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MASTER 1–ALGEBRAIC GEOMETRY

TD 7-Immersions and the geometry of valuations

Recall that a morphism of schemes $f : X \to Y$ is a **closed immersion** (resp. **open immersion**) if $|f| : |X| \to |Y|$ is a homeomorphism onto a closed (resp. open) subset of $|Y|$ and the map $O_Y \to f_*O_X$ is surjective (resp. $f^{-1}O_Y \to O_X$ is an isomorphism). If $Y = \mathrm{Spec}(R)$, any closed immersion $f : X \to Y$ is isomorphic to $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal I of R. An **immersion** is a morphism of schemes $X \to Y$ which factors $X \to Z \to Y$, where $X \to Z$ is a closed immersion and $Z \to Y$ an open immersion.

A property P of morphisms of schemes is **stable under base change (resp. composition)** if whenever $f : X \to S$ has P and S' is an S-scheme, $X \times_S S' \to S'$ has P (resp. whenever $X \to Y$ and $Y \to Z$ have P, so does $X \to Z$). Being a closed/open immersion or an immersion is stable under base change and composition. Finally, **iff** is a shortcut for "if and only if"...

0.1 Basics on closed immersions

1. Let S' be an S-scheme. If $X \to S$ is a closed immersion, prove that so is $X \times_S S' \to S'$. If $f : X \to Y$ is an S-morphism and a closed immersion, prove that $X \times_S S' \to Y \times_S S'$ is a closed immersion.
2. Let $\mathcal{I} \subset O_X$ be a quasi-coherent sheaf of ideals [1] and $V(\mathcal{I}) = \mathrm{Supp}(O_Y/\mathcal{I})$, a closed subset of $|Y|$.
   a) Letting $i : V(\mathcal{I}) \to Y$ the inclusion, prove that there is a unique sheaf of rings O on $V(\mathcal{I})$ for which $i_*O = O_Y/\mathcal{I}$, and that $V(\mathcal{I}) := (V(\mathcal{I}), O)$ is a scheme with a natural closed immersion $V(\mathcal{I}) \to Y$.
   b) If $f : X \to Y$ is a closed immersion, prove that $\mathcal{I} = \ker(O_Y \to f_*O_X)$ is a quasi-coherent sheaf of ideals of O_Y and there is a unique factorisation $X \simeq V(\mathcal{I}) \to Y$ of f.
3. Let $Z \to X$ be a closed immersion, with Z reduced. Prove that a morphism $f : Y \to X$, with Y reduced, factors through $Z \to X$ iff $f(Y) \subset \mathrm{Im}(Z \to X)$ set-theoretically.
4. Let $T \subset X$ be a closed subset of a scheme X. Prove that $U \to \mathcal{I}(U) = \{f \in O_X(U) | f(t) = 0 \in k(t),\ \forall t \in T \cap U\}$ is a quasi-coherent sheaf of ideals of O_X and that there is a natural bijection $V(\mathcal{I}) = T$ (this scheme structure on T is called the **reduced induced scheme structure on** T).

0.2 Diagonal morphism, graphs and equalizers

If X is an S-scheme, its **diagonal morphism** $\Delta_{X/S} : X \to X \times_S X$ is given on T-points s by $\Delta(s) = (s, s) : X(T) \to (X \times_S X)(T) = X(T) \times_{S(T)} X(T)$. If $f : X \to Y$ is an S-morphism of schemes, its **graph** is the morphism $\Gamma_f : X \to X \times_S Y$ given on T-points by $x \in X(T) \to (x, f(x)) \in X(T) \times_{S(T)} Y(T) = (X \times_S Y)(T)$.

