



Motivic integration and birational invariance of BCOV invariants

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Abstract

Bershadsky, Cecotti, Ooguri and Vafa constructed a real valued invariant for Calabi–Yau manifolds, which is now called the BCOV torsion. Based on it, a metric-independent invariant, called BCOV invariant, was constructed by Fang–Lu–Yoshikawa and Eriksson–Freixas i Montplet–Mourougane. The BCOV invariant is conjecturally related to the Gromov–Witten theory via mirror symmetry. Based upon previous work of the second author, we prove the conjecture that birational Calabi–Yau manifolds have the same BCOV invariant. We also extend the construction of the BCOV invariant to Calabi–Yau varieties with Kawamata log terminal singularities, and prove its birational invariance for Calabi–Yau varieties with canonical singularities. We provide an interpretation of our construction using the theory of motivic integration.

Keywords Analytic torsion · Calabi–Yau manifolds · Motivic integration · Mirror symmetry

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0 Introduction

0.1 Background: mirror symmetry

BCOV torsion, introduced by Bershadsky, Cecotti, Ooguri, and Vafa in the outstanding papers [8, 9], is a real valued invariant for Calabi–Yau manifolds equipped with Ricci-flat metrics [59]. More precisely, let X be a Calabi–Yau manifold, i.e., a compact Kähler manifold with trivial canonical bundle, and let ω be a Ricci-flat Kähler metric, the *BCOV torsion* of (X, ω) is the weighted product

$$\mathcal{T}_{\text{BCOV}}(X, \omega) := \prod_{p=1}^{\dim X} \mathcal{T}_p^{(-1)^p p}, \quad (0.1)$$

where \mathcal{T}_p is the analytic torsion, introduced by Ray–Singer [52], of the p -th exterior power of the holomorphic cotangent bundle $\wedge^p(T^*X)$ equipped with the induced metric.

The motivation of Bershadsky–Cecotti–Ooguri–Vafa [8, 9] comes from string theory and has impact on mirror symmetry, which predicts that for a family of Calabi–Yau manifolds, there is another family of Calabi–Yau manifolds with maximal degeneration, called the *mirror* family, such that the symplectic geometry (e.g. Gromov–Witten invariants) of the first family, called the *A-model*, is “equivalent” to the complex geometry (e.g. variation of Hodge structures) of the mirror family, called the *B-model*. Candelas–de la Ossa–Green–Parkes [18] conjectured a precise relation between the potential (J -function) of the genus zero Gromov–Witten invariants of quintic threefolds (*A-model*) and the potential (I -function) of the Yukawa coupling for the quintic

mirror family (B-model). Such a relation is expected to hold in general for mirror Calabi–Yau families (see [50]) and gives surprising predictions in enumerative geometry. This genus zero mirror symmetry conjecture was proved by Givental [38, 39] and Lian–Liu–Yau [46] for a large class of examples including the original case of quintic threefolds. Bershadsky–Cecotti–Ooguri–Vafa [8, 9] computed certain invariants on the B-model that conjecturally correspond to *higher genus* Gromov–Witten invariants. This allows them to put forth conjectural formulas for all genus Gromov–Witten invariants of quintic threefolds. The genus one part of this conjecture was proved by Zinger [63, 64] in the broader setting of Calabi–Yau hypersurfaces in projective spaces. A lot of progress has been made recently on the study of Gromov–Witten invariants of genus ≥ 2 (see [20–25, 40, 41] and references therein). Despite the recent increasing interest on A-model invariants in genus ≥ 2 , the research on B-model invariants is currently focused on the genus one theory and the particular case of Bershadsky–Cecotti–Ooguri–Vafa’s B-model invariant corresponding to the genus one Gromov–Witten invariant is the aforementioned BCOV torsion (0.1).

The central object in this paper is the following normalization of the BCOV torsion, called the *BCOV invariant*. Let X be an n -dimensional Calabi–Yau manifold equipped with a Ricci-flat metric of Kähler form ω , its BCOV invariant [31, 34] is defined¹ by

$$\mathcal{T}(X) := \mathcal{T}_{\text{BCOV}}(X, \omega) \left(\prod_{k=1}^n \text{covol}_{L^2} \left(H^k(X, \mathbb{Z}), \omega \right)^{(-1)^k k} \right) \left((2\pi)^{-n} \int_X \frac{\omega^n}{n!} \right)^{\frac{\chi(X)}{12}}, \tag{0.2}$$

where $\chi(X)$ is the topological Euler characteristic of X , and $\text{covol}_{L^2} \left(H^k(X, \mathbb{Z}), \omega \right)$ is the covolume of the lattice $\text{Im} \left(H^k(X, \mathbb{Z}) \rightarrow H^k(X, \mathbb{R}) \right)$ with respect to the L^2 -metric induced by ω . The virtue of the BCOV invariant is that it depends only on the complex structure, but not on the Kähler metric. Fang–Lu–Yoshikawa [34] constructed the BCOV invariant for strict Calabi–Yau threefolds and studied its asymptotic behavior along degenerations. Their work confirmed the conjectural formula of Bershadsky–Cecotti–Ooguri–Vafa [8, 9] for the BCOV invariant near the large complex structure limit of the quintic mirror family (see [34, Conjecture 1.2 (B)]). Eriksson–Freixas i Montplet–Mourougane [31] generalized the construction as well as the asymptotic study of the BCOV invariant to Calabi–Yau manifolds of arbitrary dimension. They proved in [32] a higher-dimensional generalization of the conjectured formula mentioned above, and showed the compatibility with Zinger’s result on the A-model [63, 64], thus completing the genus one mirror symmetry conjecture of Bershadsky–Cecotti–Ooguri–Vafa [8, 9] in this case.

Throughout this paper, for an n -dimensional Calabi–Yau manifold X , we will use the following “normalized logarithmic BCOV invariant”:

$$\tau(X) = \log \mathcal{T}(X) + \frac{\log(2\pi)}{2} \sum_{k=0}^{2n} (-1)^k k(k-n) b_k(X), \tag{0.3}$$

¹ Here we use the definition of [31], which differs from the one in [34] by an explicit power of 2π .

where $b_k(X)$ is the k -th Betti number of X . This normalization comes from [61, (0.13)].

0.2 Birational invariance conjecture

As two birationally isomorphic Calabi–Yau varieties share the same mirror, their BCOV invariants should coincide. This leads to the following conjecture.

Conjecture 0.1 *For birational Calabi–Yau manifolds X and X' , we have $\tau(X) = \tau(X')$.*

In view of (0.3), Conjecture 0.1 is equivalent to say that $\mathcal{T}(X) = \mathcal{T}(X')$, since birational Calabi–Yau manifolds have the same Betti numbers, by Batyrev [5].

Conjecture 0.1 was proposed in dimension 3 by Yoshikawa [60, Conjecture 2.1], and in arbitrary dimension by Eriksson–Freixas i Montplet–Mourougane [31, Conjecture B]. Fang–Lu–Yoshikawa [34, Conjecture 4.17] stated a weaker form of this conjecture.

Since the BCOV invariant can be thought as a “secondary” analogue of variation of Hodge structures associated with deformations of Calabi–Yau manifolds, Conjecture 0.1 is a “secondary” analogue of the theorem of Batyrev [5] and Kontsevich [45] that birational Calabi–Yau manifolds have the same Hodge numbers.

Several results were obtained towards Conjecture 0.1:

- Maillot and Rössler [49, Theorem 1.1] showed that for two smooth projective Calabi–Yau threefolds X, X' defined over a subfield K of \mathbb{C} such that $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are birational, then for any fixed finite set T of complex embeddings of K , there exist $n \in \mathbb{N}_{>0}$ and $\alpha \in K^{\times}$, such that

$$\tau(X'_{\sigma}) - \tau(X_{\sigma}) = \frac{1}{n} \log |\sigma(\alpha)| \quad \text{for any } \sigma \in T, \quad (0.4)$$

where $X_{\sigma} := X \otimes_{K, \sigma} \mathbb{C}$. Maillot and Rössler also proved the same result under the strictly more general hypothesis that $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are derived equivalent.²

- The second author [62, Corollary 0.5] proved Conjecture 0.1 for Atiyah flops of $(-1, -1)$ -curves in Calabi–Yau threefolds.

0.3 Main results

In this paper, we confirm Conjecture 0.1.

Theorem A *Let X and X' be projective Calabi–Yau manifolds. If X and X' are birationally isomorphic, then $\tau(X) = \tau(X')$.*

The BCOV invariants can be extended to projective manifolds with torsion canonical bundle (or equivalently, with vanishing first Chern class by [7]), see [61]. Theorem

² i.e., their bounded derived categories of coherent sheaves are equivalent as \mathbb{C} -linear triangulated categories. Note that the derived equivalence of birational Calabi–Yau threefolds was proved by Bridgeland [15, Theorem 1.1] and there are derived equivalent Calabi–Yau threefolds that are not birationally equivalent [14, 27, 56].

A still holds in this more general case. In fact, we can prove the birational invariance in a much broader setting involving singular varieties.

We call a normal projective complex variety X a *Calabi–Yau variety with canonical (resp. KLT) singularities*, if it has canonical (resp. Kawamata log terminal) singularities (cf. [44] or Definition 6.1) and $K_X \sim_{\mathbb{Q}} 0$, where $\sim_{\mathbb{Q}}$ is the linear equivalence relation for \mathbb{Q} -Cartier divisors. We will propose a natural definition of the BCOV invariant for Calabi–Yau varieties with KLT singularities (see Definition 6.9), which we still denote by τ . It coincides with the usual one in the smooth case. Theorem A admits the following extension:

Theorem B *Let X and X' be Calabi–Yau varieties with canonical singularities. If X and X' are birationally isomorphic, then $\tau(X) = \tau(X')$.*

A resolution of singularities $f: \tilde{X} \rightarrow X$ is called *crepant* if the relative canonical divisor $K_{\tilde{X}/X}$ is trivial. By Theorem B, the BCOV invariant of a canonical Calabi–Yau variety equals to the BCOV invariant of any crepant resolution. Note that neither the existence nor the uniqueness of crepant resolution is guaranteed. Bridgeland–King–Reid [16] proved its existence in dimension 3 for Gorenstein quotient singularities.

It is worth mentioning that in the recent work [28], Dai and Yoshikawa constructed examples showing that certain analytically defined BCOV invariants for orbifolds is *not* a birational invariant already in dimension 2. The orbifold surfaces in their examples have singular points worse than canonical (i.e. du Val) singularities, hence compatible with our Theorem B. Nevertheless, quotient singularities are KLT and it is highly interesting to compare our extended definition and the analytic definition for Calabi–Yau orbifolds; see Remark 6.10.

The *curvature formula* is of fundamental importance in the theory of BCOV invariants. We refer the readers to [34, Theorem 4.9], [31, Proposition 5.10] and [61, Theorem 0.4] for the precise formulation in the smooth case. We have the following curvature formula for the BCOV invariant of locally trivial deformation families (in the sense of Flenner–Kosarew [35], cf. Definition 6.11) of KLT Calabi–Yau varieties.

Theorem C *Let S be a complex manifold. Let $\pi: \mathcal{X} \rightarrow S$ be a flat family of normal projective KLT Calabi–Yau varieties. Let $X_s = \pi^{-1}(s)$ for $s \in S$. Assume that π is locally trivial. Then the following function is C^∞ ,*

$$\begin{aligned} \tau(\mathcal{X}/S): S &\rightarrow \mathbb{R} \\ s &\mapsto \tau(X_s). \end{aligned} \tag{0.5}$$

Moreover, we have the following identity of $(1, 1)$ -forms on S ,

$$\frac{\bar{\partial}\partial}{2\pi i} \tau(\mathcal{X}/S) = \omega_{\text{Hdg}, \mathcal{X}/S} - \frac{\chi(X)}{12} \omega_{\text{WP}, \mathcal{X}/S}, \tag{0.6}$$

where $\chi(X)$ is the stringy Euler characteristic of X_s (cf. Definition 6.8), $\omega_{\text{Hdg}, \mathcal{X}/S}$ is the Hodge form of the family \mathcal{X}/S (cf. Definition 6.14) and $\omega_{\text{WP}, \mathcal{X}/S}$ is the Weil–Petersson form of the family \mathcal{X}/S (cf. Definition 6.16).

It is still in active research to lay a rigorous foundation of a mathematical theory of B-model invariants in genus ≥ 2 . Once such a theory is built, its birational invariance will be of great importance. We hope that our results on genus one can serve as the first step towards the big picture.

0.4 Overview of proof

To highlight the key ideas, we only explain the proof of Theorem A here. The proof contains three main ingredients.

a) BCOV invariant for pairs The BCOV invariant for Calabi–Yau manifolds was extended by the second author [61] to all pairs (X, γ) with X a compact Kähler manifold and γ a meromorphic canonical form on X such that $\text{div}(\gamma)$ is of simple normal crossing support and without component of multiplicity -1 .

We denote $\text{div}(\gamma) = D = m_1 D_1 + \cdots + m_l D_l$ and $D_J = \bigcap_{j \in J} D_j$ for $J \subseteq \{1, \dots, l\}$. By convention, $D_\emptyset = X$ and $\prod_{j \in \emptyset} \frac{-m_j}{m_j+1} = 1$. The BCOV invariant of (X, γ) is defined by

$$\tau(X, \gamma) = \sum_{J \subseteq \{1, \dots, l\}} \left(\prod_{j \in J} \frac{-m_j}{m_j + 1} \right) \tau_{\text{BCOV}}(D_J, \omega) + \text{correction terms}, \quad (0.7)$$

where ω is a Kähler form on X , $\tau_{\text{BCOV}}(D_J, \omega)$ is the (logarithmic) BCOV torsion of $(D_J, \omega|_{D_J})$ (see Definition 1.3), and the correction terms are given by Bott–Chern forms, making $\tau(X, \gamma)$ independent of ω (see Definition 3.2).

b) Blow-up formula The second author [61] worked out the precise behavior of the extended BCOV invariant (0.7) under a blow-up (see Theorem 3.6). The formula of the second author expresses the change of the BCOV invariant under a blow-up in terms of the BCOV invariant of projective spaces endowed with some canonical form, together with certain topological data. The work of the second author is based on the technique of deformation to the normal cone of Baum–Fulton–MacPherson [6, §1.5], and on series of work of Bismut and his collaborators on the Quillen metric [10–13].

c) Birational BCOV To confirm Conjecture 0.1, by the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1, 57], it suffices to normalize the BCOV invariant $\tau(X, \gamma)$ in (0.7) in such a way that the normalized BCOV invariant does not change under blow-ups. The normalization in this paper is a linear combination of the Betti numbers of the strata $\{D_J\}_{J \subseteq \{1, \dots, l\}}$.

