Errata to "A brief introduction to Berezin-Toeplitz operators on compact Kähler manifolds"

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We list below a number of errors contained in our book "A brief introduction to Berezin-Toeplitz operators on compact Kähler manifolds". We warmly thank Bruce Bartlett for pointing out these errors. If you find more errors in the book, please send them to the email address below so that they can be added in this list.

- Lemma 2.3.2 (p. 10) is missing an hypothesis. Its statement should read

  Lemma 0.1. *A k-form α belongs to $\Omega^{k,0}(M)$ if and only if for every vector field $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, $i_X\alpha = 0$. More generally, a k-form α belongs to $\Omega^{p,q}(M)$ with $p+q = k$ if and only if for any $q+1$ vector fields $X_1, \dots, X_{q+1} \in \mathcal{C}^\infty(M, T^{0,1}M)$, $i_{X_1} \dots i_{X_{q+1}}\alpha = 0$ and for any $p+1$ vector fields $Y_1, \dots, Y_{p+1} \in \mathcal{C}^\infty(M, T^{1,0}M)$, $i_{Y_1} \dots i_{Y_{p+1}}\alpha = 0$.*

  Moreover, the remark after this statement should be changed accordingly. For the sake of completeness, we give a proof of the case $q > 0$ that was left as an exercise.

  Proof of the case $q > 0$. Let $X_1, \dots, X_q \in \mathcal{C}^\infty(M, T^{0,1}M)$. Since for any $X \in \mathcal{C}^\infty(M, T^{0,1}M)$, $i_X\left(i_{X_1} \dots i_{X_q}\beta\right) = 0$, by the $q = 0$ case, $i_{X_1} \dots i_{X_q}\beta$ belongs to $\Omega^{k-q,0}(M)$. Write

  $$\beta = \beta^{(k,0)} + \beta^{(k-1,1)} + \ldots + \beta^{(0,k)}$$

  with $\beta^{(k-m,m)} \in \Omega^{k-m,m}(M)$. Since

  $$i_{X_1} \dots i_{X_q}\beta = \sum_{m=0}^{k} i_{X_1} \dots i_{X_q}\beta^{(k-m,m)} = \sum_{m=q}^{k} \underbrace{i_{X_1} \dots i_{X_q}\beta^{(k-m,m)}}_{\in \Omega^{k-m,m-q}(M)}$$

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we obtain by uniqueness of the decomposition that for every $m \in \{q+1, \dots, k\}$, $i_{X_1} \dots i_{X_q} \beta^{(k-m,m)} = 0$. We claim that this implies that for every such m, $\beta^{(k-m,m)} = 0$ (one can for instance decompose this form in a basis of $(T^*M)^{1,0}$ as in the proof of the case $q = 0$). A similar argument shows that for every $m \in \{0, \dots, q-1\}$ and any $Y_1, \dots, Y_p \in \mathcal{C}^\infty(M, T^{1,0}M)$ (with $p = k - q$), $i_{Y_1} \dots i_{Y_p} \beta^{(k-m,m)} = 0$, which in turns yields $\beta^{(k-m,m)} = 0$. Therefore $\beta = \beta^{(p,q)}$ belongs to $\Omega^{p,q}(M)$. □

- Because of the previous error, the proof of Lemma 2.5.2 is incomplete. Here is how to fill this small gap: if $Z, W \in \mathcal{C}^\infty(M, T^{1,0}M)$, then

$$\omega(Z, W) = \overline{\omega(\overline{Z}, \overline{W})} = 0$$

by the rest of the proof, since $\overline{Z}, \overline{W}$ belong to $\mathcal{C}^\infty(M, T^{0,1}M)$.

- In the proof of Proposition 4.2.1 (p. 40), the equation

$$\frac{\partial^2 H}{\partial z_\ell \partial \bar{z}_m} = \frac{\partial^2 H}{\partial x_\ell \partial x_m} + i \frac{\partial^2 H}{\partial x_\ell \partial y_m} - i \frac{\partial^2 H}{\partial y_\ell \partial x_m} + \frac{\partial^2 H}{\partial y_\ell \partial y_m}$$

should read

$$\frac{\partial^2 H}{\partial z_\ell \partial \bar{z}_m} = \frac{1}{4} \left(\frac{\partial^2 H}{\partial x_\ell \partial x_m} + i \frac{\partial^2 H}{\partial x_\ell \partial y_m} - i \frac{\partial^2 H}{\partial y_\ell \partial x_m} + \frac{\partial^2 H}{\partial y_\ell \partial y_m} \right).$$

Consequently, the equation

$$\bar{\partial} \partial H(\partial_{x_\ell}, j \partial_{x_m}) = 2i \left(\frac{\partial^2 H}{\partial x_\ell \partial x_m} + \frac{\partial^2 H}{\partial y_\ell \partial y_m} \right)$$

becomes

$$\bar{\partial} \partial H(\partial_{x_\ell}, j \partial_{x_m}) = \frac{i}{2} \left(\frac{\partial^2 H}{\partial x_\ell \partial x_m} + \frac{\partial^2 H}{\partial y_\ell \partial y_m} \right)$$

and the final result

$$(i \bar{\partial} \partial H)_{m_0}(X, j_{m_0} X) = -2 \left(\mathrm{Hess}_H(m_0)(X, X) + \mathrm{Hess}_H(m_0)(j_{m_0} X, j_{m_0} X) \right)$$

is changed to

$$(i \bar{\partial} \partial H)_{m_0}(X, j_{m_0} X) = -\frac{1}{2} \left(\mathrm{Hess}_H(m_0)(X, X) + \mathrm{Hess}_H(m_0)(j_{m_0} X, j_{m_0} X) \right).$$

This does not affect the proof since the discussion on the signs is still valid.