}{2}\varphi = 0.$$

Hence, for $\psi : \mathbb{C} \to \mathbb{C}, z \mapsto \exp\left(-\frac{1}{2}|z|^2\right)$, the section ψ^k is a non-vanishing element of $H^0(\mathbb{R}^2, L_0^k)$ and any other holomorphic section is of the form $f\psi^k$ where f is a holomorphic function.

One can define the algebra of Toeplitz operators and the various symbols in a similar way than in the compact case; see [22] for details. We will call $\mathcal{T}_j$ the class of Toeplitz operators with symbol in $\mathcal{S}^1_j$. In what follows, Π^0_k will denote the orthogonal projector of $L^2(\mathbb{R}^2, L_0^k)$ onto $\mathcal{H}^0_k$, and we define the Toeplitz operator $\mathrm{Op}(f., k) = \Pi^0_k M_{f(.,k)}$ for $f(.,k)$ in $\mathcal{S}^1_j$.

Let us give more details about the microsupport in this setting. We start by recalling the following inequality in Bargmann spaces [1, equation (1.7)].

Lemma 2.5. *Let $\phi_k \in \mathcal{B}_k$. Then for every complex variable z*

$$|\phi_k(z)| \leq \left(\frac{k}{2\pi}\right)^{1/2} \|\phi_k\|_{\mathcal{B}_k}.$$

Similarly, for every vector fields $X_1, \ldots, X_p$ on $\mathbb{C}$, there exists a polynomial $P \in \mathbb{R}[x_1, x_2]$ with positive values such that for every $z \in \mathbb{C}$

$$|(\nabla_{X_1} \ldots \nabla_{X_p} \phi_k)(z)| \leq P(|z|, k)^{1/2} \|\phi_k\|_{\mathcal{B}_k}.$$

Proof. The first claim is proved in [1] in the case $k = 1$; the general case then comes from a change of variables. The second claim can be proved in the same way. □

Lemma 2.6. *Let u_k be a sequence of elements of $\mathcal{B}_k$ and Ω a bounded open subset of $\mathbb{C}$. Assume that $\|u_k\|_{L^2(\Omega),\exp} = O(k^{-\infty})$; then for any compact subset K of Ω, u_k and all its covariant derivatives are uniformly $O(k^{-\infty})$ on K.*

Proof. Choose a compactly supported smooth function η which is positive, vanishing outside Ω and with constant value 1 on K and set $v_k = \mathrm{Op}(\eta) u_k$. One has

$$\|v_k\|_{\mathcal{B}_k} = \|\Pi^0_k \eta u_k\|_{\mathcal{B}_k} \leq \|\eta u_k\|_{L^2,\exp} \leq \|u_k\|_{L^2(\Omega),\exp}$$

since Π^0_k is continuous $L^2(\mathbb{R}^2, L_0^k) \to L^2(\mathbb{R}^2, L_0^k)$ with norm smaller than 1. Hence, $\|v_k\|_{\mathcal{B}_k} = O(k^{-\infty})$. By lemma 2.5, this implies that v_k and its covariant derivatives are uniformly $O(k^{-\infty})$ on K; since $u_k = v_k + O(k^{-\infty})$ on K, the same holds for u_k. □

Lemma 2.7. *Let $(u_k)_{k\geq 1}$ be an admissible sequence of elements of $\mathcal{B}_k$ and $z_0 \in \mathbb{C}$. Then $z_0 \notin \mathrm{MS}(u_k)$ if and only if there exists a Toeplitz operator $T_k \in \mathcal{T}_0$, elliptic at z_0, such that $\|T_k u_k\|_{\mathcal{B}_k} = O(k^{-\infty})$.*

Proof. Assume that $z_0 \notin \mathrm{MS}(u_k)$. There exists a neighbourhood $\mathcal{U}$ of z_0 such that u_k is negligible on $\mathcal{U}$. Choose a compactly supported function

$\chi \in \mathcal{C}^\infty(\mathbb{C}, \mathbb{R})$ with support K contained in $\mathcal{U}$ and such that $\chi(z_0) = 1$, and set $T_k = \mathrm{Op}(\chi)$. One has for $z_1 \in \mathbb{C}$

$$(T_k u_k)(z_1) = \frac{k}{2\pi} \int_K \exp\left(-\frac{k}{2}\left(|z_1|^2 + |z_2|^2 - 2z_1\bar{z}_2\right)\right) \chi(z_2) u_k(z_2)\, d\mu(z_2),$$

which gives

$$|(T_k u_k)(z_1)| \leq \frac{k}{2\pi} \sup_K |u_k| \int_K \exp\left(-\frac{k}{2}|z_1 - z_2|^2\right)\, d\mu(z_2).$$

This allows to estimate the norm of $T_k u_k$:

$$\|T_k u_k\|^2_{\mathcal{B}_k} \leq \left(\frac{k}{2\pi}\right)^2 \left(\sup_K |u_k|\right)^2 \int_{\mathbb{C}} \int_K \exp\left(-k|z_1 - z_2|^2\right)\, d\mu(z_1) d\mu(z_2).$$

Hence

$$\|T_k u_k\|^2_{\mathcal{B}_k} \leq \left(\frac{k}{2\pi}\right)^2 \left(\sup_K |u_k|\right)^2 \mu(K) \int_{\mathbb{C}} \exp\left(-k|z_1|^2\right)\, d\mu(z_1)$$

and the necessary condition is proved since the integral is $O(k^{-1/2})$.

Conversely, assume that there exists a Toeplitz operator $T_k \in \mathcal{T}_0$ elliptic at z_0 such that $\|T_k u_k\|_{\mathcal{B}_k} = O(k^{-\infty})$. There exists a neighbourhood of z_0 where T_k is elliptic. Hence, by symbolic calculus, we can find a Toeplitz operator $S_k \in \mathcal{T}_0$ such that $S_k T_k \sim \Pi_k^0$ near (z_0, z_0). Thus, there exists a neighbourhood Ω of z_0 such that $S_k T_k u_k \sim u_k$ on Ω; this implies that $\|S_k T_k u_k\|_{L^2(\Omega)} = \|u_k\|_{L^2(\Omega)} + O(k^{-\infty})$. But, since S_k is bounded $\mathcal{B}_k \to \mathcal{B}_k$ by a constant $C > 0$ which does not depend on k, one has $\|S_k T_k u_k\|_{L^2(\Omega)} \leq C\|T_k u_k\|_{\mathcal{B}_k}$; this yields that $\|u_k\|_{L^2(\Omega)}$ is $O(k^{-\infty})$. Lemma 2.6 then gives the negligibility of u_k on Ω. □

Definition 2.8. A sequence $(u_k)_{k\geq 1}$ of elements of $\mathfrak{S}_k$ is said to be

- $\mathfrak{S}_k$-*admissible* if there exists N in $\mathbb{Z}$ such that every $\mathfrak{S}_k$ semi-norm of u_k is $O(k^N)$,
- $\mathfrak{S}_k$-*negligible* if every $\mathfrak{S}_k$ semi-norm of u_k is $O(k^{-\infty})$. We write $u_k = O_{\mathfrak{S}_k}(k^{-\infty})$.

Lemma 2.9. *Let $(u_k)_{k\geq 1}$ be an admissible sequence of elements of $\mathcal{B}_k$ and $z_0 \in \mathbb{C}$. Then $z_0 \notin \mathrm{MS}(u_k)$ if and only if there exists a Toeplitz operator $T_k \in \mathcal{T}_0$, elliptic at z_0, such that $T_k u_k = O_{\mathfrak{S}_k}(k^{-\infty})$.*

Proof. The proof is nearly the same as the one of lemma 2.4. One can show that if $z_0 \notin \mathrm{MS}(u_k)$, there exists a Toeplitz operator $T_k \in \mathcal{T}_0$, elliptic at z_0, such that for every polynomial function $P(z)$ of z only, $\sup_{z\in\mathbb{C}} |P(z)(T_k u_k)(z)| = O(k^{-\infty})$, using the fact that the multiplication by $P(z)$ is a Toeplitz operator. □

3 The Bargmann transform

3.1 Definition and first properties

The Bargmann transform is the unitary operator $B_k : L^2(\mathbb{R}) \to \mathcal{B}_k$ defined by

$$(B_k f)(z) = \left(\left(\frac{k}{\pi}\right)^{1/4} \int_{\mathbb{R}} \exp\left(k\left(-\frac{1}{2}(z^2+t^2)+\sqrt{2}zt\right)\right) f(t)\ dt \right) \psi^k(z).$$

We claimed earlier that the subspace $\mathfrak{S}_k$ of $\mathcal{B}_k$ defined in (1) is the analogue of the Schwartz space on the Bargmann side. The case $k=1$ is treated by the following theorem, due to Bargmann.

Theorem 3.1 ([2, theorem 1.7])**.** *The Bargmann transform B_1 is a bijective, bicontinuous mapping between $\mathscr{S}(\mathbb{R})$ and $\mathfrak{S}_1$.*

This allows us to handle the general case.

Proposition 3.2. *The Bargmann transform B_k is a bijection between $\mathscr{S}(\mathbb{R})$ and $\mathfrak{S}_k$.*

Proof. If f belongs to $\mathscr{S}(\mathbb{R})$, one has for z in $\mathbb{C}$

$$(B_k f)(z) = \left(\frac{k}{\pi}\right)^{1/4} \int_{\mathbb{R}} \exp\left(k\left(-\frac{1}{2}(z^2+t^2)+\sqrt{2}zt\right)\right) f(t)\ dt;$$

introducing the variables u and w such that $z = k^{-1/2}w$ and $t = k^{-1/2}u$, this reads

$$(B_k f)(z) = (k\pi)^{-1/4} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}(w^2+u^2)+\sqrt{2}wu\right) f(k^{-1/2}u)\ du.$$

Hence, we have $(B_k f)(z) = (k\pi)^{-1/4}\,(B_1 g)(k^{1/2}z)$, where $g(t) = f(k^{-1/2}t)$. Obviously, the function g belongs to $\mathscr{S}(\mathbb{R})$; thus, by the previous theorem, $B_1 g$ belongs to $\mathfrak{S}_1$. Hence, for $j \in \mathbb{N}$, there exists a constant $C_j > 0$ such that for every complex variable w

$$\left| (B_1 g)(w) \exp\left(-\frac{1}{2}|w|^2\right) \right| \leq C_j \left(1+|w|^2\right)^{-j/2}.$$

This implies that for every z in $\mathbb{C}$,

$$\left| (B_k f)(z) \exp\left(-\frac{k}{2}|z|^2\right) \right| \leq C_j k^{-j/2} \left(1+k|z|^2\right)^{-j/2}$$

and since $k \geq 1$, this yields

$$\left| (B_k f)(z) \exp\left(-\frac{k}{2}|z|^2\right) \right| \leq C_j \left(1+|z|^2\right)^{-j},$$

which means that $B_k f$ belongs to $\mathfrak{S}_k$. The converse is proved in the same way, using the explicit form of the inverse mapping:

$$(B_k^* g)(t) = \left(\frac{k}{\pi}\right)^{1/4} \int_{\mathbb{R}} \exp\left(k\left(-\frac{1}{2}(\bar{z}^2 + t^2) + \sqrt{2}\bar{z}t - |z|^2\right)\right) g(z)\ d\mu(z)$$

for g in $\mathfrak{S}_k$ and $t \in \mathbb{R}$. □

3.2 Action on Toeplitz operators

The Bargmann transform has the good property to conjugate a Toeplitz operator with symbol defined on $\mathbb{C}$ (thus acting on the spaces $\mathcal{B}_k$) to a pseudodifferential operator with symbol defined on $T^*\mathbb{R}$ (thus acting on $L^2(\mathbb{R})$), and conversely.

Lemma 3.3. *Let T_k be a Toeplitz operator in the class $\mathcal{T}_j$, with contravariant symbol $\sigma_{cont}(T_k) = f(.,\hbar)$; then $B_k^* T_k B_k$ is a pseudodifferential operator with Weyl symbol*

$$\sigma^W(B_k^* T_k B_k)(x,\xi) = I(f(.,\hbar))(x,\xi) = \frac{1}{\pi\hbar}\int_{\mathbb{C}} \exp(-2\hbar^{-1}|w|^2) f(w+z,\hbar) d\lambda(w),$$

where $z = \frac{1}{\sqrt{2}}(x - i\xi)$. The map I is continuous $\mathcal{S}_j \to \mathcal{S}_j$. Moreover, for any $f(.,\hbar) \in \mathcal{S}_j$ and all $p \geq 1$,

$$I(f(.,\hbar)) = \sum_{j=0}^{p-1} \left(\frac{\hbar}{2}\right)^j \frac{\Delta^j f(.,\hbar)}{j!} + h^p R_p(f(.,\hbar)). \tag{2}$$

where R_p is a continuous map from $\mathcal{S}_j$ to $\mathcal{S}_j$.

Proof. Thanks to [12, theorem 5.2], we know that the result holds when $T_k = \Pi_k^0 f \Pi_k^0$, f being a bounded function on $\mathbb{C}$ not depending on k. Now, using the stationary phase method, one can prove that the map I is continuous $\mathcal{S}_j \to \mathcal{S}_j$ with the asymptotic expansion (2), and conclude by a density argument. □

3.3 Microlocalization and Bargmann transform

Lemma 3.4. *1. B_k maps $\mathscr{S}$-admissible functions to $\mathfrak{S}_k$-admissible sections, and B_k^* maps $\mathfrak{S}_k$-admissible sections to $\mathscr{S}$-admissible functions.*

2. B_k maps $O_{\mathscr{S}}(k^{-\infty})$ into $O_{\mathfrak{S}_k}(k^{-\infty})$, and B_k^* maps $O_{\mathfrak{S}_k}(k^{-\infty})$ into $O_{\mathscr{S}}(k^{-\infty})$.

Proof. These results are proved by performing a change of variables, as in proposition 3.2. □

We are now able to prove the link between the wavefront set and the microsupport *via* the Bargmann transform.

Proposition 3.5. *Let u_k be an admissible sequence of elements of $\mathscr{S}(\mathbb{R})$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u_k)$ if and only if $z_0 = \frac{1}{\sqrt{2}}(x_0 - i\xi_0) \notin \mathrm{MS}(B_k u_k)$.*

Proof. Assume that $z_0 = \frac{1}{\sqrt{2}}(x_0 - i\xi_0)$ does not belong to $\mathrm{MS}(B_k u_k)$; by lemma 2.9, there exists a Toeplitz operator T_k, elliptic at z_0, such that $T_k B_k u_k \psi^k = O_{\mathfrak{S}_k}(k^{-\infty})$. Thanks to lemma 3.3, $P_k = B_k^* T_k B_k$ is a pseudodifferential operator elliptic at (x_0, ξ_0). Furthermore, thanks to lemma 3.4, $P_k u_k = B_k^* T_k B_k u_k \psi_k = O_{\mathscr{S}}(k^{-\infty})$; we conclude by lemma 2.4. The proof of the converse follows the same steps. □

4 The sheaf of microlocal solutions

In this section, T_k is a self-adjoint Toeplitz operator on M, with normalized symbol $f(., \hbar) = \sum_{\ell \geq 0} \hbar^\ell f^\ell$. Following Vũ Ngọc [29, 30], we introduce the sheaf of microlocal solutions of the equation $T_k \psi_k = 0$.

Let us recall the motivation for considering microlocal solutions: roughly speaking, they allow to split the eigenvalue equation $T_k \psi_k = \lambda \psi_k$ into several local problems, the Bohr-Sommerfeld rules being a necessary and sufficient condition to glue together the solutions to these problems in order to obtain a global approximate solution to this equation. For the sake of brevity, we begin with the case $\lambda = 0$, and we introduce a spectral parameter only in section 4.6.

4.1 Microlocal solutions

Let U be an open subset of M; we call a sequence of sections $\psi_k \in \mathcal{C}^\infty(U, L^k \otimes K)$ a local state over U.

Definition 4.1. We say that a local state ψ_k is a microlocal solution of

$$T_k \psi_k = 0 \tag{3}$$

on U if it is admissible and for every $x \in U$, there exists a function $\chi \in \mathcal{C}^\infty(M)$ with support contained in U, equal to 1 in a neighbourhood of x and such that

$$\Pi_k(\chi \psi_k) = \psi_k + O(k^{-\infty}), \quad T_k(\Pi_k(\chi \psi_k)) = O(k^{-\infty})$$

on a neighbourhood of x.