0.5 BCOV invariants and motivic integration

It might seem mysterious that the weighted sum in (0.7) happens to be the right object to study, which eventually allows us to prove the birational invariance of the BCOV invariant. One conceptual progress made in this paper is an explanation of the

construction of $\tau(X, \gamma)$ (see (0.7)) using Kontsevich’s motivic integration [45]. It is worth mentioning that the technique of motivic integration already appeared in a less refined form in the paper of Maillot–Rössler [49] mentioned above.

Let (X, γ) be as in a). We temporarily assume that X is projective and $D = \text{div}(\gamma)$ is effective. Let $Z(X, \mathcal{I}_D; \mathbb{L}^{-1})$ be the motivic Igusa zeta function (see Sect. 4.1) associated with (X, D) , evaluated at \mathbb{L}^{-1} . Its Hodge realization can be computed as follows:

$$H^\bullet(Z(X, \mathcal{I}_D; \mathbb{L}^{-1})) = \sum_{J \subseteq \{1, \dots, l\}} \mathbb{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - \mathbb{L}^{m_j}}{\mathbb{L}^{m_j+1} - 1} \right) H^\bullet(D_J), \tag{0.8}$$

where \mathbb{L} on the right hand side is understood as the operator of tensoring with the Lefschetz Hodge structure $\mathbb{Z}(-1)$. The following observation is crucial,

$$H^\bullet(Z(X, \mathcal{I}_D; \mathbb{L}^{-1}))|_{\mathbb{L}=1} = \sum_{J \subseteq \{1, \dots, l\}} \left(\prod_{j \in J} \frac{-m_j}{m_j + 1} \right) H^\bullet(D_J), \tag{0.9}$$

where the coefficients are exactly the same as in (0.7). Using (0.9), we will show in Sect. 4 that the BCOV invariant $\tau(X, \gamma)$ is essentially the Quillen metric on

$$\bigotimes_k \left(\det H^k(Z(X, \mathcal{I}_D; \mathbb{L}^{-1})) \right)^{(-1)^k k}. \tag{0.10}$$

By the change of variables formula in motivic integration (cf. Theorem 4.2), the virtual Hodge structure $H^\bullet(Z(X, \mathcal{I}_D; \mathbb{L}^{-1}))$ in (0.8) is a birational invariant. Hence the virtual determinant line in (0.10) is also a birational invariant, in the sense that there is a canonical isomorphism between the complex lines (0.10) associated with two birational varieties equipped with effective simple normal crossing divisors. This partially explains the reason why $\tau(X, \gamma)$ is almost a birational invariant.

This paper is organized as follows.

In Sect. 1, we give a reminder on the Quillen metric and the BCOV torsion. A discussion on simple normal crossing divisors is also included.

In Sect. 2, we develop some basic properties of the so-called localizable and log-type invariants, which will appear repeatedly throughout the paper.

In Sect. 3, we recall the construction of $\tau(X, \gamma)$ and collect several fundamental properties of the BCOV invariant.

In Sect. 4, we explain the construction of $\tau(X, \gamma)$ using motivic integration.

In Sect. 5, we construct a birational BCOV invariant.

In Sect. 6, we extend the BCOV invariant to the singular cases and prove Theorem C.

In Sect. 7, we prove Theorems A and B.

Convention: When we write a divisor $D = \sum_{j=1}^l m_j D_j$, we implicitly assume that the D_j ’s are distinct prime divisors. For a complex manifold X and a complex submanifold Y , we denote by $\text{Bl}_Y X$ the blow-up of X along Y .

1 Preliminaries

1.1 Quillen metric and topological torsion

Let X be a compact Kähler manifold of dimension n . For any holomorphic vector bundle E over X , its *determinant* line of cohomology [43] is

$$\lambda(E) = \det H^\bullet(X, E) := \bigotimes_{q=0}^n (\det H^q(X, E))^{(-1)^q}. \tag{1.1}$$

For any Kähler metric on X and any Hermitian metric on E , one can define the so-called *Quillen metric* [51] on the determinant line $\lambda(E)$, see [13, Definition 1.10].

For $p = 0, \dots, n$, set

$$\lambda_p(X) = \lambda \left(\bigwedge^p (T^*X) \right) = \bigotimes_{q=0}^n (\det H^{p,q}(X))^{(-1)^q}. \tag{1.2}$$

The determinant lines $\lambda_p(X)$, with their Quillen metrics, will be the basic building blocs in the construction of the BCOV invariant. Before doing that, let us first introduce a natural invariant that will appear later, called topological torsion, and recall its vanishing.

Set

$$\eta(X) = \det H_{\text{dR}}^\bullet(X) := \bigotimes_{k=0}^{2n} (\det H_{\text{dR}}^k(X))^{(-1)^k}. \tag{1.3}$$

By the Hodge decomposition

$$H_{\text{dR}}^k(X) = \bigoplus_{p+q=k} H^{p,q}(X), \text{ for any } 0 \leq k \leq n, \tag{1.4}$$

we have

$$\eta(X) = \bigotimes_{p=0}^n (\lambda_p(X))^{(-1)^p}. \tag{1.5}$$

We fix a square root of i . This choice will be irrelevant. We identify the de Rham cohomology $H_{\text{dR}}^k(X)$ with the singular cohomology $H_{\text{Sing}}^k(X, \mathbb{C})$ as follows,

$$\begin{aligned} H_{\text{dR}}^k(X) &\rightarrow H_{\text{Sing}}^k(X, \mathbb{C}) \\ [\alpha] &\mapsto \left[\mathfrak{a} \mapsto (2\pi i)^{-k/2} \int_{\mathfrak{a}} \alpha \right], \end{aligned} \tag{1.6}$$

where α is a closed k -form and \mathfrak{a} is a k -chain in X . The identification (1.6) endows $H_{\text{dR}}^k(X)$ with an integral structure. Let ϵ_X be a generator of the induced integral structure on $\eta(X)$. More precisely, for $k = 0, \dots, 2n$, let

$$\sigma_{k,1}, \dots, \sigma_{k,b_k} \in H_{\text{Sing}}^k(X, \mathbb{Z})_{\text{tf}} \tag{1.7}$$

be a \mathbb{Z} -basis of the quotient of $H_{\text{Sing}}^k(X, \mathbb{Z})$ modulo its subgroup of torsion elements. Then $\sigma_{k,1}, \dots, \sigma_{k,b_k}$ form a basis of $H_{\text{dR}}^k(X)$. Set

$$\epsilon_X = \bigotimes_{k=0}^{2n} (\sigma_{k,1} \wedge \dots \wedge \sigma_{k,b_k})^{(-1)^k} \in \eta(X), \tag{1.8}$$

which is well-defined up to ± 1 .

Let ω be a Kähler form on X , which induces a Hermitian metric on $\bigwedge^p(T^*X)$ for any p . Let $\|\cdot\|_{\lambda_p(X), \omega}$ be the Quillen metric on $\lambda_p(X)$ associated with ω . Let $\|\cdot\|_{\eta(X)}$ be the metric on $\eta(X)$ induced by $\|\cdot\|_{\lambda_p(X), \omega}$ via (1.5). Proceeding in the same way as in the proof of [62, Theorem 2.1], we can show that $\|\cdot\|_{\eta(X)}$ is independent of ω .

Definition 1.1 We define the *topological torsion* of X as

$$\tau_{\text{top}}(X) = \log \|\epsilon_X\|_{\eta(X)}. \tag{1.9}$$

The identification (1.6) allows us to have the following vanishing result.

Proposition 1.2 ([61, Proposition 1.24]) *For any compact Kähler manifold X , we have*

$$\tau_{\text{top}}(X) = 0. \tag{1.10}$$

1.2 BCOV torsion

Keep the same setting of Sect. 1.1. Following [9, §5.8], we consider the weighted product of determinant lines $\lambda_p(X)$ defined in (1.2).

$$\lambda(X) = \bigotimes_{0 \leq p, q \leq n} (\det H^{p,q}(X))^{(-1)^{p+q}p} = \bigotimes_{p=1}^n (\lambda_p(X))^{(-1)^p p}. \tag{1.11}$$

Set

$$\lambda_{\text{dR}}(X) = \bigotimes_{k=1}^{2n} (\det H_{\text{dR}}^k(X))^{(-1)^k k}. \tag{1.12}$$

By the Hodge decomposition (1.4), we have

$$\lambda_{\text{dR}}(X) = \lambda(X) \otimes \overline{\lambda(X)}. \tag{1.13}$$

The identity (1.13) appeared in Kato [42, last identity in §1.3] and was first applied to this setting in [31].

Let $\|\cdot\|_{\lambda(X),\omega}$ be the metric on $\lambda(X)$ induced by $\|\cdot\|_{\lambda_p(X),\omega}$ via (1.11). Let $\|\cdot\|_{\lambda_{\text{dR}}(X),\omega}$ be the metric on $\lambda_{\text{dR}}(X)$ induced by $\|\cdot\|_{\lambda(X),\omega}$ via (1.13). Let σ_X be the integral generator of $\lambda_{\text{dR}}(X)$ defined as follows, using the \mathbb{Z} -basis of $H_{\text{Sing}}^\bullet(X, \mathbb{Z})_{\text{tf}}$ in (1.7),

$$\sigma_X = \bigotimes_{k=1}^{2n} (\sigma_{k,1} \wedge \cdots \wedge \sigma_{k,b_k})^{(-1)^k k}. \tag{1.14}$$

Definition 1.3 We define the *BCOV torsion* of (X, ω) as

$$\tau_{\text{BCOV}}(X, \omega) = \log \|\sigma_X\|_{\lambda_{\text{dR}}(X),\omega}. \tag{1.15}$$

In the case where X is a Calabi–Yau manifold equipped with a Ricci-flat metric ω , this invariant $\tau_{\text{BCOV}}(X, \omega)$ is precisely the logarithm of the product of the first two factors on the right hand side of (0.2).

1.3 Divisor with simple normal crossing support

For $I \subseteq \{1, \dots, n\}$, set

$$\mathbb{C}_I^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ for } i \in I\} \subseteq \mathbb{C}^n. \tag{1.16}$$

Let X be a complex manifold of dimension n . Let $Y_1, \dots, Y_l \subseteq X$ be closed complex submanifolds.

Definition 1.4 We say that Y_1, \dots, Y_l *transversally intersect* if for any $x \in X$, there exists a holomorphic local chart $\mathbb{C}^n \supseteq U \xrightarrow{\varphi} X$ such that

- $0 \in U$ and $\varphi(0) = x$;
- for each $k = 1, \dots, l$, either $\varphi^{-1}(Y_k) = \emptyset$, or $\varphi^{-1}(Y_k) = U \cap \mathbb{C}_{I_k}^n$ for certain $I_k \subseteq \{1, \dots, n\}$.

Remark 1.5 *The definition of transversal intersection above is more general than the usual one.* Let $Y, Z \subseteq X$ be connected complex submanifold which transversally intersect in the sense of Definition 1.4. If $Y \subseteq X$ is of codimension 1, then

- either Y and Z transversally intersect in the usual sense,
- or $Z \subseteq Y$.

We denote

$$D = \sum_{j=1}^l m_j D_j, \tag{1.17}$$

where $m_j \in \mathbb{Z} \setminus \{0\}$ and $D_1, \dots, D_l \subseteq X$ are mutually distinct prime divisors.

Definition 1.6 A divisor D on X is said with *simple normal crossing support* if D_1, \dots, D_l are smooth and transversally intersect.

Now let D be a divisor on X with simple normal crossing support. Let L be the holomorphic line bundle $\mathcal{O}_X(D)$. Denote by $\mathcal{M}(X, L)$ the space of meromorphic sections of L . Let $\gamma \in \mathcal{M}(X, L)$ such that $\text{div}(\gamma) = D$. Let L_j be the normal line bundle of $D_j \hookrightarrow X$.

Definition 1.7 We define $\text{Res}_{D_j}(\gamma) \in \mathcal{M}(D_j, L \otimes L_j^{-m_j})$ as the image of $\gamma \in \mathcal{M}(X, L)$ via the canonical isomorphism

$$L(-m_j D_j)|_{D_j} \simeq L|_{D_j} \otimes L_j^{-m_j}. \tag{1.18}$$

Let $\mathbb{C}^n \supseteq U \xrightarrow{\varphi} X$ be a local chart as in Definition 1.4. Assume that

$$\gamma|_{\varphi(U)} = s\varphi_*(z_1^{m_1} \cdots z_n^{m_r}), \tag{1.19}$$

where $0 \leq r \leq n$ and $s \in H^0(\varphi(U), L)$ is nowhere vanishing. For $j = 1, \dots, r$, we have

$$\text{Res}_{D_j}(\gamma)|_{D_j \cap \varphi(U)} = s\varphi_*(z_1^{m_1} \cdots z_{j-1}^{m_{j-1}} z_{j+1}^{m_{j+1}} \cdots z_r^{m_r} (dz_j)^{m_j}), \tag{1.20}$$

where dz_j is viewed as a conormal vector of $\{z_j = 0\} \subseteq \mathbb{C}^n$.

Note that

$$\text{div}(\text{Res}_{D_1}(\gamma)) = \sum_{j=2}^l m_j (D_1 \cap D_j). \tag{1.21}$$

The following identity holds in $\mathcal{M}(D_1 \cap D_2, L \otimes L_1^{-m_1} \otimes L_2^{-m_2})$,

$$\text{Res}_{D_1 \cap D_2}(\text{Res}_{D_1}(\gamma)) = \text{Res}_{D_1 \cap D_2}(\text{Res}_{D_2}(\gamma)). \tag{1.22}$$

In other words, the order of taking $\text{Res}(\cdot)$ does not matter.

2 Localizable invariants

2.1 Definitions and examples

Definition 2.1 Let Käh be the category of compact Kähler manifolds. Let $\phi : \text{Käh} \rightarrow \mathbb{R}$ be a function that depends only on the isomorphism classes of compact Kähler manifolds.

- ϕ is called a *localizable invariant* if for any $X, X' \in \text{K\"ah}$, and closed complex submanifolds $Y \subseteq X, Y' \subseteq X'$ such that $Y \simeq Y'$ and $N_{Y/X} \simeq N_{Y'/X'}$, we have

$$\phi(\text{Bl}_Y X) - \phi(X) = \phi(\text{Bl}_{Y'} X') - \phi(X'). \tag{2.1}$$

- ϕ is called of *log-type*³ if for any $X \in \text{K\"ah}$ and V a holomorphic vector bundle of rank r over X , we have

$$\phi(\mathbb{P}(V)) = \chi(\mathbb{C}P^{r-1})\phi(X) + \chi(X)\phi(\mathbb{C}P^{r-1}). \tag{2.2}$$

- ϕ is called *additive* if for any $X \in \text{K\"ah}$ and $Y \subseteq X$ a closed complex submanifold, we have

$$\phi(\text{Bl}_Y X) - \phi(X) = \phi(\mathbb{P}(N_{Y/X})) - \phi(Y). \tag{2.3}$$

An additive invariant is clearly localizable. A linear combination of localizable (resp. log-type, additive) invariants is again localizable (resp. of log-type, additive).