1. Prove that $\Delta_{X/S}$ is a closed immersion if X, S are affine. Deduce that $\Delta_{X/S}$ is always an immersion, and it's a closed immersion iff its image is closed. **Warning** : $\mathrm{Im}(\Delta_{X/S}) \subset \{z \in X \times_S X | p(z) = q(z)\}$ (p, q are the natural projections $X \times_S X \to X$), but this inclusion may not be an equality.
2. For an S-morphism $f : X \to Y$ prove that Γ_f is the base change of $\Delta_{Y/S}$ by $X \times_S Y \to Y \times_S Y$ (given on T-points by $(x, y) \to (f(x), y)$), i.e. [2] $X \simeq Y \times_{(Y \times_S Y)} (X \times_S Y)$, in such a way that the natural projections to Y and $X \times_S Y$ are f and Γ_f. Deduce that Γ_f is always an immersion.
3. Let $f, g : X \to Y$ be S-morphisms of schemes. Define $\mathrm{eq}(f, g) = X \times_{(X \times_S Y)} X$, where the two maps $X \to X \times_S Y$ are Γ_f and Γ_g. Prove that there is a canonical immersion $\mathrm{eq}(f, g) \to X$, inducing a functorial bijection [3] $\mathrm{eq}(f, g)_S(T) = \{x \in X_S(T) | f(x) = g(x)\}$ in the S-scheme T. Moreover, $x \in X$ is in the image of $\mathrm{eq}(f, g) \to X$ iff $f(x) = g(x)$ and the two maps $k(f(x)) = k(g(x)) \to k(x)$ coincide.
4. Prove that $f : X \to S$ is a monomorphism (i.e. $X(T) \to S(T)$ is injective for all T) iff $\Delta_{X/S}$ is an isomorphism, and that f is universally injective [4] iff $\Delta_{X/S}$ is surjective.

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1. i.e. a sheaf of ideals whose restriction to any affine open U of X is a quasi-coherent O_U-module.
2. Seen as $Y \times_S Y$-schemes via $\Delta_{Y/S}$ and via the morphism $X \times_S Y \to Y \times_S Y$ above.
3. We also write f, g for the induced maps $X_S(T) \to Y_S(T)$.
4. i.e. $X \times_S S' \to S'$ is injective for all S-schemes S', or equivalently $X(K) \to S(K)$ is injective for all fields K.

0.3 The diagonal morphism, separated morphisms

We say that $f : X \to S$ is **separated** (resp. **quasi-separated**) if $\Delta_{X/S}$ is a closed immersion (resp. a **quasi-compact morphism**, i.e. the inverse image of any quasi-compact open subset is quasi-compact).

1. Prove that each of the following statements is equivalent to $f : X \to S$ being separated :
   - for any S-morphisms $f, g : T \to X$, $\mathrm{eq}(f, g) \to T$ is a closed immersion.
   - For any S-morphism $f : T \to X$ the graph Γ_f is a closed immersion.
2. a) Prove that a morphism of affine schemes is separated. Also, any immersion is separated.
   b) Prove that being separated (resp. quasi-separated) is stable under base change and composition. Moreover, an S-morphism of schemes $f : X \to Y$ is separated (resp. quasi-separated) if $X \to S$ is so.
3. a) Let $f, g : X \to Y$ be S-morphisms of schemes such that $f|_U = g|_U$ for an open dense subscheme U of X. If $Y \to S$ is separated and X is reduced, prove that $f = g$.
   b) Let $f, g : Z := \mathrm{Spec}(k[X, Y]/(XY, Y^2)) \to \mathrm{Spec}(k[T]/T^2)$ be induced by the maps on rings sending T to 0 (resp. Y). Prove that there is an open dense subset $U \subset Z$ such that $f|_U = g|_U$, yet $f \neq g$.
4. a) Prove that $X \to \mathrm{Spec}(R)$ is separated if and only if $U \cap V$ is affine and $O_X(U) \otimes_R O_X(V) \to O_X(U \cap V)$ is surjective for any affine opens U, V of X. Deduce that $\mathbf{P}^n \to \mathrm{Spec}(\mathbf{Z})$ is separated.
   b) If $X \to S$ and $S \to \mathrm{Spec}(\mathbf{Z})$ are separated, then $U \cap V$ is affine and a closed subscheme of $U \times V$ for any affine opens U, V of X. If $S \to \mathrm{Spec}(\mathbf{Z})$ is separated and $f : X \to S$ is any morphism, then $U \cap f^{-1}(V)$ is affine for any open affine subschemes $V \subset S$ and $U \subset X$.

