ENS de Lyon 2017 - 2018

Master 1–Algebraic Geometry Week 2, 22.01.2018

TD 2

If not explicitly mentioned, A is always an arbitrary ring, $X = \mathrm{Spec}(A)$ is endowed with the Zariski topology and for $f \in A$ and I an ideal of A we let

$$V(I) = \{\mathfrak{p} \in X \mid I \subset \mathfrak{p}\}, \quad D(f) = X \setminus V((f)) = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

If $\mathfrak{p} \in X$ let $k(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be its residue field. Finally, if $f : A \to B$ is a map of rings, the associated map on Spec is the (continuous) map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$. If S is a subset of a topological space, we write $\overline{S}$ for the closure of S in that space. Finally, k is always a field.

0.1 Basic things

1. Describe the spectrum of $\mathbf{Z}$, of a discrete valuation ring and of $\mathbf{Z}[T]/(T^2+1)$, $\mathbf{Z}[T]/(T^n)$ for $n \geq 1$.
2. For $\mathfrak{p} \in X$, what is the closure of $\{\mathfrak{p}\}$ in X? When is $\{\mathfrak{p}\}$ closed in X? If $\mathfrak{q} \in X$, what does it mean concretely that $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ (we say that $\mathfrak{p}$ **specializes** to $\mathfrak{q}$)? What are the closed points of $\mathrm{Spec}(k[X,Y])$?
3. Given a multiplicative subset S of A and an ideal I of A, identify (in a way compatible with topologies!) $\mathrm{Spec}(S^{-1}A)$ and $\mathrm{Spec}(A/I)$ with subspaces of $\mathrm{Spec}(A)$. Is the image of $\mathrm{Spec}(S^{-1}A)$ always open?
4. Prove that if $(f_i)_{i\in I}, g$ are elements of A, then $D(g) \subset \cup_{i\in I} D(f_i)$ if and only if $g \in \sqrt{(f_i)_{i\in I}}$. Prove that the open subsets of X that are quasi-compact[1] are exactly the subsets $X \setminus V(I)$ for finitely generated ideals I of A, or equivalently the finite unions of $D(f)$'s.

0.2 Irreducibility

A nonempty topological space X is **irreducible** if it cannot be written as the union of two closed subsets different from X. A subset of X is called irreducible it it's an irreducible space for the induced topology.

1. Prove that X is irreducible if and only if each nonempty open subset of X is dense in X. Also, prove that a subset S of X is irreducible iff its closure in X is so, and the image of an irreducible subset of X by a continuous map $f : X \to Y$ is irreducible.
2. Prove that $\mathrm{Spec}(A)$ is irreducible iff $A/\mathrm{nil}(A)$ is an integral domain, and construct a natural bijection between irreducible closed subsets of $\mathrm{Spec}(A)$ and prime ideals of A.

0.3 Fundamental tools

Keep the following properties in mind, you will use them all the time! Let $f : A \to B$ be a morphism of rings and let $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism.

1. Prove that φ has dense image if and only if $\ker(f)$ consists of nilpotent elements.
2. Prove that if $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\varphi^{-1}(\mathfrak{p})$ with the induced topology is homeomorphic to $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$. **Hint**: $k(\mathfrak{p}) \otimes_A B$ is the same as $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.
3. Prove that if φ is surjective, then $\mathrm{Spec}(A' \otimes_A B) \to \mathrm{Spec}(A')$ is surjective for all morphisms $A \to A'$. **Hint**: use the previous result.
4. Prove that if f is finite, then φ has finite fibers. What about the converse?
5. Describe $\mathrm{Spec}(\mathbf{Z}[T])$ and $\mathrm{Spec}(k[X,Y])$. **Hint**: consider the fibres of $\mathrm{Spec}(\mathbf{Z}[T]) \to \mathrm{Spec}(\mathbf{Z})$.

[1] A topological space X is quasi-compact if any open covering of X can be refined to a finite sub-covering.

0.4 Integrality: Going up

Let $f : A \to B$ be an integral map of rings, and let $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced map.