One can show that if $\psi_k \in \mathcal{H}_k$ is admissible and satisfies $T_k \psi_k = 0$, then the restriction of ψ_k to U is a microlocal solution of (3) on U. Moreover, the set $S(U)$ of microlocal solutions of this equation on U is a $\mathbb{C}_k$-module containing the set of negligible local states as a submodule (let us recall that $\mathbb{C}_k$ is the set of constant symbols, see section 2.1). We denote by $\mathrm{Sol}(U)$ the module obtained by taking the quotient of $S(U)$ by the negligible local states; the notation $[\psi_k]$ will stand for the equivalence class of $\psi_k \in S(U)$.

Lemma 4.2. *The collection of* $\mathrm{Sol}(U)$*,* U *open subset of* M*, together with the natural restrictions maps* $r_{U,V} : \mathrm{Sol}(V) \to \mathrm{Sol}(U)$ *for* U, V *open subsets of* M *such that* $U \subset V$*, define a complete presheaf.*

Thus, we obtain a sheaf Sol over M, called the sheaf of microlocal solutions on M.

4.2 The sheaf of microlocal solutions

One can show that if the principal symbol f_0 of T_k does not vanish on U, then $\mathrm{Sol}(U) = \{0\}$. Equivalently, if $\psi_k \in \mathcal{H}_k$ satisfies $T_k\psi_k = 0$, then its microsupport is contained in the level $\Gamma_0 = f_0^{-1}(0)$. This implies the following lemma.

Lemma 4.3. *Let* Ω *be an open subset of* Γ_0*; write* $\Omega = U \cap \Gamma_0$ *where* U *is an open subset of* M*. Then the restriction map*

$$r_\Omega : \mathrm{Sol}(U) \to \mathfrak{F}_U(\Omega) := r_\Omega(\mathrm{Sol}(U)), \quad [\psi_k] \mapsto [\psi_{k|\Omega}]$$

is an isomorphism of $\mathbb{C}_k$*-modules.*

We want to define a new sheaf $\mathfrak{F} \to \Gamma_0$ that still describes the microlocal solutions of (3). In order to do so, we will check that the module $\mathfrak{F}_U(\Omega)$ does not depend on the open set U such that $\Omega = \Gamma_0 \cap U$. We first prove:

Lemma 4.4. *Let* $U, \widetilde{U}$ *be two open subsets of* M *such that* $\Omega = U \cap \Gamma_0 = \widetilde{U} \cap \Gamma_0$*. Then there exists an isomorphism between* $\mathrm{Sol}(U)$ *and* $\mathrm{Sol}(\widetilde{U})$ *commuting with the restriction maps.*

Proof. Assume that U and $\widetilde{U}$ are distinct and set $V = U \cap \widetilde{U}$; of course $\Omega \subset V$. Write $\widetilde{U} = V \cup W$ where the open set W is such that there exists an open set $X \subset V$ containing Ω such that $W \cap X = \emptyset$. Let χ_V, χ_W be a partition of unity subordinate to $\widetilde{U} = V \cup W$; in particular, $\chi_V(x) = 1$ whenever $x \in X$. One can show that the class $F_{\chi_V}(\psi_k) = [\chi_V \psi_k]$ belongs to $\mathrm{Sol}(\widetilde{U})$. We claim that the map F_{χ_V} is an isomorphism with the required property. □

From these two lemmas, we deduce the:

Proposition 4.5. *Let* $U, \widetilde{U}$ *be two open subsets of* M *such that* $\Omega = U \cap \Gamma_0 = \widetilde{U} \cap \Gamma_0$*. Then* $\mathfrak{F}_U(\Omega) = \mathfrak{F}_{\widetilde{U}}(\Omega)$*.*

This allows to define a sheaf $\mathfrak{F} \to \Gamma_0$, which will be called the sheaf of microlocal solutions over Γ_0. Let us point out that so far, we have made no assumption on the structure (regularity) of the level Γ_0.

4.3 Regular case

Consider a point $m \in \Gamma_0$ which is regular for the principal symbol f_0. Then there exists a symplectomorphism χ between a neighbourhood of m in M and a neighbourhood of the origin in $\mathbb{R}^2$ such that $(f_0 \circ \chi^{-1})(x, \xi) = \xi$. We can quantize this symplectomorphism by means of a Fourier integral operator [7, 32, 9, 22]: there exists an admissible sequence of operators $U_k^{(m)} : \mathcal{C}^\infty(\mathbb{R}^2, L_0^k) \to \mathcal{C}^\infty(M, L^k \otimes K)$ such that

$$U_k^{(m)} \left(U_k^{(m)}\right)^* \sim \Pi_k \quad \text{near } m$$

and

$$\left(U_k^{(m)}\right)^* U_k^{(m)} \sim \Pi_k^0, \quad \left(U_k^{(m)}\right)^* T_k U_k^{(m)} \sim S_k \quad \text{near } 0,$$

where S_k is the Toeplitz operator

$$S_k = \frac{i}{\sqrt{2}} \left(z - \frac{1}{k} \frac{d}{dz} \right),$$

which means that $S_k u = \frac{i}{\sqrt{2}} \left(zf - \frac{1}{k} \frac{df}{dz} \right) \psi^k$ if $u = f\psi^k$. Consider the element Φ_k of $\mathcal{C}^\infty(\mathbb{R}^2, L_0^k)$ given by

$$\Phi_k(z) = \exp\left(kz^2/2\right) \psi^k(z), \qquad \psi(z) = \exp\left(-\frac{1}{2}|z|^2\right);$$

it satisfies $S_k \Phi_k = 0$. Choosing a suitable cutoff function η and setting $\Phi_k^{(m)} = \Pi_k^0(\eta \Phi_k)$, we obtain an admissible sequence $\Phi_k^{(m)}$ of elements of $\mathcal{B}_k$ microlocally equal to Φ_k near the origin and generating the $\mathbb{C}_k$-module of microlocal solutions of $S_k u_k = 0$ near the origin.

Proposition 4.6. *The $\mathbb{C}_k$-module of microlocal solutions of equation (3) near m is free of rank 1[2], generated by $U_k^{(m)} \Phi_k^{(m)}$.*

This is a slightly modified version of proposition 3.6 of [9], in which the normal form is achieved on the torus instead of the complex plane.

Thus, if Γ_0 contains only regular points of the principal symbol f_0, then $\mathfrak{F} \to \Gamma_0$ is a sheaf of free $\mathbb{C}_k$-modules of rank 1; in particular, this implies that $\mathfrak{F} \to \Gamma_0$ is a flat sheaf, thus characterised by its Čech holonomy $\mathrm{hol}_{\mathfrak{F}}$.

4.4 Lagrangian sections

In order to compute the holonomy $\mathrm{hol}_{\mathfrak{F}}$, we have to understand the structure of the microlocal solutions. For this purpose, a family of solutions of particular interest is given by Lagrangian sections; let us define these. Consider a curve $\Gamma \subset \Gamma_0$ containing only regular points, and let $j : \Gamma \to M$ be the embedding of Γ into M. Let U be an open set of M such that $U_\Gamma = j^{-1}(U \cap \Gamma)$

[2] We recall that this means that this module admits a basis with one element.

is contractible; there exists a flat unitary section t_Γ of $j^*L \to U_\Gamma$. Now, consider a formal series

$$\sum_{\ell \geq 0} \hbar^\ell g_\ell \in \mathcal{C}^\infty(U_\Gamma, j^*K)[[\hbar]].$$

Let V be an open set of M such that $\overline{V} \subset U$. Then a sequence $\Psi_k \in \mathcal{H}_k$ is a *Lagrangian section* associated to (Γ, t_Γ) with symbol $\sum_{\ell \geq 0} \hbar^\ell g_\ell$ if

$$\Psi_k(m) = \left(\frac{k}{2\pi}\right)^{1/4} F^k(m)\tilde{g}(m,k) \text{ over } V,$$

where

- F is a section of $L \to U$ such that

  $$j^*F = t_\Gamma \quad \text{and} \quad \bar{\partial}F = 0$$

  modulo a section vanishing to every order along $j(\Gamma)$, and $|F(m)| < 1$ if $m \notin j(\Gamma)$,

- $\tilde{g}(.,k)$ is a sequence of $\mathcal{C}^\infty(U, K)$ admitting an asymptotic expansion $\sum_{\ell \geq 0} k^{-\ell} \tilde{g}_\ell$ in the $\mathcal{C}^\infty$ topology such that

  $$j^*\tilde{g}_\ell = g_\ell \quad \text{and} \quad \bar{\partial}\tilde{g}_\ell = 0$$

  modulo a section vanishing at every order along $j(\Gamma)$.

Assume furthermore that Ψ_k is admissible in the sense that $\Psi_k(m)$ is uniformly $O(k^N)$ for some N and the same holds for its successive covariant derivatives. It is possible to construct such a section with given symbol $\sum_{\ell \geq 0} \hbar^\ell g_\ell$ (see [10, part 3]). Furthermore, if Ψ_k is a non-zero Lagrangian section, then the constants $c_k \in \mathbb{C}_k$ such that $c_k \Psi_k$ is still a Lagrangian section are the elements of the form

$$c_k = \rho(k) \exp(ik\phi(k)) + O(k^{-\infty}) \tag{4}$$

where $\rho(k), \phi(k) \in \mathbb{R}$ admit asymptotic expansions of the form $\rho(k) = \sum_{\ell \geq 0} k^{-\ell} \rho_\ell$, $\phi(k) = \sum_{\ell \geq 0} k^{-\ell} \phi_\ell$.

Lagrangian sections are important because they provide a way to construct microlocal solutions. Indeed, if Ψ_k is a Lagrangian section over V associated to (Γ, t_Γ) with symbol $\sum_{\ell \geq 0} \hbar^\ell g_\ell$, then $T_k \Psi_k$ is also a Lagrangian section over V associated to (Γ, t_Γ), and one can in principle compute the elements $\hat{g}_\ell$, $\ell \geq 0$ of the formal expansion of its symbol as a function of the g_ℓ, $\ell \geq 0$ (by means of a stationary phase expansion). This allows to solve equation (3) by prescribing the symbol of Ψ_k so that for every $\ell \geq 0$, $\hat{g}_\ell$ vanishes. Let us detail this for the two first terms.

Introduce a *half-form bundle* (δ, φ), that is a line bundle $\delta \to M$ together with an isomorphism of line bundles $\varphi : \delta^{\otimes 2} \to \Lambda^{1,0}T^*M$. Since the first Chern class of M, which is equal to its Euler characteristic, is even, such a couple exists. Introduce the Hermitian holomorphic line bundle L_1 such that $K = L_1 \otimes \delta$. Define the *subprincipal form* κ as the 1-form on Γ such that

$$\kappa(X_{f_0}) = -f_1$$

where X_{f_0} stands for the Hamiltonian vector field associated to f_0. Introduce the connection ∇^1 on $j^*L_1 \to \Gamma$ defined by

$$\nabla^1 = \nabla^{j^*L_1} + \frac{1}{i}\kappa,$$

with $\nabla^{j^*L_1}$ the connection induced by the Chern connection of L_1 on j^*L_1. Let δ_Γ be the restriction of δ to Γ; the map

$$\varphi_\Gamma : \delta_\Gamma^{\otimes 2} \to T^*\Gamma \otimes \mathbb{C}, \quad u \mapsto j^*\varphi(u)$$

is an isomorphism of line bundles. Define a connection ∇^{δ_Γ} on δ_Γ by

$$\nabla_X^{\delta_\Gamma}\sigma = \mathcal{L}_X^{\delta_\Gamma}\sigma,$$

where $\mathcal{L}_X^{\delta_\Gamma}$ is the first-order differential operator acting on sections of δ_Γ such that

$$\varphi_\Gamma\left(\mathcal{L}_X^{\delta_\Gamma} g \otimes g\right) = \frac{1}{2}\mathcal{L}_X \varphi_\Gamma\left(g^{\otimes 2}\right)$$

for every section g; here, $\mathcal{L}$ stands for the standard Lie derivative of forms.

It was proved in [10, theorem 3.3, theorem 3.4] that $T_k\Psi_k$ is a Lagrangian section over V associated to t_Γ with symbol $(j^*f_0)g_0 + O(\hbar) = O(\hbar)$ (so Ψ_k satisfies equation (3) up to order $O(k^{-1})$) and that the subprincipal symbol of $T_k\Psi_k$ is

$$(j^*f_1)g_0 + \frac{1}{i}\left(\nabla_{X_{f_0}}^{j^*L_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}_{X_{f_0}}^{\delta_\Gamma}\right) g_0.$$

Consequently, equation (3) is satisfied by Ψ_k up to order $O(k^{-2})$ if and only if

$$\left(f_1 + \frac{1}{i}\left(\nabla_{X_{f_0}}^{j^*L_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathcal{L}_{X_{f_0}}^{\delta_\Gamma}\right)\right) g_0 = 0 \quad \text{over } V \cap \Gamma. \tag{5}$$

This can be interpreted as a parallel transport equation: if we endow $j^*L_1 \otimes \delta_\Gamma$ with the connection induced from ∇^1 and ∇^{δ_Γ}, equation (5) means that g_0 is flat.

4.5 Holonomy

We now assume that Γ_0 is connected (otherwise, one can consider connected components of Γ_0) and contains only regular points; it is then a smooth closed curve embedded in M. We would like to compute the holonomy of the sheaf $\mathfrak{F} \to \Gamma_0$.

Proposition 4.7. *The holonomy* $\mathrm{hol}_{\mathfrak{F}}(\Gamma_0)$ *is of the form*

$$\mathrm{hol}_{\mathfrak{F}}(\Gamma_0) = \exp(ik\Theta(k)) + O(k^{-\infty}) \tag{6}$$

where $\Theta(k)$ *is real-valued and admits an asymptotic expansion of the form* $\Theta(k) = \sum_{\ell \geq 0} k^{-\ell}\Theta_\ell$.

In particular, this means that if we consider another set of solutions to compute the holonomy, we only have to keep track of the phases of the transition constants.

Proof. Cover Γ_0 by a finite number of open subsets Ω_α in which the normal form introduced before proposition 4.6 applies, and let U_k^α and Φ_k^α be as in this proposition. We obtain a family u_k^α of microlocal solutions; observe that for each α, u_k^α is a Lagrangian section associated to Γ. Hence, if $\Omega_\alpha \cap \Omega_\beta$ is non-empty, the unique (modulo $O(k^{-\infty})$) constant $c_k^{\alpha\beta} \in \mathbb{C}_k$ such that $u_k^\alpha = c_k^{\alpha\beta} u_k^\beta$ on $\Omega_\alpha \cap \Omega_\beta$ is of the form given in equation (4):

$$c_k^{\alpha\beta} = \rho^{\alpha\beta}(k)\exp(ik\phi^{\alpha\beta}(k)) + O(k^{-\infty}).$$

But if m belongs to $\Omega_\alpha \cap \Omega_\beta$, then near m we have $u_k^\alpha \sim U_k^\alpha \Phi_k^{(m)}$ and $u_k^\beta \sim U_k^\beta \Phi_k^{(m)}$ where $\Phi_k^{(m)}$ is an admissible sequence of elements of $\mathcal{B}_k$ microlocally equal to Φ_k near the origin. Therefore, we have

$$c_k^{\alpha\beta}\Phi_k^{(m)} = (U_k^\beta)^{-1} U_k^\alpha \Phi_k^{(m)} + O(k^{-\infty}),$$

and the fact that the operators U_k^α, U_k^β are microlocally unitary yields $\left|c_k^{\alpha\beta}\right|^2 = 1 + O(k^{-\infty})$. This implies that the coefficients $\rho_\ell^{\alpha\beta}$ in the asymptotic expansion of $\rho^{\alpha\beta}(k)$ vanish for $\ell \geq 1$, which gives the result. □

Let us be more specific and compute the first terms of this asymptotic expansion. Consider a finite cover $(\Omega_\alpha)_\alpha$ of Γ_0 by open subsets with $j^{-1}(\Omega_\alpha)$ contractible and endow a neighbourhood of each Ω_α in M with a non-trivial microlocal solution Ψ_k^α which is a Lagrangian section. Choose a flat unitary section t_α of the line bundle $j^*L \to j^{-1}(\Omega_\alpha)$ and write, for $m \in \Omega_\alpha$:

$$\Psi_k^\alpha(m) = \left(\frac{k}{2\pi}\right)^{1/4} g_\alpha(m,k) t_\alpha^k(m)$$

where the section $g_\alpha(.,k)$ of $j^*K \to \Omega_\alpha$ is the symbol of Ψ_k^α, whose principal symbol will be denoted by $g_\alpha^{(0)}$. Now, assume that $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$; there exists a unique (up to $O(k^{-\infty})$) $c_k^{\alpha\beta} \in \mathbb{C}_k$ such that $\Psi_k^\alpha \sim c_k^{\alpha\beta}\Psi_k^\beta$ on $\Omega_\alpha \cap \Omega_\beta$.