Let us give several examples of such invariants that will play important roles later.

Example 2.2 Let X be a compact Kähler manifold.

- For any $k \in \mathbb{N}$, the k -th Betti number $b_k(X)$ is an additive invariant. Let

$$P_t(X) = \sum_{k=0}^{2 \dim X} b_k(X)t^k \tag{2.4}$$

be the Poincaré polynomial. For any $t \in \mathbb{R}$, $P_t(X)$ is an additive invariant. In particular, the topological Euler characteristic

$$\chi(X) = P_{-1}(X) \tag{2.5}$$

is additive.

- The invariant

$$\chi'(X) = \left. \frac{d}{dt} P_t(X) \right|_{t=-1} = \dim(X)\chi(X) \tag{2.6}$$

is of log-type and additive. To show that $\chi'(X)$ is of log-type (i.e., identity (2.2)), we take the derivative of the identity $P_t(\mathbb{P}(V)) = P_t(X)P_t(\mathbb{C}P^{r-1})$.

- The invariant

$$\chi''(X) = \left. \frac{d^2}{dt^2} P_t(X) \right|_{t=-1} - \dim(X)^2\chi(X) = P_t(X) \left. \frac{d^2}{dt^2} \log P_t(X) \right|_{t=-1} \tag{2.7}$$

³ The terminology refers to the fact that if $\chi(X) \neq 0$, then $\frac{\phi(\mathbb{P}(V))}{\chi(\mathbb{P}(V))} = \frac{\phi(X)}{\chi(X)} + \frac{\phi(\mathbb{C}P^{r-1})}{\chi(\mathbb{C}P^{r-1})}$.

is of log-type and localizable (but not additive). To show that $\chi''(X)$ is of log-type (i.e., identity (2.2)), we take the second derivative of the logarithm of the identity $P_t(\mathbb{P}(V)) = P_t(X)P_t(\mathbb{C}P^{r-1})$.

2.2 Localizable invariant for pairs

Let d be a non-zero integer.

Definition 2.3 For a compact Kähler manifold X and a divisor

$$D = \sum_{j=1}^l m_j D_j \tag{2.8}$$

on X , we say that (X, D) satisfies condition (\star_d) if D is of simple normal crossing support and $m_j \neq -d$ for all j .

We will always use the following notation. For $J \subseteq \{1, \dots, l\}$, set

$$w_d^J = \prod_{j \in J} \frac{-m_j}{m_j + d}, \quad D_J = \bigcap_{j \in J} D_j. \tag{2.9}$$

By convention, $w_\emptyset^d = 1$ and $D_\emptyset = X$.

Definition 2.4 Let ϕ be a localizable invariant. Let $d \in \mathbb{Z} \setminus \{0\}$. For (X, D) satisfying the condition (\star_d) , we define

$$\phi_d(X, D) = \sum_{J \subseteq \{1, \dots, l\}} w_d^J \phi(D_J). \tag{2.10}$$

If there is a meromorphic section γ of a holomorphic line bundle over X such that $\text{div}(\gamma) = D$, we define $\phi_d(X, \gamma) = \phi_d(X, D)$.

Let $[\xi_0 : \dots : \xi_n] \in \mathbb{C}P^n$ be homogenous coordinates. For $j = 0, \dots, n$, we denote $H_j = \{\xi_j = 0\} \subseteq \mathbb{C}P^n$. For $m_0, \dots, m_n \in \mathbb{Z}$, we denote

$$D_{m_0, \dots, m_n} = \sum_{j=0}^n m_j H_j. \tag{2.11}$$

Recall that χ is the topological Euler characteristic. Replacing ϕ by χ in Definition 2.4, we get $\chi_d(\cdot, \cdot)$.

Lemma 2.5 For $d \in \mathbb{Z} \setminus \{0\}$ and $m_0, \dots, m_n \in \mathbb{Z} \setminus \{-d\}$, we have

$$\chi_d(\mathbb{C}P^n, D_{m_0, \dots, m_n}) = \left(\prod_{j=0}^n (m_j + d) \right)^{-1} d^n \sum_{j=0}^n (m_j + d). \tag{2.12}$$

It follows that $\chi_d(\mathbb{C}P^n, D_{m_0, \dots, m_n})$ vanishes if and only if D_{m_0, \dots, m_n} is a d -canonical divisor.

Proof Let w_d^J be as in (2.9). By Definition 2.4, we have

$$\chi_d(\mathbb{C}P^n, D_{m_0, \dots, m_n}) = \sum_{J \subseteq \{0, \dots, n\}} w_d^J (n + 1 - |J|). \tag{2.13}$$

Set

$$f(t) = \prod_{j=0}^n \left(t - \frac{m_j}{m_j + d} \right) = \sum_{J \subseteq \{0, \dots, n\}} w_d^J t^{n+1-|J|}. \tag{2.14}$$

By (2.13) and (2.14), we have

$$\chi_d(\mathbb{C}P^n, D_{m_0, \dots, m_n}) = f'(1), \quad \frac{f'(1)}{f(1)} = \frac{d}{dt} \log f(t) \Big|_{t=1} = \sum_{j=0}^n \frac{m_j + d}{d}. \tag{2.15}$$

From (2.14) and (2.15), we obtain (2.12). This completes the proof.

Let Y be a compact Kähler manifold and V be a holomorphic vector bundle of rank r over Y . Set $X = \mathbb{P}(V)$. Let $\pi : X \rightarrow Y$ be the canonical projection. Let D be a divisor on X . We assume that there exist a divisor D_Y on Y , non-zero integers m_1, \dots, m_s , and holomorphic sub-bundles $V_1, \dots, V_s \subseteq V$ of rank $r - 1$, such that

$$D = \pi^* D_Y + \sum_{j=1}^s m_j \mathbb{P}(V_j). \tag{2.16}$$

We further assume that $V_1, \dots, V_s \subseteq V$ transversally intersect. In particular, $s \leq r$. We will use the convention $m_{s+1} = \dots = m_r = 0$. For $y \in Y$, we denote $Z_y = \pi^{-1}(y)$. Set

$$D_{Z_y} = \sum_{j=1}^s m_j (\mathbb{P}(V_j) \cap Z_y). \tag{2.17}$$

Then (Z_y, D_{Z_y}) is isomorphic to $(\mathbb{C}P^{r-1}, D_{m_1, \dots, m_r})$ for any $y \in Y$. In the sequel, we omit the index y in (Z_y, D_{Z_y}) . Such a pair (X, D) will be called a *fibration over (Y, D_Y) with fiber (Z, D_Z)* .

Lemma 2.6 *Assume that D satisfies the condition (\star_d) in Definition 2.3. We have*

$$\chi_d(X, D) = \chi_d(Y, D_Y) \chi_d(Z, D_Z). \tag{2.18}$$

Proof It is a straightforward computation from Definition 2.4 by using the fact that $\chi(\cdot)$ is an additive invariant and is multiplicative with respect to products of varieties.

Proposition 2.7 *Let ϕ be a log-type localizable invariant and $d \in \mathbb{Z} \setminus \{0\}$. Assume that D is a d -canonical divisor and satisfies the condition (\star_d) . We have*

$$\phi_d(X, D) = \chi_d(Y, D_Y)\phi_d(Z, D_Z). \tag{2.19}$$

Proof By Definitions 2.1 and 2.4, we have

$$\phi_d(X, D) = \chi_d(Y, D_Y)\phi_d(Z, D_Z) + \phi_d(Y, D_Y)\chi_d(Z, D_Z). \tag{2.20}$$

Since (X, D) is d -canonical, so is (Z, D_Z) . By Lemma 2.5, we have

$$\chi_d(Z, D_Z) = 0. \tag{2.21}$$

From (2.20) and (2.21), we obtain (2.19). This completes the proof.

For $r \in \mathbb{N} \setminus \{0\}$ and $m_1, \dots, m_s \in \mathbb{Z}$ with $s \leq r$, let $(\mathbb{C}P^r, D_{m_1, \dots, m_s})$ be such that

$$D_{m_1, \dots, m_s} = \sum_{j=1}^s m_j H_j. \tag{2.22}$$

For $r \in \mathbb{N} \setminus \{0\}$ and $m_1, \dots, m_s \in \mathbb{Z}$ with $s \leq r$, consider a pair $(\mathbb{C}P^r, D_{d; m_1, \dots, m_s})$ with

$$D_{d; m_1, \dots, m_s} = -(m_1 + \dots + m_s + rd + d)H_0 + \sum_{j=1}^s m_j H_j. \tag{2.23}$$

We remark that $D_{d; m_1, \dots, m_s}$ is a d -canonical divisor.

Let X be a compact Kähler manifold. Let

$$D = \sum_{j=1}^l m_j D_j \tag{2.24}$$

be a divisor on X with simple normal crossing support. Let $Y \subseteq X$ be a connected complex submanifold of codimension r intersecting D_1, \dots, D_l transversally (see Definition 1.4) and

$$Y \subseteq D_j \text{ for } j = 1, \dots, s; \quad Y \not\subseteq D_j \text{ for } j = s + 1, \dots, l. \tag{2.25}$$

In particular, $s \leq r$. Set

$$D_Y = \sum_{j=s+1}^l m_j (D_j \cap Y). \tag{2.26}$$

Let $f : X' \rightarrow X$ be the blow-up along Y . Let \tilde{D} be the strict transformation of D . Let $E = f^{-1}(Y) \subseteq X$ be the exceptional divisor. Set

$$D' = \tilde{D} + m_e E, \quad \text{where } m_e = (r - 1)d + m_1 + \cdots + m_s. \tag{2.27}$$

Note that if D is a d -canonical divisor, then so is D' . Indeed,

$$\begin{aligned} dK_{X'} &= f^*K_X + d(r - 1)E \\ &= f^* \left(\sum_{j=1}^l m_j D_j \right) + d(r - 1)E \\ &= \sum_{j=1}^s m_j (\tilde{D}_j + E) + \sum_{j=s+1}^l m_j \tilde{D}_j + d(r - 1)E \\ &= \tilde{D} + m_e E. \end{aligned}$$

Proposition 2.8 *Let ϕ be a log-type localizable invariant and $d \in \mathbb{Z} \setminus \{0\}$. Assume that D is a d -canonical divisor and satisfies the condition (\star_d) in Definition 2.3. We have*

$$\begin{aligned} \chi_d(X', D') - \chi_d(X, D) &= 0, \\ \phi_d(X', D') - \phi_d(X, D) &= \chi_d(Y, D_Y) \left(\chi_d(\mathbb{C}P^{r-1}, D_{m_1, \dots, m_s}) \phi_d(\mathbb{C}P^1, D_{d; m_e}) - \phi_d(\mathbb{C}P^r, D_{d; m_1, \dots, m_s}) \right). \end{aligned} \tag{2.28}$$

Proof Denote by $\mathbb{1}$ the trivial line bundle. Set $W = \mathbb{P}(N_{Y/X} \oplus \mathbb{1})$. Let $\pi : W \rightarrow Y$ be the canonical projection. Let $\iota : Y \hookrightarrow W$ be the inclusion by the zero section of $N_{Y/X}$. Let $g : W' \rightarrow W$ be the blow-up along $\iota(Y)$. Set

$$D_W = \pi^*(D_Y) - (m_e + 2d)\mathbb{P}(N_{Y/X}) + \sum_{j=1}^s m_j \mathbb{P}(N_{Y/D_j} \oplus \mathbb{1}). \tag{2.29}$$

Let \tilde{D}_W be the strict transformation of D_W . We still use E to denote the exceptional divisor of $g : W' \rightarrow W$. Set $D_{W'} = \tilde{D}_W + m_e E$. By Definition 2.4 and (2.1), we have

$$\begin{aligned} \chi_d(X', D') - \chi_d(X, D) &= \chi_d(W', D_{W'}) - \chi_d(W, D_W), \\ \phi_d(X', D') - \phi_d(X, D) &= \phi_d(W', D_{W'}) - \phi_d(W, D_W). \end{aligned} \tag{2.30}$$

Note that (W, D_W) is a fibration over (Y, D_Y) with fiber $(\mathbb{C}P^r, D_{d; m_1, \dots, m_s})$ (in the sense explained before Lemma 2.6), by Lemmas 2.5, 2.6 and Proposition 2.7, we have

$$\begin{aligned} \chi_d(W, D_W) &= \chi_d(Y, D_Y) \chi_d(\mathbb{C}P^r, D_{d; m_1, \dots, m_s}) = 0, \\ \phi_d(W, D_W) &= \chi_d(Y, D_Y) \phi_d(\mathbb{C}P^r, D_{d; m_1, \dots, m_s}). \end{aligned} \tag{2.31}$$

We denote $D_E = \widetilde{D}_W|_E$. Note that W' is a fibration over Y with fiber $\mathrm{Bl}_0\mathbb{C}P^r$, and $\mathrm{Bl}_0\mathbb{C}P^r$ is a fibration over $\mathbb{C}P^{r-1}$ with fiber $\mathbb{C}P^1$, we can show that $(W', D_{W'})$ is fibration over (E, D_E) with fiber $(\mathbb{C}P^1, D_{d;m_e})$. By Lemma 2.5, 2.6 and Proposition 2.7, we have

$$\begin{aligned} \chi_d(W', D_{W'}) &= \chi_d(E, D_E)\chi_d(\mathbb{C}P^1, D_{d;m_e}) = 0, \\ \phi_d(W', D_{W'}) &= \chi_d(E, D_E)\phi_d(\mathbb{C}P^1, D_{d;m_e}). \end{aligned} \tag{2.32}$$

Note that (E, D_E) is fibration over (Y, D_Y) with fiber $(\mathbb{C}P^{r-1}, D_{m_1, \dots, m_s})$, by Lemma 2.6, we have

$$\chi_d(E, D_E) = \chi_d(Y, D_Y)\chi_d(\mathbb{C}P^{r-1}, D_{m_1, \dots, m_s}). \tag{2.33}$$

From (2.30)–(2.33), we obtain (2.28). This completes the proof.

3 BCOV invariants for pairs

In this section, we recall the construction of BCOV invariants, as well as their properties, such as their behavior under projective bundles and blow-ups. See [61] for more details. Moreover, in Sect. 3.2, we compute the BCOV invariants for projective spaces in dimension 1 and 2.

