Quantum propagation for Berezin-Toeplitz operators

Laurent Charles, Yohann Le Floch

November 18, 2025

Abstract

We describe the asymptotic behaviour of the quantum propagator generated by a Berezin-Toeplitz operator with real-valued principal symbol. We also give precise asymptotics for smoothed spectral projectors associated with the operator in the autonomous case; this leads us to introducing quantum states associated with immersed Lagrangian submanifolds. These descriptions involve geometric quantities of two origins, coming from lifts of the Hamiltonian flow to the prequantum bundle and the canonical bundle respectively. The latter are the main contribution of this article and are connected to the Maslov indices appearing in trace formulas, as will be explained in a forthcoming paper.

1 Introduction

In quantum mechanics, the evolution of a state Ψ_t under the influence of a Hamiltonian $\hat{H}$ can be described using Schrödinger's equation

$$i\hbar \frac{d}{dt}\Psi_t = \hat{H}\Psi_t.$$

Under suitable assumptions on $\hat{H}$, the solutions to this equation are of the form $\Psi_t = U_{\hbar,t}\Psi_0$ where $U_{\hbar,t}$ is an operator called the quantum propagator. This propagator is the quantum analogue of the Hamiltonian flow in classical Hamiltonian mechanics. This analogy can be studied rigorously by

2020 Mathematics Subject Classification. 53D50, 81Q20, 81S10.

Key words and phrases. Berezin-Toeplitz operators, Schrödinger equation, Lagrangian states, Geometric quantization, semiclassical limit.

investigating the so-called semiclassical limit $\hbar \to 0$ in which, if $\hat{H}$ quantizes the classical Hamiltonian H, $U_{\hbar,t}$ is expected to behave like the Hamiltonian flow of H. This statement has been given a precise meaning by studying the Schwartz kernel of $U_{\hbar,t}$ in different regimes of $\hbar$ and t, for semiclassical Schrödinger Hamiltonians $\hat{H} = -\hbar^2\Delta + V$ on $T^*\mathbb{R}^d$, and more generally for $\hbar$-pseudodifferential operators on $T^*\mathbb{R}^d$ or T^*X with X a compact Riemannian manifold; see Section 1.5 for a longer discussion and references.

Here we are interested in a different setting where the underlying phase space is a compact symplectic manifold; then the quantum states are sections of a power of some well-chosen line bundle, and this power is the relevant semiclassical parameter. This setting naturally appears in several problems from physics, such as the study of spin systems in the large spin limit, coherent states, and the quantum Hall effect, *cf.* for example [24, 15, 22]. The limit of large power of a suitable line bundle is also very important in complex geometry, see for instance [13, 26, 16, 1].

The aim of our work is to understand, in this context, the geometric invariants appearing in the asymptotic description of the quantum propagator (and its counterparts, smoothed spectral projectors) in the semiclassical limit. As can be seen from other results in the same direction [3, 28, 21], this is in fact non trivial and different authors have different, more or less explicit, ways to compute these invariants. Here we obtain expressions that are both completely natural and easily computable. This will be particularly important in forthcoming papers in which we revisit trace formulae: the explicit asymptotics that we obtain here will allow us to derive those in a direct way and with a precise control of the quantities they involve, in particular the Maslov-like indices contained in the subprincipal contributions.

1.1 Berezin-Toeplitz operators

Let M^n be a compact complex manifold endowed with two Hermitian holomorphic line bundles L and L'. We assume that L is positive, meaning that the curvature of its Chern connection is $\frac{1}{i}\omega$ with $\omega \in \Omega^2(M,\mathbb{R}) \cap \Omega^{(1,1)}(M)$ positive. For any positive integer k, let $\mathcal{H}_k$ be the space of holomorphic sections of $L^k \otimes L'$. The scalar product of sections of $L^k \otimes L'$ is defined as the integral of the pointwise scalar product against the Liouville volume form $\mu = \frac{\omega^n}{n!}$.

Given a function $f \in \mathcal{C}^\infty(M)$, the Berezin-Toeplitz operator $T_k(f)$ is the endomorphism of $\mathcal{H}_k$ such that

$$\langle T_k(f)u, v\rangle = \langle fu, v\rangle, \qquad \forall u, v \in \mathcal{H}_k.$$

We are interested in the semiclassical limit $k \to +\infty$ and the techniques we use allow to consider more general families $T := (T_k(f(\cdot,k)))$ where the multiplicator itself depends on k and has an expansion of the form $f(\cdot,k) = f_0 + k^{-1} f_1 + \dots$ with coefficients $f_\ell \in \mathcal{C}^\infty(M)$. We will also consider time-dependent sequences $f(\cdot,t,k)$ with an expansion with time-dependent coefficients.

We call the family T a *Toeplitz operator*, f_0 its *principal symbol* and $f_1 + \frac{1}{2}\Delta f_0$ its *subprincipal symbol*. Here Δ is the holomorphic Laplacian associated with the Kähler form ω, so $\Delta = \sum h^{ij} \partial_{z_i} \partial_{\overline{z}_j}$ when $\omega = i \sum h_{ij} dz_i \wedge d\overline{z}_j$. The reason why we introduce this subprincipal symbol is merely that it simplifies the subleading calculus.

Typically, if T and S are two Toeplitz operators with principal symbols f and g, then TS and $ik[T,S]$ are Toeplitz operators with respective principal symbols fg and the Poisson bracket $\{f,g\}$ with respect to ω [4, 2]. If now T and S have identically zero subprincipal symbols, then the subprincipal symbols of TS and $ik[T,S]$ are $\frac{1}{2i}\{f,g\}$ and $-\omega_1(X,Y)$ respectively [10], where X, Y are the Hamiltonian vector fields of f and g and ω_1 is the real two-form given by $\omega_1 = i(\Theta_{L'} - \frac{1}{2}\Theta_K)$, with $\Theta_{L'}$, Θ_K the curvatures of the Chern connections of L' and of the canonical bundle K.

1.2 The quantum propagator

It is a well-known result that the solution of the Schrödinger equation for a pseudo-differential operator is a Fourier integral operator associated with the Hamiltonian flow of its principal symbol. Our first result is the Toeplitz analogue of this fact. Consider a time-dependent Toeplitz operator $(T_{k,t})$ with principal symbol (H_t) and subprincipal symbol (H_t^{sub}). The quantum propagator generated by $T_{k,t}$ is the smooth path $(U_{k,t},\, t \in \mathbb{R})$ of (unitary in case $T_{k,t}$ is self-adjoint) maps of $\mathcal{H}_k$ satisfying the Schrödinger equation

$$(ik)^{-1} \frac{d}{dt} U_{k,t} + T_{k,t} U_{k,t} = 0, \qquad U_{k,0} = \mathrm{id}\,. \tag{1}$$

Our goal is to describe the Schwartz kernel of $U_{k,t}$, which by definition is

$$U_{k,t}(x,y) = \sum_{i=1}^{d_k} (U_{k,t}\psi_i)(x) \otimes \overline{\psi}_i(y) \in (L^k \otimes L')_x \otimes (\overline{L}^k \otimes \overline{L}')_y$$

where $d_k = \dim \mathcal{H}_k$ and (ψ_i) is any orthonormal basis of $\mathcal{H}_k$. In the sequel, we will view $U_{k,t}(x,y)$ as a map from $(L^k \otimes L')_y$ to $(L^k \otimes L')_x$ using the scalar product of $(L^k \otimes L')_y$.

As we will see, when the principal symbol (H_t) is real, this Schwartz kernel is concentrated on the graph of the Hamiltonian flow ϕ_t of H_t. Here the symplectic form ω is i times the curvature of L, and the Hamiltonian vector field X_t is such that

$$\omega(X_t, \cdot) + dH_t = 0. \tag{2}$$

To describe the asymptotic behavior of $U_{k,t}(\phi_t(x), x)$, we need to introduce two lifts of ϕ_t, the first one to L and the second one to the canonical bundle K of M. The relevant structures on L will be its metric and its connection, which is generally called the prequantum structure.

Parallel transport and prequantum lift

If $A \to M$ is a Hermitian line bundle endowed with a connection ∇, the *parallel transport* in A along a path $\gamma : [0, \tau] \to M$ is a unitary map

$$\mathcal{T}(A, \gamma) : A_{\gamma(0)} \to A_{\gamma(\tau)}$$

which can be computed as follows: if u is a frame of $\gamma^* A$, then $\mathcal{T}(A, \gamma)u(0) = \exp(i \int_\gamma \alpha) u(\tau)$, where $\alpha \in \Omega^1([0, \tau], \mathbb{R})$ is the connection one-form defined in terms of the covariant derivative of u by $\nabla u = -i\alpha \otimes u$. In particular we can lift by parallel transport the Hamiltonian flow ϕ_t. We set $\mathcal{T}_t^A(x) := \mathcal{T}(A, \phi_{[0,t]}(x)) : A_x \to A_{\phi_t(x)}$.

The *prequantum lift* of the Hamiltonian flow ϕ_t to L is defined by

$$\phi_t^L(x) = e^{\frac{1}{i} \int_0^t H_r(\phi_r(x))\, dr} \mathcal{T}_t^L(x). \tag{3}$$

This lift has an interest independently of Toeplitz operators: by the Kostant-Souriau theory, ϕ_t^L is the unique (up to a phase) lift of ϕ_t which preserves the metric and the connection of L. Furthermore, if x belongs to a contractible periodic trajectory with period T, so that we can define the action $\mathcal{A}(x, T) \in \mathbb{R}$, then $\phi_T^L(x) : L_x \to L_x$ is the multiplication by $\exp(i\mathcal{A}(x, T))$.

In our results, it is actually the k-th power $(\phi_t^L)^{\otimes k}$ that will appear, with some corrections involving the subprincipal data L' and H_t^{sub}, more precisely we will see

$$e^{\frac{1}{i} \int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr} \Big[\phi_t^L(x)\Big]^{\otimes k} \otimes \mathcal{T}_t^{L'}(x) : (L^k \otimes L')_x \to (L^k \otimes L')_{\phi_t(x)}. \tag{4}$$

So we merely replace L by $L^k \otimes L'$ and H_t by $kH_t + H_t^{\mathrm{sub}}$ in (3).

Holomorphic determinant and lift to the canonical bundle

The second ingredient we need is an invariant of the complex and symplectic structures together. If $g : S \to S'$ is a linear symplectomorphism between two $2n$-dimensional symplectic vector spaces both endowed with linear complex structures, we define an isomorphism $K(g) : K(S) \to K(S')$ between the canonical lines $K(S) = \wedge^{n,0} S^*$, $K(S') = \wedge^{n,0} (S')^*$ characterized by

$$K(g)(\alpha)(g_* u) = \alpha(u), \qquad \forall \alpha \in K(S), u \in \wedge^n S. \tag{5}$$

Equivalently, if E and E' are the $(1,0)$-spaces of S and S' respectively, we have decompositions

$$S \otimes \mathbb{C} = E \oplus \overline{E}, \qquad S' \otimes \mathbb{C} = E' \oplus \overline{E}', \qquad g \otimes \mathrm{id}_{\mathbb{C}} = \begin{pmatrix} g^{1,0} & * \\ * & * \end{pmatrix}$$

with $g^{1,0} : E \to E'$. Then $K(g)$ is the dual map of $\det g^{1,0} : \wedge^n E \to \wedge^n E'$ in the sense that $K(g)(\alpha)((\det g^{1,0})u) = \alpha(u)$ for any $\alpha \in K(S)$ and $u \in \wedge^n E$.

This holomorphic determinant has a nice structure in terms of the polar decomposition of linear symplectic maps. When $S = S' = \mathbb{R}^{2n}$ with its usual complex structure j, $g = g_1 g_2$ where g_1 and g_2 are both symplectic, g_1 commutes with j, and g_2 is symmetric positive definite. Then $K(g)$ is a complex number whose inverse is

$$\det_{\mathbb{C}} g^{1,0} = \left(\prod_{i=1}^{n} \frac{\lambda_i + \lambda_i^{-1}}{2} \right) \det_{\mathbb{C}} g_1 \tag{6}$$

with $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n < 1 < \lambda_n^{-1} \leqslant \ldots \leqslant \lambda_1^{-1}$ the eigenvalues of g_2, and $\det_{\mathbb{C}} g_1$ the determinant of g_1 viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}^n$. Indeed, $\det_{\mathbb{C}} g^{1,0} = \det_{\mathbb{C}} g_2^{1,0} \det_{\mathbb{C}} g_1$ and one readily computes $\det_{\mathbb{C}} g_2^{1,0}$ using the diagonalization of g_2 in an orthonormal basis $(e_1, \ldots, e_n, je_1, \ldots, je_n)$ with $g_2 e_\ell = \lambda_\ell e_\ell$ and $g_2 j e_\ell = \lambda_\ell^{-1} j e_\ell$. This formula generalizes for $(S, j) \neq (S', j')$, with now linear symplectic maps $g_1 : S \to S'$ and $g_2 : S \to S$, where $g_1 \circ j = j' \circ g_1$, and g_2 is positive definite for the Euclidean structure $\omega(\cdot, j \cdot)$ of S. The complex determinant of g_1 may be viewed as a map from $\det_{\mathbb{C}}(S, j)$ to $\det_{\mathbb{C}}(S', j')$ or equivalently from $\wedge^{\mathrm{top}} E$ to $\wedge^{\mathrm{top}} E'$.

This definition provides us with a lift $\mathcal{D}_t$ of the Hamiltonian flow ϕ_t to the canonical bundle $K = \wedge^{(n,0)} T^* M$, defined by

$$\mathcal{D}_t(x) = K(T_x \phi_t) : K_x \to K_{\phi_t(x)}. \tag{7}$$

We have another lift of ϕ_t to the canonical bundle which is the parallel transport $\mathcal{T}_t^K$. Define the complex number $\rho_t(x)$ such that $\mathcal{D}_t(x) = \rho_t(x) \mathcal{T}_t^K(x)$.

The result

Theorem 1.1. *Let* $(U_{k,t})$ *be the quantum propagator of a time-dependent Toeplitz operator* $(T_{k,t})$ *with real principal symbol. Then for any* $t \in \mathbb{R}$ *and* $x \in M$,

$$U_{k,t}(\phi_t(x), x) = \Big(\frac{k}{2\pi}\Big)^n \big[\rho_t(x)\big]^{\frac{1}{2}} \, e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr} \Big[\phi_t^L(x)\Big]^{\otimes k} \otimes \mathcal{T}_t^{L'}(x) + \mathcal{O}(k^{n-1}) \tag{8}$$

where ϕ_t *is the Hamiltonian flow of the principal symbol* H_t, ϕ_t^L *and* $\mathcal{T}_t^{L'}$ *are its prequantum and parallel transport lifts,* H_t^{sub} *is the subprincipal symbol and* $(\rho_t)^{1/2}$ *is the continuous square root equal to* 1 *at* $t=0$ *of the function* ρ_t *such that* $\mathcal{D}_t = \rho_t \mathcal{T}_t^K$ *with* $\mathcal{D}_t(x) = K(T_x\phi_t)$.

If $y \in M$ *is different from* $\phi_t(x)$, *then* $U_{k,t}(y,x) = \mathcal{O}(k^{-N})$ *for all* N.

The first part of the result has an alternative formulation when M has a half-form bundle, that is a line bundle δ and an isomorphism between δ^2 and the canonical bundle K. Introducing the line bundle L_1 such that $L' = L_1 \otimes \delta$ and using that $\mathcal{T}_t^{L'} = \mathcal{T}_t^{L_1} \otimes \mathcal{T}_t^{\delta}$, we obtain

$$U_{k,t}(\phi_t(x), x) \sim \Big(\frac{k}{2\pi}\Big)^n e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr} \Big[\phi_t^L(x)\Big]^{\otimes k} \otimes \mathcal{T}_t^{L_1}(x) \otimes \big[\mathcal{D}_t(x)\big]^{\frac{1}{2}}$$

where $\big[\mathcal{D}_t(x)\big]^{1/2} : \delta_x \to \delta_{\phi_t(x)}$ is the continuous square root of $\mathcal{D}_t(x)$ equal to 1 at $t = 0$. Observe that to write this equation, it is sufficient to define δ on the trajectory $t \to \phi_t(x)$, which is always possible.

In the above statement, we focused on the geometrical description of the leading order term, because it is the real novelty. The complete result, Theorem 4.2, too long for the introduction, is that $U_{k,t}(\phi_t(x), x)$ has a full asymptotic expansion in integral powers of k^{-1} and we also have a uniform description with respect to x, y and t on compact regions. Such a uniform description is not obvious because the asymptotic behavior of $U_{k,t}(y,x)$ is completely different whether $y = \phi_t(x)$ or not. We actually show that the Schwartz kernel of $U_{k,t}$ is a Lagrangian state in the sense of [8] (see the definition in Section 2), associated with the graph of ϕ_t.

In the Appendix, we investigate an explicit example in which the Hamiltonian flow does not preserve the complex structure, and verify the validity of the above theorem for the kernel of the propagator on the graph of this flow.

1.3 Smoothed spectral projector

Our second result is the asymptotic description of the Schwartz kernel of $f(k(E-T_k))$ where (T_k) is a self-adjoint Toeplitz operator, E is a regular value of the principal symbol H of (T_k) and $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is a smooth function having a compactly supported Fourier transform. For a function $g : \mathbb{R} \to \mathbb{R}$, $g(T_k)$ is merely defined as $\sum_{\lambda \in \operatorname{sp}(T_k)} g(\lambda)\Pi_\lambda$ where for each eigenvalue λ of T_k, Π_λ is the orthogonal projector onto the corresponding eigenspace. For g smooth, $g(T_k)$ is itself a Toeplitz operator with principal symbol $g \circ H$ and so its Schwartz kernel is concentrated on the diagonal; more precisely

$$\begin{aligned} g(T_k)(x,x) &= \Big(\frac{k}{2\pi}\Big)^n g(H(x)) + \mathcal{O}(k^{n-1}), \\ g(T_k)(x,y) &= \mathcal{O}(k^{-N}), \quad \forall N \qquad \text{when } x \neq y. \end{aligned}$$

In the rest of the paper we will work with a function g depending on k in the very specific way $g(\tau) = f(k(\tau - E))$, which we interpret as a focus at scale k^{-1} around E. For instance, for f the characteristic function of a subset A of $\mathbb{R}$, $f(k(\cdot - E))$ is the characteristic function of $E + k^{-1}A$. However, we will only consider very regular functions f, having a smooth compactly supported Fourier transform $\hat{f}$.

