INVERSE SPECTRAL THEORY FOR SEMICLASSICAL JAYNES-CUMMINGS SYSTEMS

YOHANN LE FLOCH ÁLVARO PELAYO SAN VŨ NGỌC

Abstract. Quantum semitoric systems form a large class of quantum Hamiltonian integrable systems with circular symmetry which has received great attention in the past decade. They include systems of high interest to physicists and mathematicians such as the Jaynes-Cummings model (1963), which describes a two-level atom interacting with a quantized mode of an optical cavity, and more generally the so-called systems of Jaynes-Cummings type. In this paper we consider the joint spectrum of a pair of commuting semiclassical operators forming a quantum integrable system of Jaynes-Cummings type. We prove, assuming the Bohr-Sommerfeld rules hold, that if the joint spectrum of two of these systems coincide up to $\mathcal{O}(\hbar^2)$, then the systems are isomorphic.

Yohann Le Floch
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
Tel Aviv 6997801, Israel
E-mail: yleflochpost.tau.ac.il
Website: https://sites.google.com/site/yohannleflochhomepage/english-version

Álvaro Pelayo
Department of Mathematics
University of California, San Diego
9500 Gilman Drive # 0112
La Jolla, CA 92093-0112, USA
E-mail: alpelayo@math.ucsd.edu

San Vũ Ngọc
Institut Universitaire de France

Institut de Recherches Mathématiques de Rennes
Université de Rennes 1
Campus de Beaulieu
F-35042 Rennes cedex, France
E-mail: san.vu-ngoc@univ-rennes1.fr
Website: http://blogperso.univ-rennes1.fr/san.vu-ngoc/

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1. Introduction

A natural question in semiclassical analysis is whether the knowledge of the joint spectrum of a quantum integrable system allows to determine the classical dynamics of the underlying integrable system. Pursuing this question in such generality has been made possible thanks to the development of semiclassical analysis with microlocal techniques (see for instance the recent books by Dimassi-Sjöstrand [17], Guillemin-Sternberg [23], and Zworski [42] and the references therein) which nowadays permits a constant interaction between symplectic geometry and spectral theory. In particular, these techniques led to the resolution of the inverse spectral question in a number of cases; for instance: (i) compact toric integrable systems, in the context of Berezin-Toeplitz quantization [11]; (ii) semiglobal inverse problem near the so called "focus-focus" singularities of 2D integrable systems, in the context of $\hbar$-pseudodifferential quantization [33]; (iii) inverse theory for the Laplacian on surfaces of revolution [40]; (iv) 1-dimensional pseudodifferential operators with Morse symbol [39]. The flexibility of microlocal analysis makes us hope that more general integrable systems will be treated in the future. An interesting step is to understand what happens for *semitoric* systems on 4-dimensional phase spaces [31], which form an important extension of toric systems.

Definition 1.1. A $\mathcal{C}^\infty$ *classical integrable system* $F := (J, H) : M \to \mathbb{R}^2$ on a connnected symplectic 4-dimensional manifold (M, ω) is *semitoric* if:

- (H.i) J is the momentum map of an effective Hamiltonian circle action.
- (H.ii) The singularities of F are non-degenerate with no hyperbolic component.
- (H.iii) J is a proper map (i.e., the preimages of compact sets are compact).

A *quantum semitoric integrable system* (P, Q) is given by two semiclassical commuting self-adjoint operators whose principal symbols form a classical semitoric integrable system. The notion of semiclassical operators that we use is defined in Section 2.4; it includes standard semiclassical pseudodifferential operators, and Berezin-Toeplitz operators.

Hypothesis (H.ii) means that if $m \in M$ is a critical point of F then there is a 2 by 2 matrix B such that the following happens: if we write $\tilde{F} = B \circ F$, then there are local symplectic coordinates near m in which :

(1) $\tilde{F}(x, y, \xi, \eta) = (\eta + \mathcal{O}(\eta^2), x^2 + \xi^2 + \mathcal{O}((x, \xi)^3))$
(2) $\mathrm{d}_m \tilde{F} = 0$ and $\mathrm{d}^2_m \tilde{F}(x, y, \xi, \eta) = (x^2 + \xi^2, y^2 + \eta^2)$
(3) $\mathrm{d}_m \tilde{F} = 0$ and $\mathrm{d}^2_m \tilde{F}(x, y, \xi, \eta) = (x\xi + y\eta, x\eta - y\xi)$

In case (1) the point m is called a *transversally elliptic singularity* (or *codimension* 1 *elliptic singularity*); in case (2) m is an *elliptic-elliptic singularity* (often the terminology *elliptic singularity* is used to refer to either of them); in case (3) m is called a *focus-focus singularity.*

When (M,ω) is four-dimensional, toric systems form a particular class of semitoric systems for which F is the momentum map of a Hamiltonian $\mathbb{T}^2$-action. The symplectic classification of toric systems was done by Delzant [16], and the quantum spectral theory in the case of Berezin-Toeplitz quantization was carried out in [11]. A simple corollary of this spectral theory is that the image $F(M)$, which is the so-called Delzant polytope, can be recovered from the joint spectrum; in view of the Delzant theorem, this implies that the joint spectrum completely determines the triple (M,ω,F) up to toric isomorphism.

The main difference with the toric case is that focus-focus singularities can appear in a semitoric system, making the system more difficult to describe. For instance, if there is at least one focus-focus singularity, the image of the moment map is no longer a convex polygon. Moreover, new symplectic invariants appear; according to [30, 31], a semitoric system is determined up to isomorphisms[1] by five symplectic invariants:

(1) the *number* of focus-focus singular values of the system;
(2) a *Taylor series* $\sum_{i,j\in\mathbb{N}} a_{ij}X^iY^j$ for each focus-focus singularity ([37, 30]);
(3) a *height invariant* $h>0$ measuring the volume of certain reduced spaces at each focus-focus singularity;
(4) a *polygonal invariant* (in fact, a family of polygons) obtained by unwinding the singular affine structure of the system;
(5) an index associated with each focus-focus singularity, called the *twisting index*.

The polygonal invariant is described in more details in Theorem 4.2 and the subsequent paragraph; it is the image of $F(M)$ by some homeomorphism (let us emphasize again that in fact, there is a family of such homeomorphisms, giving rise to a family of polygons). Given one polygon $\Delta=\Phi(F(M))$ in this family, the height invariant associated with a focus-focus value $c\in F(M)$ is simply the height of the point $\Phi(c)$ in Δ. Other invariants are more involved: the Taylor series invariant encodes the behaviour of the periods of the integrable system near the singularity, while the twisting index is an integer coming from the choice of a privileged toric momentum map near the singularity. However, the true twisting index is rather the equivalence class of the family of twisting indices associated with each singularity for the equivalence relation given by addition of a common integer. Hence this invariant is not relevant for systems displaying a single focus-focus singularity. In the case of the Jaynes-Cummings system, Figure 4 in the article [30] can help visualize invariants (1), (2), (3), (4).

Therefore, if one is able to recover these five invariants from the semiclassical joint spectrum of a quantum integrable system quantizing (J,H), then in effect one can recover the triple (M,ω,F) up to the appropriate notion of isomorphism. From [33], the invariant (2) associated with a critical

[1] The notion of isomorphism for semitoric systems is recalled in Definition 2.1.

singularity of focus-focus type can be recovered from the joint spectrum, provided that one knows the corresponding critical value. The goal of this paper is to extend this result, namely to show that one can detect in the joint spectrum the invariants (1) to (4). Let us be more precise and state our main result. We will say that a semitoric integrable system $F = (J, H)$ is *simple* if it satisfies the following: if m is a focus-focus critical point for F, then m is the unique critical point of the level set $J^{-1}(J(m))$. Our main theorem is the following.

