

Moduli spaces of stable sheaves on K3 surfaces

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Introduction

When one wants to study a scheme, one can instead study the category of (coherent) sheaves on that scheme. It is fruitful to not only study the sheaves individually, but to study families of them as well. This naturally leads to the question of the existence of a moduli space, which classifies such families. Such a classifying space exists when we restrict our attention to the so-called stable sheaves on a projective scheme X .

Stable sheaves satisfy a numerical condition in terms of the Hilbert polynomial. Therefore, they are only defined for projective schemes, with a fixed embedding in projective space. It is difficult to say when a sheaf is stable, though abstractly it can be shown that plenty of them exist, as they generate the Grothendieck group of X .

Since in a flat family of coherent sheaves the Hilbert polynomial is constant, we may as well restrict ourselves to families of stable sheaves with a fixed Hilbert polynomial. In fact, we may go even further and consider families with constant Chern character. Members of such a family share many numerical invariants, such as their rank and degree.

A scheme classifying families of stable sheaves is called a *fine moduli space*. However, such a scheme does not always exist. There is an alternative notion, that of a *coarse moduli space*, which still captures the geometric intuition. When rank and degree are coprime, the moduli space does turn out to be fine.

Another question one might ask is whether the moduli space is projective. If we consider instead of stable sheaves the semi-stable sheaves, then the resulting moduli space is projective, but it is in general not a coarse moduli space. It has the moduli space of stable sheaves as an open subscheme, which is therefore quasi-projective. When rank and degree are coprime, all semi-stable sheaves are stable. Then the two moduli spaces coincide and subsequently are projective and fine. For the rest of the introduction, let us assume that we are in this case and denote the moduli space by M .

It is possible to obtain an explicit description of the tangent bundle of the moduli space, namely, the tangent space to a point $m \in M$ corresponding to a stable sheaf E is $\text{Ext}^1(E, E)$. There also is a global description of the tangent bundle as the *relative Ext-sheaf* $\mathcal{E}xt_{\pi_M}^1(\mathcal{E}, \mathcal{E})$. Here \mathcal{E} is the *universal family*, whose existence is guaranteed by the fineness of the moduli space.

The explicit description of the tangent bundle allows us to study the smoothness of M . When X is a K3 surface, $\dim \text{Ext}^1(E, E)$ is constant for stable sheaves with a fixed Chern character, resulting in smoothness of the moduli space.

For a surface with trivial canonical bundle, Serre duality gives an alternating pairing

$$\mathrm{Ext}^1(E, E) \otimes \mathrm{Ext}^1(E, E) \rightarrow k$$

for any sheaf E . This pairing extends to a morphism of sheaves $TM \otimes TM \rightarrow \mathcal{O}_M$, resulting in an *algebraic symplectic structure* on the moduli space. In fact, one can show that M becomes a *hyperkähler variety* in this case. It is true that many examples of such varieties are constructed in this fashion [14, Sec. 6.2].

In this thesis, we investigate the construction of the moduli space and the properties mentioned above. As a result, the material presented in this thesis is not new. However, as a part of the thesis project, I obtained the results in Section 3.5 and Section 4.3 independently. The main reference for the material in this thesis is the book [14], which contains much more material on sheaves on surfaces.

Outline In Chapter 1, we introduce moduli problems and define coarse moduli spaces. We then go on to prove the basic properties of stable sheaves in Chapter 2. The main construction is outlined in Chapter 3. The construction works for all projective schemes X . Lastly, we introduce K3 surfaces in Chapter 4 and prove that the moduli space has a symplectic structure in this case.

Conventions In the entire thesis, we work over $k = \mathbb{C}$. Thus, when we say that X is a scheme, we implicitly mean a scheme over k . Also, fibre products are taken in the category of k -schemes, unless otherwise mentioned. All schemes in the thesis are in addition assumed to be Noetherian, and as such, all sheaves under consideration are coherent sheaves. We will often drop the adjective “coherent”, simply calling the objects “sheaves”.

By convention, X always denotes a projective scheme over k with fixed ample line bundle. This allows us to consider the Hilbert polynomial of a sheaf on X . In Chapter 4, we will often assume that in addition, X is a K3 surface.

We have a convention for denoting projection morphisms. The projection $X \times Y \rightarrow X$ is denoted π_X , so the subscript denotes where we are projecting *to*. This is potentially confusing when we consider a projection $X \times X \rightarrow X$, but these occasions are rare and it will still be clear from context which morphism is meant.

1. Moduli problems

In this chapter, we will introduce the notions necessary to discuss moduli problems. In the first section we recall the basic formalism of the functors of points of schemes. In the second section, we specialise to moduli problems of families of sheaves and state our main theorem.

1.1. Representability of functors

Recall the following basic definition.

Definition 1.1. Let \mathcal{C} be a category and X an object of \mathcal{C} . The functor $h_X : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ that sends $Y \in \mathcal{C}$ to $\text{Hom}(Y, X)$ is called the *functor represented by X* .

A functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ is called *representable* if it is isomorphic to h_X for some object X of \mathcal{C} . In this case, we say that X *represents F* . If \mathcal{C} is the category of schemes, we also say that X is a *fine moduli space* for F .

The following statement about representable functors is well-known.

Proposition 1.2 (Yoneda Lemma). *Let \mathcal{C} be a category, F a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ and X an object of \mathcal{C} . There is a natural isomorphism*

$$\text{Nat}(h_X, F) \rightarrow F(X).$$

In particular, the assignment $X \mapsto h_X$ extends to a fully faithful functor from \mathcal{C} to the functor category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$.

A proof can be found in many places, such as [18, Ch. III]. It is also easy to describe the isomorphism: it sends a natural transformation λ to $\lambda_X(\text{id}_X)$. This is especially interesting if λ is an isomorphism of functors, in that case the corresponding element of $F(X)$ is called a *universal object*. Sometimes, there is a preferred choice of universal object, which we then call *the* universal object.

We are mostly interested in examples arising from algebraic geometry. In the following, we consider the category of schemes over a fixed field k .

Example 1.3. • The functor $\Gamma(-, \mathcal{O})$ is represented by the affine line \mathbb{A}_k^1 . We know that $\Gamma(\mathbb{A}_k^1, \mathcal{O}_{\mathbb{A}_k^1}) = k[x]$. The universal object in this case is x , the affine coordinate of \mathbb{A}_k^1 . Indeed, every function $f \in \Gamma(X)$ induces a morphism $\varphi : X \rightarrow \mathbb{A}_k^1$, and the statement that x is universal translates to the fact that f can be recovered as φ^*x .

- A more involved example is the universal property of the projective space \mathbb{P}_k^n . It is well-known that a morphism $X \rightarrow \mathbb{P}_k^n$ is equivalent to the datum of a quotient $\mathcal{O}_X^{n+1} \rightarrow L$ where L is a line bundle (see e.g. [8, Thm. II.7.1]). Thus, projective space represents the functor which sends X to the set of quotients of \mathcal{O}_X^{n+1} which are line bundles. The universal object is the quotient $\mathcal{O}_{\mathbb{P}_k^n}^{n+1} \rightarrow \mathcal{O}(1)$, sending (a_0, \dots, a_n) to $\sum_i a_i x^i$. Here the x_i are the homogeneous coordinates of \mathbb{P}_k^n . Indeed, when $f : X \rightarrow \mathbb{P}_k^n$ is a morphism, we can recover the quotient $\mathcal{O}_X^{n+1} \rightarrow L$ that induced f by pulling back the universal quotient along f .
- Let V be a k -vector space and consider the algebraic group $\mathrm{GL}(V)$. This scheme represents the functor $X \mapsto \mathrm{Aut}(V \otimes \mathcal{O}_X)$. More explicitly, if V is n -dimensional, this is the set of invertible $n \times n$ -matrices with coefficients in $\Gamma(X, \mathcal{O}_X)$.
- The explicit description of a functor of points is very useful. For example, it is immediate to compute the k -points of \mathbb{A}_k^1 : it is just the set $\Gamma(\mathrm{Spec} k, \mathcal{O}_{\mathrm{Spec} k}) \cong k$. For projective spaces, note that a quotient $k^{n+1} \rightarrow k$ is given by $(x_0, \dots, x_n) \mapsto \sum_i a_i x_i$ for some $a_i \in k$; the condition that it is a quotient translating to the fact that not all a_i are zero. Two quotients given by $\{a_i\}$ and $\{a'_i\}$ are isomorphic if and only if there is a nonzero scalar such that $a_i = \lambda a'_i$ for all i . Thus, we recover the classical definition of projective space in this way.

Sometimes, a functor is not representable, but the next best thing is true.

Definition 1.4. A functor $F : \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathbf{Set}$ is *corepresented* by a scheme X , if there is a morphism of functors $\eta : F \rightarrow h_X$ such that for all morphisms $\theta : F \rightarrow h_Y$ there is a unique morphism $f : X \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\eta} & h_X \\ & \searrow \theta & \downarrow h_f \\ & & h_Y \end{array}$$

We also say that X *corepresents* F .

We say that X *coarsly represents* F , or that X is a *coarse moduli space* if $\eta_{\mathrm{Spec} k}$ is an isomorphism.

Note that if X corepresents F , for any scheme S and $a \in F(S)$ there is a map $\eta_S(a) : S \rightarrow X$. However, the correspondence $a \mapsto \eta_S(a)$ is not injective or surjective in general. In particular, there is no easy description of the closed points of X in terms of F . By definition, when X is a coarse moduli space, then the set of closed points can be identified with $F(\mathrm{Spec} k)$. Note that every fine moduli space is also coarse, hence we can describe their closed points as well.

It is not so difficult to give an example of a functor that is corepresented, yet not represented. Completely proving that statement is much more involved. In general, examples can be found when considering a group action on a scheme, and trying to construct a quotient. Indeed the following example is of that form. The reader can safely skip the details.

Example 1.5. Consider the category of schemes over k and let F be the functor associating to X the set of conjugacy classes of 2×2 -matrices with coefficients in $\Gamma(X, \mathcal{O}_X)$. Consider the map $F \rightarrow h_{\mathbb{A}_k^2}$, sending a matrix to its determinant and its trace (this is well-defined on the conjugacy classes). In this way, \mathbb{A}_k^2 universally corepresents F , but the morphism of functors is not an isomorphism, because there exist nontrivial matrices with vanishing determinant and trace.

To prove that the map above is indeed universal, suppose that $F \rightarrow h_X$ is any morphism. Consider the map $h_{\mathbb{A}_k^2} \rightarrow F$, sending (x, y) to the diagonal matrix with entries x and y . Then the composition $h_{\mathbb{A}_k^2} \rightarrow h_X$ is symmetric, i.e. it sends (x, y) and (y, x) to the same thing. Thus, the resulting map $\mathbb{A}_k^2 \rightarrow X$ factors through the quotient of \mathbb{A}_k^2 by the action of the symmetric group S_2 which interchanges x and y . The subring of $\Gamma(\mathbb{A}_k^2, \mathcal{O}) = k[x, y]$ fixed by the action is $k[x + y, xy]$. Then [3, Thm. 6.1] implies that $\mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$ given by $(x, y) \mapsto (xy, x + y)$ is the corresponding quotient. This factorisation $\mathbb{A}_k^2 \rightarrow X$ is the required morphism. We only need to show that composition with our universal morphism $F \rightarrow h_{\mathbb{A}_k^2}$ gives us back $F \rightarrow h_X$. This follows after observing that the set of diagonalisable matrices is dense in the set of all matrices. Details are omitted.

1.2. Moduli of sheaves

Definition 1.6. Let X be a scheme. A *family of sheaves on X parameterised by S* is a coherent sheaf E on $X \times S$ which is flat over S .

For $s \in S$, we denote by $E|_s$ the pullback of E along the inclusion $X \times \{s\} \rightarrow X \times S$.

Geometrically, we view the $E|_s$ as being glued together to obtain a single sheaf E . The flatness condition informally says that this gluing is done in a continuous manner.

Families of sheaves play a central role in the study of moduli problems of sheaves. As an example of this, we first consider the Quot-scheme. Suppose we have a projective scheme X over k with fixed ample line bundle, a sheaf F on X and a polynomial P . We define a functor $\mathbf{Quot}_F^P : \mathbf{Sch}^{\text{op}} \rightarrow \mathbf{Set}$:

$$\mathbf{Quot}_F^P(S) = \{(q, Q) \mid q : \pi_X^* F \twoheadrightarrow Q \text{ such that } Q \text{ is flat over } S \\ \text{and for all } s \in S, Q|_s \text{ has Hilbert polynomial } P.\} / \sim \quad (1.1)$$

Here $(q, Q) \sim (q', Q')$ if there is an isomorphism $\varphi : Q \rightarrow Q'$ with $\varphi \circ q = q'$. One might say that an element of $\mathbf{Quot}_F^P(S)$ is a family of quotients of F . For the required background on flatness and the Hilbert polynomial, see [8] or [25]. For a summary of properties of the Hilbert polynomial, see also Theorem 2.6.

Theorem 1.7 (Grothendieck). *The functor \mathbf{Quot}_F^P is representable by a projective scheme \mathbf{Quot}_F^P .*

A detailed construction can be found in [22] and there is also a construction in [14, Ch. 2].

Note that it may very well happen that a sheaf G is a quotient of E in multiple ways, i.e., that there are quotients $q : E \rightarrow G$ and $q' : E \rightarrow G$ which are not isomorphic. In

this sense, Quot_F^P parameterises sheaves with extra structure, the extra structure being this quotient map. In the moduli space of stable sheaves, such behaviour does not occur.

To define the moduli functor, a definition of (semi)-stability is not required. We will postpone the definition of (semi)-stability to Chapter 2.

Definition 1.8. Let X be a projective scheme with fixed very ample line bundle L . Fix again a polynomial P . We define a functor $\mathcal{M} : \text{Sch}^{\text{op}} \rightarrow \mathbf{Set}$ by

$$\mathcal{M}(S) = \{F \in \text{Coh}(X \times S) \mid F \text{ is flat over } S \text{ and for each } s \in S, F|_s \text{ is a semi-stable sheaf with Hilbert polynomial } P.\} / \sim$$

Here, we set $F \sim F'$ if there is a line bundle L on S , such that $F \cong F' \otimes \pi_S^* L$.

We also define a functor \mathcal{M}^s in the same way, except that we replace “semi-stable” by “stable”.

The main result of this thesis is that this functor has a coarse moduli space.

Theorem 1.9. *Let X be a projective variety with fixed very ample line bundle over an algebraically closed field k of characteristic zero. Then the functor \mathcal{M} is corepresented by a projective scheme M . The functor \mathcal{M}^s is corepresented by an open subscheme M^s of M . Furthermore, M^s is a coarse moduli space for \mathcal{M}^s .*

We will prove Theorem 1.9 in Chapter 3. Even though M does not represent \mathcal{M} in general, there are criteria when it does, see for example the next result.

Theorem 1.10. *Suppose that X is a smooth projective variety. Write the polynomial P as*

$$P(n) = \sum_{i=0}^d a_i \binom{n+i-1}{i}.$$

If $\gcd(a_0, a_1, \dots, a_d) = 1$, then M^s represents \mathcal{M}^s .

Proof. We will not prove this Theorem 1.10 in this thesis. The interested reader can find a proof in [14, Sec. 4.6]. \square

There is a general criterion for P such that sheaves with Hilbert polynomial P are stable if and only if they are semi-stable. This implies that $\mathcal{M} = \mathcal{M}^s$ and hence $M = M^s$. Since M is projective, this implies that M^s is projective. See Prop. 2.26 for such a criterion.

We should notice that the Quot-functor \mathbf{Quot}_F^P and \mathcal{M}^s depend on the choice of a very ample line bundle on X . Indeed, the notion of Hilbert polynomial changes if we change the line bundle. Thus, one might ask if we can do without the Hilbert polynomial, so that the moduli functor is independent of the chosen line bundle. Consider the Quot-functor. If we omit the Hilbert polynomial-condition, the resulting functor would still be representable, but it would decompose into infinitely many disjoint subschemes. Such a scheme is not of finite type. By choosing a Hilbert polynomial, we are labelling some of the components in such a way that the resulting subscheme is of finite type.

In fact, instead of fixing Hilbert polynomials one could also fix a Chern character (see Appendix C). These are independent of a chosen ample line bundle. In Chapter 4, we construct a variant of the moduli space of stable sheaves with fixed Chern character instead of the Hilbert polynomial (see Prop. 4.15). This construction can be carried out for the Quot scheme as well, making it independent of the chosen ample line bundle. However, the moduli functor of stable sheaves is still dependent on the choice of an ample line bundle, because the definition of stability depends on it.

2. Stable sheaves

In this chapter we introduce (semi-)stable sheaves. To my knowledge, the main purpose they serve is that families of stable sheaves admit a coarse moduli space in the sense of Theorem 1.9.

We first introduce the prerequisite notion of pure sheaves. After defining stable sheaves, we will prove that there exist many of them, in the sense that they satisfy a Jordan-Hölder type theorem. As such, we may view the stable sheaves as building blocks for general sheaves. After that, we discuss how stable sheaves behave in a family. The reference for this material is [14].

2.1. Pure sheaves

Recall that X is a projective scheme with fixed ample line bundle.

Definition 2.1. The *dimension* of a sheaf is the dimension of its support. A sheaf is *pure* if all of its nonzero subsheaves have the same dimension.

When X is projective (as we assume), there is another description of the dimension of a sheaf, namely the degree of its Hilbert polynomial, see Thm. 2.6.

Example 2.2. Let X be a variety of dimension d . The pure sheaves of dimension d are exactly the torsion-free sheaves.

Suppose that E is a torsion-free sheaf, F is a nonzero subsheaf and $\dim F < d$. Let U be an affine open subset of X . Then there is $f \in \Gamma(U, \mathcal{O}_X)$ such that $\text{Supp } F \cap D(f) = \emptyset$. Therefore, f annihilates F and so F is not torsion-free. This gives a contradiction, as F is a subsheaf of a torsion-free sheaf.

On the other hand, suppose that E is pure and that $s \in E(U)$ is a section. If $u \in \mathcal{O}_X(U)$ is such that $us = 0$, then the subsheaf generated by s is supported on $V(u)$, which has dimension less than d unless $u = 0$. Therefore E is torsion-free.

Lemma 2.3. *Given a coherent sheaf E of dimension d , then for each $0 \leq n \leq d$, there is a unique maximal subsheaf of E of dimension at most n .*

Proof. Consider the set \mathcal{S} of subsheaves of E of dimension at most n . Since E is a Noetherian object in the category of coherent sheaves, every set of subsheaves of E has an element maximal with respect to inclusion. Suppose \mathcal{S} has two such elements, F and

F' . Then their sum $F + F'$ also has dimension at most n . Since F and F' were maximal with respect to inclusion, it follows that $F = F'$. Hence such a maximal subsheaf is uniquely determined. \square

Definition 2.4. Let E be a coherent sheaf of dimension d . Denote by $T_n(E)$ the largest subsheaf of E with dimension at most n . The resulting filtration

$$0 \subseteq T_0(E) \subseteq T_1(E) \subseteq \dots \subseteq T_{d-1}(E) \subseteq T_d(E) = E. \quad (2.1)$$

is called the *torsion filtration* of E .

Note that a d -dimensional sheaf E is pure if and only if $T_{d-1}(E) = 0$.

Lemma 2.5. *The quotients $T_i(E)/T_{i-1}(E)$ are either zero or purely i -dimensional.*

Sketch of proof. Suppose the quotient is nonzero. Then $T_i(E)$ has dimension i . The short exact sequence

$$0 \rightarrow T_{i-1}(E) \rightarrow T_i(E) \rightarrow T_i(E)/T_{i-1}(E) \rightarrow 0$$

shows that the quotient must have dimension i as well. If F is a subsheaf of the quotient of dimension less than i , its inverse image has dimension less than i as well. But then it is contained in $T_{i-1}(E)$ and hence goes to zero. The assertions about dimensions appearing in this proof can be quickly checked using the Hilbert polynomial, defined in Section 2.2. \square

Thus, even if a sheaf is not pure, we can filter it by pure sheaves. This proposition is most useful when $i = d$, because $E/T_{d-1}(E)$ is the largest pure quotient of E . In fact, any map $E \rightarrow F$ of sheaves of equal dimension with F pure factors through $E/T_{d-1}(E)$.

2.2. Stable sheaves

Recall that $\chi(X, E) = \sum_i (-1)^i \dim H^i(X, E)$ and that $E(m)$ is an abbreviation for $E \otimes \mathcal{O}(1)^{\otimes m}$, where $\mathcal{O}(1)$ is the fixed ample line bundle on X .

Theorem 2.6. *Let E be a sheaf of dimension d on a projective scheme X . Then $\chi(X, E(m))$ is a polynomial, denoted $P(E)$, with rational coefficients and of degree d . We write $P(E, m) = \sum_{i=0}^d \alpha_i(E) \frac{x^i}{i!}$. In this case, $\alpha_d(E)$ is a positive integer.*

Whenever $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is a short exact sequence, we have $P(E) = P(F) + P(G)$ and $\alpha_i(E) = \alpha_i(F) + \alpha_i(G)$ for each i .

