ENS de Lyon 2018 - 2019

Master 1–Algebraic Geometry

TD 5-Schemes-basic properties

If X is a scheme, we denote by $|X|$ its underlying topological space. Recall that for any scheme X and any open subset U of $|X|$ the locally ringed space $(U, O_X|_U)$ is a scheme. We call such schemes $(U, O_X|_U)$ (or simply U) **open subschemes** of X. We say U is an **affine open subscheme** if U is an open subscheme of X and U is an affine scheme (warning : an open subscheme of an affine scheme has no reason to be affine!). An **irreducible component** of a topological space is a maximal irreducible closed subset (the whole space is the union of its irreducible components).

0.1 General (and very useful) principles

Let X be a scheme.

1. Prove that any irreducible closed subset Z of $|X|$ has a unique **generic point** (i.e. whose closure is Z).
2. Prove that the intersection of two affine open subsets U, V of X can be covered by sets that are principal open subsets of both U and V. Deduce that for any affine open covering $X = \cup_i U_i$ and any affine open V of X, there is a covering $V = V_1 \cup ... \cup V_n$ by principal open subsets of V, each of which is also a principal open subset of some U_j.
3. (**local properties of schemes**) A property P of rings is called local if :
   - whenever R has P, so does $R[1/f]$ for any $f \in R$
   - whenever $f_1, ..., f_n \in R$ generate R (as ideal) and $R[1/f_i]$ have P for all i, R has P.

   a) Prove that the following properties of rings are local : being noetherian, reduced, flat over a given ring, finitely generated over a given ring, Jacobson, normal [1].

   b) Let X be a scheme and let P be a local property of rings. We say that X is locally P is there is an open covering $X = \cup_i U_i$ by affine subschemes such that $O_X(U_i)$ has P for all i. Prove that if X is locally P, then for any open affine subscheme U of X the ring $O_X(U)$ has P and any open subscheme of X is locally P.

0.2 Integral and reduced schemes

A scheme X is called **integral** (resp. **reduced**) if $O_X(U)$ is an integral domain (resp. reduced, i.e. no nonzero nilpotents) for all (nonempty) open subsets U of X.

1. Prove that a scheme X is reduced if and only if $O_{X,x}$ is reduced for all $x \in X$ (the analogue for integral schemes is false!).
2. Prove that a scheme X is integral if and only if X is reduced and irreducible (i.e. $|X|$ is irreducible). If X has finitely many irreducible components, this is also equivalent to $|X|$ being connected and $O_{X,x}$ being an integral domain for all $x \in X$.
3. Let X be an integral scheme and $\eta \in X$ its unique generic point (so $\overline{\{\eta\}} = X$). Prove that for all open subsets $U \subset X$ and all $x \in X$ the natural maps $O_X(U) \to O_{X,\eta}$ and $O_{X,x} \to O_{X,\eta}$ are injective, we have $\mathrm{Frac}(O_X(V)) = \mathrm{Frac}(O_{X,x}) = O_{X,\eta}$ and $O_X(U) = \cap_{x \in U} O_{X,x}$ inside $O_{X,\eta}$.
4. (**reduced underlying scheme**) Let X be a scheme, $O_{X_{\mathrm{red}}}$ the sheafification of $U \to O_X(U)/\mathrm{Nil}(O_X(U))$, and consider the ringed space $X_{\mathrm{red}} = (|X|, O_{X_{\mathrm{red}}})$.

   a) Prove that if $X = \mathrm{Spec}(A)$, then $X_{\mathrm{red}} = \mathrm{Spec}(A/\mathrm{Nil}(A))$ (with its structure sheaf).

   b) Prove that X_{red} is a reduced scheme and that there is a natural morphism of schemes $X_{\mathrm{red}} \to X$, inducing a bijection $\mathrm{Hom}(Y, X_{\mathrm{red}}) = \mathrm{Hom}(Y, X)$ for any reduced scheme Y.