3.1 BCOV invariants

Let X be a compact Kähler manifold. Let K_X be the canonical bundle of X . Let $d \in \mathbb{Z} \setminus \{0\}$. Let $\gamma \in \mathcal{M}(X, K_X^d)$ be an invertible element (in the commutative ring $\bigoplus_{k \in \mathbb{Z}} \mathcal{M}(X, K_X^k)$). In other words, γ is non-zero on any connected component of X . We denote

$$\mathrm{div}(\gamma) = D = \sum_{j=1}^l m_j D_j, \tag{3.1}$$

where $m_j \in \mathbb{Z} \setminus \{0\}$ and $D_1, \dots, D_l \subseteq X$ are mutually distinct prime divisors.

Definition 3.1 We call (X, γ) a d -Calabi–Yau pair if $(X, \mathrm{div}(\gamma))$ satisfies the condition (\star_d) in Definition 2.3. In particular, if X is Calabi–Yau and $\gamma \in H^0(X, K_X)$ is nowhere vanishing, then (X, γ) is a 1-Calabi–Yau pair.

Now we assume that (X, γ) is a d -Calabi–Yau pair.

Let D_J be as in (2.9). For any $j \in J \subseteq \{1, \dots, l\}$, let $L_{J,j}$ be the normal line bundle of $D_J \hookrightarrow D_{J \setminus \{j\}}$. For $J \subseteq \{1, \dots, l\}$, set

$$K_J = K_X^d|_{D_J} \otimes \bigotimes_{j \in J} L_{J,j}^{-m_j} = K_{D_J}^d \otimes \bigotimes_{j \in J} L_{J,j}^{-m_j - d}. \tag{3.2}$$

which is a holomorphic line bundle over D_J . In particular, we have $K_\emptyset = K_X^d$.

Recall that $\text{Res}(\cdot)$ was defined in Definition 1.7. By (1.22), there exist

$$\left(\gamma_J \in \mathcal{M}(D_J, K_J)\right)_{J \subseteq \{1, \dots, l\}} \tag{3.3}$$

such that

$$\gamma_\emptyset = \gamma, \quad \gamma_J = \text{Res}_{D_J}(\gamma_{J \setminus \{j\}}) \quad \text{for } j \in J \subseteq \{1, \dots, l\}. \tag{3.4}$$

Let ω be a Kähler form on X . Let $|\cdot|_{K_{D_J, \omega}}$ be the metric on K_{D_J} induced by ω . Let $|\cdot|_{L_{J, j, \omega}}$ be the metric on $L_{J, j}$ induced by ω . Let $|\cdot|_{K_{J, \omega}}$ be the metric on K_J induced by $|\cdot|_{K_{D_J, \omega}}$ and $|\cdot|_{L_{J, j, \omega}}$ via (3.2).

Let $g_\omega^{TD_J}$ be the metric on TD_J induced by ω . Let $c_k(TD_J, g_\omega^{TD_J})$ be k -th Chern form of $(TD_J, g_\omega^{TD_J})$. Let n be the dimension of X . Let $|J|$ be the number of elements in J . Set

$$a_J(\gamma, \omega) = \frac{1}{12} \int_{D_J} c_{n-|J|}(TD_J, g_\omega^{TD_J}) \log |\gamma_J|_{K_{J, \omega}}^{2/d}. \tag{3.5}$$

We consider the short exact sequence of holomorphic vector bundles over D_J ,

$$0 \rightarrow TD_J \rightarrow TD_{J \setminus \{j\}}|_{D_J} \rightarrow L_{J, j} \rightarrow 0. \tag{3.6}$$

Let $g_\omega^{TD_{J \setminus \{j\}}}$ be the metric on $TD_{J \setminus \{j\}}$ induced by ω . Let

$$\tilde{c}\left(TD_J, TD_{J \setminus \{j\}}|_{D_J}, g_\omega^{TD_{J \setminus \{j\}}}|_{D_J}\right) \tag{3.7}$$

be the same Bott–Chern form as in [62, §1.1]. Set

$$b_{J, j}(\omega) = \frac{1}{12} \int_{D_J} \tilde{c}\left(TD_J, TD_{J \setminus \{j\}}|_{D_J}, g_\omega^{TD_{J \setminus \{j\}}}|_{D_J}\right). \tag{3.8}$$

Let w_d^J be as in (2.9). For ease of notations, we denote $\tau_{\text{BCOV}}(D_J, \omega) = \tau_{\text{BCOV}}(D_J, \omega)|_{D_J}$.

The second author [61, Definition 3.2] defined the following extended BCOV invariant.

Definition 3.2 The BCOV invariant of a d -Calabi–Yau pair (X, γ) is defined by

$$\tau_d(X, \gamma) = \sum_{J \subseteq \{1, \dots, l\}} w_d^J \left(\tau_{\text{BCOV}}(D_J, \omega) - a_J(\gamma, \omega) - \sum_{j \in J} \frac{m_j + d}{d} b_{J, j}(\omega) \right). \tag{3.9}$$

It is shown in [61, Theorem 3.1] that $\tau_d(X, \gamma)$ is independent of ω .

3.2 Projective spaces of dimension 1 and 2

We identify $\mathbb{C}P^n$ with $\mathbb{C}^n \cup \mathbb{C}P^{n-1}$. Let $(z_1, \dots, z_n) \in \mathbb{C}^n$ be the affine coordinates. For positive integers m_1, \dots, m_n , let $\gamma_{m_1, \dots, m_n} \in \mathcal{M}(\mathbb{C}P^n, K_{\mathbb{C}P^n}^d)$ be such that

$$\gamma_{m_1, \dots, m_n}|_{\mathbb{C}^n} = z_1^{m_1} \cdots z_n^{m_n} (dz_1 \wedge \cdots \wedge dz_n)^d. \tag{3.10}$$

Then $(\mathbb{C}P^n, \gamma_{m_1, \dots, m_n})$ is a d -Calabi–Yau pair.

We denote

$$\tau(\mathbb{C}P^n) = \tau_d(\mathbb{C}P^n, \gamma_{0, \dots, 0}). \tag{3.11}$$

By [61, Proposition 3.3], $\tau(\mathbb{C}P^n)$ is well-defined, i.e., independent of d .

Theorem 3.3 *For any $m \in \mathbb{N}$, we have*

$$\tau_d(\mathbb{C}P^1, \gamma_m) = \tau(\mathbb{C}P^1). \tag{3.12}$$

In other words, $\tau_d(\mathbb{C}P^1, \gamma_m)$ is independent of m .

Proof Let $w = 1/z$. We have

$$\gamma_m = z^m (dz)^d = \frac{(-1)^d}{w^{m+2d}} (dw)^d. \tag{3.13}$$

We have $\text{div}(\gamma_m) = m\{0\} - (m+2d)\{\infty\}$. Recall that $\text{Res}(\cdot)$ was defined in Definition 1.7. We have

$$\text{Res}_{\{0\}}(\gamma_m) = (dz)^{m+d}, \quad \text{Res}_{\{\infty\}}(\gamma_m) = (-1)^d (dw)^{-m-d}. \tag{3.14}$$

Let ω be a Kähler form on $\mathbb{C}P^1$. We will use the notations in (3.9). Since $D_J = \text{pt}$ for $|J| = 1$ and $D_J = \emptyset$ for $|J| > 1$, we have

$$b_{J,j}(\omega) = 0 \quad \text{for any } J \text{ and } j \in J, \quad \tau_{\text{BCOV}}(D_J, \omega) = 0 \quad \text{for } |J| \geq 1. \tag{3.15}$$

Let $g^{T\mathbb{C}P^1}$ (resp. $g^{T^*\mathbb{C}P^1}$) be the metric on $T\mathbb{C}P^1$ (resp. $T^*\mathbb{C}P^1$) induced by ω . Let $|dz|$ (resp. $|dw|$) be the norm of dz (resp. dw) with respect to $g^{T^*\mathbb{C}P^1}$. By (3.9) and (3.13)–(3.15), We have

$$\begin{aligned} \tau_d(\mathbb{C}P^1, \gamma_m) &= \tau_{\text{BCOV}}(\mathbb{C}P^1, \omega) - \frac{1}{12} \int_{\mathbb{C}P^1} c_1(T\mathbb{C}P^1, g^{T\mathbb{C}P^1}) \left(\log |dz|^2 + \frac{m}{d} \log |z|^2 \right) \\ &\quad + \frac{m}{12} \log |dz|_0^2 - \frac{m+2d}{12} \log |dw|_\infty^2, \end{aligned} \tag{3.16}$$

where the second term corresponds to $a_{\emptyset}(\gamma, \omega)$ and the last two terms correspond to $a_J(\gamma, \omega)$ with $|J| = 1$.

In the sequel, we take the Fubini–Study metric on $\mathbb{C}P^1$, whose Kähler form is

$$\omega = \frac{idz \wedge d\bar{z}}{(1 + |z|^2)^2}. \tag{3.17}$$

We have

$$c_1(T\mathbb{C}P^1, g^{T\mathbb{C}P^1}) = \frac{\omega}{\pi}, \quad |dz|^2 = (1 + |z|^2)^2. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\log |dz|_0^2 = \log |dw|_{\infty}^2 = 0, \quad \int_{\mathbb{C}P^1} c_1(T\mathbb{C}P^1, g^{T\mathbb{C}P^1}) \log |z|^2 = 0. \tag{3.19}$$

By (3.16) and (3.19), we obtain (3.12). This completes the proof.

Theorem 3.4 *For any $m_1, m_2 \in \mathbb{N}$, we have*

$$\tau_d(\mathbb{C}P^2, \gamma_{m_1, m_2}) = \tau(\mathbb{C}P^2) + \left(\frac{3}{2} - \frac{m_1}{m_1 + d} - \frac{m_2}{m_2 + d} - \frac{m_1 + m_2 + 3d}{m_1 + m_2 + 2d} \right) \tau(\mathbb{C}P^1). \tag{3.20}$$

Proof Let $[\xi_0 : \xi_1 : \xi_2]$ be homogenous coordinates on $\mathbb{C}P^2$. Let $H_1 \subseteq \mathbb{C}P^2$ (resp. $H_2 \subseteq \mathbb{C}P^2, H_{\infty} \subseteq \mathbb{C}P^2$) be defined by $\xi_1 = 0$ (resp. $\xi_2 = 0, \xi_0 = 0$). Set

$$z_1 = \xi_1/\xi_0, \quad z_2 = \xi_2/\xi_0, \quad w_0 = \xi_0/\xi_2, \quad w_1 = \xi_1/\xi_2, \quad t_0 = \xi_0/\xi_1, \quad t_2 = \xi_2/\xi_1. \tag{3.21}$$

Then (z_1, z_2) (resp. $(w_0, w_1), (t_0, t_2)$) are affine coordinates on $\mathbb{C}P^2 \setminus H_{\infty}$ (resp. $\mathbb{C}P^2 \setminus H_2, \mathbb{C}P^2 \setminus H_1$). We have

$$\begin{aligned} \gamma_{m_1, m_2} &= z_1^{m_1} z_2^{m_2} (dz_1 \wedge dz_2)^d \\ &= \frac{w_1^{m_1}}{w_0^{m_1+m_2+3d}} (dw_0 \wedge dw_1)^d = \frac{t_2^{m_2}}{t_0^{m_1+m_2+3d}} (dt_2 \wedge dt_0)^d. \end{aligned} \tag{3.22}$$

We remark that $\text{div}(\gamma_{m_1, m_2}) = m_1 H_1 + m_2 H_2 - (m_1 + m_2 + 3d) H_{\infty}$.

Recall that $\text{Res}(\cdot)$ was defined in Definition 1.7. We have

$$\begin{aligned}
 \text{Res}_{H_1}(\gamma_{m_1, m_2}) &= z_2^{m_2} (dz_1)^{m_1+d} (dz_2)^d, \\
 \text{Res}_{H_2}(\gamma_{m_1, m_2}) &= z_1^{m_1} (dz_2)^{m_2+d} (dz_1)^d, \\
 \text{Res}_{H_\infty}(\gamma_{m_1, m_2}) &= w_1^{m_1} (dw_0)^{-m_1-m_2-2d} (dw_1)^d, \\
 \text{Res}_{H_1 \cap H_2}(\text{Res}_{H_1}(\gamma_{m_1, m_2})) &= (dz_1)^{m_1+d} (dz_2)^{m_2+d}, \\
 \text{Res}_{H_1 \cap H_\infty}(\text{Res}_{H_\infty}(\gamma_{m_1, m_2})) &= (dw_0)^{-m_1-m_2-2d} (dw_1)^{m_1+d}, \\
 \text{Res}_{H_2 \cap H_\infty}(\text{Res}_{H_\infty}(\gamma_{m_1, m_2})) &= (dt_0)^{-m_1-m_2-2d} (dt_2)^{m_2+d}.
 \end{aligned}
 \tag{3.23}$$

We fix a Fubini–Study metric on $\mathbb{C}P^2$, whose Kähler form is as follows:

$$\omega = \frac{i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 - \bar{z}_1 z_2 dz_1 \wedge d\bar{z}_2 - z_1 \bar{z}_2 d\bar{z}_1 \wedge dz_2)}{(1 + |z_1|^2 + |z_2|^2)^2}.
 \tag{3.24}$$

We will use the notations in (3.9). With the Kähler form (3.24), we have

$$a_J(\gamma_{m_1, m_2}, \omega) = b_{J, j}(\omega) = 0 \text{ for } |J| = 2, \quad b_{J, j}(\omega) = 0 \text{ for } |J| = 1.
 \tag{3.25}$$

By (3.9), (3.24), (3.25) and the fact that $\tau_{\text{BCOV}}(\text{pt}) = 0$, we have

$$\begin{aligned}
 &\tau_d(\mathbb{C}P^2, \gamma_{m_1, m_2}) \\
 &= \tau_{\text{BCOV}}(\mathbb{C}P^2, \omega) - \frac{1}{12} \frac{1}{d} \int_{\mathbb{C}P^2} c_2(T\mathbb{C}P^2, g^{T\mathbb{C}P^2}) \log \left| z_1^{m_1} z_2^{m_2} (dz_1 \wedge dz_2)^d \right|^2 \\
 &\quad - \frac{m_1}{m_1 + d} \left(\tau_{\text{BCOV}}(H_1, \omega) \right. \\
 &\quad \left. - \frac{1}{12} \frac{1}{d} \int_{H_1} c_1(T H_1, g^{T H_1}) \log \left| z_2^{m_2} (dz_1)^{m_1+d} (dz_2)^d \right|^2 \right) \\
 &\quad - \frac{m_2}{m_2 + d} \left(\tau_{\text{BCOV}}(H_2, \omega) \right. \\
 &\quad \left. - \frac{1}{12} \frac{1}{d} \int_{H_2} c_1(T H_2, g^{T H_2}) \log \left| z_1^{m_1} (dz_2)^{m_2+d} (dz_1)^d \right|^2 \right) \\
 &\quad - \frac{m_1 + m_2 + 3d}{m_1 + m_2 + 2d} \left(\tau_{\text{BCOV}}(H_\infty, \omega) \right. \\
 &\quad \left. - \frac{1}{12} \frac{1}{d} \int_{H_\infty} c_1(T H_\infty, g^{T H_\infty}) \log \left| w_1^{m_1} (dw_0)^{-m_1-m_2-2d} (dw_1)^d \right|^2 \right).
 \end{aligned}
 \tag{3.26}$$

