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The topology of real loci of $\mathbb{R}$-varieties

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Contents

1 Introduction

Studying the topology of real algebraic varieties is an old topic. The ancient Greek mathematicians already were familiar with real algebraic curves, in particular with conics and quartics. In the 17th and 18th centuries, mathematicians like Descartes and Euler were also studying higher degree curves.

One of the earliest topological results for curves of arbitrary degree is Harnack's inequality (1876), which states that the number of connected components of a degree d curve is at most $\frac{(d-1)(d-2)}{2}+1$. Harnack, who was a student of Felix Klein, proved this for plane curves [4] and Klein [11] later proved the general version for non-planar curves.

Curves for which the number of connected components is exactly $\frac{(d-1)(d-2)}{2}+1$ are called *maximal curves*. These maximal curves became an interesting object to study in the following years.

For example, they occurred in Hilbert's famous list of problems that he published in 1900. Problem 16 of this list is to study how the connected components of maximal curves are arranged, relative to each other.

The connected components of a curve are also called ovals. Based on whether an oval 'sits inside' an even or odd number of other ovals, it is called an even or odd oval. Some important results that were developed during the 20th century were constraints on the difference between the number of even ovals P and the number of odd ovals N. The first was the Petrovskii–Oleinik inequality [16] (1938), which gave the bound

$$\frac{3}{2}d-\frac{3}{2}d^2 \leq P-N \leq \frac{3}{2}d-\frac{3}{2}d^2+1$$

for a curve of degree $2d$. For maximal curves of degree $2d$ the extra constraint $P-N \equiv d^2$ modulo 8 holds, which was shown by Rokhlin [17].

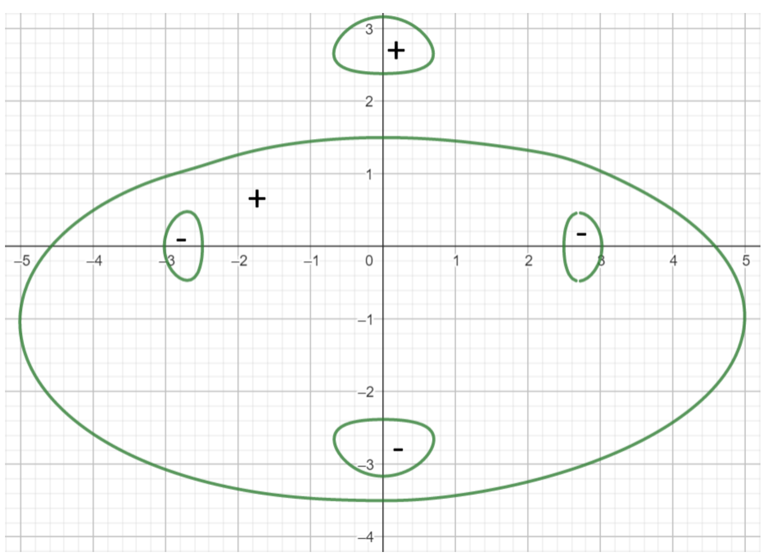

Figure 1: The curve $0 = x^3y^3 - ((x^2+3y^2-17)(3x^2+y^2-10)+15x^2)(x^2+4(y+1)^2-25)$, with 2 even ovals and 3 odd ovals.

Some of these results were also generalised to surfaces, but it wasn't until the second half of the 20th century that Harnack's inequality was generalized to arbitrary dimensions. This generalization was found by transitioning from the direct study of real algebraic varieties to the study of $\mathbb{R}$-varieties. These are complex manifolds with an anti-holomorphic involution. The fixed locus of such an involution is called the *real locus* of the $\mathbb{R}$-variety. The real algebraic varieties that were studied before correspond to the real loci of $\mathbb{R}$-varieties.

The generalized version of Harnack's inequality is called the Thom–Smith inequality and it states

$$\sum_{n\geq 0} \dim H_n(X^\sigma;\mathbb{F}_2) \leq \sum_{n\geq 0} \dim H_n(X;\mathbb{F}_2),$$

for an $\mathbb{R}$-variety X, with an anti-holomorphic involution σ.

The Petrovskii–Oleinik inequality was also generalized to higher dimensions, by Kharlamov [10] in 1974. This Petrovskii–Oleinik–Kharlamov inequalityholds for even dimensional $\mathbb{R}$-varieties X that are *Kähler* manifolds and states

$$2 - h^{n,n}(X) \leq \chi(X(\mathbb{R})) \leq h^{n,n}(X),$$

where $h^{n,n}(X)$ is the dimension of the *Dolbeault cohomology* of X in degree (n,n) and $\chi(X(\mathbb{R}))$ is the Euler characteristic of the real locus of X.

The Thom–Smith inequality also gave rise to the notion of a *maximal variety*, which is the higher dimensional analogue of a maximal curve. Apart from generalizing the known results for curves to higher dimensions, another interesting problem is the construction of maximal varieties from other (lower dimensional) maximal varieties. In 2017, Biswas and D'Mello [1] gave such a construction using the symmetric product. The n-fold symmetric product is the quotient of the n-fold Cartesian product by the natural action of the symmetric group S_n. Biswas and D'Mello showed that the nth symmetric product of a maximal genus g curve, is maximal for $n \leq 3$ or $n \geq 2g-1$. In 2018, this was generalized by Franz [3] to all n.

In this thesis we will present proofs of most of the results mentioned above. In Chapter 2 we start with some preliminary knowledge about the homology and cohomology of topological spaces. Then we continue in Chapter 3 with Smith theory, which is an important tool that will be used later on. In Chapter 4 we will introduce the main object of study, namely $\mathbb{R}$-varieties and also define the concept of a maximal variety and prove Harnack's inequality in the meantime. Then, in Chapter 5, we prove the Petrovskii–Oleinik–Kharlamov inequality, which is the generalized version of Petrovskii–Oleinkik inequality. Finally, in Chapters 6 and 7, we explore the symmetric product as a way to construct maximal varieties.

2 (Co)homology

In this thesis we will repeatedly use the singular homology and cohomology of topological spaces. This chapter will provide some required definitions and theorems about them.

Definition 2.1. Let $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}$. A *singular n-simplex* in a topological space X is a continuous function $\rho : \Delta^n \to X$. We denote the set of n-simplices on X by $S_n(X)$.

Definition 2.2. Let A be an abelian group. The *singular chain complex* $C_*(X; A)$ of a topological space X with coefficients in A is defined in index i by

$$C_i(X; A) := A[S_i(X)] = \left\{ \sum_{j \in J} a_j s_j \mid J \text{ is finite, } a_j \in A, s_j \in S_i(X) \right\}.$$

The *differentials* of the chain complex are induced by inclusions $d_j : \Delta^{i-1} \to \Delta^i$ as follows. Let

$$d_j(x_0, \ldots, x_{i-1}) = (x_0, \ldots, x_{j-1}, 0, x_j, \ldots, x_{i-1})$$

for $0 \leq j \leq i$. These inclusions induce maps $d_j^* : C_i(X; A) \to C_{i-1}(X; A)$ defined by $s \mapsto s \circ d_j$, for $s \in S_i(X)$. The differentials

$$\partial_i : C_i(X; R) \to C_{i-1}(X; R)$$

are now defined by the alternating sum of these induced maps, i.e.

$$\partial_i(x) = \sum_{j=0}^{i} (-1)^j d_j^*(x).$$

It can be checked that this makes C_* a chain complex, i.e. $\partial_i \circ \partial_{i+1} = 0$.

Definition 2.3. The ith *singular homology group* with coefficients in A of a topological space X is the ith homology of its singular chain complex with coefficients in A:

$$H_i(X; A) = H_i(C_*(X; A)) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i+1}.$$

Example 2.4.

i) Let $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sqrt{x_0^2 + \cdots + x_n^2} = 1\}$ denote the n-sphere. For $n > 0$ we have, [6, Corollary 2.14]

$$H_k(S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

ii) The homology groups of n-dimensional real projective space $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*$ and infinite dimensional real projective space $\mathbb{RP}^\infty$ are given as follows; see [6, Example 2.42].

$$H_k(\mathbb{RP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } n \text{ is odd and } k = n, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < n \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

$$H_k(\mathbb{RP}^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

iii) The homology groups of n-dimensional complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ are given as follows; see [6, page 140].

$$H_k(\mathbb{CP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \text{ and } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5. Let A be an abelian group. The *singular cochain complex* $C^*(X; A)$ of a topological space X with coefficients in abelian group A, is defined in index i by,

$$C^i(X; A) := \operatorname{Hom}(C_i(X; \mathbb{Z}), A).$$

The corresponding differentials $\delta^i : C^i(X; A) \to C^{i+1}(X; A)$ are defined by

$$\delta^i(x) = x \circ \partial_{i+1}.$$

Definition 2.6. The ith *singular cohomology* with coefficients in A of a topological space X is the ith cohomology group of its singular cochain complex with coefficients in A:

$$H^i(X; A) = H^i(C^*(X; A)) = \operatorname{Ker} \delta^i / \operatorname{Im} \delta^{i-1}.$$

Definition 2.7. Let R be a ring. The *cup product* on the singular cochain complex is a bilinear map

$$- \cup - : C^p(X; R) \times C^q(X; R) \to C^{p+q}(X; R), (f, g) \mapsto f \cup g,$$

where $f \cup g$ is defined on $(p + q)$-simplices s in X by

$$(f \cup g)(s) = f(\text{front}_p(s)) \cdot g(\text{back}_q(s)),$$

where $\text{front}_p(s)$ is the p-simplex $(x_0, \ldots, x_p) \mapsto s(x_0, \ldots, x_p, 0, \ldots, 0)$ and $\text{back}_q(s)$ is the q-simplex $(x_0, \ldots, x_q) \mapsto s(0, \ldots, 0, x_0, \ldots, x_q)$.

This induces a bilinear map on cohomology

$$H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R),$$

which makes $H^*(X; R)$ a graded ring. [6, Lemma 3.6]

If there is a bilinear map $A \times B \to M$ on R-modules A, B and M, this map can be used instead of the product on R, to get a cup product $H^p(X; A) \times H^q(X; B) \to H^{p+q}(X; M)$.

We need some more results to easily compute cohomology groups, so we will postpone examples of cohomology to Section 2.2

2.1 Tor and Ext functors

Definition 2.8. Let R be a ring and M be an R-module. A *resolution* of M is an exact sequence $\cdots \to C_2 \to C_1 \to C_0$ and an R-linear map $C_0 \to M$, such that $\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$ is also exact. A resolution is called a *projective resolution* if all C_i are projective R-modules, which means that for each i there is an R-module D_i, such that $C_i \oplus D_i$ is a free R-module.

Example 2.9. Let $R = \mathbb{Z}$, $M = \mathbb{Z}/p\mathbb{Z}$ for a prime number p. Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_1 \,=\, \cdot p} \mathbb{Z},$$

is a projective resolution of M.

Definition 2.10. Let R be a ring, and let M, N be R-modules. Let (C_*, ∂_C) be a projective resolution of M. Let $(D_*, \partial_D) = C_* \otimes_R N$, i.e. $D_i = C_i \otimes_R N$ and $\partial_{D,i} : D_i \to D_{i-1}$ is given by

$$c \otimes n \mapsto \partial_{C,i}(c) \otimes n.$$

Then we define $\operatorname{Tor}_i^R(M, N) := H_i(D_*)$, which, up to isomorphism, does not depend on the chosen resolution.

When $R = \mathbb{Z}$, we can always construct a short exact sequence

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow \mathbb{Z}[M] \xrightarrow{\phi} M \longrightarrow 0,$$

where $\mathbb{Z}[M]$ is the free abelian group generated by M and $\phi : \mathbb{Z}[M] \to M$ is the group homomorphism that sends $\sum_i a_i \cdot m_i \in \mathbb{Z}[M]$ to $\sum_i a_i \cdot m_i \in M$. Its kernel is a subgroup of a free abelian group and thus free itself [12, Page 880], so we have a projective resolution with $C_i = 0$ for $i \geq 2$. Therefore $\operatorname{Tor}_i^{\mathbb{Z}}(M, N) = 0$ for all M, N when $i \geq 2$, so instead of $\operatorname{Tor}_1^{\mathbb{Z}}$, we just write Tor.

Example 2.9 (Continued)**.** Let $N = \mathbb{F}_q$ for a prime number q. Taking the tensor product with N, we obtain the chain complex $D_* = C_* \otimes \mathbb{F}_q$,

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_q \xrightarrow{\partial_1 \,=\, \cdot p \,\otimes\, \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_q.$$

which is isomorphic to $0 \longrightarrow \mathbb{F}_q \xrightarrow{\partial_1 \,=\, \cdot p} \mathbb{F}_q$.

i) If $p = q$, then multiplication by p is the 0-map, in which case the homology is

$$\operatorname{Tor}_i^{\mathbb{Z}}(M, N) = H_i(D_*) = \begin{cases} \mathbb{F}_q & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $p \neq q$, then multiplication by p is an isomorphism, so the homology is $\operatorname{Tor}_i^{\mathbb{Z}}(M, N) = H_i(D_*) = 0$ for all i.

Definition 2.11. Let R be a ring, and let M, N be R-modules. Let C_* be a projective resolution of M. Let $D^* = \operatorname{Hom}(C_*, N)$, i.e. $D^i = \operatorname{Hom}(C_i, N)$ and $\partial^{D,i} : D^i \to D^{i+1}$ is given by $f \mapsto f \circ \partial_{C,i+1}$. Then $\operatorname{Ext}_R^i(M, N) := H^i(D^*)$, which, up to isomorphism, does not depend on the chosen resolution. Just as with $\operatorname{Tor}_i^{\mathbb{Z}}$, we have $\operatorname{Ext}_{\mathbb{Z}}^i(M, N) = 0$ for all M, N for $i \geq 2$, so instead of $\operatorname{Ext}_{\mathbb{Z}}^1$ we write Ext.

Example 2.9 (Continued)**.** Applying $\operatorname{Hom}(-, \mathbb{F}_q)$ to the resolution of $M = \mathbb{Z}/p\mathbb{Z}$, we obtain the cochain complex $D^* = \operatorname{Hom}(C_*, \mathbb{F}_q)$,

$$\operatorname{Hom}(\mathbb{Z}, \mathbb{F}_q) \xrightarrow{\delta^0 \,=\, \cdot p} \operatorname{Hom}(\mathbb{Z}, \mathbb{F}_q) \longrightarrow 0,$$

which is isomorphic to $\mathbb{F}_q \xrightarrow{\delta^0 \,=\, \cdot p} \mathbb{F}_q \longrightarrow 0.$

i) If $p = q$, then multiplication by p is the 0-map, in which case the cohomology is

$$\mathrm{Ext}^i_{\mathbb{Z}}(M,N) = H^i(D_*) = \begin{cases} \mathbb{F}_q & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $p \neq q$, then multiplication by p is an isomorphism, so the cohomology is $\mathrm{Ext}^i_{\mathbb{Z}}(M,N) = H^i(D_*) = 0$ for all i.

2.2 Universal Coefficient Theorem

For both homology and cohomology there are universal coefficient theorems, which are useful for computing them when the coefficients are not in $\mathbb{Z}$.

Theorem 2.12. *Let A be an abelian group and X a topological space. Then there is a split exact sequence*

$$0 \longrightarrow H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow H_n(X;A) \longrightarrow \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) \longrightarrow 0.$$

In particular there is an isomorphism

$$H_n(X;A) \cong H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} A \oplus \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A).$$

Proof. See Theorem 3A.3 in [6]. □

Theorem 2.13. *Let A be an abelian group and X a topological space. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{k-1}(X;\mathbb{Z}), A) \longrightarrow H^k(X;A) \longrightarrow \mathrm{Hom}(H_k(X;\mathbb{Z}), A) \longrightarrow 0.$$

In particular there is an isomorphism

$$H^n(X;A) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}), A) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}), A).$$

Proof. See Theorem 3.2 in [6]. □

In order to apply the universal coefficient theorem to some examples, it is useful to have some results on how to compute the Tor and Ext functors.

Lemma 2.14. *Let A be an abelian group. Then the following hold:*

i) $A \cong \mathbb{Z} \otimes_{\mathbb{Z}} A \cong \mathrm{Hom}(\mathbb{Z}, A)$,
ii) $0 \cong \mathrm{Tor}(\mathbb{Z}, A) \cong \mathrm{Ext}(\mathbb{Z}, A)$.

Proof.
i) The isomorphisms are given by

$$A \to \mathbb{Z} \otimes_{\mathbb{Z}} A : \; a \mapsto 1 \otimes a,$$
$$A \to \mathrm{Hom}(\mathbb{Z}, A) : \; a \mapsto (n \mapsto n \cdot a).$$

ii) This follows from the projective resolution $0 \to \mathbb{Z} \to \mathbb{Z}$ of $\mathbb{Z}$, which leads to the (co)-chaincomplex $0 \to \mathbb{Z} \to 0$, which has zero (co)homology in index 1.

$\square$

Example 2.15. The homology groups of S^n and $\mathbb{CP}^n$ are all $\mathbb{Z}$ or 0, so we can use Lemma 2.14 to compute the corresponding Ext-groups. Combined with Theorem 2.13, this gives the following computations of the cohomology of S^n and $\mathbb{CP}^n$.

$$\begin{aligned} H^k(S^n;\mathbb{Z}) &\cong \mathrm{Hom}(H_k(S^n;\mathbb{Z});\mathbb{Z}) \oplus \mathrm{Ext}(H_{k-1}(S^n;\mathbb{Z}),\mathbb{Z}) \\ &\cong H_k(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \text{ or } k=n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} H^k(\mathbb{CP}^n;\mathbb{Z}) &\cong \mathrm{Hom}(H_k(\mathbb{CP}^n;\mathbb{Z});\mathbb{Z}) \oplus \mathrm{Ext}(H_{k-1}(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z}) \\ &\cong H_k(\mathbb{CP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \le k \le 2n \text{ and } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Example 2.16. We can use the universal coefficient theorem to compute the cohomology groups $H_k(\mathbb{CP}^n;\mathbb{F}_2)$ and $H_k(\mathbb{RP}^n;\mathbb{F}_2)$. For even $0 \le k \le 2n$, we have $H_k(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ and $H_{k-1}(\mathbb{CP}^n;\mathbb{Z}) \cong 0$, so

$$H_k(\mathbb{CP}^n;\mathbb{F}_2) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(0,\mathbb{F}_2) \cong \mathbb{F}_2.$$

For odd $0 < k < 2n$, we have $H_k(\mathbb{CP}^n;\mathbb{Z}) \cong 0$ and $H_{k-1}(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$, so

$$H_k(\mathbb{CP}^n;\mathbb{F}_2) \cong 0 \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(\mathbb{Z},\mathbb{F}_2) \cong 0.$$

Therefore, the homology of complex projective space with coefficients in $\mathbb{F}_2$ is quite similar to that with coefficients in $\mathbb{Z}$:

$$H_k(\mathbb{CP}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } 0 \le k \le 2n \text{ and } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 0$ or $k = n$ with n odd, we have $H_k(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$ and $H_{k-1}(\mathbb{RP}^n;\mathbb{Z}) \cong 0$, hence

$$H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(0,\mathbb{F}_2) \cong \mathbb{F}_2.$$

For odd $0 < k < n$, we have $H_k(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{F}_2$ and $H_{k-1}(\mathbb{RP}^n;\mathbb{Z}) \cong 0$, hence

$$H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(0,\mathbb{F}_2) \cong \mathbb{F}_2.$$

For even $0 < k \le n$, we have $H_k(\mathbb{RP}^n;\mathbb{Z}) \cong 0$ and $H_{k-1}(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{F}_2$, hence

$$H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong 0 \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathbb{F}_2,$$

where we use the computation $\mathrm{Tor}(\mathbb{F}_2,\mathbb{F}_2) = \mathbb{F}_2$ of Example 2.9.
For $k = n+1$, with n odd, we have $H_k(\mathbb{RP}^n;\mathbb{Z}) \cong 0$ and $H_{k-1}(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$, hence

$$H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong 0 \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus \mathrm{Tor}(\mathbb{Z},\mathbb{F}_2) \cong 0.$$

For $k < 0$ or $k > n+1$, or $k = n+1$ with n even, we have $H_k(\mathbb{RP}^n;\mathbb{Z}) \cong H_{k-1}(\mathbb{RP}^n) \cong 0$, hence $H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong 0$. We conclude

$$H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.17. A similar computation yields

$$H^k(\mathbb{RP}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H^k(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } 0 \le k, \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Euler characteristic

Definition 2.18. Let X be a topological space and K be a field. Then the *Euler characteristic* $\chi(X;K)$ of X with coefficients in K is defined by

$$\chi(X;K) = \sum_k (-1)^k \dim_K H_k(X;K).$$

Note that this is only a well-defined integer if all $H_k(X;K)$ are finite dimensional and finitely many of them are non-zero. In particular it is well-defined if X is a *finite CW-complex*.

Example 2.19. Using the homology groups we computed in Example 2.16, we find that

$$\chi(\mathbb{CP}^n;\mathbb{F}_2) = n+1$$

and

$$\chi(\mathbb{RP}^n;\mathbb{F}_2) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

The Euler characteristic is a useful algebraic invariant for topological spaces. In this section we will prove that it is independent of the field of coefficients and it can also be defined in terms of cohomology.

Lemma 2.20. *Let K be a field of characteristic p. Let $A \cong \mathbb{Z}/q^l\mathbb{Z}$, for a prime number q. Then the following hold.*

i) If $p = q$, then $K \cong A \otimes_{\mathbb{Z}} K \cong \operatorname{Hom}(A,K) \cong \operatorname{Tor}(A,K) \cong \operatorname{Ext}(A,K)$.
ii) If $p \neq q$, then $0 \cong A \otimes_{\mathbb{Z}} K \cong \operatorname{Hom}(A,K) \cong \operatorname{Tor}(A,K) \cong \operatorname{Ext}(A,K)$.

