ENS de Lyon 2017 - 2018

MASTER 1–ALGEBRAIC GEOMETRY WEEK 5, 12.02.2018

TD 4-Sheaves on Spec(A)

Let RS be the category of **ringed spaces**, i.e. pairs $X = (X, O_X)$ with X a topological space and O_X a sheaf of rings on X, morphisms $f : X \to Y$ in RS being pairs $(f, f^\sharp)$ with $f : X \to Y$ a continuous map and $f^\sharp : O_Y \to f_* O_X$ a map of sheaves of rings. Mod(X) **is the category of sheaves of O_X-modules on** X and we set $f^*(F) = O_X \otimes_{f^{-1}O_Y} f^{-1}F$ (sheafification of the obvious presheaf) for $F \in \mathrm{Mod}(X)$.

0.1 Local-global

Let A be a ring and let $f_1, ..., f_n \in A$ be such that $\mathrm{Spec}(A) = \cup_{i=1}^n D(f_i)$, in other words $\sum_{i=1}^n Af_i = A$.

1. a) Prove that for any A-module M the natural map $M \to \prod_{i=1}^n M[1/f_i]$ is injective.
   b) Prove that a complex $0 \to M \to N \to P \to 0$ of A-modules is exact if and only if each of the complexes $0 \to M[1/f_i] \to N[1/f_i] \to P[1/f_i] \to 0$ is exact.
   c) Let B be an A-algebra, M a B-module and $g_1, ..., g_n \in B$ such that $\mathrm{Spec}(B) = \cup_i D(g_i)$. If $M[1/g_i]$ is flat as A-module for all i, then M is flat over A.
2. a) Prove that if M is an A-module such that $M[1/f_i]$ is finitely generated (resp. finitely presented) over $A[1/f_i]$ for all i, then M is finitely generated (resp...) over A.
   b) Let B be an A-algebra. If $B[1/f_i]$ is finitely generated (resp. finitely presented) over A for all i, prove that B is finitely generated (resp...) over A.
3. Prove that if $A[1/f_i]$ is noetherian (or reduced, or Jacobson) for all i, then A is noetherian (or...).
4. Let A be an integral domain and let $X = \mathrm{Spec}(A)$. Prove that if U is a nonempty subset of X, then we have a canonical isomorphism (whose meaning you will have to explain...) $O_X(U) = \cap_{x \in U} O_{X,x}$.

0.2 The category LRS of locally ringed spaces

We say $X \in \mathrm{RS}$ is a **locally ringed space** if $O_{X,x}$ is a local ring for all $x \in X$, in which case we let m_x be the unique maximal ideal of $O_{X,x}$ and $k(x) := O_{X,x}/m_x$. A morphism of locally ringed spaces $f : X \to Y$ is a morphism $(f, f^\sharp)$ of ringed spaces **such that** the map on stalks $O_{Y,f(x)} \to O_{X,x}$ is a local map of local rings, i.e. it sends $m_{f(x)}$ into m_x for all $x \in X$. Let LRS be the category of locally ringed spaces.

1. Recall quickly why there is a natural functor Rings $\to$ LRS, $A \to (\mathrm{Spec}(A), O_{\mathrm{Spec}(A)})$.
2. Let X be a topological space and define $O_X(U)$ as the ring of continuous real-valued functions on U. Prove that (X, O_X) is an object of LRS, describe m_x and $k(x)$ for $x \in X$, and show that any continuous map $f : X \to Y$ induces naturally a morphism in LRS from (X, O_X) to (Y, O_Y).
3. a) Let $X \in \mathrm{LRS}$, $x \in X$, A a ring and finally let $f = (f, f^\sharp) : X \to \mathrm{Spec}(A)$ be a morphism in LRS. Prove that $f(x)$ is the inverse image of m_x under the map $A = O_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \to O_X(X) \to O_{X,x}$.
   b) Let $X \in \mathrm{LRS}$, $f \in O_X(X)$ and let $X_f = \{x \in X | f_x \notin m_x\}$. Prove that X_f is open in X and that $f|_{X_f} \in O_X(X_f)$ is invertible in $O_X(X_f)$.
   c) Prove that $X \in \mathrm{LRS}$ is connected if and only if $O_X(X)$ has no nontrivial idempotents, if and only if $\mathrm{Spec}(O_X(X))$ is connected.