By (3.11), (3.16), (3.19) and (3.26), we have

$$\begin{aligned}
 &\tau_d(\mathbb{C}P^2, \gamma_{m_1, m_2}) \\
 &= \tau_{\text{BCOV}}(\mathbb{C}P^2, \omega) - \frac{1}{12} \int_{\mathbb{C}P^2} c_2(T\mathbb{C}P^2, g^{T\mathbb{C}P^2}) \log |dz_1 \wedge dz_2|^2 \\
 &\quad - \frac{1}{12} \frac{1}{d} \int_{\mathbb{C}P^2} c_2(T\mathbb{C}P^2, g^{T\mathbb{C}P^2}) (m_1 \log |z_1|^2 + m_2 \log |z_2|^2) \\
 &\quad - \frac{m_1}{m_1 + d} \left(\tau(\mathbb{C}P^1) - \frac{1}{12} \frac{m_1 + d}{d} \int_{H_1} c_1(TH_1, g^{TH_1}) \log |dz_1|^2 \right) \\
 &\quad - \frac{m_2}{m_2 + d} \left(\tau(\mathbb{C}P^1) - \frac{1}{12} \frac{m_2 + d}{d} \int_{H_2} c_1(TH_2, g^{TH_2}) \log |dz_2|^2 \right) \\
 &\quad - \frac{m_1 + m_2 + 3d}{m_1 + m_2 + 2d} \left(\tau(\mathbb{C}P^1) + \frac{1}{12} \frac{m_1 + m_2 + 2d}{d} \int_{H_\infty} c_1(TH_\infty, g^{TH_\infty}) \log |dw_0|^2 \right).
 \end{aligned} \tag{3.27}$$

Similarly to (3.19), we have

$$\int_{\mathbb{C}P^2} c_2(T\mathbb{C}P^2, g^{T\mathbb{C}P^2}) \log |z_1|^2 = \int_{\mathbb{C}P^2} c_2(T\mathbb{C}P^2, g^{T\mathbb{C}P^2}) \log |z_2|^2 = 0. \tag{3.28}$$

On the other hand, by (3.24), we have

$$\begin{aligned}
 \int_{H_1} c_1(TH_1, g^{TH_1}) \log |dz_1|^2 &= \int_{H_2} c_1(TH_2, g^{TH_2}) \log |dz_2|^2 \\
 &= \int_{H_\infty} c_1(TH_\infty, g^{TH_\infty}) \log |dw_0|^2.
 \end{aligned} \tag{3.29}$$

From (3.27)–(3.29), we obtain (3.20). This completes the proof.

3.3 Projective bundle

Let Y be a compact Kähler manifold. Let N be a holomorphic vector bundle of rank r over Y . Set

$$X = \mathbb{P}(N \oplus \mathbb{1}). \tag{3.30}$$

Let \mathcal{N} be the total space of N . We have $X = \mathcal{N} \cup \mathbb{P}(N)$.

Let $s \in \{0, \dots, r\}$. Let $(L_j)_{j=1, \dots, s}$ be holomorphic line bundles over Y . We assume that there is a surjective map

$$N \rightarrow L_1 \oplus \dots \oplus L_s. \tag{3.31}$$

Let N^* be the dual of N . Taking the dual of (3.31), we get

$$L_1^{-1} \oplus \cdots \oplus L_s^{-1} \hookrightarrow N^*. \tag{3.32}$$

Let m_1, \dots, m_s be positive integers. Let $d \in \mathbb{N} \setminus \{0\}$. Let

$$\gamma_Y \in \mathcal{M}(Y, K_Y^d \otimes (\det N^*)^d \otimes L_1^{-m_1} \otimes \cdots \otimes L_s^{-m_s}) \tag{3.33}$$

be an invertible element. We assume that

- $\text{div}(\gamma_Y)$ is of simple normal crossing support;
- $\text{div}(\gamma_Y)$ does not possess components of multiplicity $-d$.

We denote $m = m_1 + \cdots + m_s$. Let $S^m N^*$ be the m -th symmetric tensor power of N^* . By (3.32) and (3.33), we have

$$\gamma_Y \in \mathcal{M}(Y, K_Y^d \otimes (\det N^*)^d \otimes S^m N^*). \tag{3.34}$$

Let $\pi : X = \mathbb{P}(N \oplus \mathbb{1}) \rightarrow Y$ be the canonical projection. We have

$$K_X|_{\mathcal{N}} = \pi^*(K_Y \otimes \det N^*). \tag{3.35}$$

We may view a section of $S^m N^*$ as a function on \mathcal{N} . By (3.34) and (3.35), γ_Y may be viewed as an element of $\mathcal{M}(\mathcal{N}, K_X^d)$. Let $\gamma_X \in \mathcal{M}(X, K_X^d)$ be such that $\gamma_X|_{\mathcal{N}} = \gamma_Y$.

For $j = 1, \dots, s$, set

$$N_j = \text{Ker}(N \rightarrow L_j), \quad X_j = \mathbb{P}(N_j \oplus \mathbb{1}) \subseteq X, \quad X_\infty = \mathbb{P}(N) \subseteq X. \tag{3.36}$$

We have

$$\text{div}(\gamma_X) = \pi^* \text{div}(\gamma_Y) - (m + rd + d)X_\infty + \sum_{j=1}^s m_j X_j. \tag{3.37}$$

Hence (X, γ_X) is a d -Calabi–Yau pair.

Let Z be the fiber of $\pi : X \rightarrow Y$. Let $U \subseteq Y$ be a small open subset. We fix an identification $\pi^{-1}(U) = U \times Z$ such that there exist $\gamma_U \in \mathcal{M}(U, K_U^d)$ and $\gamma_Z \in \mathcal{M}(Z, K_Z^d)$ satisfying

$$\gamma_X|_{\pi^{-1}(U)} = \text{pr}_1^*(\gamma_U) \otimes \text{pr}_2^*(\gamma_Z). \tag{3.38}$$

Then (Z, γ_Z) is a d -Calabi–Yau pair.

The following theorem was proved by the second author [61, Theorem 3.6]

Theorem 3.5 *The following identity holds,*

$$\tau_d(X, \gamma_X) = \chi_d(Y, \gamma_Y) \tau_d(Z, \gamma_Z). \tag{3.39}$$

3.4 Blow-up

Let (X, γ) be a d -Calabi–Yau pair. We denote

$$\operatorname{div}(\gamma) = D = \sum_{j=1}^l m_j D_j. \tag{3.40}$$

Let $Y \subseteq X$ be a connected complex submanifold intersecting D_1, \dots, D_l transversally (see Definition 1.4). Assume that for $j \in \{1, \dots, l\}$ satisfying $Y \subseteq D_j$, we have $m_j > 0$. Let r be the codimension of $Y \subseteq X$. Let s be the number of D_j containing Y . We have $s \leq r$. Without loss of generality, we assume that

$$Y \subseteq D_j \text{ for } j = 1, \dots, s; \quad Y \not\subseteq D_j \text{ for } j = s + 1, \dots, l. \tag{3.41}$$

Let $f : X' \rightarrow X$ be the blow-up along Y . Let $D'_j \subseteq X'$ be the strict transformation of $D_j \subseteq X$. Set $E = f^{-1}(Y)$. We denote $D' = \operatorname{div}(f^*\gamma)$. We denote

$$m_0 = m_1 + \dots + m_s + rd - d. \tag{3.42}$$

We have

$$D' = m_0 E + \sum_{j=1}^l m_j D'_j. \tag{3.43}$$

Hence $(X', f^*\gamma)$ is a d -Calabi–Yau pair.

Set

$$D_Y = \sum_{j=s+1}^l m_j (D_j \cap Y), \quad D_E = \sum_{j=s+1}^l m_j (D'_j \cap E), \tag{3.44}$$

which are divisors with simple normal crossing support.

Now we introduce a pair involving projective spaces, which is a slightly different way of denoting the one defined in Equation (3.10). More precisely: we identify $\mathbb{C}P^r$ with $\mathbb{C}^r \cup \mathbb{C}P^{r-1}$. Let $(z_1, \dots, z_r) \in \mathbb{C}^r$ be the coordinates. Let $\gamma_{r,m_1,\dots,m_s} \in \mathcal{M}(\mathbb{C}P^r, K_{\mathbb{C}P^r}^d)$ be such that

$$\gamma_{r,m_1,\dots,m_s}|_{\mathbb{C}^r} = (dz_1 \wedge \dots \wedge dz_r)^d \prod_{j=1}^s z_j^{m_j}. \tag{3.45}$$

Then $(\mathbb{C}P^r, \gamma_{r,m_1,\dots,m_s})$ is a d -Calabi–Yau pair. Note that γ_{r,m_1,\dots,m_s} is precisely $\gamma_{m_1,\dots,m_s,0,0,\dots,0}$ with $r - s$ zeros in the end, in the notation of (3.10).

The following blow-up formula was proved by the second author [61, Theorem 0.5].

Theorem 3.6 *The following identity holds,*

$$\begin{aligned} &\tau_d(X', f^*\gamma) - \tau_d(X, \gamma) \\ &= \chi_d(E, D_E)\tau_d(\mathbb{CP}^1, \gamma_{1,m_0}) - \chi_d(Y, D_Y)\tau_d(\mathbb{CP}^r, \gamma_{r,m_1,\dots,m_s}). \end{aligned} \tag{3.46}$$

Remark 3.7 Keep the notation as is in Theorem 3.6. Let $g : \text{Bl}_0\mathbb{CP}^r \rightarrow \mathbb{CP}^r$ be the blow-up along $0 \in \mathbb{C}^r \subseteq \mathbb{CP}^r$. Since $\text{Bl}_0\mathbb{CP}^r$ is a \mathbb{CP}^1 -bundle over \mathbb{CP}^{r-1} , apply Theorem 3.5, with $N = L = \mathcal{O}_{\mathbb{CP}^{r-1}}(-1)$,

$$\tau_d(\text{Bl}_0\mathbb{CP}^r, g^*\gamma_{m_1,\dots,m_s}) = \chi_d(\mathbb{CP}^{r-1}, D_{m_1,\dots,m_s})\tau_d(\mathbb{CP}^1, \gamma_{m_0}). \tag{3.47}$$

Note that $\chi_d(E, D_E) = \chi_d(Y, D_Y)\chi_d(\mathbb{CP}^{r-1}, D_{m_1,\dots,m_s})$, we could reinterpret Theorem 3.6 as follows,

$$\begin{aligned} &\tau_d(X', f^*\gamma_X) - \tau_d(X, \gamma_X) \\ &= \chi_d(Y, D_Y)\left(\tau_d(\text{Bl}_0\mathbb{CP}^r, g^*\gamma_{m_1,\dots,m_s}) - \tau_d(\mathbb{CP}^r, \gamma_{m_1,\dots,m_s})\right). \end{aligned} \tag{3.48}$$

4 Motivic integration and BCOV invariants

In this section, we use the theory of motivic integration to explain our key construction, namely, the BCOV invariant for pairs $\tau(X, \gamma)$. We stress the fact that this section is purely heuristic and logically independent of the rest of the paper.

4.1 Motivic integration

We consider an n -dimensional smooth complex algebraic variety X and an effective divisor with simple normal crossing support $D = \sum_{j=1}^l m_j D_j$ on X . Denote by $\mathcal{L}_\infty(X)$ the space of formal arcs in X , that is, the projective limit of jet schemes $\mathcal{L}_n(X) := X(\mathbb{C}[t]/(t^{n+1}))$ (see [29, §1]). Let

$$\text{ord}_D : \mathcal{L}_\infty(X) \rightarrow \mathbb{N} \cup \{+\infty\} \tag{4.1}$$

be the function sending a formal arc to its intersection number with the divisor D .

Let $\text{Var}_{\mathbb{C}}$ be the category of complex algebraic varieties. The Grothendieck group of complex algebraic varieties, denoted by $K_0(\text{Var}_{\mathbb{C}})$, is the free abelian group generated by the isomorphism classes of objects in $\text{Var}_{\mathbb{C}}$, modulo the *scissor relation*:

$$[X] = [Y] + [X \setminus Y] \quad \text{for any } X \text{ and any closed subvariety } Y \subseteq X. \tag{4.2}$$

$K_0(\text{Var}_{\mathbb{C}})$ is endowed with a natural ring structure given by the fiber product.

Let \mathbb{L} be the class of the affine line. We denote $\mathcal{M} = K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, the localization of $K_0(\text{Var}_{\mathbb{C}})$ with respect to the multiplicative system $\{\mathbb{L}^k\}_{k \in \mathbb{N}}$. For any integer i , let $F^i\mathcal{M} \subseteq \mathcal{M}$ be the subgroup generated by elements of the form $\mathbb{L}^{-m}[Y]$ with

$m - \dim(Y) \geq i$. Then F^\bullet is a filtration on \mathcal{M} . Let $\widehat{\mathcal{M}}$ be the completion of \mathcal{M} with respect to F^\bullet . The motivic Igusa zeta function⁴ is by definition

$$Z(X, D; T) := \int_{\mathcal{L}_\infty(X)} T^{\text{ord}_D} d\mu \in \widehat{\mathcal{M}}[[T]], \tag{4.3}$$

where μ is the motivic measure constructed by Kontsevich [45] and Denef–Loeser [29, Definition-Proposition 3.2].

The following theorem gives a formula for $Z(X, D; T)$, see [19, Theorem 3.3.4].

Theorem 4.1 *The following identity holds,*

$$Z(X, D; T) = \sum_{J \subseteq \{1, \dots, l\}} \mathbb{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - T^{-m_j}}{\mathbb{L}T^{-m_j} - 1} \right) [D_J], \tag{4.4}$$

where $D_J = \bigcap_{j \in J} D_j$ with the convention that $D_\emptyset = X$.

