A REMARK ON THE HIGHER TORSION INVARIANTS FOR FLAT VECTOR BUNDLES WITH FINITE HOLONOMY

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ABSTRACT. We show that the Igusa-Klein topological torsion and the Bismut-Lott analytic torsion are equivalent for any flat vector bundle whose holonomy is a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Our proof uses Artin's induction theorem in representation theory to reduce the problem to the special case of trivial flat line bundles, which is a recent result of Puchol, Zhu and the second author. The idea of using Artin's induction theorem appeared in a paper of Ohrt on the same topic, of which our present work is an improvement.
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0. INTRODUCTION

The theory of topological torsion was developed by Franz [12], Reidemeister [30], de Rham [10], Milnor [23], Whitehead [34] and many others. The analytic torsion, which is an analogue of the topological torsion, was defined by Ray and Singer [29].

Cheeger [9] and Müller [24] independently proved that the topological torsion and the analytic torsion coincide for unitarily flat vector bundles. This result is now known as the Cheeger-Müller theorem. Bismut and Zhang [5] extended the theorem to the general case. Independently, Müller [25] extended the theorem to the unimodular case. There are also various extensions to equivariant cases [6, 20, 21].

Wagoner [33] conjectured the existence of higher topological/analytic torsion invariants. The conjectured invariant should be an invariant for pairs $(M \to S, F)$, where $M \to S$ is a smooth fibration with compact fiber and F is a flat vector bundle over M. Bismut and Lott [4] confirmed the analytic side of Wagoner's conjecture by constructing the so-called *Bismut-Lott analytic torsion*. Igusa [18] confirmed the topological side of Wagoner's conjecture by constructing the so-called *Igusa-Klein topological torsion*. Goette, Igusa and Williams [17, 16] used Igusa-Klein topological torsion to detect the exotic smooth structure of fiber bundles. Dwyer, Weiss and Williams

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[11] constructed another topological torsion. The relation among these higher torsion invariants (in the most general case) is still unknown.

Bismut and Goette [2] showed that the Bismut-Lott torsion and the Igusa-Klein torsion are equivalent if there exists a fiberwise Morse function $f : M \to \mathbb{R}$ satisfying the Morse-Smale transversality [32]. In fact, Bismut and Goette extended the Bismut-Lott torsion to the equivariant case and proved their result in the equivariant context. Goette [13, 14] extended the results in [2] to arbitrary fiberwise Morse functions. There are also related works in [3, 7]. We refer to the survey by Goette [15] for an overview on higher torsion invariants. Goette also proposed a program extending the argument in [13, 14] to functions with both non-degenerate critical points and birth-death critical points.

Igusa [19] axiomatized higher torsion invariants for trivial flat line bundles. He showed that any invariant satisfying the additivity axiom and the transfer axiom is essentially the Igusa-Klein torsion. Badzioch, Dorabiala, Klein and Williams [1] showed that the Dwyer-Weiss-Williams torsion satisfies Igusa's axioms. Using the results in [22] and [27], Puchol, Zhu and the second author [28] showed that the Bismut-Lott torsion satisfies Igusa's axioms. As a result, all these higher torsion invariants are equivalent for trivial flat line bundles.

As for arbitrary flat vector bundles, Ohrt [26] proposed a similar axiomatization approach for higher torsion invariants. Under the assumption that the fibrations under consideration have simple fibers, he showed that any invariant satisfying his axioms is essentially the Igusa-Klein torsion. Puchol, Zhu and the second author [28] showed that the Bismut-Lott torsion also satisfies Ohrt's axioms.

The purpose of this paper is to explore the relation between the Igusa-Klein torsion and the Bismut-Lott torsion without restrictions on the fibrations. Instead, we need to assume that the holonomy of the flat vector bundle in question lies in $\mathrm{GL}_n(\mathbb{Q})$. Our result is related to the transfer index conjecture proposed by Bunke and Gepner [8].

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1. MAIN RESULT

Let $M \to S$ be a smooth fibration. Let Z be the fiber. Let F be a flat vector bundle over M. We assume that

- $\pi_1(S)$ is finite;
- Z is closed and oriented;
- the holonomy group of F is finite.