Proof. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \phi\ } \mathbb{Z} \xrightarrow{a \mapsto \bar{a}} A \longrightarrow 0$$

provides a projective resolution of A, where $\phi : \mathbb{Z} \to \mathbb{Z}$ is multiplication by q^l. Leaving out A and applying the $- \otimes_{\mathbb{Z}} K$ functor, we get the chain complex $0 \to K \to K \to 0$, where the map $K \to K$ is still multiplication by q^l. If $q = p$, then this is the zero map, so $\operatorname{Tor}(A,K)$, which is the homology in index 1 of this complex, is K. If $q \neq p$, then it is an isomorphism, so the sequence is exact and the homology is zero. Leaving out A and applying the $\operatorname{Hom}(-,K)$ functor, we get the cochain complex $0 \to K \to K \to 0$, where the map $K \to K$ is again multiplication by q^l, so a similar argument shows the statement for $\mathrm{Ext}(A, K)$.

If $q \neq p$, then p is invertible in $\mathbb{Z}/q^l\mathbb{Z}$, so

$$\bar{a} \otimes b = p(p^{-1}\bar{a} \otimes b) = (p^{-1}\bar{a} \otimes pb) = 0$$

for all $(\bar{a}, b) \in A \times K$, hence $A \otimes_{\mathbb{Z}} K = 0$. Now suppose $q = p$. If $\bar{a} = \bar{a'}$ in A then $p^l \mid a - a'$, hence $a - a' = 0$ in K. So $\bar{a} \mapsto a \cdot 1 : A \to K$ is well-defined. Therefore also the homomorphisms

$$\begin{aligned} A \otimes_{\mathbb{Z}} K \to K :\ & \bar{a} \otimes b \mapsto ab, \\ K \to A \otimes_{\mathbb{Z}} K :\ & b \mapsto \bar{1} \otimes b \end{aligned}$$

are well-defined. They are each others inverse, because $\bar{a} \otimes b = a \cdot \bar{1} \otimes_{\mathbb{Z}} b = \bar{1} \otimes ab$. This shows that $A \otimes_{\mathbb{Z}} K \cong K$.

Lastly we compute $\mathrm{Hom}(A, K)$. A homomorphism $\phi : A \to K$ is completely determined by $\phi(\bar{1})$. If $q \neq p$, then $q^l \neq 0$ in K, hence $0 = \phi(0) = \phi(q^l) = \phi(1) \cdot q^l$ implies that $\phi(1) = 0$, hence $\mathrm{Hom}(A, K) = 0$. If $q = p$, then any $\phi(\bar{a}) = a \cdot \phi(\bar{1})$ is well-defined, since $p^l = 0$ in K. Therefore $\mathrm{Hom}(A, K) = K$, via $\phi \mapsto \phi(\bar{1})$ in this case. □

Proposition 2.21. *Let X be a finite CW-complex. For any field K we have $\chi(X; K) = \chi(X; \mathbb{Q})$.*

Proof. Let $p = \operatorname{char} K$. Write $H_k(X; \mathbb{Z}) \cong \mathbb{Z}^{r_k} \oplus T_k$, where T_k is the torsion subgroup of $H_k(X; \mathbb{Z})$, which is of the form

$$T_k \cong \mathbb{Z}/p_1^{l_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n^{l_n}\mathbb{Z},$$

for some prime numbers $p_1, \ldots, p_n$. We can order the p_i such that $p_1 = \cdots = p_{s_k} = p$ and $p_{s_k+1}, \ldots, p_n$ are all not equal to p, for some $s_k \geq 0$. We then have $T_k \otimes_{\mathbb{Z}} K = K^{s_k}$ and $\mathrm{Tor}(T_{k-1}, K) = K^{s_{k-1}}$ by Lemma 2.20.

By the universal coefficient theorem, we have

$$H_k(X, K) \cong H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} K \oplus \mathrm{Tor}(H_{k-1}(X; \mathbb{Z}), K).$$

If $p = 0$, in particular when $K = \mathbb{Q}$, then $s_k = 0$ for all k, so then $\mathrm{Tor}(H_{k-1}(X; \mathbb{Z}), K) = 0$ and $H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} K \cong K^{r_k}$, so

$$\chi(X; K) = \sum_k (-1)^k \dim_K K^{r_k} = \sum_k (-1)^k \dim_{\mathbb{Q}} \mathbb{Q}^{r_k} = \chi(X; \mathbb{Q}).$$

If $p > 0$, then

$$\begin{aligned} \chi(X; K) &= \sum_k (-1)^k \dim_K (K^{r_k + s_k} \oplus K^{s_{k-1}}) \\ &= \sum_k (-1)^k (r_k + s_k + s_{k-1}) \\ &= \chi(X; \mathbb{Q}) + \sum_k (-1)^k (s_k + s_{k-1}) \\ &= \chi(X; \mathbb{Q}). \end{aligned}$$

□

So we can speak of *the* Euler characteristic $\chi(X)$ of X.

Definition 2.22. The Euler characteristic in cohomology with coefficients in K is defined as

$$\chi_{co}(X;K) = \sum_k (-1)^k \dim_K H^k(X;K).$$

Just as for the Euler characteristic in homology, this is only a well-defined if there are only finitely many non-zero terms and they are all finite dimensional.

Proposition 2.23. *Let X be a finite CW-complex. For any field K, $\chi_{co}(X;K) = \chi(X)$.*

Proof. By the universal coefficient theorem, we have

$$H^k(X;K) \cong \mathrm{Ext}(H_{k-1}(X;\mathbb{Z}), K) \oplus \mathrm{Hom}(H_k(X;\mathbb{Z}), K).$$

Let $p = \operatorname{char} K$ and let r_k and s_k be as in the proof of Proposition 2.21. Then, by Lemmata 2.14 and 2.20,

$$\mathrm{Ext}(H_{k-1}(X;\mathbb{Z}), K) \cong K^{s_{k-1}}$$

and

$$\mathrm{Hom}(H_k(X;\mathbb{Z}), K) \cong K^{r_k+s_k},$$

so we see $H^k(X;K) \cong K^{r_k+s_k+s_{k-1}} \cong H_k(X;K)$. Therefore the result follows from Proposition 2.21. $\square$

3 Smith theory

In this chapter X will denote a topological space and σ a continuous involution on X. Furthermore we denote the fixed locus of σ by $X^\sigma = \{x \in X \mid \sigma(x) = x\}$. The involution σ induces a chain map $C_*(X;\mathbb{F}_2) \to C_*(X;\mathbb{F}_2)$, which we will also denote by σ. Now define

$$\rho = \mathrm{id} + \sigma : C_*(X;\mathbb{F}_2) \to C_*(X;\mathbb{F}_2).$$

Additionally we define

$$i : \rho C_*(X;\mathbb{F}_2) \oplus C_*(X^\sigma;\mathbb{F}_2) \to C_*(X;\mathbb{F}_2),$$

by $(a,b) \mapsto a+b$, where we write b for the image of b under the natural inclusion

$$C_*(X^\sigma;\mathbb{F}_2) \to C_*(X;\mathbb{F}_2).$$

Lemma 3.1. *The following sequence of chain complexes is exact.*

$$0 \longrightarrow \rho C_*(X;\mathbb{F}_2) \oplus C_*(X^\sigma;\mathbb{F}_2) \xrightarrow{\;i\;} C_*(X;\mathbb{F}_2) \xrightarrow{\;\rho\;} \rho C_*(X;\mathbb{F}_2) \longrightarrow 0.$$

Proof. We check the exactness degree-wise.

The surjectivity of $\rho : C_k(X;\mathbb{F}_2) \to \rho C_k(X;\mathbb{F}_2)$ is obvious.

Suppose $i(a,b) = 0$, then $a = b$ since we are working over $\mathbb{F}_2$, hence

$$a \in \rho C_k(X;\mathbb{F}_2) \cap C_k(X^\sigma;\mathbb{F}_2).$$

Suppose $a = \rho(x) = x + \sigma(x) \in C_k(X^\sigma;\mathbb{F}_2)$. We can write $x = \sum_i s_i$, where the s_i are distinct k-simplices in X. If $\sigma(s_i) \neq s_i$, then there must be a j, such that $s_j = \sigma(s_i)$, because otherwise $x + \sigma(x) \notin C_k(X^\sigma;\mathbb{F}_2)$. Since $\rho(x - s_i - \sigma(s_i)) = \rho(x)$, we can assume without loss of generality that $a = \rho(x)$ with $x \in C_k(X^\sigma;\mathbb{F}_2)$ and thus $b = a = 2x = 0$. Therefore i is injective.

For the exactness at $C_k(X;\mathbb{F}_2)$, we first show $\operatorname{Im} i \subseteq \operatorname{Ker} \rho$ by showing that $\rho \circ i = 0$. Let

$$(\rho(a), b) \in \rho C_k(X;\mathbb{F}_2) \oplus C_k(X^\sigma;\mathbb{F}_2),$$

then

$$\begin{aligned} \rho(i(\rho(a),b)) &= \rho(\rho(a)+b) = \rho(\rho(a)) + \rho(b) \\ &= \rho(a+\sigma(a)) + b + \sigma(b) = 2a + 2\sigma(a) + 2b = 0. \end{aligned}$$

Now for the reverse inclusion $\operatorname{Ker} \rho \subseteq \operatorname{Im} i$, let $s = \sum_i s_i \in \operatorname{Ker} \rho$. For every i we either have $s_i = \sigma(s_i)$ or $s_i \neq \sigma(s_i)$, so we can write

$$s = \sum_i (x_i a_i + y_i \sigma(a_i)) + \sum_j b_j,$$

with $x_i, y_i \in \mathbb{F}_2$ and $\sigma(b_j) = b_j$ and $a_i \neq \sigma(a_i)$. Note that by definition $\sigma(b_j) = b_j \iff \sigma(b_j(x)) = b_j(x)$ for all $x \in \Delta^k$, so the b_j are simplices $b_j : \Delta^k \to X^\sigma$ and $\sum_j b_j \in C_k(X^\sigma;\mathbb{F}_2)$.

Without loss of generality, each a_i, $\sigma(a_i)$ and b_j occur only once in the sum, i.e. $i \neq j \implies a_i \neq a_j, a_i \neq \sigma(a_j)$ and $b_i \neq b_j$. Furthermore, for all i we can assume that x_i and y_i are not both 0. Now

$$\begin{aligned} 0 = \rho(s) &= \sum_i ((x_i + y_i)a_i + (x_i + y_i)\sigma(a_i)) + \sum_j (b_j + \sigma(b_j)) \\ &= \sum_i ((x_i + y_i)a_i + (x_i + y_i)\sigma(a_i)). \end{aligned}$$

Because we assumed each a_i occurred only once in the sum, this implies that $x_i + y_i = 0$ for all i. From the assumption that x_i and y_i are not both zero, we conclude that $x_i = y_i = 1$ for all i. Therefore $s = i(\rho(\sum_i a_i), \sum_j b_j)$ and $\operatorname{Ker} \rho \subseteq \operatorname{Im} i$.

This shows that the sequence is exact. $\square$

Corollary 3.2. *There is a long exact sequence of homology groups*

$$\cdots \xrightarrow{i_k} H_k(X;\mathbb{F}_2) \xrightarrow{\rho_k} H_k(\rho C_*(X;\mathbb{F}_2)) \xrightarrow{\delta_k} H_{k-1}(\rho C_*(X;\mathbb{F}_2)) \oplus H_{k-1}(X^\sigma;\mathbb{F}_2)$$

$$\xrightarrow{i_{k-1}} H_{k-1}(X;\mathbb{F}_2) \xrightarrow{\rho_{k-1}} \cdots .$$

Proof. A short exact sequence of chain complexes induces a long exact sequence of its homology groups, see Theorem 1.3.1 in [19]. Applying this on the short exact sequence from Lemma 3.1 gives the long exact sequence above. $\square$

Corollary 3.3. *For all k, the sequence*

$$0 \to H_{k+1}(\rho C_*(X;\mathbb{F}_2))/\operatorname{Im}\rho_{k+1} \xrightarrow{\bar{\delta}_k} H_k(\rho C_*(X;\mathbb{F}_2)) \oplus H_k(X^\sigma;\mathbb{F}_2) \xrightarrow{i_k} H_k(X;\mathbb{F}_2) \xrightarrow{\rho_k} \operatorname{Im}\rho_k \to 0$$

is exact. Where i_k and ρ_k are as in Corollary 3.2 and $\bar{\delta}_k$ is the map induced by δ_k from Corollary 3.2.

Proof. The exactness in the middle two places follows immediately from Corollary 3.2. Exactness in the last place is clear, since $\rho_k : H_k(X;\mathbb{F}_2) \to \operatorname{Im}\rho_k$ is surjective. Exactness in the first place also follows from Corollary 3.2, because $\operatorname{Ker}\delta_k = \operatorname{Im}\rho_{k+1}$ implies that $\bar{\delta}_k$ is injective. $\square$

Definition 3.4. Let X be a topological space and K a field, then the kth *Betti number* with coefficients in K of X is $b_k(X;K) = \dim_K H_k(X;K)$. The sum of all Betti numbers is denoted by $b_*(X;K) = \sum_k b_k(X;K)$ and is called the *total Betti number* with coefficients in K.

Theorem 3.5 (Thom–Smith inequality, [15, Theorem 3.3.6])**.** *Let X be an n-dimensional manifold and let $\sigma : X \to X$ be an involution, then*

$$b_*(X^\sigma;\mathbb{F}_2) \leq b_*(X;\mathbb{F}_2).$$

Proof. Let $a_k = \dim_{\mathbb{F}_2}(\operatorname{Im}\rho_k)$ and $c_k = \dim_{\mathbb{F}_2}(H_k(\rho C_*(X;\mathbb{F}_2)))$. Then Corollary 3.3 implies

$$0 = (c_{k+1} - a_{k+1}) - (c_k + b_k(X^\sigma;\mathbb{F}_2)) + b_k(X;\mathbb{F}_2) - a_k,$$

so

$$b_k(X^\sigma;\mathbb{F}_2) = b_k(X;\mathbb{F}_2) - a_{k+1} - a_k + c_{k+1} - c_k.$$

Taking sums, we get

$$\begin{aligned} b_*(X^\sigma;\mathbb{F}_2) &= \sum_{k=-\infty}^{\infty} b_k(X^\sigma;\mathbb{F}_2) = \sum_{k=-\infty}^{\infty} \left(b_k(X;\mathbb{F}_2) - a_{k+1} - a_k + c_{k+1} - c_k\right) \\ &= \sum_{k=0}^{2n} \left(b_k(X;\mathbb{F}_2) - 2a_k\right). \end{aligned}$$

In particular $b_*(X^\sigma;\mathbb{F}_2) \leq b_*(X;\mathbb{F}_2)$. □

For the rest of this section we let $Y = X/\sigma$, equipped with the quotient topology. The subspace X^σ of X can also be seen as a subspace of Y, since the points in X^σ only get identified with themselves in Y.

Lemma 3.6. *Let K be a field. If $\operatorname{char} K = 2$, then*

$$\rho C_*(X;K) \cong C_*(Y, X^\sigma; K) = C_*(Y;K)/C_*(X^\sigma;K).$$

Otherwise

$$C_*(X;K)^\sigma \cong \rho C_*(X;K) \cong C_*(Y;K).$$

Here $C_*(X;K)^\sigma$ denotes the subchaincomplex of $C_*(X;K)$ that is invariant under σ.

Proof. Let $c \in \rho C_k(X;K)$, then $c = \sum_i x_i(a_i + \sigma(a_i))$, where $x_i \in K - \{0\}$ and the a_i are k-simplices in X.

Define $\phi_k : \rho C_k(X;K) \to C_k(Y;K)$, by $\rho(a) \mapsto [a]$, for k-simplices a, where $[a] \in C_k(Y;K)$ is the k-simplex that is obtained by composing a with the quotient map $X \to Y$. We check that ϕ_k is well-defined. We distinguish the cases $a = \sigma(a)$ and $a \neq \sigma(a)$. If $a \neq \sigma(a)$, then

$$\rho(a) = a + \sigma(a) = b + \sigma(b) = \rho(b)$$

implies that $a = b$ or $a = \sigma(b)$. In both cases $[a] = [b]$. If $a = \sigma(a)$ then

$$\rho(a) = 2a = b + \sigma(b) = \rho(b).$$

If $\operatorname{char} K \neq 2$, then this implies $a = b$ and thus $[a] = [b]$. If $\operatorname{char} K = 2$, then this implies $b + \sigma(b) = 0$, i.e. $b = \sigma(b)$.

We see that ϕ_k is well-defined if $\operatorname{char} K \neq 2$. There is an inverse of ϕ_k, namely $\psi_k = [a] \mapsto \rho(a)$, which is well-defined, because $\rho(a) = \rho(\sigma(a))$. Therefore ϕ_k is an isomorphism. Now we need to check that $(\phi_k)_k$ is a chain-map. This means we need to check that $\phi_{k-1} \circ \partial = \partial \circ \phi_k$. The map σ is induced by a continuous map and hence is a chain-map. As a consequence, ρ is also a chain map. The map induced by the the quotient map $X \to Y$ is also a chain map. Therefore,

$$\phi_{k-1}(\partial(\rho(a))) = \phi_{k-1}(\rho(\partial(a)) = [\partial(a)] = \partial([a]) = \partial(\phi_k(\rho(a)).$$

This shows that $(\phi_k)_k$ is an isomorphism of chain complexes.

In the other case, when $\operatorname{char} K = 2$, then ϕ_k isn't well-defined in general, because if a and b are different k-simplices in X^σ, then $\rho(a) = \rho(b) = 0$, but $a = \sigma(a) \neq b = \sigma(b)$, so $[a] \neq [b]$.

However, if we consider ϕ_k as a map $\rho C_k(X;K) \to C_k(Y, X^\sigma; K)$, then it is well-defined, since then $[a] = [b] = 0$ for all simplices a, b in X^σ. The inverse $[a] \mapsto \rho(a)$ is also still well-defined, because $\rho(a) = a + \sigma(a) = 2a = 0$ for a in X^σ. The argument that $(\phi_k)_k$ is a chain map is the same as for $\operatorname{char} K \neq 2$.

Lastly we remark that that $C_*(K;X)^\sigma = \rho C_*(X;K)$ when $\operatorname{char} K \neq 2$. The inclusion

$$C_*(K;X)^\sigma \subseteq \rho C_*(X;K)$$

holds, because ρ acts as multiplication by 2 on $C_*(K;X)^\sigma$, hence $x = \rho(\frac{1}{2}x) \in \rho C_*(X;K)$ for every $x \in C_*(X;K)^\sigma$. The other inclusion, $C_*(X;K)^\sigma \supseteq \rho C_*(X;K)$, holds because $\sigma \circ \rho = \rho$. □

Corollary 3.7. *For a field K with* $\operatorname{char} K \neq 2$ *there is an isomorphism $H_n(X;K)^\sigma \cong H_n(Y;K)$.*

Proof. We can extend the map $\phi_k : \rho C_k(X;K)^\sigma \to C_k(Y;K)$ in the proof above to a map

$$\tilde{\phi}_k : C_k(X;K) \to C_k(Y;K),$$

by setting $\tilde{\phi}_k(a) = \frac{1}{2}[a]$. On homology now $(\psi_k \circ \tilde{\phi}_k)_* = \mathrm{id}$ and $\mathrm{Im}(\psi_k)_* \subseteq H_k(X;K)^\sigma$ and

$$(\tilde{\phi}_k \circ \psi_k)_*(a) = \frac{1}{2}\rho_k(a),$$

which is the identity when restricted to $H_k(X;K)^\sigma$. This shows that we have an isomorphism $H_n(X;K)^\sigma \cong H_n(Y;K)$. □

Corollary 3.8. *There is an exact sequence*

$$\cdots \to H_{k+1}(Y, X^\sigma; \mathbb{F}_2) \to H_k(Y, X^\sigma; \mathbb{F}_2) \oplus H_k(X^\sigma; \mathbb{F}_2) \to H_k(X; \mathbb{F}_2) \to H_k(Y, X^\sigma; \mathbb{F}_2) \to \cdots$$

Proof. By Corollary 3.2 we have the exact sequence

$$H_{k+1}(\rho C(X;\mathbb{F}_2)) \to H_k(\rho C(X;\mathbb{F}_2)) \oplus H_k(X^\sigma;\mathbb{F}_2) \to H_k(X;\mathbb{F}_2) \to H_k(\rho C(X;\mathbb{F}_2)).$$

Lemma 3.6 shows that $\rho C(X;\mathbb{F}_2) \cong C(Y, X^\sigma; \mathbb{F}_2)$, from which the result follows. □

Example 3.9. If we let $X = S^2$ and $\sigma(x,y,z) = (x,y,-z)$, then $X^\sigma = \{(x,y,z) \in S^2 \mid z = 0\} \cong S^1$ and $Y = X/\sigma \cong \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \cong D^2$.

Theorem 3.5 says that $b_*(S^1;\mathbb{F}_2) \leq b_*(S^2;\mathbb{F}_2)$. And indeed, Example 2.4 combined with the Universal Coefficient Theorem, shows that

$$H_k(S^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

so $b_*(S^1;\mathbb{F}_2) = 2 = b_*(S^2;\mathbb{F}_2)$.

We can already use the long exact sequence in homology of the pair (D^2, S^1) to compute the relative cohomology groups $H_k(D^2, S^1; \mathbb{F}_2)$, but Corollary 3.8 gives an alternative way to compute these groups, given that $H_k(D^2, S^1; \mathbb{F}_2) \cong 0$ for some $k \geq 3$.

If we denote $A_k = H_k(D^2, S^1; \mathbb{F}_2)$, then for $k \geq 4$ we get the exact sequence

$$0 \to A_k \to A_{k-1} \oplus 0 \to 0,$$

so $A_k \cong A_{k-1}$ for all $k \geq 4$. Together with the assumption $A_k \cong 0$ for some $k \geq 3$, this implies that $A_k \cong 0$ for all $k \geq 3$.