Proof. This is well-known, see [25, Sec. 18.6] or [14, Sec. 1.2]. The claims about $\alpha_d(E)$ can be checked using a quite explicit formula in the second reference. \square

Definition 2.7. The polynomial $P(E)$ is called the *Hilbert polynomial* of E . The *reduced Hilbert polynomial* $p(E)$ is defined as $P(E)/\alpha_d(E)$. In addition, the integer $\alpha_d(E)$ is called the *multiplicity* of E .

For two polynomials $P, Q \in \mathbb{Q}[x]$, we say that $P < Q$ if $P(x) < Q(x)$ for sufficiently large x .

There is a more elementary way to describe the ordering. Namely, we can consider the *lexicographic ordering* of the coefficients. That is, if we have P, Q of degree at most d , we first compare their coefficients of x^d . If they are equal, we move on to x^{d-1} , etc. It is not difficult to see that this gives the same ordering.

By Theorem 2.6, the multiplicity of a sheaf E is a positive integer. It is closely related to the rank of E , see Lemma 2.22.

Definition 2.8. A sheaf E is *stable* if for each proper nonzero subsheaf $F \subseteq E$, we have that $p(E) < p(F)$. A sheaf E is *semi-stable* if for each proper nonzero subsheaf $F \subseteq E$, we have that $p(E) \leq p(F)$.

In a large amount of cases, statements about semi-stable sheaves and stable sheaves are proven almost the same way, only replacing strict inequalities by non-strict ones or the other way around. Thus, almost always it makes sense to give only one of the proofs. We trust the reader will be able to give the other proof themselves.

Lemma 2.9. *Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be a short exact sequence of d -dimensional sheaves. Then we have an equality*

$$\alpha_d(G)(p(G) - p(E)) = \alpha_d(F)(p(E) - p(F)). \quad (2.2)$$

In particular, $p(F) \leq p(E)$ if and only if $p(E) \leq p(G)$. Similarly, $p(F) < p(E)$ if and only if $p(E) < p(G)$.

Proof. The equation is equivalent to $P(G) - \alpha_d(G)p(E) = \alpha_d(F)p(E) + P(F)$, since $\alpha_d(G)p(G) = P(G)$, etc. Rewriting, we obtain

$$P(G) + F(G) = (\alpha_d(F) + \alpha_d(G))p(E).$$

This holds because $\alpha_d(F) + \alpha_d(G) = \alpha_d(E)$ and $P(F) + P(G) = P(E)$. The second statement is implied by the fact that $\alpha_d(F)$ and $\alpha_d(G)$ are both positive by Thm. 2.6, so that $p(E) - p(F) \geq 0$ if and only if $\alpha_d(F)(p(E) - p(F)) \geq 0$, etc. \square

This lemma tells us that the inequality in the definition of (semi-)stability may as well be replaced by an inequality in terms of the quotients of E . An improvement of the definition is given by the next lemma, which says that it suffices to check the inequality when the quotient is purely d -dimensional. A subsheaf F satisfying this condition is called *saturated*.

Lemma 2.10. *Let E be a purely d -dimensional sheaf. The following are equivalent:*

1. E is semi-stable.
2. For each short exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ with G d -dimensional and $F \neq 0$, we have $p(F) \leq p(E)$.
3. For each short exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ with G d -dimensional and $F \neq 0$, we have $p(E) \leq p(G)$.

4. For each short exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ with G purely d -dimensional and $F \neq 0$, we have $p(F) \leq p(E)$.
5. For each short exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ with G purely d -dimensional and $F \neq 0$, we have $p(E) \leq p(G)$.

For the analogous statement about stability, replace \leq by $<$ everywhere.

Proof. By Lemma 2.9 we see that (2) is equivalent to (3) and, similarly, (4) is equivalent to (5). We have trivial implications from (1) to (2) and from (2) to (4). Now assume (5) and consider a subsheaf F of E , with corresponding quotient G . Define $G' = G/T_{d-1}(G)$, the canonical purely d -dimensional quotient of G . If $G' = 0$, then G is of dimension less than d , so $\alpha_d(F) = \alpha_d(F) + \alpha_d(G) = \alpha_d(E)$. As a result,

$$p(F) = \frac{P(F)}{\alpha_d(F)} = \frac{P(E) - P(G)}{\alpha_d(E)} < \frac{P(E)}{\alpha_d(E)} = p(E),$$

so we are done. If G' is not zero, then it is d -dimensional. Since $T_{d-1}(G)$ has dimension less than d , $\alpha_d(G) = \alpha_d(G')$. The same calculation as the one we just did shows $p(G') = p(G) - \frac{P(T_{d-1}(G))}{\alpha_d(G)} \leq p(G)$. By assumption, $p(E) \leq p(G')$, thus we find $p(E) \leq p(G)$. Then, Lemma 2.9 again gives us that $p(F) \leq p(E)$, since F is d -dimensional by assumption. This shows (1), completing the proof. \square

The next lemma is a useful criterion for morphisms between (semi-)stable sheaves. It is often contrasted with Schur's lemma in representation theory. In this comparison, stable sheaves correspond to the irreducible representations. The analogy is strengthened by Lemma 2.12 and the results of Section 2.3.

Lemma 2.11. *Let $\psi : F \rightarrow G$ be a morphism of semi-stable sheaves.*

1. If $p(F) > p(G)$, then $\psi = 0$.
2. If $p(F) = p(G)$ and F is stable, then ψ is injective or zero.
3. If $p(F) = p(G)$ and G is stable, then ψ is surjective or zero.
4. If $P(F) = P(G)$ and either F or G is stable, then ψ is zero or an isomorphism.

Proof. For (1), consider the image $\text{im}(\psi)$. Assume $\psi \neq 0$, so that $\text{im}(\psi)$ is nonzero. We then see that $p(F) \leq p(\text{im}(\psi)) \leq p(G)$ by the semi-stability assumptions. This is a contradiction.

For (2), if ψ is not injective, then $\text{im}(\psi)$ is a proper quotient of F , so we have $p(F) < p(\text{im}(\psi)) \leq p(G)$, a contradiction. One proves (3) in a similar way.

For (4), assume also that $P(F) = P(G)$ and F is stable. By (2), $F \subseteq G$ and now $P(F) = P(G)$ implies $P(G/F) = 0$. Hence $G/F = 0$ so $F = G$. The case where G is stable is proved analogously. \square

Lemma 2.12. *Every stable sheaf is simple, i.e. $\text{End}(F, F) = k$ for each stable sheaf F .*

Proof. By Lemma 2.11, $\text{End}(F, F)$ is a finite-dimensional division algebra over k (possibly non-commutative). However, the only such division algebra is k itself because k is algebraically closed.

Another proof goes as follows: suppose $\lambda \in \text{End}(F, F)$. Pick $x \in X$ and consider an eigenvalue c of $\lambda \otimes k(x) : F \otimes k(x) \rightarrow F \otimes k(x)$. Then $\lambda - c \cdot \text{id}_F$ is not an isomorphism at x , so it must be zero. Hence $\lambda = c \cdot \text{id}_F$. \square

Example 2.13. Using Lemma 2.10, we can prove that every line bundle L on a variety of dimension d is stable. Note that L is purely d -dimensional, because it is torsion-free (see Example 2.2). Suppose $L \rightarrow G$ is a d -dimensional quotient with kernel K . Since G is d -dimensional, it must have rank at least one. However, $1 = \text{rk } L = \text{rk } G + \text{rk } K$, so $\text{rk } K = 0$. Thus $K = 0$, since L is pure. It follows that $L \rightarrow G$ is an isomorphism. Thus the stability condition of Lemma 2.10 holds vacuously and we are done.

In fact, the argument shows more generally that any torsion-free sheaf of rank one on a variety is stable.

Example 2.14. Let $x \in X$ be a closed point and denote by $k(x)$ the structure sheaf of x as a sheaf on X . Any zero-dimensional sheaf is of the form $\bigoplus_{i=1}^n k(x_i)$ for some integer n and points x_1, \dots, x_n (not necessarily distinct). Such sheaves are always pure. It has Hilbert polynomial $P = n$ and reduced Hilbert polynomial $p = 1$. Thus, every zero-dimensional sheaf is semi-stable and it is stable when $n = 1$.

Note that in both of the above cases, (semi-)stability does not depend on the chosen ample line bundle, while in general the notion of stability does depend on it.

2.3. Filtrations: Harder-Narasimhan and Jordan-Hölder

In this section we prove that every sheaf can be filtered with stable (or semi-stable) factors. We may view this result as saying that there are many (semi-)stable sheaves, or that the semi-stable sheaves are the “building blocks” of arbitrary sheaves. A more formal way of saying this is that the (semi-)stable sheaves generate the Grothendieck group $K(X)$.

Lemma 2.15. *Let E be a purely d -dimensional sheaf. There is a subsheaf F which has the property that for each subsheaf $F' \subseteq E$, $p(F') \leq p(F)$ and if $p(F') = p(F)$ then $F' \subseteq F$. This sheaf is uniquely determined and is called the maximal destabilising subsheaf. Furthermore, F is semistable.*

Proof. See [14, Sec. 1.3]. The construction is as follows: consider the set of subsheaves F of E , satisfying the additional property that any subsheaf F' of E properly containing F has $p(F') < p(F)$. Among those F , take one with minimal multiplicity. \square

The filtration in Theorem 2.16 is called the *Harder-Narasimhan filtration*. It is similar in spirit to the torsion filtration and, in fact, the two can be combined (see the remarks after Thm. 2.18).

Theorem 2.16. *Each pure sheaf E has a filtration $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_{n-1} \subseteq F_n = E$ such that each F_{i+1}/F_i is semi-stable and d -dimensional and if $p_i = p(F_{i+1}/F_i)$ then we have*

$$p_0 > p_1 > p_2 > \dots > p_{n-1}.$$

Such a filtration is uniquely determined. We define $p_{\max}(E) = p_0$ and $p_{\min}(E) = p_{n-1}$.

Proof. Let E be pure of dimension d . We construct the filtration by induction on $\alpha_d(E)$. First, we let F_1 be the maximal destabilising subsheaf of E . Then E/F_1 is pure, for if it has a subsheaf of dimension less than d , its inverse image G would satisfy $p(F_1) < p(G)$, a contradiction (see the argument of Lemma 2.10).

Now $\alpha_d(E/F_1) = \alpha_d(E) - \alpha_d(F_1) < \alpha_d(E)$, so we can apply the induction hypothesis to get a Harder-Narasimhan filtration G_\bullet for E/F_1 . We let F_i be the inverse image of G_{i-1} (note that the two definitions of F_1 coincide) and then we set $F_0 = 0$. Since the quotients do not change under taking the inverse image we obtain that the factors of the filtration are semi-stable, also using that F_1 is semi-stable. To obtain the inequality on polynomials, first we use the short exact sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_2/F_1 \rightarrow 0.$$

We know $p(F_2) < p(F_1)$ because F_1 is the maximal destabilising subsheaf. Lemma 2.9 then implies that $p(F_2) > p(F_1/F_2)$. As a result, $p(F_2/F_1) < p(F_2) < p(F_1)$. The other inequalities follow by the induction hypothesis.

For uniqueness, we again argue by induction on $\alpha_d(E)$. Suppose that F_\bullet is any such filtration. Let M be the maximal destabilising subsheaf. Consider the smallest j with $M \subseteq F_j$. Then the composition $M \rightarrow F_j \rightarrow F_j/F_{j-1}$ is non-trivial, by minimality of j . Using Schur's lemma 2.11, $p(M) \leq p(F_j/F_{j-1})$. By assumption, $p(F_j/F_{j-1}) \leq p(F_1)$ and since M is maximal destabilising, $p(F_1) \leq p(M)$. Thus we have equality everywhere. Then $p(F_j/F_{j-1}) = p(F_1)$ implies $j = 1$ by assumption on the filtration F_\bullet . Thus $M \subseteq F_1$. Also, $p(F_1) = p(M)$ implies $F_1 \subseteq M$ by definition of M . Hence $F_1 = M$. Now apply the induction hypothesis to E/F_1 . \square

Even though we do not use Theorem 2.16 explicitly in this text, it is used in the proof of Theorem 3.6.

It is possible to extend Schur's lemma for stable sheaves to a more general situation, using p_{\min} and p_{\max} instead of the usual reduced Hilbert polynomial, see [14, Sec. 1.3].

Example 2.17. We use Lemma 2.15 to construct additional examples of semi-stable sheaves. Let F and G be two semi-stable sheaves with $p(F) = p(G)$. Then any extension $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is also semi-stable. We easily calculate that $p(E) = p(F)$. If E is not semi-stable, consider its maximal destabilising subsheaf E' . We must have $p(E') > p(E)$. By Lemma 2.11, the map $E' \rightarrow G$ is zero. But then $E' \subseteq F$ and we get $p(E') > p(E) = p(F)$, a contradiction.

The filtration in the next result is called the *Jordan-Hölder filtration*. Unlike the Harder-Narasimhan filtration, this one is not canonical. However, the factors of the

filtration are in fact unique, much like the Jordan-Hölder theorem appearing in group theory and module theory, hence the name of this result.

Theorem 2.18. *Let E be a semi-stable sheaf with reduced Hilbert polynomial p . Then there is a filtration $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_{n-1} \subseteq F_n = E$, such that F_{i+1}/F_i is a stable sheaf. Moreover, the sheaf $\text{gr}(F)$, defined as $\bigoplus_i F_{i+1}/F_i$ is independent of the filtration.*

Proof. Let $\mathcal{C}(p)$ be the full subcategory of coherent sheaves consisting of the semi-stable sheaves with reduced Hilbert polynomial p , together with the zero sheaf. We will prove this category is abelian. It is closed under direct sums (see Example 2.17), so we prove that it is closed under kernels and cokernels. Let $\psi : F \rightarrow G$ be any morphism. We may assume that $\text{im}(\psi) \neq 0$. Then $p(\text{im}(\psi)) \leq p(G) = p(F) \leq p(\text{im}(\psi))$, so we have equality. Now we have a short exact sequence

$$0 \rightarrow \text{im}(\psi) \rightarrow G \rightarrow \text{coker}(\psi) \rightarrow 0.$$

If $\alpha_d(\text{coker}(\psi)) = 0$, then $\alpha_d(\text{im}(\psi)) = \alpha_d(G)$, so $P(\text{im}(\psi)) = P(G)$ which implies $P(\text{coker}(\psi)) = 0$. Hence, $\text{coker}(\psi) = 0$, which is in $\mathcal{C}(p)$. Otherwise, Lemma 2.9 implies that $p(\text{coker}(\psi)) = p$. The case for $\ker(\psi)$ is easier and omitted.

Now, we also claim that $\mathcal{C}(p)$ is a Noetherian and Artinian category, i.e. any sequence of subobjects of any object must stabilise, both increasing and decreasing sequences. To see this, we note that the map $F \mapsto \alpha_d(F)$ defined on subobjects of E preserves the ordering, so any sequence of subobjects of a semistable sheaf E has length at most $\alpha_d(E)$. Now Theorem 2.18 follows from Proposition 2.19, coupled with the observation that the simple objects in $\mathcal{C}(p)$ are exactly the stable sheaves. \square

Proposition 2.19. *Suppose \mathcal{C} is an Abelian, Noetherian and Artinian category. Then every object has a filtration whose factors are simple objects of \mathcal{C} , and these factors do not depend on the choice of filtration.*

Proof. Any proof of the Jordan-Hölder theorem for modules which is sufficiently categorical in nature immediately generalises to this case. See for example [28, Thm. 32.1]. \square

Combining the torsion filtration, the Harder-Narasimhan filtration and the Jordan-Hölder filtration, we see that it is possible to filter any sheaf with stable factors. In fact, if we choose to filter only with semi-stable factors, this filtration becomes uniquely determined.

Definition 2.20. Two semi-stable sheaves E, E' are *S-equivalent* if $\text{gr}(E) \cong \text{gr}(E')$.

Note that a stable sheaf is the only semi-stable sheaf in its S-equivalence class. Indeed, a stable sheaf F can be recovered from $\text{gr}(F)$ since $F \cong \text{gr}(F)$. This will come into play when discussing moduli spaces: it will turn out that the moduli space of semi-stable sheaves actually parameterises S-equivalence classes of sheaves. Thus, this observation implies that the closed points of the moduli space of stable sheaves are in bijection with the set of stable sheaves.

2.4. μ -stability

There is another version of stability, called μ -stability (in contrast, our notion of stability is sometimes called *Gieseker-Maruyama-stability*). The advantage of using μ -stability is that it is only in terms of a single number, instead of the more complicated Hilbert polynomial. The two notions are not equivalent in general, but see the remark after Lemma 2.25.

Definition 2.21. Let E be a coherent sheaf of dimension $d = \dim X$. Then we define $\text{rk } E = \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$. We also define $\text{deg}(E) = \alpha_{d-1}(E) - \text{rk}(E) \cdot \alpha_{d-1}(\mathcal{O}_X)$.

When X is integral, there is a more usual notion of the rank of a sheaf E : it is the dimension of E_η as a $k(\eta)$ vector space, where η is the generic point. In this case, the notions coincide. For general schemes one has to be more careful: the rank is not always an integer and it depends on the choice of very ample line bundle.

Lemma 2.22. *When X is integral and E is a coherent sheaf on X , the rank of E coincides with the usual notion of rank described above. In particular, the rank is an integer.*

Furthermore, when X is smooth and integral, $\text{deg}(E)$ is an integer.

Proof. The first statement is proven in Lemma C.9. One can prove the second statement in a similar fashion, proving that $\text{deg}(E) = c_1(\mathcal{O}(1))^{d-1} \cdot c_1(E)$. We omit this calculation. \square

Definition 2.23. Let E be a sheaf of dimension d . Define $\hat{\mu}(E) = \frac{\alpha_{d-1}(E)}{\alpha_d(E)}$. For a polynomial $P = \sum_{i=0}^d \alpha_i \frac{x^i}{i!}$ with $\alpha_i \in \mathbb{Q}$ for each i and $\alpha_d \neq 0$, we define $\hat{\mu}(P) = \frac{\alpha_{d-1}}{\alpha_d}$.

Note that by definition, $\hat{\mu}(E) = \hat{\mu}(P(E))$.

Definition 2.24. A coherent sheaf E of dimension d is called μ -semi-stable if $T_{d-1}(E) = T_{d-2}(E)$ and for every proper nontrivial subsheaf F of E with $0 < \text{rk}(F) < \text{rk}(E)$, we have $\hat{\mu}(F) \leq \hat{\mu}(E)$. It is called μ -stable if strict inequality holds for all such F .

The condition that $T_{d-1}(E) = T_{d-2}(E)$ means that every subsheaf of E of dimension less than d has also dimension less than $d - 1$. The condition that $0 < \text{rk}(F) < \text{rk}(E)$ means that F and the corresponding quotient of E are both d -dimensional. This last observation, combined with Lemma 2.10 allows us to compare the two notions.

Lemma 2.25. *Let E be a coherent sheaf. If E is semi-stable, then it is μ -semi-stable. Secondly, if E is pure and μ -stable, then it is stable.*

Proof. Observe that $\hat{\mu}(E)$ is the coefficient of x^{d-1} in $p(E)$. Also, the coefficient of x^d is fixed (it is always $\frac{1}{d!}$). Thus if $p(F) \leq p(E)$ we must in particular have $\hat{\mu}(F) \leq \hat{\mu}(E)$, implying the first statement. For the second statement, strict inequality $\hat{\mu}(F) < \hat{\mu}(E)$ implies $p(F) < p(E)$ by looking at the x^{d-1} coefficient. \square

We remark that for a one-dimensional sheaf E , the notions of μ -stability and stability coincide. Indeed, E is pure when $T_0(E) = 0$, which is equivalent to $T_0(E) = T_{-1}(E) = 0$. We also see that $p(E) = \frac{1}{2}x + \hat{\mu}(E)$, so the required inequalities are equivalent. In particular, when X is a curve, μ -stability equals stability for all sheaves.

There is also a general setting in which μ -stability and μ -semi-stability coincide. In view of Lemma 2.25, this implies that stability and semi-stability coincide as well. This we investigate next.

First we notice that, by definition, the following formula holds:

$$\frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\alpha_{d-1}(E)}{\mathrm{rk}(E)} - \alpha_{d-1}(\mathcal{O}_X) = \alpha_d(\mathcal{O}_X)\hat{\mu}(E) - \alpha_{d-1}(\mathcal{O}_X).$$

Thus, the inequality $\hat{\mu}(F) \leq \hat{\mu}(E)$ is equivalent to $\deg(F)\mathrm{rk}(E) \leq \deg(E)\mathrm{rk}(F)$ (and similar for $<$). On an integral scheme, where these are both integers, this gives us an interesting result.

Proposition 2.26. *Let X be a smooth projective variety of dimension d . If a coherent sheaf E of pure dimension d is μ -semistable and $\deg(E)$ and $\mathrm{rk}(E)$ are coprime, then E is μ -stable.*

In particular, if E is semi-stable and $\gcd(\deg(E), \mathrm{rk}(E)) = 1$, then E is stable.