Definition 4.8. Let $A, B \in M$ and γ be a piecewise smooth curve joining A and B; denote by $P_{A,B,\gamma} : L_A \to L_B$ the linear isomorphism given by parallel transport from A to B along γ. Given two sections s, t of $L \to M$ such that $s(A) \neq 0$ and $t(B) \neq 0$, define the *phase difference* between $s(A)$ and $t(B)$ along γ as the number

$$(\Phi_s(A) - \Phi_t(B))_\gamma = \arg(\lambda_{A,B,\gamma}) - c_0([A,B]) \in \mathbb{R}/2\pi\mathbb{Z},$$

where $\lambda_{A,B,\gamma}$ is the unique complex number such that $P_{A,B,\gamma}(s(A)) = \lambda_{A,B,\gamma} t(B)$ and $c_0([A,B])$ is the (phase of the) holonomy of γ in (L, ∇) (computed with respect to some fixed trivializations at A, B). Define in the same way the phase difference for two sections of $K \to M$, this time using the Chern connection of K.

Now, consider three points $A, B, C \in M$ and let γ_1 (resp. γ_2) be a piecewise smooth curve joining A and B (resp. B and C). Let γ be the concatenation of γ_1 and γ_2. It is easily checked that

$$(\Phi_s(A) - \Phi_t(B))_{\gamma_1} + (\Phi_t(B) - \Phi_u(C))_{\gamma_2} = (\Phi_s(A) - \Phi_u(C))_\gamma$$

for three sections s, t, u of L. Furthermore, if γ is a closed curve and A is a point on γ, then the phase difference between $s(A)$ and $s(A)$ along γ is

$$(\Phi_s(A) - \Phi_s(A))_\gamma = 0$$

by definition of the holonomy c_0. This is why we write this number as a difference. Note that this still holds true if we change the set of trivializations used to compute c_0.

Coming back to our problem, denote by $\Phi_\alpha^{(-1)}(A) - \Phi_\beta^{(-1)}(B)$ the phase difference between $t_\alpha(A)$ and $t_\beta(B)$ along Γ_0 in L, and by $\Phi_\alpha^{(0)}(A) - \Phi_\beta^{(0)}(B)$ the phase difference between $g_\alpha^{(0)}(A)$ and $g_\beta^{(0)}(B)$ along Γ_0 in K. Let ζ be the path in Γ_0 starting at a point $A \in \Omega_\alpha$ and ending at $B \in \Omega_\alpha \cap \Omega_\beta$. Since t_α is flat and the principal symbol g_0 of Ψ_k^α satisfies equation (5), we have

$$\begin{aligned}\arg\left(c_k^{\alpha\beta}\right) &= k\left(c_0(\zeta) + \Phi_\alpha^{(-1)}(A) - \Phi_\beta^{(-1)}(B)\right) \\ &+ c_1(\zeta) + \mathrm{hol}_{\delta_{\Gamma_0}}(\zeta) + \Phi_\alpha^{(0)}(A) - \Phi_\beta^{(0)}(B) \ + O(k^{-1}),\end{aligned}$$

where $c_1(\zeta)$ is the holonomy of ζ in (L_1, ∇^1) and $\mathrm{hol}_{\delta_{\Gamma_0}}(\zeta)$ is the holonomy of ζ in $(\delta_{\Gamma_0}, \nabla^{\delta_{\Gamma_0}})$ (both computed with respect to some fixed trivializations of L_1 and δ_{Γ_0} at A, B).

Thanks to the discussion above, we know that the term

$$k\left(\Phi_\alpha^{(-1)}(A) - \Phi_\beta^{(-1)}(B)\right) + \Phi_\alpha^{(0)}(A) - \Phi_\beta^{(0)}(B)$$

is a Čech coboundary. The values $c_0(\Gamma_0)$, $c_1(\Gamma_0)$ and $\mathrm{hol}_{\delta_{\Gamma_0}}(\Gamma_0)$ do not depend on the trivializations chosen for the computations. Moreover, one can check that $\nabla^{\delta_{\Gamma_0}}$ has holonomy in $\mathbb{Z}/2\mathbb{Z}$, represented by $\epsilon(\Gamma_0) \in \{0, 1\}$. Thus, we obtain:

Proposition 4.9. *The first two terms of the asymptotic expansion of the quantity* $\Theta(k)$ *defined in proposition 4.7 are given by*

$$\Theta_0 = c_0(\Gamma_0)$$

and

$$\Theta_1 = c_1(\Gamma_0) + \epsilon(\Gamma_0)\pi.$$

Since one can construct a non-trivial microlocal solution over Γ_0 if and only if $\Theta(k) \in 2\pi\mathbb{Z}$, we recover the usual Bohr-Sommerfeld conditions.

Let us give another interpretation of the index ϵ. Consider a smooth closed curve γ immersed in M. Denote by $\iota : \gamma \to M$ this immersion, and by $\delta_\gamma = \iota^*\delta$ the pullback bundle over γ. Let $\tilde{\iota} : \delta_\gamma \to \delta$ be the natural lift of ι, and define $\tilde{\iota}^2 : \delta_\gamma^{\otimes 2} \to \delta^{\otimes 2}$ by the formula $\tilde{\iota}^2(u \otimes v) = \tilde{\iota}(u) \otimes \tilde{\iota}(v)$. The map

$$\varphi_\gamma : \delta_\gamma^{\otimes 2} \to T^*\gamma \otimes \mathbb{C}, \quad u \mapsto \iota^*\varphi(\tilde{\iota}^2(u))$$

is an isomorphism of line bundles. The set

$$\left\{ u \in \delta_\gamma ; \varphi_\gamma(u^{\otimes 2}) > 0 \right\}$$

has one or two connected components. In the first case, we set $\epsilon(\gamma) = 1$, and in the second case $\epsilon(\gamma) = 0$. One can check that this definition coincides with the one above when γ is a smooth embedded closed curve. Notice that the value of $\epsilon(\gamma)$ only depends on the isotopy class of γ in M.

4.6 Spectral parameter dependence

For spectral analysis, one has to do the same study as above replacing the operator T_k with $T_k - \lambda$, $\lambda \in \mathbb{R}$; then it is natural to ask if the previous study can be done taking into account the dependence of the operator on the spectral parameter λ.

Assume that there exists a tubular neighbourhood Ω of Γ such that for λ close enough to 0, the intersection $\Gamma_\lambda \cap \Omega$ is regular. Then we can construct microlocal solutions of $(T_k - \lambda)u_k = 0$ as Lagrangian sections depending smoothly on a parameter (see [9, section 2.6]); these solutions are uniform in λ. We can then define all the previous objects with smooth dependence in λ. Proceeding this way, we obtain the parameter dependent Bohr-Sommerfeld conditions, that we describe below.

Let I be an interval of regular values of the principal symbol f_0 of the operator. For $\lambda \in I$, denote by $\mathcal{C}_j(\lambda)$, $1 \leq j \leq N$, the connected components of $f_0^{-1}(\lambda)$ in such a way that for j fixed and $\lambda_1 \neq \lambda_2 \in I$, $\mathcal{C}_j(\lambda_1)$ and $\mathcal{C}_j(\lambda_2)$ belong to the same connected component of $f_0^{-1}(I)$. Observe that $\mathcal{C}_j(\lambda)$ is a smooth embedded closed curve, endowed with the orientation depending continuously on λ given by the Hamiltonian flow of f_0. Define the *principal action* $c_0^{(j)} \in \mathcal{C}^\infty(I)$ in such a way that the parallel transport in L along $\mathcal{C}_j(\lambda)$

is the multiplication by $\exp(ic_0^{(j)}(\lambda))$. Define the *subprincipal action* $c_1^{(j)}$ in the same way, replacing L by L_1 and using the connection ∇^1 (depending on λ) described above. Finally, set $\epsilon_\lambda^{(j)} = \epsilon(\mathcal{C}_j(\lambda))$; in fact, $\epsilon_\lambda^{(j)}$ is a constant $\epsilon_\lambda^{(j)} = \epsilon^{(j)}$ for λ in I. Fix E in I; the Bohr-Sommerfeld conditions (see [10] for more details) state that there exists $\eta > 0$ such that the intersection of the spectrum of T_k with $[E-\eta, E+\eta]$ modulo $O(k^{-\infty})$ is the union of the spectra σ_j, $1 \leq j \leq N$, where the elements of σ_j are the solutions of

$$g^{(j)}(\lambda, k) \in 2\pi k^{-1}\mathbb{Z}$$

where $g^{(j)}(., k)$ is a sequence of functions of $\mathcal{C}^\infty(I)$ admitting an asymptotic expansion

$$g^{(j)}(., k) = \sum_{\ell \geq 0} k^{-\ell} g_\ell^{(j)}$$

with coefficients $g_\ell^{(j)} \in \mathcal{C}^\infty(I)$. Furthermore, one has

$$g_0^{(j)}(\lambda) = c_0^{(j)}(\lambda) \text{ and } g_1^{(j)}(\lambda) = c_1^{(j)}(\lambda) + \epsilon^{(j)}\pi.$$

5 Microlocal normal form

5.1 Normal form on the Bargmann side

Let P_k be the operator defined by $P_k = \frac{i}{2}\left(z^2 - \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)$ with domain $\mathbb{C}[z] \subset \mathcal{B}_k$; it is a Toeplitz operator with normalized symbol $p_0(x,\xi) = x\xi$. We will use this operator to understand the behaviour of A_k when acting on sections localized near each s_j, $1 \leq j \leq n$. In fact, we study the operator $A_k - E$, where $E \in \mathbb{R}$ is allowed to vary in a neighbourhood of zero.

Let $j \in [\![1, n]\!]$. The isochore Morse lemma [17] yields a symplectomorphism χ_E from a neighbourhood of s_j in M to a neighbourhood of the origin in $\mathbb{R}^2$, depending smoothly on E, and a smooth function g_j^E, again depending smoothly on E, such that

$$((a_0 - E) \circ \chi_E^{-1})(x, \xi) = g_j^E(x\xi)$$

and $(g_j^E)'(0) \neq 0$. Using a Taylor formula, one can write

$$g_j^E(t) = w_j^E(t)\,(t - f_j(E))$$

with w_j^E smooth, depending smoothly on E, and such that $w_j^E(0) \neq 0$, and f_j a smooth function of E with $f_j(0) = 0$. This symplectic normal form can be quantized to the following semiclassical normal form.

Proposition 5.1. *Fix $j \in [\![1, n]\!]$. There exist a smooth function f_j, a Fourier integral operator $U_k^E : \mathcal{B}_k \to \mathcal{H}_k$, a Toeplitz operator W_k^E elliptic* *at 0 and a sequence of smooth functions $\varepsilon_j(.,k)$ admitting an asymptotic expansion $\varepsilon_j(E,k) = \sum_{\ell=0}^{+\infty} k^{-\ell}\varepsilon_j^{(\ell)}(E)$ such that*

$$(U_k^E)^*(A_k - E)U_k^E \sim W_k^E \left(P_k - f_j(E) - k^{-1}\varepsilon_j(E,k)\right)$$

microlocally near s_j. Furthermore,

- *U_k and W_k depend smoothly on E,*
- *$f_j(E)$ is the value of $x\xi$ whenever $(x,\xi) = \chi_E(m)$ for $m \in \Gamma_E$,*
- *and the first term of the asymptotic expansion of $\varepsilon_j(0,k)$ is given by*

$$\varepsilon_j^{(0)}(0) = \frac{-a_1(s_j)}{|\det(\mathrm{Hess}(a_0)(s_j))|^{1/2}},$$

  where $\mathrm{Hess}(a_0)(s_j)$ is the Hessian of a_0 at s_j.

The proof is an adaptation of the one in [14, section 3] to the Toeplitz setting; see also [22, theorem 5.3] for a similar result in the elliptic case.

5.2 Link with the pseudodifferential setting

Now, we use the Bargmann transform to understand the structure of the space of microlocal solutions of $P_k - E = 0$.

Lemma 5.2. *For $u \in \mathscr{S}(\mathbb{R})$, one has*

$$B_k^* P_k B_k u = \frac{1}{ik}\left(x\partial_x + 1\right)u.$$

From now on, we will denote by S_k the pseudodifferential operator $\frac{1}{ik}\left(x\partial_x + 1\right)$. This correspondence will allow us to understand the space of microlocal solutions of $P_k - E$ on a neighbourhood of the origin. Let us recall the results of [14, 15] that will be useful to our study.

Proposition 5.3 ([14, proposition 3])**.** *Let E be such that $|E| < 1$. The space of microlocal solutions of $(S_k - E)u_k = 0$ on $Q = [-1,1]^2$ is a free $\mathbb{C}_k$-module of rank 2.*

Moreover, we know two bases of this module. Indeed, let $\mathcal{F}_k$ be the semiclassical Fourier transform:

$$(\mathcal{F}_k u)(\xi) = \frac{k}{2\pi}\int_{\mathbb{R}} \exp(-ikx\xi)u(x)\mathrm{d}x;$$

then the tempered distributions $v_{k,E}^{(j)}$, $j \in [\![1,4]\!]$ defined as:

$$v_{k,E}^{(1),(2)}(x) = \mathbf{1}_{\mathbb{R}_\pm^*}(x)\exp\left((-\tfrac{1}{2} + ikE)\ln(|x|)\right),$$

$$v_{k,E}^{(3),(4)}(x) = \mathcal{F}_k^{-1}\left(\mathbf{1}_{\mathbb{R}^*_\pm}(\xi)\exp\left((-\tfrac{1}{2}+ikE)\ln(|\xi|)\right)\right)(x),$$

are exact solutions of the equation $(S_k - E)v_{k,E}^{(j)} = 0$; better than that, the couple $(v_{k,E}^{(1)}, v_{k,E}^{(2)})$ (resp. $(v_{k,E}^{(3)}, v_{k,E}^{(4)})$) forms a basis of the space of solutions of this equation. Now, choose a compactly supported function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with constant value 1 on a neighbourhood of $I = [-1,1]$ and vanishing outside $2I$. Define the pseudodifferential operator Π_Q by

$$\Pi_Q u(x) = \frac{k}{2\pi}\int_{\mathbb{R}^2} \exp\left(ik(x-y)\xi\right)\chi(\xi)\chi(y)u(y)\; dyd\xi.$$

Then Π_Q maps $\mathscr{D}'(\mathbb{R})$ into $\mathscr{S}(\mathbb{R})$, and $\Pi_Q \sim \mathrm{Id}$ on Q. Set

$$w_{k,E}^{(j)} = \Pi_Q v_k^{(j)};$$

then the $w_{k,E}^{(j)}$, $j \in [\![1,4]\!]$, belong to $\mathscr{S}(\mathbb{R})$, and are microlocal solutions of $(S_k - E)w_{k,E}^{(j)} = 0$ on Q. The matrix of the change of basis from $(w_{k,E}^{(3)}, w_{k,E}^{(4)})_{|Q}$ to $(w_{k,E}^{(1)}, w_{k,E}^{(2)})_{|Q}$ is given by

$$M_k(E) = \mu_k(E)\begin{pmatrix} 1 & i\exp(-\pi kE) \\ i\exp(-\pi kE) & 1\end{pmatrix} + O(k^{-\infty}) \qquad (7)$$

with

$$\mu_k(E) = \frac{1}{\sqrt{2\pi}}\Gamma\left(\tfrac{1}{2}+ikE\right)\exp\left(\frac{\pi}{4}(2kE - i) - ikE\ln(k)\right).$$

5.3 Microlocal solutions of $(P_k - E)u_k = 0$

Now, consider the Bargmann transforms of the sequences $w_{k,E}^{(j)}$: $u_{k,E}^{(j)} = B_k w_{k,E}^{(j)}$. Propositions 5.3 and 3.5 yield the:

Proposition 5.4. *For E such that $|E| < 1$, the space of microlocal solutions of $(P_k - E)u_k = 0$ on $Q = [-1,1]^2 \subset \mathbb{C}$ is a free $\mathbb{C}_k$-module of rank 2. Moreover, the couples $\left(u_{k,E}^{(1)}, u_{k,E}^{(2)}\right)$ and $\left(u_{k,E}^{(3)}, u_{k,E}^{(4)}\right)$ are two bases of this module; the transfer matrix is given by equation (7).*

Remark. The sections $u_{k,E}^{(j)}$, $j = 1, \ldots, 4$, can be written in terms of parabolic cylinder functions. In the article [25], Nonnenmacher and Voros studied these functions in order to understand the behaviour of the generalized eigenfunctions of P_k; the result of this subtle analysis, based on Stokes lines techniques, was not exactly what we needed here, and this is partly why we chose to use the microlocal properties of the Bargmann transform instead.

