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MASTER 1–ALGEBRAIC GEOMETRY WEEK 3, 29.01.2018

TD 3-Sheaves on topological spaces, ringed spaces

We write Sh(X), Ab(X) for the categories of sheaves of sets, resp. abelian sheaves on a topological space X. I will simply write equality instead of "canonical isomorphism". If $F \in \mathrm{Sh}(X)$ and $x \in X$, F_x is the stalk of F at x, and for $s \in F(U)$ with $x \in U$, s_x is the image of s in F_x.

A **ringed space** $X = (X, O_X)$ is a topological space X together with a sheaf of rings O_X on it, and a morphism of ringed spaces $f = (f, f^\sharp) : X \to Y$ consists of a map of topological spaces $f : X \to Y$ and a map of sheaves of rings $f^\sharp : O_Y \to f_* O_X$. We write Mod($X$) for the category of O_X-modules (i.e. abelian sheaves F such that $F(U)$ is an $O_X(U)$-module in a way compatible with restrictions) on a ringed space (X, O_X). If $f : X \to Y$ is a morphism of ringed spaces and $F \in \mathrm{Mod}(Y)$, we define $f^*(F) \in \mathrm{Mod}(X)$ by

$$f^* F = O_X \otimes_{f^{-1}(O_Y)} f^{-1}(F),$$

the map $f^{-1}(O_Y) \to O_X$ being induced (by adjunction) by $f^\sharp$ and the tensor product sheaf being the sheafification of the obvious presheaf.

X **is always a topological space below, sometimes-always mentioned-a ringed space**.

0.1 A few concrete examples

1. Show that the rule that sends each open $U \subset \mathbf{R}$ to the set of to the set of continuous bounded maps $f : U \to \mathbf{R}$, with the restrictions being the usual restriction of functions, is a pre-sheaf, but not a sheaf. What is the sheatification ?
2. a) Let F be the sheaf of holomorphic functions on $\mathbf{C}$, and let $f : F \to F$ be the map induced by $\frac{d}{dz}$. Is the presheaf $U \to \mathrm{coker}(f(U) : F(U) \to F(U))$ a sheaf ? What is its sheafification ?

   b) Let $X = \mathbf{C}^*$ and consider the exponential as a map between the sheaf F of continuous $\mathbf{C}$-valued functions on X and the sheaf G of invertible continuous maps on X. Is the presheaf $U \to \mathrm{Im}(\exp(U) : F(U) \to G(U))$ a sheaf ? What is its sheafification ?
3. Let $X = \mathbf{C}^*$ and consider the map $f : X \to X$ sending z to z^2. Is the direct image $f_* \mathbf{C}$ of the constant sheaf $\mathbf{C}$ a constant sheaf ?

0.2 Morphisms of sheaves

Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a topological spaces X and $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

1. Show that the rule that sends each open $U \subset X$ to $ker(\phi : \mathcal{F}(U) \to \mathcal{G}(U))$ is a sheaf.
2. Show that the rule that sends each open $U \subset X$ to $coker(\phi : \mathcal{F}(U) \to \mathcal{G}(U))$ is a a pre-sheaf but not necessarily a sheaf.

Application : let X be a complex manifold, for all open subset U of X, we define $\mathcal{F}(U)$ to be the set of all holomorphic functions that admit a holomorphic square root. The restriction is the usual restriction of functions. Is $\mathcal{F}$ a pre-sheaf ? or sheaf ?

0.3 Espace étalé

Let F be a sheaf on X. Let $\tilde{F} = \coprod_{x \in X} F_x$ and $\pi : \tilde{F} \to X$ the natural map. If $s \in F(U)$, define a map $g_s : U \to \tilde{F}$, $g_s(x) = s_x \in F_x$, and consider the topology on $\tilde{F}$ having as open sets the $g_s(U)$.

1. Prove that the topology induced on F_x is the discrete topology, and that g_s is a homeomorphism from U onto its image.
2. Prove that $F(U)$ is identified with the set of continuous functions $s : U \to \tilde{F}$ such that $\pi \circ s = \mathrm{id}_U$.

0.4 Everything is seen by stalks

Let $f, g : F \to G$ be maps of presheaves on a topological space X.

1. Prove that if F is a sheaf, then for any open $U \subset X$ the natural map $F(U) \to \prod_{x\in U} F_x$ is injective.
2. If G is a sheaf and $f_x = g_x : F_x \to G_x$ for all $x \in X$, then $f = g$.
3. If F is a sheaf, prove that $f(U) : F(U) \to G(U)$ is injective for all open subsets U if and only if $f_x : F_x \to G_x$ is injective for all $x \in X$ (in which case we say that f is injective). The same happens with injective replaced by bijective, if we assume moreover that G is a sheaf.
4. Let X be an open subset of $\mathbf{C}$ and consider the sheaf O_X of holomorphic functions on (open subsets of) X. Prove that the map $D : O_X \to O_X$, $D(f) = f'$ is a surjective map of sheaves (i.e. it induces surjective maps on all stalks), and give an example where the induced map on sections over X is not surjective.
5. Consider an exact sequence of sheaves $0 \to E \to F \to G \to 0$ in $\mathrm{Ab}(X)$ (i.e. the map $F \to G$ is surjective as a map of sheaves, and its kernel-computed in the naive way-is E). Prove that for all open subsets U of X we have an exact sequence $0 \to E(U) \to F(U) \to G(U)$. Give examples where this sequence is not exact on the right.

