École Normale Supérieure de Lyon

MASTER 1 Algebraic Geometry

Mid-term Examen **12 March 2018**

Duration : 2h00

Convention : All rings are assumed to be commutative, associative and have a unit.

Questions —

1. Give an example of a presheaf which is not a sheaf.
2. Give the definition of the stalk of a sheaf at a point.
3. Give the definition of locally ringed spaces and morphisms between them.

Exercise — Let A be a ring and $f \in A$. Let D_f be the *principal* open subset of $\mathrm{Spec}(A)$ determined by f. Show that

1. $D_f = \mathrm{Spec}(A)$ if and only if f is invertible.
2. $D_f = \emptyset$ if and only if f is nilpotent.

Exercise — Let k be a field. Let $f : A \to B$ be a morphism of k-algebras of finite type. Let $\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced map. Show that the image of a closed point is closed.

Exercise — Let A be a ring and $X = \mathrm{Spec}(A)$ be its spectrum equipped with the Zariski topology.

1. Show that if A is noetherian, then all open subsets of X are quasi-compact.
2. Is the converse true ? If yes, give proof ; if no, give a counter-example.
3. Is the intersection of two quasi-compact open subsets of X always quasi-compact ?

Exercise — Let X be a topological space and F a sheaf of abelian groups on it. The *support* of F is by definition

$$Supp(F) := \{x \in X \mid F_x \neq 0\}.$$

If we suppose that $Supp(F)$ is a finite set of closed points, show that for any open subset U of X, we have $F(U) \cong \oplus_{x \in U} F_x$.

Exercise — Let k be a field, A a k-algebra of finite type and $X = \mathrm{Spec}(A)$ equipped with the Zariski topology. Let X_0 be the set of closed points of X.

1. Show that X_0 is dense in X.
2. Prove that there is a natural bijection between the open subsets of X and those of X_0.
3. Show that the restriction from X to X_0 gives a natural bijection between sheaves of abelian groups on X and those on X_0.
4. Write $A = k[T_1, ..., T_n]/(f_1, ..., f_m)$. Show that X_0 is identified with the set of solutions in $\bar{k}$ of the system of equations $f_i(T_1, \cdots, T_n) = 0,\ \forall 1 \leq i \leq m$, modulo the (diagonal) action of the Galois group $Gal(\bar{k}/k)$, where $\bar{k}$ is an algebraic closure of k.

Exercise — Let n be a positive integer. An *affine algebraic subset* is by definition the set of solutions of a system of polynomial equations in n variables.

1. Show that an affine algebraic subset in $\mathbb{C}^n$ is compact if and only if it is finite.
2. What if we replace $\mathbb{C}^n$ by $\mathbb{R}^n$?