6 Bohr-Sommerfeld conditions

To obtain the Bohr-Sommerfeld conditions, we will recall the reasoning of Colin de Verdière and Parisse [16], and will also refer to the work of Colin de Verdière and Vũ Ngọc [18]. Since the general approach is the same, we only recall the main ideas and focus on what differs in the Toeplitz setting.

6.1 The sheaf of standard basis

As in section 4, introduce the sheaf $(\mathfrak{F}, \Gamma_0)$ of microlocal solutions of $A_k \psi_k = 0$ over Γ_0; we recall that a global non-trivial microlocal solution corresponds to a global non-trivial section of this sheaf. However, since the topology of Γ_0 is much more complicated than in the regular case, the condition for the existence of such a section is not as simple as saying that a holonomy must be trivial. In particular, we have to handle what happens at critical points. To overcome this difficulty, the idea is to introduce a new sheaf over Γ_0 that will contain all the information we need to construct a global non-trivial microlocal solution; roughly speaking, this new sheaf can be thought of as the limit of the sheaf $\mathfrak{F} \to \Gamma_E$ of microlocal solutions over regular levels as E goes to 0.

Following Colin de Verdière and Parisse [16], we introduce a sheaf $(\mathfrak{L}, \Gamma_0)$ of free $\mathbb{C}_k$-modules of rank one over Γ_0 as follows: to each point $m \in \Gamma_0$, associate the free module $\mathfrak{L}(m)$ generated by *standard basis* at m. If m is a regular point, a standard basis is any basis of the space of microlocal solutions near m. At a critical point s_j, we define a standard basis in the following way. The $\mathbb{C}_k$-module of microlocal solutions near s_j is free of rank 2; moreover, it is the graph of a linear application. Indeed, number the four local edges near s_j with cyclic order $1, 3, 2, 4$, so that the edges e_1, e_2 are the ones that leave s_j. Let us denote by $\mathrm{Sol}(e_1 e_2)$ (resp. $\mathrm{Sol}(e_3 e_4)$) the module of microlocal solutions over the disjoint union of the local unstable (resp. stable) edges e_1, e_2 (resp. e_3, e_4). $\mathrm{Sol}(e_1 e_2)$ and $\mathrm{Sol}(e_3 e_4)$ are free modules of rank 2, and there exists a linear map $T_j : \mathrm{Sol}(e_3 e_4) \to \mathrm{Sol}(e_1 e_2)$ such that u is a solution near s_j if and only if its restrictions satisfy $u_{|\mathrm{Sol}(e_1 e_2)} = T_j u_{|\mathrm{Sol}(e_3 e_4)}$. Equivalently, given two solutions on the entering edges, there is a unique way to obtain two solutions on the leaving edges by passing the singularity. One can choose a basis element for each $\mathfrak{F}(e_i)$, $i \in [\![1, 4]\!]$, and express T_j as a 2×2 matrix (defined modulo $O(k^{-\infty})$); one can show that the entries of this matrix are all non-vanishing. An argument of elementary linear algebra shows that once the matrix T_j is chosen, the basis elements of the modules $\mathfrak{F}(e_i)$ are fixed up to multiplication by the same factor; this means that for T_j fixed, the $\mathbb{C}_k$-module of basis elements is of rank one. Moreover, the study of the previous section implies that there exists a choice of basis elements such that T_j has the following expression:

$$T_j = \exp\left(-\frac{i\pi}{4}\right)\mathcal{E}_k(\varepsilon_j(0,k))\begin{pmatrix} 1 & i\exp(-\pi\varepsilon_j(0,k)) \\ i\exp(-\pi\varepsilon_j(0,k)) & 1\end{pmatrix}, \quad (8)$$

where

$$\mathcal{E}_k(t) = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2}+it\right)\exp\left(t\left(\frac{\pi}{2} - i\ln k\right)\right). \quad (9)$$

This allows us to call the choice of the basis elements of $\mathfrak{F}(e_i)$ a standard basis whenever T_j is given by equation (8).

$(\mathfrak{L}, \Gamma_0)$ is a locally free sheaf of rank one $\mathbb{C}_k$-modules, and its transition functions are constants. Hence, it is flat, thus characterised by its holonomy

$$\mathrm{hol}_{\mathfrak{L}} : \mathrm{H}_1(\Gamma_0) \to \mathbb{C}_k.$$

In terms of Čech cohomology, if γ is a cycle in Γ_0, and $\Omega_1, \ldots, \Omega_\ell$ is an ordered sequence of open sets covering the image of γ, each Ω_i being equipped with a standard basis u_i (at a critical point, we make the abusive correspondance between a standard basis and its elements), then

$$\mathrm{hol}_{\mathfrak{L}}(\gamma) = x_{1,2}\ldots x_{\ell-1,\ell}x_{\ell,1}, \quad (10)$$

where $x_{i,j} \in \mathbb{C}_k$ is such that $u_i = x_{i,j}u_j$ on $\Omega_i \cap \Omega_j$.

Now, cut $n+1$ edges of Γ_0, each one corresponding to a cycle γ_i in a basis $(\gamma_1, \ldots, \gamma_{n+1})$ of $\mathrm{H}_1(\Gamma_0, \mathbb{Z})$, in such a way that the remaining graph is a tree $\mathcal{T}$. Then the sheaf $(\mathfrak{L}, \mathcal{T})$ has a non-trivial global section. The conditions to obtain a non-trivial global section of the sheaf $(\mathfrak{F}, \Gamma_0)$ of microlocal solutions on Γ_0 are given in the following theorem. They were already present in the work of Colin de Verdière and Parisse in the case of pseudodifferential operators, but the fact that they extend to our setting is a consequence of the results obtained in the previous sections.

Theorem 6.1. *The sheaf $(\mathfrak{F}, \Gamma_0)$ has a non-trivial global section if and only if the following linear system of $3n+1$ equations with $3n+1$ unknowns $(x_\alpha \in \mathbb{C}_k)_{\alpha \in \{\text{edges of } \mathcal{T}\}}$ has a non-trivial solution:*

1. *if the edges $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ connect at s_j (with the same convention as before for the labeling of the edges), then*

$$\begin{pmatrix} x_{\alpha_3} \\ x_{\alpha_4}\end{pmatrix} = T_j \begin{pmatrix} x_{\alpha_1} \\ x_{\alpha_2},\end{pmatrix}$$

2. *if α and β are the extremities of a cut cycle γ_i, then*

$$x_\alpha = \mathrm{hol}_{\mathfrak{L}}(\gamma_i)x_\beta,$$

where the following orientation is assumed: γ_i can be represented as a closed path starting on the edge α and ending on the edge β.

Proof. It follows from propositions 4.6 and 5.4 that the proof can be directly adapted from the one of [18, theorem 2.7]. □

6.2 Singular invariants

Of course, in order to use this result, it remains to compute the holonomy $\text{hol}_{\mathfrak{L}}$. For this purpose, let us introduce some geometric quantities close from the ones used to express the regular Bohr-Sommerfeld conditions. Let γ be a cycle in Γ_0, and denote by s_{j_m}, $m = 1, \ldots, p$ the critical points contained in γ.

Definition 6.2 (singular subprincipal action)**.** Decompose γ as a concatenation of smooth paths and paths containing exactly one critical point; if A and B are the ordered endpoints of a path, we will call it $[A, B]$. Define the subprincipal action $\tilde{c}_1(\gamma)$ as the sum of the contributions of these paths, given by the following rules:

- if $[A, B]$ contains only regular points, its contribution to the singular subprincipal action is

  $$\tilde{c}_1([A, B]) = c_1([A, B])$$

  as in the regular case (see section 4.5 for the definition of $c_1([A, B])$),

- if $[A, B]$ contains the singular point s and is smooth at s, then

  $$\tilde{c}_1([A, B]) = \lim_{a,b \to s} \left(c_1([A, a]) + c_1([b, B]) \right)$$

  where a (resp. b) lies on the same branch as A (resp. B),

- if $[A, B]$ contains the singular point s and is not smooth at s, we set

  $$\tilde{c}_1([A, B]) = \lim_{a,b \to s} \left(c_1([A, a]) + c_1([b, B]) \pm \varepsilon_s^{(0)} \ln \left| \int_{P_{a,b}} \omega \right| \right) \tag{11}$$

  where $P_{a,b}$ is the parallelogram (defined in any coordinate system) built on the vectors $\overrightarrow{sa}$ and $\overrightarrow{sb}$, $\pm = +$ if $[A, B]$ is oriented according to the flow of X_{a_0}, $\pm = -$ otherwise and

  $$\varepsilon_s^{(0)} = \frac{-a_1(s)}{|\det(\text{Hess}(a_0)(s))|^{1/2}},$$

  as before.

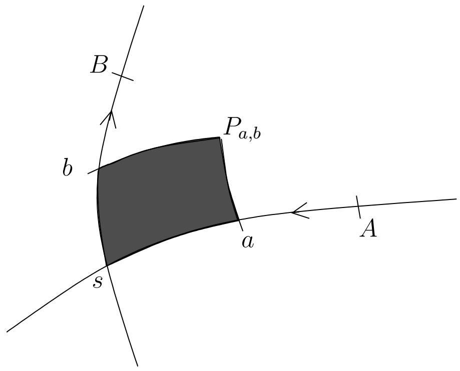


Figure 1: Computation of $\tilde{c}_1([A,B])$

Definition 6.3 (singular index)**.** Let $(\gamma_t)_t$ be a continuous family of immersed closed curves such that $\gamma_0 = \gamma$ and γ_t is smooth for $t > 0$. Then the function $t \mapsto \epsilon(\gamma_t)$, $t > 0$, is constant; we denote by ϵ its value. We define the singular index $\tilde{\epsilon}(\gamma)$ by setting

$$\tilde{\epsilon}(\gamma) = \epsilon + \sum_{m=1}^{p} \frac{\rho_m}{4} \tag{12}$$

where $\rho_m = 0$ if γ is smooth at s_{j_m}, $\rho_m = +1$ if at s_{j_m}, γ turns in the direct sense with respect to the cyclic order $(1,3,2,4)$ of the local edges, and $\rho_m = -1$ otherwise.

Observe that both $\tilde{c}_1$ and $\tilde{\epsilon}$ define $\mathbb{Z}$-linear maps on $\mathrm{H}_1(\Gamma_0, \mathbb{Z})$.

Theorem 6.4. *Let γ be a cycle in Γ_0. Then the holonomy $hol_{\mathfrak{L}}(\gamma)$ of γ in $\mathfrak{L}$ has the form*

$$\mathrm{hol}_{\mathfrak{L}}(\gamma) = \exp(ik\theta(\gamma, k)) \tag{13}$$

where $\theta(\gamma, k)$ admits an asymptotic expansion in non-positive powers of k. Moreover, if we denote by $\theta(\gamma, k) = \sum_{\ell \geq 0} k^{-\ell} \theta_\ell(\gamma)$ this expansion, the first two terms are given by the formulas:

$$\theta_0(\gamma) = c_0(\gamma), \quad \theta_1(\gamma) = \tilde{c}_1(\gamma) + \tilde{\epsilon}(\gamma)\pi. \tag{14}$$

Proof. We just prove here that the holonomy has the claimed behaviour. It is enough to show that one can choose a finite open cover $(\Omega_\alpha)_\alpha$ of γ and a section u_k^α of $\mathfrak{L} \to \Omega_\alpha$ for which the transition constants $c_k^{\alpha\beta}$ have the required form. On the edges of γ, this follows from the analysis of section 4. At a vertex, we choose the standard basis $U_k^0 u^{(j)}_{k,\varepsilon_j(0,k)}$, where $u^{(j)}_{k,E}$ is defined in section 5.3 and U_k^E is the operator of proposition 5.1; to conclude, we observe that the restrictions of these sections to the corresponding edge are Lagrangian sections. □

6.3 Computation of the singular holonomy

This section is devoted to the proof of the second part of theorem 6.4. We use the method of [18], but of course, our case is simpler, because in the latter, the authors investigated the case of singularities in (real) dimension 4 (for pseudodifferential operators). Let us work on microlocal solutions of the equation

$$(A_k - E)u_k = 0 \tag{15}$$

where E varies in a small interval I containing the critical value 0. The critical value separates I into two open sets I^+ and I^-, with the convention $I^\pm = I \cap \mathbb{R}^*_\pm$. Let $D^\pm = I^\pm \cup \{0\}$, and let $\mathfrak{C}^\pm$ be the set of connected components of the open set $a_0^{-1}(I^\pm)$. The smooth family of circles in the component $p^\pm$ is denoted by $\mathcal{C}_{p^\pm}(E)$, $E \in I^\pm$.

As in section 4, for $E \neq 0$, we denote by $(\mathfrak{F}, \Gamma_E)$ the sheaf of microlocal solutions of (15) on Γ_E; remember that it is a flat sheaf of rank 1 $\mathbb{C}_k$-modules, characterized by its Čech holonomy $\mathrm{hol}_{\mathfrak{F}}$. The idea is to let E go to 0 and compare this holonomy to the holonomy of the sheaf $\mathfrak{L} \to \Gamma_0$.

Definition 6.5. Near each critical point s_j, consider two families of points $A_j(E)$ and $B_j(E)$ in $\mathcal{C}^\infty(D^\pm, \bar{p}^\pm \setminus \{s_j\})$ lying on $\mathcal{C}_{p^\pm}(E)$ and such that $A_j(0)$ and $B_j(0)$ lie respectively in the stable or unstable manifold. Endow a small neighbourhood of A_j (resp. B_j) with a microlocal solution u_{A_j} (resp. u_{B_j}) of (15) which is a Lagrangian section uniform in $E \in D^\pm$. Define the quantity $\Theta([A_j(E), B_j(E)], k)$ as the phase of the Čech holonomy of the path $[A_j(E), B_j(E)] \subset \Gamma_E$ joining $A_j(E)$ and $B_j(E)$ in the sheaf $(\mathfrak{F}, \Gamma_E)$ computed with respect to u_{A_j} and u_{B_j}. Define in the same way the quantity $\Theta([B_j(E), A_{j'}(E)], k)$ for the path joining $B_j(E)$ and $A_{j'}(E)$.

Note that if we change the sections u_{A_j} and $u_{B_{j'}}$, the phase of the holonomy is modified by an additive term admitting an asymptotic expansion in $k\,\mathcal{C}^\infty(D^\pm)[[k^{-1}]]$. The singular behaviour of the holonomy is thus preserved; moreover, the added term is a Čech coboundary, and hence does not change the value of the holonomy along a closed path.

Then, we consider continuous families of paths $(\zeta_E)_{E \in D^\pm}$ drawn on a circle $\mathcal{C}_{p^\pm}(E)$ and whose endpoints are some of the $A_j(E)$ and $B_{j'}(E)$ of the previous definition. We say that ζ_E is

- *regular* if ζ_0 does not contain any of the critical points s_j,
- *local* if ζ_0 contains exactly one critical point,

and we consider only these two types of paths. The following proposition implies that a path that is local in the above sense can always be assumed to be local in the sense that it is included in a small neighbourhood of the critical point that it contains.