Let d be a positive integer. We define

$$F_d(X, D) := Z(X, D; \mathbb{L}^{-1/d}) = \sum_{m=0}^\infty \mu(\text{ord}_D^{-1}(m)) \mathbb{L}^{-m/d} \in \widehat{\mathcal{M}}[\mathbb{L}^{1/d}]. \tag{4.5}$$

By (4.4), we have

$$F_d(X, D) = \sum_{J \subseteq \{1, \dots, l\}} \mathbb{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - \mathbb{L}^{m_j/d}}{\mathbb{L}^{1+m_j/d} - 1} \right) [D_J]. \tag{4.6}$$

An equivalent form of (4.6) when $d = 1$ is in Craw [26, Theorem 1.1].

Now we state the formula of change of variables, due to Kontsevich [45] and Denef–Loeser [30, Theorem 1.16]), in the following form taken from Craw [26, Theorem 2.19] (when $d = 1$).

Theorem 4.2 *Let X be a projective complex manifold. Let $f : X' \rightarrow X$ be the blow-up along a smooth center. Let $K_{X'/X}$ be the relative canonical divisor. Let d be a positive integer. Let D be an effective divisor on X such that both D and $f^*(D) + dK_{X'/X}$ are of simple normal crossing support. We have*

$$F_d(X, D) = F_d(X', f^*D + dK_{X'/X}). \tag{4.7}$$

⁴ It is usually denoted by $Z(X, \mathcal{I}_D; T)$, where $\mathcal{I}_D = \mathcal{O}_X(-D)$ is the ideal sheaf of D .

4.2 From motivic integration to BCOV invariant

Let X be a smooth projective complex variety. Let γ be a d -canonical form on X such that $D = \text{div}(\gamma)$ satisfies the condition (\star_d) in Definition 2.4. Hence (X, γ) is a d -Calabi–Yau pair in the sense of Definition 3.1.

Recall that the Hodge realization is the ring homomorphism

$$\chi_{\text{Hdg}}: K_0(\text{Var}_{\mathbb{C}}) \rightarrow K_0(\text{HS}) \tag{4.8}$$

that sends the class of a smooth projective variety X to the class of its cohomology $H^\bullet(X, \mathbb{Z})$ endowed with Hodge structure. It is easy to see that $\chi_{\text{Hdg}}(\mathbb{L}) = \mathbb{Z}(-1)$ is the Lefschetz Hodge structure, which we will denote by \mathbf{L} in the sequel. Therefore, for a Hodge structure H^\bullet and $s \in \mathbb{Z}$, $\mathbf{L}^s H^\bullet$ is the Tate twist $H^\bullet(-s)$, namely,

$$\mathbf{L}^s H_{\mathbb{Z}}^k = H_{\mathbb{Z}}^{k-2s}, \quad \mathbf{L}^s H_{\mathbb{C}}^{p,q} = H_{\mathbb{C}}^{p-s,q-s}. \tag{4.9}$$

For a polynomial $f(x) = a_0 + a_1x + \dots + a_mx^m \in \mathbb{Z}[x]$, we denote

$$f(\mathbf{L})H^\bullet = \sum_{s=0}^m (\mathbf{L}^s H^\bullet)^{\oplus a_s}. \tag{4.10}$$

By (4.6), we have

$$\chi_{\text{Hdg}}\left(F_d(X, D)\mathbb{L}^{\frac{n}{2}}\right) = \sum_J \mathbf{L}^{|J|-\frac{n}{2}} \left(\prod_{j \in J} \frac{1 - \mathbf{L}^{m_j/d}}{\mathbf{L}^{1+m_j/d} - 1}\right) H^\bullet(D_J). \tag{4.11}$$

Mimicking (1.3) and (1.12), for a Hodge structure H^\bullet , we define

$$\eta(H^\bullet) = \bigotimes_k (\det H^k)^{(-1)^k}, \quad \lambda_{\text{dR}}(H^\bullet) = \bigotimes_k (\det H^k)^{(-1)^k k}. \tag{4.12}$$

We are interested in applying λ_{dR} to (4.11). First we remark that

$$\eta(\mathbf{L}H^\bullet) = \eta(H^\bullet), \quad \lambda_{\text{dR}}(\mathbf{L}H^\bullet) = \left(\eta(H^\bullet)\right)^2 \otimes \lambda_{\text{dR}}(H^\bullet). \tag{4.13}$$

Therefore, for any polynomial f , we have

$$\lambda_{\text{dR}}(f(\mathbf{L})H^\bullet) = \left(\eta(H^\bullet)\right)^{2f'(1)} \otimes \left(\lambda_{\text{dR}}(H^\bullet)\right)^{f(1)}. \tag{4.14}$$

Let w_d^J be as in (2.9). For

$$f(x) = x^{|J|-\frac{n}{2}} \prod_{j \in J} \frac{1 - x^{m_j/d}}{x^{1+m_j/d} - 1}, \tag{4.15}$$

we have

$$\lambda_{\text{dR}}(f(\mathbf{L})H^\bullet) = \left(\eta(H^\bullet)\right)^{(|J|-n)w'_d} \otimes \left(\lambda_{\text{dR}}(H^\bullet)\right)^{w'_d}. \tag{4.16}$$

By (4.11) and (4.16), we have

$$\begin{aligned} &\lambda_{\text{dR}}\left(\chi_{\text{Hdg}}\left(F_d(X, D)\mathbb{L}^{\frac{n}{2}}\right)\right) \\ &= \bigotimes_J \left(\left(\lambda_{\text{dR}}(H^\bullet(D_J))\right)^{w'_d} \otimes \left(\eta(H^\bullet(D_J))\right)^{(|J|-n)w'_d} \right). \end{aligned} \tag{4.17}$$

Observe that the BCOV invariant $\tau_d(X, D)$ (cf. Definition 3.2) is essentially the Quillen metric on $\bigotimes_J \left(\lambda_{\text{dR}}(H^\bullet(D_J))\right)^{w'_d}$. On the other hand, by Definition 1.1, the Quillen metric on $\eta(H^\bullet(D_J))$ gives rise to $\tau_{\text{top}}(D_J)$, which vanishes by (1.10). Therefore, our BCOV invariant is essentially the Quillen metric on the determinant line (4.17).

5 Birational BCOV invariants

Definition 5.1 For any d -Calabi–Yau pair (X, γ) , its *birational BCOV invariant* is

$$\begin{aligned} \tau_d^{\text{bir}}(X, \gamma) &= \tau_d(X, \gamma) - \frac{1}{2}\tau(\mathbb{CP}^1)\chi'_d(X, \gamma) \\ &\quad + \left(-\frac{1}{2}\tau(\mathbb{CP}^2) + \frac{3}{4}\tau(\mathbb{CP}^1)\right)\chi''_d(X, \gamma), \end{aligned} \tag{5.1}$$

where $\chi'_d(X, \gamma)$ and $\chi''_d(X, \gamma)$ are as in Definition 2.4, applied to the invariants χ' and χ'' in Example 2.2.

For a Calabi–Yau manifold X and a d -canonical form γ on X such that $\int_X |\gamma\bar{\gamma}|^{1/d} = (2\pi)^{\dim X}$, we have

$$\tau_d^{\text{bir}}(X, \gamma) = \tau(X) - \frac{1}{2}\tau(\mathbb{CP}^1)\chi'(X) + \left(-\frac{1}{2}\tau(\mathbb{CP}^2) + \frac{3}{4}\tau(\mathbb{CP}^1)\right)\chi''(X). \tag{5.2}$$

Lemma 5.2 *Let (X, γ_X) , (Y, γ_Y) and (Z, γ_Z) be as in Theorem 3.5 (see (3.30)–(3.38) in Sect. 3.3). Then*

$$\tau_d^{\text{bir}}(X, \gamma_X) = \chi_d(Y, \gamma_Y)\tau_d^{\text{bir}}(Z, \gamma_Z). \tag{5.3}$$

Proof Since χ' and χ'' are log-type localizable invariants (Examples 2.2), Proposition 2.7 yields $\chi'_d(X, \gamma_X) = \chi_d(Y, \gamma_Y)\chi'_d(Z, \gamma_Z)$ and $\chi''_d(X, \gamma_X) = \chi_d(Y, \gamma_Y)\chi''_d(Z, \gamma_Z)$. Combining them with Theorem 3.5 allows us to conclude.

Lemma 5.3 *Let $(X, \gamma_X), (Y, D_Y), f : X' \rightarrow X$ and $m_0, \dots, m_s \in \mathbb{Z}$, be as in Theorem 3.6. We have*

$$\begin{aligned} &\tau_d^{\text{bir}}(X', f^* \gamma_X) - \tau_d^{\text{bir}}(X, \gamma_X) \\ &= \chi_d(Y, D_Y) \left(\chi_d(\mathbb{CP}^{r-1}, D_{m_1, \dots, m_s}) \tau_d^{\text{bir}}(\mathbb{CP}^1, \gamma_{m_0}) - \tau_d^{\text{bir}}(\mathbb{CP}^r, \gamma_{m_1, \dots, m_s}) \right). \end{aligned} \tag{5.4}$$

Proof Similarly to the proof of Lemma 5.2, we use Theorem 3.6, Remark 3.7 and Proposition 2.8.

Theorem 5.4 *For any $m_1, \dots, m_n \in \mathbb{N}$, we have*

$$\tau_d^{\text{bir}}(\mathbb{CP}^n, \gamma_{m_1, \dots, m_n}) = 0. \tag{5.5}$$

Proof We have (see Example 2.2)

$$\begin{aligned} \chi(\mathbb{CP}^1) &= 2, & \chi'(\mathbb{CP}^1) &= 2, & \chi''(\mathbb{CP}^1) &= 0, \\ \chi(\mathbb{CP}^2) &= 3, & \chi'(\mathbb{CP}^2) &= 6, & \chi''(\mathbb{CP}^2) &= 2. \end{aligned} \tag{5.6}$$

By Definition 2.4 and (5.6), for any $m_1, m_2 \in \mathbb{N}$, we have

$$\begin{aligned} \chi_d''(\mathbb{CP}^1, \gamma_{m_1}) &= 0, & \chi_d''(\mathbb{CP}^2, \gamma_{m_1, m_2}) &= 2, & \chi_d'(\mathbb{CP}^1, \gamma_{m_1}) &= 2, \\ \chi_d'(\mathbb{CP}^2, \gamma_{m_1, m_2}) &= 6 - 2 \left(\frac{m_1}{m_1 + d} + \frac{m_2}{m_2 + d} + \frac{m_1 + m_2 + 3d}{m_1 + m_2 + 2d} \right). \end{aligned} \tag{5.7}$$

By Theorem 3.3, Theorem 3.4, Definition 5.1 and (5.7), we have $\tau_d^{\text{bir}}(\mathbb{CP}^1, \gamma_{m_1}) = \tau_d^{\text{bir}}(\mathbb{CP}^2, \gamma_{m_1, m_2}) = 0$. Hence (5.5) holds for $n \leq 2$. We proceed by induction. Let $r \geq 2$ be an integer. Assume that

$$\tau_d^{\text{bir}}(\mathbb{CP}^n, \gamma_{m_1, \dots, m_n}) = 0 \quad \text{for } n \leq r. \tag{5.8}$$

Let $i : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{r+1}$ be the extension of $\mathbb{C} \ni z \mapsto (z, 0, \dots, 0) \in \mathbb{C}^{r+1}$. Let $f : \text{Bl}_{\mathbb{CP}^1} \mathbb{CP}^{r+1} \rightarrow \mathbb{CP}^{r+1}$ be the blow-up along $i(\mathbb{CP}^1)$. Then $\text{Bl}_{\mathbb{CP}^1} \mathbb{CP}^{r+1}$ is a \mathbb{CP}^2 -bundle over \mathbb{CP}^{r-1} . By Lemma 5.2 and (5.8), we have

$$\tau_d^{\text{bir}}(\text{Bl}_{\mathbb{CP}^1} \mathbb{CP}^{r+1}, f^* \gamma_{m_1, \dots, m_{r+1}}) = 0. \tag{5.9}$$

By Lemma 5.3 and (5.8), we have

$$\tau_d^{\text{bir}}(\text{Bl}_{\mathbb{CP}^1} \mathbb{CP}^{r+1}, f^* \gamma_{m_1, \dots, m_{r+1}}) - \tau_d^{\text{bir}}(\mathbb{CP}^{r+1}, \gamma_{m_1, \dots, m_{r+1}}) = 0. \tag{5.10}$$

From (5.9) and (5.10), we obtain

$$\tau_d^{\text{bir}}(\mathbb{CP}^n, \gamma_{m_1, \dots, m_n}) = 0 \quad \text{for } n \leq r + 1. \tag{5.11}$$

This completes the proof by induction.

Theorem 5.5 *Let (X, γ_X) and $f : X' \rightarrow X$ be as in Theorem 3.6. Then*

$$\tau_d^{\text{bir}}(X', f^* \gamma_X) = \tau_d^{\text{bir}}(X, \gamma_X). \tag{5.12}$$

Proof This is a direct consequence of Lemma 5.3 and Theorem 5.4. □

6 Extension to the singular cases

We extend the theory of BCOV invariants to Calabi–Yau varieties with mild singularities. In this section, X is a normal projective complex variety of dimension n .

6.1 Definitions and basic properties

Recall that a variety X is called \mathbb{Q} -Gorenstein if the canonical divisor K_X is \mathbb{Q} -Cartier, i.e., there exists a positive integer d such that dK_X is a Cartier divisor. The minimal value of such d is called the *index* of X .

Definition 6.1 (Canonical and KLT singularities [44, Definition 2.34]) *Let X be a \mathbb{Q} -Gorenstein normal variety and let $f : X' \rightarrow X$ be a log-resolution, i.e., f is a resolution of singularities and the support of the exceptional divisor $E = \bigcup_{j=1}^l E_j$ is simple normal crossing. Write the equality of \mathbb{Q} -divisors*

$$K_{X'/X} = \sum_{j=1}^l a_j E_j, \tag{6.1}$$

where $K_{X'/X}$ is the relative canonical divisor $K_{X'} - \frac{1}{d} f^*(dK_X)$ for any positive integer d divisible by the index of X , where $K_{X'}$ is such that $f_* K_{X'} = K_X$. We say that X has *canonical* singularities if $a_j \geq 0$ for all j . Similarly, X is said to have *Kawamata log terminal* (KLT) singularities if $a_j > -1$ for all j . The coefficients $a_j \in \mathbb{Q}$ are called *discrepancy numbers*. The definitions are independent of the log-resolution f .

Definition 6.2 *A canonical (resp. KLT) Calabi–Yau variety is a \mathbb{Q} -Gorenstein normal projective complex variety X with canonical (resp. KLT) singularities such that $K_X \sim_{\mathbb{Q}} 0$, where $\sim_{\mathbb{Q}}$ is the linear equivalence relation for \mathbb{Q} -divisors.*

Let us record the following basic result.