0.3 A fundamental result

1. Prove that for any ring A and any $X \in \mathrm{LRS}$ there is a canonical bijection
   $$\mathrm{Hom}_{\mathrm{LRS}}(X, \mathrm{Spec}(A)) = \mathrm{Hom}_{\mathrm{rings}}(A, O_X(X)).$$
   In particular, there is a canonical map $X \to \mathrm{Spec}(O_X(X))$ for any $X \in \mathrm{LRS}$, and the functor Rings $\to$ LRS, $A \to \mathrm{Spec}(A)$, is fully faithful. **Hint** : use the previous exercise.
2. Let k be a field and let $X = \mathrm{Spec}(k[T_1, T_2])$ and $U = X \setminus V((T_1, T_2))$. What is $O_X(U)$? Deduce that $(U, O_X|_U)$ is not isomorphic in LRS to $\mathrm{Spec}(B)$ for any ring B.

0.4 Glueing

1. Let $(X_i)_{i\in I}$ be locally ringed spaces and let $U_{ij} \subset X_i$ be open subspaces[1] together with isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ in LRS such that $U_{ii} = X_i$, $\varphi_{ii} = \mathrm{Id}_{X_i}$ and for all i, j, k we have $\varphi_{ij}^{-1}(U_{ji}\cap U_{jk}) = U_{ij}\cap U_{ik}$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$. Prove that there exists $X \in$ LRS and open subspaces $U_i \subset X$, as well as isomorphisms $\varphi_i : X_i \to U_i$ in LRS such that $\varphi_i(U_{ij}) = U_i \cap U_j$ and $\varphi_{ij} = \varphi_j^{-1}|_{U_i\cap U_j} \circ \varphi_i|_{U_{ij}}$. Moreover, prove that for any $Y \in$ LRS there is a canonical bijection between $\mathrm{Hom}_{\mathrm{LRS}}(X, Y)$ and the families $(f_i)_{i\in I}$ where $f_i : X_i \to Y$ satisfy $f_j \circ \varphi_{ij} = f_i|_{U_{ij}}$.
2. Let A be a ring, $\mathrm{Spec}(A) = \cup_{i\in I} D(f_i)$ an open covering and M_i an $A[1/f_i]$-module. Suppose that $\psi_{ij} : M_i[1/f_j] \to M_j[1/f_i]$ are isomorphisms of $A[1/(f_i f_j)]$-modules such that $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$ as maps $M_i[1/(f_j f_k)] \to M_k[1/(f_i f_j)]$. Prove that there is an R-module M and identifications $M_i = M[1/f_i]$ compatible with the isomorphisms ψ_{ij}.

   Hint : try to guess the right candidates for the glued objects, and, if you have nothing better to do, check that they work...

0.5 Quasi-coherent sheaves on $\mathrm{Spec}(A)$

Let A be a ring and let $X = \mathrm{Spec}(A)$.

1. Prove that there is an exact functor $A-\text{modules} \to \mathrm{Mod}(X), M \to \tilde{M}$ such that $\tilde{M}(D(f)) = M[1/f]$ for $f \in A$ (the restriction maps being the obvious ones). What is the stalk of $\tilde{M}$ at $x \in X$?
2. Prove that for any $F \in \mathrm{Mod}(X)$ there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{Mod}(X)}(\tilde{M}, F) = \mathrm{Hom}_{A-\text{modules}}(M, F(X)).$$

   Deduce that $M \to \tilde{M}$ is fully faithful and that $\tilde{M}$ is the sheafification of the presheaf $U \to M \otimes_A O_X(U)$.
3. Let $\mathrm{Qcoh}(X)$ be the subcategory of $\mathrm{Mod}(X)$ consisting in **quasi-coherent** O_X-modules, i.e. those F isomorphic to $\tilde{M}$ for some A-module M. Prove that for $F \in \mathrm{Mod}(X)$ we have $F \in \mathrm{Qcoh}(X)$ if and only if F satisfies one of the following equivalent properties :
   - There is a covering $X = \cup_{i\in I} D(f_i)$ and $A[1/f_i]$-modules M_i such that $F|_{D(f_i)} \simeq \tilde{M}_i$.
   - The natural map $F(X) \otimes_A A[1/f] \to F(D(f))$ is an isomorphism for all $f \in A$.
4. a) Prove that kernels and cokernels of maps in $\mathrm{Qcoh}(X)$ are quasi-coherent, and also that arbitrary direct sums (or inductive limits) of quasi-coherent sheaves on X are quasi-coherent.

   b) Let $f : X := \mathrm{Spec}(B) \to Y := \mathrm{Spec}(A)$ be a morphism in LRS and let M (resp. N) be a B-module (resp. A-module). Prove that $f_*\tilde{M}$ and $f^*(\tilde{N})$ are quasi-coherent and describe the A-module (resp. B-module) to which they correspond.