We now have the exact sequences

$$0 \longrightarrow A_1 \xrightarrow{\gamma} A_0 \oplus \mathbb{F}_2 \xrightarrow{\beta} \mathbb{F}_2 \xrightarrow{\alpha} A_0 \longrightarrow 0$$

and

$$0 \longrightarrow A_2 \oplus 0 \xrightarrow{h} \mathbb{F}_2 \xrightarrow{g} A_2 \xrightarrow{f} A_1 \oplus \mathbb{F}_2 \longrightarrow 0.$$

The map β restricted to $0 \oplus \mathbb{F}_2$ is the map $H_0(S^1; \mathbb{F}_2) \to H_0(S^2; \mathbb{F}_2)$ induced by the inclusion $S^1 \to S^2$. This is not the zero map, as the class of a single 0-simplex in S^1 is also not zero in S^2. Therefore $\operatorname{Ker} \beta = A_0$ and $\operatorname{Ker} \alpha = \operatorname{Im} \beta = \mathbb{F}_2$ so $\alpha = 0$. Since α is surjective, we get $0 = A_0 = \operatorname{Ker} \beta = \operatorname{Im} \gamma$. Since γ is injective, we get $A_1 = 0$.

The second exact sequence now has the form

$$0 \longrightarrow A_2 \xrightarrow{h} \mathbb{F}_2 \xrightarrow{g} A_2 \xrightarrow{f} \mathbb{F}_2 \longrightarrow 0.$$

As h is injective and f is surjective, we get $2 \leq |A_2| \leq 2$, so $A_2 \cong \mathbb{F}_2$.

Computing relative homology groups is not the most important application of *Corollary* 3.8. Instead we will use it to derive a relation between the Euler characteristics $\chi(X)$ and $\chi(X^\sigma)$.

4 $\mathbb{R}$-varieties

Definition 4.1. An n-dimensional *complex manifold* is a second-countable, Hausdorff space X with an *atlas*, i.e. a collection of homeomorphisms $\{\phi_i : U_i \to V_i\}$, with $U_i \subseteq X$ open and $V_i \subseteq \mathbb{C}^n$ open, such that the U_i cover X and $\phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)}$ is holomorphic for all i, j.

Definition 4.2. Let X be a complex manifold, let $U \subseteq X$ be an open subset and let $f : U \to \mathbb{C}$ be a function. Then f is called *holomorphic* if for any $x \in U$ there is a chart $\phi : V \to W$ around x, such that $f \circ \phi^{-1}|_{\phi(V \cap U)}$ is holomorphic.

Definition 4.3 ([15, Definition 2.1.8])**.** Let X and Y be complex manifolds and let $\mathcal{O}_X$ denote the *sheaf of holomorphic functions* on X, i.e. $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$. Similarly, let $\mathcal{O}_Y$ be the sheaf of holomorphic functions on Y. Then a map $F : X \to Y$ is called *anti-holomorphic* if it pulls back holomorphic functions to anti-holomorphic functions. More precisely, F is anti-holomorphic if it is continuous and for every open $V \subseteq Y$ and $f \in \mathcal{O}_Y(V)$ we have $\mathrm{conj} \circ f \circ F \in \mathcal{O}_X(F^{-1}(V))$, where $\mathrm{conj} : \mathbb{C} \to \mathbb{C}$ is complex conjugation.

Proposition 4.4. *Let X and Y be n- and m-dimensional complex manifolds, respectively. Let $F : X \to Y$ be a continuous function and let $\mathrm{conj}^k : \mathbb{C}^k \to \mathbb{C}^k$ denote component-wise complex conjugation. Then the following are equivalent.*

i) F is anti-holomorphic.

ii) For all charts $\phi : U \to \tilde{U}$ of X and $\psi : V \to \tilde{V}$ of Y such that $F(U) \subseteq V$, there is a vector-valued, holomorphic function $g : \mathrm{conj}^n(\tilde{U}) \to \tilde{V}$ such that $\psi \circ F \circ \phi^{-1} = g \circ \mathrm{conj}^n$.

iii) For every $x \in X$, there is a chart $\phi : U \to \tilde{U}$ around x and a chart $\psi : V \to \tilde{V}$ around $F(x)$, such that $F(U) \subseteq V$ and $\psi \circ F \circ \phi^{-1} = g \circ \mathrm{conj}^n$, for some holomorphic, vector-valued function $g : \mathrm{conj}^n(\tilde{U}) \to \tilde{V}$.

Proof. We first note that a function $f : \mathbb{C}^a \to \mathbb{C}^b$ is holomorphic $\iff$ $\mathrm{conj}^b \circ f \circ \mathrm{conj}^a$ is holomorphic. For the implication to the right we note that f is of the form

$$f(z) = \left(\sum_I a_{I,i} z^I \right)_i ,$$

as f is holomorphic. Here we use multi-index notation, i.e. $z = (z_1, ..., z_a), I = (k_1, ..., k_n)$ and $z^I = z_1^{k_1} \cdots z_n^{k_n}$.

Therefore $\mathrm{conj}^b \circ f \circ \mathrm{conj}^a$ is of the form

$$\overline{f(\overline{z})} = \left(\sum_I \overline{a_{I,i}} z^I \right)_i$$

and thus also holomorphic. The implication to the left now follows, since

$$f = \mathrm{conj}^m \circ \mathrm{conj}^m \circ f \circ \mathrm{conj}^n \circ \mathrm{conj}^n .$$

$i) \Rightarrow ii)$: Let $\phi : U \to \tilde{U}$ and $\psi : V \to \tilde{V}$ be charts such that $F(U) \subseteq V$. Then $\psi \in \mathcal{O}_Y(V)$. Since F is anti-holomorphic, $\mathrm{conj} \circ \psi \circ F$ is holomorphic. In particular, this means that $\mathrm{conj} \circ \psi \circ F \circ \phi^{-1}$ is holomorphic. Let $g = \psi \circ F \circ \phi^{-1} \circ \mathrm{conj}^n$, then $\mathrm{conj} \circ \psi \circ F \circ \phi^{-1} = \mathrm{conj} \circ g \circ \mathrm{conj}^n$ is holomorphic,

so g is holomorphic and $\psi \circ F \circ \phi^{-1} = g \circ \mathrm{conj}^n$.
ii) $\Rightarrow$ *iii)*: For any charts U and V around x and $F(x)$ respectively, we can restrict U such that $F(U) \subseteq V$ and then apply *ii)*.
iii) $\Rightarrow$ *i)*: Let $f \in \mathcal{O}_Y(V)$, let $x \in F^{-1}(V)$ and let $\phi : U \to \tilde{U}$ be a chart around x. By assumption there exists a chart $\psi : \hat{V} \to \tilde{V}$ around $F(x)$, such that $F(U) \subseteq \hat{V}$ and $\psi \circ F \circ \phi^{-1} = g \circ \mathrm{conj}^n$. By restricting both U and $\hat{V}$ we can assume that $F(U) \subseteq \hat{V} \subseteq V$. Since f is holomorphic at $F(x) \in V$, the composition $f \circ \psi^{-1}$ is holomorphic. Therefore

$$\mathrm{conj} \circ f \circ F \circ \phi^{-1} = \mathrm{conj} \circ f \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1} = \mathrm{conj} \circ f \circ \psi^{-1} \circ g \circ \mathrm{conj}^n$$

is holomorphic. This shows that $\mathrm{conj} \circ f \circ F$ is holomorphic at x. This holds for any $x \in F^{-1}(V)$, so $\mathrm{conj} \circ f \circ F$ is holomorphic and F is anti-holomorphic. □

Definition 4.5 ([15, Definition 2.1.10])**.** A *real structure* on a complex manifold X is an anti-holomorphic involution $\sigma : X \to X$.

Example 4.6. Complex conjugation $\mathrm{conj} : \mathbb{C} \to \mathbb{C}$ is a real structure on $\mathbb{C}$. Proposition 4.4 says that conj is anti-holomorphic, since $\mathrm{conj} = \mathrm{id} \circ \mathrm{conj}$ and it is also clearly an involution.

Example 4.7. Complex conjugation also induces a real structure on the projective space $\mathbb{CP}^n$,

$$\sigma([x_0 : ... : x_n]) = [\overline{x_0} : ... : \overline{x_n}].$$

It is clear that it is an involution. To check that it is anti-holomorphic. we look at the standard affine charts

$$\phi_i : U_i = \{[x_0 : ... : x_n] \mid x_i \neq 0\} \to \mathbb{C}^n, [x_0 : ... : x_n] \mapsto \left(\frac{x_0}{x_1}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right).$$

We note that $\sigma(U_i) = U_i$ and $\phi_i \circ \sigma \circ \phi_i^{-1} = (z_1, ..., z_n) \mapsto (\overline{z_1}, ..., \overline{z_n})$, i.e. $\phi_i \circ \sigma \circ \phi_i^{-1} = \mathrm{id} \circ \mathrm{conj}^n$. By Proposition 4.4, σ is anti-holomorphic and thus is a real structure on $\mathbb{CP}^n$. This real structure is called the *standard real structure* on $\mathbb{CP}^n$.

Example 4.8. Another real structure on $\mathbb{CP}^1$ is

$$\tau([x_0 : x_1]) = [\overline{x_1} : -\overline{x_0}].$$

It is again clear that this is an involution. To check that it is anti-holomorphic, we proceed as above, but now

$$\begin{aligned} \tau(U_1) &= U_0, \\ \tau(U_0) &= U_1, \\ \phi_0 \circ \tau \circ \phi_1^{-1} &= z \mapsto \phi_0(\tau([z : 1])) = \phi_0([1 : -\overline{z}]) = -\overline{z}, \\ \phi_1 \circ \tau \circ \phi_0^{-1} &= z \mapsto -\overline{z}. \end{aligned}$$

So τ is a real structure on $\mathbb{CP}^1$. We will later see that this real structure is not isomorphic to the standard real structure. This shows that this real structure is not isomorphic to the standard real structure, because the standard real structure on $\mathbb{CP}^1$ does not have an empty real locus.

Example 4.9. Let $L = \mathbb{Z}+i\mathbb{Z}$ and let X be the complex torus $\mathbb{C}/L$. Note that $\operatorname{conj}(L) = \mathbb{Z}-i\mathbb{Z} = L$, so conj induces a well-defined involution $\sigma : X \to X$. Let $\pi : \mathbb{C} \to X, z \mapsto [z]$ be the quotient map. Then for any $z \in \mathbb{C}$ there is an open $U \ni z$, such that $\pi : U \to \pi(U)$ is a homeomorphism and $\pi^{-1}|_{\pi(U)}$ is a chart of X. For such U we have

$$(\pi^{-1} \circ \sigma \circ \pi)|_U = z \mapsto \overline{z} : U \to \operatorname{conj}(U),$$

so σ is anti-holomorphic.

Definition 4.10. An $\mathbb{R}$-*variety* (X, σ) is a complex manifold X equipped with a real structure σ. If X is a complex curve, i.e. a complex manifold of dimension 1, (X, σ) is called an $\mathbb{R}$-*curve*. The *real locus* of X, which is denoted by $X(\mathbb{R})$ or X^σ, is the fixed locus of σ,

$$X(\mathbb{R}) := \{x \in X \mid \sigma(x) = x\}.$$

Example 4.6 (Continued)**.** The real locus of $\mathbb{C}$ with complex conjugation is $\mathbb{R}$.

Example 4.7 (Continued)**.** The real locus of complex projective space $\mathbb{CP}^n$ with real structure induced by complex conjugation, is the real projective space $\mathbb{RP}^n$.

Example 4.8 (Continued)**.** The real locus of $\mathbb{CP}^1$ with the alternative real structure from Example 4.8 is empty, because if $[x_0 : x_1] = [x_1 : -\overline{x_0}]$ there must be a $\lambda \neq 0$ with $x_0 = \lambda x_1$ and $x_1 = -\lambda\overline{x_0}$. In particular we have $x_1 = -\lambda\overline{\lambda}x_1 = -|\lambda|\, x_1$, so we must have $|\lambda| = -1$, which is a contradiction.

Example 4.9 (Continued)**.** On the complex torus $X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, we have

$$\sigma([x+iy]) = [x+iy] \iff [x+iy] = [x-iy] \iff 2iy \in \mathbb{Z}+i\mathbb{Z} \iff y \in \frac{1}{2}\mathbb{Z}.$$

Hence

$$X(\mathbb{R}) = \{[x + \frac{1}{2}iy] \mid x \in \mathbb{R}, y \in \mathbb{Z}\} = \{[x] \mid x \in \mathbb{R}\} \sqcup \{[x + \frac{1}{2}i] \mid x \in \mathbb{R}\} \cong S^1 \sqcup S^1.$$

Proposition 4.11. *The real locus of an n-dimensional $\mathbb{R}$-variety (X, σ) is either empty or it is an n dimensional real manifold.*

Proof. If $X(\mathbb{R})$ is not empty, we construct an atlas for $X(\mathbb{R})$ by giving a chart around every point $p \in X(\mathbb{R})$. Without loss of generality $\sigma(U) = U$, because we can look at $U \cap \sigma(U)$, which is not empty, since $p \in X(\mathbb{R})$. Let $\phi : U \to V$ be a chart of X, centered at $p \in X(\mathbb{R})$, i.e. $\phi(p) = 0$. We write

$$\tau = \phi \circ \sigma \circ \phi^{-1} = (\tau_1, ..., \tau_{2n}) : V \to V,$$

where we view V as a subset of $\mathbb{R}^{2n}$ via $z_j = x_j + iy_j$. Since σ is anti-holomorphic, each $\tau_{2j-1} + i\tau_{2j}$ is a power series in $\bar{z} = (\bar{z_1}, ..., \bar{z_n})^T$, so every τ_j is a power series in $\vec{x} = (x_1, y_1, ..., x_n, y_n)^T$.

The Jacobian $A = J_0\tau$ of τ at 0 is determined by the coefficients of the linear terms of the τ_i, i.e. we can write

$$\tau(\vec{x}) = A\vec{x} + \text{higher order terms}.$$

Note that the constant term is 0, because $\sigma(p) = p$, hence $\tau(0) = 0$. Since σ is an involution, we have

$$\vec{x} = \tau(\tau(\vec{x})) = \tau(A\vec{x} + \text{higher order terms}) = A^2\vec{x} + \text{higher order terms}.$$

Filling in $\vec{x}_j = (0, ..., 0, x, 0, ..., 0)^T$, with x at the jth position, we find $x = a_{j,j}x + x^2 f_{j,j}(x)$ and $0 = a_{i,j}x + x^2 f_{i,j}(x)$ for $i \neq j$, for all $x \in \mathbb{C}$, where $a_{i,j}$ the element in the ith row and jth column of A^2 and $x^2 f_{i,j}$ are the higher order terms of $\tau_i \circ \tau(\vec{x}_j)$. We must have $f_{j,j}(x) = \frac{1-a_{j,j}}{x}$ and $f_{i,j}(x) = \frac{-a_{i,j}}{x}$ for $i \neq j$, but these are not holomorphic in 0, except when $a_{j,j} = 1$ and $a_{i,j} = 0$, in which case $f_{j,j} = f_{i,j} = 0$. This is true for all j and all $i \neq j$, so $A^2 = I$. In particular A is diagonalizable with eigenvalues ± 1.

Let $c_k = a_k + ib_k$ be the coefficient of $\bar{z}_k = x_k - iy_k$ in $\tau_{2j-1} + i\tau_{2j}$. We can compute the coefficients of x_k and y_k in τ_{2j-1} and τ_{2j},

$$c_k \bar{z}_k = (a_k + ib_k)(x_k - iy_k) = a_k x_k + b_k y_k + i(b_k x_k - a_k y_k).$$

Therefore A must be of the form

$$A = \begin{pmatrix} a_{r,k} & b_{r,k} \\ b_{r,k} & -a_{r,k} \end{pmatrix}_{r,k} \in \mathbb{R}^{2n \times 2n}.$$

Note that if $\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$, then $\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} d \\ -c \end{pmatrix}$. So if $(x_1, y_1, ..., x_n, y_n)^T$ is an eigenvector of A corresponding to the eigenvalue 1, then $(-y_1, x_1, ..., -y_n, x_n)^T$ is an eigenvector of A corresponding to the eigenvalue -1. This gives a bijection between the eigenspaces belonging to 1 and -1, so they both have dimension n. In particular, the rank of $A - I$ is n.

The intersection $U \cap X(\mathbb{R})$ corresponds to the fixed locus of τ, which can also be written as the zero locus of $\tau - \mathrm{id}$, which has Jacobian $A - I$ at 0. This Jacobian has rank n, so by the inverse function theorem, there is an open subset $W \subseteq V$ around 0 that is homeomorphic to an open subset of $\mathbb{R}^n$. We use $\phi|_{\phi^{-1}(W)} : \phi^{-1}(W) \to W$ as a chart around p.

The charts around different points are compatible, because the original charts are compatible. $\square$

For connected, compact, complex manifolds of dimension one, which are also called compact Riemann Surfaces, it is well-known that they can be classified topologically by their genus. That is, as a topological space any compact Riemann Surface is homeomorphic to a g-holed torus, for some g, which is called the genus of the space [9, Section 2.4.A]. The following theorem gives constraints on the real locus of genus g $\mathbb{R}$-curves.

Theorem 4.12 (Harnack's inequality, [15, Theorem 2.7.2]). *Let (X, σ) be a connected, compact $\mathbb{R}$-curve of genus g, i.e. (X, σ) is an $\mathbb{R}$-variety and X is a connected, compact curve of genus g. Let s be the number of connected components of $X(\mathbb{R})$. Then $s \leq g + 1$.*

We can prove Harnack's inequality using the Smith theory from the previous chapter. We first note that an $\mathbb{R}$-variety is a special case of a topological space with a continuous involution, so we have the following corollary from Smith theory.

Corollary 4.13. *For an $\mathbb{R}$-variety (X, σ), we have*

$$b_k(X(\mathbb{R}); \mathbb{F}_2) \leq b_k(X; \mathbb{F}_2).$$

Proof. This is an immediate consequence of Theorem 3.5 and the fact that $X(\mathbb{R}) = X^\sigma$. $\square$

This corollary is essentially a general version of Harnack's inequality, as we can see in the following proof.

Proof of Theorem 4.12. The real locus is a compact manifold of real dimension 1, so it is the disjoint union of s circles,

$$X(\mathbb{R}) \cong \bigsqcup_{i=1}^{s} S^1,$$

which has homology

$$H_k(X(\mathbb{R}); \mathbb{F}_2) = \begin{cases} (\mathbb{F}_2)^s & \text{if } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

We also know that the homology of a genus g curve is given by

$$H_k(X; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } k = 0, 2, \\ (\mathbb{F}_2)^{2g} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can apply Corollary 4.13 to get

$$2s = b_*(X(\mathbb{R}); \mathbb{F}_2) \leq b_*(X; \mathbb{F}_2) = 2 + 2g,$$

hence $s \leq g + 1$. □

We can also prove Harnack's inequality in a more geometric way when X is a plane curve.

Definition 4.14. An $\mathbb{R}$-curve X is called a plane $\mathbb{R}$-curve if $X \subseteq \mathbb{CP}^2$ is a plane projective curve and its real structure is the one inherited by the standard real structure on $\mathbb{CP}^2$. Equivalently, X is given by the zero-set of a homogeneous *real* polynomial $F \in \mathbb{R}[X, Y, Z]$; see [15, Proposition 2.1.4].

Definition 4.15. The fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$, so every closed loop in $\mathbb{RP}^2$ is either contractible or homotopic to the line $\{[x : y : 0] \in \mathbb{RP}^2\} \cong \mathbb{RP}^1$. In the first case the loop is called an *oval* and in the second case it is called a *pseudo-line.*

Lemma 4.16.

i) *The connected components of the real locus of a plane $\mathbb{R}$-curve of even degree are all ovals.*
ii) *The connected component of the real locus of a plane $\mathbb{R}$-curve of odd degree are all ovals, except for one, which is a pseudo-line.*
iii) *Any curve intersects an oval in an even number of points, counted with multiplicity.*
iv) *Given $\frac{1}{2}(d+2)(d+1) - 1$ points in $\mathbb{RP}^2$, there is a unique degree d smooth projective plane curve through these points.*
v) *A degree d curve has genus $\frac{1}{2}d(d-1)$.*

Proof.

i)-iii) See [15, Lemma 2.7.8]

iv) A general degree d curve is given by $\sum_{a+b+c=d} a_{a,b,c}X^aY^bZ^c = 0$. In total we need to find $k = \binom{d+2}{2} = \frac{1}{2}(d+2)(d+1)$ coefficients. Filling in the coordinates of $k-1$ points, gives $k-1$ linear equations for the coefficients. These equations are linearly independent if the $k-1$ points are different, so they give a rank $k-1$ system of equations. This gives the coefficients, up to a constant factor, which determines a unique curve.

v) See [15, Theorem 1.6.17].

□

Geometric proof of Theorem 4.12. A degree 1 curve has genus 0 and the real locus is a line, which has $1 \leq 0+1$ connected component. A degree 2 curve also has genus 0. The connected components of its real locus are all ovals. If there is more than 1 you can choose two points on different ovals. The line between these points then must intersect the curve in at least 4 points, counted with multiplicity. This is a contradiction with Bézout's theorem, which says that a line intersects a degree 2 curve in 2 points.

For curves of higher degree we can do something similar. Let C be a plane $\mathbb{R}$-curve of degree $d > 2$, which has genus $g = \frac{1}{2}(d-2)(d-1)$. Suppose $C(\mathbb{R})$ has more than $g+1$ connected components. Then at least $g+1$ of them are ovals. Choose one point on each of these $g+1$ ovals and $\frac{1}{2}d(d-1) - 1 - (g+1)$ on the remaining connected components. We can construct a degree $d-2$ curve through these $\frac{1}{2}d(d-1) - 1$ points. Since this curve must intersect each of the $g+1$ ovals in an even number of points, it must intersect them in at least 2, so it intersects C in at least

$$\begin{aligned}
&2g + 2 + \frac{1}{2}d(d-1) - 1 - (g+1) \\
&= \frac{1}{2}d(d-1) + g \\
&= \frac{1}{2}(d(d-1) + (d-1)(d-2)) \\
&= \frac{1}{2}(d-1)(d+d-2) \\
&= (d-1)^2
\end{aligned}$$

points. Bézout's theorem says that it can intersect in at most $d(d-2) < (d-1)^2$, which is a contradiction, so $C(\mathbb{R})$ can have at most $g+1$ connected components. □

Definition 4.17. An $\mathbb{R}$-curve X of genus g is called a *maximal curve* or *M-curve* if Harnack's inequality is an equality, i.e. if the number of connected components of $X(\mathbb{R})$ is equal to $g+1$.