Our result is that the Schwartz kernel of $f(k(E-T_k))$ is (up to normalization by some power of k, see Remark 2.1 for a discussion) a Lagrangian state associated with the Lagrangian immersion

$$j_E : \mathbb{R} \times H^{-1}(E) \to M^2, \qquad (t,x) \to (\phi_t(x), x). \tag{9}$$

Here ϕ_t is the flow of the autonomous Hamiltonian H. It is important to note that j_E is not injective and not proper in general. However only the times t in the support of the Fourier transform of f matter, so we will work with the restriction of j_E to a compact subset. Still, it is possible for j_E to have multiple points because of the periodic trajectories.

The description of the Schwartz kernel on the image of j_E will be in terms of the parallel transport lift of ϕ_t to L and L' as introduced above and a lift $\mathcal{D}'_t$ to the canonical bundle of the restriction of ϕ_t to the energy level set $H^{-1}(E)$, defined as follows.

First since E is regular, for any $x \in H^{-1}(E)$, the Hamiltonian vector field X of H is not zero at x. Second, H being time-independent, $T_x\phi_t$ sends X_x into $X_{\phi_t(x)}$, so it induces a symplectic map from $T_xH^{-1}(E)/\mathbb{R}X_x$ into $T_{\phi_t(x)}H^{-1}(E)/\mathbb{R}X_{\phi_t(x)}$. In the case x is periodic with period t, this map is the tangent linear map to the Poincaré section map.

For any $x \in H^{-1}(E)$, write $T_x M = F_x \oplus G_x$ where F_x is the subspace spanned by X_x and $j_x X_x$ and G_x is its symplectic orthogonal. Observe that $T_x H^{-1}(E) = G_x \oplus \mathbb{R}X_x$ so that $G_x = T_x H^{-1}(E)/\mathbb{R}X_x$. Furthermore, both F_x and G_x are symplectic subspaces preserved by j_x, so $K_x \simeq K(F_x) \otimes K(G_x)$. Then we define

$$\mathcal{D}'_t(x) : K_x \to K_{\phi_t(x)} \tag{10}$$

as the tensor product of the following maps:

1. $K(F_x) \to K(F_{\phi_t(x)})$, $\lambda \mapsto 2\|X_x\|^{-2}\lambda'$ where λ, λ' are normalised by $\lambda(X_x) = \lambda'(X_{\phi_t(x)}) = 1$,
2. $K(\psi) : K(G_x) \to K(G_{\phi_t(x)})$ with ψ the symplectomorphism

$$\psi : G_x \simeq T_x H^{-1}(E)/\mathbb{R}X_x \xrightarrow{T_x\phi_t} T_{\phi_t(x)} H^{-1}(E)/\mathbb{R}X_{\phi_t(x)} \simeq G_{\phi_t(x)}.$$

In the particular case where $T_x\phi_t$ sends $(jX)_x$ into $(jX)_{\phi_t(x)}$, one checks that $\mathcal{D}'_t(x) = 2\|X_x\|^{-2}\mathcal{D}_t(x)$. Otherwise, there does not seem to be any simple relation between $\mathcal{D}_t(x)$ and $\mathcal{D}'_t(x)$.

Exactly as we did for $\mathcal{D}_t$, we define $\rho'_t(x)$ as the complex number such that

$$\mathcal{D}'_t(x) = \rho'_t(x)\mathcal{T}^K_t(x).$$

We denote by $\left[\rho'_t(x)\right]^{1/2}$ the continuous square root equal to $\sqrt{2}\|X_x\|^{-1}$ at $t = 0$.

Theorem 1.2. *For any self-adjoint Toeplitz operator (T_k) and regular value E of its principal symbol H, we have for any $x, y \in H^{-1}(E)$,*

$$f(k(E - T_k))(y, x) = \left(\frac{k}{2\pi}\right)^n k^{-\frac{1}{2}}$$
$$\times \sum_{\substack{t \in \operatorname{Supp} \hat{f},\\ \phi_t(x) = y}} \hat{f}(t) \left[\rho'_t(x)\right]^{\frac{1}{2}} e^{\frac{1}{i}\int_0^t H^{\mathrm{sub}}(\phi_r(x))\, dr} \left[\mathcal{T}^L_t(x)\right]^{\otimes k} \otimes \mathcal{T}^{L'}_t(x) + \mathcal{O}(k^{n-\frac{3}{2}}).$$

Furthermore, for any $(x, y) \in M^2$ not belonging to $j_E(\operatorname{Supp}(\hat{f}) \times H^{-1}(E))$, we have $f(k(E - T_k))(x, y) = \mathcal{O}(k^{-\infty})$.

As in Theorem 1.1, in the case M has a half-form bundle δ, we can write $L' = L_1 \otimes \delta$ and replace the sum above by

$$\sum_{\substack{t \in \operatorname{Supp} \hat{f},\\ \phi_t(x) = y}} \hat{f}(t)\, e^{\frac{1}{i}\int_0^t H^{\mathrm{sub}}(\phi_r(x))\, dr} \left[\mathcal{T}^L_t(x)\right]^{\otimes k} \otimes \mathcal{T}^{L_1}_t(x) \otimes \left[\mathcal{D}'_t(x)\right]^{\frac{1}{2}}$$

where $\left[\mathcal{D}'_t(x)\right]^{1/2} : \delta_x \to \delta_{\phi_t(x)}$ is continuous and equal to $\sqrt{2}\,\|X_x\|^{-1}\,\mathrm{id}_{\delta_x}$ at $t = 0$.

Furthermore, we will give a uniform description with respect to (x, y) of the Schwartz kernel of $f(k(E - T_k))$ by showing it is a Lagrangian state associated with the Lagrangian immersion j_E.

1.4 Discussion

Let us explain more the structure of the Lagrangian states appearing in the previous results (see also Section 2 for precise definitions). Roughly, a Lagrangian state of M is a family $(\Psi_k \in \mathcal{H}_k,\ k \in \mathbb{N})$ which is $\mathcal{O}(k^{-\infty})$ outside a Lagrangian submanifold Γ of M and which has an asymptotic expansion at any point $x \in \Gamma$ of the form

$$\Psi_k(x) = k^m [t(x)]^{\otimes k} \Big(a_0(x) + k^{-1} a_1(x) + \dots \Big) \tag{11}$$

where m is some real number, $t(x) \in L_x$ has norm one, and the coefficients $a_0(x),\ a_1(x),\ \dots$ belong to L'_x. We can think of $[t(x)]^k$ as an oscillatory factor and $k^m \sum k^{-\ell} a_\ell(x)$ as an amplitude, so the right-hand side of (11) is completely analogous to the well-known WKB ansatz. Indeed, locally in a trivialization open set for M, L and L', sections of L and L' can be identified with functions, which yields $t(x)^{\otimes k} = e^{ik\phi(x)}$ for some phase ϕ which is real on Γ. Furthermore, $t(x)$ and the $a_\ell(x)$ all depend smoothly on x so that they define sections of L and L' respectively over Γ. The section t has the important property to be flat. Regarding the leading order term $a_0(x)$ of the amplitude, it is often useful to think about it as a product $t_1(x) \otimes \nu(x)$ where $t_1(x) \in (L_1)_x$ and $\nu(x) \in \delta_x$. Here δ is a half-form bundle, which can be introduced at least locally, and $L' = L_1 \otimes \delta$. Then $[t(x)]^{\otimes k} \otimes t_1(x)$ may be viewed as a deformation of $[t(x)]^{\otimes k}$, whereas $\nu(x)$ is a square root of a volume element of Γ. Indeed, Γ being Lagrangian, there is a natural pairing between the restriction of the canonical bundle to Γ and the determinant bundle $\det T\Gamma \otimes \mathbb{C}$.

In our results, the Lagrangian states, which are Schwartz kernels of operators, are defined on M^2, with the prequantum bundle $L \boxtimes \overline{L}$. Here, the symplectic and prequantum structures are such that the graphs of symplectomorphisms are Lagrangian submanifolds and their prequantum lifts define flat sections. In Theorem 1.1, the Schwartz kernel of the quantum propagator is defined as a Lagrangian state associated with the graph of the Hamiltonian flow and its prequantum lift. As was already noticed, the prequantum lift appears with correction terms $\exp\big(i \int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr\big)$ and

$\mathcal{T}_t^{L_1}(x)$, which are the contributions of the corrections $H^{\rm sub}$ to H and L_1 to L. Then the last term $\mathcal{D}_t^{\frac{1}{2}}$ is merely the square root of the image of the Liouville volume form by the map $M \to \operatorname{graph} \phi_t$ sending x into $(\phi_t(x), x)$.

The relation between Theorems 1.1 and 1.2 relies on the time/energy duality. Roughly, for a time-independent operator $\hat{H}$, we pass from the quantum propagator $(\exp(-i\hbar^{-1}t\hat{H}),\ t \in \mathbb{R})$ to the smoothed spectral projector $(f(\hbar^{-1}(\hat{H} - E)),\ E \in \mathbb{R})$, by multiplying by $\hat{f}(t)$ and then doing a partial $\hbar$-Fourier transform with respect to the variables t, E (here k plays the part of $\hbar^{-1}$). In the microlocal point of view, the variables t and E are equivalent and we can view the quantum propagator and the spectral projector as two facets of the same object.

In our results, this duality is expressed by the fact that the graph of ϕ_t and the Lagrangian immersion j_E are obtained in a symmetric way from the Lagrangian submanifold

$$\widetilde{\Gamma} = \{(t, H(x), \phi_t(x), x)/x \in M, t \in \mathbb{R}\} \tag{12}$$

of $T^*\mathbb{R} \times M \times M^-$. Indeed, the graph of ϕ_t and the image of j_E are the projections onto M^2 of the slices

$$\widetilde{\Gamma}_t = \widetilde{\Gamma} \cap (\{t\} \times \mathbb{R} \times M^2), \qquad \widetilde{\Gamma}^E = \widetilde{\Gamma} \cap (\mathbb{R} \times \{E\} \times M^2).$$

The prequantum lifts and the volume elements can also be incorporated in this picture. In particular, we pass from $\mathcal{D}_t$ to $\mathcal{D}'_t$ by canonical isomorphisms between volume elements of $\widetilde{\Gamma}$, $\widetilde{\Gamma}_t$ and $\widetilde{\Gamma}^E$.

As we will see in a next paper, the quantum propagator viewed as a function of time is actually a Lagrangian state associated with $\widetilde{\Gamma}$ (and we will particularly focus on the computation of the precise geometric quantities involved in its principal symbol). This statement is delicate because here we mix real and complex variables, cotangent bundles and Kähler manifolds, and the description of Lagrangian states is rather different in these two settings. To give a sense to this, we will perform a Bargmann transform so that the quantum propagator will become a holomorphic function of the complex variable $t + iE$. This point of view will be interesting, even for the proof of Theorem 1.1, to understand the transport equation satisfied by the leading order term of the amplitude.

1.5 Comparison with earlier results

The introduction of Fourier integral operators with application to the Schrödinger equation and spectral properties of pseudodifferential operators has its origin in the seminal Hörmander [19] and Duistermaat-Guillemin [14] papers, cf. the survey [17]. In these first developments, the operator under study is the Laplace-Beltrami operator and the corresponding classical flow is the geodesic flow.

The transcription of these results to Berezin-Toeplitz operators has been done in the paper [3] by Bothwick-Paul-Uribe, by applying the Boutet de Monvel-Guillemin approach of [4]. Similar results have been proved in a recent paper [28] by Zelditch-Zhou where the application to spectral densities has been pushed further. These papers both rely on the Boutet de Monvel-Guillemin book [4]. In particular the properties of Berezin-Toeplitz operators are deduced from the pseudodifferential calculus, and the quantum propagator is viewed as a Fourier integral operator. From this is deduced the asymptotics of the smoothed spectral projector on the diagonal, [3, Theorem 1.1] and [28, Theorem 2.2]. The leading order term is computed in terms of the symbolic calculus of Fourier integral operator of Hermite type in [3], or with a non-linear problem in Bargmann space in [28]. Another description of the kernel of the quantum propagator associated with an autonomous Hamiltonian was obtained by Ioos [21]; this description involves quantities related with parallel transport in the canonical bundle with respect to a connection induced by the tranport of the initial complex structure by the Hamiltonian flow, and computing these coefficients appears to be quite complicated in general, relatively to our formulas. In fact, in all these works the analysis is well-understood but our main addition, apart from obtaining a direct derivation in our context, is the precise computation of the geometric quantities contained in the principal symbol of the kernel of the quantum propagator seen as a Lagrangian state. In particular these quantities have a very natural interpretation in terms of half-forms, and can be easily computed for concrete examples.

The techniques that we use come from the work of the first author where a direct definition of Lagrangian states on a Kähler manifold is introduced [8]. As explained in the discussion following Equation (11), these Lagrangian states locally look like WKB functions with complex phase. The microlocal toolbox for complex phase WKB states was developed in the seminal paper [25] in the homogeneous case. However our Lagrangian states are specific to the Kähler setting, for instance, the states being defined directly on phase space, there are no caustics. Moreover the relevant symplectic geometry is not the geometry of the cotangent space of the base but the Kähler geometry of the base itself.

In the first author's PhD thesis [6, Section 3.5.2], it is shown that the quantum propagator is a Lagrangian state, but without the precise computation of the principal symbol that we obtain here. The use of half-form bundles for Berezin-Toeplitz operators started in [9, 10] and here we apply them to the description of the quantum propagator. The isomorphisms (5) have been introduced in [10], [11] where their square roots are called half-form bundle morphisms. A similar invariant appears in [28] under the form (6). Again, we insist that the main novelty in our results is the precise description and computation of the coefficients $\rho_t(x)$ and $\rho'_t(x)$ appearing in Theorems 1.1, 1.2.

Whereas the relation between the quantum propagator and the Hamiltonian flow is a classical result, the similar statement for the smoothed spectral projector and the Lagrangian immersion (9) seems to be new. In [3] and [28], only the diagonal behavior of the Schwartz kernel is described. To state our result, we will introduce a general class of Lagrangian states associated with Lagrangian immersions.

1.6 Outline of the paper

Section 2 is devoted to time-dependent Lagrangian states, that we call Lagrangian state families. The main result is that these states provide solutions to the Schrödinger equation with quantum Hamiltonian a Toeplitz operator and initial data a Lagrangian state, cf. Theorem 2.6. The principal symbol of these solutions satisfies a transport equation, that is solved in Section 2.3 (up to a rather technical part which is postponed to Section 7 for the sake of clarity), while in Section 3, we give an elegant expression in the context of metaplectic quantization. These results will be applied in Section 4 to the quantum propagator, where Theorem 1.1 is proved.

In Section 5, we prove that the Fourier transform of a time-dependent Lagrangian state is a Lagrangian state as well, with an underlying Lagrangian manifold which is in general only immersed and not embedded, cf. Theorem 5.4. The needed adaptations in the Lagrangian state definition for immersed manifolds are given in Section 5.2. In Section 6, we deduce Theorem 1.2 on the smoothed spectral projector.

Acknowledgments. We thank two anonymous referees for useful comments.

2 Propagation of Lagrangian states

In this section, we introduce some one-parameter families of Lagrangian states which are relevant to our setting and study how they evolve under the

Schrödinger equation. The definition of these states is new and builds on the standard definition of Lagrangian states introduced in [8], which we briefly recall now. It will also be useful to have the standard definition in mind when introducing Lagrangian states associated with immersed Lagrangians, see Section 5.2.

Let M, L and L' be as in Section 1.1. Let Γ be a Lagrangian submanifold of M equipped with a flat unitary section $s \in \mathcal{C}^\infty(\Gamma, L)$. A *Lagrangian state* associated with (Γ, s) is a sequence $(\Psi_k \in \mathcal{H}_k)_{k\geq 1}$ of the form

$$\Psi_k(x) = \left(\frac{k}{2\pi}\right)^{\frac{n}{4}} F^k(x)a(x,k) + \mathcal{O}(k^{-\infty})$$

where

- $F \in \mathcal{C}^\infty(M, L)$ is such that $\bar{\partial}F$ vanishes to infinite order along Γ,
- $F_{|\Gamma} = s$ and $|F(x)| < 1$ for $x \notin \Gamma$,
- $a(\cdot, k)$ is a sequence of smooth sections of $L' \to M$ with an asymptotic expansion $a(\cdot, k) = \sum_{\ell\geq 0} k^{-\ell} a_\ell$ for the $\mathcal{C}^\infty$ topology, where each section a_ℓ, for $\ell \geq 0$, is such that $\bar{\partial}a_\ell$ vanishes to infinite order along Γ,
- the $\mathcal{O}$ is for the pointwise norm and uniform on M.

For any sequence $(b_\ell)_{\ell\geq 0}$ of elements of $\mathcal{C}^\infty(\Gamma, L')$, there exists a Lagrangian state Ψ_k such that for every $\ell \geq 0$, $b_\ell = a_{\ell|\Gamma}$. The *full symbol* of Ψ_k is the formal series $\sum_{\ell\geq 0} \hbar^\ell b_\ell$, and uniquely determines Ψ_k up to $\mathcal{O}(k^{-\infty})$. The first term b_0 in this full symbol is called the *principal symbol* of Ψ_k.