Proposition 6.3 (Integrability of volume form [53, Thm. 2.1], [36, Prop. 1.17]) *Let X be an n -dimensional variety with KLT singularities. Let d be a positive integer divisible by the index of X . Then for any $\gamma \in H^0(X, \mathcal{O}_X(dK_X))$, the integral*

$$\int_{X_{\text{reg}}} |\gamma \bar{\gamma}|^{1/d} \tag{6.2}$$

converges, where X_{reg} is the regular part of X and $|\gamma\bar{\gamma}|^{1/d}$ is the unique positive volume form on X whose d -th power equals to $i^{n^2}\gamma \wedge \bar{\gamma}$.

We extend the birational BCOV invariant studied in Sect. 5 to varieties with KLT singularities, equipped with a pluricanonical form or a pluricanonical effective divisor.

Definition 6.4 Let X be an n -dimensional variety with KLT singularities. Let $d \in \mathbb{N}_{>0}$ divisible by the index of X . Let $D \in |dK_X|$. Let $f: X' \rightarrow X$ be a resolution of singularities such that $\tilde{D} \cup E$ is of simple normal crossing support, where \tilde{D} is the strict transform of D and E is the exceptional divisor. For any $\gamma \in H^0(X, \mathcal{O}_X(dK_X))$ such that $D = \text{div}(\gamma)$, we define

$$\tau_d^{\text{bir}}(X, \gamma) := \tau_d^{\text{bir}}(X', f^*\gamma). \tag{6.3}$$

Here the right hand side is defined in Definition 5.1. Note that X having KLT singularities implies that $(X', f^*\gamma)$ is indeed a d -Calabi–Yau pair (i.e. Condition (\star_d) is verified). We also define

$$\tau_d^{\text{bir}}(X, D) := \tau_d^{\text{bir}}(X, \gamma) + \frac{\chi_d(X, D)}{12} \log \left((2\pi)^{-n} \int_{X_{\text{reg}}} |\gamma\bar{\gamma}|^{1/d} \right). \tag{6.4}$$

Note that the integral on the right hand side converges by Proposition 6.3.

In the next two lemmas, we show that $\tau_d^{\text{bir}}(X, \gamma)$ and $\tau_d^{\text{bir}}(X, D)$ are well-defined.

Lemma 6.5 *The quantity $\tau_d^{\text{bir}}(X, \gamma)$ is independent of the resolution f .*

Proof As any two resolutions are dominated by a third one, it suffices to show that for a further blow-up $g: X'' \rightarrow X'$ satisfying the same properties, we have

$$\tau_d^{\text{bir}}(X', f^*\gamma) = \tau_d^{\text{bir}}(X'', g^*f^*(\gamma)). \tag{6.5}$$

But this follows from Theorem 5.5.

Lemma 6.6 *For any $z \in \mathbb{C}^*$, the following identity holds,*

$$\tau_d^{\text{bir}}(X, z\gamma) = \tau_d^{\text{bir}}(X, \gamma) - \frac{\chi_d(X, D)}{12} \log |z|^{2/d}. \tag{6.6}$$

Consequently, $\tau_d^{\text{bir}}(X, D)$ is independent of γ .

Proof This follows directly from [61, Proposition 3.4].

In the sequel, we use the same notation as in Sect. 4.1.

Definition 6.7 Let X be a variety with KLT singularities. In the situation of Definition 6.1, the *Gorenstein volume* of X is by definition

$$\mu^{\text{Gor}}(X) := \sum_{J \subseteq \{1, \dots, l\}} \mathbb{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - \mathbb{L}^{a_j}}{\mathbb{L}^{a_j+1} - 1} \right) [E_J] \in \widehat{\mathcal{M}}. \tag{6.7}$$

In other words, for any $d \in \mathbb{N}_{>0}$ divisible by the index of X , $\mu^{\text{Gor}}(X)$ is equal to $F_d(X', dK_{X'/X})$ defined in (4.6). By Theorem 4.2, the definition is independent of the choice of d and the log-resolution X' . Note that when X is smooth, $\mu^{\text{Gor}}(X) = \mathbb{L}^{-n}[X]$.

Let $P_t : K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}[t]$ be the ring homomorphism sending a smooth projective variety to its Poincaré polynomial, which extends to a ring homomorphism $P_t : \widehat{\mathcal{M}} \rightarrow \mathbb{Z}[[t, t^{-1}]]$ (cf. [19, §3.4.7]).

Definition 6.8 Let X be an n -dimensional variety with KLT singularities. The *stringy Poincaré polynomial* of X is defined as

$$P_t(X) := P_t(\mathbb{L}^n \mu^{\text{Gor}}(X)). \tag{6.8}$$

Following Batyrev [4], we define the *stringy Betti numbers* of X as the coefficients of $P_t(X)$. The quantities $\chi(X)$, $\chi'(X)$ and $\chi''(X)$ are defined by the same formulas (2.5)–(2.7) with Betti numbers replaced by stringy Betti numbers. If X admits a crepant resolution Y , the stringy invariants of X equal to the corresponding invariants of Y .

Definition 6.9 Let X be an n -dimensional KLT Calabi–Yau variety (Definition 6.2). We define the (stringy) BCOV invariant of X as

$$\tau(X) := \tau_d^{\text{bir}}(X, \emptyset) + \frac{1}{2} \tau(\mathbb{C}P^1) \chi'(X) + \left(\frac{1}{2} \tau(\mathbb{C}P^2) - \frac{3}{4} \tau(\mathbb{C}P^1) \right) \chi''(X), \tag{6.9}$$

where $d \in \mathbb{N}_{>0}$ is divisible by the index of X and such that $|dK_X| \neq \emptyset$, $\tau_d^{\text{bir}}(X, \emptyset)$ is defined in (6.4) (it is independent of d by [61, Proposition 3.3]), $\chi'(X)$ and $\chi''(X)$ are the stringy invariants introduced in Definition 6.8. By (5.2), we recover the BCOV invariant when X is smooth.

Remark 6.10 Quotient singularities form one of the most important instances of KLT singularities, see [44, Proposition 5.20]. In particular, (complex effective) orbifolds, or equivalently, V-manifolds in the sense of Satake [54, 55], are KLT. A compact Kähler orbifold X is called Calabi–Yau if $c_1(X) = 0 \in H^2(X, \mathbb{R})$. On one hand, thanks to the orbifold version of the Beauville–Bogomolov decomposition theorem, due to Campana [17] and Fujiki [37], Calabi–Yau orbifolds have torsion canonical divisor. Therefore Calabi–Yau orbifolds are special cases of KLT Calabi–Yau varieties in the sense of Definition 6.2, hence their BCOV invariants can be defined as in Definition 6.9. On the other hand, the Quillen metric can be extended to orbifolds (see Ma [47, §2]) and enjoys similar properties as in the smooth case (see Ma [47, 48]). Hence the definition of BCOV invariant (see Definition 3.2) can be directly extended to Calabi–Yau orbifolds. We plan to compare the two definitions in the future.

6.2 Curvature formula

We extend the curvature formula [61, Theorem 0.4] to locally trivial families (in the sense of Flenner–Kosarew [35, Page 627]) of KLT Calabi–Yau varieties.

Definition 6.11 Let S be a complex manifold and \mathcal{X} a complex space. Let $\pi : \mathcal{X} \rightarrow S$ be a flat and proper morphism, viewed as a family of complex spaces $(X_t := \pi^{-1}(t))_{t \in S}$. The family π is called *locally trivial* if for any $t \in S$ and any $x \in X_t$, there are analytic open neighborhoods $\Delta \subset S$ of t and $\mathcal{U} \subset \pi^{-1}(\Delta) \subset \mathcal{X}$ of x such that we have a Δ -isomorphism $(\mathcal{U} \cap X_t) \times \Delta \simeq \mathcal{U}$.

Locally trivial families admit (strong) simultaneous resolution.

Lemma 6.12 (Simultaneous resolution [3, Lemma 4.9 and Remark 4.10]) *Let $\pi : \mathcal{X} \rightarrow S$ be a locally trivial family. Then there is a proper bimeromorphic S -morphism $\mathcal{Y} \rightarrow \mathcal{X}$, which is a composition of blow-ups along locally trivial centers which are smooth over S and disjoint from the smooth locus of π , such that the composed map $\pi' : \mathcal{Y} \rightarrow S$ is a submersion and for any $t \in S$, the map $Y_t := (\pi')^{-1}(t) \rightarrow X_t$ is a log resolution.*

In the sequel, a Hodge structure (resp. variation of Hodge structures) is a finite direct sum of pure polarizable \mathbb{Q} -Hodge structures (resp. variations of pure polarizable \mathbb{Q} -Hodge structures), possibly of different weights. Following [33, Proposition 2.8], [31, (5.6)] and [62, (0.6)], we make the following definition:

Definition 6.13 Let S be a complex manifold. Let $(\mathbb{H}, \mathcal{H}, F^\bullet)$ be a variation of Hodge structures over S , where \mathbb{H} is a local system, $\mathcal{H} = \mathbb{H} \otimes \mathcal{O}_S$ and F^\bullet is a Hodge filtration on \mathcal{H} . For any $k \in \mathbb{Z}$, denote by \mathbb{H}^k the weight- k part of the variation. For any $p, q \in \mathbb{Z}$, denote $\mathcal{H}^{p,q} := \text{Gr}_F^p \mathcal{H}^{p+q}$, which we view as a holomorphic vector bundle over S . The *Hodge form* of the variation is the following $(1, 1)$ -form on S :

$$\omega_{\mathbb{H}} := \frac{1}{2} \sum_{p,q} (-1)^{p+q} (p - q) c_1(\mathcal{H}^{p,q}, g^{\mathcal{H}^{p,q}}) \in A^{1,1}(S). \tag{6.10}$$

Here $g^{\mathcal{H}^{p,q}}$ is a Hermitian metric on $\mathcal{H}^{p,q}$ such that $g^{\mathcal{H}^{p,q}}(u, u) = g^{\mathcal{H}^{q,p}}(\bar{u}, \bar{u})$. One can show (cf. [62, Proposition 1.1]) that $\omega_{\mathbb{H}}$ is independent of the Hermitian metrics $g^{\mathcal{H}^{p,q}}$. Clearly, ω is additive with respect to short exact sequences of variations of Hodge structures, hence it gives rise to a group homomorphism:

$$\omega : K_0(\text{VHS}_S) \rightarrow A^{1,1}(S), \tag{6.11}$$

where VHS_S is the category of variations of Hodge structures over S .

On the other hand, the Hodge realization can be performed in the relative setting: given a complex variety S , there is a group homomorphism

$$\begin{aligned} \chi_{\text{Hdg}, S} : K_0(\text{Var}_S) &\rightarrow K_0(\text{MHM}_S) \\ (\pi : \mathcal{X} \rightarrow S) &\mapsto \mathbb{R} \pi_! \mathbb{Q}_{\mathcal{X}}, \end{aligned} \tag{6.12}$$

where MHM_S is the category of mixed Hodge modules over S . However, note that if we start with a smooth proper morphism $\pi : \mathcal{X} \rightarrow S$, then the image $\mathbb{R}\pi_!\mathbb{Q}_{\mathcal{X}}$ lies in $K_0(\text{VHS}_S)$, the subgroup generated by variations of Hodge structures over S .

Now let $\pi : \mathcal{X} \rightarrow S$ be a locally trivial family of KLT Calabi–Yau varieties. The *Gorenstein volume* in Definition 6.7 can be extended as follows. Take a simultaneous resolution (see Lemma 6.12) $f : \mathcal{X}' \rightarrow \mathcal{X}$ with simple normal crossing exceptional divisor $\mathcal{E} = \bigcup_{j=1}^l \mathcal{E}_j$, which is locally trivial over S . For any $1 \leq j \leq l$, let $a_j \in \mathbb{Q}_{>-1}$ be the discrepancy number of the resolution f along \mathcal{E}_j . Then define the *relative Gorenstein volume*

$$\mu^{\text{Gor}}(\mathcal{X}/S) := \sum_{J \subseteq \{1, \dots, l\}} \mathbb{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - \mathbb{L}^{a_j}}{\mathbb{L}^{a_j+1} - 1} \right) [\mathcal{E}_J/S] \in \widehat{\mathcal{M}}_S[\mathbb{L}^{1/d}], \tag{6.13}$$

where $\widehat{\mathcal{M}}_S$ is the completion of $K_0(\text{Var}_S)[\mathbb{L}^{-1}]$ with respect to the dimension filtration. Similarly to Theorem 4.2, $\mu^{\text{Gor}}(\mathcal{X}/S)$ is independent of the resolution f .

Taking the Hodge realization and denoting $\mathbf{L} := \mathbb{Q}_S(-1)$ the Lefschetz variation of Hodge structure over S , we get

$$\chi_{\text{Hdg},S}(\mu^{\text{Gor}}(\mathcal{X}/S)) = \sum_{J \subseteq \{1, \dots, l\}} \mathbf{L}^{|J|-n} \left(\prod_{j \in J} \frac{1 - \mathbf{L}^{a_j}}{\mathbf{L}^{a_j+1} - 1} \right) H^\bullet(\mathcal{E}_J/S), \tag{6.14}$$

where $H^\bullet(\mathcal{E}_J/S)$ denotes the variation of Hodge structures $\mathbb{R}\pi_{J,*}\mathbb{Q}_{\mathcal{E}_J} := \bigoplus_k \mathbf{R}^k \pi_{J,*}\mathbb{Q}_{\mathcal{E}_J}$, where $\pi_J : \mathcal{E}_J \rightarrow S$ is the natural projection.

Definition 6.14 Let $\pi : \mathcal{X} \rightarrow S$ be as above. Its (stringy) *Hodge form*, denoted by $\omega_{\text{Hdg},\mathcal{X}/S}$, is the image of $\chi_{\text{Hdg},S}(\mu^{\text{Gor}}(\mathcal{X}/S))$ via (6.11).

Lemma 6.15 *Notation is as before. Taking a simultaneous resolution $f : \mathcal{X}' \rightarrow \mathcal{X}$ as above, the Hodge form can be computed as*

$$\omega_{\text{Hdg},\mathcal{X}/S} = \sum_{J \subseteq \{1, \dots, l\}} \left(\prod_{j \in J} \frac{-a_j}{a_j + 1} \right) \omega_{H^\bullet(\mathcal{E}_J/S)}. \tag{6.15}$$

Proof It suffices to apply the homomorphism (6.11) to the right hand side of (6.14), and use the fact that $\omega_{H^\bullet} = \omega_{\mathbf{L}H^\bullet}$. □

Definition 6.16 Let $\pi : \mathcal{X} \rightarrow S$ be as above and d an integer divisible by the index of fibers. The *Weil–Peterson form* of $\pi : \mathcal{X} \rightarrow S$, denoted by $\omega_{\text{WP},\mathcal{X}/S}$, is defined as follows: for any open subset $U \subseteq S$ and any nowhere vanishing section $\gamma \in H^0(U, \pi_*\mathcal{O}(dK_{\mathcal{X}/S}))$ viewed as holomorphic family $(\gamma_s)_{s \in U}$, we define

$$\omega_{\text{WP},\mathcal{X}/S}|_U = -\frac{\bar{\partial}\partial}{2\pi i} \log \int_{X_{\text{reg}}} |\gamma \bar{\gamma}|^{1/d}, \tag{6.16}$$

where $\int_{X_{s,\text{reg}}} |\gamma \overline{\gamma}|^{1/d}$ is the function $s \mapsto \int_{X_{s,\text{reg}}} |\gamma_s \overline{\gamma_s}|^{1/d}$ whose convergence is guaranteed by Proposition 6.3. Thanks to the next lemma, the partial derivatives make sense in (6.16).