Example 4.18. $\mathbb{CP}^1$ is a genus 0 curve. With the standard real structure, its real locus is $\mathbb{RP}^1$, which is non-empty and connected, so it has $1 = g+1$ connected component. Therefore $\mathbb{CP}^1$ is a maximal curve.

Example 4.19. The complex torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ has genus $g = 1$. With the real structure from Example 4.9, we saw that its real locus is the disjoint union of two circles, so it has $2 = g+1$ connected components and is thus a maximal curve.

Example 4.20. We can also equip the complex torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with the involution $[x+yi] \mapsto [y+xi]$. This locally is $z \mapsto i\bar{z}$, so it indeed is a real structure. Its real locus is

$$\{[x+yi] \mid x, y \in \mathbb{R} x - y = \in \mathbb{Z}\} \cong \{[x+xi] \mid x \in \mathbb{R}\} \cong S^1,$$

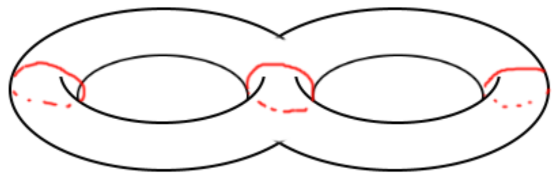

Figure 2: Sketch of a genus 2 surface with a real locus consisting of 3 connected components.

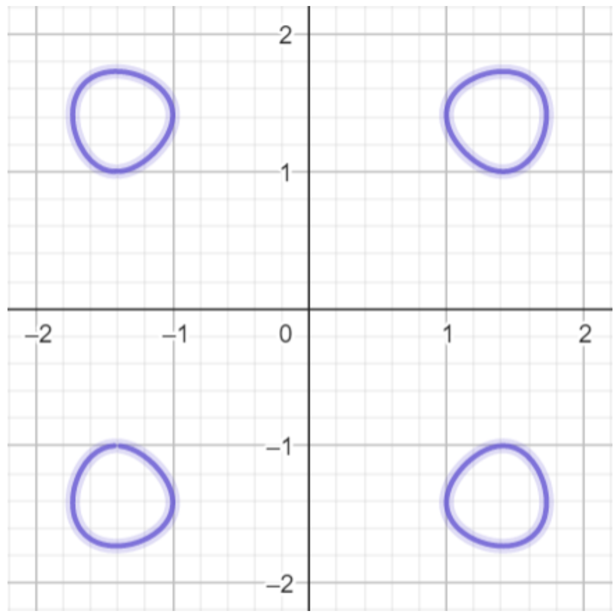

Figure 3: The curve $0 = (x^2 - 2)^2 + (y^2 - 2)^2 - 1$ with genus $(4-2)(4-1)/2 = 3$.

which only has one connected component, so with this involution the complex torus is not a maximal curve.

In the proof of Theorem 4.12, we see that being an M-curve is equivalent to the Thom–Smith inequality being an equality. This motivates the definition of a generalization of M-curves.

Definition 4.21. An $\mathbb{R}$-variety (X, σ) is called a *maximal variety* or *M-variety* if the Thom–Smith inequality (Corollary 4.13) is an equality, i.e. if $b_*(X(\mathbb{R}); \mathbb{F}_2) = b_*(X; \mathbb{F}_2)$.

Example 4.22. Using the homology groups computed in Example 2.16, we see that

$$b_*(\mathbb{CP}^n; \mathbb{F}_2) = b_*(\mathbb{RP}^n; \mathbb{F}_2) = n + 1,$$

so n-dimensional complex projective space with the standard real structure is a maximal variety for any n.

5 Petrovskii–Oleinik–Kharlamov inequality

In this chapter we will prove the Petrovskii–Oleinik–Kharlamov inequality. The proof relies on the Lefschetz fixed point theorem and on the Hodge decomposition. We will not prove the latter, but only give the required definitions to state the theorem.

Theorem 5.1 (Lefschetz fixed point theorem for involutions, [15, Theorem 3.4.23])**.** *Let $\sigma : X \to X$ be an involution on a topological space X, then σ induces linear maps $T_k : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})$ for all k, which satisfy*

$$\chi(X^\sigma) = \sum_k (-1)^k \operatorname{Tr}(T_k).$$

Proof. The exact sequence of Corollary 3.8 implies that

$$\chi(Y, X^\sigma) + \chi(X^\sigma) - \chi(X) + \chi(Y, X^\sigma) = 0,$$

where $Y = X/\sigma$ and

$$\chi(Y, X^\sigma) = \sum_{k\geq 0} (-1)^k \dim H_k(Y, X^\sigma; \mathbb{Q})$$

is the *relative Euler characteristic* of the pair (Y, X^σ). Therefore,

$$\chi(X) = \chi(X^\sigma) + 2\chi(Y, X^\sigma).$$

The exact sequence of the pair (Y, X^σ) implies that

$$\chi(Y, X^\sigma) - \chi(Y) + \chi(X^\sigma) = 0.$$

Hence $\chi(X) = \chi(X^\sigma) + 2(\chi(Y) - \chi(X^\sigma)) = 2\chi(Y) - \chi(X^\sigma)$, i.e.

$$\chi(X^\sigma) = 2\chi(Y) - \chi(X).$$

The T_k are themselves involutions, i.e. $T_k^2 = I$, so they are diagonalizable. The eigenvalues λ of T_k are ± 1, since for an eigenvector v we have $T_k^2(v) = \lambda^2 v = v$. Without loss of generality $T_k = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$ and $\operatorname{Tr}(T_k) = a - b$. Where $a = \dim H_k(X;\mathbb{Q})^{T_k}$ and $b = \dim H_k(X;\mathbb{Q}) - a$. So

$$\operatorname{Tr}(T_k) = 2\dim H_k(X;\mathbb{Q})^{T_k} - H_k(X;\mathbb{Q}) = 2\dim H_k(Y;\mathbb{Q}) - H_k(X;\mathbb{Q}),$$

where the second equality follows from Corollary 3.7. Taking alternating sum we get

$$\sum_k (-1)^k \operatorname{Tr}(T_k) = 2\chi(Y) - \chi(X) = \chi(X^\sigma).$$

□

Remark 5.2. *The previous theorem is a specific version of the more general Lefschetz fixed point theorem, which states that if $f : X \to X$ is a continuous map, then it has a fixed point if $\sum_k (-1)^k \operatorname{Tr}(T_k) \neq 0$, where $T_k : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})$ is the map induced by f. [14, Theorem 2]*

Remark 5.3. *Because the Euler characteristic can also be computed using cohomology and does not depend on the field of coefficients, we can also use the maps that are induced by σ on cohomology $T_k : H^k(X;\mathbb{C}) \to H^k(X;\mathbb{C})$.*

5.1 The Hodge decomposition

The second important prerequisite for the proof of the Petrovskii–Oleinik–Kharlamov inequality is the Hodge decomposition. We first need some more definition before we can state what Hodge decomposition is.

Definition 5.4. Let X be an n-dimensional complex manifold. Let TX denote the *real* tangent bundle of X and T^*X the real cotangent bundle. If $z_1 = x_1 + iy_1, ..., z_n = x_n + iy_n$ are local coordinates at $p \in X$, then $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ is a basis of T_pX and dx_i, dy_i is a basis of T_p^*X. These also form a basis of the complexified (co)tangent spaces, but additionally

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}),$$

form a basis of the *complexified tangent space* $T_{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$ and similarly $dz_j = \frac{1}{2}(dx_j - idy_j)$ and $d\bar{z}_j = \frac{1}{2}(dx_j + idy_j)$ form a basis of the *complexified cotangent space* $T^*_{\mathbb{C}}X = T^*X \otimes_R \mathbb{C}$.

Definition 5.5. The *exterior algebra* $\bigwedge^* V$ of a real or complex vector space V is the quotient of the graded $\mathbb{R}$- or $\mathbb{C}$-algebra $\mathrm{T}(V) = \bigoplus_{k\geq 0} V^{\otimes k}$, by the two-sided ideal generated by elements of the form $x \otimes x$.

The kth *exterior power* $\bigwedge^k V$ of a vector space V is the degree k part of $\mathrm{T}(V)$. The equivalence class of $x_1 \otimes ... \otimes x_n$ is denoted by $x_1 \wedge ... \wedge x_n$. Since

$$\begin{aligned} x \otimes y &= x \otimes (x+y) - x \otimes x \\ &= (x+y) \otimes (x+y) - y \otimes (x+y) - x \otimes x \\ &= (x+y) \otimes (x+y) - y \otimes x - y \otimes y - x \otimes x, \end{aligned}$$

we have $x \wedge y = -y \wedge x$.

Let X be a manifold, then for a vector bundle $E \to X$, its kth exterior power $\bigwedge^k E$ is defined by $(\bigwedge^k E)_p = \bigwedge^k E_p$ for all $p \in X$.

Definition 5.6. A *complex differential k-form* is a C^∞-section of the vector bundle $\bigwedge^k T^*_{\mathbb{C}}X$, where $T^*_{\mathbb{C}}X$ is the *complexified cotangent bundle* of X. Locally a differential k-form α is of the form

$$\alpha = \sum_I \alpha_I dx_{I_1} \wedge ... \wedge dx_{I_k},$$

where $I = (I_1, ..., I_k)$, all I_j are different and the α_I are smooth functions. We denote the set of complex differential k-forms on X by $\mathcal{A}^k(X)$.

Another way to describe k-forms is as smooth sections of $(T^*_{\mathbb{C}}X)^k$, that are alternating k-linear maps at each point.

Definition 5.7. The exterior derivative $d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$ is defined by

$$d\left(\sum_I \alpha_I dx_{I_1} \wedge ... \wedge dx_{I_k}\right) = \sum_I \sum_i \frac{\partial \alpha_I}{\partial x_i} dx_i \wedge dx_{I_1} \wedge ... \wedge dx_{I_k}.$$

Definition 5.8. The exterior derivative d satisfies $d \circ d = 0$, so it makes $\mathcal{A}^*(X)$ a cochain complex. The cohomology of this complex is called the *de Rham cohomology* with coefficients in $\mathbb{C}$ and is denoted by

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) = \frac{\operatorname{Ker}\left(d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)\right)}{\operatorname{Im}\left(d : \mathcal{A}^{k-1}(X) \to \mathcal{A}^k(X)\right)}.$$

Theorem 5.9 (de Rham)**.** *De Rham cohomology with coefficients in $\mathbb{C}$ is isomorphic to singular cohomology with coefficients in $\mathbb{C}$:*

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong H^k(X;\mathbb{C}).$$

The isomophism is given by $[\alpha] \mapsto [c \mapsto \int_c \alpha]$.

Proof. See [13, Theorem 18.14]. $\square$

Definition 5.10. A *Riemannian metric* on a smooth manifold X is a section g of the vector bundle $T^*X \otimes T^*X$, such that $g_p : T_pX \otimes T_pX \to \mathbb{R}$ is a scalar product for every $p \in X$.

Definition 5.11. An *almost complex structure* I on a real manifold X is a morphism of vector bundles $I : TX \to TX$, such that $I^2 = -\operatorname{id}$.

Definition 5.12. A complex manifold with a Riemannian metric is called a *Hermitian manifold* if the Riemannian metric is compatible with its almost complex structure, i.e. $g_x(I_x(v), I_x(w)) = g_x(v,w)$. In this case we define the *fundamental form* ω by $\omega_x(v,w) = g_x(I_x(v), w)$. By extension of scalars, we can view g and ω as sections of $T^*_{\mathbb{C}}X \otimes T^*_{\mathbb{C}}X$.

Definition 5.13. On a complex manifold we have the *canonical almost complex structure* $\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$. On the cotangent bundle this induces $dx_i \mapsto dy_i$ and $dy_i \mapsto -dx_i$. On the complexified cotangent bundle, I acts via $I(dz_j) = i dz_j$ and $I(d\bar{z}_j) = -i d\bar{z}_j$.

Definition 5.14. Let X be a complex manifold and let $(z_1, ..., z_n)$ be local (holomorphic) coordinates, then we call a k-form a *form of type* (p,q), or simply a (p,q)-form, if it is of the form

$$\sum_I \alpha_I dz_{I_1} \wedge ... \wedge dz_{I_p} \wedge d\bar{z}_{I_{p+1}} \wedge ... \wedge d\bar{z}_{I_{p+q}}.$$

We denote the set of forms of type (p,q) on X by $\mathcal{A}^{p,q}(X)$. Clearly we have a decomposition $\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$.

In this situation the exterior derivative decomposes as $d = \partial + \bar{\partial}$, where

$$\partial(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial z_i} dz_i \wedge dx_{I_1} \wedge ... \wedge dx_{I_k}$$

and

$$\bar{\partial}(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial \bar{z_i}} d\bar{z}_i \wedge dx_{I_1} \wedge ... \wedge dx_{I_k}.$$

They satisfy $\partial \circ \partial = 0$ and $\bar{\partial} \circ \bar{\partial} = 0$, and when restricted to $\mathcal{A}^{p,q}(X)$ they give $\mathbb{C}$-linear maps $\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)$.

Definition 5.15. In particular we have a cochain complex $\mathcal{A}^{p,*}$. The cohomology of this complex is called the *Dolbeault cohomology*, which is denoted by

$$H^{p,q}(X) = \frac{\operatorname{Ker}\left(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right)}{\operatorname{Im}\left(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\right)}.$$

We would like to relate Dolbeault cohomology to singular cohomology. For a special type of complex manifold, called a *Kähler manifold*, we indeed have such a relation.

Proposition 5.16. *The fundamental form ω of an Hermitian manifold X is a $(1,1)$-form.*

Proof. By abuse of notation we also write g, ω and I when we mean the forms g_p, ω_p and I_p at a point $p \in X$.

Since g is bilinear and I is linear and

$$\omega(w,v) = g(I(w),v) = g(I^2(w), I(v)) = g(-w, I(v)) = -g(I(v),w) = -\omega(v,w),$$

ω is locally an alternating bilinear map, hence $\omega \in \mathcal{A}^2(X)$. The canonical almost complex structure I acts as multiplication by i on $\mathcal{A}^{1,0}(X)$, since $\mathcal{A}^{1,0}(X)$ is generated by dz_j and similarly it acts as multiplication by $-i$ on $\mathcal{A}^{0,1}(X)$. Therefore it acts as multiplication by -1 on $\mathcal{A}^{2,0}(X)$ and $\mathcal{A}^{0,2}$ and as the identity on $\mathcal{A}^{1,1}(X)$. We have

$$\omega(I(v), I(w)) = g(I^2(v), I(w)) = g(I(v), w) = \omega(v,w),$$

So $I(\omega) = \omega$ and we must have $\omega \in \mathcal{A}^{1,1}(X)$. $\square$

Proposition 5.17. *The fundamental form ω of an Hermitian manifold X is locally of the form*

$$\omega = i \sum_{j,k} w_{j,k} dz_j \wedge d\bar{z}_k,$$

where the matrix $W = (w_{j,k})_{j,k}$ is a positive definite Hermitian matrix. Conversely, every $(1,1)$-form α of this form induces a Hermitian metric, for which α is the corresponding fundamental form.

Proof. Write $\omega = \sum_{j,k} l_{j,k} dz_i \wedge d\bar{z}_j$. In terms of dx_j and dy_j we get

$$\omega = \sum_{j,k} \frac{1}{4} l_{j,k}(dx_j - i dy_j) \wedge (dx_k + i dy_k).$$

The total coefficient of $dx_j \wedge dx_k$ and $dy_j \wedge dy_k$ in ω is $\frac{1}{4}(l_{j,k} - l_{k,j})$ and of $dx_j \wedge dy_k$ is $\frac{1}{4}i(l_{j,k} + l_{j,k})$. These coefficients must be real, so we must have $l_{j,k} = i w_{j,k}$ with $\bar{w}_{j,k} = w_{k,j}$. This shows that W is Hermitian.

Let $0 \neq v = \sum_j \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right)$, with $a_j, b_j \in \mathbb{R}$. Since $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j})$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j})$, we can also write $v = \sum_j \left(c_j \frac{\partial}{\partial z_j} + \bar{c}_j \frac{\partial}{\partial \bar{z}_j} \right)$, with $c_j = a_j + i b_j$.

We know that $g(v,v) > 0$. We write

$$\begin{aligned}
g(v,v) = \omega(v, I(v)) &= \omega(\sum_j \left(c_j \frac{\partial}{\partial z_j} + \bar{c}_j \frac{\partial}{\partial \bar{z}_j} \right), i \sum_k \left(c_k \frac{\partial}{\partial z_j} - \bar{c}_k \frac{\partial}{\partial \bar{z}_j} \right)) \\
&= i\omega(\sum_j \left(c_j \frac{\partial}{\partial z_j} + \bar{c}_j \frac{\partial}{\partial \bar{z}_j} \right), \sum_k \left(c_k \frac{\partial}{\partial z_k} - \bar{c}_k \frac{\partial}{\partial \bar{z}_k} \right)) \\
&= -i \sum_{j,k} \left(c_j \bar{c_k} \omega(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}) + \bar{c}_j c_k \omega(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_j}) \right) \\
&= \sum_{j,k} (c_j \bar{c_k} w_{j,k} + c_k \bar{c}_j w_{k,j}) = 2\bar{c}^T W c > 0,
\end{aligned}$$

where $c = (c_1, ..., c_n)^T$. This must be true for all v, hence for all c, so this shows W is positive definite.

Similarly, if α is of this form and we define $g(v,w) = \alpha(v, I(w))$, then $g(v,v) = 2\bar{c}^T W c > 0$ for all v, hence g is positive definite. It is also clear that g is a symmetric bilinear form that is compatible with I. □

Definition 5.18. A Hermitian manifold is called a *Kähler manifold* if its fundamental form is closed. This means that $d\omega = 0$.

Example 5.19 (Fubini-Study metric)**.** The complex projective space $\mathbb{CP}^n$ has a canonical Kähler metric. It is induced by the fundamental form that is locally defined on $U_i = \{[z_0 : ... : z_n] \mid z_i \neq 0\}$ by

$$\frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{k=0}^{n} \left| \frac{z_k}{z_i} \right|^2 \right).$$

See [8, Example 3.1.9.i)] for the details showing that its fundamental form is closed.

As a consequence, every smooth projective complex variety is a Kähler manifold, as it inherits a metric from the Fubini-Study metric on $\mathbb{CP}^n$.

Definition 5.20. We define the operator $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$, where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ with respect to the Riemannian metric on X. This means that $\bar{\partial}^*$ is determined by $\langle \bar{\partial}\alpha, \beta \rangle = \langle \alpha, \bar{\partial}^* \beta \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\bigwedge^* T_p^* X$ induced by the inner product g_p on $\bigwedge^* T_p X$.
A form is called $\bar{\partial}$*-harmonic* if $\Delta_{\bar{\partial}}(\alpha) = 0$. This is equivalent to $\bar{\partial}^*(\alpha) = \bar{\partial}(\alpha) = 0$, because

$$\langle \Delta_{\bar{\partial}}(\alpha), \alpha \rangle = \langle \bar{\partial}^* \bar{\partial} \alpha, \alpha \rangle + \langle \bar{\partial} \bar{\partial}^* \alpha, \alpha \rangle = \langle \bar{\partial}\alpha, \bar{\partial}\alpha \rangle + \langle \bar{\partial}^* \alpha, \bar{\partial}^* \alpha \rangle = \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\| = 0.$$

Similarly we define $\Delta_d = d^* d + d d^*$ and a form is called d-harmonic if $\Delta_d(\alpha) = 0$, which is equivalent to $d^*\alpha = d\alpha = 0$.

We define

$$\mathcal{H}^k(X) = \{\alpha \in \mathcal{A}^k(X) \mid \alpha \text{ is } d\text{-harmonic}\}$$

and

$$\mathcal{H}^{a,b}(X) = \{\alpha \in \mathcal{A}^{a,b}(X) \mid \alpha \text{ is } \bar{\partial}\text{-harmonic}\}.$$

Proposition 5.21. *Let X be a Kähler manifold, then $\Delta_d = 2\Delta_{\bar{\partial}}$, so the notions of d-harmonic and $\bar{\partial}$-harmonic coincide.*

Proof. See [8, Proposition 3.1.12.iii)]. $\square$

Corollary 5.22. *If X is a compact Kähler manifold then*

$$\mathcal{H}^k(X) = \bigoplus_{a+b=k} \mathcal{H}^{a,b}(X)$$

and complex conjugation exchanges $\mathcal{H}^{a,b}(X)$ and $\mathcal{H}^{b,a}(X)$

Proof. The decomposition follows from $\mathcal{A}^k(X) = \bigoplus_{a+b=k} \mathcal{A}^{a,b}(X)$. Complex conjugation acts on $\mathcal{A}^{a,b}(X)$, via $dz_i \mapsto d\bar{z}_i$ and vice versa, so it exchanges $\mathcal{A}^{p,q}(X)$ and $\mathcal{A}^{q,p}(X)$. Since $\Delta_{\bar{\partial}}$ is a linear operator, we also have $\Delta_{\bar{\partial}}(\bar{\alpha}) = \overline{\Delta_{\bar{\partial}}(\alpha)}$, so complex conjugation preserves $\bar{\partial}$-harmonicity, hence it exchanges $\mathcal{H}^{p,q}(X)$ and $\mathcal{H}^{q,p}(X)$. $\square$

Theorem 5.23. *The maps*

$$\phi = \alpha \mapsto [\alpha] : \mathcal{H}^k(X) \to H^k_{\mathrm{dR}}(X)$$

and

$$\psi = \alpha \mapsto [\alpha] : \mathcal{H}^{a,b}(X) \to H^{a,b}(X)$$

are isomorphisms, such that $\phi \circ \psi^{-1}$ sends $H^{a,b}(X)$ to

$$\{a \in H^k_{\mathrm{dR}}(X) \mid \exists \alpha \in \mathcal{A}^{a,b}(X) \text{ such that } a = [\alpha]\}.$$

Proof. See [8, Corollary 3.2.12]. $\square$

Corollary 5.24 (Hodge decomposition)**.** *Let X be a compact Kähler manifold, then*

$$H^k(X) \cong \bigoplus_{a+b=k} H^{a,b}(X)$$

and complex conjugation exchanges $H^{a,b}(X)$ and $H^{b,a}(X)$

Proof. The decomposition is obtained by combining Theorem 5.9, Corollary 5.22 and Theorem 5.23. $\square$

Definition 5.25. Let X be a Hermitian manifold. The *Lefschetz operator* $L : \bigwedge^* T^*X \to \bigwedge^* T^*X$ is defined locally by $\alpha \mapsto \alpha \wedge \omega$. The *dual Lefschetz operator* Λ is the adjoint of L with respect to the Riemannian metric. A differential form α is called a *primitive form* if $\Lambda\alpha = 0$.