Since we will use later some generalisations of this construction, let us briefly recall the proof, the details being in [8, Section 2]. First, since Γ is a totally real submanifold, any smooth function of Γ has an extension f to M such that $\overline{\partial} f$ vanishes to infinite order along Γ. The same holds for the sections of a holomorphic line bundle. In this way we construct F and the a_ℓ's from s and the b_ℓ's respectively. These sections are not uniquely determined, but their Taylor expansion along Γ is. In particular, a computation shows that $\ln|F|$ has a non degenerate minimum along Γ, so modifying F away from Γ if necessary, the condition $|F| < 1$ on $M \setminus \Gamma$ is satisfied. The Lagrangian state Ψ_k is then obtained by projecting the smooth section $\widetilde{\Psi}_k = (k/2\pi)^{n/4} F^k a(\cdot, k)$ onto $\mathcal{H}_k$. We claim that $\Psi_k = \widetilde{\Psi}_k + \mathcal{O}(k^{-\infty})$. The proof of this fact was obtained by stationary phase computations in [8]. Alternatively, this follows from the fact that $\overline{\partial}\Psi_k = \mathcal{O}(k^{-\infty})$ and the Kodaira-Hörmander estimates [23, 18].

Remark 2.1. The normalization factor $\left(\frac{k}{2\pi}\right)^{\frac{n}{4}}$ is somewhat arbitrary. First, the power of 2π could be included in the symbol of the Lagrangian state. Second, the choice of the power of k is more or less convenient depending on the context, since Lagrangian states appear in different situations (for instance as approximate eigenvectors for Berezin-Toeplitz operators, or as an ansatz for the Schwartz kernel of such an operator). Here the choice of normalization yields a L^2-norm of order $\mathcal{O}(1)$ for the Lagrangian states. □

2.1 Families of Lagrangian states

As explained above, to define a Lagrangian state, we need a Lagrangian submanifold of M equipped with a flat unitary section of the prequantum bundle L. Let us consider a one-parameter family of such pairs. More precisely, let $I \subset \mathbb{R}$ be an open interval, $\mathbb{C}_I$ be the trivial complex line bundle over I, Γ be a closed submanifold of $I \times M$ and $s \in \mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L)$ be such that

1. the map $q : \Gamma \to I$, $q(t,x) = t$ is a proper submersion. So for any $t \in I$, the fiber $\Gamma_t := \Gamma \cap (\{t\} \times M)$ is a submanifold of M,
2. for any $t \in I$, Γ_t is a Lagrangian submanifold of M and the restriction of s to Γ_t is flat and unitary.

Remark 2.2. a. Since q is a proper submersion, by Ehresmann's lemma, Γ is diffeomorphic to $I \times \mathcal{N}$ for some manifold $\mathcal{N}$ in such a way that q becomes the projection onto I.

b. Given a proper submersion $q : \Gamma \to I$ and a map $f : \Gamma \to M$, it is equivalent that the map $\Gamma \to I \times M$, $x \to (q(x), f(x))$ is a proper embedding and that for any t, $f(t, \cdot) : \Gamma_t \to M$ is an embedding. We decided to start from a closed submanifold of $I \times M$ to be more efficient in the definition of Lagrangian states below. □

We will consider states Ψ_k in $\mathcal{H}_k$ depending smoothly on $t \in I$, so Ψ_k belongs to $\mathcal{C}^\infty(I, \mathcal{H}_k)$. Equivalently Ψ_k is a smooth section of $\mathbb{C}_I \boxtimes (L^k \otimes L')$ such that $\overline{\partial}\Psi_k = 0$. Here the $\overline{\partial}$ operator only acts on the M factor. Similarly, it makes sense to differentiate with respect to $t \in I$ a section of $\mathbb{C}_I \boxtimes A$, where A is any vector bundle over M.

Definition 2.3. A *Lagrangian state family* associated with (Γ, s) is a family $(\Psi_k \in \mathcal{C}^\infty(I, \mathcal{H}_k),\ k \in \mathbb{N})$ such that for any N,

$$\Psi_k(t,x) = \left(\frac{k}{2\pi}\right)^{\frac{n}{4}} F^k(t,x) \sum_{\ell=0}^{N} k^{-\ell} a_\ell(t,x) + R_N(t,x,k) \tag{13}$$

where

- F is a section of $\mathbb{C}_I \boxtimes L$ such that $F|_\Gamma = s$, $\overline{\partial} F$ vanishes to infinite order along Γ and $|F| < 1$ outside of Γ,
- (a_ℓ) is a sequence of sections of $\mathbb{C}_I \boxtimes L'$ such that $\overline{\partial} a_\ell$ vanishes to infinite order along Γ,
- for any p and N, $\partial_t^p R_N = \mathcal{O}(k^{p-N-1})$ in pointwise norm uniformly on any compact subset of $I \times M$.

It is not difficult to adapt the argument of [8, Section 2] and to prove the following facts. First, the section F exists. Second, we can specify arbitrarily the coefficients a_ℓ of the asymptotic expansion along Γ and this determines (Ψ_k) up to $\mathcal{O}(k^{-\infty})$. More precisely, for any sequence $(b_\ell) \in \mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L')$, there exists a Lagrangian state (Ψ_k) satisfying for any $y \in \Gamma$

$$\Psi_k(y) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} s^k(y) \sum_{\ell=0}^{N} k^{-\ell} b_\ell(y) + \mathcal{O}(k^{-N-1}) \qquad \forall N.$$

Furthermore, (Ψ_k) is unique up to a family $(\Phi_k \in \mathcal{C}^\infty(I, \mathcal{H}_k),\ k \in \mathbb{N})$ satisfying $\|(\frac{d}{dt})^p \Phi_k(t)\| = \mathcal{O}(k^{-N})$ for any p and N uniformly on any compact subset of I.

We will call the formal series $\sum \hbar^\ell b_\ell$ the *full symbol* of (Ψ_k). The first coefficient b_0 will be called the *principal symbol*.

It could be interesting to define Lagrangian state families with a different regularity with respect to t. Here, our ultimate goal is to solve a Cauchy problem, so we will differentiate with respect to t and in a later proof, we will use that $(k^{-1}\partial_t \Psi_k)$ is still a Lagrangian state family. So we need to consider states which are smooth in t. Observe that in the estimate satisfied by R_N we lose one power of k for each derivative; this is consistent with the fact that $\partial_t(F^k) = kF^k f$ where f is the logarithmic derivative of F, that is $\partial_t F = fF$.

A last result which is an easy adaptation of [8, section 2.4] is the action of Toeplitz operators on Lagrangian state families. Let $(T_{k,t})$ be a time-dependent Toeplitz operator and $(\Psi_{k,t})$ be a Lagrangian state as above. Then $(T_{k,t}\Psi_{k,t})$ is a Lagrangian state family associated with (Γ, s) as well. Furthermore, its full symbol is equal to $\sum \hbar^{\ell+m} Q_m(b_\ell)$ where $\sum \hbar^\ell b_\ell$ is the full symbol of (Ψ_k) and the Q_m are differential operators acting on $\mathcal{C}^\infty(\Gamma, \mathbb{C}_I \otimes L')$ and depending only on $(T_{k,t})$. In particular, Q_0 is the multiplication by the restriction to Γ of the principal symbol of (T_k).

2.2 Propagation

Consider the same data $\Gamma \subset I \times M$ and $s \in \mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L)$ as in the previous section. We claim that the covariant derivative of s has the form

$$\nabla s = i\tau dt \otimes s \qquad \text{with} \qquad \tau \in \mathcal{C}^\infty(\Gamma, \mathbb{R}). \tag{14}$$

Here the covariant derivative is induced by the trivial derivative of $\mathbb{C}_I$ and the connection of L. To prove (14), use that the restriction of s to each Γ_t is flat. So $\nabla s = i\alpha \otimes s$, with $\alpha \in \Omega^1(\Gamma, \mathbb{R})$ vanishing in the vertical directions of $q : \Gamma \to I$. So $\alpha = \tau dt$ for some function $\tau \in \mathcal{C}^\infty(\Gamma, \mathbb{R})$.

In the following two propositions, (Ψ_k) is a Lagrangian state family associated with (Γ, s) with full symbol $b(\hbar) = \sum \hbar^\ell b_\ell$.

Proposition 2.4. $((ik)^{-1}\partial_t \Psi_k)$ *is a Lagrangian state family associated with* (Γ, s) *with full symbol* $(\tau + \hbar P) b(\hbar)$*, where* P *is a differential operator of* $\mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L')$*.*

Proof. Differentiating the formula (13) with respect to t, we obtain on a neighborhood of Γ that

$$\partial_t \Psi_k = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} F^k \sum_\ell k^{-\ell} (k f a_\ell + \partial_t a_\ell)$$

where $f \in \mathcal{C}^\infty(I \times M)$ is the logarithmic derivative of F with respect to time, so $\partial_t F = fF$. Using that $\overline{\partial}$ and the derivative with respect to t commute, we easily prove that $\overline{\partial} f$ and $\overline{\partial}(\partial_t a_\ell)$ both vanish to infinite order along Γ. This shows that $(k^{-1}\partial_t \Psi_k)$ is a Lagrangian state associated with (Γ, s).

Its full symbol is the restriction to Γ of the series $\sum \hbar^\ell (f a_\ell + \hbar \partial_t a_\ell)$. We claim that $f|_\Gamma = i\tau$. Indeed, at any point of Γ, ∇F vanishes in the directions tangent to M, because it vanishes in the directions of type $(0,1)$ and in the directions tangent to Γ_t as well. So $\nabla F = f dt \otimes F$ along Γ. The restriction of F to Γ being s, (14) implies that $f|_\Gamma = i\tau$.

Using similarly that at any $x \in \Gamma_t$, $(T_x\Gamma_t \otimes \mathbb{C}) \oplus T^{0,1}_x M = T_x M \otimes \mathbb{C}$, and $\overline{\partial} a_\ell = 0$ along Γ, it comes that $\partial_t a_\ell = \nabla_Z a_\ell$ along Γ, where $Z(t,x) \in T_{(t,x)}\Gamma$ is the projection of $\partial/\partial t$ onto $T_{(t,x)}\Gamma$ parallel to $T^{0,1}_x M$. This concludes the proof with P the operator $\frac{1}{i}\nabla_Z$. $\square$

We now assume that (Γ, s) is obtained by propagating a Lagrangian submanifold Γ_0 of M and a flat section s_0 of $L \to \Gamma_0$ by a Hamiltonian flow and its prequantum lift. Let (H_t) be the time-dependent Hamiltonian generating our flow (ϕ_t) and denote by ϕ_t^L its prequantum lift defined as in the introduction by (3). So we set

$$\Gamma_t = \phi_t(\Gamma_0), \qquad s_t(\phi_t(x)) = \phi_t^L(x) s_0(x).$$

Let Y be the vector field of $\mathbb{R} \times M$ given by $Y(t,x) = \frac{\partial}{\partial t} + X_t(x)$ where X_t is the Hamiltonian vector field of H_t.

Introduce a time-dependent Toeplitz operator $(T_{k,t})$ with principal symbol (H_t).

Proposition 2.5. *$(\frac{1}{ik}\partial_t \Psi_k + T_{k,t}\Psi_k)$ is a Lagrangian state family associated with (Γ, s) with full symbol $\hbar(\frac{1}{i}\nabla_Y + \zeta)b_0 + \mathcal{O}(\hbar^2)$ for some $\zeta \in \mathcal{C}^\infty(\Gamma)$.*

Proof. By Proposition 2.4 and the last paragraph of section 2.1, we already know that $(\frac{1}{ik}\partial_t \Psi_k + T_{k,t}\Psi_k)$ is a Lagrangian state with full symbol

$$(\tau(t,x) + H_t(x))(b_0 + \hbar b_1) + \hbar Q b_0 + \mathcal{O}(\hbar^2), \tag{15}$$

where Q is a differential operator acting on $\mathcal{C}^\infty(\Gamma, \mathbb{C}_I \otimes L')$. By differentiating (3) in the definition of s_t and by the fact that $\nabla s = i\tau dt \otimes s$, it comes that

$$\tau(t, \phi_t(x)) + H_t(\phi_t(x)) = 0, \tag{16}$$

so the leading order term in (15) is zero. Consider $f \in \mathcal{C}^\infty(\mathbb{R} \times M)$ and compute the commutator

$$\begin{aligned} \left[\tfrac{1}{ik}\partial_t + T_{k,t}, T_k(f)\right] &= \tfrac{1}{ik}\big(T_k(\partial_t f) + T_k(\{H_t, f\}\big) + \mathcal{O}(k^{-2}) \\ &= \tfrac{1}{ik} T_k(Yf) + \mathcal{O}(k^{-2}). \end{aligned}$$

Letting this act on our Lagrangian state family Ψ_k, it comes that $[Q, f|_\Gamma] = \frac{1}{i}(Yf)|_\Gamma$. Since this holds for any f, this proves that iQ is a derivation in the direction of Y so $iQ = \nabla_Y + i\zeta$ for some function ζ. ☐

Theorem 2.6. *For any Lagrangian state $(\Psi_{0,k} \in \mathcal{H}_k)$ associated with (Γ_0, s_0), the solution of the Schrödinger equation*

$$\tfrac{1}{ik}\partial_t \Psi_k + T_{k,t}\Psi_k = 0, \qquad \Psi_k(0, \cdot) = \Psi_{0,k} \tag{17}$$

is a Lagrangian state family associated with (Γ, s) with symbol $b_0 + \mathcal{O}(\hbar)$ where b_0 satisfies the transport equation $\frac{1}{i}\nabla_Y b_0 + \zeta b_0 = 0$.

Since the integral curves of Y are $t \mapsto (t, \phi_t(x))$, the solution of the transport equation is

$$b_0(t, \phi_t(x)) = e^{\frac{1}{i}\int_0^t \zeta(r, \phi_r(x))\, dr} \mathcal{T}_t^{L'}(x)\, b_0(0,x). \tag{18}$$

In the next section, we will give a geometric formula for ζ in terms of the canonical bundle.

Proof. The proof is the same as for differential operators (see the proof of Theorem 20.1 in [27] for instance), so we only sketch it. We successively construct the coefficients b_ℓ to solve $\frac{1}{ik}\partial_t \Psi_k + T_{k,t}\Psi_k = \mathcal{O}(k^{-N-1})$ with initial condition $\Psi_k(0,\cdot) = \Psi_{0,k} + \mathcal{O}(k^{-N})$. At each step, we have to solve a transport equation $\nabla_Y b_N + db_N = r_N$ with initial condition $b_N(0,\cdot) = b_{0,N}$, which has a unique solution. This provides us with a Lagrangian state (Ψ_k) such that both equations of (17) are satisfied up to a $\mathcal{O}(k^{-\infty})$. Then, applying Duhamel's principle, we show that the difference between $\partial_t\Psi_k$ and the actual solution of (17) is a $\mathcal{O}(k^{-\infty})$ uniformly on any bounded interval. $\square$

2.3 Transport equation

We will now give a formula for the function ζ and solve the above transport equation. Essential to our presentation are line bundle isomorphisms involving the canonical bundle K of M and the determinant bundles $\bigwedge^n T^*\Gamma_t$ and $\bigwedge^{n+1} T^*\Gamma$.

First, for any $t \in I$, let K_t be the restriction of K to Γ_t. Then we have an isomorphism

$$K_t \simeq \det(T^*\Gamma_t) \otimes \mathbb{C} \tag{19}$$

defined by sending $\Omega \in (K_t)_x = \bigwedge^{n,0} T^*_x M$ to its restriction to $T_x\Gamma_t \subset T_x M$. This is an isomorphism because $(T_x\Gamma_t \otimes \mathbb{C}) \cap T^{0,1}_x M = \{0\}$, which follows from the fact that Γ_t is Lagrangian.

Second, Γ_t being a fiber of $\Gamma \to I$, the linear tangent maps to the injection $\Gamma_t \to \Gamma$ and the projection $\Gamma \to \mathbb{R}$ give an exact sequence

$$0 \to T_x\Gamma_t \to T_{(t,x)}\Gamma \to \mathbb{R} = T^*_t I \to 0.$$

Since $\mathbb{R}$ has a canonical volume element, we obtain an isomorphism

$$\det(T^*\Gamma_t) \simeq \det(T^*\Gamma)|_{\Gamma_t} \tag{20}$$

defined in the usual way: for any $\alpha \in \bigwedge^n T^*_{(t,x)}\Gamma$, one sends $dt \wedge \alpha \in \bigwedge^{n+1} T^*_{(t,x)}\Gamma$ into the restriction of α to $T_x\Gamma_t$.

Gathering these two isomorphisms , we get a third one:

$$K_\Gamma := (\mathbb{C}_I \boxtimes K)|_\Gamma \xrightarrow{\simeq} \det(T^*\Gamma) \otimes \mathbb{C}, \qquad (1 \boxtimes \alpha)|_\Gamma \mapsto j^*(dt \wedge \alpha) \tag{21}$$

for any $\alpha \in \Omega^{n,0}(M)$ with j the embedding $\Gamma \to I \times M$. On the one hand, K_Γ has a natural connection induced by the Chern connection of K, which gives us a derivation ∇_Y acting on sections of K_Γ. On the other hand, the Lie derivative $\mathcal{L}_Y$ acts on the differential forms of Γ, and in particular on the sections of $\det(T^*\Gamma)$. Under the isomorphism (21),

$$\mathcal{L}_Y = \nabla_Y + i\theta$$

where $\theta \in \mathcal{C}^\infty(\Gamma)$ since $\mathcal{L}_Y$ and ∇_Y are derivatives in the same direction Y.