---

1. A ring R is normal if $R_{\mathfrak{p}}$ is an integrally closed integral domain for all primes $\mathfrak{p}$ of R.

0.3 Points of a scheme

Let X be a scheme. If S is a scheme, we write $X(S)$ for the set of morphisms of schemes $S \to X$, and we call elements of $X(S)$ **S-points of** X. If $S = \mathrm{Spec}(R)$, we also write $X(R)$ for $X(S)$.

1. Suppose that $X = \mathrm{Spec}(\mathbf{Z}[T_1, ..., T_n]/(f_1, ..., f_k))$. Describe the S-points of X in terms of the ring $O_S(S)$. In particular, what are the S-points of $\mathrm{Spec}(\mathbf{Z})$?
2. a) If K is a field, give a bijection between $X(K)$ and the set of pairs (x, ι), with $x \in X$ and $\iota : k(x) \to K$ a morphism of fields. Describe an equivalence relation on $\coprod_{K \text{ field}} X(K)$, such that the set of equivalence classes is X.

   b) Prove that for any **local** ring R there is a natural bijection between $X(R)$ and the set of pairs (x, φ), with $x \in X$ and $\varphi : O_{X,x} \to R$ a local morphism of local rings.

   c) Deduce that for any $x \in X$ there is a canonical morphism $\mathrm{Spec}(O_{X,x}) \to X$, which is a homeomorphism onto the intersection of all open subsets of X containing x, which is also the set of points specializing to x, i.e. those points y for which $x \in \overline{\{y\}}$.

0.4 Gluing schemes

1. Consider a **gluing datum**, i.e. a family of schemes $(U_i)_{i \in I}$, together with open subschemes $U_{ij} \subset U_i$, as well as isomorphisms $\varphi_{ji} : U_{ij} \simeq U_{ji}$ such that $U_{ii} = U_i$ and $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $U_{ij} \cap U_{ik}$ for all i, j, k (in particular $\varphi_{ji}(U_{ij} \cap U_{ik}) \subset U_{jk}$). Prove that there is a scheme X and morphisms $\psi_i : U_i \to X$ such that ψ_i is an isomorphism onto an open subscheme $V_i = \psi_i(U_i)$ of X, the V_i's form an open covering of X, we have $V_i \cap V_j = \psi_i(U_{ij}) = \psi_j(U_{ij})$ for all i, j and finally $\psi_j \circ \varphi_{ji} = \psi_i$ on U_{ij}. Moreover, X together with the ψ_i has the following universal property : for any scheme T and morphisms $f_i : U_i \to T$ which are isomorphisms onto open subschemes of T and such thay $f_j \circ \varphi_{ji} = f_i$ on U_{ij}, there is a unique morphism $f : X \to T$ such that $f \circ \psi_i = f_i$ for all i.
2. a) By letting $U_{ij} = \emptyset$ for $i \neq j$ in the situation above, we obtain a scheme $X := \coprod_{i \in I} U_i$ called the **disjoint union** of the schemes U_i. What is the topological space and the structure sheaf of X ?

   b) Prove that if I is finite and U_i are affine schemes, then $\coprod_{i \in I} U_i$ is affine, but that this is no longer the case if I is infinite and the U_i's are nonempty.

   c) Describe "concretely" (sic !) what is happening when I has two elements.

0.5 The projective space

Let R be a ring. We define a gluing datum [2] by taking $U_i = \mathrm{Spec}(R[X_j/X_i]_{0 \le j \le n, j \neq i})$ for $0 \le i \le n$ (all rings live inside $R[X_k, 1/X_k]_{0 \le k \le n}$), then set $U_{ij} = D(X_j/X_i) \subset U_i$ for $i \neq j$ and $U_{ii} = U_i$. Finally, set $\varphi_{ii} = \mathrm{id}$ and for $i \neq j$ let $\varphi_{ji} : U_{ij} \to U_{ji}$ be the obvious map.