Proof. Suppose that $\deg(F)\mathrm{rk}(E) = \deg(E)\mathrm{rk}(F)$ for some subsheaf $F \subseteq E$. Then $\mathrm{rk}(E) \mid \deg(E)\mathrm{rk}(F)$ and so by assumption, $\mathrm{rk}(E) \mid \mathrm{rk}(F)$. Since $\mathrm{rk}(F) \leq \mathrm{rk}(E)$, we must have equality or $\mathrm{rk}(F) = 0$. But we are allowed to assume that this is not the case.

The second statement is immediate when applying Lemma 2.25. \square

2.5. Stability is an open condition

In this section we sketch the proof of a difficult but crucial result. Near the end of the proof, we use the notion of bounded families, to be introduced in Section 3.1. We also use the existence of the relative Quot-scheme. This is a generalisation of Thm. 1.7, which can also be found in [22].

Theorem 2.27. *Let $X \rightarrow Y$ be a projective morphism and L a relatively very ample line bundle. Suppose E is a sheaf on X , flat over Y . Then the set of $y \in Y$ for which $E|_y$ is stable (resp. semi-stable) with respect to $L|_y$ is open.*

Sketch of proof. Assume for simplicity that Y is connected, so that E has a constant Hilbert polynomial P and reduced Hilbert polynomial p , both of degree d . Now the idea is not so difficult: for every $y \in Y$ we see that E_y is *not* semi-stable if and only if there is a polynomial P' of degree d with $p' < p$ and a pure quotient $E_s \rightarrow G$ such that $P(G) = P'$ (in this proof, p' is the reduced Hilbert polynomial corresponding to P' , etc.). We will construct a closed subset $S_{P'}$ of Y which consists of those $s \in S$ for which such a quotient exists.

Consider the relative Quot scheme $Q_{P'} = \mathrm{Quot}_{E/X/Y}^{P'}$. The fibre above a point $y \in Y$ is the scheme $\mathrm{Quot}_{E_y}^{P'}$, which is nonempty if and only if a quotient with Hilbert polynomial

P' exists. But saying that the fibre above y is nonempty is the same as saying that y is in the image of the structure map $Q_{P'} \rightarrow Y$. The image of this map is closed since $Q_{P'}$ is projective (hence proper). Thus $S_{P'}$ is the image of $Q_{P'}$.

Thus, the open subset we are looking for is

$$\bigcap_{P'} Y \setminus S_{P'}$$

where P' runs over the polynomials such that $p' < p$. This intersection is potentially infinite. To make it finite, we reduce to the case that $Y = \text{Spec } A$ is affine. Then $L|_Y$ allows us to embed X in some \mathbb{P}_A^n , so we reduce to the case $X = \mathbb{P}_A^n$. Note that there is m such that each $E|_y$ is m -regular by Theorem 3.4, since the family of $E|_y$ is bounded by definition. Now we use Theorem 2.28 to see that the set

$$\{ G \text{ is a purely } d\text{-dimensional sheaf with } \hat{\mu}(G) \leq \hat{\mu}(P) \\ \text{and a quotient of } E|_y \text{ for some } y \in Y \}$$

is bounded. Since $p(G) < p$ implies that $\hat{\mu}(G) \leq \hat{\mu}(P)$, the above set contains

$$\{ G \text{ is a pure } d\text{-dimensional sheaf with } p(G) < p \text{ and a quotient of } E|_y \text{ for some } y \in Y \}$$

as a subset, hence the latter set is bounded as well. By Theorem 3.4 in a bounded family only finitely many Hilbert polynomials occur. \square

Theorem 2.28 (Grothendieck). *Let P be a polynomial of degree d , m an integer and μ a number. Let X be a projective scheme with fixed very ample line bundle. The set of purely d -dimensional sheaves F on X which satisfy*

1. *there is a m -regular sheaf E with Hilbert polynomial P and a surjection $E \rightarrow F$,*
2. *$\hat{\mu}(F) \leq \mu$.*

is bounded.

Sketch of proof. See [14, Lemma 1.7.9] for the case of quotients of a fixed E . To get this statement, reduce to this case by noting that all such E can be written as quotient of a fixed sheaf by Theorem 3.4. \square

3. Moduli spaces of stable sheaves

In this chapter, we give the construction of the moduli space of stable sheaves. We will start with some generalities on families of sheaves. The subsequent three sections describe the construction. Lastly, we describe the tangent sheaf of the moduli space in the presence of a universal family.

The reference for the first four sections is again [14]. The last section is original, though the results are already known, see e.g. [16].

3.1. Castelnuovo-Mumford regularity

Recall that a family of sheaves on X , parameterised by S is a sheaf E on $X \times S$, which is flat over S . If F is such a family, we have for each point s of S a sheaf $F|_s$ on $X \times \{s\} \cong X$.

Serre's theorems say that when some integer m is large enough, $F(m)$ is globally generated and its higher cohomology vanishes. A basic question is whether we can find such m which work for all sheaves in a family. The next notion is due to Mumford. Besides answering this question, it also gives a condition for sheaves to appear in a family.

Definition 3.1. Let X be a projective scheme with ample line bundle L . A sheaf E is called *m -regular* if for each $i > 0$, $H^i(X, E(m - i)) = 0$.

Lemma 3.2. *Every sheaf is m -regular for some m .*

Proof. Use Serre vanishing: $H^i(X, E(N))$ vanishes for $i > 0$ and N large enough. \square

Prop. 3.3 makes precise the claim that when E is m -regular, then m is large enough in the sense of Serre vanishing.

Proposition 3.3. *If E is m regular, then E is also m' regular, for $m' \geq m$. Furthermore, whenever $m' \geq m$, $E(m')$ is generated by its global sections and its higher cohomology vanishes.*

Proof. See [22, Sec. 2]. \square

Theorem 3.4. *Let \mathcal{S} be a set of sheaves on a projective scheme X over k with fixed very ample line bundle. The following are equivalent:*

- (i) There is a scheme S of finite type over k and a sheaf E on $S \times X$ such that $\mathcal{S} \subseteq \{E|_s \mid s \in S\}$.
- (ii) There is a scheme S of finite type over k and a family E parameterised by S such that $\mathcal{S} \subseteq \{E|_s \mid s \in S\}$.
- (iii) The set $\{P(E) \mid E \in \mathcal{S}\}$ is finite and there is an integer m such that each $E \in \mathcal{S}$ is m -regular.
- (iv) The set $\{P(E) \mid E \in \mathcal{S}\}$ is finite and there is a sheaf G on X , such that every $E \in \mathcal{S}$ is a quotient of G .

Definition 3.5. A set of sheaves satisfying the equivalent conditions above is called *bounded*.

Sketch of proof. We observe that (i) to (ii) follows by the use of a *flattening stratification* (see [14, Sec. 2.1]), which applies since X is projective.

The implication (ii) to (iii) follows if we use [22, Thm. 2.3]. Indeed, first notice that in any flat family only finitely many Hilbert polynomials occur. Now reduce to $S = \text{Spec } A$ and $X = \mathbb{P}_A^n$. Then our family is a quotient of some $\mathcal{O}(-m)^n$. The cited result allows us to calculate a number N such that the kernel is fibrewise N -regular. Here we used that only a finite amount of Hilbert polynomials occur. Using the long exact sequence of cohomology, we can now find an integer N' such that each $E|_s$ is N' -regular.

If we assume (iii), then Prop. 3.3 implies that $E(m)$ is a quotient of \mathcal{O}_X^M , where M is the maximum of the numbers $P(E, m)$. Note that M is finite because there are only finitely many $P(E)$.

For (iv) implies (i), we take S to be the disjoint union of Quot_G^P , where P ranges over the finitely many $P(E)$ that occur. \square

Thus, if there is a fine moduli space of stable sheaves with Hilbert polynomial P which is of finite type, there must be an integer m such that all stable sheaves with Hilbert polynomial P are m -regular. In fact, we will prove this first. This will supply us with a scheme S as in (i), which we can then manipulate further to construct the moduli space.

Theorem 3.6. *Fix a polynomial P . There exists a number m , such that all semi-stable sheaves with Hilbert polynomial P are m -regular.*

Sketch of proof. By a result in [15], a family \mathcal{S} of sheaves with constant Hilbert polynomial is bounded if there is a constant C such that for each E in \mathcal{S} we can find a E -regular¹ sequence of hyperplane sections H_j in X , such that $h^0(E|_{\bigcap_{j \leq i} H_j}) \leq C$ holds for each i . It turns out that for purely d -dimensional sheaves one can in fact bound

¹The definition of an E -regular sequence can be found in [14, Sec. 1.1], but will not be important for us. It turns out that almost every sequence of hyperplane sections is E -regular, hence there exists at least one. Our argument works for any such sequence.

$h^0(E|_{\bigcap_{j \leq i} H_j})$ in terms of $\alpha_d(E)$, i , d and $\hat{\mu}(p_{\max}(E))$ (see Theorem 2.16 for the definition of p_{\max}). For semi-stable sheaves, $p_{\max}(E) = p(E)$, thus this is a constant as well. The proof of this estimate is difficult and involves the Grauert-Müllich theorem. For the full details, see [14, Ch. 3]. \square

In the proof of the Grauert-Müllich theorem, another estimate comes up, which we can use to prove the following alternative definition of stability. It depends on a large integer m , but the advantage is that we only have to consider global sections instead of Hilbert polynomials. See [14, Sec. 4.4] for a proof.

Proposition 3.7. *Let p be a polynomial of degree d and let r be a positive integer. Then for sufficiently large m , the following properties are equivalent for a purely d -dimensional sheaf F with multiplicity r and reduced Hilbert polynomial p :*

1. F is semi-stable (resp. stable).
2. $r \cdot p(m) \leq h^0(F(m))$ and for all subsheaves $F' \subseteq F$ of multiplicity $0 < r' < r$, we have $h^0(F'(m)) \leq r' \cdot p(m)$ (resp. $h^0(F'(m)) < r' \cdot p(m)$).
3. For all quotient sheaves $F \rightarrow F''$ with multiplicity $0 < r'' < r$, we have $r'' \cdot p(m) \leq h^0(F''(m))$ (resp. $r'' \cdot p(m) < h^0(F''(m))$).

Moreover, if in (2) we have an F' such that we have equality, $h^0(F'(m)) = r' \cdot p(m)$ then $p(F') = p(F)$.

3.2. The construction part 1: functors

Let X be a projective scheme over k with fixed ample line bundle. Fix a polynomial P . For convenience, we repeat the definition of the moduli functor:

$$\mathcal{M}(S) = \{F \in \text{Coh}(X \times S) \mid F \text{ is flat over } S \text{ and for each } s \in S, F|_s \text{ is a semi-stable sheaf with Hilbert polynomial } P.\} / \sim$$

where $E \sim E'$ if there exists a line bundle L on S with $E \cong E' \otimes \pi_S^* L$. Furthermore, we have the subfunctor \mathcal{M}^s where we replace semi-stable by stable. We will now start proving Theorem 1.9, the proof of which will take three sections.

The entire construction will depend on a large integer m . By Theorem 3.6, we know that there is an integer m such that the family of semi-stable sheaves with Hilbert polynomial P is m -regular. This is what we will use in this section. Later, in Section 3.4, we will need stronger assumptions. In order to avoid changing our definition of m , we specify it now. Write $P = r \cdot p$, where p is the reduced polynomial corresponding to P . We pick m such that for $i = 1, 2, \dots, r$ the semi-stable sheaves with Hilbert polynomial $i \cdot p$ are all m -regular and furthermore that the conditions of Proposition 3.7 hold for p and r .

Definition 3.8. Let V be a vector space of dimension $P(m)$. Define $\mathcal{H} = V \otimes \mathcal{O}_X(-m)$. Let Q be $\text{Quot}_{\mathcal{H}}^P$, the Quot-scheme with universal quotient $q : \mathcal{H} \otimes \mathcal{O}_Q \rightarrow \mathcal{E}$.

Recall that every closed point of the Quot-scheme corresponds to a quotient of \mathcal{H} . Every semi-stable sheaf E is m -regular, and from this it follows that E can be written as a quotient of \mathcal{H} . This can be extracted from the proof of Thm. 3.4. In other words, Q is an explicit example of a scheme which parameterises a family \mathcal{E} which includes the set of semi-stable sheaves with Hilbert polynomial P . However, a stable sheaf may occur multiple times in the family, see the remark after Def. 3.11.

The next technical result is easy, but I do not know of an explicit reference. It is essentially a relative version of [25, Ex. 24.3.C] and can be proven in the same fashion.

Lemma 3.9. *Suppose that $Y \rightarrow Z$ is a morphism and $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ is an exact sequence of sheaves on Y , with G flat over Z . Let $f : Z' \rightarrow Z$ be any morphism and let $f' : Y' = Z' \times_Z Y \rightarrow Y$ denote the projection. Then $0 \rightarrow f'^*E \rightarrow f'^*F \rightarrow f'^*G \rightarrow 0$ is still exact.*

We abbreviate $\mathcal{E} \otimes \pi_X^* \mathcal{O}(m)$ by $\mathcal{E}(m)$. This is a sheaf satisfying $\mathcal{E}(m)|_x = \mathcal{E}|_x(m)$.

Lemma 3.10. *The set of points $x \in Q$ such that \mathcal{E}_x is semi-stable and $V = H^0(V \otimes \mathcal{O}_X) \rightarrow H^0(\mathcal{E}|_x(m))$ is a bijection is open.*

Proof. We know that those x where \mathcal{E}_x is semi-stable is open by Thm. 2.27. We prove that the subset where the above map is a bijection is an open subset.

Denote by K the kernel of $V \otimes \mathcal{O}_{X \times Q} \rightarrow \mathcal{E}(m)$. Since the latter sheaf is flat over Q , Lemma 3.9 says that $0 \rightarrow K|_x \rightarrow V \otimes \mathcal{O}_X \rightarrow \mathcal{E}|_x(m) \rightarrow 0$ is still exact. The associated long exact sequence shows that $h^0(K|_x) = 0$ if and only if the map $V \rightarrow H^0(\mathcal{E}|_x(m))$ is an injection. But $\dim V = h^0(\mathcal{E}|_x(m)) = P(m)$, so this means it is a bijection. Thus our statement now follows from the semi-continuity theorem, [25, Thm. 28.1.1]. \square

Definition 3.11. We denote this open subset by U , i.e. U consists of those $x \in Q$ for which $\mathcal{E}|_x$ is semi-stable and such that $H^0(\mathcal{H}(m)) \rightarrow H^0(\mathcal{E}|_x(m))$ is a bijection. Furthermore, let U^s be the open subset where $\mathcal{E}|_x$ is in addition stable and let \bar{U} be the closure of U in Q .

Let F be a stable sheaf. Then F is a quotient of $V \otimes \mathcal{O}_X$ in multiple ways. First, note that by Lemma 2.12, any automorphism of F is scalar. Therefore, if τ is any nonscalar automorphism of V and $p : V \otimes \mathcal{O}_X \rightarrow F(m)$ is a quotient, the quotients p and $p \circ \tau$ are not isomorphic. We next introduce a group action such that these points are in the same orbit. We then need to investigate whether there exists a quotient of our action.

To define the action we use a functor-of-points approach. Recall from Example 1.3 that the functor of points of the algebraic group $\mathrm{GL}(V)$ is the functor $S \mapsto \mathrm{Aut}(V \otimes \mathcal{O}_S)$.

Definition 3.12. We have a natural map $Q(S) \times \mathrm{GL}(V)(S) \rightarrow Q(S)$ by sending (q, τ) to $q \circ \pi_S^* \tau$. By the Yoneda Lemma, this gives us an action of $\mathrm{GL}(V)$ on Q . We denote this action by σ .

Lemma 3.13. *The subschemes U , U^s and \bar{U} are all preserved under σ .*

Proof. For U and U^s , this follows because the action preserves the sheaves \mathcal{E}_q (but possibly changes the quotient maps). For \bar{U} , it follows because U is preserved. \square

Lemma 3.14. *A scheme corepresents $h_U/\mathrm{GL}(V)$ if and only if it corepresents \mathcal{M} .*

Here $(h_U/\mathrm{GL}(V))(S) = h_U(S)/\mathrm{GL}(V)(S)$, i.e., we take pointwise quotients.

Proof. We prove that \mathcal{M} and $h_U/\mathrm{GL}(V)$ have equal Zariski sheafifications. Note first that there is a canonical map $\beta : h_U/\mathrm{GL}(V) \rightarrow \mathcal{M}$. Indeed, any morphism $S \rightarrow U$ induces a family of stable sheaves on S by pulling back the universal family, and this is invariant under the $\mathrm{GL}(V)$ -action.

We first prove that β is surjective, after sheafifying. Let F be a family of semi-stable sheaves parameterised by S . Abbreviate $F \otimes \pi_X^* \mathcal{O}(1)$ by $F(m)$. Then by the Cohomology and Base change theorem ([25, Thm. 28.1.6]), $\pi_{S,*} F(m)$ is locally free of rank $P(m)$. Thus, locally we can write F as a quotient of $V \otimes \mathcal{O}_S$. This implies that F is in the image of β after sheafifying.

We also see that β is injective. Suppose that $f, g : S \rightarrow U$ are two maps such that when pulling back the universal sheaf we get two flat families differing by a line bundle on S (recall that we modded out by this equivalence relation). Locally this line bundle is trivial, thus we may assume that when pulling back the universal sheaf we get the same family twice. Using that every quotient induces an isomorphism on global sections, we can build an automorphism of $V \otimes \mathcal{O}_S$ relating the two quotients. Thus β is an isomorphism after sheafification.

This implies that corepresenting $h_U/\mathrm{GL}(V)$ is the same as corepresenting \mathcal{M} , because this is the same as corepresenting the sheafification. \square

Lemma 3.15. *A scheme corepresenting $h_U/\mathrm{GL}(V)$ is a categorical quotient of U by $\mathrm{GL}(V)$.*

Proof. A map $U \rightarrow Y$ is equivariant if and only if the corresponding natural transformation $h_U \rightarrow h_Y$ factors via $h_U/\mathrm{GL}(V)$. Now combine the two universal properties. \square

To construct such a categorical quotient, we will use results from GIT (see Appendix A). Thus, we need to construct a linearised line bundle of Q and we need to show that GIT-(semi-)stability is the same as our notion of (semi-)stability.

There is a problem still. Let λI denote the automorphism $v \mapsto \lambda v$ of V (in other words, it is the matrix with only λ 's on the diagonal). Then λI acts as the identity on Q . Thus, the stabiliser of any point is infinite, and hence no point can be GIT-stable. Therefore, we want to instead consider the action of $\mathrm{SL}(V)$. The discussion before Definition 3.12 implies that the stabiliser of a stable sheaf then becomes finite.

I included all details of the following proposition, because I could not find them in the literature.

Proposition 3.16. *Let X be a scheme on which $\mathrm{GL}(V)$ acts in such a way that the group $D = \{\lambda I \mid \lambda \in \mathbb{G}_m\}$ acts trivially. We have an induced action of $\mathrm{SL}(V)$ on X by restriction. Then a quotient of X by $\mathrm{GL}(V)$ exists if and only if a quotient by $\mathrm{SL}(V)$ exists and in that case they coincide.*

Proof. Consider the group homomorphism $\varphi : \mathbb{G}_m \times \mathrm{SL}(V) \rightarrow \mathrm{GL}(V)$ defined by sending (λ, M) to $\lambda \cdot M$. This homomorphism is surjective on k -points: if M is an arbitrary matrix, then M can be written as $\varphi(\zeta, \frac{1}{\zeta}M)$, where ζ satisfies $\zeta^n = \det M$.

We will show that φ is in addition flat, which does imply that it is an epimorphism, see below. If we can show this we are done. Indeed, if $f : X \rightarrow Y$ is $\mathrm{GL}(V)$ -invariant, then it is trivially $\mathrm{SL}(V)$ -invariant. If f is $\mathrm{SL}(V)$ -invariant, then it is also $\mathbb{G}_m \times \mathrm{SL}(V)$ -invariant. Denote the action of $\mathrm{GL}(V)$ by σ , then we need to show that $f \circ \pi_X = f \circ \sigma$. However, we know this is true when composing with φ , which is an epimorphism.

To show that φ is flat, we use the generic flatness theorem [25, Ex. 24.5.M]. It follows that there is some closed point M in $\mathrm{GL}(V)$ where φ is flat. Now we use a standard translation trick to show that φ is flat at every point of $\mathrm{GL}(V)$.

From the general theory of Grothendieck topologies, flat morphisms between finite-type schemes over k which are surjective on closed points are epimorphisms. Indeed, such morphisms are coverings in the fppf-topology. See [24, tag 020K] for the results on Grothendieck topologies. \square

3.3. The construction part 2: linearised line bundles

The theory of GIT requires a linearised line bundle in order to take a quotient (see Def. A.8 and Thm. A.11). In this section we will construct such a line bundle. We will in fact construct an ample linearised line bundle, which will result in a projective quotient.