Theorem 2.7. *The function ζ defined in Proposition 2.5 satisfies the equality $\zeta = \frac{1}{2}\theta + H^{\mathrm{sub}}|_\Gamma$.*

The proof is postponed to Section 7 since it does not help to understand what follows and it is quite technical. On the one hand, we can compute θ in terms of second derivatives of H_t, cf. Proposition 7.1. On the other hand, we directly compute the function ζ, cf. Proposition 7.2.

We will now give an explicit expression for the term involving ζ in the solution (18) of the transport equation in light of Theorem 2.7. For any $t \in I$, the tangent map to ϕ_t restricts to an isomorphism from $T\Gamma_0$ to $T\Gamma_t$. By the identification (19), we get an isomorphism $\mathcal{E}_t$ from $K|_{\Gamma_0}$ to $K|_{\Gamma_t}$ lifting ϕ_t. More precisely, for any $x \in \Gamma_0$, $u \in K_x$ and $v \in \det(T_x\Gamma_0)$, we define $\mathcal{E}_t(x)u \in K_{\phi_t(x)}$ so that

$$(\mathcal{E}_t(x)u)\big((T_x\phi_t)_*v\big) = u(v). \tag{22}$$

The parallel transport $\mathcal{T}_t^K$ restricts as well to an isomorphism $K|_{\Gamma_0} \to K|_{\Gamma_t}$. Define the complex number $C_t(x)$ by $\mathcal{E}_t(x) = C_t(x)\mathcal{T}_t^K(x)$.

Proposition 2.8. *The solution of the transport equation $\frac{1}{i}\nabla_Y b + \zeta b = 0$ with $b \in \mathcal{C}^\infty(\Gamma, L')$ is*

$$b(t, \phi_t(x)) = C_t(x)^{\frac{1}{2}}\, e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\,dr}\, \mathcal{T}_t^{L'}(x)\, b(0,x) \tag{23}$$

with the square root of $C_t(x)$ chosen continuously and $C_0 = 1$.

Proof. In view of Equation (18) and Theorem 2.7, it suffices to deal with the case $H^{\mathrm{sub}}|_\Gamma = 0$. Moreover, observe that if $\tilde{b}$ satisfies $\nabla_Y \tilde{b} = 0$, then $b = f\tilde{b}$ solves $\frac{1}{i}\nabla_Y b + \zeta b = 0$ if and only if $\frac{1}{i}Y.f + \zeta f = 0$. So it suffices to prove that $f : (t, \phi_t(x)) \mapsto C_t(x)^{1/2}$ is a solution of the latter equation.

First the isomorphism $I \times \Gamma_0 \simeq \Gamma$, $(t,x) \to (t, \phi_t(x))$ sends the vector field ∂_t to Y. The solutions of $\mathcal{L}_{\partial_t}\beta = 0$ with $\beta \in \Omega^{n+1}(I \times \Gamma_0)$ have the form $\beta = dt \wedge \beta_0$ with $\beta_0 \in \Omega^n(\Gamma_0)$. So the solutions of $\mathcal{L}_Y\alpha = 0$ with $\alpha \in \Omega^{n+1}(\Gamma)$ are parametrised by $\alpha_0 \in \Omega^n(\Gamma_0)$ and given by

$$\alpha|_{(t,\phi_t(x))} = dt \wedge (\phi_t^*)^{-1}\alpha_0|_x.$$

Now, identify K_Γ and $\det(T^*\Gamma)\otimes\mathbb{C}$ through (21). Then by (22), the previous equation becomes

$$\alpha|_{(t,\phi_t(x))} = \mathcal{E}_t(x)\,\alpha|_{(0,x)}.$$

Second, the solutions of $\nabla_Y \alpha' = 0$ with now $\alpha' \in \mathcal{C}^\infty(\Gamma, K_\Gamma)$ are given by

$$\alpha'|_{(t,\phi_t(x))} = \mathcal{T}_t^K(x)\,\alpha'|_{(0,x)}.$$

Assume that $\alpha'|_{(0,x)} = \alpha|_{(0,x)}$; then we have $\alpha = C\alpha'$ with $C \in \mathcal{C}^\infty(\Gamma)$ defined by $C(t,\phi_t(x)) = C_t(x)$. Therefore

$$0 = \mathcal{L}_Y \alpha = \mathcal{L}_Y(C\alpha') = (Y.C)\alpha' + C\mathcal{L}_Y\alpha' = (Y.C)\alpha' + C\underbrace{\nabla_Y\alpha'}_{=0} + 2i\zeta C\alpha'$$

so $Y.C + 2i\zeta C = 0$, hence $\frac{1}{i}Y.C^{1/2} + \zeta C^{1/2} = 0$. □

3 Metaplectic correction

It is useful to reformulate the previous results with a half-form bundle.

3.1 Definitions

Recall first some definitions. A *square root* (B,φ) of a complex line bundle $A \to \mathcal{N}$ over a manifold $\mathcal{N}$ is a complex line bundle $B \to \mathcal{N}$ with an isomorphism $\varphi : B^{\otimes 2} \to A$. A *half-form bundle* of a complex manifold is a square root of its canonical bundle. Since the group of isomorphism classes of complex line bundles of a manifold $\mathcal{N}$ is isomorphic to $H^2(\mathcal{N})$, the isomorphism being the Chern class, a sufficient condition for a complex manifold to have a half-form bundle is that its second cohomology group is trivial. This condition will be sufficient for our purposes. Before we discuss the uniqueness, let us explain how derivatives and connections can be transferred from a bundle to its square roots.

Assume that (B,φ) is a square root of A. Then any derivative D_B acting on sections of B induces a derivative D_A acting on sections of A such that the Leibniz rule is satisfied

$$D_A(u\otimes v) = D_B(u)\otimes v + u \otimes D_B(v), \qquad \forall\, u,v \in \mathcal{C}^\infty(B)$$

The converse is true as well: any derivative D_A of A determines a derivative D_B of B such that the above identity is satisfied. Similarly a covariant derivative ∇^B of B induces a covariant derivative ∇^A of A such that $\nabla^A(u\otimes v) = \nabla^B(u)\otimes v + u\otimes\nabla^B(v)$, and the converse holds as well.

Two square roots (B, φ) and (B', φ') of A are isomorphic if there exists a line bundle isomorphism $\psi : B \to B'$ such that $\varphi' \circ \psi^2 = \varphi$. The isomorphism classes of square roots of the trivial line bundle $\mathbb{C}_{\mathcal{N}}$ of $\mathcal{N}$ are in bijection with $H^1(\mathcal{N}, \mathbb{Z}_2)$. Indeed, each square root of $\mathbb{C}_{\mathcal{N}}$ has a natural flat structure with holonomy in $\{-1, 1\} \subset \mathrm{U}(1)$, induced by the flat structure of $\mathbb{C}_{\mathcal{N}}$. We easily check this determines the square root up to isomorphism. Furthermore, the tensor product of line bundles defines an action of square roots of $\mathbb{C}_{\mathcal{N}}$ on the space of square roots of a given line bundle A. This makes the set of isomorphism classes of square root of A a homogeneous space for the group $H^1(\mathcal{N}, \mathbb{Z}_2)$.

3.2 Propagation in terms of half-form bundle

When M has a half-form bundle δ, we can reformulate the previous results by introducing a new line bundle L_1 such that $L' = L_1 \otimes \delta$. The relevant structures of L_1 and δ have a different nature:

- L_1 has a natural connection, its Chern connection,
- the restriction of δ to a Lagrangian submanifold $\mathcal{N}$ of M is a square root of $\det(T^*\mathcal{N}) \otimes \mathbb{C}$, through the isomorphism $K|_{\mathcal{N}} \simeq \det(T^*\mathcal{N}) \otimes \mathbb{C}$.

For instance, in our propagation results, on the one hand, the tangent map to the flow defines a map from $\det(T^*\Gamma_0)$ to $\det(T^*\Gamma_t)$, which gives the map $\mathcal{E}_t : K|_{\Gamma_0} \to K|_{\Gamma_t}$. We then introduce the square root of $\mathcal{E}_t$

$$[\mathcal{E}_t(x)]^{\frac{1}{2}} : \delta_x \to \delta_{\phi_t(x)}, \qquad x \in \Gamma_0$$

which is equal to the identity at $t = 0$. On the other hand, we can define the parallel transport $\mathcal{T}_t^{L_1}$ from the connection of L_1. Then (23) writes equivalently

$$b(t, \phi_t(x)) = e^{\frac{1}{i} \int_0^t H^{\mathrm{sub}}(r, \phi_r(x))\, dr}\, \mathcal{T}_t^{L_1}(x) \otimes [\mathcal{E}_t(x)]^{\frac{1}{2}}\, b(0, x) \tag{24}$$

The transport equation $(\nabla_Y + i\zeta)b = 0$ has a similar formulation in terms of the decomposition $L' = L_1 \otimes \delta$. Here it is convenient to lift everything to Γ. So we consider $\mathbb{C}_I \boxtimes L' \to \Gamma$ as the tensor product of $\mathbb{C}_I \boxtimes L_1 \to \Gamma$ and $\delta_\Gamma := (\mathbb{C}_I \boxtimes \delta \to \Gamma)$. Then the transport equation is

$$((\nabla_Y^{L_1} \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_Y^{\delta}) + iH^{\mathrm{sub}})b = 0 \tag{25}$$

On the one hand, ∇^{L_1} is the Chern connection of L_1 with derivative $\nabla^{L_1}_Y$ acting on $\mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L_1)$. On the other hand, $\mathcal{L}^\delta_Y$ is the derivative of $\mathcal{C}^\infty(\Gamma, \delta_\Gamma)$ induced by the Lie derivative $\mathcal{L}_Y$ of Γ through the isomorphism

$$\delta_\Gamma^2 \simeq K_\Gamma \simeq \det(T^*\Gamma) \otimes \mathbb{C}$$

defined by (21). More precisely, $\mathcal{L}^\delta_Y$ is the unique derivative such that $\mathcal{L}_Y(s^2) = 2s \otimes \mathcal{L}^\delta_Y s$ for any section $s \in \mathcal{C}^\infty(\Gamma, \delta_\Gamma)$. Then Formula (25) follows from the relation between ζ and θ and the fact that $\nabla^{L'}_Y b = (\nabla^{L_1}_Y \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^\delta_Y) b$ where ∇^δ is the connection on δ induced by the one on K, which satisfies $\nabla^\delta_Y = \mathcal{L}^\delta_Y - \frac{i}{2}\theta$.

Interestingly, these formulations can be used even when M has no half-form bundle. To give a meaning to Equation (24), we need a square root δ of the restriction of K to the trajectory $\phi_{[0,t]}(x)$ of x on the interval $[0, t]$. This trajectory being an arc or a circle, such a square root exists. In the circle case, there are two square roots up to isomorphism, but it is easy to see that the right-hand side of (24) does not depend on the choice. Similarly we can give a meaning to the transport equation (25) even when M has no half-form bundle. Indeed a differential operator of Γ is determined by its restriction to the open sets of any covering of Γ. And we can always introduce a half-form bundle on the neighborhood of each point of M.

3.3 Norm estimates

The introduction of half-form bundles is also useful when we estimate the norm of a Lagrangian state. For instance, consider a Lagrangian state $\Psi_k(t)$ as in (13). Then, by [9, Theorem 3.2],

$$\|\Psi_k(t)\|^2_{\mathcal{H}_k} = \int_{\Gamma_t} \Omega_t + \mathcal{O}(k^{-1}) \tag{26}$$

where Ω_t is a density on Γ_t, which is given in terms of the principal symbol $b_0(\cdot, t)$ of $\Psi_k(t)$ as follows. We assume that $L' = L_1 \otimes \delta$ with δ a half-form bundle. Again we treat δ and L_1 in completely different ways. On the one hand, L_1 has a natural metric so $L_1 \otimes \overline{L}_1 \simeq \mathbb{C}$. On the other hand, $\delta|_{\Gamma_t}$ being a square root of $\det(T^*\Gamma_t) \otimes \mathbb{C}$, the identity $z\overline{z} = |z^2|$ induces an isomorphism between $\delta|_{\Gamma_t} \otimes \overline{\delta}|_{\Gamma_t}$ and the bundle $|\bigwedge| T^*\Gamma_t \otimes \mathbb{C}$ of densities. So we have an isomorphism

$$L'|_{\Gamma_t} \otimes \overline{L'}_{\Gamma_t} \simeq |\textstyle\bigwedge| T^*\Gamma_t \otimes \mathbb{C}. \tag{27}$$

Then Ω_t is the image of $b_0(\cdot, t) \otimes \overline{b}_0(\cdot, t)$ by (27). When M does not have a half-form bundle, we can still define the isomorphism (27) by working locally and the global estimate (26) still holds. The normalization $(k/2\pi)^{\frac{n}{4}}$ in the definition (13) has been chosen to obtain this formula.

Interestingly the isomorphism (21) is also meaningful for our norm estimates. Indeed, consider now b_0 as a section of $\mathbb{C}_I \boxtimes L' \to \Gamma$; repeating the previous considerations to $(\mathbb{C}_I \boxtimes \delta)|_\Gamma$ and $(\mathbb{C}_I \boxtimes L_1)|_\Gamma$, we define a density Ω on Γ such that

$$\int_I f(t)\, \|\Psi_k(t)\|^2_{\mathcal{H}_k}\, dt = \int_\Gamma (f \circ q)\, \Omega + \mathcal{O}(k^{-1}), \qquad \forall\, f \in \mathcal{C}_0^\infty(I). \tag{28}$$

where q is the projection $\Gamma \to I$. This follows from (26), because Ω_t is the restriction of $\iota_{\partial_t}\Omega$ to Γ_t and a geometric version of Fubini's theorem tells us that $\int_I f(t) \int_{\Gamma_t} \Omega_t = \int_\Gamma (f \circ q)\, \Omega$.

4 The quantum propagator

In this section, we prove Theorem 1.1. We will apply the previous considerations to $M \times \overline{M}$, $L \boxtimes \overline{L}$ and $L' \boxtimes \overline{L}'$ instead of M, L and L'. The holomorphic sections of $(L \boxtimes \overline{L})^k \otimes (L' \boxtimes \overline{L}')$ are the Schwartz kernels of the endomorphisms of $\mathcal{H}_k$.

The symplectic structure of $\overline{M}$ being the opposite of ω, the diagonal Δ_M is a Lagrangian submanifold of $M \times \overline{M}$. There is a canonical flat section $s : \Delta_M \to L \boxtimes \overline{L}$ defined by $s(x, x) = u \otimes \overline{u}$ where $u \in L_x$ is any vector of norm 1. The Lagrangian states corresponding to (Δ_M, s) are the Toeplitz operators up to a factor $(\frac{k}{2\pi})^{\frac{n}{2}}$. More precisely, the Schwartz kernel of $(\frac{k}{2\pi})^{-\frac{n}{2}} T_k(f)$ is a Lagrangian state associated with (Δ_M, s) with principal symbol f, where we identify the restriction of $L' \boxtimes \overline{L}'$ to the diagonal with the trivial line bundle $\mathbb{C}_M = L' \otimes \overline{L}'$ by using the Hermitian metric of L'. This applies in particular to the identity of $\mathcal{H}_k$, which is the Toeplitz operator $T_k(1)$ and is actually a reformulation of a theorem by Boutet de Monvel-Sjöstrand [5, 7].

By Theorem 2.6, the Schwartz kernel of the quantum propagator $(U_{k,t})$ multiplied by $(\frac{k}{2\pi})^{-\frac{n}{2}}$ is a Lagrangian state family, associated with the graph of ϕ_t and its prequantum lift. Indeed, in the Schrödinger equation (1), we can interpret the product $T_{k,t} U_{k,t}$ as the action of the Toeplitz operator $T_{k,t} \boxtimes \mathrm{id}$ on $U_{k,t}$. Its principal symbol is $H_t \boxtimes 1$, so its Hamiltonian flow is $\phi_t \boxtimes \mathrm{id}$. There is no difficulty to deduce Formula (8) from Proposition 2.8 except for the relation between $\mathcal{E}_t$ and $\mathcal{D}_t$.

Lemma 4.1. *$\mathcal{E}_t(x, x)(\mathrm{id}_{K_x}) = \mathcal{D}_t(x)$.*

Everything relies on the identification (19) which in our case is an isomorphism between $K_{\phi_t(x)} \otimes \overline{K}_x$ and the space of volume forms on the graph of $T_x\phi_t$. On the one hand, the elements of $K_{\phi_t(x)} \otimes \overline{K}_x$ will be viewed as morphisms from K_x to $K_{\phi_t(x)}$. On the other hand, the graph of $T_x\phi_t$ is naturally isomorphic with T_xM through the map $\xi \to (T_x\phi_t(\xi), \xi)$. So (19) becomes an isomorphism

$$\mathrm{Mor}(K_x, K_{\phi_t(x)}) \simeq \det(T_x^*M) \otimes \mathbb{C} \tag{29}$$

Now the tangent map to the flow $\phi_t \boxtimes \mathrm{id}$ sends the graph of $T_x\phi_0$ to the graph of $T_x\phi_t$, and with our identifications, it becomes the identity of T_xM. So the map $\mathcal{E}_t(x,x)$ is the isomorphism

$$\mathrm{Mor}(K_x, K_x) \simeq \mathrm{Mor}(K_x, K_{\phi_t(x)})$$

otained by applying (29) with $t = 0$ and then the inverse of (29).