Theorem A. *Let (P, Q) be a quantum simple semitoric system on M for which the Bohr-Sommerfeld rules hold. Then from the knowledge of the semiclassical joint spectrum* $\mathrm{JointSpec}(P, Q) + \mathcal{O}(\hbar^2)$*, one can recover the four following invariants of the associated classical semitoric system:*

(1) *the number m_f of focus-focus values,*
(2) *the Taylor series associated with each focus-focus value,*
(3) *the height invariant associated with each focus-focus value,*
(4) *the polygonal invariant of the system.*

Of course, Theorem A is not entirely satisfactory if one has in mind the problem of completely recovering the classical system from the joint spectrum of its quantum counterpart. However, there is one case where we can say more: for the simplest examples of semitoric integrable systems, which we call systems *of Jaynes-Cummings type*. The characteristic of such a system is to display only one focus-focus singularity. One of the simplest yet most important models in classical and quantum mechanics was proposed by Jaynes and Cummings [26, 14] in 1963, and it is now known as the *Jaynes--Cummings model*.[2]The Jaynes-Cummings model is obtained by coupling a spin with a harmonic oscillator. In this way one obtains a physical system with phase space $S^2 \times \mathbb{R}^2$ and Hamiltonian functions $J := \frac{u^2+v^2}{2} + z$, $H = \frac{ux+vy}{2}$, where (x, y, z) denotes the point in the 2-sphere $S^2 \subset \mathbb{R}^3$ and (u, v) denote points in $\mathbb{R}^2$ (J is the momentum map for the combined rotational S^1-actions about the origin in $\mathbb{R}^2$ and about the vertical axes on S^2). Recently the second and third authors described in full the semiclassical spectral theory of this system [32].

Definition 1.2. A classical integrable system $F := (J, H)\colon M \to \mathbb{R}^2$ on a symplectic 4-manifold (M, ω) is of *Jaynes-Cummings type* if:

(a) F is a semitoric system;
(b) F has one, and only one, singularity of focus-focus type.

[2]The Jaynes-Cummings model was initially introduced to describe the interaction between an atom prepared in a mixed state with a quantum particle in an optical cavity. It was found to apply to many physical situations (quantum chemistry, quantum optics, quantum information theory, etc.) because it represents the easiest way to have a finite dimensional state (like a spin) interact with an oscillator.

A *quantum integrable system* (P, Q) *of Jaynes-Cummings type* is given by two semiclassical commuting self-adjoint operators whose principal symbols form a classical integrable system of Jaynes-Cummings type.

The Jaynes-Cummings model is a particular example of a system of Jaynes-Cummings type. Jaynes-Cummings type systems form a large class of integrable Hamiltonian systems because the structure of the singularity in part (b) is extremely rich, and it is classified by a Taylor series $\sum_{i,j\in\mathbb{N}} a_{ij}X^iY^j$, according to [37] (two such singularities are symplectically equivalent if and only if each and everyone of the coefficients in the Taylor series coincide for both singularities). Moreover, by [37, 31] every such Taylor series can be realized as the Taylor series invariant of an integrable system (in fact, of *many* inequivalent such systems). Accordingly, the moduli space of Jaynes-Cummings type systems is, from the point of view of Hamiltonian dynamics, extremely rich. As a corollary of Theorem A, we solve the inverse spectral problem for quantum integrable systems of Jaynes-Cummings type. Indeed, as we explained above, the twisting index invariant is always trivial for such systems.

Theorem B. *Let (P, Q) be a quantum integrable system of Jaynes-Cummings type on M for which the Bohr-Sommerfeld rules hold. Then from the knowledge of* $\mathrm{JointSpec}(P, Q)+\mathcal{O}(\hbar^2)$*, one can recover the principal symbol $\sigma(P, Q)$ up to isomorphisms of semitoric integrable systems.*

This theorem gives the first global inverse spectral result that the authors are aware of for integrable Hamiltonian systems with focus-focus singularities (and hence no global action-angle variables). In the context of Hamiltonian toral actions (eg. toric integrable systems), all singularities are of elliptic type, which is strongly related to the dynamical and spectral rigidity of such systems [11]. We believe that allowing focus-focus singularities, which have a much larger moduli space, is an important step forward in the study of the inverse spectral problem for general integrable systems.

The problem treated in this paper belongs to a class of semiclassical inverse spectral questions which has attracted much attention in recent years, e.g. [21, 24, 25, 15, 29, 34, 39], which goes back to pioneer works of Bérard [1], Brüning-Heintze [3], Colin de Verdière [12, 13], Duistermaat-Guillemin [19], and Guillemin-Sternberg [22], in the 1970s/1980s, and are closely related to inverse problems that are not directly semiclassical but do use similar microlocal techniques for some integrable systems, as in [40] (see also [41] and references therein).

We conclude this section by a natural question. The following corollary of Theorem A directly follows from the symplectic classification [30] of semitoric systems:

Corollary 1.3. *Let (P, Q) and (P', Q') be quantum simple semitoric systems on M and M', respectively, for which the Bohr-Sommerfeld rules hold.*

If

$$\text{JointSpec}(P, Q) = \text{JointSpec}(P', Q') + \mathcal{O}(\hbar^2), \tag{1}$$

and if the twisting index invariants of $\sigma(P,Q)$ and $\sigma(P',Q')$ are equal, then $\sigma(P,Q)$ and $\sigma(P',Q')$ are isomorphic as semitoric integrable systems.

In view of this result, one question remains: can one obtain the twisting index invariant of a semitoric system from the data of the joint spectrum of the corresponding quantum system? A positive answer to this question would lead to the definition of a new quantum invariant which would be quite robust (since the twisting index between two focus-focus singularities is just an integer).

2. Preliminaries

Let (M, ω) be a smooth, connected 4-dimensional symplectic manifold.

2.1. Integrable systems.

An *integrable system* (J, H) on (M, ω) consists of two Poisson commuting functions $J, H \in \mathcal{C}^\infty(M; \mathbb{R})$ i.e. :

$$\{J, H\} := \omega(\mathcal{X}_J, \mathcal{X}_H) = 0,$$

whose differentials are almost everywhere linearly independent 1-forms. Here $\mathcal{X}_J, \mathcal{X}_H$ are the Hamiltonian vector fields induced by J, H, respectively, via the symplectic form ω: $\omega(\mathcal{X}_J, \cdot) = -\mathrm{d}J$, $\omega(\mathcal{X}_H, \cdot) = -\mathrm{d}H$. Moreover, the function $F = (J, H)$ will be assumed to be proper throughout this paper.

For instance, let $M_0 = \mathrm{T}^*\mathbb{T}^2$ be the cotangent bundle of the torus $\mathbb{T}^2$, equipped with canonical coordinates (x_1, x_2, ξ_1, ξ_2), where $x \in \mathbb{T}^2$ and $\xi \in \mathrm{T}^*_x\mathbb{T}^2$. The linear system

$$(J_0, H_0) := (\xi_1, \xi_2)$$

is integrable.

