École Normale Supérieure de Lyon

MASTER 1 Algebraic Geometry

Final Exam **30 April 2018**

Duration : 3h00

Convention : All rings are assumed to be commutative, associative and have a unit.
Questions —

1. Recall the definition of irreducibility of a scheme.
2. Let A be a ring. Show that the scheme $X = \mathrm{Spec}(A)$ is irreducible if and only if A has a unique minimal prime ideal (which is therefore the nil-radical of A).
3. Give an example of a morphism of schemes which is of finite type but not finite.
4. Give an example of non-affine morphism.

Exercise 1 — Prove that if k is a field, then any affine open subset of the affine n-space $\mathbb{A}^n_k$ is principal.

Exercise 2 — Let $f : X \to Y$ be a morphism of integral schemes, and let η_X, η_Y be the generic points of X, Y (so that the closure of $\{\eta_X\}$ is X, similarly for Y).

1. Prove that if $f(X)$ is dense in Y then $f(\eta_X) = \eta_Y$.
2. Recall why the natural map $O_Y(U) \to O_{Y,\eta_Y}$ is injective and identifies O_{Y,η_Y} with the fraction field of $O_Y(U)$ for any nonempty affine open subset U of Y.
3. Deduce that if $f(\eta_X) = \eta_Y$, then the map $f^\sharp : O_Y \to f_* O_X$ is injective.
4. Prove that if $f^\sharp$ is injective, then $f(X)$ is dense in Y.

Exercise 3 — A scheme is called *quasi-separated* if the intersection of any two quasi-compact open subsets is quasi-compact.

1. Show that affine schemes are quasi-separated.
2. (Seen in the lecture.) Let X be a quasi-compact and quasi-separated scheme and $f \in \mathcal{O}_X(X)$. Let X_f be the open subscheme where the value of f does not vanish. Show that the natural morphism $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ is an isomorphism.

Exercise 4 — Let ϕ be the automorphism of $X = \mathbb{A}^1_{\mathbf{Q}} = \mathrm{Spec}(\mathbf{Q}[T])$ induced by the morphism of $\mathbf{Q}$-algebras $\mathbf{Q}[T] \to \mathbf{Q}[T]$ sending T to $T+1$.

1. Prove that if U is a nonempty open subset of X such that $\phi(U) \subset U$, then $U = X$.
2. Let Y be a scheme and let $f : X \to Y$ be a morphism of schemes such that $f = f \circ \phi$. Prove that f factors uniquely through the natural map $X \to \mathrm{Spec}(\mathbf{Q})$.
3. Does the result in 2) hold if instead of schemes (and morphisms of schemes) Y we consider arbitrary ringed spaces and morphisms of such spaces ?

Exercise 5 — A morphism $f : X \to Y$ of schemes is *integral* if f is affine, and for every affine open subset $\mathrm{Spec}\, B \subset Y$, with $f^{-1}(\mathrm{Spec}\, B) = \mathrm{Spec}\, A \subset X$, the induced ring homomorphism $B \to A$ is integral.

1. Show that the composition of two integral morphisms is integral.

2. Show that finite morphisms are integral. Does the converse hold ?
3. Show that integral morphisms are closed, i.e., the images of closed subsets are closed.
4. Show that an affine morphism f is integral if there is an open affine cover $\{U_i = \operatorname{Spec} B_i\}_i$ of Y, with $f^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} A_i \subset X$ such that the induced map $B_i \to A_i$ is an integral ring homomorphism for all i.

Exercise 6 — Let p be a prime number. If X is a $\mathbf{F}_p$-scheme of finite type, let X_0 be the set of closed points of X. If L is an extension of $\mathbf{F}_p$, let $X(L)$ be the set of morphisms of schemes $\operatorname{Spec}(L) \to X$.

1. Prove that for any $x \in X_0$, the residue field $\kappa(x)$ is a finite extension of $\mathbf{F}_p$. We denote $\deg(x) = [\kappa(x) : \mathbf{F}_p]$ its degree.
2. Let a_n be the number of points $x \in X_0$ such that $\deg(x) = n$. Prove that $|X(\mathbf{F}_{p^m})| = \sum_{d|m} da_d$ for all m, where d runs over all positive divisors of m.
3. Define the *Zeta function of* X to be the following formal series :
$$Z(X,t) = exp\left(\sum_{m\geq 1} |X(\mathbf{F}_{p^m})| \cdot \frac{t^m}{m}\right).$$
Prove that $Z(X,t) = \prod_{x\in X_0}(1 - t^{\deg x})^{-1}$. Deduce that $Z(X,t)$ has integer coefficients.
4. Compute $Z(X,t)$ explicitly for $X = \mathbb{A}^n_{\mathbf{F}_p}$ the affine n-space and $X = \mathbb{P}^n_{\mathbf{F}_p}$ the projective n-space.