0.6 The sheaf of differentials

Let $R \to A$ be a map of rings. If M is an A-module, let $\mathrm{Der}_R(A, M)$ be the A-module of M-valued derivations of A over R, i.e. R-linear maps $d : A \to M$ such that $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$.

1. Prove that there is an A-module $\Omega^1_{A/R}$ and a derivation $d : A \to \Omega^1_{A/R}$ which is universal, i.e. it induces a canonical bijection $\mathrm{Hom}_A(\Omega^1_{A/R}, M) \to \mathrm{Der}_R(A, M), f \to f \circ d$ for all A-modules M. Moreover, if $I = \ker(A \otimes_R A \to A)$ (the map being multiplication in the R-algebra A) then we have a canonical isomorphism of A-modules $I/I^2 \simeq \Omega^1_{A/R}$ compatible with d and with the derivation $d : A \to I/I^2$, $d(a) = 1 \otimes a - a \otimes 1$. **Hint** : there are many things to be checked...
2. Suppose that $A = R[X_i|i \in I]$ is a polynomial ring over R. Prove that $\Omega^1_{A/R}$ is a free A-module on the basis $dX_i := d(X_i)$. If $A = R[X_i|i \in I]/(f_j|j \in J)$, prove that $\Omega^1_{A/R} \simeq (\oplus_i AdX_i)/(d(f_j))_{j\in J}$.
3. a) Prove that $\Omega^1_{A/R}$ commutes with localization : if $S \subset A$ is a multiplicative subset, then we have a canonical isomorphism $\Omega^1_{A/R} \otimes_A S^{-1}A \simeq \Omega^1_{S^{-1}A/R}$.

   b) Let $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$. Deduce that there is a quasi-coherent sheaf $\Omega^1_{X/S}$ on X such that $\Omega^1_{X/S}(D(a)) = \Omega^1_{A[1/a]/R}$ for $a \in A$.

1. I.e. U_{ij} is an open subset of X_i, endowed with $O_{X_i}|_{U_{ij}}$.

4. Prove that if $R \to A \to B$ and $R \to S$ are maps of rings, then we have a canonical isomorphism $\Omega^1_{A\otimes_R S/S} = \Omega^1_{A/R} \otimes_R S$ and a canonical exact sequence

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

and if $A \to B$ is surjective with kernel I, then $\Omega^1_{B/A} = 0$ and we have a canonical exact sequence

$$I/I^2 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to 0.$$

0.7 Serre's vanishing theorem

Do this exercise (which is completely bonus) only if you did the last exercise in the previous sheet ! We want to prove the following fundamental theorem of Serre : if A is a ring and $X = \mathrm{Spec}(A)$, then for any $F \in \mathrm{Qcoh}(X)$ we have $H^i(X, F) = 0$ for all $i \geq 1$.

1. Prove the following lemma of Kempf : let X be a topological space and $\mathcal{B}$ a basis for X that is closed under finite intersections. Let $n \geq 1$ be an integer and suppose that F is an abelian sheaf on X such that $H^i(U, F|_U) = 0$ for $0 < i < n$ and $U \in \mathcal{B}$. Then for any $c \in H^n(X, F)$ there is a covering of X by open sets $V \in \mathcal{B}$ such that the image of c in $H^n(X, i_*(i^{-1}F))$ is 0, where $i : V \to X$ is the inclusion (note that $i^*(i^{-1}F)(U) = F(U \cap V)$). **Hint** : argue by induction on n, distinguishing the cases $n = 1$ and $n > 1$, and using the exact sequence $0 \to F \to G(F) \to H \to 0$, where $G(F)$ is the Godement sheaf (it is flasque, so has no cohomology in degree > 0).
2. Prove Serre's theorem. **Hint** : argue by induction on i and take for $\mathcal{B}$ the basis of $D(f)$, with $f \in A$.
3. Prove that if $0 \to F \to G \to H \to 0$ is an exact sequence in $\mathrm{Mod}(X)$ and two of the sheaves F, G, H are in $\mathrm{Qcoh}(X)$, then so is the third.