0.5 Flasque sheaves

An abelian sheaf F on a topological space X is called **flasque** if the restriction map $F(X) \to F(U)$ is surjective for all open subsets U of X.

1. Check that F is flasque iff $F(V) \to F(U)$ is surjective for all open subsets $U \subset V$ of X.
2. Prove that being flasque is stable under restriction to an open subset and under direct image by a continuous map.
3. Prove that a constant sheaf on an irreducible space is flasque.
4. a) Prove that if $0 \to E \to F \to G \to 0$ is an exact sequence in $\mathrm{Ab}(X)$, with E flasque, then $0 \to E(U) \to F(U) \to G(U) \to 0$ is exact for all open subsets U of X. Moreover, if E and F are flasque, then so is G.

   b) Consider a long exact sequence of flasque sheaves $0 \to E \to F^0 \to F^1 \to F^2 \to ...$ in $\mathrm{Ab}(X)$. Prove that for all open subsets U of X we have a long exact sequence $0 \to E(U) \to F^0(U) \to F^1(U) \to ....$

0.6 Restriction to open and closed subspaces

Let $j : U \to X$, resp. $i : Z \to X$ be the inclusion of an open, resp. closed subset of a topological space X.

1. Describe concretely $j^{-1}F$ (often written $F|_U$) and prove that $j^{-1}j_*(F) = F$ for $F \in \mathrm{Ab}(U)$.
2. Let $j_! : \mathrm{Ab}(U) \to \mathrm{Ab}(X)$ be the **extension by zero** functor, so $j_!(F)$ is the sheafification of the presheaf sending V to $\{0\}$ when V is not contained in U, and to $F(V)$ otherwise.

   a) Prove that
   $$\mathrm{Hom}_{\mathrm{Ab}(X)}(j_!(F), G) = \mathrm{Hom}_{\mathrm{Ab}(U)}(F, j^{-1}(G)).$$
   b) Prove that $(j_!F)_x$ is 0 when $x \notin U$ and F_x otherwise. Deduce that $j_!$ is an exact functor, identifying $\mathrm{Ab}(U)$ with the category of abelian sheaves on X whose stalks vanish outside U (start by checking that $j^{-1}j_!(F) = F$ for $F \in \mathrm{Ab}(U)$).
3. Prove that $(i_*F)_x$ is 0 when $x \notin Z$ and F_x otherwise. Deduce that i_* is an exact functor, identifying $\mathrm{Ab}(Z)$ with the category of abelian sheaves on X whose stalks vanish outside Z (start by checking that that $i^{-1}i_*(F) = F$ for $F \in \mathrm{Ab}(Z)$).
4. Suppose that $Z = X \setminus U$. Prove that for any $F \in \mathrm{Ab}(X)$ one has a canonical exact sequence
   $$0 \to j_!j^{-1}(F) \to F \to i_*i^{-1}(F) \to 0.$$
5. a) Prove that any $F \in \mathrm{Ab}(X)$ has a largest abelian subsheaf $\mathcal{H}_Z(F)$ whose support is contained in Z. **Hint** : the sections of $\mathcal{H}_Z(F)$ are those $s \in F(U)$ whose support is contained in $Z \cap U$.

   b) Define $i^! : \mathrm{Ab}(X) \to \mathrm{Ab}(Z)$ by $i^!(F) = i^{-1}\mathcal{H}_Z(F)$. Prove that $i^!$ is right adjoint to i_*, i.e.
   $$\mathrm{Hom}_{\mathrm{Ab}(X)}(i_*G, F) = \mathrm{Hom}_{\mathrm{Ab}(Z)}(G, i^!F).$$

0.7 New sheaves out of old ones

Let X be a topological space.