Proof of Lemma 4.1, technical part. First we claim that (29) sends a morphism $\psi : K_x \to K_{\phi_t(x)}$ to

$$((T_x\phi_t)^*\psi(\alpha)) \wedge \overline{\alpha}$$

where $\alpha \in K_x$ is any vector with norm 1. Indeed, ψ is first identified with $\psi(\alpha)\otimes\overline{\alpha} \in K_{\phi_t(x)}\otimes\overline{K}_x$. Then it is viewed as the $2n$-form of $T_xM\oplus T_xM$ given by $p_1^*\psi(\alpha) \wedge p_2^*\overline{\alpha}$ where p_1 and p_2 are the projections $T_xM \oplus T_xM \to T_xM$ onto the first and the second factor respectively. Then it is restricted to the graph of $T_x\phi_t$ which is identified with T_xM via the map $h(\xi) = (T_x\phi_t(\xi), \xi)$, so we obtain

$$h^*(p_1^*\psi(\alpha) \wedge p_2^*\overline{\alpha}) = ((T_x\phi_t)^*\psi(\alpha)) \wedge \overline{\alpha}$$

because $p_1 \circ h = T_x\phi_t$ and $p_2 \circ h = \mathrm{id}$.

For $t = 0$ and $\psi = \mathrm{id}$, we have $((T_x\phi_t)^*\psi(\alpha)) \wedge \overline{\alpha} = \alpha \wedge \overline{\alpha}$. So we have to prove that for $\beta = \mathcal{D}_t(x)(\alpha)$

$$((T_x\phi_t)^*\beta) \wedge \overline{\alpha} = \alpha \wedge \overline{\alpha} \tag{30}$$

This is equivalent to $j^*(T_x\phi_t)^*\beta = \alpha$ where j is the injection $T_x^{1,0}M \to T_xM \otimes \mathbb{C}$. Since $\beta \in K_{\phi_t(x)}$, we have $\pi^*\beta = \beta$ where π is the projection of $T_{\phi_t(x)}M \otimes \mathbb{C}$ onto the $(1,0)$-subspace with kernel the $(0,1)$-subspace. So we have to show that $(\pi \circ (T_x\phi_t) \circ j)^*\beta = \alpha$. But $\pi \circ (T_x\phi_t) \circ j = (T_x\phi_t)^{1,0}$, so (30) is equivalent to $((T_x\phi_t)^{1,0})^*\beta = \alpha$. And this last equality is actually the definition of $\beta = \mathcal{D}_t(x)(\alpha)$. ∎

Theorem 4.2. *Let $(T_{k,t},\ t \in I)$ be a smooth family of Toeplitz operators with real principal symbol H_t and subprincipal symbol H_t^{sub}. Then the Schwartz kernel of the quantum propagator of $(T_{k,t})$ multiplied by $(\frac{k}{2\pi})^{-\frac{n}{2}}$ is a Lagrangian state family associated with (Γ, s, σ) given by $\Gamma = \{(t, \phi_t(x), x)/\ t \in I,\ x \in M\}$ and*

$$\begin{cases} s(t, \phi_t(x), x) = \phi_t^L(x) : L_x \to L_{\phi_t(x)}, \\ \sigma(t, \phi_t(x), x) = \left[\rho_t(x)\right]^{\frac{1}{2}} e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr} \mathcal{T}_t^{L'}(x) : L'_x \to L'_{\phi_t(x)}, \end{cases}$$

where (ϕ_t) is the Hamiltonian flow of H_t, ϕ_t^L its prequantum lift, $\mathcal{T}_t^{L'}$ its parallel transport lift to L' and $\mathcal{D}_t(x) = \rho_t(x)\mathcal{T}_t^K(x)$ with $\mathcal{D}_t(x) = K(T_x\phi_t) : K_x \to K_{\phi_t(x)}$.

As explained in the introduction, it is very natural to express the symbol by using a half-form bundle:

$$\sigma(t, \phi_t(x), x) = e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\, dr} \mathcal{T}_t^{L_1}(x) \otimes \left[\mathcal{D}_t(x)\right]^{\frac{1}{2}}$$

where $L' = L_1 \otimes \delta$ and $\left[\mathcal{D}_t(x)\right]^{\frac{1}{2}} : \delta_x \to \delta_{\phi_t(x)}$ is the continuous square root of $\mathcal{D}_t(x)$ equal to 1 at $t = 0$.

Remark 4.3. In our next paper on trace formulas, we will use the following expression for $\rho_t(x)$. Denote by $\gamma : \mathbb{R} \to M$, $t \mapsto \phi_t(x)$ the trajectory of x. Choose a unitary frame s_K of $\gamma^* K$ and write $\nabla s_K = \frac{1}{i} f_K dt \otimes s_K$. Then

$$\rho_t(x) = c_t e^{-i\int_0^t f_K(r)\, dr} \qquad \text{where} \qquad c_t \Omega_t(\xi_t u) = \Omega_0(\xi_0 u). \tag{31}$$

Here u is any generator of $\wedge^{\mathrm{top}} T_x M$, ξ_t is the linear map $T_x M \to T_{\gamma(t)}M \oplus T_x M$ sending X into $(T_x\phi_t(X), X)$, and Ω_t is the $2n$-form of $T_{\gamma(t)}M \oplus T_x M$ equal to $\Omega_t = p_1^* s_K(t) \wedge p_2^* \overline{s}_K(0)$, p_1 and p_2 being the projection on $T_{\gamma(t)}M$ and $T_x M$ respectively.

The proof of (31) is that on the one hand $\mathcal{E}_t(x,x)\Omega_0 = c_t\, \Omega_t$ and on the other hand $\mathcal{T}_t^{K \boxtimes \bar{K}}(x,x)\Omega_0 = e^{i\int_0^t f_K(r) dr}\Omega_t$. ☐

5 Fourier Transform of Lagrangian state families

In this section, we investigate how the (inverse) semiclassical Fourier transform acts on the Lagrangian state families introduced in Section 2.1. It turns out that the outcomes are states which are associated with Lagrangians that are only immersed; hence we need to generalize the usual definition of Lagrangian states recalled at the beginning of Section 2.

5.1 Symplectic preliminaries

Consider the same data $\Gamma \subset I \times M$ and $s \in \mathcal{C}^\infty(\Gamma, \mathbb{C}_I \boxtimes L)$ as in Section 2.1. So we assume that $\Gamma \to I$, $(t,x) \mapsto t$ is a proper submersion and that for any $t \in I$, Γ_t is a Lagrangian submanifold of M and the restriction of s to Γ_t is flat and unitary. Recall that $\nabla s = i\tau dt \otimes s$ for a function $\tau \in \mathcal{C}^\infty(\Gamma)$, cf (14). For any E in $\mathbb{R}$, introduce

$$\Gamma^E := \{(t,x) \in \Gamma \mid \tau(t,x) + E = 0\}. \tag{32}$$

Proposition 5.1. *Let E be a regular value of $-\tau$. Then*

1. *Γ^E is a submanifold of Γ and $j_E : \Gamma^E \to M$, $(t,x) \mapsto x$ is a Lagrangian immersion,*
2. *$j : \Gamma \to \mathbb{R} \times M$, $(t,x) \mapsto (\tau(t,x), x)$ is an immersion at any $(t_0, x_0) \in \Gamma^E$,*
3. *the section s^E of $(j^E)^* L$ given by $s^E(t,x) = e^{itE} s(t,x)$ is flat.*

Proof. For any tangent vectors $Y_1 = (a_1, \xi_1)$, $Y_2 = (a_2, \xi_2)$ in $T_{(t_0,x_0)}\Gamma \subset \mathbb{R} \oplus T_{x_0}M$, we have

$$\omega(\xi_1, \xi_2) = a_1 d\tau(Y_2) - a_2 d\tau(Y_1) \tag{33}$$

To prove this, we extend Y_1 and Y_2 to vector fields of Γ on a neighborhood of (t_0, x_0) so that $[Y_1, Y_2] = 0$. Then, the curvature of ∇ being $\frac{1}{i}\omega$, we have that

$$[\nabla_{Y_1}, \nabla_{Y_2}] = \tfrac{1}{i}\omega(\xi_1, \xi_2).$$

Furthermore, since $dt(Y_j) = a_j$, we have that $\nabla_{Y_j} s = i a_j \tau \otimes s$ so $\nabla_{Y_1}\nabla_{Y_2} s = i((Y_1.a_2)\tau + a_2(Y_1.\tau))s - a_1 a_2 \tau^2 s$. Using that $Y_1.a_2 - Y_2.a_1 = [Y_1, Y_2].t = 0$, it comes that

$$[\nabla_{Y_1}, \nabla_{Y_2}]s = i(a_2(Y_1.\tau) - a_1(Y_2.\tau))s.$$

Comparing with the previous expression for the curvature, we obtain (33).

We prove the second assertion. Assume $Y_1 = (a_1, \xi_1)$ is in the kernel of the tangent map of $j : \Gamma \to \mathbb{R} \times M$, that is $d\tau(Y_1) = 0$ and $\xi_1 = 0$. Then (33) writes $0 = a_1 d\tau(Y_2)$. If $a_1 \neq 0$, this implies that $d\tau(Y_2) = 0$ for any $Y_2 \in T_{(t_0,x_0)}\Gamma$, which contradicts the assumption that $-E$ is a regular value of τ.

This implies that j_E is an immersion. It is Lagrangian by (33) again because if Y_1, Y_2 are tangent to Γ^E, then $d\tau(Y_1) = d\tau(Y_2) = 0$, so $\omega(\xi_1, \xi_2) = 0$. Finally, $\nabla(e^{itE}s) = (iEdt + i\tau dt) \otimes e^{itE}s = 0$ on Γ^E. ☐

5.2 Immersed Lagrangian states

We will adapt the definition of Lagrangian states for immersed manifolds. Suppose we have a Lagrangian immersion $j : \mathcal{N} \to M$, a flat unitary section s of $j^* L$ and a formal series $\sum \hbar^\ell b_\ell$ with coefficients $b_\ell \in \mathcal{C}^\infty(j^* L')$.

First, for any $y \in \mathcal{N}$, we will define a germ of Lagrangian state at $j(y)$, uniquely defined up to $\mathcal{O}(k^{-\infty})$ as follows. Let us assume temporarily that there exists an open set V in M such that $j : \mathcal{N} \to V$ is a proper embedding, so that $j(\mathcal{N})$ is a closed submanifold of V. Then we can introduce sections $F : V \to L$ and $a_\ell : V \to L'$ such that $\overline{\partial} F$ and $\overline{\partial} a_\ell$ vanish to infinite order along $j(\mathcal{N})$, $j^* F = s$ and $j^* a_\ell = b_\ell$ and $|F| < 1$ on $V \setminus j(\mathcal{N})$. These sections are not unique but if $(F', a'_\ell,\ \ell \in \mathbb{N})$ satisfy the same condition, then for any N,

$$F^k \sum_{\ell=0}^{N} k^{-\ell} a_\ell = (F')^k \sum_{\ell=0}^{N} k^{-\ell} a'_\ell + \mathcal{O}(k^{-N-1}) \tag{34}$$

the $\mathcal{O}$ being uniform on any compact set of V. This follows on one hand from the fact that $|F|$ and $|F'|$ are < 1 on $V \setminus j(\mathcal{N})$, so that both sides of (34) are in $\mathcal{O}(k^{-N-1})$ uniformly on any compact set of $V \setminus j(\mathcal{N})$. On the other hand, the sections F, F' and a_ℓ, a'_ℓ have the same Taylor expansions along $j(\mathcal{N})$ which implies (34) on a neighborhood of $j(\mathcal{N})$, (see [8, Section 2.2] for details).

Back to a general immersion $\mathcal{N} \to M$, for any $y \in \mathcal{N}$, by the local normal form for immersions, there exists open neighborhoods U and V of y and $j(y)$ respectively such that $j(U) \subset V$ and j restricts to a closed embedding from U into V. Then we can introduce the sections F and a_ℓ, $\ell \in \mathbb{N}$ as above on V, which extend the restrictions of s and b_ℓ to U. This defines the expansion

$$\Psi_{N,k} := F^k \sum_{\ell=0}^{N} k^{-\ell} a_\ell \tag{35}$$

on V. If we have another set of data (U', V', F', a'_ℓ), we obtain another sequence $\Psi'_{N,k} := (F')^k \sum_{\ell=0}^{N} k^{-\ell} a'_\ell$ on V'.

Lemma 5.2. *For any N, $\Psi_{N,k} = \Psi'_{N,k} + \mathcal{O}(k^{-N-1})$ on a neighborhood of $j(y)$.*

So we have a well defined germ of Lagrangian states at $j(y)$.

Proof. Choose open sets W and W' of V and V' respectively such that $j(U \cap U') = j(U) \cap W = j(U') \cap W'$. Set $U'' = U \cap U'$ and $V'' = W \cap W'$. Then

j restricts to an embedding from U'' into V'' and $j(U'') = j(U') \cap V''$. So the restriction of F, a_ℓ to V'' gives us a new set of data $(U'', V'', F|_{V''}, a_\ell|_{V''})$. The fact that $j(U'') = j(U') \cap V''$ is used to see that $|F| < 1$ on $V'' \setminus j(U'')$. Similarly, we can restricts F', a'_ℓ to V'' and get $(U'', V'', F'|_{V''}, a'_\ell|_{V''})$. The final result follows from our initial remark (34). □

Now assume that there exists a compact subset K of $\mathcal{N}$ such that for each ℓ, a_ℓ is supported in K. Our goal is to construct a Lagrangian state which, on a neighborhood of each $x \in M$, is equal to the sum of the local Lagrangian states defined previously for each $y \in j^{-1}(x) \cap K$. An essential observation is that $j^{-1}(x)$ is discrete in $\mathcal{N}$, so $j^{-1}(x) \cap K$ is finite since K is compact.

Lemma 5.3. *There exists a family $(\Psi_k \in \mathcal{H}_k)$ such that for any $x \in M$ and any N:*

1. *if $x \notin j(K)$, $|\Psi_k| = \mathcal{O}(k^{-N})$ on a neighborhood of x,*
2. *if $j^{-1}(x) \cap K = \{y_i,\ i \in I\}$, then $\Psi_k = \sum_i \Psi^i_{N,k} + \mathcal{O}(k^{-N})$ on a neighborhood of x, where each $\Psi^i_{N,k}$ is defined as in (35) with $y = y_i$.*

We will call (Ψ_k) a Lagrangian state associated with the Lagrangian immersion $j : \mathcal{N} \to M$, the flat unitary section s of j^*L and the formal series $\sum \hbar^\ell b_\ell$ with coefficients in $\mathcal{C}^\infty(\mathcal{N}, j^*L')$. (Ψ_k) is unique up to $\mathcal{O}(k^{-\infty})$. But unlike the case of a Lagrangian submanifold, we can not recover the symbol $\sum \hbar^\ell b_\ell$ from the state by taking the restriction to $j(\mathcal{N})$ because of the possible multiple points.

Proof. Consider an open set V of M and a finite family $(U_i)_{i \in I}$ of disjoint open sets of $\mathcal{N}$ such that for any $i \in I$, j restricts to a proper embedding from U_i into V and $K \cap j^{-1}(V) \subset \bigcup U_i$. Then introduce sections F_i, $a_{i,\ell}$ on V as above associated with each submanifold $j(U_i)$. Consider the sum

$$\left(\frac{k}{2\pi}\right)^{\frac{n}{4}} \sum_{i \in I} F_i^k(x) \sum_{\ell=0}^{N} k^{-\ell} a_\ell(x), \qquad x \in V. \tag{36}$$

Then for any $x \in V$, each $y \in j^{-1}(x) \cap K$ belongs to one of the U_i, so on a neighborhood of x, (36) is equal to the sum of the Lagrangian state germs associated with the $y \in j^{-1}(x) \cap K$. So by the previous discussion, the state defined by (36) on a neighborhood of x does not depend, up to $\mathcal{O}(k^{-\infty})$, on the choice of V and of $(U_i, F_i, a_{i,\ell})$ for $i \in I$ and $\ell \in \mathbb{N}$. It is not difficult to prove that any point x of M has an open neighborhood

V admitting a family (U_i) as above. Indeed, if we set $I = j^{-1}(x) \cap K$, then for any $y \in I$, there exists a pair $(U_y \ni y, V_y)$ such that j restricts to a proper embedding from U_y into V_y. Then we choose for V a sufficiently small neighborhood of x in $\cap_y V_y$ such that $j^{-1}(V) \cap K \subset \cup_y U_y$ and we restrict the U_y accordingly. So with a partition of unity, we can construct global states $\Psi_k \in \mathcal{C}^\infty(M, L^k \otimes L'), k \in \mathbb{N}$, such that for any data $(V, U_i, i \in I)$ as above, Ψ_k is equal to (36) on V up to a $\mathcal{O}(k^{-\infty})$, uniform on any compact subset of V. Since $\overline{\partial}\Psi_k$ is in $\mathcal{O}(k^{-\infty})$, we can replace Ψ_k by its projection onto $\mathcal{H}_k$, which only modifies it by a $\mathcal{O}(k^{-\infty})$ by Kodaira-Hörmander estimates [23, 18]. ☐

5.3 Fourier transform

Introduce the $\hbar$-Fourier transform and its inverse with parameter $k = \hbar^{-1}$

$$\mathcal{F}_k(f)(t) = \Big(\frac{k}{2\pi}\Big)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-iktE} f(E)\, dE, \quad \mathcal{F}_k^{-1}(g)(E) = \Big(\frac{k}{2\pi}\Big)^{\frac{1}{2}} \int_{\mathbb{R}} e^{iktE} g(t)\, dt.$$

We are now ready to state the main result of this section.