These assumptions appear in [26]. For $M \to S$ and F as above, we denote by

$$\tau^{\mathrm{BL}}(M/S, F) \in H^{\mathrm{even}\geqslant 2}(S) \tag{1.1}$$

its Bismut-Lott analytic torsion class [4] (cf. [28, Definition 2.1]), and denote by

$$\tau^{\mathrm{IK}}(M/S, F) \in H^{\mathrm{even}\geqslant 2}(S) \tag{1.2}$$

its Igusa-Klein topological torsion class [18]. For $k \in \mathbb{N}$ and a class $a \in H^\bullet(S)$, let $a^{[k]} \in H^k(S)$ be its component of degree k. Set

$$\begin{aligned} \tau^{\mathrm{an}}(M/S,F) &= \sum_k \left\{ \frac{2^{2k}\big(k!\big)^2}{(2k+1)!} \tau^{\mathrm{BL}}(M/S,F) \right\}^{[2k]} , \\ \tau^{\mathrm{top}}(M/S,F) &= \sum_k \left\{ - \frac{k!}{(2\pi)^k} \tau^{\mathrm{IK}}(M/S,F) + \frac{\zeta'(-k)\mathrm{rk}F}{2} \int_Z \mathrm{e}(TZ)\mathrm{ch}(TZ) \right\}^{[2k]} , \end{aligned} \tag{1.3}$$

where $\int_Z : H^\bullet(M) \to H^\bullet(S)$ is the integration along the fiber, $\mathrm{e}(TZ)$ (resp. $\mathrm{ch}(TZ)$) is the Euler class of the relative tangent bundle TZ (resp. the Chern character of $TZ \otimes_{\mathbb{R}} \mathbb{C}$) , and ζ is the Riemann zeta function. The first identity in (1.3) is the Chern normalization introduced by Bismut and Goette [2, Defintion 2.37].

Let $\pi_1(M)$ be the fundamental group of M. Let $\widetilde{M}$ be the universal cover of M, which is canonically equipped with a right group action of $\pi_1(M)$. For a group homomorphism $\rho : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$ with finite image, set

$$F_\rho = \widetilde{M} \times_\rho \mathbb{C}^n , \tag{1.4}$$

which is a flat vector bundle over M with finite holonomy. For convenience, we denote

$$\tau^{\mathrm{an/top}}(M/S,\rho) = \tau^{\mathrm{an/top}}(M/S,F_\rho) . \tag{1.5}$$

For a finite Galois extension $K/\mathbb{Q}$, we denote by $\mathrm{Gal}(K/\mathbb{Q})$ its Galois group. For $g \in \mathrm{Gal}(K/\mathbb{Q})$ and a group homomorphism $\rho : \pi_1(M) \to \mathrm{GL}_n(K)$, we define

$$\begin{aligned} g.\rho : \pi_1(M) &\to \mathrm{GL}_n(K) \\ \gamma &\mapsto \big(g\big(\rho(\gamma)_{i,j}\big)\big)_{1\leqslant i,j\leqslant n} , \end{aligned} \tag{1.6}$$

where $\rho(\gamma)_{i,j} \in K$ are the entries of the matrix $\rho(\gamma) \in \mathrm{GL}_n(K)$.

Theorem 1. *For a smooth manifold S with finite fundamental group, a smooth fibration $M \to S$ with closed oriented fiber, a finite Galois extension $K/\mathbb{Q}$ and a group homomorphism $\rho : \pi_1(M) \to \mathrm{GL}_n(K)$ with finite image, we have*

$$\sum_{g\in\mathrm{Gal}(K/\mathbb{Q})} \tau^{\mathrm{an}}(M/S,g.\rho) = \sum_{g\in\mathrm{Gal}(K/\mathbb{Q})} \tau^{\mathrm{top}}(M/S,g.\rho) . \tag{1.7}$$

In particular, for a homomorphism $\rho : \pi_1(M) \to \mathrm{GL}_n(\mathbb{Q})$ with finite image, we have

$$\tau^{\mathrm{an}}(M/S,\rho) = \tau^{\mathrm{top}}(M/S,\rho) . \tag{1.8}$$

2. A consequence of Artin's induction theorem

Let G be a finite group. Let $R(G)$ be its representation ring with rational coefficients. In other words, as a $\mathbb{Q}$-vector space,

$$R(G) = \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathbb{Q}\,\rho , \tag{2.1}$$

where $\mathrm{Irr}(G)$ is the set of isomorphism classes of irreducible (complex) representations of G. The ring structure of $R(G)$ is given by tensor products.

For $\rho \in R(G)$, let $\chi_\rho : G \to \mathbb{C}$ be its character. We denote

$$R_{\mathrm{rat}}(G) = \big\{\rho \in R(G) \ : \ \chi_\rho(g) \in \mathbb{Q} \text{ for any } g \in G\big\} \ . \tag{2.2}$$

For any subgroup $H \leq G$, let $\mathbb{1} \in R(H)$ be the one-dimensional trivial representation, let $\mathrm{Ind}_H^G \mathbb{1} \in R(G)$ be the induced representation. Clearly, we have

$$\mathrm{Ind}_H^G \mathbb{1} \in R_{\mathrm{rat}}(G) \ . \tag{2.3}$$

Lemma 2. *The vector space $R_{\mathrm{rat}}(G)$ is spanned by $\big(\mathrm{Ind}_H^G \mathbb{1}\big)_{H \leq G}$.*

The lemma above is a consequence of [31, Exercise 13.8]. We still give a proof for the sake of completeness.