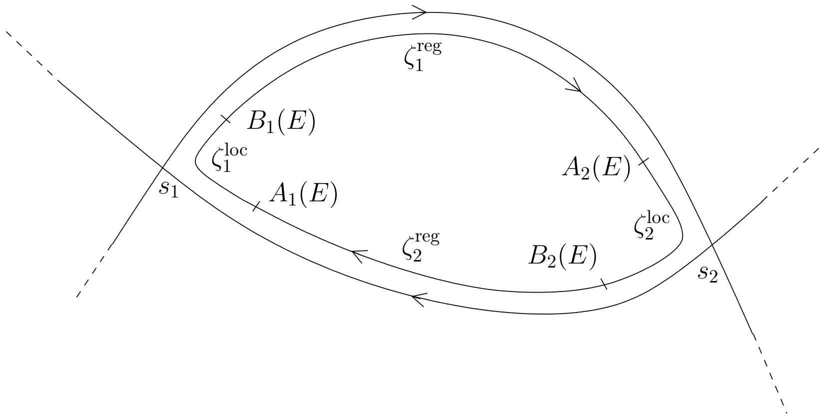


Figure 2: Regular and local paths

Proposition 6.6. *If $\zeta_E = [B_j(E), A_{j'}(E)]$ is a regular path, then the map $E \mapsto \Theta(\zeta_E, k)$ belongs to $\mathcal{C}^\infty(D^\pm)$ and admits an asymptotic expansion in $k\,\mathcal{C}^\infty(D^\pm)[[k^{-1}]]$. This expansion starts as follows:*

$$\begin{aligned}\Theta(\zeta_E, k) = k\left(c_0(\zeta_E) + \Phi^{(-1)}_{B_j(E)}(B_j(E)) - \Phi^{(-1)}_{A_{j'}(E)}(A_{j'}(E))\right)\\ + c_1(\zeta_E) + \mathrm{hol}_{\delta_{\zeta_E}}(\zeta_E) + \Phi^{(0)}_{B_j(E)}(B_j(E)) - \Phi^{(0)}_{A_{j'}(E)}(A_{j'}(E)) + O(k^{-1});\end{aligned} \tag{16}$$

see section 4.5 for the notations.

In order to study the behaviour of the holonomy of a local path with respect to E, we use the parameter dependent normal form given by proposition 5.1. Using the notations of this proposition, we will write $\mathfrak{e}_j(E,k) = f_j(E) + k^{-1}\varepsilon_j(E,k)$. Introduce the Bargmann transform $w^i_{k,E}$ of $v^i_{k,E}$, where

$$v^{1,2}_{k,E}(x) = \mathbf{1}_{\mathbb{R}^*_\pm}(x)|x|^{-1/2}\exp(ik\mathfrak{e}_j(E,k)\ln|x|);$$

$$v^{3,4}_{k,E}(x) = \mathcal{F}_k^{-1}\left(\mathbf{1}_{\mathbb{R}^*_\pm}(x)(\xi)|\xi|^{-1/2}\exp(ik\mathfrak{e}_j(E,k)\ln|\xi|)\right)(x).$$

Let $\tilde{w}^i_{k,E}$ be a sequence having microsupport in a sufficiently small neighbourhood of the origin and microlocally equal to $w^i_{k,E}$ on it; then $\tilde{w}^i_{k,E}$ is a basis of the module of microlocal solutions of $P_k - \mathfrak{e}_j(E,k)$ near the image of the edge with label i by the symplectomorphism χ_E. Consequently, the section $\phi^{(i)}_{k,E} = U^E_k \tilde{w}^i_{k,E}$, where U^E_k is the operator used for the normal form, is a basis of the module of microlocal solutions of equation (15) near the edge e_i. Moreover, it displays a good behaviour with respect to the spectral parameter.

Lemma 6.7. *The restriction of $\phi^{(i)}_{k,E}$ to a neighbourhood of the edge number i is a Lagrangian section uniformly for $E \in D^\pm$.*

Proof. First, we prove using a parameter dependant stationary phase lemma that $w^i_{k,E}$ is a Lagrangian section associated to the image of the i-th edge, uniformly in $E \in D^{\pm}$. We conclude by the fact that the image of a Lagrangian section depending smoothly on a parameter by a Fourier integral operator is a Lagrangian section depending smoothly on this parameter. □

We also recall the following useful lemma.

Lemma 6.8 ([18, lemma 2.18])**.** *Set* $\beta_j(E,k) = \frac{1}{2} + ik\mathfrak{e}_j(E,k)$ *and*

$$\nu_j^+ = \left(\tfrac{k}{2\pi}\right)^{1/2} \Gamma(\beta_j) \exp\left(-\beta_j \ln k - i\beta_j \tfrac{\pi}{2}\right);$$

$$\nu_j^- = \left(\tfrac{k}{2\pi}\right)^{1/2} \Gamma(\beta_j) \exp\left(-\beta_j \ln k + i\beta_j \tfrac{\pi}{2}\right),$$

so that

$$M_k(\mathfrak{e}_j(E,k)) = \begin{pmatrix} \nu_j^+(E,k) & \nu_j^-(E,k) \\ \nu_j^-(E,k) & \nu_j^+(E,k) \end{pmatrix}$$

where M_k was defined in equation (7). Then for any $E \in I^{\pm}$,

$$-i \ln \nu_j^{\pm} = k\left(f_j(E) \ln |f_j(E)| - f_j(E)\right) + \varepsilon_j^{(0)}(E) \ln |f_j(E)| \mp \frac{\pi}{4} + O_E(k^{-1}).$$

The following proposition shows that the holonomy $\Theta(\zeta_E, k)$, which has a singular behaviour as E tends to 0, can be regularized.

Proposition 6.9. *Fix a component $p^{\pm} \in \mathfrak{C}^{\pm}$, and let $\zeta_E = [A_j(E), B_j(E)]$ be a local path near the critical point s_j. Assume moreover that ζ_E is oriented according to the flow of a_0. Then there exists a sequence of $\mathbb{R}/2\pi\mathbb{Z}$-valued functions $g_\zeta(.,k) \in \mathcal{C}^\infty(D^{\pm})$, $E \mapsto g_{\zeta_E}(k)$, admitting an asymptotic expansion in $k\,\mathcal{C}^\infty(D^{\pm})[[k^{-1}]]$ of the form*

$$g_\zeta(E,k) = \sum_{\ell=-1}^{+\infty} k^{-\ell} g_\zeta^{(\ell)}(E),$$

such that

$$\forall E \in I^{\pm}, \quad g_\zeta(E,k) = \Theta(\zeta_E, k) - i \ln(\nu_j^{\pm}(E)) \pmod{2\pi\mathbb{Z}}.$$

The first terms of the asymptotic expansion of $g_\zeta(.,k)$ are given, for $E \in I^{\pm}$, by:

$$g_\zeta^{(-1)}(E) = c_0(\zeta_E) + (f_j(E) \ln |f_j(E)| - f_j(E)) + \Phi^{(-1)}_{A_j(E)}(A_j(E)) - \Phi^{(-1)}_{B_j(E)}(B_j(E))) \tag{17}$$

and

$$g_\zeta^{(0)}(E) = c_1(\zeta_E) + \mathrm{hol}_{\delta_{\zeta_E}}(\zeta_E) \mp \frac{\pi}{4} + \varepsilon_j^0(E) \ln |f_j(E)| + \Phi^{(0)}_{A_j(E)}(A_j(E)) - \Phi^{(0)}_{B_j(E)}(B_j(E)). \tag{18}$$

Proof. We can assume that the paths ζ_E, $E \in D^{\pm}$ all entirely lie in the open set Ω_{s_j} where the normal form of proposition 5.1 is valid. Endow each edge e_i with the section $\phi^{(i)}_{k,E}$ defined earlier; by lemma 6.7, these sections can be used to compute a new holonomy $\widetilde{\Theta}(\zeta_E, k)$. But we know how the different sections $\phi^{(i)}_{k,E}$ are related: equation (7) shows that $\widetilde{\Theta}(\zeta_E, k) - i \ln \nu_j^{\pm}(E) = 0$. Now, coming back to the microlocal solutions u_{A_j}, u_{B_j}, we have that $\Theta(\zeta_E, k) = \widetilde{\Theta}(\zeta_E, k) + c(E, k)$, where $c(E, k)$ admits an asymptotic expansion in $k\, \mathcal{C}^{\infty}(D^{\pm})[[k^{-1}]]$. □

Since the sections $\phi^{(i)}_{k,E}$, $i = 1 \dots 4$, form a standard basis at s_j, they can also be used to compute the holonomy $\mathrm{hol}_{\mathfrak{L}}$. Of course, for this choice of sections, one has $\mathrm{hol}_{\mathfrak{L}}(\zeta_0) = 1$. This allows to obtain the following result.

Proposition 6.10. *Let γ be a cycle in Γ_0, oriented according to the Hamiltonian flow of a_0, and of the form $\gamma = \zeta_1^{\mathrm{loc}}(0)\zeta_1^{\mathrm{reg}}(0)\zeta_2^{\mathrm{loc}}(0)\zeta_2^{\mathrm{reg}}(0)\dots\zeta_p^{\mathrm{loc}}(0)\zeta_p^{\mathrm{reg}}(0)$, where ζ_j^{loc} and ζ_j^{reg} are respectively local and regular paths in the sense introduced earlier. Define*

$$g(0, k) \sim \sum_{\ell=-1}^{+\infty} g^{(\ell)}(0) k^{-\ell}$$

as the sum

$$g(0, k) = g_{\zeta_1^{\mathrm{loc}}}(0, k) + g_{\zeta_1^{\mathrm{reg}}}(0, k) + \dots + g_{\zeta_p^{\mathrm{loc}}}(0, k) + g_{\zeta_p^{\mathrm{reg}}}(0, k),$$

where $g_{\zeta_j^{\mathrm{loc}}}$ is given by proposition 6.9 and $g_{\zeta_j^{\mathrm{reg}}}(E, k) = \Theta(\zeta_j^{\mathrm{reg}}(E), k)$. Then

$$\mathrm{hol}_{\mathfrak{L}}(\gamma) = \exp\left(i g(0, k)\right) + O(k^{-\infty}).$$

Proof. Notice that $\tilde{g}_{\zeta_j^{\mathrm{loc}}}(0, k) = 0$, where $\tilde{g}_{\zeta_j^{\mathrm{loc}}}(\cdot, k)$ is defined as $g_{\zeta_j^{\mathrm{loc}}}(\cdot, k)$ replacing $\Theta(\zeta_j^{\mathrm{loc}}, k)$ by $\widetilde{\Theta}(\zeta_j^{\mathrm{loc}}, k)$. Hence $\mathrm{hol}_{\mathfrak{L}}(\zeta_j^{\mathrm{loc}}(0)) = \exp(i\tilde{g}_{\zeta_j^{\mathrm{loc}}}(0, k))$. As in the previous proof, come back to the solutions u_{A_j}, u_{B_j} and set

$$c_j(E, k) = g_{\zeta_j^{\mathrm{loc}}}(E, k) - \tilde{g}_{\zeta_j^{\mathrm{loc}}}(0, k).$$

Putting $\tilde{g}_{\zeta_j^{\mathrm{reg}}}(E, k) = \widetilde{\Theta}(\zeta_j^{\mathrm{reg}}(E), k)$, a simple computation shows that

$$\sum_{j=1}^{p} \tilde{g}_{\zeta_j^{\mathrm{reg}}}(E, k) = \sum_{j=1}^{p} \left(g_{\zeta_j^{\mathrm{reg}}}(E, k) + c_j(E, k) \right)$$

and the conclusion follows. □

This is enough to prove the second part of theorem 6.4, recalled in the following corollary.

Corollary 6.11. *The first two terms in the asymptotic expansion of the phase of* $\mathrm{hol}_{\mathfrak{L}}(\gamma)$ *are given by formula (14).*

Note that γ cannot always be obtained as a limit of smooth families of regular cycles; consider for instance the cycles $\gamma_1, \gamma_2, \gamma_3$ in the example treated in section 7.3 (see figures 13, 14). This is why the proof of this result requires some care.

Proof. We start by the case of a cycle γ oriented according to the Hamiltonian flow of a_0. Since $\mathfrak{e}_j^0(0) = 0$, formula (17) gives for $j \in [\![1, p]\!]$

$$g^{(-1)}_{\zeta_j^{\mathrm{loc}}}(0) = c_0\left(\zeta_j^{\mathrm{loc}}(0)\right) + \Phi^{(-1)}_{A_j}(A_j(0)) - \Phi^{(-1)}_{B_j}(B_j(0))$$

while proposition 6.6 shows that (identifying $j = p+1$ with $j = 1$)

$$g^{(-1)}_{\zeta_j^{\mathrm{reg}}}(0) = c_0\left(\zeta_j^{\mathrm{reg}}(0)\right) + \Phi^{(-1)}_{B_j}(B_j(0)) - \Phi^{(-1)}_{A_{j+1}}(A_{j+1}(0)).$$

Consequently,

$$g^{(-1)}(0) = c_0(\gamma).$$

Let us now compute the subprincipal term $g^{(0)}_{\zeta_j^{\mathrm{loc}}}(0)$. Recall that it is equal to the limit of

$$c_1(\zeta_j^{\mathrm{loc}}(E)) + \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{loc}}(E)}}(\zeta_j^{\mathrm{loc}}(E)) \mp \frac{\pi}{4} + \varepsilon_j^{(0)}(E)\ln|f_j(E)| + \Phi^{(0)}_{A_j(E)}(A_j(E)) - \Phi^{(0)}_{B_j(E)}(B_j(E))$$

as E goes to 0, which is equal to

$$\Phi^{(0)}_{A_j(0)}(A_j(0)) - \Phi^{(0)}_{B_j(0)}(B_j(0)) \mp \frac{\pi}{4} + \lim_{E\to 0}\left(c_1(\zeta_j^{\mathrm{loc}}(E)) + \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{loc}}(E)}}(\zeta_j^{\mathrm{loc}}(E)) + \varepsilon_j^{(0)}(E)\ln|f_j(E)|\right).$$

First, we show that

$$\lim_{E\to 0}\left(c_1(\zeta_j^{\mathrm{loc}}(E)) + \varepsilon_j^{(0)}(E)\ln|f_j(E)|\right) = \tilde{c}_1(\zeta_j^{\mathrm{loc}}(0)), \tag{19}$$

where $\tilde{c}_1$ was introduced in definition 6.2. Decompose

$$c_1(\zeta_j^{\mathrm{loc}}(E)) = \int_{\zeta_j^{\mathrm{loc}}(E)} \nu + \int_{\zeta_j^{\mathrm{loc}}(E)} \kappa_E,$$

where we recall that $-i\nu$ stands for the local connection 1-form associated to the Chern connection of L_1, and κ_E is such that $\kappa_E(X_{a_0}) = -a_1$. Of course, the term $\int_{\zeta_j^{\mathrm{loc}}(E)} \nu$ converges to $\int_{\zeta_j^{\mathrm{loc}}(0)} \nu$ as E tends to 0. Moreover, we have seen that there exists a symplectomorphism χ_E and a smooth function g_j^E such that $(g_j^E)'(0) > 0$ and

$$(a_0 \circ \chi_E^{-1})(x, \xi) - E = g_j^E(x\xi). \tag{20}$$

Hence, if we denote by $\tilde{a}_0$ (resp. $\tilde{a}_1$, $\tilde{\kappa}_E$) the pullback of a_0 (resp. a_1, κ_E) by χ_E^{-1}, we have

$$X_{\tilde{a}_0}(x,\xi) = (g_j^E)'(x\xi) X_{x\xi}(x,\xi),$$

so that $\tilde{\kappa}_E$ is characterized by

$$\tilde{\kappa}_E(X_{x\xi}) = \frac{-\tilde{a}_1(x,\xi)}{(g_j^E)'(x\xi)}.$$

Since $(g_j^E)'(0) \neq 0$, the function $b(x,\xi) = \frac{-\tilde{a}_1(x,\xi)}{(g_j^E)'(x\xi)}$ is smooth (considering a smaller neighbourhood of s_j for the definition of ζ_j^{loc} if necessary). Moreover, from equation (20), one finds that $(g_j^E)'(0) = |\det(\mathrm{Hess}(a_0)(s))|^{-1/2}$, which yields the fact that $b(0) = \varepsilon_j^{(0)}(0)$. Using a known result (see [20, theorem 2, p.175] for instance), we can construct smooth functions $F : \mathbb{R}^2 \to \mathbb{R}$ and $K : \mathbb{R} \to \mathbb{R}$ such that

$$b(x,\xi) = K(x\xi) - L_{X_{x\xi}} F(x,\xi);$$

since $x\xi = f_j(E)$ whenever $\chi_E^{-1}(x,\xi)$ belongs to Γ_E, this can be written $b(x,\xi) = K(f_j(E)) - L_{X_{x\xi}} F(x,\xi)$. Therefore, the function

$$G = K(f_j(E)) \ln|x| - F \quad (\text{or } -K(f_j(E)) \ln|\xi| - F \text{ where } x = 0)$$

restricted to $\chi(\Gamma_E)$ is a primitive of $\tilde{\kappa}_E$.This yields

$$\int_{\zeta_j^{\mathrm{loc}}(E)} \kappa_E = G(\tilde{B}_j) - G(\tilde{A}_j) = K(f_j(E)) \left(\ln|x_{B_j}| - \ln|x_{A_j}| \right) + F((\tilde{A}_j) - F(\tilde{B}_j) \tag{21}$$

where $\tilde{m} = \chi_E(m)$ for any point $m \in M$, and (x_m, ξ_m) are the coordinates of $\tilde{m}$ (E being implicit to simplify notations). Writing $\ln|x_{B_j}| - \ln|x_{A_j}| = \ln|x_{B_j}\xi_{A_j}| - \ln|x_{A_j}\xi_{A_j}|$, we obtain

$$\int_{\zeta_j^{\mathrm{loc}}(E)} \kappa_E + \varepsilon_j^{(0)}(E) \ln|f_j(E)| = F(\tilde{A}_j) - F(\tilde{B}_j) + K(f_j(E)) \ln|x_{B_j}\xi_{A_j}| + \left(\varepsilon_j^{(0)}(E) - K(f_j(E)) \right) \ln|f_j(E)|.$$