An *isomorphism* of integrable systems (J, H) on (M, ω) and (J', H') on (M', ω') is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi^*\omega' = \omega$ and

$$\varphi^*(J', H') = (f_1(J, H), f_2(J, H))$$

for some local diffeomorphism (f_1, f_2) of $\mathbb{R}^2$. This same definition of isomorphism extends to any open subsets $U \subset M$, $U' \subset M'$ (and this is the form in which we will use it later). Such an isomorphism will be called *semiglobal* if U, U' are respectively saturated by level sets $\{J = \text{const}_1, H = \text{const}_2\}$ and $\{J' = \text{const}'_1, H' = \text{const}'_2\}$.

If $F = (J, H)$ is an integrable system on (M, ω), consider a point $c \in \mathbb{R}^2$ that is a *regular value* of F, and such that the fiber $\Lambda_c = F^{-1}(c)$ is compact and connected. Then, by the action-angle theorem [18], a saturated neighborhood of the fiber is *isomorphic* in the previous sense to the above linear model on $M_0 = \mathrm{T}^*\mathbb{T}^2$. Therefore, all such regular fibers (called *Liouville tori*) are isomorphic in a neighborhood. The two Hamiltonians given in the corresponding action-angle coordinates by the variables (ξ_1, ξ_2) will be called *a basis of action variables.*

However, the situation changes drastically when the condition that c be regular is violated. For instance, it has been proved in [37] that, when c is a so-called *focus-focus* critical value, an infinite number of equations has to be satisfied in order for two systems to be semiglobally isomorphic near the critical fiber.

2.2. **Semitoric systems.** Semitoric systems (Definition 1.1) form a particular class of integrable systems admitting an S^1 symmetry. It is therefore natural to introduce a suitable notion of isomorphism for such systems, which mixes the general notion defined in the previous section with the more rigid one coming from Hamiltonian S^1-manifolds.

Definition 2.1. The semitoric systems $(M_1, \omega_1, F_1 := (J_1, H_1))$ and $(M_2, \omega_2, F_2 := (J_2, H_2))$ are *isomorphic* if there exists a symplectomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*(J_2, H_2) = (J_1, h(J_1, H_1))$ for a smooth h such that $\frac{\partial h}{\partial H_1} > 0$.

2.3. **The period lattice.** Let $F = (J, H)$ be an integrable system on a 4-dimensional symplectic manifold. For any regular value c of F, the set of points $(t, u) \in \mathbb{R}^2$ such that the vector field $t\mathcal{X}_J + u\mathcal{X}_H$ has a 2π-periodic flow on Λ_c is a sublattice of $\mathbb{R}^2$ called the *period lattice* [18]. When c varies in the set of regular values of F, the collection of the period lattices is a Lagrangian subbundle of $T^*\mathbb{R}^2$, called the period bundle.

Coming back to our case where $F = (J, H)$ is a semitoric system, there is a natural way to construct a basis of this lattice. Firstly, since J generates a S^1-action, $(1, 0)$ belongs to the period lattice. Secondly, define two real numbers $\tau_1(c), \tau_2(c)$ as follows: choose a point $m \in \Lambda_c$, and define $\tau_2(c) > 0$ as the time of first return for the Hamiltonian flow associated with H to the trajectory of the Hamiltonian flow of J passing through m. Let $\tau_1(c) \in [0, 2\pi)$ be the time that it takes to come back to m following the flow of $\mathcal{X}_J$. Because of the commutativity of the Hamiltonian flows of J and H, the values of $\tau_1(c), \tau_2(c)$ do not depend on the choice of the starting point $m \in \Lambda_c$. The vector field $\tau_1(c)\mathcal{X}_J + \tau_2(c)\mathcal{X}_H$ defines a 1-periodic flow; hence, if we define

$$\zeta_1(c) = \frac{\tau_1(c)}{2\pi}, \quad \zeta_2(c) = \frac{\tau_2(c)}{2\pi}, \tag{2}$$

then $(\zeta_1(c), \zeta_2(c))$ and $(1, 0)$ form a basis of the period lattice.

2.4. **Semiclassical operators.** Let $I \subset (0, 1]$ be any set which accumulates at 0. If $\mathcal{H}$ is a complex Hilbert space, we denote by $\mathcal{L}(\mathcal{H})$ the set of linear (possibly unbounded) self-adjoint operators on $\mathcal{H}$ with a dense domain.

A space Ψ of *semiclassical operators* is a subspace of $\prod_{\hbar \in I} \mathcal{L}(\mathcal{H}_\hbar)$, containing the identity, and equipped with a weakly positive principal symbol map, which is an $\mathbb{R}$-linear map

$$\sigma \colon \Psi \to \mathcal{C}^\infty(M; \mathbb{R}),$$

with the following properties:

(1) $\sigma(I) = 1$; (*normalization*)
(2) if $P \in \Psi$ then $P^2 \in \Psi$; (*square*)
(3) if P, Q are in Ψ and if the composition $P \circ Q$ is well defined and is in Ψ, then $\sigma(P \circ Q) = \sigma(P)\sigma(Q)$; (*product formula*)
(4) if $\sigma(P) \geqslant 0$, then there exists a function $\hbar \mapsto \epsilon(\hbar)$ tending to zero as $\hbar \to 0$, such that $P \geqslant -\epsilon(\hbar)$, for all $\hbar \in I$. (*weak positivity*)

If $P = (P_\hbar)_{\hbar \in I} \in \Psi$, the image $\sigma(P)$ is called the *principal symbol of* P.

There are two major examples of such semiclassical operators. One is given by semiclassical pseudodifferential operators, as described for instance in [17] or [42], when the symbols are assumed to be uniformly bounded, together with all their derivatives. The boundedness assumption is needed for axiom (4); we could extend the validity to much larger classes by considering the full symbol instead of the principal symbol, since by Gårding's inequality, the statement (4) for the Weyl symbol holds for very general classes of pseudodifferential operators. However, this would add technicalities which are not really necessary, because in many cases, when studying the spectrum of an elliptic operator below a certain threshold, one is able to truncate the symbol in order to make it bounded, without changing the spectrum modulo $\mathcal{O}(\hbar^\infty)$ (see for instance [17, Chapter 10]).

The second category of semiclassical operators is less well known, but developing very fast: semiclassical (or Berezin) - Toeplitz operators, as described in [4, 6, 7, 8, 9, 27, 35] following the pioneer work [2].

2.5. **Semiclassical spectrum.** Recall that when A and B are unbounded self-adjoint operators, they are said to commute when their projector-valued spectral measures commute.

If $P = (P_\hbar)_{\hbar \in I}$ and $Q = (Q_\hbar)_{\hbar \in I}$ are semiclassical operators on $(\mathcal{H}_\hbar)_{\hbar \in I}$, in the sense of Section 2.4, we say that they commute if for each $\hbar \in I$ the operators $P_\hbar$ and $Q_\hbar$ commute.