Theorem 5.4. *Let $(\Psi_k \in \mathcal{C}^\infty(I, \mathcal{H}_k),\ k \in \mathbb{N})$ be a Lagrangian state family associated with (Γ, s) and such that the Ψ_k are supported in a compact set of I independent of k. Let $-E$ be a regular value of τ and $\Gamma^E := \tau^{-1}(-E)$, using the notation introduced before and in Equation (32).*

Then $\mathcal{F}_k^{-1}(\Psi_k)(E)$ is a Lagrangian state associated with the Lagrangian immersion $j_E : \Gamma^E \to M$, $(t,x) \mapsto x$, the section $s^E \in \mathcal{C}^\infty(\Gamma^E, j_E^* L)$ given by $s^E(t,x) = e^{itE} s(t,x)$ and the principal symbol

$$\sigma^E(t,x) = B(t,x)^{-\frac{1}{2}} \sigma(t,x), \qquad (t,x) \in \Gamma^E$$

where σ is the principal symbol of (Ψ_k) and $B(t,x)$ is such that $d\tau \wedge \alpha = iB(t,x)\, dt \wedge \alpha$ on $T_{(t,x)}\Gamma$ for any $\alpha \in K_x$, the square root $B(t,x)^{1/2}$ having a non negative real part.

We already explained that for a non-zero $\alpha \in K_x$, $dt \wedge \alpha$ is nonzero on $T_{(t,x)}\Gamma$. By the second assertion of Proposition 5.1, the same argument shows that $d\tau \wedge \alpha$ is non zero on $T_{(t,x)}\Gamma$ when $(t,x) \in \Gamma^E$. This proves that $B(t,x)$ is uniquely determined and nonzero as well.

Proof. Introduce a local unitary frame u of L and write $F(t,x) = e^{if(t,x)} u(x)$, where F is the section appearing in the definition (13) of Ψ_k. Then

$$\mathcal{F}_k^{-1}(\Psi_k)(E)(x) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4} + \frac{1}{2}} u^k(x) \int_{\mathbb{R}} e^{ik\phi(t,x)} a(t,x,k)\, dt \qquad (37)$$

with $\phi(t,x) = tE + f(t,x)$. The imaginary part of ϕ is nonnegative. It vanishes when $|F(t,x)| = 1$, that is when $(t,x) \in \Gamma$. We have

$$\phi'_t(t,x) = E + f'_t(t,x), \qquad \phi''_{tt}(t,x) = f''_{tt}(t,x).$$

Here $g \mapsto g'_t$ means differentiation with respect to t.

We claim that the function f'_t is an extension of τ such that $\overline{\partial} f'_t$ vanishes to infinite order along Γ. Indeed, by taking the restriction of ∇F to Γ, we obtain that $f'_t = \tau$ on Γ (see also the argument in the proof of Proposition 2.4). Then since $\overline{\partial} F$ vanishes to infinite order along Γ, the same holds for $\overline{\partial} f + \frac{\overline{\partial} u}{u}$, and by taking the derivative with respect to t, the same holds for $\overline{\partial} f'_t$.

So for $(t,x) \in \Gamma$, $\phi'_t(t,x) = 0$ if and only if $(t,x) \in \Gamma^E$. So by Lemma 7.7.1 in [20], if $(t_0, x_0) \notin \Gamma^E$, then the integral in (37) restricted to a neighborhood of t_0 is in $\mathcal{O}(k^{-\infty})$ on a neighborhood of x_0. So to estimate (37) on a neighborhood of a point x_0, it suffices to integrate on a neighborhood of $j_E^{-1}(x_0)$. Let V, U be neighborhoods of x_0 and $t_0 \in j_E^{-1}(x_0)$ respectively such that $W = \Gamma^E \cap (U \times V)$ is a graph $\{(t(x), x),\, x \in j_E(W) \cap V\}$.

Since $f'_t = \tau$ and $\overline{\partial} f'_t = 0$ on Γ, we have $f''_{tt} dt + \partial f'_t = d\tau$ on Γ. Mutiplying by $\alpha \in \Omega^{n,0}(M)$, we get $f''_{tt} dt \wedge \alpha = d\tau \wedge \alpha$ on Γ. As explained before the proof, $d\tau \wedge \alpha$ does not vanish on $T_{t,x}\Gamma$, so f''_{tt} does not vanish on Γ^E and we can apply the stationary phase lemma for a complex valued phase, see [25, Theorem 2.3] or [20, Theorem 7.7.12]. This theorem implies that on a neighborhood of $j_E(W)$

$$\Big(\frac{k}{2\pi}\Big)^{\frac{1}{2}} \int_U e^{ik\phi(t,x)} a(t,x,k)\, dt = e^{ik\phi_E(x)} \sum_{\ell=0}^{N} k^{-\ell} a_{E,\ell}(x) + \mathcal{O}(k^{-N-1}) \qquad (38)$$

for any N, where

$$\phi_E(x) = \phi(T(x), x), \qquad a_{E,0}(x) = (-i\phi''_{tt}(t,x))^{-\frac{1}{2}} a_0(T(x), x),$$

the square root having a non negative real part. Here $T : U \to \mathbb{C}$ is an extension of $x \to t(x)$, that is $(T(x), x) \in \Gamma^E$ when $x \in j_E(W)$. The extension is chosen so that $\phi'_t(T(x), x) = 0$, where ϕ itself has been extended almost analytically to a neighborhood of $\mathbb{R} \times M$ in $\mathbb{C} \times M$. We claim that $F(x) = e^{i\phi_E(x)} u(x)$ is adapted to $(j_E(W), s^E|_W)$. First if $x \in j_E(W)$, then

$$e^{i\phi_E(x)} u(x) = e^{it(x)E + if(t(x),x)} u(x) = e^{it(x)E} s(t(x), x) = s^E(t(x), x)$$

It remains to show that $\overline{\partial} F$ vanishes to infinite order along $j_E(W)$. Assume first that we can choose the section F to be holomorphic, so that ϕ'_t depends holomorphically on x. If furthermore we can extend ϕ'_t so that it depends holomorphically on t, then by the holomorphic version of the implicit function theorem T is holomorphic and F is holomorphic as well. In general, we only know that $\overline{\partial} F$ vanishes to infinite order along Γ and by adapting the previous argument, we conclude that $\overline{\partial} F$ vanishes to infinite order along $j_E(W)$.

With a similar proof, we can also show that the same holds for the coefficients: $\overline{\partial} a_{E,\ell} \equiv 0$ along $j_E(W)$ to infinite order. However, it is actually easier to use the following fact: $F^k \sum k^{-\ell} b_\ell = \mathcal{O}(k^{-\infty})$ if and only if all the coefficients b_ℓ vanish to infinite order along $j_E(W)$. And here we know that $\overline{\partial}(F^k \sum k^{-\ell} a_{E,\ell}) = \mathcal{O}(k^{-\infty})$ by differentiating (38) under the integral sign. $\square$

6 Spectral projector

Consider a self-adjoint operator $\hat{H}$ acting on a finite dimensional Hilbert space $\mathcal{E}$. Here it is important that $\hat{H}$ is time-independent. Introduce a smooth function $f : \mathbb{R} \to \mathbb{C}$ with smooth compactly supported Fourier transform $\hat{f}$. We will work with the unitary Fourier transform, so

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itE} f(E)\, dE, \qquad f(E) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itE} \hat{f}(t)\, dt$$

The second formula directly gives

$$f(\hbar^{-1}(E-\lambda)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{itE}{\hbar}}\, e^{-\frac{it\lambda}{\hbar}}\, \hat{f}(t)\, dt$$

with $\hbar$ and λ two real parameters. Doing a spectral decomposition $\hat{H} = \sum \lambda \Pi_\lambda$ where the λ and Π_λ are the eigenvalues and spectral projectors, and introducing the quantum propagator $U_t = \exp(-\frac{it\hat{H}}{\hbar}) = \sum e^{-\frac{it\lambda}{\hbar}} \Pi_\lambda$, we obtain

$$f(\hbar^{-1}(E-\hat{H})) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{itE}{\hbar}}\, U_t\, \hat{f}(t)\, dt$$

We can apply this to our Toeplitz operator ($T_k : \mathcal{H}_k \to \mathcal{H}_k$) with quantum propagator $U_{k,t}$, which gives

$$f\big(k(E-T_k)\big) = k^{-\frac{1}{2}} \mathcal{F}_k^{-1}(\hat{f}(t) U_{k,t})(E) \tag{39}$$

If E is a regular value of the principal symbol H of T_k, we deduce by Theorem 5.4 that the Schwartz kernel of this operator is a Lagrangian state. This will be done in Section 6.2 and will prove Theorem 1.2 of the introduction. Before that, we will consider the simpler case of a state ($\Psi_k \in \mathcal{H}_k$):

$$f((k(E - T_k))\Psi_k = k^{-\frac{1}{2}} \mathcal{F}_k^{-1}(\hat{f}(t)\Psi_{k,t})(E) \tag{40}$$

where $\Psi_{k,t}$ is the solution of $(\frac{1}{ik}\partial_t + T_k)\Psi_{k,t} = 0$ with initial condition Ψ_k.

6.1 Lagrangian state spectral decomposition

Let Γ_0 be Lagrangian submanifold of M and let H be an autonomous Hamiltonian with flow ϕ_t. Set $\Gamma_t = \phi_t(\Gamma_0)$ and $\Gamma = \{(t,x) \mid x \in \Gamma_t\}$. Let $E \in \mathbb{R}$ be a regular value of the restriction of H to Γ_0 and define

$$\Gamma^E = \{(t,x) \mid x \in \Gamma_t \cap H^{-1}(E)\}.$$

This submanifold is the same as the submanifold Γ^E defined in (32) from a (local) flat section s_0 of $L \to \Gamma_0$, because the corresponding function τ is the restriction of $-H$ to Γ, see (16). Furthermore our assumption on E is equivalent to the fact that $-E$ is regular value of τ.

The computation of the symbol of $f(k(E-T_k))\Psi_k$ in terms of the symbol of $\Psi_{k,0}$ will amount to transforming a volume element of Γ_0 into a volume element of Γ^E. Let us explain this. We denote by X the Hamiltonian vector field of H and by $X_x^{\perp_\omega}$ the symplectic orthogonal of $\mathbb{R}X_x$ in T_xM. For any $(t,x) \in \Gamma^E$, the Lagrangian space

$$\mathfrak{L}_{(t,x)} := \mathbb{R}X_x \oplus (T_x\Gamma_t \cap X_x^{\perp_\omega}) \tag{41}$$

is the image of $T_{(t,x)}j_E$ with $j_E : \Gamma^E \to M$ the projection $(t,x) \mapsto x$. Observe that X_x does not belong to $T_x\Gamma_t$ because E is a regular value of $H|_{\Gamma_0}$, so it a regular value of $H|_{\Gamma_t}$ as well.

Assume now $t = 0$ and $(0,x) \in \Gamma^E$. Choose $\eta \in T_x\Gamma_0$ such that $\omega(X_x, \eta) = 1$. Then we have

$$T_x\Gamma_0 = \mathbb{R}\eta \oplus (T_x\Gamma_0 \cap X_x^{\perp_\omega}), \qquad \mathfrak{L}_{(0,x)} = \mathbb{R}X_x \oplus (T_x\Gamma_0 \cap X_x^{\perp_\omega}).$$

Starting from $v \in \det T_x\Gamma_0$, we write $v = \eta \wedge w$ with $w \in \det((T_x\Gamma_0) \cap X_x^{\perp_\omega})$ and we set $v(0,x) := X_x \wedge w \in \det \mathfrak{L}_{(0,x)}$. This definition makes sense because η is unique modulo $(T_x\Gamma_0 \cap X_x^{\perp_\omega})$ so that w is unique. More generally, if t is any real and $(0,x) \in \Gamma^E$, we set $v(t,x) := (T_x\phi_t)_* v(0,x) \in \det \mathfrak{L}_{t,\phi_t(x)}$, viewing $T_x\phi_t$ as a map from $\mathfrak{L}_{(0,x)}$ to $\mathfrak{L}_{(t,\phi_t(x))}$. Equivalently

$$v(t,x) = X_{\phi_t(x)} \wedge (T_x\phi_t)_* w. \tag{42}$$

We now define a map $\mathcal{E}'_t(x) : K_x \to K_{\phi_t(x)}$ for any $x \in \Gamma_0 \cap H^{-1}(E)$ by

$$(\mathcal{E}'_t(x)\alpha)(v(t,x)) = -i\alpha(v), \qquad \forall v \in \det T_x\Gamma_0. \tag{43}$$

We define the function C'_t by the equality $\mathcal{E}'_t(x) = C'_t(x)\mathcal{T}_t^K(x)$.

Proposition 6.1. *Let (Ψ_k) be a Lagrangian state of M associated with (Γ_0, s_0) with symbol $\sigma_0 \in \mathcal{C}^\infty(\Gamma_0, L')$, (T_k) a self-adjoint Toeplitz operator with principal and subprincipal symbol H, H^{sub}, and $f \in \mathcal{C}^\infty(\mathbb{R})$ having a smooth compactly supported Fourier transform.*

If E is a regular value of $H|_{\Gamma_0}$, then $\Psi'_k = k^{\frac{1}{2}} f((k(E - T_k))\Psi_k$ is a Lagrangian state associated with the Lagrangian immersion $j_E : \Gamma^E \to M$, the flat section s^E of $j_E^* L$ given by $s^E(t, \phi_t(x)) = \mathcal{T}_t^L(x)s_0(x)$ and the symbol $\sigma^E \in \mathcal{C}^\infty(j_E^* L')$ defined as

$$\sigma^E(t, \phi_t(x)) = \hat{f}(t)\, C'_t(x)^{\frac{1}{2}}\, e^{\frac{1}{i}\int_0^t H^{\mathrm{sub}}(\phi_r(x))\, dr}\, \mathcal{T}_t^{L'}(x)\, \sigma_0(x).$$

where the square root is chosen so as to be continuous and to have a positive real part at $t = 0$.

Proof. The solution of the Schrödinger equation with initial condition Ψ_k is described as a Lagrangian state associated with $(\Gamma, s(t,x) = \phi_t^L(x)s_0(x))$ in Theorem 2.6. Then Ψ'_k is the k-Fourier transform of this solution (40), so by Theorem 5.4, it is a Lagrangian state associated with the immersion $j_E : \Gamma^E \to M$ and the section $s^E(t,x) = e^{itE}s(t,x) = \mathcal{T}_t^L(x)s_0(x)$ because for an autonomous Hamiltonian, $\phi_t^L = e^{-itH}\mathcal{T}_t^L$, see (3).

It remains to check the formula for the principal symbol. By Proposition 2.8, we have to prove that $C'_t(x) = \frac{C_t(x)}{B(t,\phi_t(x))}$, that is $\mathcal{E}'_t(x) = \frac{\mathcal{E}_t(x)}{B(t,\phi_t(x))}$, with B the function of Theorem 5.4. Comparing the definitions (22) and (43) of $\mathcal{E}_t(x)$ and $\mathcal{E}'_t(x)$, we have to show that for any $\beta \in K_{\phi_t(x)}$ and $v \in \det(T_x\Gamma_0)$,

$$B(t, \phi_t(x)) = \frac{i\beta(v(t,x))}{\beta((T_x\phi_t)_* v)} \tag{44}$$

where $v(t,x)$ is as in Equation (42). Let us first explain the proof at $t = 0$. Recall that $v(0,x) = X_x \wedge w$ and $v = \eta \wedge w$. Now B is defined by the relation $d\tau \wedge \beta = iB\, dt \wedge \beta$ on Γ for every $\beta \in K$. We have

$$T_{(0,x)}\Gamma = \mathbb{R}(1, X_x) \oplus \mathbb{R}(0, \eta) \oplus \{(0,\xi),\, \xi \in (T_x\Gamma_0) \cap X_x^{\perp_\omega}\}$$

and $d\tau(1, X_x) = 0$ so that $d\tau(0,\eta) = 1$ and $d\tau(0,\xi) = 0$ for any $\xi \in T_x\Gamma_0 \cap X_x^{\perp_\omega}$ by (33). So evaluating the relation $d\tau \wedge \beta = iB\, dt \wedge \beta$ on $(1, X_x) \wedge (0,\eta) \wedge (0,\xi_2) \wedge \ldots \wedge (0,\xi_n)$ where $w = \xi_2 \wedge \ldots \wedge \xi_n$, we get

$$-\beta(X_x \wedge w) = iB(0,x)\beta(\eta \wedge w), \tag{45}$$

which gives (44). The proof for $t \neq 0$ is exactly the same where all the symplectic data X_x, η, w, Γ_0 are replaced by their image under ϕ_t.

The last point is the determination of the square root: we have $(C_t'(x))^{1/2} = C_t(x)^{1/2}/B(t,\phi_t(x))^{1/2}$, with $C_0(x)^{1/2} = 1$ and the square root $B(t,x)^{1/2}$ has a non negative real part by Theorem 5.4. It is even positive as explained in Remark 6.2. ☐

Remark 6.2. The quantity $C_0'(x) = B(0,x)^{-1}$ can be computed explicitly as follows. For $x \in H^{-1}(E) \cap \Gamma_0$,

$$B(0,x) = \|X_1\|^2 + i\omega(X_1, X_2) \tag{46}$$

where $X_x = X_1 + X_2$ with $X_1 \in j_x(T_x\Gamma_0)$ and $X_2 \in T_x\Gamma_0$. Recall that $X_x \notin T_x\Gamma_0$, so $\|X_1\|^2 \neq 0$.

Proof of (46). We set $\eta = \|X_1\|^{-2} j_x X_1$ and compute $B(0,x)$ from (45). On the one hand, β being a $(n,0)$ form, $\beta(X_1 \wedge w) = -i\beta(jX_1 \wedge w) = -i\|X_1\|^2 \beta(\eta \wedge w)$. On the other hand, $X_2 = \omega(X_1, X_2)\eta$ plus a linear combination of the ξ_i, so $X_2 \wedge w = \omega(X_1,X_2)\eta \wedge w$. Gathering these equalities we get

$$\beta(X_x \wedge w) = \Big(\omega(X_1, X_2) - i\|X_1\|^2\Big)\, \beta(\eta \wedge \omega)$$

and the conclusion follows. ☐

6.2 Smoothed spectral projector

Recall that (T_k) is a self-adjoint Toeplitz operator with principal symbol H and subprincipal symbol H^{sub} and that $f \in \mathcal{C}^\infty(\mathbb{R})$ has a smooth compactly supported Fourier transform.