By definition of K, $b(0) - K(0) = 0$, hence $K(f_j(E)) = b(0) + O(f_j(E)) = \varepsilon_j^{(0)}(0) + O(f_j(E))$. Thus, the term $\left(\varepsilon_j^{(0)}(E) - K(f_j(E)) \right) \ln|f_j(E)|$ tends to zero as E tends to zero; this induces

$$\lim_{E \to 0} \int_{\zeta_j^{\mathrm{loc}}(E)} \kappa_E + \varepsilon_j^{(0)}(E) \ln|f_j(E)| = F(\tilde{A}_j) - F(\tilde{B}_j) + K(f_j(E)) \ln|x_{B_j}\xi_{A_j}|$$

(one must keep in mind that in this formula, we should write $\tilde{A}_j = \tilde{A}_j(0)$, etc.). Now, if a, b are points on $\zeta_j^{\mathrm{loc}}(0)$ located respectively in $[A_j, s_{m_j}]$ and

$[s_{m_j}, B_j]$, then the term on the right hand side of the previous equation is equal to

$$I = \lim_{a,b\to s_j} \left(F(\tilde{A}_j) - F(\tilde{a}) + F(\tilde{b}) - F(\tilde{B}_j) + K(f_j(E)) \ln |x_{B_j} \xi_{A_j}| \right).$$

Using equation (21), it is easily seen that

$$I = \lim_{a,b\to s_j} \left(\int_{[A_j,a]} \kappa_E + \int_{[b,B_j]} \kappa_E + \varepsilon_j^{(0)}(0) \ln |x_b \xi_a| \right).$$

Remembering definition 6.2, this proves equation (19). Since $g^{(0)}_{\zeta_j^{\mathrm{loc}}}$ and the quantities $\Phi^{(-1)}_{A_j}(A_j) - \Phi^{(-1)}_{B_j}(B_j)$ and $\Phi^{(0)}_{A_j}(A_j) - \Phi^{(0)}_{B_j}(B_j)$ are continuous at $E = 0$, the term

$$\mathrm{hol}_{\delta_{\zeta_j^{\mathrm{loc}}(E)}}(\zeta_j^{\mathrm{loc}}(E))$$

is continuous at $E = 0$. Hence, if we sum up all the contributions from regular and local paths, we finally obtain

$$g^{(0)}(\gamma) = \tilde{c}_1(\gamma) + \sum_{m=1}^{p} \frac{\rho_m \pi}{4} + \ell(\gamma)$$

where ρ_m (respectively $\tilde{c}_1$) was introduced in definition 6.3 (respectively definition 6.2) and $\ell(\gamma)$ is the quantity

$$\ell(\gamma) = \sum_{j=1}^{p} \left(\mathrm{hol}_{\delta_{\zeta_j^{\mathrm{reg}}(0)}}(\zeta_j^{\mathrm{reg}}(0)) + \lim_{E\to 0} \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{loc}}(E)}}(\zeta_j^{\mathrm{loc}}(E)) \right);$$

it is not hard to show that $\ell(\gamma)$ is independent of the choice of the local and regular paths. Furthermore, let ϵ be the index of any smooth embedded cycle which is a continuous deformation of γ. If the regular and local paths can be chosen so that they all lie in the same connected component γ_E of Γ_E, it is clear that $\ell(\gamma) = \epsilon$, because for $E \neq 0$

$$\sum_{j=1}^{p} \left(\mathrm{hol}_{\delta_{\zeta_j^{\mathrm{reg}}(E)}}(\zeta_j^{\mathrm{reg}}(E)) + \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{loc}}(E)}}(\zeta_j^{\mathrm{loc}}(E)) \right) = \epsilon(\gamma_E) = \epsilon.$$

If it is not the case, we remove a small path $\eta_j(E)$ of $\zeta_j^{\mathrm{reg}}(E)$ at any point A_j (resp. B_j) where there is a change of connected component, and replace it by a smooth path $\nu_j(E)$ connecting $\zeta_j^{\mathrm{reg}}(E)$ and $\zeta_j^{\mathrm{loc}}(E)$ (see figure 3). We obtain a smooth path $\tilde{\gamma}(E)$; on the one hand, one has $\epsilon(\tilde{\gamma}(E)) = \epsilon$. On the other hand, $\epsilon(\tilde{\gamma}(E))$ is the sum of the holonomies of the paths composing $\tilde{\gamma}(E)$. But, if we denote by $\tau_j(E)$ the part of $\zeta_j^{\mathrm{reg}}(E)$ that remains when we remove $\eta_j(E)$, we have

$$\mathrm{hol}_{\delta_{\tau_j(E)}}(\tau_j(E)) = \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{reg}}(E)}}(\zeta_j^{\mathrm{reg}}(E)) - \mathrm{hol}_{\delta_{\eta_j(E)}}(\eta_j(E))$$

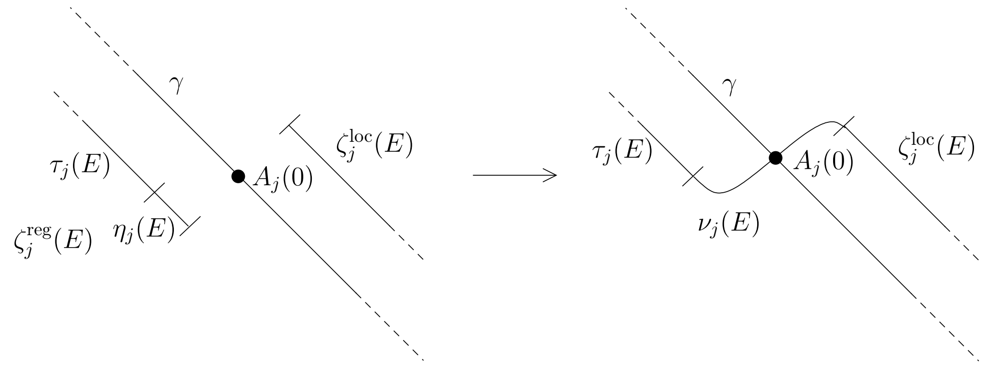


Figure 3: Computation of $\ell(\gamma)$

which implies

$$\mathrm{hol}_{\delta_{\tau_j(E)}}(\tau_j(E)) + \mathrm{hol}_{\delta_{\nu_j(E)}}(\nu_j(E)) \underset{E\to 0}{\longrightarrow} \mathrm{hol}_{\delta_{\zeta_j^{\mathrm{reg}}(0)}}(\zeta_j^{\mathrm{reg}}(0))$$

because

$$\mathrm{hol}_{\delta_{\nu_j(E)}}(\nu_j(E)) - \mathrm{hol}_{\delta_{\eta_j(E)}}(\eta_j(E)) \underset{E\to 0}{\longrightarrow} 0.$$

This shows that $\ell(\gamma) = \epsilon$, which concludes the proof for this first case, where γ is oriented according to the Hamiltonian flow of a_0.

If the orientation of the cycle γ is opposite to the one of the flow of X_{a_0}, we only have to change the sign of the holonomy.

It remains to investigate the case where there are some paths in γ oriented according to the flow of X_{a_0} and some which orientation is opposite to the latter, which means γ is smooth at some critical point s. We can use the analysis above by introducing two local paths ζ_1^{loc} and ζ_2^{loc} at s as in figure 4 (we make a small move forwards and backwards on an edge added to γ); one can obtain the claimed result by looking carefully at the obtained holonomies, remembering that the two paths have opposite orientation on the added edge.

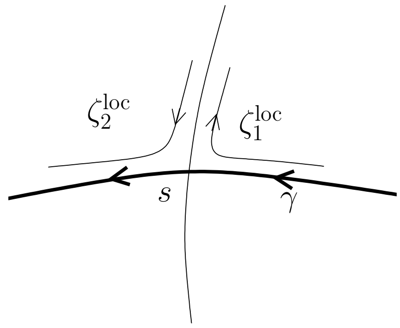


Figure 4: Case of a cycle γ smooth at s

Note that the choice of the added edge does not change the result. $\square$

6.4 Derivation of the Bohr-Sommerfeld conditions

The previous results allow to compute the spectrum of A_k in an interval of size $O(1)$ around the singular energy. Indeed, let γ_E, $E \in I^{\pm}$ be a connected component of the level $a_0^{-1}(E)$ and γ be the cycle in Γ_0 obtained by letting E go to 0. Then one can choose the local and regular paths used to compute the holonomy $\mathrm{hol}_{\mathfrak{L}}(\gamma)$ so that they all lie on γ_E, and define $g(E,k)$ as the sum

$$g(E,k) = g_{\zeta_1^{\rm loc}}(E) + g_{\zeta_1^{\rm reg}}(E) + \ldots + g_{\zeta_p^{\rm loc}}(E) + g_{\zeta_p^{\rm reg}}(E).$$

Furthermore, the matrix of change of basis associated to the sections $\phi_{k,E}^{(i)}$ is given by

$$T_j(E) = \exp\left(-\frac{i\pi}{4}\right)\mathcal{E}_k(k\mathfrak{e}_j(E,k))\begin{pmatrix} 1 & i\exp(-k\pi\mathfrak{e}_j(E,k)) \\ i\exp(-k\pi\mathfrak{e}_j(E,k)) & 1 \end{pmatrix},$$

where the function $\mathcal{E}_k$ is defined in equation (9). To compute eigenvalues near E, apply theorem 6.1 where T_j is replaced by $T_j(E)$ and $\mathrm{hol}_{\mathfrak{L}}(\gamma)$ by $\exp(ig(E,k))$. Applying Stirling's formula, we obtain

$$T_j(E) = \exp(ik\theta(E,k))\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(k^{-1}), \quad f_j(E) > 0$$

and

$$T_j(E) = \exp(ik\theta(E,k))\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + O(k^{-1}), \quad f_j(E) < 0.$$

with $\theta(E,k) = f_j(E)\ln|f_j(E)| - f_j(E) + k^{-1}\left(\varepsilon_j^{(0)}(E)\ln|f_j(E)| - \frac{\pi}{4}\right)$. Together with equations (17) and (18), this ensures that we recover the usual Bohr-Sommerfeld conditions away from the critical energy.

In the rest of the paper, we will look for eigenvalues of the form $k^{-1}e + O(k^{-2})$, where e is allowed to vary in a compact set. Hence, we have to replace A_k by $A_k - k^{-1}e$; this operator still has principal symbol a_0, but its subprincipal symbol is $a_1 - e$. Thanks to theorem 6.4, we are able to compute the singular holonomy and the invariants ε_j up to $O(k^{-2})$; hence, we approximate the spectrum up to an error of order $O(k^{-2})$.

6.5 The case of a unique saddle point

In the case of a unique saddle point in Γ_0, it is not difficult to write the Bohr-Sommerfeld conditions in a more explicit form. The critical level Γ_0 looks like a figure eight. We choose the convention for the cut edges and cycles as in figure 5.

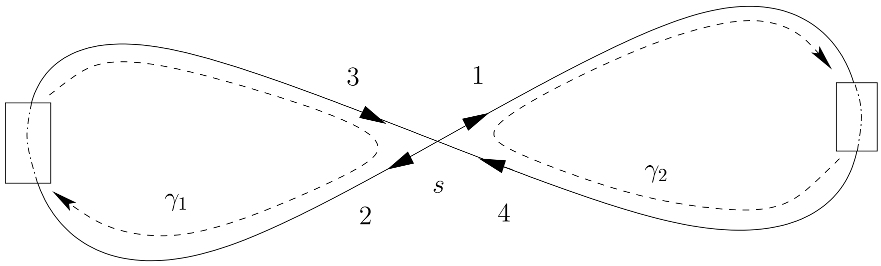


Figure 5: The singular level $\Gamma_0 = a_0^{-1}(0)$ and the choice of cut edges and cycles.

Let s be the saddle point, and let $\varepsilon(e, k)$ be the invariant associated to the operator $A_k - k^{-1}e$ at s; one has $\varepsilon^{(0)}(e) = \varepsilon^{(0)}(0) + e|\det(\text{Hess}(a_0)(s))|^{-1/2}$. Denote by $h_j(e, k) = \exp(i\theta_j(e, k))$ the holonomy of the loop γ_j in $\mathfrak{L}$; remember that θ_j is given by

$$\theta_j(e, k) = k\, c_0(\gamma_j) + \tilde{c}_1(\gamma_j) + \tilde{\epsilon}(\gamma_j)\pi + O(k^{-1}).$$

The Bohr-Sommerfeld conditions are given by the holonomy equations

$$x_4 = h_2 x_1, \quad x_3 = h_1 x_2$$

and by the transfer relation at the critical point

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $T = T(\varepsilon)$ is defined in equation (8). Using lemma 2 of [15], the quantization rule can in fact be written as a real scalar equation.

Proposition 6.12. *The equation $A_k u_k = k^{-1} e\, u_k + O(k^{-\infty})$ has a normalized eigenfunction if and only if e satisfies the condition*

$$\frac{1}{\sqrt{1+\exp(2\pi\varepsilon)}} \cos\left(\frac{\theta_1-\theta_2}{2}\right) = \sin\left(\frac{\theta_1+\theta_2}{2} + \frac{\pi}{4} + \varepsilon \ln(k) - \arg\left(\Gamma\left(\tfrac{1}{2}+i\varepsilon\right)\right)\right), \tag{22}$$

where we wrote for the sake of brevity θ_j, ε instead of $\theta_j(e,k), \varepsilon(e,k)$ (see definitions above).

7 Examples

We conclude by investigating two examples on the torus and one on the sphere; these examples present various topologies. More precisely, using the terminology of Bolsinov, Fomenko and Oshemkov [26, 5] for atoms (neighbourhoods of singular levels of Morse functions), we provide an example of a type B atom–the only type in complexity 1 (here, complexity means the number of critical points on the singular level) in the orientable case–and two examples of atoms of complexity 2: one is of type C_2 (xy on the sphere S^2) and the other is of type C_1 (Harper's Hamiltonian on the torus $\mathbb{T}^2$). It is a remarkable fact that these two examples are natural not only as the canonical realization of the atom on a surface but also because they come from the simplest possible Toeplitz operators with critical level of given type.

Note that there are two other types of atoms of complexity 2 in the orientable case (more precisely, types D_1 and D_2); it would be interesting to realize each of them as a hyperbolic level of the principal symbol of a selfadjoint Toeplitz operator and to complete this study. Note that in the context of pseudodifferential operators, Colin de Verdière and Parisse [16] treated the case of a type D_1 atom (the triple well potential) among some other examples. More generally, one could use the classification of Bolsinov, Fomenko and Oshemkov to write the Bohr-Sommerfeld conditions for all cases in low complexity (≤ 3 for instance); however, the case of two critical points already gives rise to rather tedious computations.