Proposition 5.26 (Primitive decomposition)**.** *There is a decomposition*

$$H^{a,b}(X) \cong \bigoplus_{r=0}^{N} \omega^r \wedge P^{a-r,b-r},$$

where $N = \min(a, b)$ and $P^{a,b}$ denotes the space of primitive $\bar{\partial}$-harmonic forms of type (a, b). Furthermore $P^{a,b} = 0$ if $a+b > 0$ and the map $L^r : P^{a,b} \to \omega^r \wedge P^{a,b} \subseteq H^{a+r,b+r}$ is an isomorphism for $a + b < n$ and $r \leq n - a - b$. For $r > n - a - b$ we have $\omega^r \wedge P^{a,b} = 0$.

Proof. See [8, Proposition 1.2.30 and 3.2.2]. $\square$

5.2 The Petrovskii–Oleinik–Kharlamov inequality

We now have everything that is required to state and prove the Petrovskii–Oleinik–Kharlamov inequality.

Theorem 5.27 (Petrovskii–Oleinik–Kharlamov inequality, [21, Theorem 4.2])**.** *Let (X, σ) be a connected $\mathbb{R}$-variety, for which X is a Kähler manifold of dimension $2n$. Then*

$$2 - h^{n,n}(X) \leq \chi(X^\sigma) \leq h^{n,n}(X).$$

Proof. Let g be a Kähler metric on X, then $h(u,v) = g(u,v) + g(\sigma(u), \sigma(v))$ is also a Kähler metric, with the property that $h(\sigma(u), \sigma(v)) = h(u,v)$. For the corresponding fundamental form, this gives $h(I(\sigma(u)), \sigma(v)) = h(-\sigma(I(u)), \sigma(v) = -h(I(u), v)$, so without loss of generality we can assume that $\sigma(\omega) = -\omega$. Let $T_k : H^k(X) \to H^k(X)$ be the involution induced by σ. The Hodge decomposition and Proposition 5.26 give

$$H^k(X) \cong \bigoplus_{a+b=k} H^{a,b}(X) \cong \bigoplus_{a+b+2r=k} \omega^r \wedge P^{a,b}.$$

We can write

$$T_k = \begin{pmatrix} T_{0,0} & \dots & T_{0,k} \\ \vdots & \dots & \vdots \\ T_{k,0} & \dots & T_{k,k} \end{pmatrix},$$

where $T_{a,b} : H^{a,k-a} \to H^{b,k-b}$. Then $\operatorname{Tr}(T_k) = \sum_{i=0}^k \operatorname{Tr}(T_{i,i})$.

Because σ is anti-holomorphic, it is locally of the form $g \circ \operatorname{conj}^n$ with $g = (g_j)_{j=1}^n$ holomorphic. Therefore it acts on $\mathcal{H}^{a,b}(X)$ via $dz_k \mapsto \sum_{j=1}^n \frac{\partial \bar{g_j}}{\partial \bar{z_k}} d\bar{z}_k$ and $d\bar{z_k} \mapsto \sum_{j=1}^n \frac{\partial g_j}{\partial z_k} dz_k$. Analogous to the proof of Proposition 4.11, we see that the Jacobian of g at 0 is the identity. Therefore, the action of σ on $\mathcal{H}^{a,b}$ is the same as that of complex conjugation and thus, by Corollary 5.24, $T_k(H^{a,b}(X)) = H^{b,a}(X)$. So if $i \neq k-i$, then $T_{i,i} : H^{i,k-i} \to H^{i,k-i}$ is the zero map. Hence $\operatorname{Tr}(T_k) = 0$ if k is odd and $\operatorname{Tr}(T_k) = \operatorname{Tr}(T_{l,l})$ if $k = 2l$.

Let $S_{l,r} = T_{l+r,l+r}|_{\omega^r \wedge P^{l,l}}$. Then

$$\operatorname{Tr}(T_{2l}) = \operatorname{Tr}(T_{l,l}) = \sum_{r=0}^{l} \operatorname{Tr} S_{l-r,r}.$$

For $r+1 > n-2l$, we have $S_{l,r+1} = 0$, since $\omega^{r+1} \wedge P^{l,l} = 0$ in this case. Otherwise, let $\{a_i\}_i$ be a basis of $\omega^r \wedge P^{l,l}$, then $\{\omega \wedge a_i\}_i$ is a basis for $\omega^{r+1} \wedge P^{l,l}$. Let $S_{l,r}(a_i) = \sum_j b_{ij} a_i$. Since $\sigma(\omega) = -\omega$,

$$S_{l,r+1}(\omega \wedge a_i) = -\omega \wedge S_{l,r}(a_i) = -\omega \wedge \sum_j b_{ij} a_i = \sum_j -b_{ij}(\omega \wedge a_i).$$

It follows that $\operatorname{Tr}(S_{l,r+1}) = -\operatorname{Tr}(S_{l,r})$ for $r+1 \leq n-2l$. Therefore

$$\begin{aligned} \sum_{l=0}^{4n} (-1)^l \operatorname{Tr}(T_l) &= \sum_{l=0}^{2n} \operatorname{Tr}(T_{2l}) = \sum_{l=0}^{2n} \sum_{a+b=l} \operatorname{Tr}(S_{a,b}) \\ &= \sum_{l=0}^{2n} \sum_{r=0}^{2n-l} \operatorname{Tr}(S_{l,r}) = \sum_{l=0}^{n} \operatorname{Tr}(S_{2l,0}). \end{aligned}$$

Theorem 5.1 and Remark 5.3 now say that $\chi(X^\sigma) = \sum_{l=0}^{n} \mathrm{Tr}(S_{2l,0})$. On $(0,0)$-forms, σ acts trivially, so $\mathrm{Tr}(S_{0,0}) = \dim P^{0,0} = h^{0,0}(X) = 1$, since we assumed X is connected. Hence $\chi(X^\sigma) = 1 + \sum_{l=1}^{n} \mathrm{Tr}(S_{2l,0})$ and $|\chi(X^\sigma) - 1| \leq \sum_{l=1}^{n} |\mathrm{Tr}\, S_{2l,0}|$. Since $S_{2l,0}$ is an involution, it is diagonalizable with eigenvalues ± 1, so its trace is between $-\dim P^{l,l}$ and $\dim P^{l,l}$. So $|\chi(X^\sigma) - 1| \leq \sum_{l=1}^{n} \dim P^{l,l} = h^{n,n}(X) - 1$, from which the result follows. □

Corollary 5.28. *There is no real structure on $\mathbb{CP}^{2n}$ that has an empty real locus.*

Proof. First note that $\mathbb{CP}^{2n}$ is a complex projective algebraic variety, so it is a Kähler manifold and we can apply the Hodge decomposition and the Petrovskii-Oleinik-Kharlamov inequality. The computation of $H^k(\mathbb{CP}^n; \mathbb{Z})$ in 2.15 combined with the universal coefficient theorem gives $H^{2n}(\mathbb{CP}^{2n}; \mathbb{C}) \cong \mathbb{C}$. The Hodge decomposition gives

$$H^{2n}(\mathbb{CP}^{2n}; \mathbb{C}) \cong \bigoplus_{p+q=2n} H^{p,q}(\mathbb{CP}^{2n}),$$

so $\sum_{p+q=2n} h^{p,q}(\mathbb{CP}^{2n}) = 1$. Since $h^{a,b}(\mathbb{CP}^{2n}) = h^{b,a}(\mathbb{CP}^{2n}) \geq 0$, we must have $h^{n,n}(\mathbb{CP}^{2n}) = 1$. Theorem 5.27 gives $1 \leq \chi(\mathbb{CP}^{2n}(\mathbb{R})) \leq 1$. In particular $\chi(\mathbb{CP}^{2n}(\mathbb{R})) \neq 0$, hence $\mathbb{CP}^{2n}(\mathbb{R}) \neq \emptyset$. The inequality in Theorem 5.27 holds for any real structure on $\mathbb{CP}^{2n}$, so we can conclude that $\mathbb{CP}^{2n}(\mathbb{R}) \neq \emptyset$ for any real structure. □

5.3 Arrangement of ovals of real loci of plane curves

As we have seen before, the real locus of a genus g $\mathbb{R}$-curve is the disjoint union of at most $g+1$ circles. In the case of plane curves, these circles are embedded in $\mathbb{RP}^2$. Recall from Definition 4.15 and Lemma 4.16 that these circles can either be ovals or pseudo-lines and if the degree of the curve is even, then they are all ovals. We are interested in the arrangement of ovals of plane curves of even degree.

Definition 5.29. Let $O \subseteq \mathbb{RP}^2$ be an oval. Then $\mathbb{RP}^2 \setminus O$ consists of two connected components. One of which is contractible and homeomorphic to a disk, the other is a Möbius band. We call the connected component that is a disk the *interior* of the oval and the Möbius band the *exterior*.

For the remainder of this section we let $C \subseteq \mathbb{RP}^2$ be a smooth, projective, plane curve given by the homogeneous polynomial $F \in \mathbb{R}[X, Y, Z]$ of degree $2d$. We denote the set of ovals of C by $\mathcal{O}$. This is a finite set by Harnack's inequality and because C is smooth all ovals in $\mathcal{O}$ are disjoint.

Definition 5.30. An oval $O \in \mathcal{O}$ is called an *even* oval if it is contained in the interior of an even number of ovals in $\mathcal{O}$. Similarly an *odd* oval is an oval that is contained in the interior of an odd number of ovals in $\mathcal{O}$.

Proposition 5.31. *The intersection of the exteriors of a finite number of disjoint ovals is not empty.*

Proof. Without loss of generality, we can assume that none of the ovals is contained in the interior of another oval, because that does not affect the intersection of the exteriors. In particular this means that the closures of the interiors of the ovals are all disjoint and closed. Suppose the intersection of

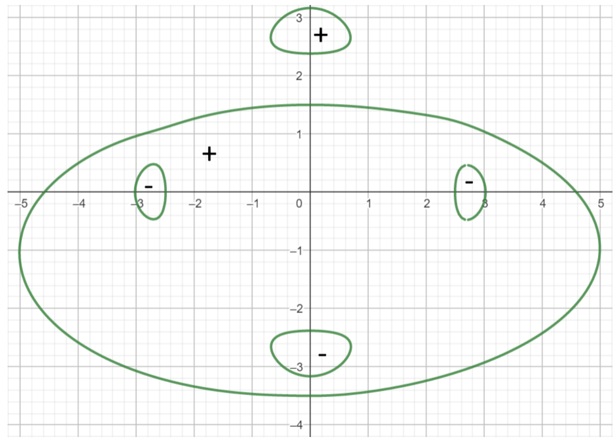

Figure 4: The curve $0 = x^3y^3 - ((x^2 + 3y^2 - 17)(3x^2 + y^2 - 10) + 15x^2)(x^2 + 4(y + 1)^2 - 25)$.

the exteriors is empty. Then there is an oval O, such that the closures of the interiors of the other ovals cover the exterior of O. But this means that the exterior of O is the union of finitely many disjoint closed sets, which is a contradiction, because the exterior is connected. $\square$

This means that we can speak of the exterior of $\mathcal{O}$.

Definition 5.32. For any point $p \in \mathbb{RP}^2$, the sign of $F(p)$ is well-defined, because the degree of F is even. This can be used to define an *inside* and an *outside* of an even degree plane curve, namely as the set of points where $F(p)$ is positive or negative respectively. We can multiply F by -1 without affecting C, so without loss of generality, the exterior of $\mathcal{O}$ is also considered the outside of C by this definition.

Lemma 5.33. *Let*

$$B_+ = \{[x : y : z] \in \mathbb{RP}^2 \mid F(x, y, z) \geq 0\}$$

be the inside of curve C and let P be the number of even ovals and N be the number of odd ovals. Then $\chi(B_+) = P - N$.

Proof. Let $X = \mathbb{RP}^2 \setminus C$. Every oval forms a part of the boundary between a connected component of X that is inside the curve and a connected component of X that is outside the curve. For an even oval, the component that is outside the curve is also in the exterior of the oval whereas for an odd oval the component that is outside the curve is in the interior of the oval. So every even oval provides a connected component of B_+, whereas every odd oval cuts out a disk of B_+. Therefore, B_+ is the disjoint union of P disks with a total of N holes, hence its Euler characteristic is $P - N$. $\square$

Definition 5.34. The *Veronese embedding* $v_{n,k} : \mathbb{C}^n \to \mathbb{C}^{\binom{n+k-1}{n-1}}$ is defined by

$$(x_1, ..., x_n) \mapsto (x_1^{\alpha_1} \cdots x_n^{\alpha_n})_{\sum_i \alpha_i = k}.$$

The induced map $\mathbb{CP}^{n-1} \to \mathbb{CP}^{\binom{n+k-1}{n-1}-1}$ is also denoted by $v_{n,k}$ or just v.

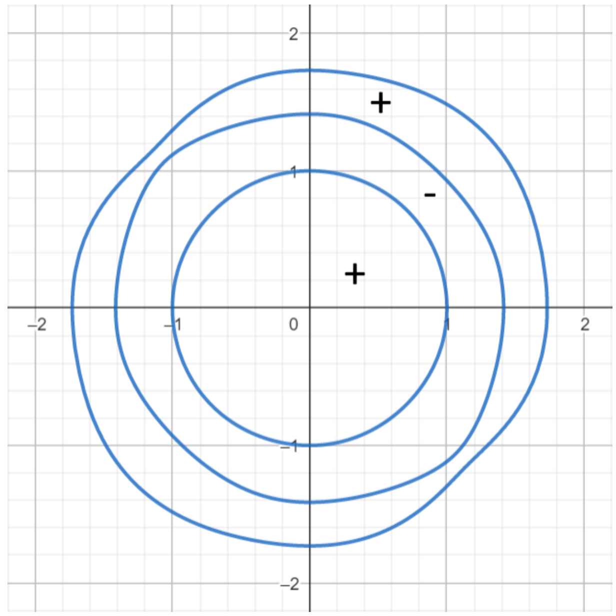

Figure 5: The curve $0 = x^3y^3 - 6(x^2+y^2-1)(x^2+y^2-2)(x^2+y^2-3)$.

Theorem 5.35 (Petrovskii inequality)**.** *Let P denote the number of even ovals of C and let N denote the number of odd ovals of C. Then*

$$\frac{3}{2}d - \frac{3}{2}d^2 \leq P - N \leq \frac{3}{2}d^2 - \frac{3}{2}d + 1.$$

Proof. Let B_+ be the inside of the curve as in Lemma 5.33. We want to apply Theorem 5.27 to find constraints on $\chi(B_+) = P - N$. For this we need an $\mathbb{R}$-variety with B_+ as real locus. We will now construct such a space using the Veronese embedding.

Let $L = \binom{2+d}{2}$. There exists a $G \in \mathbb{R}[X_1, ..., X_L]_2$ such that $G \circ v = F$, where $v = v_{3,d} : \mathbb{CP}^2 \to \mathbb{CP}^{L-1}$ is the Veronese embedding. This is because each monomial in F has degree $2d$, so it can be written as the product of two monomials of degree d.

We now define $Y = \{[z_0 : ... : z_L] \in \mathbb{CP}^L \mid [z_1 : ... : z_L] \in v(\mathbb{CP}^2) \text{ and } z_0^2 = G(z_1, ..., z_L)\}$, which is well-defined, because $z_0^2 - G(z_1, ..., z_L)$ is a homogeneous polynomial of degree 2. We equip Y with the real structure that is inherited by the standard real structure on $\mathbb{CP}^L$. We find

$$Y(\mathbb{R}) = \{[z_0 : ... : z_L] \in \mathbb{RP}^L \mid [z_1 : ... : z_L] \in v(\mathbb{RP}^2) \text{ and } z_0^2 = G(z_1, ..., z_L)\}$$

We can define $\pi : Y \to \mathbb{CP}^n$ by $[z_0 : ... : z_L] \mapsto v^{-1}([z_1 : ... : z_L])$. If $\pi([z_0 : ... : z_L]) = x$, then by definition of G, we have $z_0^2 = G(v(x)) = F(x)$. Note that for $x \in \mathbb{RP}^n$ we have

$$z_0 \in \mathbb{R} \iff F(x) \geq 0 \iff x \in B_+.$$

So $Y(\mathbb{R})$ is a double cover of B_+ that is ramified along the curve $F = 0$. Let C be a connected component of B_+, then C is a disk with a number of holes. Let k be this number. Then $\chi(C) = 1-k$ and $\pi^{-1}(C)$ must be a k-holed torus, so $\chi(\pi^{-1}(C)) = 2-2k$. Summing over all components of B_+, we get $2\chi(B_+) = \chi(Y(\mathbb{R}))$.

Y is a smooth projective variety, so it is a Kähler manifold and we can apply Theorem 5.27 on Y, which gives $2 - h^{1,1}(Y) \leq 2(P - N) \leq h^{1,1}(Y)$.

By [10, Section 4], $h^{1,1}(Y) = 3d^2 - 3d + 2$. Combined, this gives

$$\frac{3}{2}d - \frac{3}{2}d^2 \leq P - N \leq \frac{3}{2}d^2 - \frac{3}{2}d + 1.$$

$\square$

Example 5.36. Let C be a maximal plane curve of degree 6. Then $g = 10$, so $C(\mathbb{R})$ consists of 11 ovals, i.e. $P+N = 11$. The inequality now gives $P-N \leq 10$, so $2P = P-N+P+N \leq 10+11 = 21$, hence $P \leq 10\frac{1}{2}$. So there can be at most 10 even ovals. In particular, there must be at least one odd oval, so the ovals cannot all lie outside each other.

6 Symmetric products of maximal curves

In this chapter and the next, we will see how maximal varieties can be constructed by taking the symmetric product of a maximal curve.

Definition 6.1. Let X be a topological space. The symmetric group S_n acts naturally on X^n via

$$\pi(x_1, \dots, x_n) = (x_{\pi(1)}, \dots, x_{\pi(n)}).$$

The nth *symmetric product* of X is the quotient space

$$X^{(n)} = X^n / S_n.$$

In other words, $X^{(n)}$ consists of unordered n-tuples $\{x_1, \dots, x_n\}$ of elements in X. An alternative notation for $X^{(n)}$ is $\mathrm{Sym}^n(X)$.

Example 6.2. The 2nd symmetric product of $\mathbb{R}$ is $\mathbb{R}^{(2)} = \{\{x, y\} \mid x, y \in \mathbb{R}\}$.
Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$, then $\phi : \mathbb{R}^{(2)} \to X$, given by

$$\phi(\{x, y\}) = (\min(x, y), \max(x, y))$$

is a homeomorphism. Note that $\mathbb{R}$ is a smooth manifold without boundary, whereas X is a smooth manifold with boundary.

Example 6.3. The nth symmetric product of $\mathbb{C}$ is $\mathbb{C}^{(n)} = \{\{z_1, \dots, z_n\} \mid z_1, \dots, z_n \in \mathbb{C}\}$. Let $R = \{f \in \mathbb{C}[X] \mid \deg f = n \text{ and } f \text{ is monic}\}$. We have an isomorphism $R \cong \mathbb{C}^n$, namely

$$X^n + a_{n-1}X^{n-1} + \dots + a_1 X + a_0 \mapsto (a_0, a_1, \dots, a_{n-1}).$$

Let $\phi : \mathbb{C}^{(n)} \to \mathbb{C}^n$ be the composition of this isomorphism with the map

$$\{z_1, \dots, z_n\} \mapsto (X - z_1) \cdot \dots \cdot (X - z_n).$$

Coordinate-wise ϕ is a polynomial in $z_1, \dots, z_n$, so ϕ is continuous, open and closed. The fundamental theorem of algebra shows that it is a bijection. Therefore ϕ is a bijective, open and closed map, so it is a homeomorphism.

So we find that $\mathbb{C}^{(n)} \cong \mathbb{C}^n$. In this case the symmetric product of the complex manifold $\mathbb{C}$ is again a complex manifold.

In fact, as a consequence we get that the symmetric product of any complex curve is again a complex manifold.

Proposition 6.4. *The symmetric product $X^{(n)}$ of a complex curve X is a complex manifold.*

Proof. For $U_1, \dots, U_n \subseteq X$, we define

$$U_1 * \dots * U_n = \{\{x_1, \dots, x_n\} \in X^{(n)} \mid \exists \pi \in S_n : \forall i : x_i \in U_{\pi(i)}\}.$$

By reordering the U_i or x_i, we can often assume that $x_i \in U_i$ for all i. Note that for disjoint U_i, this is homeomorphic to the Cartesian product and for all U_i equal it is the nth symmetric product.