The construction depends on an integer ℓ . We will choose a specific ℓ in Section 3.4. Our line bundles will be linearised for the action of $\mathrm{GL}(V)$, but this makes them automatically linearised for the action of $\mathrm{SL}(V)$.

Definition 3.17. Let ℓ be an integer. We construct a line bundle on Q by defining

$$L_\ell = \det(\pi_{Q,*}(\mathcal{E}(\ell))).$$

Our proof strategy is as follows: first we prove that \mathcal{E} is linearised. Then we show step by step how this induces a linearisation of L_ℓ when ℓ is large enough.

Proposition 3.18. *For ℓ large enough, the line bundle L_ℓ is ample on Q .*

Proof. See [14, Sec. 2.2]. The proof depends on the construction of the Quot-scheme. \square

Next, we define the action of $\mathrm{GL}(V)$ on Q in a slightly different way, using the universal automorphism τ on $\mathrm{GL}(V)$. On $X \times Q \times \mathrm{GL}(V)$ we have the following morphism of sheaves, which is surjective, being a composition of surjections.

$$V \otimes \mathcal{O} \xrightarrow{\pi_{\mathrm{GL}(V)}^* \tau} V \otimes \mathcal{O} \xrightarrow{\pi_{X \times Q}^*} \pi_{X \times Q}^* \mathcal{E}. \quad (3.1)$$

Any quotient of $V \otimes \mathcal{O}$ with constant Hilbert polynomial P defines a morphism $Q \times \mathrm{GL}(V) \rightarrow Q$, by the universal property of Q .

Lemma 3.19. *The above map is equal to $\sigma : Q \times \mathrm{GL}(V) \rightarrow Q$, the action we defined in Def. 3.12.*

Proof. The morphisms agree on their universal elements, hence they agree in general by Yoneda's lemma. \square

The universal property of Q implies that the quotient (3.1) is isomorphic to the quotient $V \otimes \mathcal{O} \rightarrow \sigma^* \mathcal{E}$. That means that there is a isomorphism ρ making the below diagram commute.

$$\begin{array}{ccc} V \otimes \mathcal{O} & \xrightarrow{\pi_{\mathrm{GL}(V)}^* \tau} & V \otimes \mathcal{O} \\ \downarrow \sigma^* q & & \downarrow \pi_{X \times Q}^* q \\ \sigma^* \mathcal{E} & \xrightarrow{\rho} & \pi_{X \times Q}^* \mathcal{E} \end{array}$$

Lemma 3.20. *The isomorphism ρ is a linearisation of \mathcal{E} .*

Proof. On $X \times Q \times \mathrm{GL}(V) \times \mathrm{GL}(V)$ define automorphisms τ_1 and τ_2 of $V \otimes \mathcal{O}$ respectively by pulling back τ from the first and second factor of $\mathrm{GL}(V)$, respectively. Consider the quotient $(\pi_{X \times Q}^* q) \circ \tau_1 \circ \tau_2$. The functor of points of $\mathrm{GL}(V)$ gives that $(\mathrm{id}_{X \times Q} \times \mu)^* \tau = \tau_1 \circ \tau_2$. The universal property of Q now states that the following diagram commutes. We suppress subscripts for readability.

$$\begin{array}{ccc} V \otimes \mathcal{O} & \xrightarrow{\tau_1 \circ \tau_2} & V \otimes \mathcal{O} \\ \downarrow (\mathrm{id} \times \mu)^* \sigma^* q & & \downarrow (\mathrm{id} \times \mu)^* \pi^* q \\ (\mathrm{id} \times \mu)^* \sigma^* \mathcal{E} & \xrightarrow{(\mathrm{id} \times \mu)^* \rho} & (\mathrm{id} \times \mu)^* \pi^* \mathcal{E} \end{array}$$

In a similar fashion, we get the following diagrams. Again we have suppressed various subscripts.

$$\begin{array}{ccc} V \otimes \mathcal{O} & \xrightarrow{\tau_2} & V \otimes \mathcal{O} & & V \otimes \mathcal{O} & \xrightarrow{\tau_1} & V \otimes \mathcal{O} \\ \downarrow (\sigma \times \mathrm{id})^* \sigma^* q & & \downarrow (\sigma \times \mathrm{id})^* \pi^* q & & \downarrow \pi^* \sigma^* q & & \downarrow \pi^* \pi^* q \\ (\sigma \times \mathrm{id})^* \sigma^* \mathcal{E} & \xrightarrow{(\sigma \times \mathrm{id})^* \rho} & (\sigma \times \mathrm{id})^* \pi^* \mathcal{E} & & \pi^* \sigma^* \mathcal{E} & \xrightarrow{\pi^* \rho} & \pi^* \pi^* \mathcal{E}. \end{array}$$

Note that the last two diagrams paste together to form the first. This implies that $(\mathrm{id} \times \mu)^* \rho = \pi^* \rho \circ (\sigma \times \mathrm{id})^* \rho$, which is the cocycle condition from Def. A.8. \square

Note that $\mathcal{E}(\ell)$ is also linearised, this follows because the action of $\mathrm{GL}(V)$ on $X \times Q$ fixes X .

Lemma 3.21. *For ℓ large enough, $\pi_{Q,*} \mathcal{E}(\ell)$ is a linearised sheaf.*

Proof. By the Cohomology and Base Change theorem, we have that

$$\sigma^* \pi_{Q,*} \mathcal{E}(\ell) \cong \pi_{Q \times \mathrm{GL}(V),*} \sigma_X^* \mathcal{E}(\ell) \cong \pi_{Q \times \mathrm{GL}(V),*} \pi_{Q \times X}^* \mathcal{E}(\ell) \cong \pi_Q^* \pi_{Q,*} \mathcal{E}(\ell)$$

when ℓ is large enough. Here the second isomorphism comes from the fact that $\mathcal{E}(\ell)$ is linearised and the first and third come from Cohomology and Base Change. The above isomorphism gives a linearisation of $\pi_{Q,*}\mathcal{E}(\ell)$. We omit the complete verification of the cocycle condition; it follows from the cocycle condition for $\mathcal{E}(\ell)$ and various natural isomorphisms coming from the theory of Cohomology and Base Change. \square

Lemma 3.22. *For ℓ large enough, L_ℓ is a linearised line bundle.*

Proof. Again by Cohomology and Base change, $\pi_{Q,*}\mathcal{E}(\ell)$, is locally free for ℓ large enough. Its determinant then becomes linearised as well, because taking the determinant is a functor which commutes with pullback. Thus L_ℓ is a linearised line bundle on Q . \square

3.4. The construction part 3: stability

Lastly, we need to analyse which points of \bar{U} are GIT-stable under the action of $\mathrm{SL}(V)$. For this, we plan to use the Hilbert-Mumford criterion A.15.

Consider 1-PS λ of $\mathrm{GL}(V)$, which is by definition a homomorphism $\lambda : \mathbb{G}_m \rightarrow \mathrm{GL}(V)$. In other words, this is an action of \mathbb{G}_m on V . It turns out that it is not difficult to describe all such actions. Let us first give an example: given $t \in \mathbb{G}_m$, we set $t \cdot v = t^n v$, where on the left, we use the multiplication by scalars. We call this the *action of weight n* . These are not all the actions, but this is almost true:

Lemma 3.23. *Suppose \mathbb{G}_m acts on V . Then there exists a decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where each V_n is closed under the action and \mathbb{G}_m acts on V_n with weight n .*

Proof. See [11, Prop. 3.12]. \square

Now of course, almost all V_n are zero. To make the above result more concrete: it implies that there exists a basis e_i of V and integers n_i such that $t \cdot e_i = t^{n_i} e_i$. Thus for this basis, the action looks like a diagonal matrix. Hence, one can view Lemma 3.23 as a diagonalisation of the action of λ .

If we have a 1-PS $\mathbb{G}_m \rightarrow \mathrm{SL}(V)$, we get the same results, with the additional requirement that $\prod_i t^{n_i} = 1$, or in other words, $\sum_i n_i = 0$. More abstractly, we may phrase this as $\sum_n n \cdot \dim(V_n) = 0$. From now on we will restrict to the action of $\mathrm{SL}(V)$.

Suppose now that q is a point of Q , represented by a quotient sheaf F . The decomposition of V does not carry over to F . Lemma 3.25 below implies that this only happens if F is fixed under the action of \mathbb{G}_m . Instead, we get a filtration, as follows.

Definition 3.24. Let $F_{\leq n}$ be the filtration of F given by the images of $\bigoplus_{i \leq n} V_i \otimes \mathcal{O}(-m)$. Define $F_n = F_{\leq n} / F_{\leq n-1}$.

It is not hard to see that F_n is a quotient of $V_n \otimes \mathcal{O}(-m)$, thus, $\bigoplus_{n \in \mathbb{Z}} F_n$ is a quotient of $V \otimes \mathcal{O}(-m)$. So instead of a decomposition of F , we have this associated sheaf which has a natural decomposition. The relation between F and this associated sheaf can be clearly expressed in terms of GIT.

Lemma 3.25. *The sheaf $\bigoplus_{n \in \mathbb{Z}} F_n$ is the limit of $t \cdot F$ as $t \rightarrow 0$. The weight of the action with respect to L_ℓ is $-\sum_{n \in \mathbb{Z}} n \cdot P(F_n, \ell)$.*

Proof. Omitted, see [14]. □

Of course, for this to work, ℓ needs to be large enough in the sense of Section 3.3. The weight is now expressed in terms of F_n , but we prefer that it is expressed in terms of $F_{\leq n}$. Indeed, the latter are subsheaves of F and we hope to be able to use the (semi-)stability condition. Let us explain how to do this. First, we use that $\sum_n n \cdot \dim(V_n) = 0$, which we know because our action is by $\mathrm{SL}(V)$. Then we get

$$\begin{aligned} \sum_{n \in \mathbb{Z}} n \cdot P(F_n, \ell) - \frac{P(F, \ell)}{\dim V} \sum_n n \cdot \dim(V_n) \\ = \frac{1}{\dim V} \sum_{n \in \mathbb{Z}} n \cdot (\dim(V)P(F_n, \ell) - \dim(V_n)P(F, \ell)). \end{aligned}$$

The next rewrite step becomes easier if we introduce some notation. So, we define $a_n = \dim(V)P(F_n, \ell) - \dim(V_n)P(F, \ell)$ and $b_n = \dim(V)P(F_{\leq n}, \ell) - \dim(V_{\leq n})P(F, \ell)$. Then $a_n = b_n - b_{n-1}$. Furthermore, for $|n|$ large enough, $a_n = b_n = 0$. Therefore, the all sums appearing in the next calculation are actually finite.

$$\sum_{n \in \mathbb{Z}} n \cdot a_n = \sum_{n \in \mathbb{Z}} n \cdot (b_n - b_{n-1}) = \sum_{n \in \mathbb{Z}} n \cdot b_n - \sum_{n \in \mathbb{Z}} (n+1)b_n = - \sum_{n \in \mathbb{Z}} b_n.$$

Thus, we can now conclude that the Hilbert-Mumford weight of λ for a point represented by F is $\sum_n \dim(V)P(F_{\leq n}, \ell) - \dim(V_{\leq n})P(F, \ell)$. In the next proposition we give a criterion for this to be non-negative.

Lemma 3.26. *Let F be a quotient of $V \otimes \mathcal{O}(-m)$ with quotient map q . Then F is GIT-semi-stable with respect to L_ℓ if and only if for each subspace $V' \subseteq V$, the induced subsheaf $F' := q(V' \otimes \mathcal{O}(-m))$ satisfies*

$$\dim V \cdot P(F', \ell) \geq \dim V' \cdot P(F, \ell). \quad (3.2)$$

For GIT-stability, replace \geq by $>$ in the above inequality.

Proof. This will be an application of Theorem A.15. Note that it applies since \bar{U} is projective, hence proper.

If every such subsheaf satisfies the condition, then our formula of the Hilbert-Mumford weight above clearly implies that F is semi-stable.

On the other hand, suppose that $V' \subseteq V$ violates the condition. Choose a complement V'' of V' in V . Define an action of \mathbb{G}_m on V by setting $t \cdot v = t^{-\dim(V'')}v$ on V' and $t \cdot v = t^{\dim(V')}v$ on V'' . This defines a 1-PS of $\mathrm{SL}(V)$. Plugging this into the formula above shows that the Hilbert-Mumford weight is negative. □

Definition 3.27. Let $\rho : V \otimes \mathcal{O}(-m) \rightarrow F$ be a quotient and let $F' \subseteq F$ be a subsheaf. We write $V \cap H^0(F'(m))$ for $H^0(\rho(m))^{-1}H^0(F'(m))$.

This is just a convenient notation. When $H^0(\rho(m))$ is an isomorphism, as it is for sheaves in U , the notation makes the most sense, since there is an isomorphism between $H^0(F'(m))$ and $V \cap H^0(F'(m))$. In general, one needs to be a bit more careful.

Lemma 3.28. *For ℓ large enough, a point ρ is GIT-semi-stable with respect to L_ℓ if and only if for all proper subsheaves $F' \subseteq F$ and $V' = V \cap H^0(F'(m))$, we have that*

$$\dim(V) \cdot P(F') \geq \dim(V') \cdot P(F). \quad (3.3)$$

For stability, again replace \geq by $>$.

Proof. Again omitted, see [14, Sec 4.4]. Note that Lemma 3.28 holds for arbitrary F' , not just ones which are induced by subspaces of V . \square

Now we can fix ℓ , we pick it large enough so that the above Lemma holds. (It is implicit in the Lemma that for such ℓ , L_ℓ is an ample linearised bundle.)

Theorem 3.29. *A point $q \in \bar{U}$ is GIT-semi-stable if and only if $q \in U$. It is GIT-stable if and only if $q \in U^s$.*

Proof. We first prove the reverse direction. Let $q : \mathcal{H} \rightarrow E$ be a point in U , i.e., E is semi-stable. Suppose $F \subseteq E$ is a subsheaf with multiplicity r' . Let $V' = V \cap H^0(F(m))$, as in Lemma 3.28. Because $q(m)$ is an isomorphism by assumption, $\dim(V') = h^0(F(m))$.

We have picked m such that Prop. 3.7 holds, and thus we can conclude that $h^0(F(m)) \leq r'p(m)$. If we have strict inequality, then we get that

$$\dim(V') \cdot r = h^0(F(m)) \cdot r < r'p(m) \cdot r = \dim(V) \cdot r'.$$

These are the leading coefficients of the polynomials in (3.3). This implies that the strict inequality holds in (3.3). In particular, if E is stable, the point corresponding to q is GIT-stable. However, if E is only semi-stable we might have equality in $h^0(F(m)) \leq r'p(m)$. In that case, Prop. 3.7 implies that F is semi-stable. By our choice of m , F is m -regular. Thus

$$\dim(V) \cdot P(F) = (rp(m)) \cdot (r'p) = (r'p(m)) \cdot (rp) = \dim(V') \cdot P(E).$$

This implies that the inequality of Lemma 3.28 still holds. We conclude that the point corresponding to q is GIT-semi-stable. Note however that if E is properly semi-stable, i.e. semi-stable but not stable, then the point q is properly GIT-semi-stable, for the argument above shows that a stable subsheaf of E (which exists by Theorem 2.18) gives equality in Lemma 3.28. Thus, for the converse it suffices to show that any GIT-semi-stable point in \bar{U} is contained in U .

For the converse, suppose that $\rho : \mathcal{H} \rightarrow E$ is a point in \bar{U} that is GIT-semi-stable. Before we go on to prove that E is semi-stable, let us state a technical lemma. Here $T_{d-1}(E)$ refers to the torsion filtration, which we introduced in Section 2.1.

Lemma 3.30. *Let $\rho : \mathcal{H} \rightarrow E$ be a point in \bar{U} . Then there is a pure sheaf F with Hilbert polynomial P and a map $E \rightarrow F$ whose kernel is $T_{d-1}(E)$.*

Proof. See [14, Sec. 4.4]. There it is shown that the result holds for a sheaf which can be deformed to a pure sheaf. Then it is shown that each quotient in \bar{U} can be deformed to a semi-stable sheaf, which is pure by definition. \square

Let F be the pure sheaf from the above Lemma. We will first prove that F is semi-stable. Let $F \rightarrow F'$ be a quotient with multiplicity r'' . We define E' to be the kernel of the composition $E \rightarrow F \rightarrow F'$. Then we set $V' = V \cap H^0(E'(m))$ and let r' be the multiplicity of E' .

The first observation we make is that E/E' is a subsheaf of F' , which implies $h^0(F'(m)) \geq h^0(E(m)/E'(m))$. Now the short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E/E' \rightarrow 0$ gives us an exact sequence

$$0 \rightarrow H^0(E'(m)) \rightarrow H^0(E(m)) \rightarrow H^0(E(m)/E'(m)) \rightarrow Q \rightarrow 0$$

for some vector space Q . In particular,

$$\begin{aligned} h^0(E(m)/E'(m)) &= h^0(E(m)) + \dim(Q) - h^0(E'(m)) \\ &\geq h^0(E(m)) - h^0(E'(m)). \end{aligned}$$

Now we use that V' is constructed as a pullback of V and $H^0(E')$ over $H^0(E)$. In particular, the sequence

$$0 \rightarrow V' \rightarrow H^0(E'(m)) \oplus V \rightarrow H^0(E(m)) \rightarrow 0$$

is exact, giving $\dim V' + h^0(E(m)) = \dim V + h^0(E'(m))$. Rewriting this, we obtain $h^0(E(m)) - h^0(E'(m)) = \dim V - \dim V'$. By definition, $\dim V = P(m) = rp(m)$. Comparing the leading coefficients in Lemma 3.28 gives us that

$$\dim(V') \cdot r \leq \dim(V) \cdot r' = rp(m) \cdot r'.$$

Thus $\dim(V') \leq r'p(m)$. Combining what we got so far, we find that

$$h^0(F'(m)) \geq h^0(E(m)) - h^0(E'(m)) = \dim(V) - \dim(V') \geq rp(m) - r'p(m).$$

If we can show that $r - r' = r''$ then Prop. 3.7 implies that F is semi-stable. This follows from a dimension analysis. From the exact sequence

$$0 \rightarrow T_{d-1}(E) \rightarrow E \rightarrow F \rightarrow F/E \rightarrow 0$$

it follows that $P(T_{d-1}(E)) = P(F/E)$ and hence that these sheaves are both of dimension less than d . If K is the kernel of $F \rightarrow F'$, then it follows that the kernel and cokernel of the induced map $E' \rightarrow K$ are also of dimension less than d . Hence, $\alpha_d(E') = \alpha_d(K)$. By definition, $r' = \alpha_d(E')$ and it is immediate that $r - r'' = \alpha_d(K)$. This shows that F is semi-stable.

Now we prove that $V \rightarrow H^0(E(m)) \rightarrow H^0(F(m))$ is injective. The kernel of this map is $V' = V \cap H^0(T_{d-1}(E(m)))$, since the kernel of $E \rightarrow F$ is $T_{d-1}(E)$. Lemma 3.28 implies that

$$\dim(V) \cdot P(T_{d-1}(E)) \geq \dim V' \cdot P(E),$$

but when $\dim V' \neq 0$, the right hand side has degree d and the left hand side has degree less than d . This is a contradiction, so our map is injective. Since F is semi-stable with Hilbert polynomial P it is m -regular and so, $h^0(F(m)) = P(m) = \dim V$. But then the composition $V \rightarrow H^0(E(m)) \rightarrow H^0(F(m))$ is an isomorphism. Since $F(m)$ is generated by its global sections, it follows that

$$V \otimes \mathcal{O}_X \rightarrow E \rightarrow F$$

is surjective. Then $E \rightarrow F$ is surjective as well, so since $P(E) = P(F)$, we find that $E = F$. Thus, E is semi-stable, as we wanted. This argument also shows that $V \rightarrow H^0(E(m))$ is an isomorphism. The proof is now finished. \square

This finishes the hard part of the proof of Theorem 1.9. We state one more result which allows us to compute the closed points of the moduli space.

Theorem 3.31. *The orbits of two points $p, q \in Q$ intersect if and only if $\mathcal{E}|_p$ and $\mathcal{E}|_q$ are S -equivalent.*

Proof. This is proven in the very last part of [14, Sec. 4.4] \square

To complete the construction, we summary what we have done in a proof of our main theorem of Section 1.2.

Proof of Thm. 1.9. Let M be the GIT-quotient of the action of $\mathrm{SL}(V)$ on the semi-stable points of \bar{U} . By Theorem 3.29, the set of GIT-semi-stable points is exactly U , thus, M is a categorical quotient of U . By Lemma 3.15, M corepresents $h_U/\mathrm{GL}(V)$ and by Lemma 3.14, this implies that M corepresents \mathcal{M} .

The proof that M^s corepresents \mathcal{M}^s is the same. To show that M^s is a coarse moduli space, we use that M^s is a geometric quotient. Theorem 3.31 and the discussion after Def. 2.20 imply that each orbit in U^s corresponds to a unique stable sheaf. Thus, M^s is a coarse moduli space. \square

We end with an example in which we can describe the moduli space explicitly.