1. Check that this is indeed a gluing datum. The resulting scheme is called **the projective space $\mathbf{P}^n_R$ over** R. We identify U_i with their (open) images in $\mathbf{P}^n_R$ and denote them $D_+(X_i)$. These $D_+(X_i)$ form an affine open covering of $\mathbf{P}^n_R$.
2. Prove that $O_{\mathbf{P}^n_R}(\mathbf{P}^n_R)$ is naturally isomorphic to R. Deduce that $\mathbf{P}^n_R$ is not affine for $n > 0$.
3. Let k be a field. Prove that there is a natural bijection (cf. exercise 0.3 for the left-hand side)
$$(\mathbf{P}^n_{\mathbf{Z}})(k) = (k^{n+1} \setminus \{0\})/k^*.$$
4. Let I be a homogeneous ideal [3] of $R[X_0, ..., X_n]$. Let $U_i = D_+(X_i)$ and let I_i be the ideal of $O_{U_i}(U_i)$ generated by the polynomials $f(X_0/X_i, ..., X_{i-1}/X_i, 1, ..., X_n/X_i)$ for all homogeneous polynomials f in I. Prove that one can glue the schemes $V_i = \mathrm{Spec}(O_{U_i}(U_i)/I_i)$ along their open subschemes $V_{ij} = D(X_j/X_i) \subset V_i$ to get a scheme $V_+(I)$, coming with a natural morphism $\iota : V_+(I) \to \mathbf{P}^n_R$ (the **vanishing scheme of** I), which identifies $|V_+(I)|$ with a closed subspace of $|\mathbf{P}^n_R|$.
5. (difficult) Let R be a ring. An R-module M is called **invertible** if there is an R-module N such that $M \otimes_R N \simeq R$(this is equivalent to the existence of a covering $\mathrm{Spec}(R) = \cup_{i=1}^n D(f_i)$ such that $M[1/f_j]$ is free of rank 1 over $R[1/f_j]$). Consider the set $X(R)$ of all surjective R-linear maps $\phi : R^{n+1} \to L$,

2. Cf. previous exercise
3. i.e. I is generated by homogeneous polynomials

where L is an invertible R-module, and say that $\phi : R^{n+1} \to L$ is equivalent to $\phi' : R^{n+1} \to L'$ if there is an isomorphism of R-modules $u : L \to L'$ such that $\phi' = u \circ \phi$. Prove that there is a natural bijection between $(\mathbf{P}^n_{\mathbf{Z}})(R)$ and the set of equivalence classes of elements of $X(R)$.

0.6 Dimension theory I

The **dimension** of a topological space X is the supremum of the lengths n of strictly increasing chains $X_0 \subset ... \subset X_n$ of irreducible closed subsets of X ($\dim \emptyset = -\infty$). If X is a scheme, define $\dim X := \dim |X|$, and write $\dim A = \dim(\mathrm{Spec}(A))$. k will always be a field below.

1. Express $\dim A$ in terms of chains of prime ideals in A. What is $\dim k[T]$?
2. a) Prove that if Y is a subspace of a topological space X, then $\dim Y \leq \dim X$. Moreover, if X is irreducible, $\dim X < \infty$ and Y is a proper closed subspace of X, then $\dim Y < \dim X$.

   b) Prove that if $X = \cup_i U_i$ is either an open covering or a **finite** covering by closed subsets, then $\dim X = \sup_i \dim U_i$. Moreover, $\dim X$ is the sup of $\dim C$ over all irreducible components C of X.

   c) Prove that $\dim A = \sup_{\mathfrak{p}} \dim A/\mathfrak{p} = \sup_m \dim A_m$, over all minimal prime ideals $\mathfrak{p}$, resp. maximal ideals m of A. Also, if X is a scheme, then $\dim X = \sup_{x \in X} \dim O_{X,x}$.
3. If $f : A \to B$ is an injective integral morphism of rings, prove that $\dim A = \dim B$.
4. a) Prove that if $f \in k[T_1, ..., T_n]$ is nonconstant, then there is $d < n$ such that $\dim k[T_1, ..., T_n]/(f) = \dim k[X_1, ..., X_d]$. **Hint** : remember the proof of Noether normalization ?