We construct a chart around every $p = \{p_1, \dots, p_n\} \in X^{(n)}$. Choose charts $\phi_i : U_i \to V_i$ of X, such that $p_i \in U_i$ and such that $U_i \cap U_j = \emptyset$ and $V_i \cap V_j = \emptyset$ if $p_i \neq p_j$ and such that $\phi_i = \phi_j$ if $p_i = p_j$. Let

$$\phi = \phi_p = \phi_1 * \cdots * \phi_n = \{x_1, \dots, x_n\} \mapsto \{\phi_{\pi(1)}(x_1), \dots, \phi_{\pi(n)}(x_n)\} : U_1 * \cdots * U_n \to V_1 * \cdots * V_n,$$

where $\pi \in S_n$, such that $x_i \in U_{\pi(i)}$. Let $\psi_n : \mathbb{C}^{(n)} \to \mathbb{C}^n$ be the homeomorphism from Example 6.3. We claim that these $\psi_n \circ \phi_p$ form an atlas on $X^{(n)}$. First we show that ϕ is well-defined. If π_1 and π_2 are both permutations such that $x_i \in U_{\pi_1(i)}$ and $x_i \in U_{\pi_1(i)}$, then $U_{\pi_1(i)} \cap U_{\pi_2(i)} \neq \emptyset$, so $\phi_{\pi_1(i)} = \phi_{\pi_2(i)}$. Therefore the value of $\phi(\{x_1, \dots, x_n\})$ does not depend on the chosen permutation. Furthermore, the value of $\phi(\{x_1, \dots, x_n\})$ does not depend on the order of the x_i, so ϕ is well-defined. To show that ϕ is continuous, it is enough to show that $\phi^{-1}(W_1 * \cdots * W_n)$ is open, for $W_i \subseteq V_i$ open, because such opens form a basis for the topology on $V_1 * \cdots * V_n$.

$$\begin{aligned}
\phi^{-1}(W_1 * \cdots * W_n) &= \{\{x_1, \dots, x_n\} \in U_1 * \cdots * U_n \mid \phi(\{x_1, \dots, x_n\}) \in W_1 * \cdots * W_n\} \\
&= \{\{x_1, \dots, x_n\} \in U_1 * \cdots * U_n \mid \{\phi_1(x_1), \dots, \phi_n(x_n)\} \in W_1 * \cdots * W_n\} \\
&= \{\{x_1, \dots, x_n\} \in U_1 * \cdots * U_n \mid \exists \pi \in S_n : \phi_{\pi(i)}(x_i) \in W_i \text{ for all } i\} \\
&= \bigcup_{\pi \in S_n} \{\{x_1, \dots, x_n\} \in U_1 * \cdots * U_n \mid \phi_{\pi(i)}(x_i) \in W_i \text{ for all } i\} \\
&= \bigcup_{\pi \in S_n} \left(\phi_{\pi(1)}^{-1}(W_1) * \cdots * \phi_{\pi(n)}^{-1}(W_n)\right).
\end{aligned}$$

Here without loss of generality we assumed that $x_i \in U_i$ for all i. Similarly $\phi^{-1} = \{v_1, \dots, v_n\} \mapsto \left\{\phi_1^{-1}(v_1), \dots, \phi_n^{-1}(v_n)\right\}$ is well-defined and continuous, so ϕ is even a homeomorphism.

Lastly we check that the charts are compatible. Let $q = \{q_1, \dots, q_n\}$ and $p = \{p_1, \dots, p_n\} \in X^{(n)}$ and let $\phi_q : U_q = U_{q_1} * \cdots * U_{q_n} \to V_{q_1} * \cdots * V_{q_n}$ and $\phi_q : U_p = U_{p_1} * \cdots * U_{p_n} \to V_{p_1} * \cdots * V_{p_n}$ be the charts around q and p as constructed above. Then we need to show that $\psi_n \circ \phi_q \circ \phi_p^{-1} \circ \psi_n^{-1}|_{\psi_n(\phi_p(U_q \cap U_p))}$ is holomorphic.

Let $r = \{r_1, \dots, r_n\} \in U_p \cap U_q$, we will show that $\psi_n \circ \phi_q \circ \phi_p^{-1} \circ \psi_n^{-1}$ is holomorphic at $\psi_n(\phi_p(r))$. By reordering, we can assume without loss of generality, that $r_i \in U_i = U_{p_i} \cap U_{q_i}$ for all i. Since $U_{p_i} = U_{p_j}$ or $U_{p_i} \cap U_{p_j} = \emptyset$ and similarly for U_{q_i}, we have $U_i = U_j$ or $U_i \cap U_j = \emptyset$. Again by reordering, we can assume that there are $0 = m_0 < m_1 < \cdots < m_l = n$, such that

$$U_1 = \cdots = U_{m_1}, U_{m_1+1} = \cdots = U_{m_2}, \dots, U_{m_{l-1}+1} = \cdots = U_{m_l}$$

and $i \neq j \implies U_{m_i} \cap U_{m_j} = \emptyset$. When we restrict ϕ_p and ϕ_q to $U_1 * \cdots * U_n$, the same reordering yields $\phi_{p_{m_{i-1}+1}} = \cdots = \phi_{p_{m_i}}$ and $i \neq j \implies V_{p_{m_i}} \cap V_{p_{m_j}} = \emptyset$. Because of this, there is a homeomorphism

$$\chi_p = \{v_1, \dots, v_n\} \mapsto (\{v_1, \dots, v_{m_1}\}, \dots, \{v_{m_{l-1}+1}, \dots, v_n\}) : V_{p_1} * \cdots * V_{p_n} \to V_{p_{m_1}}^{(k_1)} \times \cdots \times V_{p_{m_l}}^{(k_l)}.$$

Furthermore, we have homeomorphisms

$$V_{p_{m_1}}^{(k_1)} \times \cdots \times V_{p_{m_l}}^{(k_l)} \xrightarrow[\sim]{(\psi_{k_1}, \dots, \psi_{k_l})} W_{p_1} \times \cdots \times W_{p_l} \subseteq \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_l}$$

and

$$\mathbb{C}^{(n)} \supseteq V_{p_1} * \cdots * V_{p_n} \xrightarrow[\sim]{\psi_n} V_p \subseteq \mathbb{C}^n.$$

Let π_p be the quotient map $V_{p_{m_1}}^{k_1} \times \cdots \times V_{p_{m_l}}^{k_l} \to V_{p_{m_1}}^{(k_1)} \times \cdots \times V_{p_{m_l}}^{(k_l)}$. Then the composition $\psi_n \circ \chi_p^{-1} \circ \pi_p$ is given by

$$((v_1, \ldots, v_{m_1}), \ldots, (v_{m_{l-1}+1}, \ldots, v_n)) \mapsto (f_{n,1}(v_1, \ldots, v_n), \ldots, f_{n,n}(v_1, \ldots, v_n)).$$

This is holomorphic and invariant under permutations of $x_1, \ldots, x_n$. In particular it is invariant under permutations of $x_{m_{i-1}+1}, \ldots, x_{m_i}$, so by Lemma 6.5, there is a holomorphic function

$$g_p : W_{p_1} \times \cdots \times W_{p_l} \to V_p,$$

such that $g_p \circ (\psi_{k_1}, \ldots, \psi_{k_l}) \circ \pi_p = \psi_n \circ \chi_p^{-1} \circ \pi_p$, hence $g_p = \psi_n \circ \chi_p^{-1} \circ (\psi_{k_1}, \ldots, \psi_{k_l})^{-1}$ is holomorphic.

Replacing p with q, we get similar maps and spaces. If we write $T_i = \phi_{q_{m_i}} \circ \phi_{p_{m_i}}^{-1}$, the composition $(\psi_{k_1}, \ldots, \psi_{k_l}) \circ \chi_q \circ \phi_q \circ \phi_p^{-1} \circ \chi_p^{-1} \circ \pi_p$ is given by

$$((v_1, \ldots, v_{m_1}), \ldots, (v_{m_{l-1}+1}, \ldots, v_n)) \mapsto (\psi_{k_1}(T_1(v_1), \ldots, T_1(v_{m_1})), \ldots, \psi_{k_l}(T_l(v_{m_{l-1}+1}), \ldots, T_l(v_n))),$$

which is holomorphic and invariant under permutations of $x_{m_{i-1}+1}, \ldots, x_{m_i}$. By Lemma 6.5, there is a holomorphic function $g : W_{p_1} \times \cdots \times W_{p_l} \to W_{q_1} \times \cdots \times W_{q_l}$ that commutes.

$$\begin{tikzcd}
 & U_1 * \cdots * U_n \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_p", "\wr"'] & U_1 * \cdots * U_n \arrow[d, "\phi_q", "\wr"'] & \\
\mathbb{C}^n \supseteq V_p & V_{p_1} * \cdots * V_{p_n} \arrow[l, "\psi_n"'] \arrow[d, "\chi_p", "\wr"'] & V_{q_1} * \cdots * V_{q_n} \arrow[r, "\psi_n"] \arrow[d, "\chi_q", "\wr"'] & V_q \subseteq \mathbb{C}^n \\
 & V_{p_{m_1}}^{(k_1)} \times \cdots \times V_{p_{m_l}}^{(k_l)} \arrow[d, "{(\psi_{k_1},\ldots,\psi_{k_l})}", "\wr"'] & V_{q_{m_1}}^{(k_1)} \times \cdots \times V_{q_{m_l}}^{(k_l)} \arrow[d, "{(\psi_{k_1},\ldots,\psi_{k_l})}"', "\wr"] & \\
V_{p_{m_1}}^{k_1} \times \cdots \times V_{p_{m_l}}^{k_l} \arrow[ur, "\pi_p"'] & W_{p_1} \times \cdots \times W_{p_{m_l}} \arrow[r, dashed, "g"] \arrow[uul, dashed, "g_p"] & W_{q_1} \times \cdots \times W_{q_{m_l}} \arrow[uur, dashed, "g_q"'] & V_{q_{m_1}}^{k_1} \times \cdots \times V_{q_{m_l}}^{k_l} \arrow[ul, "\pi_q"]
\end{tikzcd}$$

In total we find $\psi_n \circ \phi_q \circ \phi_p^{-1} \circ \psi_n^{-1} = g_q \circ g \circ g_p^{-1}$ is holomorphic. This shows that the charts are compatible and thus that $X^{(n)}$ is a complex manifold.

$\square$

Lemma 6.5. *Let $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a holomorphic function that is symmetric in $x_1, ..., x_n$, i.e. $f(x_1, \ldots, x_n, y_1, \ldots, y_m) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}, y_1, \ldots, y_m)$ for all $\pi \in S_n$. Then there is a holomorphic function $g(x_1, \ldots, x_n, y_1, \ldots, y_m)$, such that*

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = g(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n), y_1, \ldots, y_m),$$

where

$$h_k = f_{n,k} = (-1)^k \sum_{(a_i)_i \in \{0,1\}^n, \sum_{i=1}^n a_i = k} \prod_{i=1}^n x_i^{a_i}.$$

Proof. In this proof we will use the notation X^I to denote $x_1^{a_1} \cdots x_n^{a_n}$, where $I = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ and similarly for H^I and Y^I.

To show this, we will show that any symmetric polynomial in $x_1, \ldots, x_n$ is a polynomial in $h_1, \ldots, h_n$. Let F be a be a symmetric polynomial that is homogeneous of degree d. Then $F = \sum_I c_i X^I$. Because F is invariant under S_n, we have $a_I = a_{\pi(I)}$, where $\pi(a_1, \ldots, a_n) = (a_{\pi(1)}, \ldots, a_{\pi(n)})$, for all $\pi \in S_n$. This means that we can write

$$F = \sum_I b_I \sum_{\pi \in S_n} X^{\pi(I)}.$$

We call $\sum_{\pi \in S_n} X^{\pi(I)}$ a symmetric monomial of multi-degree I. When we write $I = (a_1, \ldots, a_n)$, we can assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_n$. We order the multi-degrees lexicographically. Now we define multideg(F) to be the largest multi-degree for which the monomial has non-zero coefficient in F, or $(-\infty, \ldots, -\infty)$ if $F = 0$. The coefficient of this monomial is called the leading coefficient of F. If F and G are non-zero symmetric polynomials that are homogeneous of degree d and that have the same multi-degree and leading coefficient a and b respectively, then $F - \frac{a}{b}G$ is a symmetric polynomial of smaller multi-degree that is homogeneous of degree d.

Using induction on multideg F, we will now show that any symmetric polynomial F in $x_1, \ldots, x_n$ that is homogeneous of degree d, is a polynomial in $h_1, \ldots, h_n$, such that

$$h_1^{a_1} \cdots h_n^{a_n} \text{ has non-zero coefficient } \implies a_1 + 2a_2 + \cdots + na_n = d. \tag{$*$}$$

If multideg $F = (-\infty, \ldots, \infty)$, then $F = 0$, so it clearly is a polynomial in $h_1, \ldots, h_n$ that satisfies $(*)$.
Now let multideg$(F) = (a_1, \ldots, a_n)$. Since F is homogeneous of degree d, we have $a_1 + \cdots + a_n = d$. Let $H = h_1^{a_1 - a_2} h_2^{a_2 - a_3} \cdots h_n^{a_n}$. Note that multideg $H = (a_1, \ldots, a_n)$ and that H is homogeneous of degree $(a_1 - a_2) + 2(a_2 - a_3) + \cdots + na_n = a_1 + \cdots + a_n = d$, so H satisfies $(*)$. There is a constant c, such that multideg$(F - cH) < (a_1, \ldots, a_n)$. Now by the induction hypothesis, $F - cH$ is a polynomial in $h_1, \ldots, h_n$ that satisfies $(*)$, hence so is F.

Now let $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a holomorphic function that is symmetric in $x_1, \ldots, x_n$, then

$$f = \sum_J \sum_I a_{I,J} \sum_{\pi \in S_n} X^{\pi(I)} Y^J.$$

Every $\sum_{\pi \in S_n} X^{\pi(I)}$ is a homogeneous, symmetric polynomial, so it can be written as a polynomial F_I in $h_1, \ldots, h_n$.

$$f = \sum_J \sum_I a_{I,J} F_I Y^J.$$

Let $b_{I,K}$ denote the coefficient of H^K in F_I, then

$$f = \sum_J Y^J \sum_K H^K \sum_I a_{I,J} b_{I,K}.$$

Since the F_I satisfy $(*)$ and there only finitely many $I = (a_1, \ldots, a_n)$ with $a_1 + 2a_2 + \cdots + na_n = d$, the sums $c_{J,K} = \sum_K a_{I,J} b_{I,K}$ are well-defined, hence

$$g = \sum_J \sum_K c_{J,K} H^K Y^K$$

is holomorphic and $f = g(h_1, \ldots, h_n, y_1, \ldots, y_m)$. $\square$

Proposition 6.6. *A real structure σ on X induces a real structure $\sigma^{(n)}$ on $X^{(n)}$.*

Proof. The real structure σ induces the map $\sigma^{(n)}$, given by

$$\{x_1, \dots, x_n\} \mapsto \{\sigma(x_1), \dots, \sigma(x_n)\}.$$

It is clear that this is a well-defined, continuous involution. We still need to check that it is anti-holomorphic. Let $f \in \mathcal{O}_{X^{(n)}}(U)$, let $\tau^n : \mathbb{C}^n \to \mathbb{C}^n$ be complex conjugation and let $\phi : U_1 * \cdots * U_n \to U$ and $\psi : \sigma(U_1) * \cdots * \sigma(U_n) \to V$ be charts, with $U_1 * \cdots * U_n \subseteq U$. We need to show that $\overline{f} \circ \sigma^{(n)} \in \mathcal{O}_{X^{(n)}}(\sigma(U))$. For this it suffices to show that $\overline{f} \circ \sigma^{(n)} \circ \psi^{-1}$ is holomorphic, for all such ϕ and ψ. Note that

$$\begin{aligned}
\overline{f} \circ \sigma^{(n)} \circ \psi^{-1} &= \overline{f} \circ \phi^{-1} \circ \phi \circ \sigma^{(n)} \circ \psi^{-1} \\
&= \overline{f} \circ \phi^{-1} \circ \tau^n \circ \tau^n \circ \phi \circ \sigma^{(n)} \circ \psi^{-1} \\
&= \overline{f} \circ \phi^{-1} \circ \tau^n \circ (\overline{\phi_1} * \cdots * \overline{\phi_n}) \circ \sigma^{(n)} \circ \psi^{-1} \\
&= \overline{f} \circ \phi^{-1} \circ \tau^n \circ ((\overline{\phi_1} \circ \sigma) * \cdots * (\overline{\phi_n} \circ \sigma)) \circ \psi^{-1},
\end{aligned}$$

where $\overline{f} \circ \phi^{-1} \circ \tau^n$ is holomorphic, because f is, and all $\overline{\phi_i} \circ \sigma$ are holomorphic, because σ is anti-holomorphic. Therefore, $\overline{f} \circ \sigma^{(n)}$ is holomorphic and $\sigma^{(n)}$ is anti-holomorphic. ☐

Example 6.7 ($\mathrm{Sym}^n(\mathbb{CP}^1)$)**.** Let $\phi_n : \mathbb{C}^{(n)} \to \mathbb{C}^n$ be the homeomorphism from Example 6.3. Then $\phi = (f_{n,1}, \dots, f_{n,n})$, with $f_{n,i} \in \mathbb{C}[X_1, \dots, X_n]$ as in Lemma 6.5. We use this to define $\psi_n : (\mathbb{CP}^1)^{(n)} \to \mathbb{CP}^n$, by

$$\{[x_1 : y_1], \dots, [x_n : y_n]\} \mapsto [1 : f_{n,1}(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n}) : \cdots : f_{n,n}(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n})].$$

This is not well-defined, but by clearing denominator, i.e. multiplying by $y_1 \cdots y_n$, we obtain $\psi_n = [F_{n,0} : \cdots : F_{n,n}]$, with $F_{n,k} \in \mathbb{C}[X_1, Y_1, \dots, X_n, Y_n]$. If we write $A_{i,0} = Y_i$ and $A_{i,1} = X_i$, then they are given by

$$F_{n,k} = (-1)^k \sum_{(a_i)_i \in \{0,1\}^n, \sum_{i=1}^n a_i = k} \prod_{i=1}^n A_{i,a_i}.$$

Because each $F_{n,k}$ is continuous, open and closed, so is ψ_n. Note that

$$\begin{aligned}
F_{n,k}(1, 0, X_2, \dots, Y_n) &= (-1)^k \left(\sum_{\substack{(a_i)_i \in \{0,1\}^n, a_1 = 0 \\ \sum_{i=1}^n a_i = k}} \left(\prod_{i=1}^n A_{i,a_i} \right) + \sum_{\substack{(a_i)_i \in \{0,1\}^n, a_1 = 1 \\ \sum_{i=1}^n a_i = k}} \left(\prod_{i=1}^n A_{i,a_i} \right) \right) \\
&= (-1)^k \left(0 + \sum_{\substack{(a_i)_{i=2}^n \in \{0,1\}^{n-1} \\ \sum_{i=2}^n a_i = k-1}} \left(\prod_{i=2}^n A_{i,a_i} \right) \right) \\
&= -F_{n-1,k-1}(X_2, \dots, Y_n).
\end{aligned}$$

Let

$$\begin{aligned}
C_1 &= \{\{[x_1 : y_1], \ldots, [x_n : y_n]\} \in \mathrm{Sym}^n(\mathbb{CP}^1) \mid \forall i : y_i \neq 0\} \cong \mathrm{Sym}^n(\mathbb{C}),\\
C_2 &= \{\{[x_1 : y_1], \ldots, [x_n : y_n]\} \in \mathrm{Sym}^n(\mathbb{CP}^1) \mid \exists i : y_i = 0\}\\
&= \{\{[1 : 0], [x_2 : y_2], \ldots, [x_n : y_n]\} \in \mathrm{Sym}^n(\mathbb{CP}^1)\} \cong \mathrm{Sym}^{n-1}(\mathbb{CP}^1),\\
D_1 &= \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_0 \neq 0\} \cong \mathbb{C}^n,\\
D_2 &= \{[0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n\} \cong \mathbb{CP}^{n-1},
\end{aligned}$$

then $\mathrm{Sym}^n(\mathbb{CP}^1) = C_1 \sqcup C_2$ and $\mathbb{CP}^n = D_1 \sqcup D_2$.

Because $F_{n,0}(x_1, \ldots, y_n) = y_1 \cdots y_n \neq 0$ on C_1, the function ψ_n is given by

$$\begin{aligned}
\{[x_1 : y_1], \ldots, [x_n : y_n]\} &\mapsto [F_{n,0}(x_1, \ldots, y_n) : \cdots : F_{n,n}(x_1, \ldots, y_n)]\\
&= [1 : f_{n,1}(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}) : \cdots : f_{n,n}(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n})],
\end{aligned}$$

on C_1. We see that $\psi_n(C_1)$ is contained in D_1. Composing with the homeomorphisms $\mathbb{C}^{(n)} \cong C_1$ and $D_1 \cong \mathbb{C}^n$, the restriction $\psi_n|_{C_1}$ is given by

$$(z_1, \ldots, z_n) \mapsto (f_{n,1}(z_1, \ldots, z_n), \ldots, f_{n,n}(z_1, \ldots, z_n)) = \phi_n(z_1, \ldots, z_n).$$

Since ϕ_n is a bijection, $\psi_n|_{C_1}$ is a bijection onto its image. On C_2, the function ψ_n is given by

$$\{[1 : 0], [x_2 : y_2], \ldots, [x_n : y_n]\} \mapsto [0 : F_{n,1}(1, 0, x_2, \ldots, y_n) : \cdots : F_{n,n}(1, 0, x_2, \ldots, y_n)].$$

Composing with the homeomorphisms $\mathrm{Sym}^{n-1}(\mathbb{CP}^1) \cong C_2$ and $D_2 \cong \mathbb{CP}^{n-1}$ and using the observation that $F_{n,k}(1, 0, X_2, \ldots, Y_n) = -F_{n-1,k-1}(X_2, \ldots, Y_n)$, we see that the function ψ_n restricts to $\psi_{n-1} : \mathrm{Sym}^{n-1}(\mathbb{CP}^1) \to \mathbb{CP}^{n-1}$. By induction, this is a homeomorphism. We conclude that ψ_n is continuous, open, closed and bijective and thus a homeomorphism.

Note that every $F_{n,k}$ satisfies $F_{n,k}(\overline{X}_1, \ldots, \overline{Y}_k) = \overline{F_{n,k}(X_1, \ldots, Y_n)}$, so if we equip $\mathbb{CP}^1$ and $\mathbb{CP}^n$ with the standard real structures σ and σ^n respectively and $\mathrm{Sym}^n(\mathbb{CP}^1)$ with the induced involution $\sigma^{(n)}$, then

$$\begin{aligned}
\psi_n(\sigma^{(n)}(\{[x_1 : y_1], \ldots, [x_n : y_n]\}) &= [F_{n,0}(\overline{x}_1, \ldots, \overline{y}_n) : \cdots : F_{n,1}(\overline{x}_1, \ldots, \overline{y}_n)]\\
&= [\overline{F_{n,0}(x_1, \ldots, y_n)} : \cdots : \overline{F_{n,1}(x_1, \ldots, y_n)}]\\
&= \sigma^n(\psi_n(\{[x_1 : y_1], \ldots, [x_n : y_n]\})).
\end{aligned}$$

Therefore the induced real structure on $\mathrm{Sym}^n(\mathbb{CP}^1)$ agrees with the standard real structure on $\mathbb{CP}^n$.