Example 3.32. Even though it is not easy to write a moduli space down explicitly, we will provide an example when this is the case. Consider the constant polynomial $P = 1$. A sheaf E with $P(E) = 1$ is of the form $k(x)$ for some $x \in X$. Then $M = X$, with the universal family on $X \times X$ being given by \mathcal{O}_Δ .

We indicate how to construct a morphism $S \rightarrow X$ given a family E . The support of such a family is proper over S . Furthermore, above $s \in S$, the support of E consists of a single point. Thus, $\mathrm{Supp} E \rightarrow S$ is a finite map of degree one (see [25, Thm. 29.6.2]) and hence an isomorphism. Now $S \rightarrow X$ is given by the composition

$$S \rightarrow \mathrm{Supp} E \rightarrow S \times X \rightarrow X.$$

3.5. The tangent sheaf of the moduli space

Denote by M the moduli space of stable sheaves on a projective scheme X with Hilbert polynomial P and assume that M is fine, i.e., that M represents \mathcal{M}^s . Then there exists a universal family \mathcal{E} parameterised by M . We will compute the tangent bundle of M . The result, Thm. 3.41 does not require the assumption that M is fine, see [14, Sec. 10.2].

Definition 3.33. For any scheme Y , denote by $Y[\epsilon]$ the scheme $Y \times \text{Spec } k[\epsilon]$.

It is well-known that $\text{Spec } k[\epsilon]$ is closely related to the tangent space at a point, see [7, p. VI.1.3]. The schemes $Y[\epsilon]$ can be used to compute the tangent bundle.

Lemma 3.34. *Let Y be a scheme over k and U an open subset of Y . There is a natural bijection*

$$\mathcal{T}_Y(U) \cong \{\phi : U[\epsilon] \rightarrow U \mid \phi \circ i = \text{id}_U\}.$$

Here ϕ is a morphism of k -schemes and $i : U \rightarrow U[\epsilon]$ is the inclusion.

Sketch of proof. A section s of the tangent sheaf over U corresponds to a map of sheaves $\Omega_U^1 \rightarrow \mathcal{O}_U$. By the universal property of Ω_U^1 (see [4, Ch. 16]), this corresponds to a derivation $\delta : \mathcal{O}_U \rightarrow \mathcal{O}_U$. This gives a ring morphism $\mathcal{O}_U \rightarrow \mathcal{O}_U[\epsilon]$ by $x \mapsto x + \delta(x)\epsilon$. In fact, all such ring morphisms are obtained in this way. By considering affine patches, we see that these correspond to maps $U[\epsilon] \rightarrow U$. \square

Definition 3.35. If $f : Y \rightarrow Z$ is a morphism of schemes, E is a sheaf on Y and $U \subseteq Z$ is an open or closed subscheme, we write $E|_U$ for the restriction of E to $f^{-1}(U)$.

This extends the notation $E|_z$ for $z \in Z$ of Def. 1.6.

Lemma 3.36. *Let $U \subseteq M$ be open. There is a natural isomorphism between $\mathcal{T}_M(U)$ and the set of equivalence classes of sheaves F on $X \times U[\epsilon]$, which are flat over $U[\epsilon]$ and restrict to $\mathcal{E}|_U$ above U , where two sheaves F, F' are equivalent if they differ by a line bundle on $U[\epsilon]$.*

Proof. This follows by the previous proposition, combined with the universal property of M . \square

In order to describe the tangent sheaf, we have to introduce a relative version of Ext-sheaves. The reader may well be familiar with the Ext-groups and sheaves from [8, Sec. III.6], which correspond to the cases f being the structure morphism $Y \rightarrow \text{Spec}(k) = Z$ or $f = \text{id}_Y$ and $Y = Z$, respectively.

Definition 3.37. Let $f : Y \rightarrow Z$ be a morphism of schemes and suppose that E is an \mathcal{O}_Y -module on Y . Then we define the *relative Ext sheaves* $\mathcal{E}xt_f^i(E, -)$ as the i -th right derived functor of $f_* \mathcal{H}om(E, -)$ as functor between categories of \mathcal{O} -modules.

Lemma 3.38. *Suppose $f : Y \rightarrow Z$ is a morphism of schemes and E and F are sheaves on Y . We have a presheaf on Z sending U to $\text{Ext}^i(E|_U, F|_U)$. Then $\mathcal{E}xt_f^i(E, F)$ is the sheafification of this presheaf.*

Second, for any open affine subset $U = \text{Spec } A$ of Z , we have that $\mathcal{E}xt_f^i(E, F)|_U$ is the module $\text{Ext}^i(E|_U, F|_U)$ as an A -module.

If in addition f is projective and E and F are coherent, then $\mathcal{E}xt_f^i(E, F)$ is coherent.

Proof. See [1, Ch. 1]. □

We will now relate the modules of Lemma 3.36 and extensions of $\mathcal{E}|_U$ on $X \times U$. It may be helpful to recall that U and $U[\epsilon]$ share the same topological space. The same holds for $X \times U[\epsilon] = (X \times U)[\epsilon]$ and $X \times U$. So, sheaves of sets (or, sheaves on groups) on both spaces coincide. In particular, the only difference between a $\mathcal{O}_{X \times U}$ -module and a $\mathcal{O}_{X \times U[\epsilon]}$ -module is an action of ϵ .

Definition 3.39. Let $U \subseteq M$ be open. If we have an extension

$$0 \longrightarrow \mathcal{E}|_U \xrightarrow{i} F \xrightarrow{\pi} \mathcal{E}|_U \longrightarrow 0$$

of sheaves on $X \times U$, define a $\mathcal{O}_{X \times U[\epsilon]}$ -module structure on F by letting multiplication by ϵ be given by $i \circ \pi$.

Note that i and π commute with multiplication by ϵ .

Lemma 3.40. *With this definition, F is flat over $U[\epsilon]$ and restricts to $\mathcal{E}|_U$ over U .*

Proof. The restriction statement is immediate: the image of multiplication by ϵ is the image of i , and when modding out we get $\mathcal{E}|_U$ by definition.

For flatness, we have an exact sequence

$$0 \rightarrow \mathcal{O}_{U \times X} \rightarrow \mathcal{O}_{U \times X[\epsilon]} \rightarrow \mathcal{O}_{U \times X} \rightarrow 0. \quad (3.4)$$

When we tensor by F , we get the sequence

$$0 \rightarrow \text{Tor}_{X \times U[\epsilon]}^1(F, \mathcal{O}_{X \times U}) \rightarrow \mathcal{E}|_U \rightarrow F \rightarrow \mathcal{E}|_U \rightarrow 0.$$

We know that the end of the sequence is exact, so the Tor group vanishes. We have shown that F becomes flat when restricted to U . These two conditions guarantee that F is flat over $U[\epsilon]$, by [20, Thm. 49]. □

This gives us a map $\psi_U : \text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow \mathcal{T}_M(U)$. It clearly extends to a morphism of presheaves, and then to a morphism of sheaves $\psi : \mathcal{E}xt_{\pi_M}^1(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{T}_M$ by the universal property of sheafification. In fact, ψ_U is an $\mathcal{O}(U)$ -module homomorphism, see Prop. 3.44 at the end of this section. This implies that our map of sheaves is in fact an \mathcal{O}_M -module homomorphism. Now we need to show that it is an isomorphism.

Theorem 3.41. *The morphism ψ defines an isomorphism between the tangent sheaf \mathcal{T}_M and $\mathcal{E}xt_{\pi_M}^1(\mathcal{E}, \mathcal{E})$.*

Proof. We first show that ψ is surjective. Consider an element of $\mathcal{T}_M(U)$, i.e., a sheaf F on $X \times U[\epsilon]$, flat over $U[\epsilon]$, which restricts to $\mathcal{E}|_U$ over U . If we tensor (3.4) with F , we find an exact sequence $0 \rightarrow \mathcal{E}|_U \rightarrow F \rightarrow \mathcal{E}|_U \rightarrow 0$, since we assumed that F restricts to $\mathcal{E}|_U$ over U . I claim that F is isomorphic to ψ_U applied to the exact sequence. Proving this means checking that multiplication on F is given by the map $F \rightarrow \mathcal{E}|_U \rightarrow F$. This is true, since multiplication by ϵ is obtained by tensoring

$$\mathcal{O}_{X \times U[\epsilon]} \rightarrow \mathcal{O}_{X \times U} \rightarrow \mathcal{O}_{X \times U[\epsilon]}$$

with F , which is what we wanted. Thus the map is surjective on the level of presheaves, and sheafification preserves arbitrary colimits and thus preserves epimorphisms (see [19, Thm. III.1]).

Next we show that ψ is injective. Suppose we have two extensions F and F' in $\text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U)$ which map to the same element in $\mathcal{T}_M(U)$. Then by definition, they differ by a line bundle L on $U[\epsilon]$. Pick a Zariski cover \mathcal{U} of U such that L is trivial on each element of \mathcal{U} . Then we find that when restricted to elements of \mathcal{U} , F and F' are equal. Hence F and F' are equal in the sheafification $\mathcal{E}\text{xt}_{\pi_M}^1(\mathcal{E}, \mathcal{E})$. This implies that the map $\mathcal{E}\text{xt}_{\pi_M}^1(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{T}_M(U)$ is injective. \square

Proposition 3.42. *Let $m \in M$ be a point representing a stable sheaf F . Then $T_m M$ is naturally isomorphic to $\text{Ext}^1(F, F)$.*

The proof is exactly the same as the proof of Theorem 3.41. In fact, it is a little easier, because there are no non-trivial line bundles on $k[\epsilon]$, so we won't have to worry about those.

Let us relate the Proposition 3.42 and Theorem 3.41.

Proposition 3.43. *Let m be a point in M and let $U = \text{Spec } A$ be an open affine subset of M . The map $\text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow \text{Ext}^1(\mathcal{E}|_m, \mathcal{E}|_m)$ is given by tensoring by $k(m)$.*

More explicitly, the sequence $0 \rightarrow \mathcal{E}|_U \rightarrow \mathcal{F} \rightarrow \mathcal{E}|_U \rightarrow 0$ is sent to the sequence $0 \rightarrow \mathcal{E}|_m \rightarrow \mathcal{F}|_m \rightarrow \mathcal{E}|_m \rightarrow 0$. This sequence is again exact because of Lemma 3.9.

Proof. We have an inclusion $k[\epsilon] \rightarrow U[\epsilon]$. Indeed, any $\mathcal{O}_{X \times U[\epsilon]}$ -algebra \mathcal{F} just gets pulled back to $X[\epsilon]$. This is exactly what we have written above. \square

Proposition 3.44. *The map ψ_U is a $\mathcal{O}_M(U)$ -module homomorphism.*

Before we go on with the proof, let us first explain how addition of sections of $\mathcal{T}_M(U)$ is defined. Consider the space $U[\epsilon, \epsilon']$, which is formed by adjoining two elements of square zero. There are three possible maps $U[\epsilon] \rightarrow U[\epsilon, \epsilon']$, which correspond to modding out by ϵ , modding out by ϵ' and modding out by $\epsilon - \epsilon'$. Let's call these maps i_0 , i_1 and δ respectively. One can verify that the following diagram is a pushout:

$$\begin{array}{ccc} U & \longrightarrow & U[\epsilon] \\ \downarrow & & \downarrow i_0 \\ U[\epsilon] & \xrightarrow{i_1} & U[\epsilon, \epsilon'] \end{array}$$

Given two sections, which correspond to two maps $f, g : U[\epsilon] \rightarrow U$, they induce a map $(f \amalg g) : U[\epsilon, \epsilon'] \rightarrow U$ by the above pushout diagram. The sum of f and g is then defined as $\delta \circ (f \amalg g)$. One can derive this addition formula explicitly from Lemma 3.34, or one can perform a similar calculation as in [7, Sec. VI.1.3].

Sketch of proof. Suppose we have two extensions $F, F' \in \text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U)$. Then we have an extension $F \oplus F' \in \text{Ext}^1(\mathcal{E}|_U \oplus \mathcal{E}|_U, \mathcal{E}|_U \oplus \mathcal{E}|_U)$. Composing with the canonical map $\mathcal{E}|_U \oplus \mathcal{E}|_U \rightarrow \mathcal{E}|_U$, we obtain an extension $F'' \in \text{Ext}^1(\mathcal{E}|_U \oplus \mathcal{E}|_U, \mathcal{E}|_U)$. In other words, we have an exact sequence

$$0 \longrightarrow \mathcal{E}|_U \oplus \mathcal{E}|_U \xrightarrow{j_0 + j_1} F'' \xrightarrow{\pi} \mathcal{E}|_U \longrightarrow 0 .$$

We give this sheaf a multiplication by ϵ and ϵ' . Indeed, we let multiplication by ϵ be given by $j_0 \circ \pi$ and multiplication by ϵ' be given by $j_1 \circ \pi$. Now it can be verified that modding out by ϵ gives us back F' and modding out by ϵ' gives us back F . This implies the following: the induced pushout $f \amalg g$ corresponding to F and F' is given by F'' .

Furthermore, modding out by the image of $j_0 - j_1$ corresponds to modding out by $\epsilon - \epsilon'$. Doing this gives us again an extension of $\mathcal{E}|_U$ by $\mathcal{E}|_U$. The process we have gone through is exactly the definition of the Baer sum, i.e., the sum defined on the Ext-group (see [26, Sec. 3.4]). But it is also the module corresponding to $\delta \circ (f \amalg g)$. Using our correspondence between multiplication by ϵ and module structures, this implies that ψ_U is a group homomorphism.

The fact that ψ_U commutes with multiplication is easier and is omitted. \square

4. Sheaves on K3 surfaces

In this chapter, we will consider the special case where X is a K3 surface. The main tools to study them are Serre duality and the Hirzebruch-Riemann-Roch theorem. A general reference on K3 surfaces is [13], which also includes a chapter on their moduli spaces of sheaves.

It also seems to be impossible to write about K3 surfaces without mentioning their name origins. André Weil named them after Kodaira, Kummer and Kähler and, in addition, “*la belle montagne K2 au Cachemire*” [27]. Thus, contrary to popular belief among Dutch students, they are *not* related to the Belgian-Dutch kids’ music group of the same name.

4.1. K3 surfaces and their basic properties

In this section we introduce K3 surfaces and explain some of their cohomological properties. Since we are working over $k = \mathbb{C}$, we may also consider the associated complex manifold. This allows us to use singular cohomology and the Hirzebruch-Riemann-Roch theorem C.8.

Definition 4.1. A *K3 surface* over k is a smooth projective surface X over k such that $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$.

A *complex K3 surface* is a two-dimensional, compact connected complex manifold such that $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$.

From now on, we let X denote a K3 surface with a fixed very ample line bundle. Usually, we abuse notation and denote the associated manifold with X as well.

Let us give a little intuition about the condition $H^1(X, \mathcal{O}_X) = 0$. On a simply connected complex manifold X , $\pi_1(X) = 0$ and hence $H_1(X, \mathbb{Z})$, the abelianisation of $\pi_1(X)$, vanishes. By the universal coefficient theorem for cohomology, $0 \cong H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})$. By Thm. B.10, $H^1(X, \mathcal{O}_X) = 0$. Thus this condition may be viewed as an algebraic analogue of being simply connected. In fact, all complex K3 surfaces are simply connected, but this is not easy to prove, see [13, Ch. 7].

Example 4.2. Let X be a smooth quartic in \mathbb{P}^3 , i.e. X is defined by a smooth polynomial, homogeneous of degree 4. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_X \rightarrow 0.$$

This induces a long exact sequence in cohomology, a part of which is

$$H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^2(\mathbb{P}^3, \mathcal{O}(-4)).$$

Since the outer two terms are zero, we find that the middle term is zero. Secondly, the adjunction formula [25, Ex. 21.5.B] implies that

$$\omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}(4))|_X \cong (\mathcal{O}(-4) \otimes \mathcal{O}(4))|_X \cong \mathcal{O}_X.$$

Thus, X is a K3 surface. To give an explicit example, we may take the zero set of $x_0^4 + x_1^4 + x_2^4 + x_3^4$, a scheme which is known as the *Fermat quartic*.

As mentioned above, every K3 surface gives rise to a complex K3 surface, by the GAGA correspondence (for more about GAGA, see [23]). All complex K3 surfaces obtained in this way are projective. Conversely, suppose X is a scheme whose analytification is a complex K3 surface. Then X must be smooth and proper over k , but then X it follows that X is projective [17, Sec. 9.3]. So X must be a K3 surface. As a result, non-projective complex K3 surfaces do not correspond to K3 surfaces. There are examples of complex K3 surfaces, the *Kummer surfaces*, which are sometimes non-projective, see [13, Sec. 1.3].

Lemma 4.3. *When X is a K3 surface, $\Omega_X^1 \cong \mathcal{T}_X$ and there is an alternating nowhere degenerate pairing $\mathcal{T}_X \times \mathcal{T}_X \rightarrow \mathcal{O}_X$.*

Proof. There is an alternating nowhere degenerate morphism $\Omega_X^1 \times \Omega_X^1 \rightarrow \Omega_X^2 = \omega_X$ defined by sending (x, y) to $x \wedge y$. Since $\omega_X \cong \mathcal{O}_X$, this shows $\Omega_X^1 \cong (\Omega_X^1)^\vee = \mathcal{T}_X$. We had an alternating nowhere degenerate pairing on Ω_X^1 which induces one on \mathcal{T}_X using this isomorphism. \square

The condition $\omega_X \cong \mathcal{O}_X$ makes Serre duality even more useful. Recall that Serre duality states that the Yoneda cup product composed with the trace map

$$\mathrm{Ext}^i(E, F) \times \mathrm{Ext}^{n-i}(F, E \otimes \omega_X) \rightarrow \mathrm{Ext}^n(E, E \otimes \omega_X) \rightarrow k$$

is a perfect pairing (here n is $\dim X$). For K3 surfaces, the ω_X disappears and $n = 2$. Rewriting this in terms of the dual vector space, this gives us

$$\mathrm{Ext}^i(E, F) \cong \mathrm{Ext}^{2-i}(F, E \otimes \omega_X)^\vee \cong \mathrm{Ext}^{2-i}(F, E)^\vee. \quad (4.1)$$

Since X is a complex manifold as well, we can consider its singular cohomology. We can calculate the dimensions of the cohomology groups of X . We start with some of the singular cohomology groups.

Lemma 4.4. *We have that $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$. Lastly, $H^2(X, \mathbb{Z})$ is torsion-free, hence a free group.*

Proof. Most of these statements follow from Poincaré duality [9, Thm. 3.30], combined with the long exact sequence associated to

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0. \quad (4.2)$$

Indeed, $H^0(X, \mathbb{Z}) = \mathbb{Z}$ since X is connected. Second, we know $H^4(X, \mathbb{Z}) \cong H_0(X, \mathbb{Z}) \cong \mathbb{Z}$. Also, we know that $H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X^*)$ is surjective, so the long exact sequence shows that $H^1(X, \mathbb{Z})$ is a subgroup of $H^1(X, \mathcal{O}_X) = 0$. Using another form of Poincaré duality [9, Prop. 3.38] shows that $H^3(X, \mathbb{Z})$ is zero up to torsion.

To deal with the torsion, we refer to [13, Ch. 1]. This involves first proving that the Picard group is torsion-free. \square

Lemma 4.5. *The holomorphic Euler characteristic of X , $\chi(X, \mathcal{O}_X)$, is equal to 2.*

Proof. This follows from Serre duality (4.1). Indeed, we know $H^0(X, \mathcal{O}_X) = k$, as is true for all projective schemes and $H^1(X, \mathcal{O}_X) = 0$. Now Serre duality gives

$$k \cong H^0(X, \mathcal{O}_X) \cong H^2(X, \mathcal{O}_X)^*.$$

which implies $\chi(X, \mathcal{O}_X) = 1 - 0 + 1 = 2$. \square

In order to extract more information about the cohomology of X , we will use the Chern classes and Chern characters from Appendix C. The results there hold for general projective smooth schemes. In case of a surface, it is easy to explicitly describe the Chern character and Todd class in terms of the Chern classes.

Lemma 4.6. *On a surface X , we have, for a coherent sheaf E :*

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{1}{2}c_1(E)^2 - c_2(E).$$

Furthermore, we can express the Todd class of X as:

$$\mathrm{td}(X) = 1 + \frac{1}{2}c_1(\mathcal{T}_X) + \frac{c_1(\mathcal{T}_X)^2 + c_2(\mathcal{T}_X)}{12}.$$

Sketch of proof. We already know first formula when E is a line bundle by Prop. C.3. For the general case, use the splitting principle C.5 together with Prop. C.4.

The second formula is just (C.1) when X is a surface. \square

In general, the Chern classes are rational cohomology classes. For a K3 surface, one can in fact show that they are integral. For this, one needs that $c_1(E)^2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ is an even integer. This follows from [13, Prop. 1.2.4]. We remark that from the formula it follows that knowing $\mathrm{ch}(E)$ is equivalent to knowing $\mathrm{rk}(E)$ and $c(E)$.