If P and Q commute, we may define for fixed $\hbar$, the *joint spectrum* of $(P_\hbar, Q_\hbar)$ to be the support of the joint spectral measure. It is denoted by $\mathrm{JointSpec}(P_\hbar,\, Q_\hbar)$. If $\mathcal{H}_\hbar$ is finite dimensional (or, more generally, when the joint spectrum is discrete), then

$$\mathrm{JointSpec}(P_\hbar,\, Q_\hbar) = \Big\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \exists v \neq 0,\ P_\hbar v = \lambda_1 v,\ Q_\hbar v = \lambda_2 v \Big\}.$$

The *joint spectrum* of P, Q is the collection of all joint spectra of $(P_\hbar, Q_\hbar)$, $\hbar \in I$. It is denoted by $\mathrm{JointSpec}(P,\, Q)$. For convenience of the notation, we will also view the joint spectrum of P, Q as a set depending on $\hbar$.

2.6. **Joint spectrum and image of the joint principal symbol.**

Proposition 2.2. *If $F := (J, H): M \to \mathbb{R}^2$ is an integrable system on a 4-dimensional connected symplectic manifold and P, Q are commuting semiclassical operators with principal symbols $J, H: M \to \mathbb{R}$, then*

$$E \notin F(M) \Rightarrow \quad \exists \varepsilon > 0 \quad \exists \hbar_0 \in I \quad \forall \hbar \leqslant \hbar_0 \in I, \\ \mathrm{JointSpec}(P_\hbar, Q_\hbar) \cap B(E, \varepsilon) = \emptyset.$$

This proposition is well-known for pseudodifferential and Toeplitz operators; it is interesting to notice that, in fact, it directly follows from the axioms we chose for semiclassical operators in Section 2.4.

Proof. If $E = (E_1, E_2)$ does not belong to $F(M)$, then the function

$$f = (J - E_1)^2 + (H - E_2)^2$$

never vanishes. Thus, by the normalization, the product rule and the weak positivity of the principal symbol (items 1, 3 and 4 in section 2.4), we have

$$(P - E_1)^2 + (Q - E_2)^2 \geqslant C, \tag{3}$$

for some constant $C > 0$, when $\hbar$ is small enough. If fact, since $F(M)$ is closed (because F is proper), the same holds uniformly when E varies is a small ball. Let $\Pi_Q(d\lambda)$ and $\Pi_P(d\mu)$ be the spectral measures of P and Q respectively (now $\hbar$ is fixed). We have

$$(P - E_1)^2 + (Q - E_2)^2 = \int (\lambda - E_1)^2 \Pi_P(\mathrm{d}\lambda) + \int (\mu - E_2)^2 \Pi_Q(\mathrm{d}\mu).$$

Suppose that (E_1, E_2) belongs to the joint spectrum of (P, Q). Then for each $n \geqslant 0$ one can find a vector u_n of norm 1 such that

$$u_n \in \mathrm{Ran}(\Pi_P([E_1 - \tfrac{1}{n}, E_1 + \tfrac{1}{n}])) \cap \mathrm{Ran}(\Pi_Q([E_2 - \tfrac{1}{n}, E_2 + \tfrac{1}{n}])).$$

Then

$$\left|\langle (P - E_1)^2 u_n, u_n\rangle\right| = \left| \int_{[E_1 - \frac{1}{n}, E_1 + \frac{1}{n}]} (\lambda - E_1)^2 \langle \Pi_P(\mathrm{d}\lambda) u_n, u_n\rangle \right| \\ \leqslant \frac{1}{n^2} \int |\langle \Pi_P(\mathrm{d}\lambda) u_n, u_n\rangle| \leqslant \frac{1}{n^2}.$$

Similarly, $\left|\langle (Q - E_2)^2 u_n, u_n\rangle\right| \leqslant \frac{1}{n^2}$. Letting $n \to \infty$, we contradict (3). Thus $E \notin$ JointSpec(P,Q), which proves the proposition. ∎

2.7. **Bohr-Sommerfeld rules.** Recall that the *Hausdorff distance* between two bounded subsets A and B of $\mathbb{R}^2$ is

$$\mathrm{d}_H(A, B) := \inf\{\epsilon > 0 \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\},$$

where for any subset X of $\mathbb{R}^2$, the set X_ϵ is

$$X_\epsilon := \bigcup_{x \in X} \{m \in \mathbb{R}^2 \mid \|x - m\| \leqslant \epsilon\}.$$

If $(A_\hbar)_{\hbar\in I}$ and $(B_\hbar)_{\hbar\in I}$ are sequences of uniformly bounded subsets of $\mathbb{R}^2$, we say that

$$A_\hbar = B_\hbar + \mathcal{O}(\hbar^N)$$

if there exists a constant $C > 0$ such that

$$\mathrm{d}_H(A_\hbar,\, B_\hbar) \leqslant C\hbar^N$$

for all $\hbar \in I$. If A or B are not uniformly bounded, we shall say that $A_\hbar = B_\hbar + \mathcal{O}(\hbar^N)$ *on a ball* D if there exists a sequence of sets $D_\hbar$, all diffeomorphic to D, such that $D_\hbar = D + \mathcal{O}(\hbar^N)$ and

$$A_\hbar \cap D = B_\hbar \cap D_\hbar + \mathcal{O}(\hbar^N).$$

Definition 2.3. Let $F := (J, H)\colon M \to \mathbb{R}^2$ be an integrable system on a 4-dimensional connected symplectic manifold, with connected regular fibers. Let P and Q be commuting semiclassical operators with principal symbols $J, H\colon M \to \mathbb{R}$. We say that $\mathrm{JointSpec}(P,\, Q)$ *satisfies the Bohr-Sommerfeld rules* if for every regular value c of F there exists a small ball $\mathrm{B}(c, \epsilon_c)$ centered at c, such that,

$$\mathrm{JointSpec}(P,\, Q) = g_\hbar(2\pi\hbar\mathbb{Z}^2 \cap D) + \mathcal{O}(\hbar^2) \quad \text{on } \mathrm{B}(c, \epsilon_c), \tag{4}$$

with

$$g_\hbar = g_0 + \hbar g_1,$$

where g_0, g_1 are smooth maps defined on a bounded open set $D \subset \mathbb{R}^2$, g_0 is a diffeomorphism into its image, $c \in g_0(D)$ and the components of $g_0^{-1} = (\mathcal{A}_1,\, \mathcal{A}_2)$ are such that $(\mathcal{A}_1 \circ F, \mathcal{A}_2 \circ F)$ form a basis of action variables, see Section 2.1

In this situation, if $\hbar$ is small enough, then $g_\hbar$ is a diffeomorphism into its image, and its inverse admits an asymptotic expansion in non-negative powers of $\hbar$ for the $\mathcal{C}^\infty$ topology; we call $(g_\hbar)^{-1}$ an *affine chart* for $\mathrm{JointSpec}(P,\, Q)$.

Bohr-Sommerfeld rules are known to hold for integrable systems of pseudodifferential operators (thus M is a cotangent bundle) [5, 36], or for integrable systems of Toeplitz operators on prequantizable compact symplectic manifolds [10]. It would be interesting to formalize the minimal semiclassical category where Bohr-Sommerfeld rules are valid.

Note that action variables are not unique. Thus, if $(g_\hbar)^{-1}$ is an affine chart for $\mathrm{JointSpec}(P, Q)$ and $B \in \mathrm{GL}(2, \mathbb{Z})$ then $B \circ (g_\hbar)^{-1}$ is again an affine chart. In view of the discussion in Section 2.3, this remark implies the following proposition.