1. Prove that if f is injective, then φ is surjective. **Hint**: reduce to the case when A is local.

2. Prove that φ is a closed map, and more precisely for all ideals I of B we have $\varphi(V(I)) = V(f^{-1}(I))$.

3. Prove the **going up theorem**: $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is specializing, i.e. if $\mathfrak{p} \subset \mathfrak{p}'$ are prime ideals of A and $\mathfrak{q}$ is a prime of B over $\mathfrak{p}$, then we can find a prime $\mathfrak{q} \subset \mathfrak{q}'$ of B over $\mathfrak{p}'$.

4. Prove that the map on spectra induced by $k[X,Y]/(X^2-Y^3) \to k[t]$, $X \to t^3, Y \to t^2$ is a homeomorphism. Is this map of rings an isomorphism?

0.5 Finitely generated algebras over a field

Let k be a field and let A be a finitely generated k algebra. Set $X = \mathrm{Spec}(A)$.

1. Prove that any nonempty locally closed subset of X meets the set X_0 of closed points of X, and give a natural bijection between the open subsets of X_0 (with induced topology) and those of X.

2. Prove that X is a finite set iff A is finite dimensional over k, iff X is discrete.

0.6 Products of varieties

1. Let B, C be A-algebras. Prove that $\mathrm{Spec}(B/nil(B) \otimes_A C) \to \mathrm{Spec}(B \otimes_A C)$ is a homeomorphism.

2. (important) Let k be an algebraically closed field and let A, B be finitely generated k-algebras.

   a) Prove that if A, B are reduced (resp. integral domains), then so is $A \otimes_k B$.

   b) Prove that if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are irreducible (resp. connected), then so is $\mathrm{Spec}(A \otimes_k B)$. Give counter-examples when k is no longer algebraically closed.

3. (more difficult) Let A, B be k-algebras (not necessarily finitely generated), with k a separably closed field. Prove that if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are both connected (resp. both irreducible), then so is $\mathrm{Spec}(A \otimes_k B)$.

0.7 (faithful) Flatness/integrality: Going down

1. Let $A \to B$ be a flat map of rings. Prove that the following statements are equivalent (we then say that the map is **faithfully flat**) and are satisfied for a flat local map of local rings:

   a) For any nonzero A-module M we have $M \otimes_A B \neq 0$.

   b) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

   c) The image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains all closed points of $\mathrm{Spec}(A)$

   A map of rings $A \to B$ has the **going down property** if the map on Spec is generalizing, i.e. if whenever $\mathfrak{p} \subset \mathfrak{p}'$ are prime ideals of A and $\mathfrak{q}'$ is a prime over $\mathfrak{p}'$, there is a prime $\mathfrak{q} \subset \mathfrak{q}'$ over $\mathfrak{p}$.

2. Prove that any flat map of rings has the going down property. **Hint**: use the previous item.

3. (difficult) Let $A \to B$ be an integral injection of integral domains, with A normal (i.e. integrally closed in its field of fractions). Prove that $A \to B$ has the going down property, as follows: first reduce to proving that $ab \notin \mathfrak{p}B$ when $b \notin \mathfrak{q}'$ and $a \notin \mathfrak{p}$, then show that any element of $\mathfrak{p}B$ satisfies an equation of the form $x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$ with $a_i \in \mathfrak{p}$, and finally study the minimal polynomial of b over $\mathrm{Frac}(A)$.

0.8 Quotients by finite groups

Let A be a ring and let G be a finite group acting by automorphisms of rings on A. Let A^G be the subring of invariant elements and let $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ be the natural map.

1. Prove that the natural morphism $A^G \to A$ is integral. Deduce that π is surjective.
2. Prove that the fibers of π are precisely the G-orbits for the natural action of G on $\mathrm{Spec}(A)$. **Hint**: use the map $a \to \prod_{g \in G} g(a)$ and the prime avoidance lemma.
3. Prove that π is open. **Hint**: if $a \in A$, prove that $\pi(D(a)) = \cup_i D(b_i)$, where

$$\prod_{g \in G} (T - g.a) = T^d + b_{d-1}T^{d-1} + ... + b_0.$$