Lemma 4.7. *We have that $c_1(\mathcal{T}_X) = 0$ and $c_2(\mathcal{T}_X) = 24$. As a result, $\mathrm{td}(X) = (1, 0, 2)$.*

Proof. On a K3 surface, $\mathcal{T}_X = (\Omega_X^1)^\vee \cong (\mathcal{T}_X)^\vee$ by Lemma 4.3. Thus, $\text{ch}(\mathcal{T}_X) = \text{ch}(\mathcal{T}_X)^\vee$. Considering the component in $H^2(X, \mathbb{Z})$ gives us $c_1(\mathcal{T}_X) = -c_1(\mathcal{T}_X)$. Since $H^2(X, \mathbb{Z})$ is torsion-free by Lemma 4.4, $c_1(\mathcal{T}_X) = 0$.

Now, we use the Hirzebruch-Riemann-Roch formula with $E = \mathcal{O}_X$. It gives us that

$$2 = \chi(X, \mathcal{O}_X) = \int_X \text{ch}(\mathcal{O}_X) \text{td}(X).$$

By the discussion after Prop. C.3, $\text{ch}(\mathcal{O}_X) = 1$. Thus we find that

$$2 = \int_X \text{td}(X) = \frac{c_1(\mathcal{T}_X)^2 + c_2(\mathcal{T}_X)}{12} = \frac{c_2(\mathcal{T}_X)}{12}.$$

Thus, $c_2(\mathcal{T}_X) = 24$. Now $\text{td}(X) = (1, 0, 2)$ follows from the formula of Lemma 4.6. \square

Using these results, we can now compute a more explicit version of the Hirzebruch-Riemann-Roch theorem C.8. We leave the easy verification to the reader.

$$\chi(X, E) = \frac{1}{2}c_1(E)^2 - c_2(E) + 2 \text{rk}(E). \quad (4.3)$$

Recall the notation $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ for the Hodge numbers of X . When X is a K3 surface, we can calculate these numbers.

Lemma 4.8. *We have that $h^{1,1}(X) = 20$ and*

$$\begin{aligned} h^{0,0}(X) &= h^{0,2}(X) = h^{2,0}(X) = h^{2,2}(X) = 1 \\ h^{1,0}(X) &= h^{1,2}(X) = h^{0,1}(X) = h^{2,1}(X) = 0. \end{aligned}$$

Proof. In Lemma 4.5, we already computed $h^{0,0}(X) = h^{0,2}(X) = 1$ and $h^{0,1}(X) = 0$. Since ω_X is trivial, this also implies $h^{2,0}(X) = h^{2,2}(X) = 1$ and $h^{2,1}(X) = 1$. Now we use Hodge decomposition, Theorem B.10, to see that $h^{1,0}(X) = h^{0,1}(X) = 0$ and $h^{1,2}(X) = h^{2,1}(X) = 0$. Lastly, we need to compute $h^{1,1}(X)$. For this, we use Hirzebruch-Riemann-Roch. We have calculated the Chern classes of $\mathcal{T}_X \cong \Omega_X^1$ in Lemma 4.7, namely $c_1(\Omega_X^1) = 0$ and $c_2(\Omega_X^1) = 24$. Plugging this in (4.3), we find that

$$-20 = \frac{1}{2}c_1(\Omega_X^1)^2 - c_2(\Omega_X^1) + 2 \text{rk}(\Omega_X^1) = \chi(X, \Omega_X^1) = -h^1(X, \Omega_X^1) = -h^{1,1}(X).$$

Thus $h^{1,1}(X) = 20$. \square

As a result, we also find the rank of $H^2(X, \mathbb{Z})$ (see Lemma 4.4), by Theorem B.10: its rank is $h^{0,2} + h^{1,1} + h^{2,0} = 22$. Thus, for a K3 surface the main interesting structure on $H^{2\bullet}(X, \mathbb{Z})$ is its multiplication structure and the Hodge decomposition of $H^2(X, \mathbb{C})$ given by Thm. B.10. The Global Torelli theorem states that a K3 surface can in fact be recovered from this decomposition and the multiplication on $H^2(X, \mathbb{Z})$ (see [13, Ch. 7]).

Now, we turn to defining another cohomological invariant, the Mukai vector.

Definition 4.9. When X is a K3 surface, denote by $\sqrt{\mathrm{td}(X)}$ the cohomology class $(1, 0, 1)$.

Note that $\sqrt{\mathrm{td}(X)}$ has indeed the property that $\sqrt{\mathrm{td}(X)}^2 = \mathrm{td}(X)$. As with the other cohomological notions in this chapter, $\sqrt{\mathrm{td}(X)}$ exists for more general X and this definition only works for K3 surfaces. We note that $\sqrt{\mathrm{td}(X)}^\vee = \sqrt{\mathrm{td}(X)}$.

Definition 4.10. Let E be a coherent sheaf on X . The *Mukai vector* $v(E)$ of E is $\mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(E)}$.

More explicitly, let $\mathrm{ch}(E)$ correspond to the vector $(\mathrm{rk}(E), c_1(E), \frac{1}{2}(c_1(E)^2 - 2c_2(E)))$. If we abbreviate this as $(r, c_1, \frac{1}{2}(c_1^2 - 2c_2))$, then $v(E)$ just becomes $(r, c_1, \frac{1}{2}(c_1^2 - 2c_2) + r)$. As for the Chern character, knowing the Mukai vector in this case is equivalent to knowing the Chern classes and the rank.

Definition 4.11. For two vectors $v, w \in H^{2\bullet}(X, \mathbb{Z})$ we define

$$(v, w) = - \int_X v^\vee \cdot w.$$

This is a bilinear form on $H^{2\bullet}(X, \mathbb{Z})$ called the *Mukai pairing*.

Again we can give a more explicit formula: if $v = (v_0, v_2, v_4)$ and $w = (w_0, w_2, w_4)$ then $(v, w) = -v_0 \cdot w_4 + v_2 \cdot w_2 - v_4 \cdot w_0$.

Definition 4.12. Given coherent sheaves E, F we define their *Euler characteristic* $\chi(E, F)$ as $\sum_i (-1)^i \dim \mathrm{Ext}^i(E, F)$.

Note that $\chi(E) = \chi(\mathcal{O}_X, E)$, as $\mathrm{Ext}^i(\mathcal{O}_X, E) = H^i(X, E)$.

Proposition 4.13. *The Euler characteristic is additive in both arguments and it is symmetric.*

Proof. The fact that it is symmetric follows immediately from Serre duality (4.1). To prove additivity in the second argument we use the long exact sequence associated to the Ext-groups. This is similar to the proof that $\chi(E)$ is additive, see [25, Ex. 18.4.A]. Additivity in the first argument follows using symmetry. \square

Proposition 4.14. *For coherent sheaves E, F , we have the following relationship between the Mukai pairing and the Euler characteristic:*

$$\chi(E, F) = -(v(E), v(F)).$$

Proof. Expanding the right hand side, we must prove

$$\chi(E, F) = \int_X v(E)^\vee \cdot v(F) = \int_X \mathrm{ch}(E)^\vee \cdot \mathrm{ch}(F) \cdot \mathrm{td}(X).$$

Suppose that E is locally free. Then since $\text{Ext}^i(E, F) \cong \text{Ext}^i(\mathcal{O}_X, F \otimes E^\vee)$, which is isomorphic to $H^i(X, F \otimes E^\vee)$ (see [8, Sec. III.6]), we find that $\chi(E, F) = \chi(F \otimes E^\vee)$. Also, $v(E)^\vee = \text{ch}(E)^\vee \cdot \sqrt{\text{td}(E)^\vee} = \text{ch}(E^\vee) \cdot \sqrt{\text{td}(E)}$, so we can expand $v(E)^\vee \cdot v(F)$ and see that it equals $\text{ch}(E^\vee) \cdot \text{ch}(F) \cdot \text{td}(X) = \text{ch}(F \otimes E^\vee) \cdot \text{td}(X)$. Thus when E is locally free, we must prove

$$\chi(F \otimes E^\vee) = \int_X \text{ch}(F \otimes E^\vee) \cdot \text{td}(X).$$

This is the Hirzebruch-Riemann-Roch formula, so this holds. For arbitrary E we choose a locally free resolution of E and then use additivity of the left and right hand side. \square

4.2. Variants of the moduli space

Proposition 4.15. *Let $c \in H^\bullet(X, \mathbb{Z})$ be a cohomology class. The functor*

$$S \mapsto \{E \in \text{Coh}(X \times S) \mid E \text{ is flat over } S \\ \text{and for each } s \in S, E|_s \text{ has Chern character } c \text{ and is semi-stable}\} / \sim$$

is corepresentable. Here $E \sim E'$ if there is $L \in \text{Pic}(S)$ such that $E \cong E' \otimes \pi_S^ L$.*

Also, this holds when semi-stable is replaced by stable.

Sketch of proof. The Chern character of a sheaf determines its Hilbert polynomial, by the Hirzebruch-Riemann-Roch formula:

$$P(E, m) = \int_X \text{ch}(E) \cdot \text{ch}(\mathcal{O}(m)) \cdot \text{td}(X).$$

Thus, given a cohomology class c , let P be the corresponding polynomial and let M be the moduli space of stable sheaves with Hilbert polynomial P . Suppose for simplicity that there is a universal family \mathcal{E} on M . By the above lemma, the Chern characters are locally constant, thus we let U be the open (and closed) subset of $m \in M$ with $\text{ch}(\mathcal{E}|_m) = c$. Then it is clear that U represents the above functor.

In general, if there is no universal family, one has to use the universal family of the Quot-scheme instead. \square

This proof also shows that if M is a fine moduli space for the stable sheaves with Hilbert polynomial P , then we also have a fine moduli space for the sheaves with Chern character c .

Instead of fixing a Chern character c , we could also fix a Mukai vector v . Indeed, for a K3 surface, the Chern character and the Mukai vector determine each other, see the remarks after Lemma 4.6 and Def. 4.10.

Proposition 4.16. *Suppose the moduli functor of stable sheaves with Chern character c is represented by M with universal family \mathcal{E} . There is a open subset U of M which represents the moduli functor of stable vector bundles on X . Furthermore, $\mathcal{E}|_U$ is locally free.*

The moduli functor of vector bundles is simply given by the subfunctor consisting of those families E , parameterised by S , such that $E|_s$ is in addition locally free for each $s \in S$.

Proof. Consider the integer-valued function on $X \times M$ given by $x \mapsto \dim \mathcal{E} \otimes k(x)$. This function is always at least the rank of \mathcal{E} and if for some x we have equality, then \mathcal{E} is locally free at x , see [25, Sec. 13.7.4]. The rank of all \mathcal{E}_m is fixed by c , and this is the rank of \mathcal{E} as well. We call this number r .

Suppose we have $m \in M$ with $\mathcal{E}|_m$ locally free. Then the rank of $\mathcal{E}|_m$ is r at all points. Thus there is an open set $V \subseteq X \times M$ on which \mathcal{E} is locally free, and $X_m \subseteq V$. Properness of the projection to M now implies that there is an open subset W around m such that $\mathcal{E}|_W$ is locally free. Thus, the set of points in m such that $\mathcal{E}|_m$ is locally free is open and we define this to be U . Clearly U represents the intended functor.

Since $\mathcal{E}|_U$ has rank r at all points, this immediately implies that it is locally free. \square

Next, we inspect a case when the M has all desired properties: given that certain numbers are relatively prime, the moduli space of stable sheaves with Chern class c becomes fine and projective. Note that c determines the Hilbert polynomial, and hence the degree and rank. For this result, we only need that X is a smooth surface, not that it is a K3 surface.

Proposition 4.17. *Let X be a smooth projective surface. Suppose $r \in \mathbb{Z}$ and $c_1 \in H^2(X, \mathbb{Z})$ satisfy $\gcd(r, c_1 \cdot c_1(\mathcal{O}(1))) = 1$. Then for any $c_2 \in H^4(X, \mathbb{Z})$ the moduli functor \mathcal{M} of semi-stable sheaves with Chern character $c = r + c_1 + c_2$ is equal to \mathcal{M}^s and M represents both functors.*

Proof. If E is semi-stable with Chern character c , then $r = \text{rk}(E)$ by definition and $c_1 \cdot c_1(\mathcal{O}(1)) = \text{deg}(E)$ (see the proof of Lemma 2.22). Hence $\gcd(\text{rk}(E), \text{deg}(E)) = 1$ and so E is stable.

The statement that M in fact represents \mathcal{M} is closely related to Thm. 1.10. Indeed, there we also assumed that some numbers are coprime. See [14, Sec. 4.6] for the details on how to derive the rest of Proposition 4.17 from Theorem 1.10. \square

In fact, in the reference, they get away with a slightly weaker condition for this proposition, namely that

$$\gcd\left(r, c_1 \cdot c_1(\mathcal{O}(1)), \frac{1}{2}c_1(c_1 - c_1(\omega_X)) - c_2\right) = 1.$$

Of course, for a K3 surface X , $c_1(\omega_X) = 0$.

Theorem 4.18. *Under the assumptions of Prop. 4.17, $r > 1$ and X is a K3 surface, then M is irreducible and \mathcal{E} is locally free, i.e., all stable sheaves with Chern character c are locally free.*

Proof. See [14, Sec. 6.1]. \square

Example 4.19. Let $c_1 \in H^2(X, \mathbb{Z})$ be a cohomology class. We consider the moduli space of stable sheaves with Chern character $c = 1 + c_1 + \frac{1}{2}c_1^2$. This is a fine, projective moduli space by Prop. 4.17. Consider the open subset U of locally free sheaves with Chern character c (see Prop. 4.16). These vector bundles are all of rank one. Since all line bundles are stable, U represents a variant of the *Picard functor*. It is defined much like our moduli functor \mathcal{M} :

$$S \mapsto \{E \in \text{Coh}(X \times S) \mid E \text{ is flat over } S \\ \text{and for each } s \in S, E|_s \text{ is a line bundle with Chern character } c\} / \sim .$$

Here we have again $E \sim E'$ if $E \cong E' \otimes \pi_S^* L$ for some $L \in \text{Pic}(S)$. In this case, U is denoted Pic_X^c and is called the *Picard scheme* of X with Chern character c . This scheme comes up often in algebraic geometry. Here we have used that X is a surface, but the argument works for all smooth projective varieties, using the results of [14, Sec. 4.6].

For a construction of the Picard scheme over arbitrary base schemes which does not use the smoothness assumption, see [2, Ch. 8].

4.3. Moduli of stable sheaves on K3 surfaces

In this section we will show that, in the presence of a universal bundle, the moduli space of stable vector bundles on a K3 surface is always smooth and carries a natural holomorphic nowhere degenerate two-form. For this, fix a Mukai vector v , let M be the moduli space of stable sheaves with Mukai vector v . Assume M is fine and let \mathcal{E} be the universal bundle on $X \times M$.

We will only construct the two-form on the open subset of M of locally free sheaves, see Prop. 4.16. Note that this open subset is sometimes equal to M , see Thm. 4.18. The result is in fact more general. The two-form not only exists on all of M , but the assumption that M is fine is not required, see [14, Ch. 10].

Proposition 4.20. *The moduli space M of stable sheaves is smooth of dimension $2 + (v, v)$.*

Proof. Let m be a point in M and let $E = \mathcal{E}_m$. First, we find by Schur's lemma 2.11 that $\text{Hom}(E, E) \cong k$. By Serre duality,

$$\text{Ext}^2(E, E) \cong \text{Ext}^2(E, E \otimes \omega) \cong \text{Hom}(E, E)^* \cong k.$$

Thus, we find that $\chi(E, E) = 2 - \dim \text{Ext}^1(E, E)$. But $\chi(E, E) = -(v, v)$, see Prop. 4.14. Since $\dim \text{Ext}^1(E, E) = \dim T_m M$ by Prop. 3.42, the dimension of the tangent space is $2 + (v, v)$ at each point of M , which is what we want. \square

Recall that $\text{Ext}^2(E, F)$ is the set of *2-extensions* $0 \rightarrow F \rightarrow H \rightarrow G \rightarrow E \rightarrow 0$. Two such sequences are equivalent if there is a morphism between them. This is not an equivalence relation, but it generates one. We will drop the “2” in 2-extensions, simply calling such sequences extensions.

Definition 4.21. Let Y and Z be schemes and $f : Y \rightarrow Z$ a morphism. Let E, F be sheaves on Y , flat over Z . An extension

$$0 \rightarrow F \rightarrow H \rightarrow G \rightarrow E \rightarrow 0 \quad (4.4)$$

is called *flat over Z* if H and G are flat over Z .

We will mostly be interested in the case $Z = M$ and $Y = X \times M$. Note that the terminology does not mention f , but this will not cause confusion. In fact, we will usually call such extensions just *flat* and it will be clear from context what is meant.

Suppose we are in the setting of Def. 4.21. Denote by Ext' the set of all flat 2-extensions of E through F , where two extensions are equivalent if there is a morphism between them (again this generates an equivalence relation). There is a natural map $\text{Ext}' \rightarrow \text{Ext}^2(E, F)$ by sending an extension to itself. This is clearly well-defined. We will formulate a condition which implies it is an isomorphism.

Lemma 4.22. *Let $f : Y \rightarrow Z$ be a morphism, as above. The map $\text{Ext}' \rightarrow \text{Ext}^2(E, F)$ is an isomorphism if for every sheaf G on Y , there is a surjection $G' \rightarrow G$ with G' flat over Z .*

Proof. We first prove surjectivity of our map. Let

$$0 \rightarrow F \rightarrow H \rightarrow G \rightarrow E \rightarrow 0$$

be an extension. Let $G' \rightarrow G$ be a surjection from a sheaf flat over Z to G . Then, let H' be the fibre product of H and G' over G . It is easy to see that the following sequence is still exact:

$$0 \rightarrow F \rightarrow H' \rightarrow G' \rightarrow E \rightarrow 0.$$

Here $F \rightarrow H'$ is defined by the universal property of the fibre product. Now G' is flat over Z by assumption, and the formal properties of flatness imply H' is flat over Z as well. The maps $G' \rightarrow G$ and $H' \rightarrow H$ give a morphism of exact sequences, thus our old and new sequence are equivalent. This shows that $\text{Ext}' \rightarrow \text{Ext}^2(E, F)$ is surjective.

To prove injectivity, we first show that any morphism between extensions is actually constructed similarly to the fibre product we saw above. To this end, suppose that $0 \rightarrow F \rightarrow H \rightarrow G \rightarrow E \rightarrow 0$ and $0 \rightarrow F \rightarrow H' \rightarrow G' \rightarrow E \rightarrow 0$ are extensions and there is a morphism from the second to the first. We let H'' be the fibre product of H and G' over G . Then we can draw the following large diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & H' & \longrightarrow & G' & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \vdots & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & H'' & \longrightarrow & G' & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & H & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

The dotted arrow exists by universal property of the fibre product. Now we apply a variant of the five lemma to the top two rows: except for the dotted arrow, we know that all other arrows from the top row to the second are isomorphisms. This implies that the dotted arrow is an isomorphism.

Now we can show injectivity. The reader is warned that this proof is notationally very inconvenient. It suffices to show the following: if we consider extensions $0 \rightarrow F \rightarrow H \rightarrow G \rightarrow E \rightarrow 0$ (which we denote by S) and $0 \rightarrow F \rightarrow H' \rightarrow G' \rightarrow E \rightarrow 0$ (which we denote by T), a map $S \rightarrow T$, and surjections $\bar{G} \rightarrow G$ and $\bar{G}' \rightarrow G'$ from sheaves flat over Z which give rise to extensions \bar{S} and \bar{T} , then \bar{S} and \bar{T} are equivalent.

Let G'' be the fibre product of \bar{G} and \bar{G}' over G' . Applying our construction above, we get another extension $0 \rightarrow F \rightarrow H'' \rightarrow G'' \rightarrow E \rightarrow 0$, which we denote by U . The statement about morphisms of extensions we proved above shows that there are morphisms of sequences $U \rightarrow \bar{S}$ and $U \rightarrow \bar{T}$. However, U does not have to be a flat extension anymore. To solve this, we take a map $\bar{G}'' \rightarrow G''$ with \bar{G}'' flat, giving us a flat extension $\bar{U} \rightarrow U$. The compositions $\bar{U} \rightarrow \bar{S}$ and $\bar{U} \rightarrow \bar{T}$ then show that \bar{S} and \bar{T} are equivalent in Ext' . \square

Lemma 4.23. *Let $f : Y \rightarrow Z$ again be a morphism and assume that Lemma 4.22 holds. If F and G are sheaves on Y flat over Z and $z \in Z$ is a point, then there is a natural map $\text{Ext}^2(G, F) \rightarrow \text{Ext}^2(G|_z, F|_z)$, which is natural in G and F .*

Furthermore, if E is locally free, then our restriction map commutes with the map $\text{Ext}^2(G, F) \rightarrow \text{Ext}^2(G \otimes E, F \otimes E)$ given by tensoring an extension by E .