Proposition 2.4. *If F is a semitoric system, then in Definition 2.3, we can assume that $\mathcal{A}_1(c_1, c_2) = c_1$. Therefore, there exists an integer k such that the actions $\mathcal{A}_1, \mathcal{A}_2$ satisfy:*

$$\mathrm{d}\mathcal{A}_1 = \mathrm{d}c_1, \quad \mathrm{d}\mathcal{A}_2 = (\zeta_1 + k)\mathrm{d}c_1 + \zeta_2 \mathrm{d}c_2, \tag{5}$$

where ζ_1, ζ_2 are defined in (2).

3. Main Result

We state in this section a more precise version of our main result, Theorem A, which explicitly indicates what we mean by "from the knowledge of the semiclassical joint spectrum $\mathrm{JointSpec}(P, Q) + \mathcal{O}(\hbar^2)$, one can recover the four following invariants of the associated classical semitoric system". Let $\mathcal{M}_{\mathrm{ST}}$ be the set of semitoric systems (*i.e.* triples (M, ω, F) satisfying Definition 1.1) modulo isomorphisms (as defined in Definition 2.1).

For each of the four invariants (1), (2), (3), or (4) mentioned in Section 1, we may define a map $\mathcal{I}_j$, $j = 1, 2, 3, 4$, from $\mathcal{M}_{\mathrm{ST}}$ with value in the appropriate space corresponding to the invariant (we refer to [30] for these spaces; here we simply denote them by $\mathcal{B}_j$, $j = 1, 2, 3, 4$, as their precise definition is not important for our purpose).

Let $\mathcal{Q}_{\mathrm{ST}}$ be the set of all quantum *simple* semitoric systems *for which the Bohr-Sommerfeld rules hold*, equipped with the natural arrow

$$\sigma : \mathcal{Q}_{\mathrm{ST}} \to \mathcal{M}_{\mathrm{ST}}$$

induced by the principal symbol map. We introduce now the *joint spectrum map*

$$\begin{aligned} \mathrm{JS} : \mathcal{Q}_{\mathrm{ST}} &\longrightarrow \mathcal{P}(\mathbb{R}^2)^I \\ (P, Q) &\longmapsto \mathrm{JointSpec}(P, Q), \end{aligned}$$

where we recall that I is the set where the semiclassical parameter $\hbar$ varies. Let us denote by $\mathcal{P}_2$ the set of equivalence classes of $\hbar$-dependent subsets of $\mathbb{R}^2$ with respect to the equality modulo $\mathcal{O}(\hbar^2)$ on every ball, and $\overline{\mathrm{JS}} : \mathcal{Q}_{\mathrm{ST}} \to \mathcal{P}_2$ the quotient map of JS. Let $\Sigma \subset \mathcal{P}_2$ be the range of $\overline{\mathrm{JS}}$, *i.e.* the subset of all joint spectra of semitoric systems, modulo $\mathcal{O}(\hbar^2)$. Then Theorem A can be rephrased as follows:

Theorem 3.1. *For each $j = 1, 2, 3, 4$, there exists a map $\hat{\mathcal{I}}_j : \Sigma \to \mathcal{B}_j$ such that the following diagram*

$$\begin{array}{ccc} \mathcal{Q}_{\mathrm{ST}} & \xrightarrow{\overline{\mathrm{JS}}} & \Sigma \\ \downarrow{\scriptstyle\sigma} & & \Downarrow{\scriptstyle\hat{\mathcal{I}}_j} \\ \mathcal{M}_{\mathrm{ST}} & \xrightarrow{\mathcal{I}_j} & \mathcal{B}_j \end{array}$$

is commutative.

Corollary 3.2. *If two quantum simple semitoric systems for which the Bohr-Sommerfeld rules hold have the same joint spectrum modulo $\mathcal{O}(\hbar^2)$, then the underlying classical systems have the same set of invariants* (1)*,* (2)*,* (3)*,* (4)*. In particular, if two quantum Jaynes-Cummings type systems for which the Bohr-Sommerfeld rules hold have the same joint spectrum modulo $\mathcal{O}(\hbar^2)$, then the underlying classical systems are isomorphic.*

4. Proof of Theorem 3.1

Let P, Q be a quantum simple semitoric system with joint principal symbol $F = (J, H)$. Remember that we want to prove that the knowledge of the joint spectrum of P, Q modulo $\mathcal{O}(\hbar^2)$ allows to recover invariants (1) to (4). For the sake of clarity, we divide the proof into five steps.

Step 1. We recover the image $F(M)$ thanks to Proposition 2.2. Indeed, choose a point $E = (E_1, E_2)$ in $\mathbb{R}^2$; assume that the following condition holds:

(C) for every $\varepsilon > 0$ and for every $\hbar_0 \in I$, there exists $\hbar \leqslant \hbar_0$ in I such that $\mathrm{JointSpec}(P_\hbar, Q_\hbar) \cap B(E, \varepsilon) \neq \emptyset$.

Then Proposition 2.2 implies that E belongs to $F(M)$. Conversely, assume $E \in B_r$, where B_r is the set of regular values of F. Because of the Bohr-Sommerfeld rules, there exists a small ball around E in $\mathbb{R}^2$ in which the joint spectrum is a deformation of the lattice $2\pi\hbar\mathbb{Z}^2$. Hence when $\hbar$ is small enough, this ball always contains some element of the joint spectrum (the number of joint eigenvalues grows like $\hbar^{-2}$), which says that Condition (C) holds. Let S be the set of $E \in \mathbb{R}^2$ for which (C) holds. We have

$$B_r \subset S \subset F(M).$$

But we know from [38, Proposition 2.9] that the closure of B_r equals $F(M)$. Therefore, $\overline{S} = F(M)$, which proves that the image $F(M)$ can be recovered from the joint spectrum.

Note that this step would also work with a weaker hypothesis than the Bohr-Sommerfeld rules. For instance, having a $\mathcal{C}_0^\infty$ functional calculus for the semiclassical operators, or being able to construct microlocal quasimodes (which is common in pseudodifferential or Toeplitz analysis) would be sufficient for recovering $F(M)$.

Step 2. In this step, we show how to recover the periods of the classical system at regular values from the knowledge of the joint spectrum. In order to do so, we adapt an argument from [39] for the resolution of a similar inverse problem in dimension 2. Although in our case we are working in dimension 4, which makes the study more difficult, the situation is also simpler by some aspects, because we know from [38, Theorem 3.4] that the regular fibers are connected.

Let c_0 be a regular value of F, and let B be a ball centered at c_0 in which the joint spectrum is described by the Bohr-Sommerfeld rules (4). Let D and $g_\hbar$ be as in the statement of the latter. We can assume that $g_\hbar$ is a diffeomorphism from $g_\hbar^{-1}(B)$ into B. We recall that

$$\mathrm{JointSpec}(P_\hbar,\, Q_\hbar) \cap B = g_\hbar(2\pi\hbar\mathbb{Z}^2 \cap D) \cap B_\hbar + \mathcal{O}(\hbar^2),$$

where $B = B_\hbar + \mathcal{O}(\hbar^2)$.