Theorem 6.3. *Let E be a regular value of H. Then the Schwartz kernel of $f(k(E - T_k))$ is a Lagrangian state associated with the Lagrangian immersion $j_E : \Gamma^E \to M^2$, the flat section $s^E \in \mathcal{C}^\infty(j^*\Gamma^E)$ and the symbol $\sigma^E \in \mathcal{C}^\infty(j_E^* L')$ given by $\Gamma^E = \mathbb{R} \times H^{-1}(E)$, $j_E(t,x) = (\phi_t(x), x)$, $s^E(t,x) = \mathcal{T}_t^L(x)$ and*

$$\sigma_E(t,x) = \hat{f}(t)\big[\rho_t'(x)\big]^{\frac{1}{2}} e^{\frac{1}{i}\int_0^t H^{\mathrm{sub}}(\phi_r(x))\, dr} \mathcal{T}_t^{L'}(x)$$

where the function $\rho_t'(x)$ is defined below.

Recall from the introduction the decomposition in symplectic subspaces $T_xM = F_x \oplus G_x$ where $F_x = \mathrm{Vect}(X_x, j_x X_x)$ and $G_x = F_x^{\perp_\omega}$. F_x and G_x are both preserved by j_x and we denote by $K(F_x)$, $K(G_x)$ their canonical lines. We define

$$\Phi_F : K(F_x) \to K(F_{\phi_t(x)}), \qquad \Phi_F(\lambda_x) = 2\|X_x\|^{-2}\lambda_{\phi_t(x)} \tag{47}$$

where $\lambda_x \in K(F_x)$ is normalised by $\lambda_x(X_x) = 1$. Furthermore Φ_G is the map $K(G_x) \to K(G_{\phi_t(x)})$ such that

$$\Phi_G(\alpha)(\psi u) = \alpha(u), \qquad \forall \alpha \in K(G_x),\, \forall u \in \wedge^n T_x M \tag{48}$$

where ψ is the symplectic map $G_x \to G_{\phi_t(x)}$ induced by $T_x\phi_t$ and the isomorphism $G_x \simeq T_x H^{-1}(E)/\mathbb{R}X_x$.

Then we set $\mathcal{D}'_t(x) := \Phi_F \otimes \Phi_G : K_x \to K_{\phi_t(x)}$ and we denote by $\rho'_t(x)$ the complex number such that

$$\mathcal{D}'_t(x) = \rho'_t(x)\mathcal{T}^K_t(x).$$

We denote by $\left[\rho'_t(x)\right]^{\frac{1}{2}}$ the continuous square root equal to $\sqrt{2}\|X_x\|^{-1}$ at $t = 0$.

Proof. This is a particular case of Proposition 6.1 just as Theorem 4.2 on the quantum propagator was a particular case of Theorem 2.6. Let us compute the coefficient $\mathcal{E}'_t(x)(\mathrm{id}_{K_x})$. We first describe the image (41) of $T_{(t,x)}j_E$:

$$\mathfrak{L}_{t,x} = \mathbb{R}(X_x, 0) \oplus \{(T_x\phi_t(\xi), \xi),\, \xi \in T_x H^{-1}(E)\}$$

and its volume (42). Set $\eta = \|X_x\|^{-2} j_x X_x$ so that (X_x, η) is a symplectic basis of F_x. Let (ξ_i) be a symplectic basis of G_x. Then if the volume of $\operatorname{diag} T_x M$ is $v = v_F \wedge v_G$ with

$$v_F = (X_x, X_x) \wedge (\eta, \eta), \qquad v_G = (\xi_1, \xi_1) \wedge \ldots \wedge (\xi_m, \xi_m),$$

then we have $v(0, x) = -(X_x, 0) \wedge (X_x, X_x) \wedge (\xi_1, \xi_1) \wedge \ldots \wedge (\xi_m, \xi_m)$ so that

$$\begin{aligned} v(t,x) &= -(X_{\phi_t(x)}, 0) \wedge (X_{\phi_t(x)}, X_x) \wedge (T_x\phi_t(\xi_1), \xi_1) \wedge \ldots \wedge (T_x\phi_t(\xi_m), \xi_m) \\ &= -(X_{\phi_t(x)}, 0) \wedge (0, X_x) \wedge (\psi(\xi_1), \xi_1) \wedge \ldots \wedge (\psi(\xi_m), \xi_m) \end{aligned}$$

because $T_x\phi_t(\xi) = \psi(\xi)$ modulo $\mathbb{R}X_{\phi_t(x)}$. Then $\mathcal{E}'_t(x)(\mathrm{id}_{K_x}) = \Phi_F \otimes \Phi_G$ where $\Phi_F : K(F_x) \to K(F_{\phi_t(x)})$ is such that

$$\big\langle \Phi_F, (X_{\phi_t(x)}, 0) \wedge (0, X_x) \big\rangle = i \langle \mathrm{id}_{K(F_x)}, v_F \rangle, \tag{49}$$

and $\Phi_G : K(G_x) \to K(G_{\phi_t(x)})$ is such that

$$\langle \Phi_G, (\psi(\xi_1), \xi_1) \wedge \ldots \wedge (\psi(\xi_m), \xi_m) \rangle = \langle \mathrm{id}_{K(G_x)}, v_G \rangle, \tag{50}$$

Here the pairings are based on the identifications $\mathrm{Mor}(K(S), K(S')) \simeq K(S') \otimes \overline{K(S)} \simeq K(S' \oplus \overline{S})$. Now Φ_G is the application satisfying (48) by Lemma 4.1. And Φ_F is the application (47) by a straightforward computation. □

7 Proof of theorem 2.7

We choose complex normal coordinates (z_i) of M centered at $x_0 \in M$. So $G_{ij}(x_0) = \delta_{ij}$ and $\partial_{z_i} G_{jk}(x_0) = \partial_{\overline{z}_i} G_{jk}(x_0) = 0$. We may assume that $T_{x_0}\Gamma_{t_0}$ is spanned by the vectors $\partial_{z_i} + \partial_{\overline{z}_i}$, $i = 1, \ldots, n$. Recall that Y is the vector field (∂_t, X_t) of $I \times M$ where X_t is the Hamiltonian vector field of H_t. Since $\omega = i \sum_{j,k} G_{jk} dz^j \wedge d\overline{z}^k$, we have

$$X_t = i \sum_{j,k} (-G^{jk} H_{z_j} \partial_{\overline{z}_k} + G^{jk} H_{\overline{z}_k} \partial_{z_j}) \tag{51}$$

where we use the notation $H_{z_j} = \partial_{z_j} H_t$ and $H_{\overline{z}_k} = \partial_{\overline{z}_k} H_t$ (and below we will use similar notation for higher order derivatives). As explained before the statement of Theorem 2.7, we have two derivatives ∇_Y and $\mathcal{L}_Y$ acting on $(\mathbb{C}_I \boxtimes K)|_\Gamma$ in the same direction Y, so $\theta := \frac{1}{i}(\mathcal{L}_Y - \nabla_Y)$ is a function in $\mathcal{C}^\infty(\Gamma)$.

Proposition 7.1. $\theta(x_0) = \sum_j (H_{z_j \overline{z}_j}(x_0) + H_{\overline{z}_j \overline{z}_j}(x_0))$

Proof. Let $\alpha = dz_1 \wedge \ldots \wedge dz_n$. First we have the section $1 \boxtimes \alpha$ of $(\mathbb{C}_I \boxtimes K)|_\Gamma$ and we compute its covariant derivative with respect to Y. We claim that this derivative vanishes at x_0. This follows from the fact that $|\alpha|^{-2} = \det G_{ij}$, so the Chern connection of K (given near x_0 by the one-form $\frac{\partial(|\alpha|^2)}{|\alpha|^2}$) is zero at x_0 because the coordinates are normal at x_0. Second we have to compute the Lie derivative with respect to Y of $j^*(dt \wedge \alpha)$ with j the embedding $\Gamma \to I \times M$. We have $\mathcal{L}_Y j^*(dt \wedge \alpha) = j^*(\mathcal{L}_Y(dt \wedge \alpha))$. Furthermore $\mathcal{L}_{\partial_t} dt = \mathcal{L}_{\partial_t} dz_i = 0$ and by (51), we have

$$\mathcal{L}_{X_t} dz_j = d(X_t.z_j) = i \sum_k (H_{\overline{z}_j z_k} dz_k + H_{\overline{z}_j \overline{z}_k} d\overline{z}_k)$$

at x_0 because the coordinates are normal. Furthermore $j^*(dt \wedge d\overline{z}_k) = j^*(dt \wedge dz_k)$ at x_0 by the assumption on $T_{x_0}\Gamma_{t_0}$. Collecting all these informations, we deduce that

$$\mathcal{L}_Y(j^*(dt \wedge \alpha)) = i \sum_j \left(H_{z_j\bar{z}_j}(x_0) + H_{\bar{z}_j\bar{z}_j}(x_0)\right) j^*(dt \wedge \alpha)$$

at x_0. The conclusion follows. □

Introduce the Szegö projector Π_k which is the orthogonal projector of $\mathcal{C}^\infty(M, L^k \otimes L')$ onto $\mathcal{H}_k$.

Proposition 7.2. *For any Lagrangian state (Ψ_k) associated with Γ_{t_0} with symbol σ and any function $f \in \mathcal{C}^\infty(M)$, $\Pi_k(f\Psi_k)$ is a Lagrangian state with symbol*

$$f|_{\Gamma_{t_0}} \sigma + \hbar\left(\tfrac{1}{i}\nabla^{L'}_U + \Box f\right) \sigma + \mathcal{O}(\hbar^2) \tag{52}$$

where U is the vector field of Γ_{t_0} such that $U(x) = X_f(x) \mod T^{0,1}_x M$, X_f being the Hamiltonian vector field of f, and $\Box f = \sum_j (f_{z_j\bar{z}_j} + \frac{1}{2} f_{\bar{z}_j\bar{z}_j})$ at x_0.

Proof. We already know that $\Pi_k(f\Psi_k)$ is a Lagrangian state associated with Γ_{t_0}. We compute its symbol up to $\mathcal{O}(\hbar^2)$ at x_0. It suffices to prove that this symbol has the form

$$(c_0 + \hbar c_1) f(x_0)\sigma(x_0) + \hbar\left(\tfrac{1}{i}\nabla^{L'}_U + \Box f\right)\sigma(x_0) + \mathcal{O}(\hbar^2) \tag{53}$$

where c_0 and c_1 are independent of f. Since for $f = 1$, we have to recover $\sigma(x_0)$, $c_0 = 1$ and $c_1 = 0$ necessarily.

Besides our normal coordinates (z_i) and our assumption on $T_{x_0}\Gamma_{t_0}$, let us introduce two holomorphic normal frames v and v' of L and L' respectively. So $|v| = e^{-\frac{\varphi}{2}}$ with φ a real function such that

$$\varphi(x_0) = \partial_{z_j}\varphi(x_0) = \partial_{z_j}\partial_{z_k}\varphi(x_0) = 0, \qquad \forall j, k.$$

Similarly $|v'| = e^{-\frac{\varphi'}{2}}$ with φ' satisfying the same conditions. Notice that the curvature of L is equal to both $\partial\bar{\partial}\varphi$ and $-i\omega$, so $\partial_{z_i}\partial_{\bar{z}_j}\varphi = G_{ij}$ for every i, j. We can assume that $v(x_0) = s(x_0)$ where s is the section over Γ_{t_0} associated with our Lagrangian state. In the rest of the proof we write all the sections of $L^k \otimes L'$ in the frame $v^k \otimes v'$.

More details on the computations to come can be found in [8, Sections 2.4, 2.5]. We have

$$\Pi_k(x_0, x) = \left(\frac{k}{2\pi}\right)^n e^{k\psi(x) + \psi'(x)} p(x, k) + \mathcal{O}(k^{-\infty})$$

where ψ has the following Taylor expansion at x_0: $\psi(x) = \sum_{\beta\in\mathbb{N}^n} \varphi_{0,\beta}\frac{\overline{z}^\beta}{\beta!}$ with the notations $\varphi_{\alpha,\beta} = \partial_z^\alpha \partial_{\overline{z}}^\beta \varphi(x_0)$, ψ' has the same Taylor expansion in terms of φ' and $p(x,k) = 1 + k^{-1}p_1(x) + k^{-2}p_2(x) + \dots$.

We have a similar expression for Ψ_k:

$$\Psi_k(x) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} e^{k\rho(x)} a(x,k) + \mathcal{O}(k^{-\infty})$$

where ρ has the Taylor expansion $\rho(x) = \frac{1}{2}\sum z_i^2 + \sum_{|\alpha|\geqslant 3} \rho_{\alpha,0}\frac{z^\alpha}{\alpha!}$. This follows from the fact that the section F entering in the definition (13) of Ψ_k satisfies $F(x_0) = s(x_0)$ so that we can assume that $\rho(x_0) = 0$, $\nabla F|_{x_0} = 0$ so that the first derivatives of ρ all vanish at x_0 and finally the second order derivatives of F at x_0 depend only on the linear data at x_0 [8, Proposition 2.2] which leads to the expression above.

So it comes that

$$\Pi_k(f\Psi_k)(x_0) = \Big(\frac{k}{2\pi}\Big)^{\frac{5n}{4}} \int e^{-k\phi} f(x) a(x,k) p(x,k) D(x)\, dz d\overline{z} + \mathcal{O}(k^{-\infty}) \tag{54}$$

where

$$\phi(x) = -\psi(x) + \varphi(x) - \rho(x) = |z|^2 - \tfrac{1}{2}\sum z_i^2 + R(x)$$

with R having the Taylor expansion at x_0

$$R(x) = \sum_{\substack{\alpha\neq 0,\beta\neq 0\\ |\alpha|+|\beta|\geqslant 3}} \varphi_{\alpha,\beta}\frac{z^\alpha \overline{z}^\beta}{\alpha!\beta!} + \sum_{|\alpha|\geqslant 3} (-\rho_{\alpha,0} + \varphi_{\alpha,0})\frac{z^\alpha}{\alpha!}.$$

Furthermore $D(x) = e^{\psi'(x)-\varphi'(x)} \det(G_{ij}) = 1 + \mathcal{O}(|z|^2)$.

By applying the stationary phase method, we obtain the asymptotic expansion of (54). At first order:

$$\Pi_k(f\Psi_k)(x_0) = C\Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} \Big(f(x_0)a(x_0,k) + \mathcal{O}(k^{-1})\Big)$$

where C can be computed in terms of the Hessian determinant of ϕ at x_0. Actually it is shorter to compare with (53), which gives $C = c_0 = 1$. We now compute the second order term,

$$\Pi_k(f\Psi_k)(x_0) = \Big(\frac{k}{2\pi}\Big)^{\frac{n}{4}} \Big(f(x_0)a(x_0,k) + k^{-1}\varepsilon_k + \mathcal{O}(k^{-2})\Big)$$

with

$$\varepsilon_k = \sum_{\ell=1}^{3} (\ell!\,(\ell-1)!)^{-1} P^\ell((-R)^{\ell-1} f a(\cdot,k) D)(x_0)$$

where $P = \sum(\partial_{z_j}\partial_{\overline{z}_j} + \frac{1}{2}\partial_{\overline{z}_j}\partial_{\overline{z}_j})$.

Recall that we do not try to compute the terms c_0 and c_1 in (53), which means that we do not take into account the terms without derivative on $a(x,k)$. Considering the form of the Taylor expansions of R and D given above and the fact that $\partial_{z_i}\partial_{\overline{z}_j}\varphi = G_{ij}$ so that $\varphi_{\alpha,\beta} = 0$ when $(|\alpha|,|\beta|) = (1,2)$ or $(2,1)$, we deduce after some investigations that we only have a single term to consider which is $k^{-1}P(f(x)a(x,k))$. Now $a(x,k) = a_0(x) + \mathcal{O}(k^{-1})$ and a_0 has the Taylor expansion of a holomorphic function at x_0 because $\overline{\partial}(a_0 v')$ vanishes to infinite order along Γ_{t_0} and the frame v' is holomorphic. So

$$P(fa_0) = (Pf)a_0 + \sum_j (\partial_{\overline{z}_j} f)(\partial_{z_j} a_0). \tag{55}$$

The Hamiltonian vector field of f at x_0 being $\sum_j(-if_{z_j}\partial_{\overline{z}_j} + if_{\overline{z}_j}\partial_{z_j})$ (see (51)), and by using again that $\overline{\partial}a_0 = 0$ at x_0, we obtain that the sum in (55) is $\frac{1}{i}X_f a_0 = \frac{1}{i}Ua_0$. Now the frame v' being normal, its covariant derivative is 0 at x_0, so $\nabla_U(a_0v') = (U.a_0)v'$. $\square$

Proof of Theorem 2.7. It suffices to consider the case where $T_{k,t} = T_k(H_t + k^{-1}H'_t)$ for some $H_t, H'_t \in \mathcal{C}^\infty(M)$. But the symbol of $k^{-1}T_k(H'_t)\Psi_k$ is $\hbar H'_t\sigma + \mathcal{O}(\hbar^2)$, so in fact we only need to consider the case $H'_t = 0$.