Example 6.8.

$$\begin{aligned}
\mathrm{Sym}^2(S^1) &\cong (([0,1]/\sim) \times ([0,1]/\sim))/S_2 \text{ where } \sim \text{ is the equivalence relation generated by } 0 \sim 1\\
&\cong (([0,1] \times [0,1])/\sim)/S_2 \text{ where } \sim \text{ is generated by } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,y)\\
&\cong ([0,1] \times [0,1])/\sim \text{ where } \sim \text{ is generated by } (x,0) \sim (x,1) \text{ and } (x,y) \sim (y,x)\\
&\cong \{(x,y) \in [0,1] \times [0,1] \mid x \leq y\}/\sim \text{ where } \sim \text{ is generated by } (0,y) \sim (y,1)\\
&\cong \{(x,y) \in [0,1] \times [0,1]\}/\sim \text{ where } \sim \text{ is generated by } (0,y) \sim (1,1-y)\\
&= \text{Möbius strip.}
\end{aligned}$$

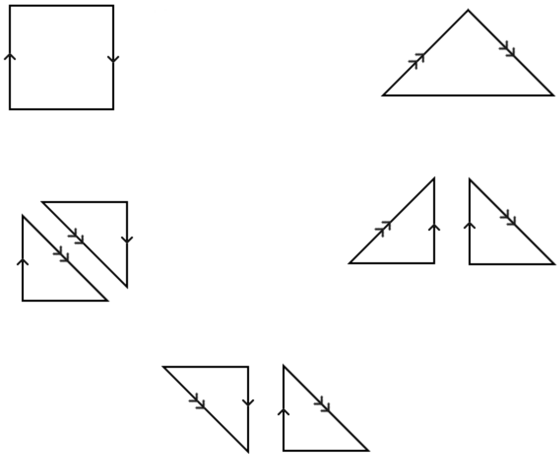

Figure 6: Visualization of the isomorphism ($\star$) in Example 6.8.

The last homeomorphism is given by

$$(x,y) \mapsto \begin{cases} (1-x-y, y-x) & \text{if } x+y \leq 1, \\ (2-x-y, 1+x-y) & \text{if } x+y \geq 1. \end{cases} \tag{$\star$}$$

Example 6.9. Let $X = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ be a complex torus. We have seen that X is a maximal curve when equipped with the real structure $\sigma([x+iy]) = [x-iy]$ and that the real locus consists of two circles $\{[x] \mid x \in \mathbb{R}\}$ and $\{[x + \frac{1}{2}i] \mid x \in \mathbb{R}\}$.

$X^{(2)}$ is a 2 dimensional complex manifold with total Betti number $b_*(X^{(2)}; \mathbb{F}_2) = 8$, [1, Lemma 2.1]. The real locus of $X^{(2)}$ is given by

$$\begin{aligned} X^{(2)}(\mathbb{R}) &= \{\{x,y\} \in X^{(2)} \mid \{x,y\} = \{\sigma(x), \sigma(y)\}\} \\ &= \{\{x,y\} \in X^{(2)} \mid x = \sigma(x) \text{ and } y = \sigma(y)\} \cup \{\{x,y\} \in X^{(2)} \mid x = \sigma(y)\} \\ &= X(\mathbb{R})^{(2)} \cup Y \\ &\cong (S^1 \sqcup S^1)^{(2)} \cup Y. \end{aligned}$$

Where $Y = \{\{x,y\} \in X^{(2)} \mid x = \sigma(y)\} \cong X/\sigma \cong \{[x+iy] \mid 0 \leq y \leq \frac{1}{2}\}$. Note that

$$(S^1 \sqcup S^1)^{(2)} \cong \mathrm{Sym}^2(S^1) \sqcup \mathrm{Sym}^2(S^1) \sqcup S^1 \times S^1,$$

and $Y \cap X(\mathbb{R})^{(2)} = \{\{x,x\} \in X^{(2)} \mid x = \sigma(x)\}$ corresponds to the circles $\{[x] \mid x \in \mathbb{R}\}$ and $\{[x + \frac{1}{2}i \mid x \in \mathbb{R}]\}$ in X/σ and the diagonals of $\mathrm{Sym}^2(S^1)$ in $(S^1 \sqcup S^1)^{(2)}$. So $X^{(2)}(\mathbb{R})$ is the disjoint union of a torus and two Möbius strips that are glued to a cylinder. Since a cylinder is homotopy equivalent to a circle, the latter is homotopy equivalent to two Möbius strips that are glued along their edge, which is a Klein bottle. The Klein bottle K has total Betti number $b_*(K; \mathbb{F}_2) = 4$, so in total we find

$$b_*(X^{(2)}(\mathbb{R}); \mathbb{F}_2) = 4 + 4 = 8 = b_*(X^{(2)}; \mathbb{F}_2),$$

hence $X^{(2)}$ is a maximal curve.

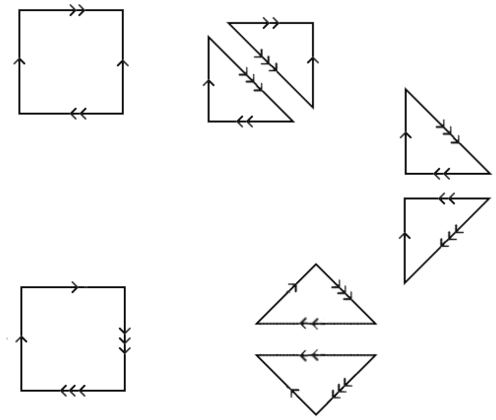

Figure 7: Two Möbius bands that are glued together form a Klein bottle.

Theorem 6.10 ([1, Proposition 2.3])**.** *Let (X, σ) be a maximal curve, then $(X^{(2)}, \sigma^{(2)})$ is a maximal variety.*

Proof. Let g be the genus of X. By [1, Lemma 2.1], the total Betti number $b_*(X^{(2)}; \mathbb{F}_2) = 3+3g+2g^2$, so we need to show that $b_*(X^{(2)}(\mathbb{R}); \mathbb{F}_2) = 3+3g+2g^2$. Since X is maximal, $X(\mathbb{R}) \cong \bigsqcup_{i=1}^{g+1} S^1$. Similar to the previous example we compute the real locus of $X^{(2)}$,

$$\begin{aligned} X^{(2)}(\mathbb{R}) &= \{\{x,y\} \in X^{(2)} \mid \{x,y\} = \{\sigma(x), \sigma(y)\}\} \\ &= \{\{x,y\} \in X^{(2)} \mid x = \sigma(x) \text{ and } y = \sigma(y)\} \cup \{\{x,y\} \in X^{(2)} \mid x = \sigma(y)\} \\ &= X(\mathbb{R})^{(2)} \cup Y \\ &\cong \mathrm{Sym}^2\left(\bigsqcup_{i=1}^{g+1} S^1\right) \cup Y. \end{aligned}$$

Where $Y = \{\{x,y\} \in X^{(2)} \mid x = \sigma(y)\} \cong X/\sigma$. The symmetric product of the disjoint union $\bigsqcup_{i=1}^{g+1} S^1$ is given by

$$\mathrm{Sym}^2\left(\bigsqcup_{i=1}^{g+1} S^1\right) \cong \bigsqcup_{i=1}^{g+1} \mathrm{Sym}^2(S^1) \sqcup \bigsqcup_{i=1}^{\binom{g+1}{2}} S^1 \times S^1,$$

and $Y \cap X(\mathbb{R})^{(2)} = \{\{x,x\} \in X^{(2)} \mid x = \sigma(x)\}$ corresponds to the $g+1$ circles $X(\mathbb{R})$ in $\{[x] \mid x \in \mathbb{R}\}$ and $\{[x + \frac{1}{2}i \mid x \in \mathbb{R}]\}$ in X/σ and the diagonals of $\mathrm{Sym}^2(S^1)$ in $\mathrm{Sym}^2\left(\bigsqcup_{i=1}^{g+1} S^1\right)$.

So $X^{(2)}(\mathbb{R})$ is the disjoint union of $\binom{g+1}{2}$ tori, each of which have $b_*(S^1 \times S^1; \mathbb{F}_2) = 4$ and $g+1$ Möbius strips that are glued to Y. We will compute the homology of the latter, which we call A, using cellular homology.

The ith Möbius strips can be realized as a CW-complex, using two 0-cells, v_i and w_i, three 1-cells, a_i and b_i from v_i to w_i and c_i from w_i to v_i, and one 2-cell f_i glued along the path $a_i c_i b_i d_i$.

Finally the surface Y is glued to these Möbius strips with extra 1-cells d_i from w_i to v_{i+1}, where

$v_{g+1} = v_0$, and two 2-cells, h_1 glued along the path $a_1 d_1 a_2 d_2 \dots a_{g+1} d_{g+1}$ and h_2 glued along the path $b_1 d_1 b_2 d_2 \dots b_{g+1} d_{g+1}$.

This gives the cellular chain complex

$$0 \longrightarrow \mathbb{Z}^{3+g} \xrightarrow{\beta} \mathbb{Z}^{4(g+1)} \xrightarrow{\alpha} \mathbb{Z}^{2(g+1)} \longrightarrow 0,$$

where

$$\begin{aligned} &\alpha(a_0, b_0, c_0, d_0, \dots, a_g, b_g, c_g, d_g) \\ &\qquad = (a_0 + b_0 - c_0 - d_g, d_0 + c_0 - a_0 - b_0, \dots, a_g + b_g - c_g - d_{g-1}, d_g + c_g - a_g - b_g) \end{aligned}$$

and

$$\beta(h_1, h_2, f_0, \dots, f_g) = (f_0 + h_1, f_0 + h_2, 2f_0, h_1 + h_2, \dots, f_g + h_1, f_g + h_2, 2f_g, h_1 + h_2).$$

The kernel of α is given by

$$\{(a_0, b_0, c_0, d_0, \dots, a_g, b_g, c_g, d_g) \in \mathbb{Z}^{4(g+1)} \mid d_0 = \dots = d_g = a_0 + b_0 - c_0 = \dots = a_g + b_g - c_g\}.$$

We will identify $\operatorname{Ker}\alpha$ with $\mathbb{Z}^{2(g+1)+1}$ via

$$\begin{aligned} \mathbb{Z}^{2(g+1)+1} \ni (c, a_0, b_0, a_1, b_1, \dots, a_g, b_g) \mapsto (&a_0, b_0, c, a_0 + b_0 - c, \\ &a_1, b_1, a_0 + b_0 - c - a_1 - b_1, a_0 + b_0 - c, \\ &\dots, \\ &a_g, b_g, a_0 + b_0 - c - a_g - b_g, a_0 + b_0 - c) \in \operatorname{Ker}\alpha. \end{aligned}$$

Via this identification, β is given by

$$(h_1, h_2, f_0, \dots, f_g) \mapsto (2f_0, f_0 + h_1, f_0 + h_2, \dots, f_g + h_1, f_g + h_2).$$

An element $(c, x_0, y_0, \dots, x_g, y_g)$ belongs to the image of β if and only if $x_0 - y_0 = \dots = x_g - y_g$ and c is even, so in $\operatorname{Ker}\alpha / \operatorname{Im}\beta$ we have

$$\begin{aligned} (c, x_0, y_0, \dots, x_g, y_g) &= (c, 0, y_0 - x_0, 0, y_1 - x_1, \dots, 0, y_g - x_g) \\ &= (c, 0, 0, 0, y_1 - x_1 - y_0 + x_0, \dots, 0, y_g - x_g - y_0 + x_0). \end{aligned}$$

So we get an isomorphism $\mathbb{Z}^g \oplus \mathbb{Z}/2\mathbb{Z} \to \operatorname{Ker}\alpha / \operatorname{Im}\beta = H_1(A; \mathbb{Z})$, given by

$$(a_1, \dots, a_g, \bar{b}) \mapsto (b, 0, 0, 0, a_1, \dots, 0, a_g).$$

Furthermore we have $\operatorname{Ker}\beta = 0$, so $H_2(A; \mathbb{Z}) = 0$ and $H_0(A; \mathbb{Z}) = \mathbb{Z}$, because Y is connected. Now we can use the universal coefficient theorem to compute

$$\begin{aligned} H_0(A; \mathbb{F}_2) &= \mathbb{F}_2, \\ H_1(A; \mathbb{F}_2) &= \mathbb{F}_2^{g+1}, \\ H_2(A; \mathbb{F}_2) &= \mathbb{F}_2, \end{aligned}$$

hence $b_*(A; \mathbb{F}_2) = g + 3$. In total we find that

$$b_*(X^{(2)}(\mathbb{R}); \mathbb{F}_2) = 3 + g + 4\binom{g+1}{2} = 3 + g + 2g(g+1) = 3 + 3g + 2g^2 = b_*(X^{(2)}; \mathbb{F}_2).$$

Therefore $(X^{(2)}, \sigma^{(n)})$ is a maximal variety. $\square$

7 Higher symmetric products

We're going to show that Theorem 6.10 holds in a much more general case. We follow the proof by Franz [3]. This proof uses that an $\mathbb{R}$-variety is maximal if and only if it is *equivariantly formal.*

7.1 Equivariant cohomology

Definition 7.1. [18, Section 2.8] Let G be a group. Let B be a topological space and let E be a topological space on which G acts freely. A *principal G-bundle* is a continuous map $p : E \to B$, such that $p(eg) = p(e)$ for all $e \in E, g \in G$ and such that there exists an open cover $\{U_i\}_i$ of E, such that $p^{-1}(U_i) \cong U_i \times G$ and the diagram below commutes. Such an open cover is called a *trivializing cover.*

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr, "p"'] & & U_i \times G \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

In other words, a principal G-bundle is a *fibre bundle* with fibre G that is invariant under the G action on E.

Definition 7.2. Given a principal G-bundle $p : E \to B$, a topological space X and a continuous map $f : X \to B$, there exists a principal G-bundle $f^*p : E \times_B X \to X$, where $E \times_B X = \{(e, x) \in E \times X \mid p(e) = f(x)\}$, with G-action $g(e, x) = (eg, x)$ and $f^*p(e, x) = x$. This bundle is called the *pullback of p along f.*

Proposition 7.3. *The pullback of a principal G-bundle $p : E \to B$ along $f : X \to B$ is again a principal G-bundle.*

Proof. Since G acts freely on E it also acts freely on $E \times_B X$. It is also clear that $f^*p(g(e, x)) = f^*p(e, x)$ for all g. Let $U \subseteq B$, such that there is an isomorphism $\phi : p^{-1}(U) \to U \times G$ and $pr_1 \circ \phi = p$, then

$$(f^*p)^{-1}(f^{-1}(U)) = \{(e, x) \in E \times f^{-1}(U) \mid p(e) = f(x)\} \subseteq p^{-1}(U) \times f^{-1}(U).$$

Now $\psi : (f^*p)^{-1}(f^{-1}(U)) \to f^{-1}(U) \times G, (e, x) \mapsto (x, pr_2(\phi(e))$ is a homeomorphism, with inverse $(x, g) \mapsto (\phi^{-1}(f(x), g), x)$. It satisfies $pr_1 \circ \psi = f^*p$, so if $\{U_i\}_i$ is a trivializing cover of $E \to B$, then this makes $\{f^{-1}(U_i)\}_i$ a trivializing cover of $E \times_B X \to X$, so the pullback of p along f is a principal G-bundle. □

Definition 7.4. A principal G-bundle $p : E \to B$ is called a *universal G-bundle* if for every principal G-bundle $p' : E' \to B'$, where B' is a CW-complex, is obtained, up to isomorphism, by taking the pullback of p along some function $B' \to B$.

Theorem 7.5. *Let G be a topological group that is homotopy equivalent to a CW-complex. Then a univeral G-bundle $p : EG \to BG$ exists and EG is contractible. Conversely, if $p : E \to B$ is a principal G-bundle and E is contractible, then p is a universal G-bundle.*

Proof. See [2, Theorem 2.5] □

Example 7.6. The quotient map $S^\infty \to \mathbb{RP}^\infty$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, where the action of the generator of $\mathbb{Z}/2\mathbb{Z}$ on S^∞ is given by the antipodal map $x \mapsto -x$. Since S^∞ is contractible, it is a universal $\mathbb{Z}/2\mathbb{Z}$-bundle.

Definition 7.7. Let K be a field and let G be a group and $EG \to BG$ the corresponding universal G-bundle. Let X be a G-space, i.e. a topological space with a continuous G-action. Then the *equivariant cohomology* of X is defined as

$$H_G^*(X; K) = H^*(X_G; K),$$

where $X_G = (EG \times X)/G$. Here G acts on $EG \times X$ via $g(e, x) = (eg, g^{-1}x)$.

Definition 7.8. A G-space X is called *equivariantly formal* over K if for any $e \in EG$ the inclusion

$$j_e : X \to X_G, x \mapsto [e, x]$$

induces a surjective map $j^* : H_G^*(X; K) \to H^*(X; K)$. Note that j^* does not depend on e, as all j_e are homotopic, because EG is contractible.

We are interested in the case that X is an $\mathbb{R}$-variety. The real structure on an $\mathbb{R}$-variety equips it with a $\mathbb{Z}/2\mathbb{Z}$-action, so we can indeed use equivariant cohomology on them.

7.2 Cohomology with local coefficients

To connect the maximality of $\mathbb{R}$-varieties to equivariant formality, we need the *Leray-Serre spectral sequence*. This uses cohomology with local coefficients, so before we look at spectral sequences, we first state the definition and some properties of cohomology with local coefficients.

Definition 7.9. Let G be a group. We define the *group ring* $\mathbb{Z}[G]$. As an abelian group, it is the free abelian group generated by the elements of G. The multiplication on $\mathbb{Z}[G]$ on the generators is induced by the group structure on G. Explicitly, this means that multiplication is given by

$$(\sum_i a_i g_i)(\sum_i b_i g_i) = \sum_i \sum_j a_i b_j (g_i g_j).$$

We call $\mathbb{Z}[G]$-modules, G-modules and use the notation $\mathrm{Hom}_G(M, N)$ for G-modules M and N to denote the $\mathbb{Z}[G]$-linear maps from M to N.

Definition 7.10. Let G be a group, let $EG \to BG$ be a universal G-bundle and let M be a G-module. The *cohomology with local coefficients* $H^i(BG; \underline{M})$ is defined as the cohomology in the cochain complex $C^i(EG; \underline{M}) = \mathrm{Hom}_G(C_i(EG; \mathbb{Z}), M)$, where $C_i(EG; \mathbb{Z})$ is the singular chain complex of EG, which is a chain complex of G-modules for which the G-action is induced by that on EG. [6, Section 3.H].

Proposition 7.11. *Let G be a group and let L be a G-module, then*

$$H^0(BG; \underline{L}) \cong L^G.$$

Proof. By definition $H^0(BG; \underline{L}) = \operatorname{Ker} \delta^0$, where

$$\delta^0 : \operatorname{Hom}_G(C_0(EG; \mathbb{Z}), L) \to \operatorname{Hom}_G(C_1(EG; \mathbb{Z}), L), \alpha \mapsto (s \mapsto \alpha(\partial_1(s)).$$

Note that elements $\alpha \in \operatorname{Hom}_G(C_k(EG), L)$ are completely determined by $\alpha(s)$, for k-simplices $s \in S_k(EG)$. Also note that 0-simplices are points in EG and that 1-simplices are paths.

Let $\alpha \in \operatorname{Ker} \delta^0$ and let $a, b \in EG$. Since EG is contractible, it is path-connected, so there is a path $s \in S_1(EG)$ from a to b. Hence $0 = \delta^0(\alpha(s)) = \alpha(\partial_1(s)) = \alpha(a-b) = \alpha(a) - \alpha(b)$. This shows that α is constant on $S_0(EG)$. Let $l \in L$ be the constant value of α. Then for any $g \in G$ and $x \in EG$ we must have $gl = g\alpha(x) = \alpha(gx) = l$, so $l \in L^G$. Conversely any α that is constant on $S_0(EG)$ satisfies $\delta^0\alpha = 0$. Therefore $\operatorname{Ker} \delta^0 \cong L^G$. □

Lemma 7.12. *Let G be a group and let L be a G-module on which G acts trivially, then the cohomology with local coefficients $H^k(BG; \underline{L})$ is isomorphic to the ordinary singular cohomology $H^i(BG; L)$*

Proof. Let $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(C_i(EG), L)$, then $\alpha \in \operatorname{Hom}_G(C_i(EG), L)$ if and only if for all $g \in G$ and $s \in S_i(EG)$: $\alpha(gs) = g\alpha(s)$. If g acts trivially on L, this means that $\alpha(gs) = \alpha(s)$ for all s. Maps that satisfy this are exactly the maps that factor through $S_i(BG)$, so we can identify $\operatorname{Hom}_G(C_i(EG), L)$ with $\operatorname{Hom}_{\mathbb{Z}}(C_i(BG), L)$, which proves the lemma. □

Definition 7.13. Let G be a group and M be a G-module. The group cohomology $H^a(G; M)$ is defined as $H^a(G; M) = \operatorname{Ext}^a_{\mathbb{Z}[G]}(\mathbb{Z}; M)$.

Proposition 7.14. *Let G be a cyclic group of order p and let g be a generator of G. Let M be a G-module and let $\phi, \psi : M \to M$ be given by $\phi(m) = m - mg$ and $\psi(m) = m + mg + ... + mg^{p-1}$, then*

$$H^n(G; M) = \begin{cases} \operatorname{Ker} \phi & \text{if } n = 0, \\ \operatorname{Ker} \phi / \operatorname{Im} \psi & \text{if } n > 0 \text{ is odd,} \\ \operatorname{Ker} \psi / \operatorname{Im} \phi & \text{if } n > 0 \text{ is even.} \end{cases}$$

Proof. Let $A, B : \mathbb{Z}[G] \to \mathbb{Z}[G]$ be given by $A(x) = x - xg$ and $B(x) = x + xg + ... + xg^{p-1}$. Consider the following sequence

$$\dots \longrightarrow \mathbb{Z}[G] \xrightarrow{\alpha_3} \mathbb{Z}[G] \xrightarrow{\alpha_2} \mathbb{Z}[G] \xrightarrow{\alpha_1} \mathbb{Z}[G] \xrightarrow{\alpha_0} \mathbb{Z} \longrightarrow 0$$

Where $\alpha_0(\sum_i a_i g^i) = \sum_i a_i$ and $\alpha_{2n+1} = A$ for $n \geq 0$ and $\alpha_{2n} = B$ for $n > 0$. We view $\mathbb{Z}$ as a G-module, where the G-action is the trivial action. It is clear that all α_n are $\mathbb{Z}[G]$-linear.