Now, let χ be a non-negative smooth function with compact support $K \subset B$, equal to 1 on a compact subset of B. We consider the spectral measure

$$D(\lambda, \hbar) = \sum_{c \in \mathrm{JointSpec}(P,Q) \cap B} \chi(c) \delta_c(\lambda)$$

where δ_c is the Dirac distribution at c. Let $\mathcal{F}_\hbar$ stand for the semiclassical Fourier transform, so that

$$\mathcal{F}_\hbar(f)(\xi) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \exp\left(-i\hbar^{-1}\langle x, \xi \rangle\right) f(x) \mathrm{d}x,$$

for smooth, compactly supported functions f, and introduce

$$Z(t,\hbar) = (2\pi\hbar)^2 \mathcal{F}_\hbar(D(\cdot,\hbar))(t) = \sum_{c \in \mathrm{JointSpec}(P,Q) \cap B} \chi(c) \exp\left(-i\hbar^{-1}\langle c, t \rangle\right).$$

Thanks to the Bohr-Sommerfeld conditions, we may estimate this quantity as

$$Z(t,\hbar) = \sum_{s \in 2\pi\hbar\mathbb{Z}^2 \cap D} \varphi_t(g_\hbar(s), \hbar) + \mathcal{O}(\hbar)$$

with

$$\varphi_t(u,\hbar) = \chi(u) \exp\left(-i\hbar^{-1}\langle u, t \rangle\right).$$

Because $\chi(g_\hbar(s)) = 0$ if $s \notin D$, this yields

$$Z(t,\hbar) = \sum_{\alpha \in \mathbb{Z}^2} \varphi_t(g_\hbar(2\pi\hbar\alpha), \hbar) + \mathcal{O}(\hbar).$$

By the Poisson summation formula, we thus obtain

$$Z(t,\hbar) = \sum_{\beta \in \mathbb{Z}^2} Z_\beta(t,\hbar) + \mathcal{O}(\hbar)$$

with

$$Z_\beta(t,\hbar) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \exp\left(-i\hbar^{-1}\left(\langle \beta, s \rangle + \langle g_\hbar(s), t \rangle\right)\right) \chi(g_\hbar(s)) \mathrm{d}s.$$

Since $g_\hbar$ is a diffeomorphism from $g_\hbar^{-1}(B)$ into B, we can use the change of variables $c = g_\hbar(s)$, $s = f_\hbar(c)$, which yields:

$$Z_\beta(t,\hbar) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \exp\left(-i\hbar^{-1}\left(\langle \beta, f_\hbar(c) \rangle + \langle c, t \rangle\right)\right) \chi(c) |\det J_{f_\hbar}(c)| \mathrm{d}c,$$

which means that $Z_\beta(t,\hbar) = \mathcal{F}_\hbar(\psi_\beta)(t)$ where

$$\psi_\beta(c) = \exp\left(-i\hbar^{-1}\langle \beta, f_\hbar(c) \rangle\right) \chi(c) |\det J_{f_\hbar}(c)|$$

is a WKB function with phase

$$\theta_\beta(c) = -\langle \beta, f_0(c) \rangle = -\beta_1 \mathcal{A}_1(c) - \beta_2 \mathcal{A}_2(c).$$

Since by equation (5)

$$\nabla \theta_\beta(c) = -\begin{pmatrix} \beta_1 + \beta_2(\zeta_1(c) + k) \\ \beta_2 \zeta_2(c) \end{pmatrix},$$

the associated Lagrangian submanifold is the set

$$\left\{(c,t)\in\mathbb{R}^4 \mid \quad (t_1,t_2) = -(\beta_1+\beta_2(\zeta_1(c)+k),\beta_2\zeta_2(c))\right\}. \tag{6}$$

One can easily check that this submanifold is indeed Lagragian: the 1-form $\nu = (\beta_1+\beta_2(\zeta_1(c)+k))\,\mathrm{d}c_1 + \beta_2\zeta_2(c)\mathrm{d}c_2$ is closed, as

$$\nu = d(\beta_1\mathcal{A}_1+\beta_2\mathcal{A}_2).$$

Since the Jacobian $|\det J_{f_\hbar}(c)|$ does not vanish in the support K of χ, this implies that the semiclassical wavefront set of $Z_\beta(\cdot,\hbar)$ is

$$\begin{aligned}\mathrm{WF}_\hbar(Z_\beta(\cdot,\hbar)) &= \left\{(c,t)\in\mathbb{R}^4 \mid (t_1,t_2) = -(\beta_1+\beta_2(\zeta_1(c)+k),\beta_2\zeta_2(c)),\ c\in K\right\}\\ &= \mathcal{L}_\beta(K)\end{aligned}$$

To obtain a similar result on $Z(\cdot,\hbar)$, we still need to sum over $\beta\in\mathbb{Z}^2$. Let $t^0=(t^0_1,t^0_2)\in(\mathbb{R}^*)^2$, $\varepsilon>0$, and let $\rho\in\mathcal{C}^\infty_0(B(t_0,\varepsilon))$.

Lemma 4.1. *If there exists a solution (c,t) of (6) with t in the support of ρ, then β is such that*

$$\max(|\beta_1|,|\beta_2|)\leqslant M$$

where M is defined as

$$M = \frac{\varepsilon+\|t^0\|}{\min_K|\zeta_2|}\max\left(1,\min_K|\zeta_2|+|k|+\max_K|\zeta_1|\right).$$

Proof. For such a solution, we have $\|t\|\leqslant\varepsilon+\|t^0\|$, thus

$$(\beta_1+\beta_2(\zeta_1(c)+k))^2+\beta_2^2\zeta_2(c)^2\leqslant(\varepsilon+\|t^0\|)^2,$$

which implies that

$$|\beta_2|\leqslant\frac{\varepsilon+\|t^0\|}{\min_K|\zeta_2|}. \tag{7}$$

Since we also have

$$|\beta_1+\beta_2(\zeta_1(c)+k)|\leqslant\varepsilon+\|t^0\|,$$

we deduce from the previous inequality that

$$|\beta_1|\leqslant(\varepsilon+\|t^0\|)\left(1+\frac{|k|+\max_K|\zeta_1|}{\min_K|\zeta_2|}\right), \tag{8}$$

which proves the result. ∎

Using the proof of the non-stationary phase lemma, we can write for such a β and any $N\geqslant 1$

$$(2\pi\hbar)^2Z_\beta(t,\hbar)$$

equals

$$\left(\frac{i\hbar}{\max(|\beta_1|,|\beta_2|)}\right)^N\int_{\mathbb{R}^2}\exp\left(-i\hbar^{-1}(\langle\beta,f_\hbar(c)\rangle+\langle c,t\rangle)\right)L^N(a(c,\hbar))\mathrm{d}c,$$

where $a(\cdot,\hbar)$ is compactly supported and admits an asymptotic expansion in non-negative powers of $\hbar$ in the $\mathcal{C}^\infty$ topology, and L is the differential operator defined as

$$Lu = \nabla\left(\max(|\beta_1|,|\beta_2|)\frac{u}{|V|^2}V\right)$$

with

$$V(c) = -\begin{pmatrix} t_1+\beta_1+\beta_2(\zeta_1(c)+k) \\ t_2+\beta_2\zeta_2(c)\end{pmatrix}.$$

Introduce the function $b = (\max(|\beta_1|,|\beta_2|)/|V|^2)V$; one has

$$|b(c)| = \left(\left(\frac{t_1+\beta_1+\beta_2(\zeta_1(c)+k)}{\max(|\beta_1|,|\beta_2|)}\right)^2 + \left(\frac{t_2+\beta_2\zeta_2(c)}{\max(|\beta_1|,|\beta_2|)}\right)^2\right)^{-1/2}.$$

