ENS de Lyon 2018 - 2019

MASTER 1–ALGEBRAIC GEOMETRY WEEK 1, 16.01.2018

TD 1-Noether normalization, Nullstellensatz and applications

All rings will be commutative with 1, all ring maps send 1 to 1. k will always be a field. A map of rings $f : A \to B$ is called

• **integral** if any $b \in B$ is integral over A, i.e. satisfies an equation $b^n + a_{n-1}b^{n-1} + ... + a_1 b + a_0 = 0$ with $a_i \in A$ (we use f to see B as A-algebra).

• **finite** (resp. **of finite type**) if it turns B into a finitely generated A-module (resp. A-algebra).

0.1 Warming up

1. Describe the following set :
   (a) $Spec(A)$, where A is a principle ring (i.e A is an integral domain and every ideals of A is principle).
   (b) $Spec(\mathbb{R}[T])$ and $Spec(\mathbb{Z}[i])$.
   (c) $Spec(k[[T]])$ where k is any field.
2. Let A be any ring, for any ideal I of A, show that : $\sqrt{I} = \bigcap_{I \subset \mathfrak{p} | \mathfrak{p} \in Spec(A)} \mathfrak{p}$

0.2 Connectedness of Spec(A)

Let A be a ring. Show that the followings are equivalent :

1. $Spec(A)$ is not connected.
2. There exists non zero idempotent elements $e_1, e_2 \in A$ (i.e $e_1^2 = e_1; e_2^2 = e_2$) such that $e_1 e_2 = 0$.
3. A can be written as a product of two non trivial rings.

0.3 Basic things on integrality

Make sure you know the following things **very well**, since we will use them constantly.

1. Prove that any finite map of rings is integral. Moreover, an integral map $f : A \to B$ is finite if and only if it is of finite type, if and only if B is generated as A-algebra by finitely many elements integral over A.
2. Prove that a composition of finite (resp. integral, resp. of finite type) rings maps is also finite (resp...). Moreover, if $A \to B$ is integral (resp...), then for any map $A \to C$ the natural map $C \to C \otimes_A B$ is integral (resp...).
3. (a recurrent trick) Suppose that $f : A \to B$ is an injective integral map of rings. Prove that A is a field if and only if B is a field.

0.4 Noether's normalization lemma

This result is absolutely fundamental, make sure you understand it !

1. (preparation) Let R be a ring and let $f \in R[X_1, ..., X_n] \setminus R$. Prove that if e is large enough, then we can find $d > 0$, $a \in R \setminus \{0\}$ and $a_0, ..., a_{d-1} \in R[X_1, ..., X_{n-1}]$ such that

$$f(X_1 + X_n^{e^{n-1}}, X_2 + X_n^{e^{n-2}}, ..., X_{n-1} + X_n^e, X_n) = aX_n^d + a_{d-1}X_n^{d-1} + ... + a_0.$$

2. Prove **Noether's normalization lemma** : let A, B be integral domains and let $f : A \to B$ be an injective map of finite type. Then we can find $x_1, ..., x_d \in B$ and $f \in A \setminus \{0\}$ such that the natural $A[1/f]$-algebra map $A[1/f][T_1, ..., T_d] \to B[1/f], T_i \mapsto x_i$ is injective and finite.

   Hint : pick the smallest d for which there are $f \in A\setminus\{0\}$ and $x_1, ..., x_d \in B$ such that the previous map is integral, and show that up to replacing f by a multiple of it, $x_1, ..., x_d$ work, using the preparation result.
3. State in a comprehensible way what this result says about finitely generated algebras over a field and over $\mathbf{Z}$.

0.5 Zariski's lemma

Let k be a field. We want to prove (in two different ways) **Zariski's lemma** : if a finitely generated k-algebra A is a field, then A is a finite extension of k.

1. Give a proof using Noether normalization and the recurrent trick in exercise 1.
2. For a second proof, we will prove by induction on n the following statement : if A is a k-algebra generated by n elements, say $x_1, ..., x_n$, and if A is a field, then $x_1, ..., x_n$ are algebraic over k.

   a) Treat the case $n = 1$. Suppose now that the result holds for algebras generated by at most $n-1$ elements (over any field!) and that x_1 is not algebraic over k.

   b) Prove that we can find $a \in k[x_1] \subset A$ nonzero such that $ax_2, ..., ax_n$ are integral over $k[x_1]$. **Hint** : note that $A = k(x_1)[x_2, ..., x_n]$.

   c) Prove that if $b \in k[x_1]$ is nonzero, then for d large enough $\frac{a^d}{b}$ belongs to $k[x_1, ax_2, ..., ax_n]$. Conclude that $\frac{a^d}{b} \in k[x_1]$ and finish the proof.

0.6 Maximal ideals in $k[X_1, ..., X_n]$ and $\mathbf{Z}[X_1, ..., X_n]$

Let k be a field.

1. Prove that if m is a maximal ideal of $k[X_1, ..., X_n]$, then $k[X_1, ..., X_n]/m$ is a finite extension of k.
2. If k is algebraically closed, give a natural bijection between maximal ideals of $k[X_1, ..., X_n]$ and k^n. In general, prove that the maximal ideals of $k[X_1, ..., X_n]$ are in canonical bijection with the G-orbits on $\bar{k}^n$, where $\bar{k}$ is an algebraic closure of k and $G = \mathrm{Gal}(\bar{k}/k)$.
3. (**weak Nullstellensatz**) The polynomials $f_1, ..., f_d \in k[X_1, ..., X_n]$ have no common zero in an algebraic closure of k. Prove that we can find polynomials $g_1, ..., g_d \in k[X_1, ..., X_n]$ such that

$$f_1 g_1 + ... + f_d g_d = 1.$$

4. Prove that if m is a maximal ideal of $\mathbf{Z}[X_1, ..., X_n]$, then $\mathbf{Z}[X_1, ..., X_n]/m$ is a finite field.
5. Prove that a family of polynomials $f_i \in \mathbf{Z}[X_1, ..., X_n]$ generates the unit ideal in $\mathbf{Z}[X_1, ..., X_n]$ if and only if the equations $f_i(x_1, ..., x_n) = 0$ have no common solution in any finite field.

0.7 The Nullstellensatz

Let k be a field.

1. Let $f : A \to B$ be a morphism of finitely generated k-algebras. Prove that the inverse image of any maximal ideal of B is a maximal ideal of A.
2. Prove that if A is a finitely generated k-algebra, then for all ideals I of A, $\sqrt{I}$ is the intersection of all maximal ideals of A containing I. **Hint** : reduce to $I = 0$, then apply the first item to the map $A \to A[1/f]$ (for suitable f).
3. Deduce **Hilbert's Nullstellensatz** : if $f, g_1, ..., g_d \in k[X_1, ..., X_n]$ and f vanishes at each common zero of $g_1, ..., g_d$ in an algebraically closed extension of k, then for N large enough $f^N \in (g_1, ..., g_d)$.