0.4 Geometry of valuation rings

A **valuation ring** is an integral domain V such that for all nonzero $x \in \mathrm{Frac}(V)$ we have $x \in V$ or $1/x \in V$.

1. Prove that any valuation ring is normal (i.e. integrally closed in its field of fractions) and local.
2. (hard) Let K be a field and let $A \subset K$ be a local subring such that $\mathrm{Frac}(A) = K$.
   a) Prove that A is a valuation ring if and only if for any local ring B with $A \subset B \subset K$ and such that $A \to B$ is a local map we have $A = B$. **Hint** : use your favorite commutative algebra book !
   b) Deduce that there is a valuation ring V such that $A \subset V \subset K$ and such that $A \to V$ is a local map.
   If X is a scheme and $x, x' \in X$ we say that x **is a specialization of** x' and write $x' \rightsquigarrow x$ if $x \in \overline{\{x'\}}$. A morphism $f : X \to S$ is **specializing** if any specialization of $f(x')$ (with $x' \in X$) is of the form $f(x)$ for a specialization x of x'.
3. a) If X is a scheme and $x' \rightsquigarrow x$ in X, prove that there is a valuation ring A with $\mathrm{Frac}(A) = k(x')$ and a morphism $\mathrm{Spec}(A) \to X$ sending the closed point to x and the generic point to x'.
   b) If $f : X \to S$ is a quasi-compact morphism of schemes [5], then $f(X)$ is closed in S iff $f(X)$ is stable under specialization (i.e. if $s' \rightsquigarrow s$ and $s' \in f(X)$ then $s \in f(X)$), and f is closed iff f is specializing.
4. (**valuative criteria**) Let $f : X \to S$ be a morphism of schemes.
   a) Prove that $X \times_S S' \to S'$ is specializing for any S-scheme S' iff $X_S(V) \to X_S(\mathrm{Frac}(V))$ is surjective for any valuation ring V with a map $\mathrm{Spec}(V) \to S$. If f is quasi-compact, this is also equivalent to f being **universally closed** (i.e. $X \times_S S' \to S'$ is closed for all S-schemes S').
   b) Prove that $f : X \to S$ is separated iff f is quasi-separated and $X_S(V) \to X_S(\mathrm{Frac}(V))$ is injective for any valuation ring V with a map $\mathrm{Spec}(V) \to S$.

0.5 Properness

A morphism of schemes $f : X \to S$ is called **proper** if f is of finite type [6], separated and universally closed.

1. a) Prove that being proper is stable under base change and composition. Moreover, $f : X \to S$ is proper if and only if $f^{-1}(U_i) \to U_i$ are proper, for an open covering $S = \cup_i U_i$.
   b) Let $f : X \to Y$ be an S-morphism, with $Y \to S$ separated. If $X \to S$ is proper, then f is proper.
   c) "The image of a proper scheme is proper" : if $f : X \to Y$ is a surjective S-morphism, with $X \to S$ proper and $Y \to S$ separated and of finite type, then $Y \to S$ is proper.

5. i.e. the inverse image of any quasi-compact open is quasi-compact.
6. i.e. f is quasi-compact and $O_S(V) \to O_X(U)$ is of finite type whenever U, V are affine opens of X, S such that $f(U) \subset V$.