Proof. Given any extension in $\text{Ext}^2(G, F)$, we pick an equivalent flat extension, and restrict that extension. The flatness property ensures that the sequence remains exact. Again, Lemma 4.22 implies that this is well-defined. To prove naturality in G and F , note that maps of extensions are defined in terms of pullbacks and pushforwards. Since restriction is a left adjoint, it preserves pushforwards. Also, it preserves a pullback over a flat base, since the pullback of $p_1 : E \rightarrow G$ and $p_2 : E' \rightarrow G$ is the kernel of

$$E \oplus E' \xrightarrow{p_1 - p_2} G,$$

which is preserved because G is flat. The second statement is obvious given our explicit description of the restriction map. \square

Lemma 4.24. *When $Z = M$, $Y = X \times M$ and f is the projection, the conditions of Lemma 4.22 are satisfied.*

Proof. Note that M is quasi-projective over k . Since π_M is projective¹, $X \times M$ can be embedded in \mathbb{P}_M^n for some integer n . This shows that any sheaf G on $X \times M$ can be written as a quotient of $\mathcal{O}(-N)^m$ for some integers N and m . These are flat over M since they are locally free and π_M is a flat morphism. So, we can take $\bar{G} = \mathcal{O}(-N)^m$. \square

¹There are multiple notions of projective morphisms. However, when the target is quasi-projective over an affine scheme, they all coincide. See also [22]

Theorem 4.25. *Let M be the moduli space of stable vector bundles, and assume that there is a universal family \mathcal{E} on M . There is a natural alternating nowhere degenerate pairing $\mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{O}_Y$.*

Recall from Theorem 3.41 that $\mathcal{T}_M \cong \mathcal{E}xt^1(\mathcal{E}, \mathcal{E})$. This allows us to use the constructions on Ext-groups in this section.

Proof. We construct the morphism locally on affine patches of M . So, let $U = \text{Spec } A$ be an affine open of M . We construct the morphism in four stages, investigating each time what its restriction to a point $m \in U$ is.

For the first stage, we use the Yoneda cup product

$$\text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \times \text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow \text{Ext}^2(\mathcal{E}|_U, \mathcal{E}|_U).$$

To be explicit, given 1-extensions F and F' of $\mathcal{E}|_U$, we have a composition $F \rightarrow \mathcal{E}|_U \rightarrow F'$ and then we send these two extensions to $0 \rightarrow \mathcal{E}|_U \rightarrow F \rightarrow F' \rightarrow \mathcal{E}|_U \rightarrow 0$. Notice that this last extension is always flat over U . Thus, it is easy to see that when we restrict to $m \in M$, we get a map which is defined in exactly the same way. In other words, the Yoneda cup product commutes with restriction to a point $m \in M$.

In the second stage, we have a composition

$$\text{Ext}^2(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow \text{Ext}^2(\mathcal{E}|_U \otimes \mathcal{E}|_U^\vee, \mathcal{E}|_U \otimes \mathcal{E}|_U^\vee) \rightarrow \text{Ext}^2(\mathcal{O}, \mathcal{O})$$

The first is obtained by tensoring extensions with $\mathcal{E}|_U^\vee$ and the second by using functoriality for Ext with the canonical maps $\mathcal{O} \rightarrow \mathcal{E}|_U \otimes \mathcal{E}|_U^\vee$ and $\mathcal{E}|_U \otimes \mathcal{E}|_U^\vee \rightarrow \mathcal{O}$. By Lemma 4.23, we find that when we define an analogous map at a point $m \in M$, it commutes with restriction.

The third stage is a bit less intuitive, as it involves Cohomology and Base change. For this, we first note that $\text{Ext}^2(\mathcal{O}, \mathcal{O}) = H^2(X \times U, \mathcal{O})$. We now apply Flat base change [25, Thm. 24.2.8] to the structure map $U \rightarrow \text{Spec } k$ to see that

$$H^2(X \times U, \mathcal{O}) \cong H^2(X, \mathcal{O}_X) \otimes A.$$

Thus we have an isomorphism $\text{Ext}^2(\mathcal{O}, \mathcal{O}) \cong H^2(X, \mathcal{O}_X) \otimes A$. If we restrict to $m \in M$, we get a map $\text{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X)$. The theory of Cohomology and Base change ensures that this is the usual, canonical identification. (Use that $\{m\} \rightarrow U \rightarrow \text{Spec } k$ is the identity and that base-change maps can be composed.)

For the last stage, we use that $\mathcal{O}_X \cong \omega_X$. We know that there is a trace map $H^2(X, \omega_X) \rightarrow k$. We use this trace map to find a morphism $H^2(X, \mathcal{O}_X) \otimes A \rightarrow A$. Clearly, when restricted to a point $m \in M$, we get the usual trace map back.

Composing our four stages, we find a morphism $\mathcal{T}_U \times \mathcal{T}_U \rightarrow A$. At a point m , we have made sure that the resulting morphism

$$\text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \times \text{Ext}^1(\mathcal{E}|_U, \mathcal{E}|_U) \rightarrow k$$

equals the Serre duality pairing. As a consequence, our pairing is alternating and non-degenerate. Thus, we have a symplectic structure on U . Our construction clearly extends to a pairing on M . This proves Theorem 4.25. \square

As mentioned before, Theorem 4.25 is also true when M is the moduli space of stable sheaves, even in the absence of a universal family. For this, see [14, Thm. 10.4.3].

This is especially interesting when there are no strictly semi-stable sheaves with Mukai vector v . For we find that M is a Kähler manifold by Prop. B.5, and for Kähler manifolds, a symplectic structure is always closed, by Prop. B.11. A more difficult statement is the following:

Theorem 4.26. *Write $v = (v_0, v_2, v_4)$. Suppose v_2 is indivisible as a cohomology class in $H^2(X, \mathbb{Z})$. Then there exists an ample line bundle on X such that M is simply connected and the symplectic form ω spans $H^2(X, \Omega_X^2)$. That is, M is irreducible symplectic.*

Proof. See [14, Sec. 6.2]. □

In fact there are many ample line bundles which make Theorem 4.26 true. The set of these H forms an open chamber, a term we informally describe by saying that H satisfies some inequalities. See the reference for the details.

Theorem 4.27. *Suppose X is a K3 surface. Suppose $v = (v_0, v_2, v_4) \in H^{2\bullet}(X, \mathbb{Z})$ such that $(v, v) = 0$, $\gcd(v_0, v_2 \cdot c_1(\mathcal{O}(1))) = 1$ and $v_0 > 1$. Then M is a K3 surface.*

Sketch of proof. By Prop. 4.20, M is smooth of dimension 2. Also, by Prop. 4.17 and Thm. 4.18, M is a smooth projective surface and parameterises only locally free sheaves. Now Theorem 4.25 shows that M admits a nowhere degenerate two-form ω . This implies that $\Omega_M^1 \cong \mathcal{T}_M$, from which it follows that the canonical bundle ω_M satisfies $\omega_M = \omega_M^\vee$. Note that ω is a global section of ω_M . But on a projective scheme, there is only one line bundle L such that L and L^\vee have a nonzero global section, namely, $L = \mathcal{O}$. Thus, $\omega_M \cong \mathcal{O}_M$.

It is more difficult to prove $H^1(M, \mathcal{O}_M) = 0$. We sketch a proof found in [14, Sec. 6.1]. First one proves that M is a variety. Then one introduces a *Fourier-Mukai transform*, namely the function $f_{\mathcal{E}} : H^\bullet(M, \mathbb{Z}) \rightarrow H^\bullet(X, \mathbb{Z})$ by $c \mapsto \pi_{X,*}(\pi_M^* c \cdot v(\mathcal{E}))$. Here $v(\mathcal{E})$ is the Mukai vector of \mathcal{E} (as mentioned in Sec. 4.1, the Mukai vector is defined for any smooth projective variety). Just as for K3 surfaces, $v(\mathcal{E})$ takes only values in even degrees. This implies that $f_{\mathcal{E}}$ preserves the odd and even parts of $H^\bullet(M, \mathbb{Z})$ and $H^\bullet(X, \mathbb{Z})$. Since the odd part of the latter space is zero by Lemma 4.4, so is the odd part of $H^\bullet(M, \mathbb{Z})$. Now we apply Thm. B.10 to see that $H^1(M, \mathcal{O}) \subseteq H^1(M, \mathbb{Z}) = 0$ to prove what we want. □

The proof not only shows that M is a K3 surface, but also that the cohomology rings of M and X are isomorphic, that is, $f_{\mathcal{E}}$ preserves the Hodge structure and the multiplication, but not necessarily the gradation. Such K3 surfaces are called *Fourier-Mukai partners*. For a K3 surface, this is equivalent to showing that there is an equivalence between the *bounded derived categories* $\mathcal{D}(M) \rightarrow \mathcal{D}(X)$. In this case, the equivalence is given by $\mathcal{F} \mapsto \pi_{X,*}(\pi_M^* \mathcal{F} \otimes \mathcal{E})$, which is also called a *Fourier-Mukai transform*. We conclude by mentioning that a K3 surface only has finitely many Fourier-Mukai partners. We refer to [13, Ch. 16] for the details.

A. Geometric Invariant Theory

The theory of GIT is one of the main construction tools in Chapter 3 for the moduli space of stable sheaves. Here, we will summarise the important definitions and results from GIT, without proofs. The original reference for GIT is [21], in which the proofs of all statements in this chapter can be found. However, there is also an introduction, in the language of varieties, in [3]. There are also the lecture notes [11].

Definition A.1. A *group scheme over k* is a scheme G over k equipped with a *multiplication* $\mu : G \times_k G \rightarrow G$, a *unit* $e : \text{Spec } k \rightarrow G$ and an *inverse* $i : G \rightarrow G$ such that the group axioms hold (they can be stated using only commutative diagrams).

For example, the axiom $e \cdot g = e$ of groups can be stated as the following commutative diagram:

$$\begin{array}{ccccc}
 G & \xrightarrow{\cong} & G \times \text{Spec } k & \xrightarrow{\text{id}_G \times e} & G \times G \\
 & \searrow & & & \downarrow \mu \\
 & & & & G \\
 & & & \text{id}_G & \nearrow \\
 & & & & G
 \end{array}$$

By the Yoneda Lemma, this is the same as saying that the functor of points h_G of G factors through the category of groups, i.e., for each scheme T , $h_G(T)$ is a group and morphisms of schemes induce group homomorphisms.

Example A.2. It is easy to give examples of group schemes. For instance, \mathbb{A}_k^1 is a group scheme, with μ being defined by addition. Similarly, $\mathbb{A}_k^1 \setminus \{0\}$ is a group scheme with μ being multiplication. When considered as group schemes, these are usually written as \mathbb{G}_a and \mathbb{G}_m for *additive* and *multiplicative*, respectively.

We also encountered the general linear group $\text{GL}(V)$ in Chapter 1. Note that \mathbb{G}_m is a special case of $\text{GL}(V)$; it can be identified with $\text{GL}(k)$.

We have seen the functor of points of these schemes before, and indeed, they all carry natural group structures. For example, the functor of points of \mathbb{A}_k^1 is $\Gamma(-, \mathcal{O})$, which takes schemes to rings, which can be considered groups under addition.

Definition A.3. An *action* of a group scheme G on a scheme X is a morphism $X \times G \rightarrow X$, which makes the usual diagrams commute.

Suppose G acts on X and Y . A morphism $X \rightarrow Y$ is called *G -equivariant* if it commutes with the action of G .

In particular, if G acts on X and Y is an arbitrary scheme then a morphism $X \rightarrow Y$ is called *invariant* if it is equivariant with respect to the trivial action of G on Y .

The definition given here correspond to a *right action* of G on X . Left actions also exist, but in this text, all actions will be right actions. The more general notion of equivariant morphisms will not be so important to us, but invariant morphisms will be. Indeed, the goal of GIT is to find a universal invariant morphism. By definition, a morphism $f : X \rightarrow Y$ is invariant if the following diagram commutes:

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \\ \downarrow \pi_X & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

where we have denoted the action of G on X by σ .

Definition A.4. Suppose G acts on X . A *categorical quotient* of this action is a scheme Y with a map $q : X \rightarrow Y$ which is invariant and universal with respect to all invariant morphisms. In other words, if $f : X \rightarrow Z$ is invariant there is a unique $g : Y \rightarrow Z$ such that $f = g \circ q$.

Before we go on, we say what it means for various notions to be invariant.

Definition A.5. Suppose G acts on X via $\sigma : X \times G \rightarrow X$. A function $s \in \Gamma(X, \mathcal{O}_X)$ is called *invariant* if the corresponding morphism $X \rightarrow \mathbb{A}_k^1$ is invariant.

Let Y be an open or closed subset of X . We say that Y is *invariant* if the morphism $Y \times G \rightarrow X$ factors through Y .

When Y is invariant, we have an induced action of G on Y . Note that when s is an invariant function, that X_s is an invariant open subset of X and $V(s)$ is an invariant closed subset of X .

When G is an affine group scheme, and $X = \text{Spec } A$ is an affine scheme, there is an algebraic construction of a quotient, namely, we take the *ring of invariants* A^G . The following definition, taken from [21, Ch. 0], attempts to mimic this construction for schemes (although the terminology *good quotient* is not used).

Definition A.6. Suppose G acts on X . A *good quotient* of this action is a scheme Y with an invariant morphism $q : X \rightarrow Y$ satisfying:

1. For every open subset $U \subseteq Y$, the map $\Gamma(U, \mathcal{O}_Y) \rightarrow \Gamma(q^{-1}(U), \mathcal{O}_X)$ is injective and its image is the ring of invariant functions.
2. If W is a closed invariant subset of X , then $q(W)$ is closed in Y .
3. If $W_i, i \in I$ is a set of closed invariant subsets satisfying $\bigcap_{i \in I} W_i = \emptyset$, then $\bigcap_{i \in I} q(W_i) = \emptyset$.

We call q a *geometric quotient* if in addition:

4. The induced map $\Psi : X \times G \rightarrow X \times_Y X$ given by $(x, g) \mapsto (x \cdot g, x)$ is an isomorphism.

When considering a set X and an action by a group G , the points of the quotient X/G can be identified with the orbits of the action on X . For schemes, such a statement cannot be true. See Ex. 1.5 for a counterexample. The notion of a geometric quotient makes precise when this will hold. Indeed, it says that two points $x, y \in X$ map to the same point in Y if and only if there is $g \in G$ with $gx = y$. This observation allows us to work with orbits in the scheme-theoretic setting.

The next result verifies for us that the method above works, i.e., produces a categorical quotient. It is still not clear when such a quotient exists or when it is geometric.

Lemma A.7. *Any good quotient is a categorical quotient.*

Definition A.8. Let G act on X via $\sigma : X \times G \rightarrow X$. A *linearisation* of a sheaf F on X is an isomorphism $\rho : \sigma^*F \rightarrow \pi_X^*F$ on $X \times G$, such that the following diagram of sheaves on $X \times G \times G$ commute:

$$\begin{array}{ccc} (\mathrm{id}_X \times \mu)^* \sigma^* F & \xrightarrow{\hspace{10em}} & (\mathrm{id}_X \times \mu)^* \pi_X^* F \\ \parallel & & \parallel \\ (\sigma \times \mathrm{id}_G)^* \sigma^* F & \longrightarrow (\sigma \times \mathrm{id}_G)^* \pi_X^* F \xlongequal{\hspace{1em}} \pi_{12}^* \sigma^* F \longrightarrow & \pi_{12}^* \pi_X^* F \end{array} .$$

Here π_{12} denotes the projection $X \times G \times G \rightarrow X \times G$ to the first two factors. The equalities above hold because we pullback along equal maps.

A global section s of F is called *invariant* if the two sections σ^*s and π_X^*s get identified under ρ .

As we will see below, the GIT-construction requires a linearised line bundle. We will need the more general notion of a linearised sheaf in the construction of the linearised line bundle in Section 3.3.

Note that the structure sheaf \mathcal{O}_X is trivially linearised and that a function on X is invariant in this sense if and only if it is invariant in the sense of Def. A.5. When L is a linearised line bundle and s is a global section of L , then X_s is an invariant subset of X . In particular, the open subset of GIT-(semi-)stable points (see below) is invariant.

Definition A.9. An action of G on X is *closed* if for every $x \in X$, the set-theoretic image of $G \rightarrow X$ which maps g to $x \cdot g$ is closed.

Definition A.10. Let L be a linearised line bundle on X . A point $x \in X$ is called *GIT-semi-stable* if there is an integer n and an invariant section $s \in L^n$ such that X_s is affine and $x \in X_s$. If in addition we can pick s such that the action of G on X_s is closed and the stabiliser of x is finite, x is called *GIT-stable*.

The set of GIT-semi-stable points is denoted $X^{ss}(L)$ and the set of GIT-stable points is denoted $X^s(L)$.

The set of GIT-semi-stable and GIT-stable subsets is an open subset of X . An interesting result is that L is ample on $X^{ss}(L)$, by [25, Thm. 16.6.2].

The next statement is the main result of GIT. In it, there appears one notion we did not explain, the notion of a *linearly reductive group*. This is a delicate notion, with subtle

variations once the ground field is not of characteristic zero. We content ourselves with stating here that $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ are linearly reductive, as we will only need to take a quotient of an action by $\mathrm{SL}(V)$ in the main text.

Theorem A.11. *Let X be a finite-type scheme over k and let G be a linearly reductive group acting on it. Let L be a G -linearised line bundle. Then a good quotient Y of $X^{ss}(L)$ exists, where the quotient map is affine and submersive (i.e. the quotient has the quotient topology). Furthermore, there is an ample invertible sheaf M on Y which pulls back to L^n for some n , so Y is quasi-projective over k . Lastly, there is an open $U \subseteq Y$ whose inverse image is $X^s(L)$ and U is a geometric quotient of $X^s(L)$.*

Furthermore, if X is projective and L is ample on X , then Y is projective.

Note that $X^{ss}(L)$ and $X^s(L)$ are indeed invariant, so that it makes sense to take a quotient of it.

We are now left with the problem of find the GIT-(semi-)stable points. There is a numerical criterion to determine GIT-(semi-)stability, which we explain now.

Definition A.12. A 1-parameter subgroup (or 1-PS) λ of G is a morphism $\lambda : \mathbb{G}_m \rightarrow G$.

The terminology is slightly confusing: we do not actually require λ to be injective. So, the image of λ might not be \mathbb{G}_m . In particular, we allow λ to be the “zero morphism” $t \mapsto e_G$.

Definition A.13. Suppose that X is proper over k . If λ is a 1-PS of G which acts on X and $x \in X$, then the morphism $\mathbb{G}_m \rightarrow X$ given by $t \mapsto x \cdot \lambda(t)$ extends uniquely to a morphism $\mathbb{A}^1 \rightarrow X$. The image of 0 is called the *limit of x* and denoted $\lim_{t \rightarrow 0} x \cdot \lambda(t)$.

The fact that this extends follows from the properness of X , in particular from the valuative criterion of properness.

The limit $y = \lim_{t \rightarrow 0} x \cdot \lambda(t)$ is a fixed point for the induced action of \mathbb{G}_m on X . It therefore induces an action of \mathbb{G}_m on the fibre of L at y . Since the fibre at y is simply a one-dimensional vector space, this amounts to giving a morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m$. Such a morphism is simply given by an integer r , the morphism is then given by sending t to t^r (see also Lemma 3.23).

Definition A.14. We define the *Hilbert-Mumford weight* $\mu^L(x, \lambda)$ to be $-r$, where r is as above.

The next result allows us to prove stability by calculating the Hilbert-Mumford weight of various 1-PS’s λ .

Theorem A.15. *Let X be proper over k . A point x is stable if and only if $\mu^L(x, \lambda) > 0$ for each 1-PS λ . A point x is semi-stable if and only if $\mu^L(x, \lambda) \geq 0$ for each 1-PS λ .*

B. The Hodge decomposition theorem

Every smooth scheme X over \mathbb{C} gives rise to a complex manifold X^{an} . This formation gives rise to a functor and this assignment also induces an equivalence between the category of coherent sheaves on X and the analytic coherent sheaves on X^{an} . Furthermore, the sheaf cohomology of a sheaf on X and the corresponding analytic coherent sheaf coincide. For all these and related statements, see [23]. Here we state aspects of the analytic theory of complex manifolds. A reference on complex geometry is [12].

We start by considering a smooth n -dimensional manifold X . We denote its structure sheaf of smooth functions by \mathcal{A} . Recall that any such manifold has a *tangent bundle* TX , which is a vector bundle of rank n . We also recall the notion of a *differential k -form*: this is an \mathcal{A} -linear map $\bigwedge^k TX \rightarrow \mathcal{A}$. We denote by \mathcal{A}^k the sheaf of differential k -forms. Note that $\mathcal{A} = \mathcal{A}^0$. It is well-known that we can identify \mathcal{A}^k with $\bigwedge^k TX^\vee$, the exterior power of the cotangent bundle.

The sheaves of differential forms fit into the *de Rham complex*:

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \dots \rightarrow \mathcal{A}^n \rightarrow 0, \quad (\text{B.1})$$

where the differentials are given by the *exterior derivative* d . This complex is useful, because it gives rise to the *de Rham cohomology*.