Since

$$A(a_0 + ... + a_{p-1}g^{p-1}) = (a_0 - a_{p-1}) + (a_1 - a_0)g + ... + (a_{p-1} - a_{p-2})g^{p-1}$$

and

$$B(a_0 + ... + a_{p-1}g^{p-1}) = (a_0 + ... + a_{p-1}) + ... + (a_0 + ... + a_{p-1})g^{p-1},$$

we easily see that $\operatorname{Im} A \subseteq \operatorname{Ker} B$, that $\operatorname{Im} B = \operatorname{Ker} A$ and that $\operatorname{Ker} B = \operatorname{Ker} \alpha_0$.

Let $x \in \operatorname{Ker} B$, then we can write $x = a_0 + a_1 g + ... + a_{p-1} g^{p-1}$, with $a_{p-1} = -(a_0 + ... + a_{p-2})$. Now

$$A(a_0 + ... + (a_0 + ... + a_{p-2})g^{p-2}) = a_0 + ... + a_{p-2}g^{p-2} - (a_0 + ... + a_{p-2})g^{p-1} = x,$$

so $\operatorname{Ker} B = \operatorname{Im} A = \operatorname{Ker} \alpha_0$.

This shows that the sequence is a free resolution of the G-module $\mathbb{Z}$. Applying the $\operatorname{Hom}_G(-, M)$-functor, we obtain the sequence

$$\cdots \longleftarrow M \xleftarrow{\beta_3} M \xleftarrow{\beta_2} M \xleftarrow{\beta_1} M \xleftarrow{\beta_0} M \ ,$$

where $\beta_{2n} = \phi$ for and $\beta_{2n+1} = \psi$ for $n > 0$. Taking cohomology finishes the proof. □

Proposition 7.15. *For the classifying space BG of a group G, the cohomology with local coefficients $H^a(BG; M)$ is given by the group cohomology $H^a(G; M)$.*

Proof. The universal cover EG of BG is contractible, so the G-module chain complex

$$\cdots \longrightarrow C_2(EG; \mathbb{Z}) \xrightarrow{\partial_2} C_1(EG; \mathbb{Z}) \xrightarrow{\partial_1} C_0(EG; \mathbb{Z}) \longrightarrow 0$$

is exact in all C_i, except for C_0. Because EG is path-connected, the differential ∂_1 has image

$$\operatorname{Im} \partial_1 = \left\{ \sum_i a_i(x_i + y_i) \mid a_i \in \mathbb{Z} \text{ and } x_i, y_i \in EG \right\} = \left\{ \sum_i a_i x_i \mid x_i \in EG, \sum_i a_i = 0 \right\}.$$

Define $\alpha : C_0 \to \mathbb{Z}$, by $\alpha(\sum_i a_i s_i) = \sum_i a_i$, then $\operatorname{Ker} \alpha = \operatorname{Im} \partial_1$, so $C_*(EG; \mathbb{Z})$ is a free resolution of $\mathbb{Z}$, hence

$$H^n(BG; \underline{M}) = H^n(\operatorname{Hom}_G(C_*(EG; \mathbb{Z}), M) = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}; M) = H^n(G; M).$$

□

Example 7.16. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}$ with the trivial G-action. Then the maps ϕ and ψ in Proposition 7.14 are given by $\phi(m) = 0$ and $\psi(m) = 2m$, so

$$H^n(G; M) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n > 0 \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.17. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{F}_2$ with the trivial G-action. Then the maps ϕ and ψ in Proposition 7.14 are given by $\phi(m) = \psi(m) = 0$, so $H^n(G; M) = M$ for all $n \geq 0$.

7.3 Spectral sequences

Let K be a field. Throughout this section, we will write $H^*(X)$ instead of $H^*(X; K)$.

Definition 7.18. A *spectral sequence* is a collection of abelian groups $E_r^{p,q}$ and maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$, i.e. they form cochain complexes. Furthermore they satisfy $E_{r+1}^{p,q} = \operatorname{Ker}(d_r^{p,q}) / \operatorname{Im}(d_r^{p+r,q-r+1})$, i.e. $E_{r+1}^{p,q}$ is the cohomology of the chain complex at $E_r^{p,q}$.

Definition 7.19. If the only non-trivial groups occur when $p, q \geq 0$, then for $r > p$ the incoming differential in $E_r^{p,q}$ is

$$d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q},$$

which is trivial, since $E_r^{p-r,q+r-1} = 0$ in this case. For $r > q+1$ the outgoing differential from $E_r^{p,q}$ is

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

which is trivial, since $E_r^{p+r,q-r+1} = 0$ in this case. So for $r \geq \max(p, q+1)$, both the incoming and outgoing differentials in $E_r^{p,q}$ are trivial and thus $E_{r+1}^{p,q} = E_r^{p,q}$. As a consequence, for all p, q there exist an abelian group $E_\infty^{p,q}$, such that there is an R such that $E_r^{p,q} = E_\infty^{p,q}$ for all $r > R$. We say that the spectral sequence *degenerates at page* r if $E_r = E_\infty$.

Definition 7.20. [6]A *fibration* is a continuous map $p : X \to B$ that satisfies the *homotopy lifting property* for every topological space Y. This means that for every homotopy $H : Y \times [0,1] \to B$ and $f : X \times \{0\}$ such that $p \circ f = H|_{Y \times \{0\}}$ there exists a homotopy $\tilde{H} : Y \times [0,1] \to X$, such that $H = p \circ \tilde{H}$ and $\tilde{H}|_{Y \times \{0\}} = f$.

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "H"] \arrow[ur, dotted, "\tilde{H}"] & B
\end{tikzcd}$$

Proposition 7.21. *Let $p : X \to B$ be a fibration. Write $F_b = p^{-1}(b)$ for the fiber over b. For any $b_0, b_1 \in B$ a path between b_0 and b_1 induces a homotopy equivalence between F_{b_0} and F_{b_1}, such that composing paths corresponds to composing homotopy equivalences.*

If B is path-connected, all fibers are homotopy equivalent. We write $F \to X \to B$ for such a fibration, where $F = F_b$ is any fiber.

A loop with basepoint $b \in B$ induces a map $F \to F$, which in turn induces a map on cohomology $H^*(F) \to H^*(F)$, that only depends on the homotopy class of the loop. This makes $H^*(F)$ a $\pi_1(B)$-module.

Proof. See [6, Proposition 4.61]. □

Proposition 7.22. *The projection $X_G \to BG, [e, x] \mapsto [e]$ is a fibration with fiber X. The action on $H^*(X)$ induced by $\pi_1(BG) = G$ is the same as the one induced by the G-action on X.*

Proof. See [18, Page 182]. □

Theorem 7.23. *Let $F \to X \to B$ be a fibration with B path-connected and let $G = \pi_1(B)$ be the fundamental group of B. There exists a spectral sequence $\{E_r^{p,q}\}_{r \geq 2}$, called the* Leray-Serre *spectral sequence, for which $E_2^{p,q} = H^p(B; \underline{H^q(F)})$, where $H^q(F)$ is a G-module as described in 7.21. There are filtrations*

$$0 \subseteq A_{n,n} \subseteq ... \subseteq A_{n,0} = H^n(X),$$

such that $E_\infty^{p,n-p} \cong A_{n,p}/A_{n,p+1}$.

Proof. See [5, Theorem 5.15] □

For the remainder of this section, let $F \to X \to B$ be a fibration, let $G = \pi_1(B)$ and let $E_r^{p,q}$ be the corresponding Leray-Serre spectral sequence.

Corollary 7.24. *$H^n(X) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$ and there is a surjection $H^q(X) \to E_\infty^{0,q}$.*

Proof. By Theorem 7.23, $E_\infty^{p,n-p} \cong A_{n,p}/A_{n,p+1}$, so there are short exact sequences

$$0 \to A_{n,p+1} \to A_{n,p} \to E_\infty^{p,n-p} \to 0.$$

For $n = q$ and $p = 0$, this gives the surjection $H^q(X) = A_{q,0} \to E_\infty^{0,q}$. Since we are working over a field, these short exact sequences are split exact and thus $A_{n,p} \cong E_\infty^{p,n-p} \oplus A_{n,p+1}$. By induction we get $A_{n,p} \cong \bigoplus_{q=p+1}^n E^{q,n-q}$ and in particular $H(X) \cong A_{n,0} = \bigoplus_{p+q=n} E_\infty^{p,q}$. $\square$

Corollary 7.25. *There is a natural injection $E_\infty^{0,q} \to H^q(X)$.*

Proof. The incoming maps in $E_r^{0,q}$ are all 0, so $E_r^{0,q}$ is a subgroup of $E_{r-1}^{0,q}$. Therefore $E_\infty^{0,q}$ is a subgroup of $E_2^{0,q}$ and using Proposition 7.11 we obtain a sequence of injections

$$E_\infty^{0,q} \to E_2^{0,q} \cong H^q(X)^G \to H^q(X).$$

$\square$

Theorem 7.26. *There exist bilinear maps $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$, that we write as a product $(x, y) \mapsto xy$, with the following properties:*

i) For $r = 2$, these bilinear maps are given by $(-1)^{qs}$ times the cup product

$$H^p(B; \underline{H^q(F)}) \times H^s(B; \underline{H^t(F)}) \to H^{p+s}(B; \underline{H^{q+t}(F)}).$$

ii) They satisfy $d(xy) = d(x)y + (-1)^{p+q} x d(y)$, for $x \in E_r^{p,q}$.
iii) For $r > 2$, the product on E_r is induced by that on E_{r-1}, via $[x][y] = [xy]$.

Proof. See [5, Section 5.1] $\square$

7.4 Equivariant cohomology of $\mathbb{R}$-varieties

In this section we will study the equivariant cohomology of a topological space X with a continuous G-action by looking at the Leray–Serre spectral sequence corresponding to the fibration $X \to X_G \to BG$ and apply this to the case where X is an $\mathbb{R}$-variety.

Proposition 7.27. *The map $j^* : H^q(X_G; K) \to H^q(X; K)$ factors as $j^* = i \circ s$, where $s : H^q(X_G; K) \to E_\infty^{0,q}$ is the surjection from Corollary 7.24 and $i : E_\infty^{0,q} \to H^q(X; K)$ is the injection from Corollary 7.25.*

Proof. See [20, Theorem 7.6*]. $\square$

Theorem 7.28 (Künneth formula)**.** *Let K be a field.*

i) Let C_* and D_* be chain complexes of K-vector spaces, then $H_*(C_* \otimes_K D_*) \cong H_*(C_*) \otimes_K H_*(D_*)$, where the isomorphism is given by

$$[c \otimes d] \mapsto [c] \otimes [d].$$

ii) Let X and Y be topological spaces, then $H^*(X \times Y; K) \cong H^*(X; K) \otimes_K H^*(Y; K)$.

Here we take tensor product of chain complexes and graded modules. The analogous statements for cohomology and cohain complexes also holds.

Proof.

i) See [7, Theorem 5.7.12].

ii) See [6, Theorem 3.16].

$\square$

Proposition 7.29. *[18, Proposition III.1.18] Let G be a group, let X be a path-connected, finite CW-complex with a continous G-action and let K be a field. Then X is equivariantly formal over K if and only if G acts trivially on $H^*(X; K)$ and the Leray-Serre spectral sequence E_* of $X \to X_G \to BG$ degenerates at page 2.*

Proof. Consider the composition $j^* = i \circ s$ from Proposition 7.27. Then j^* is surjective if and only if i is surjective. Since i factors as

$$E_\infty^{0,q} \to E_2^{0,q} \cong H^q(X; K)^G \to H^q(X; K),$$

this happens exactly when $E_\infty^{0,q} = E_2^{0,q}$ and G acts trivially on $H^q(X; K)$. What is left to show is that $E_\infty^{0,q} = E_2^{0,q}$ for all q implies that $E_\infty^{p,q} = E_2^{p,q}$ for all p and q.

Let R be a ring and let N be an R-module and let M be a free R-module of finite rank, with basis $m_1, ..., m_n$. Then $\mathrm{Hom}_R(N, R) \otimes_R M \cong \mathrm{Hom}_R(N, M)$, where the isomorphism is given by $f \otimes m \mapsto (a \mapsto f(a) \cdot m)$ and the inverse is given by sending f to $\sum_i (f_i \otimes m_i)$, where f_i is given by sending a to the coefficient of m_i in $f(a)$.

If N_* is a chain complex, than this isomorphism commutes with the induced differentials on the cochain complexes $\mathrm{Hom}_R(N_*, M)$ and $\mathrm{Hom}_R(N_*, R) \otimes M$. We can view any R-module M as a cochain complex M^* with $M^0 = M$ and $M^k = 0$ for all $k \neq 0$ and with trivial differentials. So we get an isomorphism of cochain complexes.

Because X is path-connected, $H^0(X; K) \cong K$ is a field, so any $H^0(X; K)$-module is free. The cup product on $H^*(X; K)$ makes $H^q(X; K)$ a $H^0(X; K)$-module. Since BG is path-connected, elements of $H^0(BG; M)$ are constant on 0-simplices of BG for any K-vector space M. This gives an isomorphism $H^0(BG; M) \to M, f \mapsto f(x)$, where x is an arbitrary 0-simplex in BG. Because $H^q(X; K)$ is finite dimensional, the previous isomorphisms and Künneth's theorem give the sequence of isomorphisms

$$\begin{aligned} H^p(BG; H^0(X; K)) \otimes H^0(BG; H^q(X; K)) &\to H^p(BG; H^0(X; K)) \otimes H^q(X) \\ &\to H^p(\mathrm{Hom}(C_*(BG; \mathbb{Z}), H^0(X; K)) \otimes H^q(X; K)) \\ &\to H^p(BG; H^q(X; K)), \end{aligned}$$

given by

$$\begin{aligned}
[f] \otimes [g] &\mapsto [f] \otimes g(x) \mapsto [f \otimes g(x)] \\
&\mapsto \left[\sum_i a_i x_i \mapsto \sum_i a_i f(x_i) g(x) \right] \\
&= \left[\sum_i a_i x_i \mapsto \sum_i a_i f(x_i) g(x_i) \right],
\end{aligned}$$

which coincides with the cup product.

Because G acts trivially on $H^*(X)$, we obtain an isomorphism

$$\begin{aligned}
E_2^{p,0} \otimes E_2^{0,q} &\cong H^p(BG; \underline{H^0(X;K)}) \otimes H^0(BG; \underline{H^q(X;K)}) \\
&\cong H^p(BG; H^0(X;K)) \otimes H^0(BG; H^q(X;K)) \\
&\cong H^p(BG; H^q(X;K)) \\
&\cong H^p(BG; \underline{H^q(X;K)}) = E_2^{p,q}
\end{aligned}$$

that is given, up to sign, by the product on E_2. In particular the product on E_2 is surjective, so we can write any element of $E_2^{p,q}$ as finite sum of products $x_i y_i$, with $x_i \in E_2^{p,0}$ and $y_i \in E_2^{0,q}$. The differentials on E_r satisfy

$$d(x_i y_i) = d(x_i) y_i \pm x_i d(y_i).$$

The assumption that $E_\infty^{0,q} = E_2^{0,q}$ is equivalent to $d(y) = 0$ for all $y \in E_r^{0,q}$ and all $r \geq 2$, so $d(y_i) = 0$. Furthermore, $d(x_i) = 0$ because $d(x_i) \in E_2^{p+2,-1} = 0$. Therefore $d(x_i y_i) = 0$ for all x_i and y_i. Hence $d(z) = 0$ for all $z \in E_2^{p,q}$, so $E_3 = E_2$. This also means that the product on E_3 is the same as the product on E_2 and in particular also surjective, so we can repeat the same argument. This gives $E_2 = E_\infty$. □

Lemma 7.30. *Let G be a cyclic group of order p, with p prime, and let X be a topological space for which there exists an n such that $H^k(X; \mathbb{F}_p) = 0$ for all $k > n$. Then the inclusion $X^G \to X$ induces an isomorphism $H_G^k(X; \mathbb{F}_p) \to H_G^k(X^G; \mathbb{F}_p)$ for $k > n$.*

Proof. See [18, Proposition III.4.9]. □

Theorem 7.31. *[18, Proposition III.4.16] Let (X, σ) be an n-dimensional $\mathbb{R}$-variety and let G be the cyclic group of order 2, generated by σ, which acts naturally on X. Then X is maximal if and only if X is equivariantly formal over $\mathbb{F}_2$.*

Proof. By Proposition 7.29 it is enough to show that X is maximal if and only if G acts trivially on $H^*(X; \mathbb{F}_2)$ and the Leray–Serre spectral sequence E_* of $X \to X_G \to BG$ degenerates at page 2. In the rest of this proof, we leave out the coefficients and write $H^*(X)$ instead of $H^*(X; \mathbb{F}_2)$.

By Corollary 7.24 and the fact that $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, we have

$$\dim H_G^k(X) = \sum_{p+q=k} \dim E_\infty^{p,q} \leq \sum_{p+q=k} \dim E_2^{p,q} = \sum_{p+q=k} \dim H^p(BG; \underline{H^q(X)})$$

with equality if and only if the spectral sequence degenerates at page 2.

Applying Proposition 7.14 on G, combined with Proposition 7.15 gives

$$H^p(BG; \underline{H^q(X)}) = \begin{cases} \operatorname{Ker}\rho & \text{if } p = 0 \\ \operatorname{Ker}\rho / \operatorname{Im}\rho & \text{if } p > 0 \end{cases}$$

with $\rho = \mathrm{id} + \sigma : H^q(X) \to H^q(X)$.

Note that $\operatorname{Ker}\rho = H^q(X)^G$ and that

$$\dim(\operatorname{Ker}\rho / \operatorname{Im}\rho) \leq \dim(\operatorname{Ker}\rho) = \dim(H^q(X))^G.$$

With equality if and only if $\rho = 0$, which is exactly the case when G acts trivially on $H^q(X)$. Therefore we have $\dim E_2^{p,q} = \dim H^p(BG; H^q(X)) \leq \dim H^q(X)$ with equality if and only if G acts trivially on $H^q(X)$. In total we get for $k > 2n$

$$\dim H^k_G(X) = \sum_{p+q=k} \dim E_\infty^{p,q} \leq \sum_{p+q=k} \dim E_2^{p,q} \leq \sum_{b=0}^{k} \dim H^b(X) = b_*(X; \mathbb{F}_2),$$

with equality if and only if X is equivariantly formal.

Since G acts trivially on X^G, the space $X^G_G = EG \times_G X^G$ is just the Cartesian product $BG \times X^G$. Therefore we can use the Künneth formula and get

$$H^k_G(X^G) = \bigoplus_{p+q=k} H^p(BG) \otimes H^q(X^G) = \bigoplus_{q=0}^{k} H^q(X^G),$$

the latter equality follows from $H^p(BG) = H^p(\mathbb{RP}^\infty) = \mathbb{F}_2$ for all $p \geq 0$.

As X is n dimensional, $H^k(X) = 0$ for $k > 2n$, so we can use Lemma 7.30 and get

$$b_*(X^G; \mathbb{F}_2) = \sum_{q=0}^{k} \dim H^q(X^G) = \dim H^k_G(X^G) = \dim H^k_G(X) \leq b_*(X; \mathbb{F}_2)$$

for $k > 2n$, with equality if and only if X is equivariantly formal over $\mathbb{F}_2$. Note that we have also reproven the Thom-Smith inequality. $\square$

7.5 Γ-products

Definition 7.32. Let X be a topological space and $\Gamma \subseteq S_n$ a subgroup, then the Γ-product X^Γ of X is the quotient of X^n, by the natural Γ action on X^n. For $\Gamma = S_n$, this is the n-fold symmetric product of X.

Theorem 7.33. *[3, Theorem 1.1] Let K be a field, let G be a group and let X be a topological space with continuous G-action that is equivariantly formal over K. Then X^Γ is also equivariantly formal over K.*

Proof. By assumptio,n the map $j_e : X \to X_G, x \mapsto [e, x]$ induces a surjection in cohomology. Lemma 2.4 in [3] implies that $j_e^\Gamma : X^\Gamma \to (X_G)^\Gamma, [x_1, ..., x_n] \mapsto [[e, x_1], ..., [e, x_n]]$ also induces a surjection in cohomology. The map j_e^Γ factors as $j_e = \phi \circ \psi_e$, with

$$\psi_e : X^\Gamma \to (X^\Gamma)_G, [x_1, ..., x_n] \mapsto [e, [x_1, ..., x_n]]$$

and

$$\phi : (X^\Gamma)_G \to (X_G)^\Gamma, [e, [x_1, ..., x_n]] \mapsto [[e, x_1], ..., [e, x_n]].$$

This induces a factorization of the map in cohomology and shows that ψ_e induces a surjection in cohomology, which proves that X^Γ is equivariantly formal over K.

$$\begin{tikzcd} X^\Gamma \arrow[rr, "j_e^\Gamma"] \arrow[dr, "\psi_e"'] & & (X_G)^\Gamma \\ & (X^\Gamma)_G \arrow[ur, "\phi"'] & \end{tikzcd} \qquad \begin{tikzcd} H^*(X^\Gamma; K) & & H^*((X_G)^\Gamma; K) \arrow[ll, two heads, "(j_e^\Gamma)^*"'] \arrow[dl, "\phi^*"] \\ & H_G^*(X^\Gamma; K) \arrow[ul, two heads, "\psi_e^*"] & \end{tikzcd}$$

□

Corollary 7.34. *Let X be a maximal curve, then $X^{(n)}$ is a maximal variety.*

Proof. By Propositions 6.4 and 6.6 $X^{(n)}$ is an $\mathbb{R}$-variety. Since X is maximal, it is equivariantly formal over $\mathbb{F}_2$, by Theorem 7.31. By Theorem 7.33 $X^{(n)}$ is also equivariantly formal over $\mathbb{F}_2$ and finally by Theorem 7.31 $X^{(n)}$ is maximal. □

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