The details of the quantization of the torus and the sphere are quite standard. Nevertheless, for the sake of completeness, we will recall a few of them at the beginning of each paragraph.

7.1 Height function on the torus

Firstly, we consider the quantization of the height function on the torus. This is one of the first examples in Morse theory, perhaps because this is the simplest and most intuitive example with critical points of each type. In particular, the description of the two hyperbolic levels is quite simple.

Endow $\mathbb{R}^2$ with the linear symplectic form ω_0 and let $L_0 \to \mathbb{R}^2$ be the complex line bundle with Hermitian form and connection defined in section

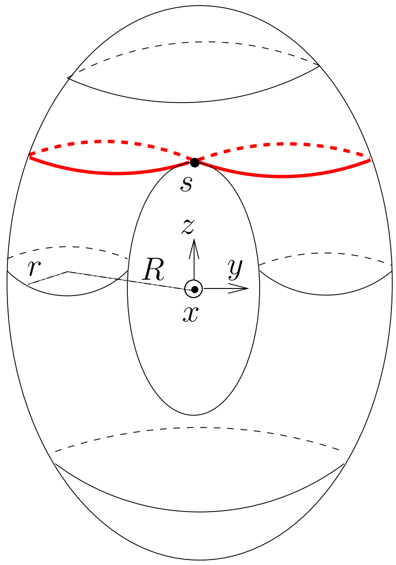


Figure 6: Height function on the torus

2.4.3. Let K be the canonical line of $\mathbb{R}^2$ with respect to its standard complex structure j: $K = \{\alpha \in (\mathbb{R}^2)^* \otimes \mathbb{C};\ \alpha(j\cdot) = i\alpha\}$. Choose a half-form line, that is a complex line δ with an isomorphism $\varphi : \delta^{\otimes 2} \to K$. K has a natural scalar product such that the square of the norm of α is $i\alpha \wedge \bar{\alpha}/\omega_0$; endow δ with the scalar product $\langle .,. \rangle_\delta$ such that φ is an isometry. The half-form bundle we work with, that we still denote by δ, is the trivial line bundle with fiber δ over $\mathbb{R}^2$.

Consider a lattice Λ with symplectic volume 4π. The Heisenberg group $H = \mathbb{R}^2 \times U(1)$ with product

$$(x,u).(y,v) = \left(x + y, uv \exp\left(\frac{i}{2}\omega_0(x,y)\right)\right)$$

acts on the bundle $L_0 \to \mathbb{R}^2$, with action given by the same formula. This action preserves the prequantum data, and the lattice Λ injects into H; therefore, the fiber bundle L_0 reduces to a prequantum bundle L over $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. The action extends to the fiber bundle L_0^k by

$$(x,u).(y,v) = \left(x + y, u^k v \exp\left(\frac{ik}{2}\omega_0(x,y)\right)\right).$$

We let the Heisenberg group act trivially on δ. We obtain a half-form bundle $\tilde{\delta}$ over $\mathbb{T}^2$ and an action

$$T^* : \Lambda \to \operatorname{End}\left(\mathcal{C}^\infty\left(\mathbb{R}^2, L_0^k \otimes \delta\right)\right), \quad u \mapsto T_u^*.$$

The Hilbert space $\mathcal{H}_k = H^0(M, L^k \otimes \tilde{\delta})$ can naturally be identified to the space $\mathcal{H}_{\Lambda,k}$ of holomorphic sections of $L_0^k \otimes \delta \to \mathbb{R}^2$ which are invariant under the action of Λ, endowed with the hermitian product

$$\langle \varphi, \psi \rangle = \int_D \langle \varphi, \psi \rangle_\delta \ |\omega_0|$$

where D is the fundamental domain of the lattice. Furthermore, $\Lambda/2k$ acts on $\mathcal{H}_{\Lambda,k}$. Let e and f be generators of Λ satisfying $\omega_0(e,f) = 4\pi$; one can show that there exists an orthonormal basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ of $\mathcal{H}_{\Lambda,k}$ such that

$$\forall \ell \in \mathbb{Z}/2k\mathbb{Z} \qquad \begin{cases} T^*_{e/2k}\psi_\ell = w^\ell \psi_\ell \\ T^*_{f/2k}\psi_\ell = \psi_{\ell+1} \end{cases}$$

with $w = \exp\left(\frac{i\pi}{k}\right)$. The sections ψ_ℓ can be expressed in terms of Θ functions.

Set $M_k = T^*_{e/2k}$ and $L_k = T^*_{f/2k}$. Let (q,p) be coordinates on $\mathbb{R}^2$ associated to the basis (e,f) and $[q,p]$ be the equivalence class of (q,p). Both M_k and L_k are Toeplitz operators, with respective principal symbols $[q,p] \mapsto \exp(2i\pi p)$ and $[q,p] \mapsto \exp(2i\pi q)$, and vanishing subprincipal symbols. For more details, see for instance [11, sections 2.2, 3.1].

It is a well-known fact that $\mathbb{T}^2$ is diffeomorphic to the surface shown in figure 6 above, which is obtained by rotating a circle of radius r around a circle of radius $R > r$ contained in the yz plane; the diffeomorphism is given by the explicit formulas

$$x = r\sin(2\pi q), \ y = (R + r\cos(2\pi q))\cos(2\pi p), \ z = (R + r\cos(2\pi q))\sin(2\pi p).$$

Hence, the Hamiltonian that we consider is

$$a_0(q,p) = (R + r\cos(2\pi q))\sin(2\pi p)$$

on the fundamental domain D. We try to quantize it, *id est* find a selfadjoint Toeplitz operator A_k with principal symbol a_0. The Toeplitz operators

$$B_k = \frac{1}{2i}(M_k - M_k^*), \quad C_k = R\Pi_k + \frac{r}{2}(L_k + L_k^*)$$

are selfadjoint and

$$\sigma_{\text{norm}}(B_k) = \sin(2\pi p) + O(\hbar^2), \quad \sigma_{\text{norm}}(C_k) = R + r\cos(2\pi q) + O(\hbar^2).$$

Hence $A_k = \frac{1}{2}(B_k C_k + C_k B_k)$ is a selfadjoint Toeplitz operator with normalized symbol $a_0 + O(\hbar^2)$. Its matrix in the basis $(\psi_\ell)_{\ell \in \mathbb{Z}/2k\mathbb{Z}}$ writes

$$\begin{pmatrix} R\alpha_0 & \frac{r}{4}(\alpha_0+\alpha_1) & 0 & \dots & 0 & \frac{r}{4}(\alpha_{2k-1}+\alpha_0) \\ \frac{r}{4}(\alpha_0+\alpha_1) & R\alpha_1 & \frac{r}{4}(\alpha_1+\alpha_2) & 0 & \dots & 0 \\ 0 & \frac{r}{4}(\alpha_1+\alpha_2) & R\alpha_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & R\alpha_{2k-2} & \frac{r}{4}(\alpha_{2k-2}+\alpha_{2k-1}) \\ \frac{r}{4}(\alpha_0+\alpha_{2k-1}) & 0 & \dots & 0 & \frac{r}{4}(\alpha_{2k-2}+\alpha_{2k-1}) & R\alpha_{2k-1} \end{pmatrix} \tag{23}$$

with $\alpha_\ell = \sin(\ell\pi/k)$.

The level $\Gamma_{R-r} = a_0^{-1}(R-r)$ contains one hyperbolic point $s = (1/2, 1/4)$. It is the union of the two branches

$$p = \frac{1}{2\pi}\arcsin\left(\frac{R-r}{R+r\cos(2\pi q)}\right) \text{ and } p = \frac{1}{2} - \frac{1}{2\pi}\arcsin\left(\frac{R-r}{R+r\cos(2\pi q)}\right).$$

The Hamiltonian vector field associated to a_0 is given by

$$X_{a_0}(q,p) = \frac{1}{2}\left(R + r\cos(2\pi q)\right)\cos(2\pi p)\frac{\partial}{\partial q} + \frac{r}{2}\sin(2\pi q)\sin(2\pi p)\frac{\partial}{\partial p}.$$

Moreover, one has

$$\varepsilon^{(0)} = \frac{e}{\pi\sqrt{r(R-r)}} \tag{24}$$

We choose the cycles γ_1 and γ_2 with the convention given in section 6.5. We have to compute the principal and subprincipal actions of γ_1, γ_2 and their indices $\tilde{\epsilon}$. Let us detail the calculations in the case of γ_1.

We parametrize γ_1 by $q \mapsto \left(q, \frac{1}{2} - \frac{1}{2\pi}\arcsin\left(\frac{R-r}{R+r\cos(2\pi q)}\right)\right)$. The principal action is given by

$$c_0(\gamma_1) = 2I(R,r) - 2\pi, \tag{25}$$

where $I(R,r)$ is the integral

$$I(R,r) = \int_0^1 \arcsin\left(\frac{R-r}{R+r\cos(2\pi q)}\right)\, dq;$$

unfortunately, we do not know any explicit expression for this integral, so for numerical computations, once fixed the radii R and r, we obtain the value of $I(R,r)$ thanks to numerical integration routines.

On γ_1, the subprincipal form reads

$$\kappa_0 = \frac{-2e\ dq}{\sqrt{(R + r\cos(2\pi q))^2 - (R-r)^2}}.$$

One can obtain an explicit primitive thanks to any computer algebra system. Furthermore, some computations show that the symplectic area of the parallelogram $R_{a,b}$ is equal to

$$\int_{R_{a,b}} \omega = 8\pi\sqrt{\frac{r}{R-r}}\left(q_a - \frac{1}{2}\right)\left(\frac{1}{2} - q_b\right).$$

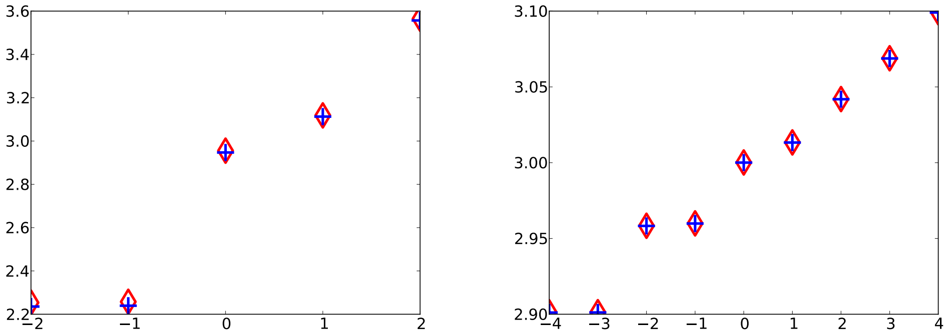
This yields the following value for the subprincipal action:

$$\tilde{c}_1(\gamma_1) = \varepsilon^{(0)} \ln\left(\frac{32}{\pi}\sqrt{\frac{r}{R}\left(1 - \frac{r}{R}\right)}\right). \tag{26}$$

Finally, the index associated to half-forms is $\tilde{\epsilon}(\gamma_1) = 1/4$. For γ_2, one can check that

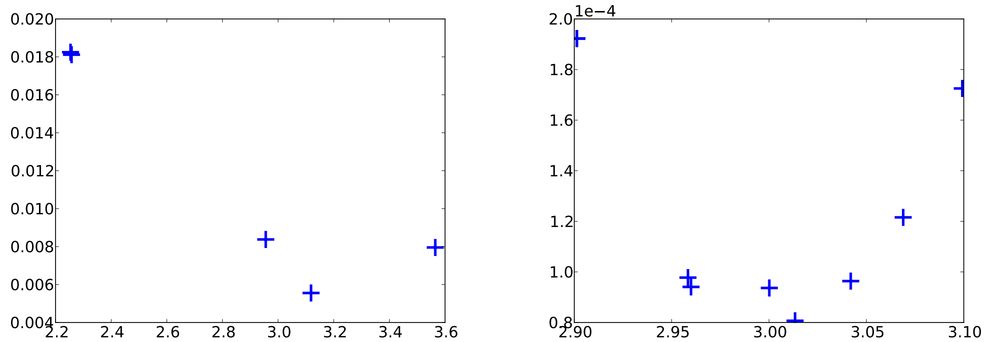
$$c_0(\gamma_2) = 2I(R,r), \quad \tilde{c}_1(\gamma_2) = \varepsilon^{(0)} \ln\left(\frac{32}{\pi}\sqrt{\frac{r}{R}\left(1-\frac{r}{R}\right)}\right), \quad \tilde{\epsilon}(\gamma_2) = 1/4. \tag{27}$$

With this data, one can test the Bohr-Sommerfeld condition for different couples (R,r). We illustrate this with $(R,r) = (4,1)$ (note that we have tested several couples). We compare the eigenvalues obtained numerically from the matrix (23) and the ones derived from the Bohr-Sommerfeld conditions (22) in the interval $I = [R-r-10k^{-1}, R-r+10k^{-1}]$. In figure 7, we plotted the theoretical and numerical eigenvalues; figure 8 shows the error between the eigenvalues and the solutions of the Bohr-Sommerfeld conditions for fixed k, while figure 9 is a graph of the logarithm of the maximal error in the interval I as a function of $\ln(k)$.



(a) $k = 10$ (b) $k = 100$

Figure 7: Eigenvalues in $[R-r-10k^{-1}, R-r+10k^{-1}]$; in red diamonds, the eigenvalues of A_k obtained numerically; in blue crosses, the theoretical eigenvalues derived from the Bohr-Sommerfeld conditions. The results are indexed with respect to the eigenvalue closest to the critical energy, labeled as 0. Observe that even for $k = 10$, the method is very precise.



(a) $k = 10$ (b) $k = 100$

Figure 8: Absolute value of the difference between the numerical and theoretical eigenvalues; the error is smaller near the critical energy ($R - r = 3$ in this case).

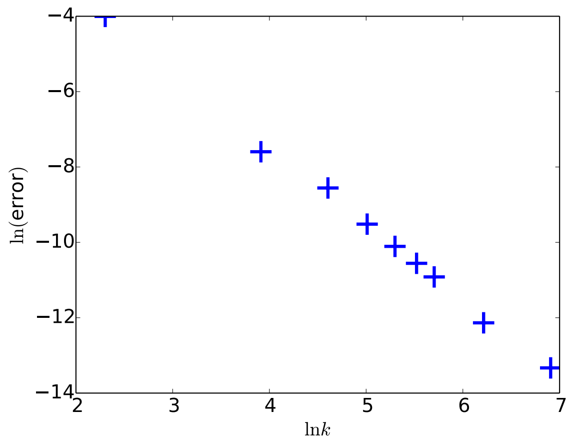


Figure 9: Logarithm of the maximal error as a function of the logarithm of k; the error displays a behaviour in $O(k^{-2})$, as expected.

7.2 xy on the 2-sphere

Let us consider another simple example, but this time with two saddle points on the critical level. We will quantize the Hamiltonian $a_0(x, y, z) = xy$ on the sphere S^2. Let us briefly recall the details of the quantization of this surface.