Theorem B.1 (de Rham). *The cohomology of the de Rham complex (B.1) is isomorphic to the singular cohomology $H^k(X, \mathbb{R})$ of X .*

It is useful to complexify the above notions. We denote by $\mathcal{A}_{\mathbb{C}}$ the sheaf of smooth complex-valued functions on X . We define $TX_{\mathbb{C}} = TX \otimes \mathcal{A}_{\mathbb{C}}$, the *complexified tangent bundle*. Similarly, the sheaf of complex-valued differential forms $\mathcal{A}_{\mathbb{C}}^k$ is defined as the sheaf of $\mathcal{A}_{\mathbb{C}}$ -linear maps $\bigwedge TX_{\mathbb{C}} \rightarrow \mathcal{A}_{\mathbb{C}}$. We have the alternative definition $\mathcal{A}^k = \bigwedge^k (TX_{\mathbb{C}})^\vee$. Here we by $-\vee$ we mean the dual with respect to $\mathcal{A}_{\mathbb{C}}$. This discussion leads to the *complexified de Rham complex*, the differential of which we denote by $d_{\mathbb{C}}$.

Suppose now that X is a complex manifold (to prevent confusion, we do not mean the related notion of an *almost complex manifold*). If X is of complex dimension n , it is of real dimension $2n$. There are the following structure sheaves on X : \mathcal{A} and $\mathcal{A}_{\mathbb{C}}$ introduced above and the structure sheaf of holomorphic functions \mathcal{O}_X . Interestingly, the sheaves $\mathcal{A}_{\mathbb{C}}$ and \mathcal{O}_X are both defined by being sections of the trivial bundle $X \times \mathbb{C}$, the former with respect to the smooth structure and the latter with respect to the holomorphic structure.

On a complex manifold, TX is already an $\mathcal{A}_{\mathbb{C}}$ -module. Then sections of $TX_{\mathbb{C}}$ can be multiplied by i in two ways. The bundle splits into two factors

$$TX_{\mathbb{C}} = TX^{1,0} \oplus TX^{0,1}.$$

Here $TX^{1,0}$ is the subbundle where the two multiplications by i coincide and $TX^{0,1}$ is the subbundle where they differ by a sign. As complex vector bundles, $TX^{1,0} \cong TX$. We also have an isomorphism $TX^{0,1} \cong \overline{TX}$. The latter bundle is defined as having TX as underlying real vector bundle, but with multiplication by i replace by multiplication by $-i$.

Now we introduce $\mathcal{A}_{\mathbb{C}}^{p,q}$ as $\bigwedge^p (TX^{1,0})^{\vee} \otimes \bigwedge^q (TX^{0,1})^{\vee}$ (again the dual is over $\mathcal{A}_{\mathbb{C}}$ here). This sheaf is called the *sheaf of (p, q) -forms*. The reason for this definition is that we now have a decomposition

$$\mathcal{A}_{\mathbb{C}}^n = \bigwedge^n (TX^{1,0} \oplus TX^{0,1})^{\vee} = \bigoplus_{p+q=n} \bigwedge^p (TX^{1,0})^{\vee} \otimes \bigwedge^q (TX)^{\vee} = \bigoplus_{p+q=n} \mathcal{A}_{\mathbb{C}}^{p,q}. \quad (\text{B.2})$$

The complexified de Rham complex behaves well with this composition, in a way which is described in the next proposition.

Lemma B.2. *For any p, q there are maps $\partial : \mathcal{A}_{\mathbb{C}}^{p,q} \rightarrow \mathcal{A}_{\mathbb{C}}^{p+1,q}$ and $\bar{\partial} : \mathcal{A}_{\mathbb{C}}^{p,q} \rightarrow \mathcal{A}_{\mathbb{C}}^{p,q+1}$, such that $d_{\mathbb{C}} = \partial + \bar{\partial}$.*

These maps ∂ and $\bar{\partial}$ satisfy $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

Note that the second statement of the lemma just follows from the relation $d_{\mathbb{C}}^2 = 0$. Thus, the lemma tells us that the sheaves $\mathcal{A}_{\mathbb{C}}^{p,q}$ form a double complex with ∂ and $\bar{\partial}$ as differentials. This double complex is called the Dolbeault complex.

Before we go on to describe to cohomology of this complex, we first introduce the notion of *holomorphic differential forms*. These are analogous to the above notions as differential forms; a holomorphic k -form is a map of \mathcal{O}_X -modules $\bigwedge^k TX \rightarrow \mathcal{O}_X$. The corresponding sheaf is denoted by Ω_X^k . Note that Ω_X^k is naturally a subsheaf of $\mathcal{A}_{\mathbb{C}}^{k,0}$. Indeed, the former are defined by being holomorphic sections of the bundle $\bigwedge^k (TX)^{\vee}$, while the latter corresponds to smooth sections of the same bundle (here we think of $\bigwedge^k (TX)^{\vee}$ as a manifold).

We will see that the cohomology of the Dolbeault complex is related to the cohomology of the Ω_X^k . We fix an integer p and consider the complex $\mathcal{A}_{\mathbb{C}}^{p,\bullet}$ with differential $\bar{\partial}$. The resulting cohomology theory is called *Dolbeault cohomology*.

Theorem B.3 (Dolbeault). *The cohomology of $\mathcal{A}_{\mathbb{C}}^{p,\bullet}(X)$ with differential $\bar{\partial}$ is canonically isomorphic to $H^q(X, \Omega_X^p)$.*

Sketch of proof. The idea of the proof is that $\mathcal{A}^{p,\bullet}$ is an acyclic resolution of Ω_X^p , thus computing its cohomology. This statement consists of three ingredients. (1) The complex is exact at $\mathcal{A}_{\mathbb{C}}^{p,q}$ for $q > 0$. This statement is now as the *Poincaré $\bar{\partial}$ -lemma*. (2) The subsheaf $\Omega_X^p \subseteq \mathcal{A}_{\mathbb{C}}^{p,0}$ can be identified with the kernel of $\bar{\partial}$. This essentially follows from the Cauchy-Riemann equations. (3) The sheaves $\mathcal{A}_{\mathbb{C}}^{p,q}$ are all acyclic. This one can show using a partition of unity argument. \square

Definition B.4. A *Kähler structure* on a complex manifold is a hermitian form h on X , such that the associated real form $g = \operatorname{Re} h$ is closed.

A manifold with a Kähler structure is called a *Kähler manifold*.

We will not go into the details of this definition. It turns out that for some results, the specific Kähler structure is not important, just that there exists one. Therefore, some authors (for example, [12]) define a Kähler manifold as a manifold admitting a Kähler structure, without choosing a preferred one. We prefer not to go into the subtleties and instead just state when a result does not depend on the metric chosen.

Proposition B.5. *The projective space \mathbb{P}^n admits a Kähler structure. Any closed submanifold of a Kähler manifold is naturally again a Kähler manifold.*

This example is crucial for us. It implies that the manifold corresponding to a smooth projective scheme is a Kähler manifold. Thus the results of this section apply to them as well.

Using a Kähler structure, one can define additional operations. For example, there turns out to be an inner product on the space of n -forms, such that the decomposition (B.2) is orthogonal. We will not need the specific product, but we will need the next operations.

Lemma B.6. *There exists operations $\partial^* : \mathcal{A}_{\mathbb{C}}^{p,q} \rightarrow \mathcal{A}_{\mathbb{C}}^{p-1,q}$ and $\bar{\partial}^* : \mathcal{A}_{\mathbb{C}}^{p,q} \rightarrow \mathcal{A}_{\mathbb{C}}^{p,q-1}$. These are adjoint to ∂ and $\bar{\partial}$, respectively, with respect to the inner product mentioned above.*

Here, we define $d^* = \partial^* + \bar{\partial}^*$.

Proposition B.7. *Let X be a compact Kähler manifold. For a form $\alpha \in \mathcal{A}_{\mathbb{C}}^{p,q}(X)$, the following are equivalent:*

1. $\partial\alpha = \partial^*\alpha = 0$.
2. $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$.
3. $d\alpha = d^*\alpha = 0$.

Definition B.8. Forms satisfying the above equivalent conditions are called *harmonic*. The set of global harmonic forms on X of degree (p, q) is denoted $\mathcal{H}_X^{p,q}$.

The notion of harmonic forms is important, since it appears in the next result, which is known as the *Hodge decomposition theorem*.

Theorem B.9. *Let X be compact Kähler. There exist two natural orthogonal decompositions:*

$$\mathcal{A}_{\mathbb{C}}^{p,q}(X) = \partial\mathcal{A}_{\mathbb{C}}^{p-1,q}(X) \oplus \mathcal{H}_X^{p,q} \oplus \partial^*\mathcal{A}_{\mathbb{C}}^{p+1,q}(X)$$

and

$$\mathcal{A}_{\mathbb{C}}^{p,q}(X) = \bar{\partial}\mathcal{A}_{\mathbb{C}}^{p,q-1}(X) \oplus \mathcal{H}_X^{p,q} \oplus \bar{\partial}^*\mathcal{A}_{\mathbb{C}}^{p,q+1}(X).$$

If one has slightly more theory at their disposal, Theorem B.9 has an interesting corollary. One can show that $\bar{\partial}$ is injective on $\bar{\partial}^* \mathcal{A}_{\mathbb{C}}^{p,q+1}$ and zero on the other summand. Thus, the cohomology of the Dolbeault complex is equal to $\mathcal{H}_X^{p,q}$. However, we already know that it is also equal to $H^q(X, \Omega_X^p)$ by Theorem B.3. Hence these vector spaces are naturally isomorphic.

For the next result, we recall the operation of complex conjugation on cohomology. We can write $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$. Then we define the complex conjugation of $c \otimes z$, with $c \in H^k(X, \mathbb{R})$ and $z \in \mathbb{C}$, as $\overline{c \otimes z} = c \otimes \bar{z}$. This gives a conjugation operation on $H^k(X, \mathbb{C})$.

Theorem B.10 is sometimes also known as the *Hodge decomposition theorem*.

Theorem B.10. *Let X be a compact Kähler manifold. Then there exists a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

which does not depend on the chosen Kähler metric. Furthermore, in this decomposition we have $H^q(X, \Omega_X^p) = \overline{H^p(X, \Omega_X^q)}$.

In particular, the second part of Theorem B.10 implies that $h^{p,q}(X) = h^{q,p}(X)$, where $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$.

Proposition B.11. *Let X be a Kähler manifold and let ω be a holomorphic p -form. Then ω is closed.*

Proof. By the proof of Theorem B.3, $\bar{\partial}\omega = 0$. Also, $\bar{\partial}^*\omega = 0$, because it lives in $\mathcal{A}_{\mathbb{C}}^{p,-1}$, which is a zero sheaf. Therefore, by Prop. B.7, $\partial\omega = 0$. Then by Lemma B.2, we find that $d\omega = \partial\omega + \bar{\partial}\omega = 0$. \square

C. Chern classes and characters

In this section we introduce the basic properties of Chern characters. These are defined on a complex manifold X . Because of GAGA, [23], we might also take X to be a smooth scheme of finite type over k . (Recall that $k = \mathbb{C}$.)

We assume their basic properties and the Hirzebruch-Riemann-Roch formula. We will provide proofs for the other statements.

Definition C.1. We denote the i -th Chern class of a vector bundle E on X by $c_i(E)$, which is an element of $H^{2i}(X, \mathbb{Z})$. The *total Chern class* is defined as $c(E) = \sum_i c_i(E)$.

We denote the *Chern character* of a vector bundle E on X by $\text{ch}(E) = \sum_i \text{ch}_i(E)$ where $\text{ch}_i(E) \in H^{2i}(X, \mathbb{Q})$.

For a construction of the Chern classes and characters, see [12, Ch. 4]. There is also an algebraic approach, for which see [5, Ch. 3] and a topological approach, see [10, Ch. 3]

Proposition C.2. *Let $f : X \rightarrow Y$ be a flat morphism. If E is a vector bundle on Y , then $f^*c(E) = c(f^*E)$ and $f^*\text{ch}(E) = \text{ch}(f^*E)$.*

Proof. See [5, Ch. 3]. □

It is well-known that the Picard group $\text{Pic}(X)$ of a scheme or manifold can be identified with $H^1(X, \mathcal{O}_X^*)$, see e.g. [8, Ex. III.4.5]. Now we consider the exponential exact sequence.

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0.$$

The associated long exact sequence of cohomology gives us a connecting homomorphism $\delta : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})$.

Proposition C.3. *The total Chern class of a line bundle L is $1 + \delta(L)$, in particular, $c_1(L) = \delta(L)$. The Chern character of a line bundle is*

$$\text{ch}(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2!}c_1(L)^2 + \frac{1}{3!}c_1(L)^3 + \dots$$

Proof. See [12, Ch. 4]. □

The above power series is finite since $c_1(L)^n = 0$ for $n > \dim X$. We remark that Prop. C.3 implies that $c(\mathcal{O}_X) = 1$ and $c_1(L^\vee) = -c_1(L)$.

Proposition C.4. *Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be an exact sequence of vector bundles on X . Then $\text{ch}(E) = \text{ch}(F) + \text{ch}(G)$. We call this property additivity. In this case, we also have that $c(F) \cdot c(G) = c(E)$.*

We also have $\text{ch}(E \otimes F) = \text{ch}(E) \cdot \text{ch}(F)$. We call this property multiplicativity.

Proof. See [5, Ch. 3]. □

When X is projective, every coherent sheaf has a finite resolution by locally free sheaves. This can be used to define the Chern classes and character for a coherent sheaf, by forcing the above proposition to be true.

Theorem C.5 (Splitting principle). *Given any scheme X over k and a vector bundle E on X , then there is a scheme X' and a smooth projective morphism $f : X' \rightarrow X$ such that f^*E admits a filtration*

$$0 = E_0 \subseteq E_1 \subseteq \dots \subseteq E_n = f^*E$$

such that the factors E_i/E_{i-1} are line bundles. In particular, when X is smooth, so is X' and on singular cohomology $H^\bullet(X, \mathbb{Z}) \rightarrow H^\bullet(X', \mathbb{Z})$ is injective.

Proof. See [5, Ch. 3]. □

Using Theorem C.5, we see that Prop. C.2, Prop. C.3 and Prop. C.4 determine the Chern character of any vector bundle on any smooth scheme. This observation is called the *axiomatic characterisation* of the Chern characters. A similar characterisation exists for the Chern classes, see [6].

In Lemma C.7 one can find a typical application of Theorem C.5. In fact, it is so typical that the splitting principle is often informally stated by saying that one can assume that for the purpose of Chern classes, every vector bundle has a filtration by line bundles (or, even more extreme, that any vector bundle is a direct sum of line bundles).

Denote by $H^{2\bullet}(X, \mathbb{Z})$ the subring of $H^\bullet(X, \mathbb{Z})$ of elements of even degree. It will be convenient to denote an element $v \in H^{2\bullet}(X, \mathbb{Z})$ by (v_0, v_2, \dots) .

Definition C.6. Let $v = (v_0, v_2, v_4, v_6, \dots) \in H^{2\bullet}(X, \mathbb{Z})$ be a cohomology class. Then we define $v^\vee = (v_0, -v_2, v_4, -v_6, \dots)$.

The same definition also applies if we take cohomology with \mathbb{Q} -coefficients rather than \mathbb{Z} -coefficients. We notice also that $v \mapsto v^\vee$ is a ring homomorphism and that it commutes with pullback.

Lemma C.7. *For E a vector bundle on X , $\text{ch}(E^\vee) = \text{ch}(E)^\vee$.*

Proof. First we prove it when E is a line bundle. In that case, Prop. C.3 gives that $c_1(E^\vee) = -c_1(E)$. Now we use the second formula of Prop. C.3:

$$\text{ch}(E^\vee) = \sum_{i=0}^{\infty} \frac{c_1(E^\vee)^i}{i!} = \sum_{i=0}^{\infty} (-1)^i \frac{c_1(E)^i}{i!} = \text{ch}(E)^\vee.$$

Now assume that E is a vector bundle admitting a filtration by line bundles. Denote the line bundles by L_i . Then

$$\mathrm{ch}(E^\vee) = \sum_i \mathrm{ch}(L_i^\vee) = \sum_i \mathrm{ch}(L_i)^\vee = \mathrm{ch}(E)^\vee.$$

Lastly, we take E to be an arbitrary vector bundle. By the splitting principle C.5, we can find a $f : X' \rightarrow X$ such that f^*E admits a filtration by line bundles. Then we get

$$f^* \mathrm{ch}(E^\vee) = \mathrm{ch}(f^*E^\vee) = \mathrm{ch}(f^*E)^\vee = f^* \mathrm{ch}(E)^\vee.$$

Since f^* is injective on cohomology in this case, we find the general result. \square

We would also like to state the Hirzebruch-Riemann-Roch theorem. For this, we need the *Todd class* of X . We refrain from defining this class for arbitrary X . Its first few terms are

$$\mathrm{td}(X) = 1 + \frac{1}{2}c_1(\mathcal{T}_X) + \frac{1}{12}(c_1(\mathcal{T}_X)^2 + c_2(\mathcal{T}_X)) + \frac{1}{24}c_1(\mathcal{T}_X)c_2(\mathcal{T}_X) + \dots \quad (\text{C.1})$$

This determines the Todd class for schemes up to dimension three. For a general formula, see [5, Ch. 3] or [12, Ch. 4].

When X is a projective variety, Poincaré duality implies that $H^{2 \dim X}(X, \mathbb{Q}) \cong \mathbb{Q}$ in a canonical way. For $c \in H^{2 \bullet}(X, \mathbb{Q})$, we define $\int_X c$ as the rational number corresponding to the component in degree $2 \dim X$.

Theorem C.8 (Hirzebruch-Riemann-Roch). *Let X be a projective variety and let E be a coherent sheaf on X . Then we have an equality*

$$\chi(E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X).$$

Proof. [5, Ch. 15]. \square

Lemma C.9. *When X is a projective variety, the zeroth Chern class $\mathrm{ch}_0(E)$ of a coherent sheaf E is equal to $\mathrm{rk} E$.*

Furthermore, if $\dim X = d$, $\mathrm{rk} E \cdot \alpha_d(\mathcal{O}_X) = \alpha_d(E)$.

Proof. For the first statement, note that the rank is an additive function in exact sequences. Using this, we reduce to the case that E is a vector bundle. Using the splitting principle C.5, we see that we can reduce to the case that E is a line bundle. But for line bundles we already know it by Prop. C.3.

For the second statement, we use the Hirzebruch-Riemann-Roch theorem C.8. Denote $c_1 = c_1(\mathcal{O}(1))$. Note that

$$\mathrm{ch}(E(m)) = \mathrm{ch}(E) \cdot \mathrm{ch}(\mathcal{O}(m)) = \mathrm{ch}(E) \cdot e^{mc_1}.$$

Thus, we now find

$$P(E, m) = \int_X \mathrm{ch}(E(m)) \cdot \mathrm{td}(X) = \int_X \mathrm{ch}(E) \cdot \left(1 + c_1 m + \frac{c_1^2}{2!} m^2 + \dots + \frac{c_1^d}{d!} m^d\right) \cdot \mathrm{td}(X).$$

We are interested in the coefficient of m^d . By considering the degrees, we see that

$$\alpha_d(E) = c_1^d \cdot \text{ch}_0(E) \cdot \text{td}_0(X) = c_1^d \cdot \text{rk}(E).$$

Indeed, this is the only term with m^d which lives in $H^{2\dim X}(X, \mathbb{Q})$. Plugging in $E = \mathcal{O}_X$ gives $\alpha_d(\mathcal{O}_X) = c_1^d$ and hence $\alpha_d(E) = \alpha_d(\mathcal{O}_X) \cdot \text{rk } E$. This is what we want. \square

We lastly have a result stating that Chern classes behave well in families.

Lemma C.10. *Let S be a smooth connected scheme of finite type and let E be a coherent sheaf on $X \times S$, flat over S . Then for each $s, s' \in S$, the Chern characters of $E|_s$ and $E|_{s'}$ coincide.*

Proof. We may reduce to $S = \text{Spec } A$ is affine. Since $X \times S$ is projective over S , we may choose a locally free resolution of E . Since E is flat over S , this locally free resolution restricts to a locally free resolution of E_s . Thus, if we can prove the result when E is a vector bundle, additivity gives the result for general E .

When E is a vector bundle, we invoke the splitting principle C.5. Then we can assume that E has a filtration by line bundles. Again by additivity, we may now assume that E is a line bundle.

Let s, s' be points in S . Since S is connected, there is a path $\gamma : [0, 1] \rightarrow S$ connecting s and s' . Then the inclusion $\text{id} \times \gamma : X \times [0, 1] \rightarrow X \times S$ gives a homotopy between the inclusions $i_s : X = X \times \{s\} \rightarrow X \times S$ and $i_{s'} : X = X \times \{s'\} \rightarrow X \times S$. By functoriality, Prop. C.2, we find that

$$c_1(E|_s) = c_1(i_s^* E) = i_s^* c_1(E) = i_{s'}^* c_1(E) = c_1(i_{s'}^* E) = c_1(E|_{s'})$$

since homotopic maps induce the same pullback map on cohomology. Now the result follows by the explicit formula of Prop. C.3. \square

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