   b) Deduce that $\dim k[T_1, ..., T_n] = n$. Hint : show first that $\dim k[T_1, ..., T_n] \geq n$. For the opposite inequality, argue by induction on n, starting with a chain $\mathfrak{p}_0 \subset ... \subset \mathfrak{p}_d$ of prime ideals in $k[T_1, ..., T_n]$ in which $\mathfrak{p}_0 = 0$, choosing $f \in \mathfrak{p}_1$ nonconstant and using a).
5. a) The **transcendence degree** of an extension K of k is the maximal number of elements of K that are algebraically independent over k(it equals n for $k(T_1, ..., T_n)$). Prove that if A is an integral domain which is finitely generated over k, then $\dim A$ is the transcendence degree over k of $\mathrm{Frac}(A)$.

   b) Prove that if $f \in k[X_1, ..., X_n]$ is irreducible, then $\dim k[X_1, ..., X_n]/(f) = n - 1$.
6. Let A be a finitely generated k-algebra which is an integral domain and let $f \in A$ be a nonzero and non-invertible element. We want to prove **Krull's Hauptidealsatz** [4] : $\dim(A/\mathfrak{p}) = \dim A - 1$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$ which is minimal among primes containing f.

   i) Prove that this is equivalent to : any irreducible component of $V(f)$ has dimension $\dim A - 1$.

   ii) Prove that it suffices to prove the result when $V(f)$ is irreducible. **Hint** : localize with respect to a function $g \in A$ vanishing on all irreducible components of $V(f)$ except the given one.

   iii) (difficult) We assume now that $V(f)$ is irreducible, hence $\sqrt{(f)} = \mathfrak{p}$. Let B be a polynomial ring for which there is a finite injective morphism $B \to A$ (it exists by Noether normalization). Let $\mathfrak{q}$ be the prime ideal of B induced by $\mathfrak{p}$ and let $g = \mathrm{Norm}_{\mathrm{Frac}(A)/\mathrm{Frac}(B)}(f) \in \mathrm{Frac}(B)$. Prove that $g \in B$ and that $\sqrt{(g)} = \mathfrak{q}$. Finish the proof of Krull's theorem.
7. Let A be an integral domain which is finitely generated over k and let $f_1, ..., f_n \in A$. If $B = A/(f_1, ..., f_n)$ is nonzero, then $\dim C \geq \dim A - n$ for any irreducible component C of $\mathrm{Spec}(B)$.
8. Let A be a k-algebra of finite type.

   a) Let $\mathfrak{p}_0 \subset \mathfrak{p}_1$ be different prime ideals of A. Prove that $\dim(A/\mathfrak{p}_1) \leq \dim(A/\mathfrak{p}_0) - 1$, with equality if $\mathfrak{p}_1$ is minimal among prime ideals containing $\mathfrak{p}_0$ properly. Deduce that if $\mathfrak{p}_0 = \mathfrak{q}_0 \subset ... \subset \mathfrak{q}_r = \mathfrak{p}_1$ is a strictly increasing chain of prime ideals of A, then $r \leq \dim(A/\mathfrak{p}_1) - \dim(A/\mathfrak{p}_0)$, with equality if the chain cannot be refined (we say that A is **catenary**).

   b) Let $\mathfrak{p}$ be a prime of A and let $\mathfrak{p} = \mathfrak{p}_0 \subset ... \subset \mathfrak{p}_d$ be a strictly increasing chain of prime ideals. Prove that $d \leq \dim(A/\mathfrak{p})$, with equality if the chain cannot be refined nor extended beyond $\mathfrak{p}_d$.

4. Actually a rather special case of it...