Then b is uniformly bounded on K for β such that $\max(|\beta_1|,|\beta_2|) > M$ and, for every $\ell\in\mathbb{N}^2$, there exists a constant C_ℓ such that

$$|\partial_{c^\ell} b| = |\partial_{{c_1}^{\ell_1}}\partial_{{c_2}^{\ell_2}} b| \leqslant C_\ell$$

on K. Consequently, there exists a constant $\tilde{C}_N > 0$ such that

$$|\rho(t)Z_\beta(t,\hbar)| \leqslant \tilde{C}_N\left(\frac{\hbar}{\max(|\beta_1|,|\beta_2|)}\right)^N$$

when $\max(|\beta_1|,|\beta_2|) > M$. Therefore, for $N \geqslant 4$, we have

$$\sum_{\substack{\beta\in\mathbb{Z}^2\\ \max(|\beta_1|,|\beta_2|)>M}} |\rho(t)Z_\beta(t,\hbar)| \leqslant \hat{C}_N\hbar^N$$

for some constant $\hat{C}_N > 0$. This shows that only a finite number of terms contribute to $\rho(t)Z(t,\hbar)$ up to $\mathcal{O}(\hbar^\infty)$, hence

$$\begin{aligned}\mathrm{WF}_\hbar(\rho Z(\cdot,\hbar)) \subset \big\{&(c_1,c_2,-\beta_1-\beta_2(\zeta_1(c)+k),-\beta_2\zeta_2(c))\in\mathbb{R}^4 \mid \\ &(c_1,c_2)\in K, \max(|\beta_1|,|\beta_2|) > M\}\end{aligned}$$

and finally

$$\begin{aligned}\mathrm{WF}_\hbar(Z(\cdot,\hbar)) &= \big\{(c_1,c_2,-\beta_1-\beta_2(\zeta_1(c)+k),-\beta_2\zeta_2(c))\in\mathbb{R}^4 \mid \\ &\qquad (c_1,c_2)\in K, \beta\in\mathbb{Z}^2\big\} \\ &= \mathcal{L}(K),\end{aligned}$$

which is exactly the restriction of the period bundle over K (see Section 2.3).

The last part of this step is to explain how one can extract the functions (τ_1,τ_2) from the data of $\mathcal{L}(K) = \bigcup_{\beta\in\mathbb{Z}^2}\mathcal{L}_\beta(K)$, which is the disjoint union of smooth surfaces in $\mathbb{R}^4$. Endow $\mathbb{R}^4$ with the coordinates (x_1,x_2,x_3,x_4), and introduce the plane $\Pi = \{x\in\mathbb{R}^4 \mid x_1 = c_1^0,\ x_2 = c_2^0\}$, for a fixed $c^0\in K$. Then the set

$$\mathcal{E} = \mathcal{L}(K)\cap\Pi = \{(c_1^0, c_2^0, -\beta_1-\beta_2(\zeta_1(c^0)+k), -\beta_2\zeta_2(c^0)) \mid \beta\in\mathbb{Z}^2\}$$

is discrete, and the set $\{x_4 \mid x \in \mathcal{E}\} \cap \mathbb{R}_+^*$ is bounded from below. Let $\mathcal{F}$ be the set of points in $\mathcal{E}$ with minimal coordinate x_4; then

$$\mathcal{F} = \{(c_1^0, c_2^0, \zeta_1(c^0) + k - \beta_1, \zeta_2(c^0)) \mid \beta_1 \in \mathbb{Z}\}.$$

Again, the set $\{x_3 \mid x \in \mathcal{F}\} \cap \mathbb{R}_+^*$ is bounded from below, and the point of this set with minimal coordinate x_3 is $(c_1^0, c_2^0, \zeta_1(c^0), \zeta_2(c^0))$. The connected component of this point in $\mathcal{L}(K)$ is the graph of the function

$$c \in K \mapsto (\zeta_1(c), \zeta_2(c)) = \frac{1}{2\pi}(\tau_1(c), \tau_2(c)).$$

Step 3. Let us now explain how to recover the position of the focus-focus values from the joint spectrum. Thanks to step 1, we know $F(M)$. By [38, Theorem 3.4], we know that the boundary of $F(M)$ consists of the singularities of elliptic-elliptic and transversally elliptic type, and that the only singular values in the interior of $F(M)$ are the images of the focus-focus singularities. Let A be any point lying on the boundary $\partial F(M)$. Let $C_1, \ldots, C_{m_f}$ be the images of the focus-focus points in $F(M)$, labelled in such a way that

$$J(m_1) < J(m_2) < \ldots < J(m_{m_f}),$$

where for i in $\{1, \ldots, m_f\}$, m_i is the only focus-focus-point in $F^{-1}(C_i)$. Consider the distance $d = \min_{1 \leqslant i \leqslant m_f} \|A - C_i\|$ and let $j \in \{1, \ldots, m_f\}$ be such that $d = \|A - C_j\|$; since C_j lies in the interior of $F(M)$, we have that $d > 0$. Let B_r be the set of regular values of F; for every ε in $(0, d]$, the intersection

$$X_\varepsilon = B(A, \varepsilon) \cap F(\overset{\circ}{M})$$

of the ball of radius ε centered at A with the interior of $F(M)$ is contained in B_r. Thus, from step 2, we can compute the function $\tau_{2|X_\varepsilon}$ from the joint spectrum. It follows from [37, proposition 3.1] that τ_2 has a logarithmic behavior near C_j. Hence, if $\tau_{2|X_\varepsilon}$ can be extended to a continuous function on $\bar{B}(A, \varepsilon) \cap F(\overset{\circ}{M})$, then necessarily $\varepsilon < d$. This allows to find d; the point C_j belongs to the circle $\mathcal{C}$ of radius d centered at A. Furthermore, the only points in $\mathcal{C} \cap F(\overset{\circ}{M})$ where τ_2 admits a logarithmic singularity are some of the C_i (including C_j), that we recover this way.

We obtain the positions of the other focus-focus values by applying this method recursively. For instance, we recover another point C_k by considering circles of growing radius centered at C_j, and so on (let us recall that m_f is finite).

Step 4. Since we now know precisely the position of the focus-focus values, [33, Theorem 3.3] implies that the Taylor series invariant associated with each focus-focus singularity can be recovered from the joint spectrum.

Step 5. In this step, we prove that from the data of the joint spectrum, one can deduce the polygonal invariant introduced in [38].

Recall that a map $U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ is *integral affine* on U if it is of the form $x \in U \mapsto Ax + b$, where $A \in \mathrm{GL}(n, \mathbb{Z})$ and $b \in \mathbb{R}^n$. An *integral affine structure* on a smooth n-dimensional manifold is the data of an atlas (U_i, φ_i) such that for all i, the transition function $\varphi_i \circ \varphi_j^{-1}$ is integral affine.

As a consequence of the action-angle theorem, the integrable system (J, H) induces an integral affine structure on the set B_r of regular values of F. The charts are action variables, that is maps $\varphi : U \to \mathbb{R}^2$ where U is a small open subset of B_r and $\varphi \circ F$ generates a $\mathbb{T}^2$-action.

Let $\varepsilon_i \in \{-1, 1\}$ and let $\ell_i^{\varepsilon_i}$ be the vertical segment starting at the focus-focus value C_i, going upwards (respectively downwards) if $\varepsilon_i = 1$ (respectively $\varepsilon_i = -1$), and ending at the boundary of $F(M)$. Set $\ell^{\vec{\varepsilon}} = \bigcup_i \ell_i^{\varepsilon_i}$.