1. (products/direct sums) Let $(F_i)_{i\in I}$ be a family of abelian sheaves on X.
   a) Prove that $U \to \prod_{i\in I} F_i(U)$ is an abelian sheaf on X and has the expected universal property.
   b) Prove that $U \to \oplus_{i\in I} F_i(U)$ is not always a sheaf on X, but its sheafification, denoted $\oplus_{i\in I} F_i$, has the expected universal property. Describe the stalks of $\oplus_{i\in I} F_i$. Also, prove that if U is a quasi-compact open subset of X, the natural map $\oplus_i F_i(U) \to (\oplus_i F_i)(U)$ is bijective.
2. (tensor product) Let (X, O_X) be a ringed space. Define, for $F, G \in \mathrm{Mod}(X)$, $F \otimes_{O_X} G \in \mathrm{Mod}(X)$ as the sheafification of $U \to F(U) \otimes_{O_X(U)} G(U)$.
   a) Prove that $(F \otimes_{O_X} G)_x = F_x \otimes_{O_{X,x}} G_x$ for all $x \in X$.
   b) Let $f : X \to Y$ be a morphism of ringed spaces. See the introduction for the functor f^*.
   i) Prove that for $F, G \in \mathrm{Mod}(Y)$ we have $f^*(F \otimes_{O_Y} G) = f^*F \otimes_{O_X} f^*G$.
   ii) Let $F \in \mathrm{Mod}(X)$ and let $G \in \mathrm{Mod}(Y)$, with G locally free of finite rank (i.e. each $y \in Y$ has an open neighborhood U on which $G|_U$ is isomorphic to $O_U^{n_U}$ for some integer n_U). Prove the **projection formula**
   $$f_*(F \otimes_{O_X} f^*(G)) = f_*(F) \otimes_{O_Y} G.$$
3. (Hom sheaf) a) Let $F, G \in \mathrm{Ab}(X)$. Prove that $U \to \mathrm{Hom}_{\mathrm{Ab}(U)}(F|_U, G|_U)$ is again an abelian sheaf, called $\mathit{Hom}(F, G)$. What happens if we try to consider instead $U \to \mathrm{Hom}_{\mathrm{Ab}}(F(U), G(U))$?
   b) Suppose now that $X = (X, O_X)$ is a ringed space. We have an obvious variant of a), that we still call $\mathit{Hom}(F, G)$, sending U to $\mathrm{Hom}_{O_U}(F|_U, G|_U)$.
   i) Prove that if $F \in \mathrm{Mod}(X)$ is finitely presented (i.e. any point $x \in X$ has an open neighborhood U in X for which there is an exact sequence of O_U-modules $O_U^m \to O_U^n \to 0$ for some integers m, n depending on U), then for all $G \in \mathrm{Mod}(X)$ and $x \in X$ we have $\mathit{Hom}(F, G)_x = \mathrm{Hom}_{O_{X,x}}(F_x, G_x)$.
   ii) Prove that for any **flat** morphism of ringed spaces $f : X \to Y$ (flatness means that the map of rings $O_{Y,f(x)} \to O_{X,x}$ is flat for all $x \in X$) and any $F, G \in \mathrm{Mod}(X)$, with F finitely presented
   $$f^* \mathit{Hom}(F, G) = \mathit{Hom}_{O_X}(f^*F, f^*G).$$

0.8 Godement resolution, flasque sheaves, cohomology

Let X be a topological space. If $F \in \mathrm{Ab}(X)$, set $G(F)(U) = \prod_{x\in U} F_x$ for $U \subset X$ open.

1. Prove that $G(F)$ is a flasque sheaf, and that the natural map $F \to G(F)$ is injective.
2. Define a sequence of sheaves $Q^n(F), G^n(F)$ for $n \geq 0$, with natural injective maps $Q^n(F) \to G^n(F)$ as follows : $G^0(F) = G(F)$, $Q^0(F) = F$, and for $n \geq 1$ set
   $$Q^n(F) = \mathrm{coker}(Q^{n-1}(F) \to G^{n-1}(F)), \quad G_n(F) = G(Q^n(F)),$$
   the map $Q^n(F) \to G^n(F)$ being the natural one.
   a) Prove that $F \to G^n(F)$ are exact functors for $n \geq 0$, and that $G^n(F)$ is a flasque sheaf for all n.
   b) Prove that there is a long exact sequence of sheaves, called the **Godement resolution of** F
   $$0 \to F \to G^0(F) \to G^1(F) \to G^2(F) \to ...$$
3. If U is an open subset of X and $F \in \mathrm{Ab}(X)$, we define the cohomology groups of F over U $H^n(U, F)$ as the cohomology groups of the induced complex $0 \to G^0(F) \to G^1(F) \to ...$ i.e. ($G^n(F) = 0$ for $n < 0$)
   $$H^n(U, F) = \frac{\ker(G^n(F) \to G^{n+1}(F))}{\mathrm{Im}(G^{n-1}(F) \to G^n(F))}.$$
   a) Check that $H^0(U, F) = F(U)$ and that if F if flasque, then $H^n(U, F) = 0$ for $n \geq 1$.
   b) Prove that if $0 \to E \to F \to G \to 0$ is an exact sequence in $\mathrm{Ab}(X)$, then for any open subset U of X we obtain a long exact sequence
   $$0 \to E(U) \to F(U) \to G(U) \to H^1(U, E) \to H^1(U, F) \to H^1(U, G) \to H^2(U, E) \to H^2(U, F) \to ...$$