By Proposition 7.2 and the proof of Proposition 2.4, the symbol of $\frac{1}{ik}\partial_t\Psi_k + T_k(H_t)\Psi_k$ is

$$\hbar\big(\tfrac{1}{i}(\nabla_Z + \nabla_U) + \Box H_t\big)\,\sigma + \mathcal{O}(\hbar^2), \tag{56}$$

where Z and U are the vector fields of Γ such that $Z(t,x) = \partial_t \mod T^{0,1}_x M$ and $U(t,x) = X_t(x) \mod T^{0,1}_x M$. Since here $Y(t,x) = (\partial_t, X_t)$ is tangent to Γ, we have $Z + U = Y$ on Γ. By Proposition 7.1, $\Box H_t = \frac{1}{2}\theta + \frac{1}{2}\Delta H_t$ with $\Delta = \sum_{i,j} G^{ij}\partial_{z_i}\partial_{\overline{z}_j}$ (indeed, recall that the coordinates are normal at x_0 so $\Delta H_t(x_0) = \sum_i H_{z_i\overline{z}_i}(x_0)$). So the symbol in (56) is

$$\hbar\big(\tfrac{1}{i}\nabla_Y + \tfrac{1}{2}\theta + \tfrac{1}{2}\Delta H_t\big)\,\sigma + \mathcal{O}(\hbar^2)$$

which concludes the proof. $\square$

A Appendix: an explicit example

Let $(M,\omega) = (\mathbb{T}^2 = \mathbb{R}^2/\Lambda, \omega_{\mathbb{T}^2})$ where $\Lambda \subset \mathbb{R}^2$ is a lattice of symplectic volume 4π. (M,ω) is naturally endowed with a prequantum line bundle L

induced by the line bundle $\mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$ with connection $\nabla = d - i\alpha$ where $\alpha = 2\pi(p\ dq - q\ dp)$. Here (p, q) are coordinates associated with a basis (e, f) of Λ with $\omega(e, f) = 4\pi$. In other words $\omega = 4\pi dp \wedge dq$; we will also work with the holomorphic coordinate $z = p + iq$, for which $\omega = 2i\pi dz \wedge d\bar{z}$.

For $k \geq 1$, the quantum space $\mathcal{H}_k = H^0(M, L^k)$ identifies with the space of Λ-invariant sections of $\mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$, which is a space of theta functions with dimension $2k$. More precisely, the family $(\Psi_\ell)_{0 \leq \ell \leq 2k-1}$ given by

$$\Psi_\ell(z) = \frac{k^{\frac{1}{4}}}{\sqrt{2\pi}} \exp(2i\pi(\ell + k\Im(z))) \exp\left(-\frac{\pi\ell^2}{2k}\right) \vartheta_3(\pi(2kz + i\ell), \exp(-2k\pi))$$

for all $z \in \mathbb{C}$ and $\ell \in \{0, \ldots, 2k-1\}$, where $\vartheta_3(w, q) = 1 + 2\sum_{n=1}^{+\infty} q^{n^2} \cos(2nw)$ is the Jacobi theta function, forms an orthonormal basis of $\mathcal{H}_k$. The diagonal operator defined as $T_k \Psi_\ell = \cos(\frac{\pi\ell}{k})\Psi_\ell$ for every $\ell \in \{0, \ldots, 2k-1\}$ is a Berezin-Toeplitz operator with principal symbol $H : (p, q) \mapsto \cos(2\pi q)$ and vanishing subprincipal symbol. For more details, see [12, Section 3.1, Appendix].

On the one hand, one can easily compute numerically the kernel of the quantum propagator $U_{k,t} = \exp(-iktT_k)$ by using the formula

$$U_{k,t}(w, z) = \sum_{\ell=0}^{2k-1} \exp\left(-ikt \cos\left(\frac{\pi\ell}{k}\right)\right) \Psi_\ell(w)\overline{\Psi_\ell(z)} \tag{57}$$

and the above expression of Ψ_ℓ (in practice, we use the built-in commands for Jacobi theta functions in the mpmath library for Python). On the other hand, the coefficients in Equation (8) can be explicitly computed as follows. Since the subprincipal symbol of T_k vanishes, it suffices to compute ρ_t and ϕ_t^L. First, the parallel transport term reads

$$\mathcal{T}_t^L(p, q) = \exp\left(i \int_0^t \alpha_{\phi_s(p,q)}(X(\phi_s(p, q)))ds\right) = \exp\left(-i\pi tq \sin(2\pi q)\right)$$

since $X(p, q) = \frac{1}{2}\sin(2\pi q)\partial_p$ and $\phi_t(p, q) = [p + \frac{t}{2}\sin(2\pi q), q]$, and we obtain

$$\phi_t^L(p, q) = \exp(-it\cos(2\pi q))\mathcal{T}_t^L(p, q) = \exp\left(-it\left(\cos(2\pi q) + \pi q \sin(2\pi q)\right)\right).$$

Second, one readily checks that $(T_{(p,q)}\phi_t)^{1,0}$ is the operator of multiplication by $1 - \frac{i\pi t}{2}\cos(2\pi q)$. Since the connection on the canonical bundle is trivial, this yields

$$\rho_t(p, q)^{\frac{1}{2}} = \left(1 - \frac{i\pi t}{2}\cos(2\pi q)\right)^{-\frac{1}{2}} = \frac{\exp\left(\frac{i}{2}\arctan\left(\frac{\pi t}{2}\cos(2\pi q)\right)\right)}{\sqrt{1 + \frac{\pi^2 t^2}{4}\cos^2(2\pi q)}}.$$

So we finally obtain that for $z = p + iq$,

$$U_{k,t}(\phi_t(z), z) \sim \frac{k \exp\left(i\left(\frac{1}{2}\arctan\left(\frac{\pi t}{2}\cos(2\pi q)\right) - kt\left(\cos(2\pi q) + \pi q \sin(2\pi q)\right)\right)\right)}{2\pi\sqrt{1 + \frac{\pi^2 t^2}{4}\cos^2(2\pi q)}}. \tag{58}$$

We compare this theoretical equivalent with the numerical value in Figures 1, 2, 3, 4 and 5. In these computations, we fix k and (p, q), and plot the real part of the kernel of the propagator evaluated at $(\phi_t(z), z)$ with $z = p + iq$, as a function of t; we also plot the imaginary part of this kernel only for one set of parameters, since the behaviour is very similar to the one of the real part. In all these figures, the blue diamonds represent the numerical values obtained from Equation (57) while the solid red line corresponds to the right hand side of Equation (58). Note that a priori the $\mathcal{O}(k^{-1})$ remainder may depend on t, so once k and (p, q) are fixed, the approximation may become less precise as t increases. In Figures 4 and 5, we display the behaviour at small and (relatively) large times. Investigating the k-dependent times up to which the approximation in Equation (8) remains valid is a classical topic in the semiclassical literature, that we do not consider in the present paper.

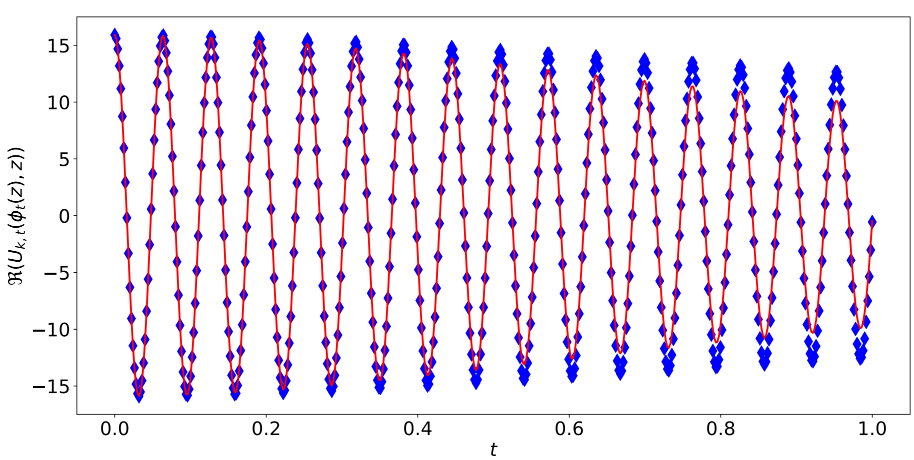

Figure 1: Real part of $U_{k,t}(\phi_t(z), z)$ for $k = 100$ and $z = p + iq$ with $(p, q) = (0.3, 0.1)$, for $0 \le t \le 1$.

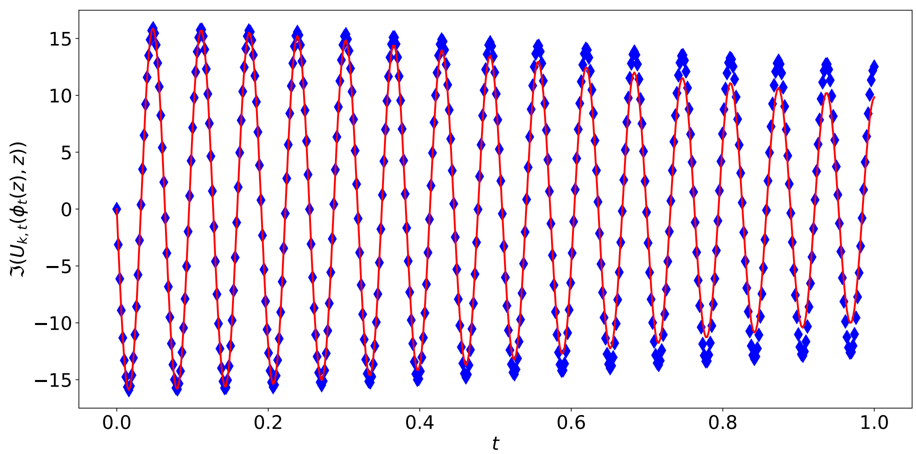

Figure 2: Imaginary part of $U_{k,t}(\phi_t(z), z)$ for $k = 100$ and $z = p + iq$ with $(p, q) = (0.3, 0.1)$, for $0 \leq t \leq 1$.

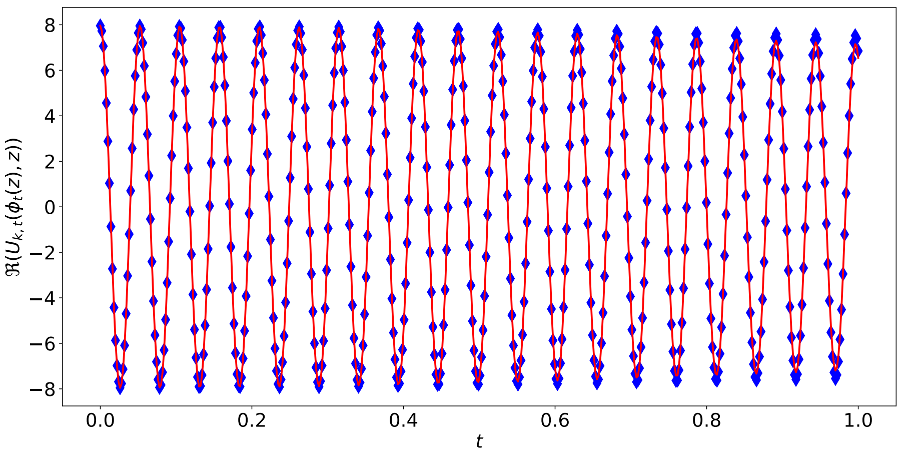

Figure 3: Real part of $U_{k,t}(\phi_t(z), z)$ for $k = 50$ and $z = p + iq$ with $(p, q) = (0.5, 0.7)$, for $0 \leq t \leq 1$.

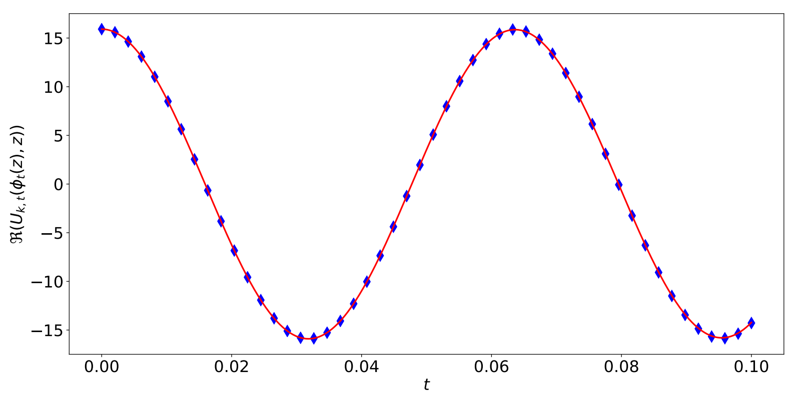

Figure 4: Real part of $U_{k,t}(\phi_t(z), z)$ for $k = 100$ and $z = p + iq$ with $(p, q) = (0.3, 0.1)$, for $0 \leq t \leq 0.1$.

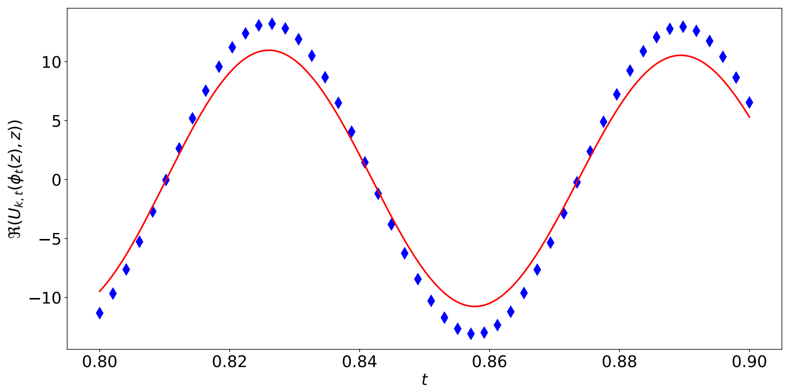

Figure 5: Real part of $U_{k,t}(\phi_t(z), z)$ for $k = 100$ and $z = p + iq$ with $(p, q) = (0.3, 0.1)$, for $0.8 \leq t \leq 0.9$.

References

[1] O. Biquard. $SL(\infty, \mathbb{R})$, Higgs bundles and quantization. In *Geometry and physics. Vol. II*, pages 419–431. Oxford Univ. Press, Oxford, 2018.

[2] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits. *Comm. Math. Phys.*, 165(2):281–296, 1994.

[3] D. Borthwick, T. Paul, and A. Uribe. Semiclassical spectral estimates for Toeplitz operators. *Ann. Inst. Fourier (Grenoble)*, 48(4):1189–1229, 1998.

[4] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.

[5] L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegő. In *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, pages 123–164. Astérisque, No. 34–35. Soc. Math. France, Paris, 1976.

[6] L. Charles. *Aspects semi-classiques de la quantification géométrique*. Theses, Université Paris Dauphine - Paris IX, Dec. 2000. https://theses.hal.science/tel-00001289.

[7] L. Charles. Berezin-Toeplitz operators, a semi-classical approach. *Comm. Math. Phys.*, 239(1-2):1–28, 2003.

[8] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10):1527–1566, 2003.

[9] L. Charles. Symbolic calculus for Toeplitz operators with half-form. *J. Symplectic Geom.*, 4(2):171–198, 2006.

[10] L. Charles. Semi-classical properties of geometric quantization with metaplectic correction. *Comm. Math. Phys.*, 270(2):445–480, 2007.

[11] L. Charles. A Lefschetz fixed point formula for symplectomorphisms. *J. Geom. Phys.*, 60(12):1890–1902, 2010.

[12] L. Charles and J. Marché. Knot state asymptotics I: AJ conjecture and Abelian representations. *Publ. Math. Inst. Hautes Études Sci.*, 121:279–322, 2015.

[13] S. K. Donaldson. Scalar curvature and projective embeddings. I. *J. Differential Geom.*, 59(3):479–522, 2001.

[14] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[15] R. J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev. (2)*, 131:2766–2788, 1963.

[16] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67(3):515–538, 1982.

[17] V. W. Guillemin. 25 years of Fourier integral operators. In *Mathematics past and present*, pages 1–21. Springer, Berlin, 1994.

[18] L. Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.

[19] L. Hörmander. The spectral function of an elliptic operator. *Acta Math.*, 121:193–218, 1968.

[20] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[21] L. Ioos. Geometric quantization of Hamiltonian flows and the Gutzwiller trace formula. *Lett. Math. Phys.*, 110(7):1585–1621, 2020.

[22] S. Klevtsov, X. Ma, G. Marinescu, and P. Wiegmann. Quantum Hall effect and Quillen metric. *Comm. Math. Phys.*, 349(3):819–855, 2017.

[23] K. Kodaira. On a differential-geometric method in the theory of analytic stacks. *Proc. Nat. Acad. Sci. U.S.A.*, 39:1268–1273, 1953.

[24] E. H. Lieb. The classical limit of quantum spin systems. *Comm. Math. Phys.*, 31:327–340, 1973.

[25] A. Melin and J. Sjöstrand. Fourier integral operators with complex-valued phase functions. pages 120–223. Lecture Notes in Math., Vol. 459, 1975.

[26] Y. A. Rubinstein and S. Zelditch. The Cauchy problem for the homogeneous Monge-Ampère equation, I. Toeplitz quantization. *J. Differential Geom.*, 90(2):303–327, 2012.

[27] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[28] S. Zelditch and P. Zhou. Pointwise Weyl law for partial Bergman kernels. In *Algebraic and analytic microlocal analysis*, volume 269 of *Springer Proc. Math. Stat.*, pages 589–634. Springer, Cham, 2018.

Laurent Charles
Sorbonne Université, Université de Paris, CNRS
Institut de Mathématiques de Jussieu-Paris Rive Gauche
F-75005 Paris, France
E-mail: `laurent.charles@imj-prg.fr`

Yohann Le Floch
Institut de Recherche Mathématique Avancée
UMR 7501, Université de Strasbourg et CNRS
7 rue René Descartes
67000 Strasbourg, France
E-mail: ylefloch@unistra.fr