Proof. Let $\mathcal{S}(G) \subseteq R(G)$ be the vector subspace spanned by $\big(\mathrm{Ind}_H^G \mathbb{1}\big)_{H \leq G}$.
Claim 1. We have

$$\mathrm{Ind}_H^G \mathcal{S}(H) \subseteq \mathcal{S}(G) \ \text{ for any } H \leq G \ . \tag{2.4}$$

This is an immediate consequence of the identity $\mathrm{Ind}_H^G \mathrm{Ind}_J^H = \mathrm{Ind}_J^G$ for $J \leq H \leq G$.
Claim 2. For a surjective homomorphism $f : G \to G'$, we have

$$f^* \mathcal{S}(G') \subseteq \mathcal{S}(G) \ , \tag{2.5}$$

where $f^* : R(G') \to R(G)$ is defined by $f^*\rho = \rho \circ f$. This is an immediate consequence of the identity $f^* \mathrm{Ind}_{H'}^{G'} = \mathrm{Ind}_H^G$ for any $H' \leq G'$ and $H = f^{-1}(H')$.
Claim 3. For finite groups G and G', we have

$$\begin{aligned} & R(G \times G') = R(G) \otimes R(G') \ , \\ & R_{\mathrm{rat}}(G \times G') = R_{\mathrm{rat}}(G) \otimes R_{\mathrm{rat}}(G') \ , \quad \mathcal{S}(G \times G') \supseteq \mathcal{S}(G) \otimes \mathcal{S}(G') \ . \end{aligned} \tag{2.6}$$

Now we are ready to prove the lemma by induction. If $|G| = 1$, the lemma obviously holds. Assume that

$$\mathcal{S}(H) = R_{\mathrm{rat}}(H) \ \text{ for any finite group } H \text{ with } |H| < N \ . \tag{2.7}$$

We consider a finite group G with $|G| = N$. We need to show that $R_{\mathrm{rat}}(G) \subseteq \mathcal{S}(G)$. Let $K/\mathbb{Q}$ be a finite Galois extension such that all the representations of all the groups of order $\leqslant N$ may take values in $\mathrm{GL}_n(K)$. There are three cases.
Case 1. The group G is not cyclic.
Let $\rho \in R_{\mathrm{rat}}(G)$. We obviously have

$$\rho = \frac{1}{[K : \mathbb{Q}]} \sum_{g \in \mathrm{Gal}(K/\mathbb{Q})} g.\rho \ . \tag{2.8}$$

On the other hand, by Artin's induction theorem, there exist cyclic subgroups $H_1, \cdots, H_m \leq G$ and $\big(\varphi_k \in R(H_k)\big)_{k=1,\cdots,m}$ such that

$$\rho = \sum_{k=1}^m \mathrm{Ind}_{H_k}^G \varphi_k \ . \tag{2.9}$$

By (2.8) and (2.9), we have

$$\rho = \frac{1}{[K:\mathbb{Q}]} \sum_{k=1}^{m} \mathrm{Ind}_{H_k}^{G} \Big(\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k \Big) \ . \tag{2.10}$$

Note that H_k is cyclic while G is not, by our hypothesis (2.7), we have

$$\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k \in R_{\mathrm{rat}}(H_k) = \mathcal{S}(H_k) \ . \tag{2.11}$$

From (2.10), (2.11) and Claim 1, we obtain $\rho \in \mathcal{S}(G)$. Hence $R_{\mathrm{rat}}(G) \subseteq \mathcal{S}(G)$.

Case 2. The group G is cyclic, and there exist non trivial cyclic groups G' and G'' such that $G = G' \times G''$.

By our hypothesis (2.7) and Claim 3, we have

$$R_{\mathrm{rat}}(G) = R_{\mathrm{rat}}(G') \otimes R_{\mathrm{rat}}(G'') = \mathcal{S}(G') \otimes \mathcal{S}(G'') \subseteq \mathcal{S}(G) \ . \tag{2.12}$$

Case 3. The group G is cyclic, and $|G| = p^r$ where p is a prime number.