Start from the complex projective plane $\mathbb{CP}^1$ and let $L = \mathcal{O}(1)$ be the dual bundle of the tautological bundle

$$\mathcal{O}(-1) = \left\{(u,v) \in \mathbb{CP}^1 \times \mathbb{C}^2; \quad v \in u\right\}$$

with natural projection. L is a Hermitian, holomorphic line bundle; let us denote by ∇ its Chern connection. The 2-form $\omega = i \ \mathrm{curv}(\nabla)$ is the symplectic form on $\mathbb{CP}^1$ associated with the Fubini-Study Kähler structure, and $L \to \mathbb{CP}^1$ is a prequantum bundle. Moreover, the canonical bundle naturally identifies to $\mathcal{O}(-2)$, hence one can choose the line bundle $\delta = \mathcal{O}(-1)$ as a half-form bundle. The state space $\mathcal{H}_k = H^0(\mathbb{CP}^1, L^k \otimes \delta)$ can be identified with the space $\mathbb{C}_{p_k}[z_1, z_2]$ of homogeneous polynomials of degree $p_k = k - 1$ in two variables. The polynomials

$$P_\ell(z_1, z_2) = \sqrt{\frac{(p_k+1)\binom{p_k}{\ell}}{2\pi}} \ z_1^\ell z_2^{p_k - \ell}, \quad 0 \leq \ell \leq p_k,$$

form an orthonormal basis of $\mathcal{H}_k$. The sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ is diffeomorphic to $\mathbb{CP}^1$ *via* the stereographic projection (from the north pole to the plane $z = 0$). The symplectic form ω on $\mathbb{CP}^1$ is carried to the symplectic form $\omega_{S^2} = -\frac{1}{2}\Omega$, with Ω the usual area form on S^2 (the one which gives the area 4π). The operator A_k acting on the basis $(P_\ell)_{0 \leq \ell \leq p_k}$ by

$$A_k P_\ell = \frac{i}{p_k^2} \left(\alpha_{\ell,k} P_{\ell-2} - \beta_{\ell,k} P_{\ell+2}\right),$$

with

$$\alpha_{\ell,k} = \sqrt{\ell(\ell-1)(p_k - \ell + 1)(p_k - \ell + 2)}$$

and

$$\beta_{\ell,k} = \sqrt{(\ell+1)(\ell+2)(p_k - \ell - 1)(p_k - \ell)},$$

is a Toeplitz operator with principal symbol $a_0(x,y,z) = xy$ and vanishing subprincipal symbol (for more details, one can consult [4, section 3] for instance). Note that $\alpha_{\ell,k} = \beta_{p_k - \ell, k}$, which implies that if λ is an eigenvalue of A_k, then $-\lambda$ also is.

The level $a_0^{-1}(0)$ is critical, and contains two saddle points: the poles N (north) and S (south). It is the union of the two great circles $x = 0$ and $y = 0$. We choose the cut edges and cycles as indicated in figure 10. Set $h_j = \mathrm{hol}_{\mathfrak{L}}(\gamma_j) = \exp(i\theta_j)$; remember that $\theta_j = kc_0(\gamma_j) + \tilde{c}_1(\gamma_j) + \tilde{\epsilon}(\gamma_j)\pi + O(k^{-1})$. The holonomy equations read

$$y_2 = x_3, \quad y_4 = h_1 x_1, \quad y_3 = h_2 x_2, \quad x_4 = h_3 y_1 \tag{28}$$

while the transfer equations are given by

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = T_S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = T_N \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{29}$$

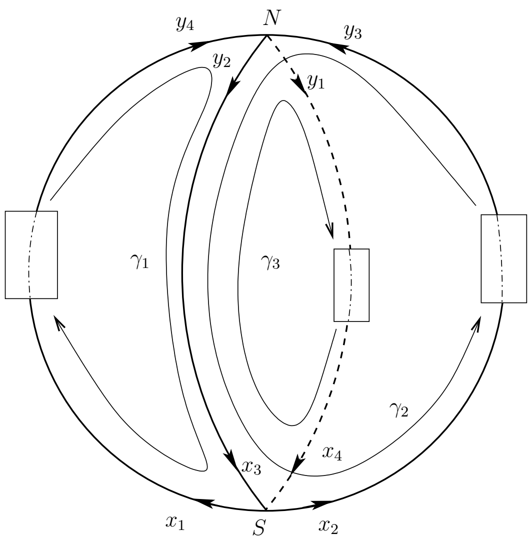


Figure 10: Choice of the cycles and cut edges

The system (28) + (29) has a solution if and only if the matrix

$$U = T_S \begin{pmatrix} 0 & \exp(-i\theta_1) \\ \exp(-i\theta_2) & 0 \end{pmatrix} T_N \begin{pmatrix} 0 & \exp(-i\theta_3) \\ 1 & 0 \end{pmatrix}$$

admits 1 as an eigenvalue. The matrix U is unitary, and if we write $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a straightforward computation shows that

$$|a|^2 = |d|^2 = \frac{1 - 2\cos(\theta_2 - \theta_1)\exp(-\pi(\varepsilon_S + \varepsilon_N)) + \exp(-2\pi(\varepsilon_S + \varepsilon_N))}{(1 + \exp(-2\pi\varepsilon_S))(1 + \exp(-2\pi\varepsilon_N))};$$

hence, by lemma 2 of [15], 1 is an eigenvalue of U if and only if

$$|a| \sin\left(\frac{\arg(ad) - \pi}{2} - \arg(a)\right) = \sin\left(\frac{\arg(ad) - \pi}{2}\right).$$

This amounts to the equation

$$|a| \cos\left(\frac{\arg(z) - \arg(w)}{2}\right)$$
$$= \sin\left(\frac{\arg(z) + \arg(w)}{2} + \arg\left(\Gamma(\tfrac{1}{2} + i\,\varepsilon_N)\right) + \arg\left(\Gamma(\tfrac{1}{2} + i\,\varepsilon_S)\right) - (\varepsilon_S + \varepsilon_N)\ln(k)\right)$$

with

$$z = \exp(-i(\theta_2 + \theta_3)) - \exp(-\pi(\varepsilon_S + \varepsilon_N) - i(\theta_1 + \theta_3))$$

and

$$w = \exp(-i\theta_1) - \exp(-\pi(\varepsilon_N + \varepsilon_S) - i\theta_2).$$

One has

$$\varepsilon_S^{(0)} = \varepsilon_N^{(0)} = \varepsilon^{(0)} = \frac{e}{2}. \tag{30}$$

Moreover, the principal actions are

$$c_0(\gamma_1) = -\frac{\pi}{2}, \quad c_0(\gamma_2) = \frac{\pi}{2}, \quad c_0(\gamma_3) = \pi. \tag{31}$$

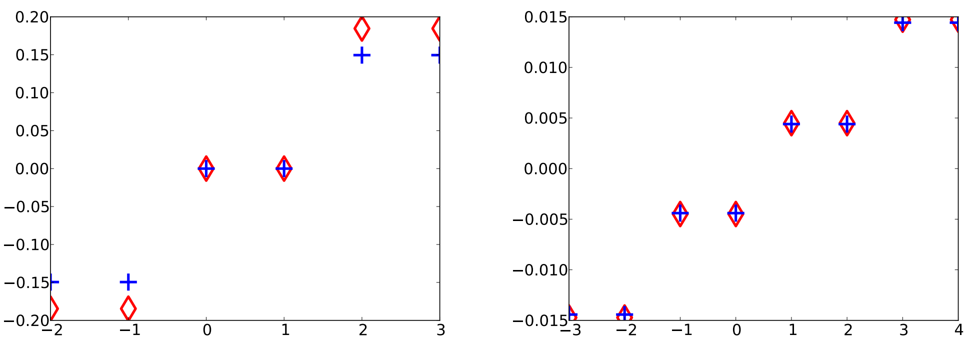
Then, one finds for the subprincipal actions

$$\tilde{c}_1(\gamma_1) = 2\varepsilon^{(0)} \ln 2 = \tilde{c}_1(\gamma_2), \quad \tilde{c}_1(\gamma_3) = 0. \tag{32}$$

Finally, the indices $\tilde{\epsilon}$ are the following:

$$\tilde{\epsilon}(\gamma_1) = \frac{3}{2}, \quad \tilde{\epsilon}(\gamma_2) = \frac{1}{2}, \quad \tilde{\epsilon}(\gamma_3) = 1. \tag{33}$$

Figure 11 shows the theoretical eigenvalues obtained by using this results, as well as the numerical evaluation of the eigenvalues of A_k from its matrix form.



(a) $k = 10$ (b) $k = 100$

Figure 11: Eigenvalues in $[-2k^{-1}, 2k^{-1}]$; in red diamonds, the eigenvalues of A_k obtained numerically; in blue crosses, the theoretical eigenvalues derived from the Bohr-Sommerfeld conditions.

7.3 Harper's Hamiltonian on the torus

Keeping the conventions and notations of the first example, we consider the Hamiltonian (sometimes known as Harper's Hamiltonian since it is related to Harper's equation [21])

$$a_0(q, p) = 2(\cos(2\pi p) + \cos(2\pi q))$$

on the torus. The operator $A_k = M_k + M_k^* + L_k + L_k^*$ is a Toeplitz operator with principal symbol a_0 and vanishing subprincipal symbol. Its matrix in the basis $(\psi_\ell)_{\ell\in\mathbb{Z}/2k\mathbb{Z}}$ is

$$\begin{pmatrix} 2\alpha_0 & 1 & 0 & \dots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 1 & 2\alpha_{2k-1} \end{pmatrix}$$

where

$$\alpha_\ell = \cos\left(\frac{\ell\pi}{k}\right), \quad 0 \le \ell \le 2k-1.$$

The critical level $\Gamma_0 = a_0^{-1}(0)$ contains two hyperbolic points: $s_1 = (0, 1/2)$ and $s_2 = (1/2, 0)$. On the fundamental domain, it is the union of the four segments described in figure 12; hence, its image on the torus it is the union of two circles that intersect at two points. We choose the cycles and cut edges

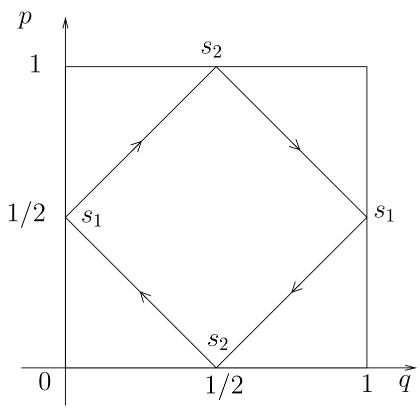


Figure 12: Critical level Γ_0 on the fundamental domain; the arrows indicate the direction of the Hamiltonian flow of a_0.

as in figure 13 (for a representation of the two circles in a two-dimensional view) and 14 (for a representation of the cycles on the fundamental domain).

We write the holonomy equations

$$y_1 = x_3, \quad y_3 = h_1 x_2, \quad y_4 = h_2 x_1, \quad x_4 = h_3 y_1 \tag{34}$$

and the transfer equations

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = T_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = T_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{35}$$

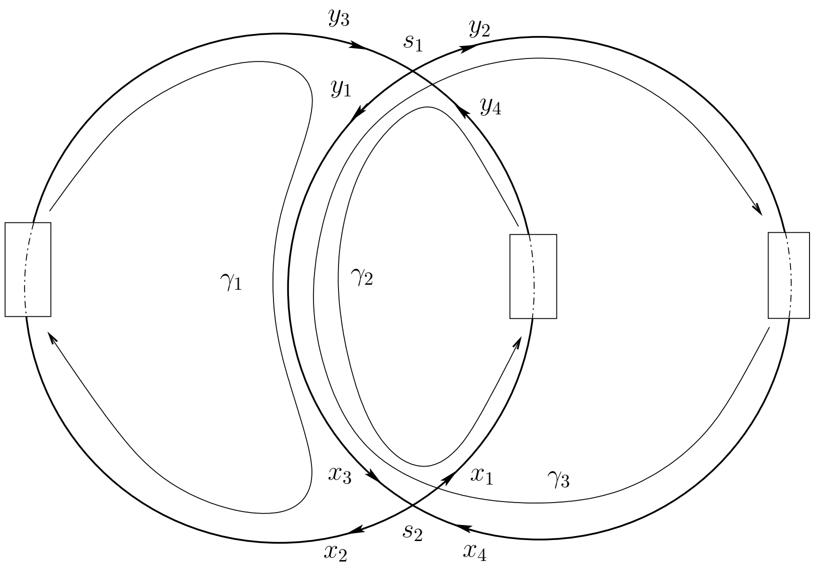


Figure 13: Choice of the cycles and cut edges

where $h_j = \mathrm{hol}_{\mathfrak{L}}(\gamma_j) = \exp(i\theta_j)$. Following the same steps as in the previous example, one can show that the system (34) + (35) has a solution if and only if e is a solution of the scalar equation

$$|a| \cos\left(\frac{\arg(w) - \arg(z)}{2}\right)$$
$$= \cos\left(\frac{\arg(z) + \arg(w)}{2} + \arg\left(\Gamma(\tfrac{1}{2} + i\ \varepsilon_1)\right) + \arg\left(\Gamma(\tfrac{1}{2} + i\ \varepsilon_2)\right) - (\varepsilon_1 + \varepsilon_2)\ln(k)\right)$$

with

$$|a|^2 = \frac{\exp(-2\pi\varepsilon_1) + \exp(-2\pi\varepsilon_2) + 2\cos(\theta_2 - \theta_1)\exp(-\pi(\varepsilon_1 + \varepsilon_2))}{(1 + \exp(-2\pi\varepsilon_1))(1 + \exp(-2\pi\varepsilon_2))},$$

$$w = \exp(-\pi\varepsilon_2 - i(\theta_2 + \theta_3)) + \exp(-\pi\varepsilon_1 - i(\theta_1 + \theta_3))$$

and

$$z = \exp(-\pi\varepsilon_1 - i\theta_2) + \exp(-\pi\varepsilon_2 - i\theta_1).$$

Moreover, one has

$$\varepsilon_1^{(0)} = \varepsilon_2^{(0)} = \frac{e}{2\pi} := \varepsilon^{(0)}. \tag{36}$$

It remains to compute the quantities θ_j (up to $O(k^{-1})$). The principal actions are easily computed:

$$c_0(\gamma_1) = -\pi, \quad c_0(\gamma_2) = 3\pi, \quad c_0(\gamma_1) = -2\pi. \tag{37}$$

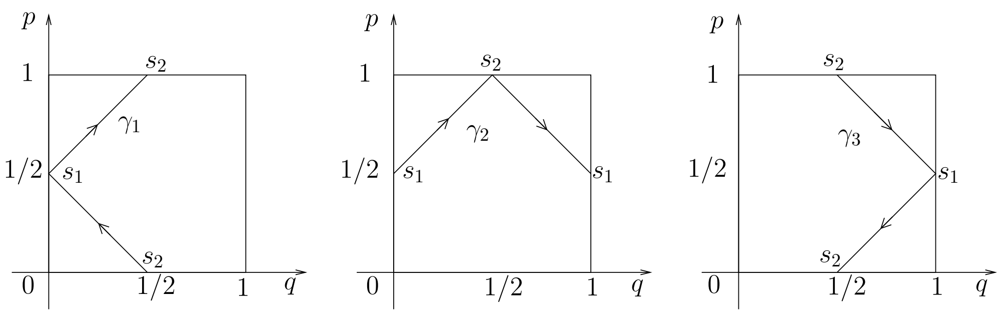


Figure 14: Cycles on the fundamental domain

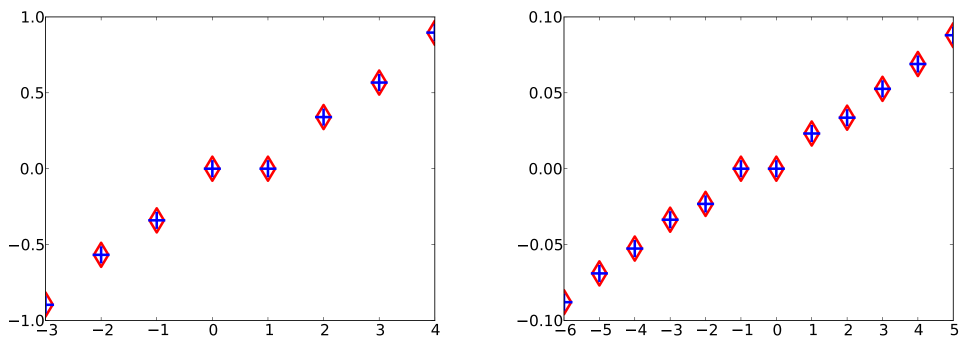
Furthermore, one can check that the subprincipal actions are given by

$$\tilde{c}_1(\gamma_1) = 2\varepsilon^{(0)} \ln\left(\frac{8}{\pi}\right) = \tilde{c}_1(\gamma_2), \quad \tilde{c}_1(\gamma_3) = 0. \tag{38}$$

Finally, one has

$$\tilde{\epsilon}(\gamma_1) = \tilde{\epsilon}(\gamma_2) = \tilde{\epsilon}(\gamma_3) = 0. \tag{39}$$

The results thus obtained are displayed in Figure 15.



(a) $k = 10$

(b) $k = 100$

Figure 15: Eigenvalues in $[-10k^{-1}, 10k^{-1}]$; in red diamonds, the eigenvalues of A_k obtained numerically; in blue crosses, the theoretical eigenvalues derived from the Bohr-Sommerfeld conditions.

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