2. a) (**valuative criterion**) Prove that f is proper iff f is quasi-separated, of finite type and $X_S(V) \to X_S(\mathrm{Frac}(V))$ is bijective for any valuation ring V with a map $\mathrm{Spec}(V) \to S$.
   b) If R is a Dedekind domain and $X \to \mathrm{Spec}(R)$ is proper, prove that $X(R) \to X(\mathrm{Frac}(R))$ is bijective.
3. a) Prove that any closed immersion, as well as $\mathbf{P}^n \to \mathrm{Spec}(\mathbf{Z})$ is proper.
   b) Prove that $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[T]) \to \mathrm{Spec}(\mathbf{Z})$ is not proper.
   c) (hard) If $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is universally closed (e.g. proper), then[7] $B \to A$ is integral.
4. (hard) Let k be an algebraically closed field and $X \to \mathrm{Spec}(k)$ a proper morphism, with X reduced and connected. Prove that $O_X(X) = k$. **Hint** : see f as a morphism $X \to \mathbf{A}^1_k \subset \mathbf{P}^1_k$.

0.6 Proper normal curves over a field

This exercise is hard and fully uses the results in exercises 4,5. Fix a field k. A **curve** over k is a reduced, irreducible scheme C of dimension 1, with a morphism of finite type $C \to \mathrm{Spec}(k)$. Its function field $K(C) = O_{C,\eta}$ ($\eta \in C$ being the generic point of C) has transcendence degree 1 over k. Call C **normal** if $O_{C,x}$ is a discrete valuation ring (i.e. a noetherian valuation ring, equivalently[8] normal) for all $x \in C \setminus \{\eta\}$. C **will always denote a normal curve over k and K will always denote an extension of k of transcendence degree** 1. For such K/k, its **Riemann-Zariski space** $RZ(K/k)$ is the set of valuation rings V such that $k \subset V \subset K$ and $\mathrm{Frac}(V) = K$, with the the topology for which a nonempty $U \subset RZ(K/k)$ is open if $K \in U$ (note that $K \in RZ(K/k)$) and $RZ(K/k) \setminus U$ is finite.

1. a) Prove that $\{K\}$ is dense in $RZ(K/k)$, and any other point of $RZ(K/k)$ is closed. Moreover, for any $f \in K$ the set $\{V \in RZ(K/k) | f \in V\}$ is open. Deduce that a nonempty $U \subset RZ(K/k)$ is open if and only if U is a union of sets of the form $\{V \in RZ(K/k) | f_1, ..., f_n \in V\}$, with $f_1, ..., f_n \in K$. **Hint** : if $V \in RZ(K/k)$ and $f \notin V$, study the integral closure of $k[1/f]$ in K.
   b) Prove that if $V \in RZ(K/k)$, then either $V = K$ or V is a discrete valuation ring.
2. Let C be a normal curve over k and let $K = K(C)$ be its function field.
   a) Prove that $x \to O_{C,x}$ induces a continuous open map $\iota : |C| \to RZ(K/k)$.
   b) Prove that $C \to \mathrm{Spec}(k)$ is separated (resp. proper) if and only if ι is an open embedding (resp. a homeomorphism). **Hint** : use several times the valuative criteria.
3. Fix K/k of transcendence degree 1 and let $X = RZ(K/k)$, a topological space. Define a pre-sheaf of rings O_X on X by setting $O_X(U) = \cap_{V \in U} V \subset K$ for $U \subset X$ nonempty.
   a) Prove that O_X is a sheaf of k-algebras on X, and that $O_{X,V} \simeq V$ for all $V \in X$, in particular $X := (X, O_X)$ is a locally ringed space.
   b) Let C be a normal curve over k, such that $K(C) = K$. Prove that there is a unique morphism of locally ringed spaces $f : C \to X$ compatible with the natural maps to $\mathrm{Spec}(k)$. Moreover, this map is an open immersion if $C \to \mathrm{Spec}(k)$ is separated.
   c) Prove that X is a normal curve over k, $X \to \mathrm{Spec}(k)$ is proper and $K(X) \simeq K$. Conversely, if C is a normal curve over k, there is a natural isomorphism of k-schemes $C \simeq RZ(K(C))$.

7. It is actually enough to assume that $\mathrm{Spec}(A[T]) \to \mathrm{Spec}(B[T])$ is closed.
8. Though this is not really trivial...