Let $a \in G$ be a generator of G. For $k = 0, \cdots, p^r - 1$, let φ_k be the one dimensional representation defined by $\varphi_k(a) = \exp\big(2k\pi i/p^r\big)$. Then $\big(\varphi_k\big)_{k=0,\cdots,p^r-1}$ is a basis of $R(G)$. Thus $R_{\mathrm{rat}}(G)$ is spanned by

$$\Big(\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k \Big)_{k=0,\cdots,p^r-1} \ . \tag{2.13}$$

If k is a multiple of p, then $\varphi_k : G \to \mathbb{C}^*$ is not injective. There exists a surjective group homomorphism $f : G \to G'$ with $|G'| < |G|$ and $\varphi'_k \in R(G')$ such that $\varphi_k = f^*\varphi'_k$. We have

$$\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k = f^* \Big(\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi'_k \Big) \in f^* R_{\mathrm{rat}}(G') \ . \tag{2.14}$$

Then, by our hypothesis (2.7) and Claim 2, we have

$$\sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k \in f^*\mathcal{S}(G') \subseteq \mathcal{S}(G) \ . \tag{2.15}$$

If k is not a multiple of p, we can directly verify that

$$\begin{aligned} \frac{1}{[K:\mathbb{Q}]} \sum_{g\in \mathrm{Gal}(K/\mathbb{Q})} g.\varphi_k &= \frac{1}{p^{r-1}(p-1)} \Big(\sum_{k=0}^{p^r-1} \varphi_k - \sum_{k=0}^{p^{r-1}-1} \varphi_{kp} \Big) \\ &= \frac{1}{p^{r-1}(p-1)} \Big(\mathrm{Ind}_{\{e\}}^{G} \mathbb{1} - \mathrm{Ind}_{\langle a^{p^{r-1}} \rangle}^{G} \mathbb{1} \Big) \in \mathcal{S}(G) \ , \end{aligned} \tag{2.16}$$

where $\{e\}$ is the trivial subgroup and $\langle a^{p^{r-1}} \rangle$ is the subgroup generated by $a^{p^{r-1}}$. In conclusion, each element in (2.13) lies in $\mathcal{S}(G)$. Hence $R_{\mathrm{rat}}(G) \subseteq \mathcal{S}(G)$. ☐

The key ingredient in the proof of Lemma 2 is Artin's induction theorem, which is also used in [26, §5].

3. Proof of the main result

Now we assume that there is a surjective group homomorphism $\mu : \pi_1(M) \to G$. Then any linear representation of G may be viewed a representation of $\pi_1(M)$. Since

$$\tau^{\mathrm{an/top}}(M/S, \rho \oplus \rho') = \tau^{\mathrm{an/top}}(M/S, \rho) + \tau^{\mathrm{an/top}}(M/S, \rho') \tag{3.1}$$

for ρ, ρ' linear representations of G, we have a $\mathbb{Q}$-linear map

$$\begin{aligned} \delta : R(G) &\to H^{\mathrm{even}\geqslant 2}(S) \\ \rho &\mapsto \tau^{\mathrm{an}}(M/S, \rho) - \tau^{\mathrm{top}}(M/S, \rho)\ . \end{aligned} \tag{3.2}$$

Lemma 3. *For any subgroup $H \leq G$, we have $\mathrm{Ind}_H^G \mathbb{1} \in \ker \delta$.*

Proof. Let $M' = \widetilde{M} \times_\mu (G/H)$, which is a finite covering of M. By [26, Remark 5.2] and the induction formula for Bismut-Lott torsion [28, Theorem 2.4], we have

$$\tau^{\mathrm{an}}(M'/S, \mathbb{1}) = \tau^{\mathrm{an}}\big(M/S, \mathrm{Ind}_H^G \mathbb{1}\big)\ . \tag{3.3}$$

By [26, Remark 5.2] and the induction formula for Igusa-Klein torsion [26, Definition 2.1, Theorem 3.1], we have

$$\tau^{\mathrm{top}}(M'/S, \mathbb{1}) = \tau^{\mathrm{top}}\big(M/S, \mathrm{Ind}_H^G \mathbb{1}\big)\ . \tag{3.4}$$

On the other hand, by the higher Cheeger-Müller/Bismut-Zhang theorem for trivial flat line bundles [28, Theorem 0.1], we have

$$\tau^{\mathrm{an}}(M'/S, \mathbb{1}) = \tau^{\mathrm{top}}(M'/S, \mathbb{1})\ . \tag{3.5}$$

From (3.2)-(3.5), we obtain $\delta\big(\mathrm{Ind}_H^G \mathbb{1}\big) = 0$. □

Proof of Theorem 1. Let $\rho : \pi_1(M) \to \mathrm{GL}_n(K)$ be as in Theorem 1. Denote its image by G, which is a finite group by assumption. We may and we will view ρ as a representation of G. We obviously have

$$\sum_{g \in \mathrm{Gal}(K/\mathbb{Q})} g.\rho \in R_{\mathrm{rat}}(G)\ . \tag{3.6}$$

On the other hand, by Lemma 2 and Lemma 3, we have

$$R_{\mathrm{rat}}(G) \subseteq \ker \delta\ . \tag{3.7}$$

From (3.2), (3.6), (3.7) and the linearity of δ, we obtain (1.7). □

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