Theorem 4.2 ([38, Theorem 3.8])**.** *For $\vec{\varepsilon} \in \{-1, 1\}^{m_f}$, there exists a homeomorphism Φ from $B = F(M)$ to $\Delta = \Phi(B) \subset \mathbb{R}^2$ such that:*

(1) $\Phi_{|B\setminus\ell^{\vec{\varepsilon}}}$ *is a diffeomorphism into its image,*
(2) $\Phi_{|B_r\setminus\ell^{\vec{\varepsilon}}}$ *is affine: it sends the integral affine structure of B_r to the standard integral affine structure of $\mathbb{R}^2$,*
(3) Φ *preserves J: $\Phi(x, y) = (x, \Phi_2(x, y))$,*
(4) $\Phi_{|B_r\setminus\ell^{\vec{\varepsilon}}}$ *extends to a smooth multi-valued map from B_r to $\mathbb{R}^2$ and for any $i \in \{1, \dots, m_f\}$ and any $c \in \mathring{\ell}_i$, then*

$$\lim_{\substack{(x,y)\to c \\ x<x_i}} \mathrm{d}\Phi(x, y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \lim_{\substack{(x,y)\to c \\ x>x_i}} \mathrm{d}\Phi(x, y),$$

(5) Δ *is a rational convex polygon.*

The polygon Δ is the sought invariant; in fact, the real invariant is a family of such polygons, more precisely the set of all such Δ for all possible choices of $\vec{\varepsilon} \in \{-1, 1\}^{m_f}$ and all their images by linear maps leaving the vertical direction invariant. We refer the reader to [30, Section 4.3] for more precise statements.

Proposition 4.3. *Given any $\vec{\varepsilon} \in \{-1, 1\}^{m_f}$, the corresponding polygon $\Delta = \Delta_{\vec{\varepsilon}}$ is determined by the integral affine structure of B_r.*

Proof. Once a starting point $c_0 \in B_r$ is chosen (which, by convention, is taken to be on the left of the first focus-focus critical value, when these values are ordered by non-decreasing abscissae), the affine map $\Phi_{|B_r\setminus\ell^{\vec{\varepsilon}}}$ is uniquely determined by the affine structure. Indeed, the set $B_r \setminus \ell^{\vec{\varepsilon}}$ is simply connected and Φ is the developing map of the induced affine structure. The crucial observation is that it follows from the construction in [38] that the map $\Phi_{|B\setminus\ell^{\vec{\varepsilon}}}$ is the natural extension of $\Phi_{|B_r\setminus\ell^{\vec{\varepsilon}}}$ to the boundary of $B_r \setminus \ell^{\vec{\varepsilon}}$ away from the half-lines $\ell^{\vec{\varepsilon}}$, and this boundary consists of elliptic (or transversally elliptic) singularities. Precisely, the extension is obtained as follows. Near a

1-dimensional family of transversally elliptic singularities, we use the normal form due to Miranda and Zung [28]: if c_e is a transversally elliptic value, there exist a symplectomorphism φ from a neighborhood of $F^{-1}(c_e)$ in M to a neighborhood of $\{I = x = \xi = 0\}$ in $\mathrm{T}^*S^1 \times \mathbb{R}^2$ with coordinates $((\theta, I), (x, \xi))$ and standard symplectic form $\mathrm{d}I \wedge \mathrm{d}\theta + \mathrm{d}\xi \wedge \mathrm{d}x$, which sends the set $\{F = \text{constant}\}$ to the set $\{I = \text{constant}, x^2 + \xi^2 = \text{constant}\}$, and a smooth function g such that

$$(F \circ \varphi^{-1})(\theta, I, x, \xi) = g(I, x^2 + \xi^2)$$

where φ^{-1}, g are defined. Let $(\mathcal{A}_1, \mathcal{A}_2)$ be an affine chart for B_r inside this neighborhood where the normal form holds. Since $(I, (x^2 + \xi^2)/2)$ is also an affine chart, there exists a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\mathcal{A}_1(c) = \alpha I + \beta(x^2 + \xi^2)/2; \quad \mathcal{A}_2(c) = \gamma I + \delta(x^2 + \xi^2)/2 \tag{9}$$

where $m = \varphi^{-1}(\theta, I, x, \xi) \in M$ is a regular point for F and $c = F(m)$. Since $\alpha, \beta, \gamma, \delta$ are constant, Formula (9) naturally gives the required extension of $(\mathcal{A}_1, \mathcal{A}_2)$ (and hence Φ) to the boundary near c_e. Near an elliptic-elliptic point, we can apply the same reasoning, using Eliasson's normal form [20]. □

In view of the proposition, step 5 will be treated as soon as we show that the integral affine structure on B_r can be recovered from the joint spectrum JointSpec(P, Q) up to $\mathcal{O}(\hbar^2)$, which can be done as follows. From the previous steps, we can recover $F(M)$ and the position of the focus-focus values $C_i = (x_i, y_i)$, $1 \leqslant i \leqslant m_f$. Therefore, we know the set of regular values B_r, which is the interior of $F(M)$ minus the focus-focus critical values.

In a small ball $B_0 \subset B_r$, we can construct action variables $(\mathcal{A}_1, \mathcal{A}_2)$. Indeed, from step 2 we can recover the functions τ_1, τ_2 on B_0. Fixing a point $s \in B_0$, we can pick for every point $c \in B_0$ a smooth path $\gamma_c : [0, 1] \to B_0$ such that $\gamma_c(0) = s$, $\gamma_c(1) = c$ and compute

$$\mathcal{A}_1^{(0)}(c) = c_1; \quad \mathcal{A}_2^{(0)}(c) = \int_0^1 \left\langle \begin{pmatrix} \zeta_1(\gamma_c(t)) \\ \zeta_2(\gamma_c(t)) \end{pmatrix}, \gamma_c'(t) \right\rangle \mathrm{d}t,$$

where we recall that $\tau_i = 2\pi\zeta_i$ for $i = 1, 2$. In this way, we have constructed the integral affine structure of B_r from the joint spectrum. In remains to apply Proposition 4.3 to construct Φ, and hence Δ by Theorem 4.2.

Step 6. It only remains to prove that we can recover the height invariant associated with each focus-focus singularity from the joint spectrum. In order to do so, let $i \in \{1, \ldots, m_f\}$ and consider a sequence $(Y_n)_{n \in \mathbb{N}}$ of points of B such that every Y_n has the same abscissa as C_i and ordinate smaller than the one of C_i, and such that $Y_n \underset{n \to +\infty}{\longrightarrow} C_i$. We may assume that Y_0 lies on the boundary of B. Let Φ be a homeomorphism from B to Δ as in the previous step. Then the point

$$P = \lim_{n\to+\infty} \Phi(Y_n)$$

is well-defined and P is the image of the focus-focus value in the polygon Δ. The height invariant that we seek is the difference between the ordinate of P and the ordinate of $\Phi(Y_0)$.

Remark 4.4. Another way of obtaining the height invariant associated with C_i would have been to count the joint eigenvalues lying on a vertical line below C_i and use a Weyl law to relate this number to the volume of the set $J^{-1}(C_i) \cap \{H < H(m_i)\}$. Although it may seem more natural than our method, it is also more technical, and that is why we have chosen not to treat the problem this way.

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