Lemma 6.17 *The function $S \ni s \mapsto \int_{X_{s,\text{reg}}} |\gamma_s \overline{\gamma_s}|^{1/d}$ is smooth.*

Proof The question is local on S . We may assume that $\gamma \in H^0(S, \pi_* \mathcal{O}(dK_{\mathcal{X}/S}))$, i.e. a holomorphic family of d -canonical forms on the fibers. By Lemma 6.12, there is a simultaneous log-resolution $f: \mathcal{X}' \rightarrow \mathcal{X}$ with simple normal crossing exceptional divisor $\mathcal{E} = \bigcup_{j=1}^l \mathcal{E}_j$, which is also locally trivial over S . Set $\gamma' = f^* \gamma$, which is a family of meromorphic d -canonical forms on the fibers of \mathcal{X}'/S . We obviously have

$$\int_{X_{s,\text{reg}}} |\gamma_s \overline{\gamma_s}|^{1/d} = \int_{X'_s} |\gamma'_s \overline{\gamma'_s}|^{1/d}. \tag{6.17}$$

Let

$$\left(\alpha_k : \mathbb{C}^m \supseteq U_k \hookrightarrow S \right)_{k=1,\dots,p}, \quad \left(\beta_k : \mathbb{C}^m \times \mathbb{C}^n \supseteq V_k \hookrightarrow \mathcal{X}' \right)_{k=1,\dots,p} \tag{6.18}$$

be holomorphic local charts such that

- for each k , we have $(\pi \circ f)(\beta_k(V_k)) = \alpha_k(U_k)$;
- for each k , the map $\alpha_k^{-1} \circ \pi \circ f \circ \beta_k$ is given by the canonical projection $\mathbb{C}^m \times \mathbb{C}^n \rightarrow \mathbb{C}^m$;
- for each k and each \mathcal{E}_j , either $\beta_k^{-1}(\mathcal{E}_j) = \emptyset$ or $\beta_k^{-1}(\mathcal{E}_j) = \{(z_1, \dots, z_m, w_1, \dots, w_n) \in V_k \subseteq \mathbb{C}^m \times \mathbb{C}^n : w_q = 0\}$, where $q \in \{1, \dots, n\}$ depends on j, k .

Let

$$\left(\eta_k : \beta_k(V_k) \rightarrow [0, 1] \right)_{k=1,\dots,p} \tag{6.19}$$

be a partition of unity, i.e., each η_k is of compact support and $\eta_1 + \dots + \eta_p = 1$. We denote $\eta_k|_{X'_s} = \eta_{k,s}$. It is sufficient to show that for each k , the function

$$s \mapsto \int_{X'_s} \eta_{k,s} |\gamma'_s \overline{\gamma'_s}|^{1/d} \tag{6.20}$$

is smooth. On the other hand, by the KLT condition, the order of poles of γ' along each \mathcal{E}_j is at most $d - 1$. Hence we have

$$\beta_k^* \gamma' = \frac{g}{w_1^{d-1} \dots w_n^{d-1}} (dw_1 \wedge \dots \wedge dw_n)^d, \tag{6.21}$$

where g is a holomorphic function of $z_1, \dots, z_m, w_1, \dots, w_n$. Then a straightforward computation shows that the integration (6.20) results in a smooth function in z_1, \dots, z_m . □

We are now ready to prove Theorem C, which we state again for convenience.

Theorem 6.18 *Let $\pi : \mathcal{X} \rightarrow S$ be a flat family of KLT Calabi–Yau varieties. Assume that π is locally trivial. Then the function $\tau(\mathcal{X}/S) : S \ni t \mapsto \tau(X_t)$ is smooth, where τ is the BCOV invariant in Definition 6.9. Moreover, we have*

$$\frac{\bar{\partial}\partial}{2\pi i} \tau(\mathcal{X}/S) = \omega_{\text{Hdg}, \mathcal{X}/S} - \frac{\chi(X)}{12} \omega_{\text{WP}, \mathcal{X}/S}, \tag{6.22}$$

where $\chi(X)$ is the stringy Euler characteristic of a fiber of π , $\omega_{\mathcal{X}/S}$ is the Hodge form in Definition 6.14, and $\omega_{\text{WP}, \mathcal{X}/S}$ is the Weil–Petersson form in Definition 6.16.

Proof The smoothness of the function $\tau_{\mathcal{X}/S}$ comes from the existence of simultaneous resolutions.

Let $d \in \mathbb{N}_{>0}$ be such that $|dK_{X_t}| \neq \emptyset$. Notation being as before, we have

$$\begin{aligned} \frac{\bar{\partial}\partial}{2\pi i} \tau(\mathcal{X}/S) &= \frac{\bar{\partial}\partial}{2\pi i} \tau_d^{\text{bir}}(\mathcal{X}/S, \emptyset) \\ &= \frac{\bar{\partial}\partial}{2\pi i} \tau_d^{\text{bir}}(\mathcal{X}/S, \sum_j da_j \mathcal{E}_j/S) = \frac{\bar{\partial}\partial}{2\pi i} \tau_d(\mathcal{X}/S, \sum_j da_j \mathcal{E}_j/S), \end{aligned} \tag{6.23}$$

Here the first and the third equalities come from the fact that $\tau_d^{\text{bir}} - \tau_d$ consists of topological invariants, which are constant in locally trivial families. The second equality comes from (6.3). By Definition 6.4, Definition 6.9 and [61, Theorem 0.4], we have

$$\begin{aligned} \frac{\bar{\partial}\partial}{2\pi i} \tau_d(\mathcal{X}/S, \sum_j da_j \mathcal{E}_j/S) &= \sum_{J \subseteq \{1, \dots, l\}} \left(\prod_{j \in J} \frac{-a_j}{a_j + 1} \right) \omega_{H^\bullet(\mathcal{E}_J/S)} + \frac{1}{12} \chi_d(X', \sum_j da_j E_j) \frac{\bar{\partial}\partial}{2\pi i} \int_{X'} |\gamma' \bar{\gamma}'|^{1/d}. \end{aligned} \tag{6.24}$$

By Lemma 6.15, we have

$$\sum_{J \subseteq \{1, \dots, l\}} \left(\prod_{j \in J} \frac{-a_j}{a_j + 1} \right) \omega_{H^\bullet(\mathcal{E}_J/S)} = \omega_{\text{Hdg}, \mathcal{X}/S}. \tag{6.25}$$

Using Definition 6.16, we can show that

$$\frac{\bar{\partial}\partial}{2\pi i} \int_{X'} |\gamma' \bar{\gamma}'|^{1/d} = \frac{\bar{\partial}\partial}{2\pi i} \int_{X_{\text{reg}}} |\gamma \bar{\gamma}|^{1/d} = -\omega_{\text{WP}, \mathcal{X}/S}. \tag{6.26}$$

By the definition of the stringy Euler characteristic, we have

$$\chi_d(X', \sum_j da_j E_j) = \chi(X). \tag{6.27}$$

Combining (6.23)–(6.27), we obtain (6.22).

7 Birational invariance

In this section, we prove our main result Theorem B. Although it clearly contains Theorem A as the smooth case, we nevertheless choose to give first the proof in this special case to highlight the main idea:

Proof of Theorem A Let X and X' be n -dimensional birationally isomorphic Calabi–Yau manifolds. By the weak factorization theorem of Abramovich, Karu, Matsuki, and Włodarczyk [1, Theorem 0.3.1] (see also [57]), there is a sequence of blow-ups and blow-downs along smooth centers:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{r-1} \dashrightarrow X_r = X', \tag{7.1}$$

such that for each $0 \leq i \leq r$, the unique canonical divisor $D_i \in |K_{X_i}|$ is of simple normal crossing support. For each i , let $\gamma_i \in H^0(X_i, \mathcal{O}_{X_i}(D_i)) = H^0(X_i, \mathcal{O}_{X_i}(K_{X_i}))$ be such that $\int_{X_i} |\gamma_i \bar{\gamma}_i| = (2\pi)^n$. By Theorem 5.5 and [61, Proposition 3.4], we have

$$\tau_1^{\text{bir}}(X_i, \gamma_i) = \tau_1^{\text{bir}}(X_{i+1}, \gamma_{i+1}) \tag{7.2}$$

for all i . Hence

$$\tau_1^{\text{bir}}(X, \gamma_0) = \tau_1^{\text{bir}}(X', \gamma_r)$$

Combining this with (5.2), we see that in order to prove $\tau(X) = \tau(X')$, it is enough to show that $\chi'(X) = \chi'(X')$ and $\chi''(X) = \chi''(X')$. However, $\chi'(\cdot)$ and $\chi''(\cdot)$, defined in (2.6) and (2.7) respectively, are certain linear combinations of Betti numbers, hence are birational invariant for Calabi–Yau manifolds by Batyrev [5]. □

Now let us proceed to the proof in the general case.

Proof of Theorem B Recall that $\mu^{\text{Gor}}(\cdot)$ was defined in Definition 6.7. By a result of Kontsevich [45] and Yasuda [58, Proposition 1.2], which extends a result of Batyrev [5], we have $\mu^{\text{Gor}}(X) = \mu^{\text{Gor}}(X')$. Then, by Definition 6.8, we have

$$\chi'(X) = \chi'(X'), \quad \chi''(X) = \chi''(X'). \tag{7.3}$$

Let $f: \tilde{X} \rightarrow X$ and $f': \tilde{X}' \rightarrow X'$ be log-resolutions. Let $d \in \mathbb{N}_{>0}$ be such that $dK_X \sim 0$ and $dK_{X'} \sim 0$ as Cartier divisors. Note that the hypothesis that X and X'

have canonical singularities implies that any smooth birational model of X and X' admits a unique d -canonical divisor.

By Abramovich–Temkin [2, Theorem 1.2.1 and §1.6], there is a sequence of blow-ups and blow-downs along smooth centers:

$$\tilde{X} = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{r-1} \dashrightarrow X_r = \tilde{X}', \tag{7.4}$$

such that for each $0 \leq i \leq r$, the (unique) d -canonical divisor $D_i \in |dK_{X_i}|$ is of simple normal crossing support. For each i , let $\gamma_i \in H^0(X_i, \mathcal{O}_{X_i}(D_i)) = H^0(X_i, \mathcal{O}_{X_i}(dK_{X_i}))$ be such that $\int_{X_i} |\gamma_i \bar{\gamma}_i|^{1/d} = (2\pi)^n$. By Theorem 5.5, we have

$$\tau_d^{\text{bir}}(X_i, \gamma_i) = \tau_d^{\text{bir}}(X_{i+1}, \gamma_{i+1}) \tag{7.5}$$

for all i . Let $\gamma \in H^0(X, \mathcal{O}_X(dK_X))$ and $\gamma' \in H^0(X', \mathcal{O}_{X'}(dK_{X'}))$ be such that $\int_X |\gamma \bar{\gamma}|^{1/d} = \int_{X'} |\gamma' \bar{\gamma}'|^{1/d} = (2\pi)^n$. By Definition 6.4 and [61, Proposition 3.4], we have

$$\begin{aligned} \tau_d^{\text{bir}}(X, \emptyset) &= \tau_d^{\text{bir}}(X, \gamma) = \tau_d^{\text{bir}}(X_0, \gamma_0), \\ \tau_d^{\text{bir}}(X', \emptyset) &= \tau_d^{\text{bir}}(X', \gamma') = \tau_d^{\text{bir}}(X_r, \gamma_r). \end{aligned} \tag{7.6}$$

From Definition 6.9, (7.3), (7.5) and (7.6), we obtain $\tau(X) = \tau(X')$. □

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References

1. Abramovich, D., Karu, K., Matsuki, K., Włodarczyk, J.: Torification and factorization of birational maps. *J. Am. Math. Soc.* **15**(3), 531–572 (2002)
2. Abramovich, D., Temkin, M.: Functorial factorization of birational maps for qc schemes in characteristic 0. *Algebra Number Theory* **13**(2), 379–424 (2019)
3. Bakker, B., Lehn, C.: The global moduli theory of symplectic varieties. *J. für die Reine und Angew. Math.* [Crelle’s Journal] (2022)
4. Batyrev, V.V.: Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In: *Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997)*, pp. 1–32. World Sci. Publ., River Edge, NJ (1998)
5. Batyrev, V.V.: Birational Calabi–Yau n -folds have equal Betti numbers. In: *New trends in algebraic geometry (Warwick, 1996)*, vol. 264 of London Math. Soc. Lecture Note Ser., pp. 1–11. Cambridge Univ. Press, Cambridge (1999)
6. Baum, P., Fulton, W., MacPherson, R.: Riemann–Roch for singular varieties. *Inst. Hautes Études Sci. Publ. Math.* **45**, 101–145 (1975)
7. Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differ. Geom.* **18**(4), 755–782 (1984), 1983

8. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Holomorphic anomalies in topological field theories. *Nuclear Phys. B* **405**(2–3), 279–304 (1993)
9. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes. *Commun. Math. Phys.* **165**(2), 311–427 (1994)
10. Berthomieu, A., Bismut, J.-M.: Quillen metrics and higher analytic torsion forms. *J. Reine Angew. Math.* **457**, 85–184 (1994)
11. Bismut, J.-M.: Quillen metrics and singular fibres in arbitrary relative dimension. *J. Algebraic Geom.* **6**(1), 19–149 (1997)
12. Bismut, J.-M.: Holomorphic and de Rham torsion. *Compos. Math.* **140**(5), 1302–1356 (2004)
13. Bismut, J.-M., Lebeau, G.: Complex immersions and Quillen metrics. *Inst. Hautes Études Sci. Publ. Math.* (74):ii+298 pp. (1992), 1991
14. Borisov, L., Căldăraru, A.: The Pfaffian-Grassmannian derived equivalence. *J. Algebraic Geom.* **18**(2), 201–222 (2009)
15. Bridgeland, T.: Flops and derived categories. *Invent. Math.* **147**(3), 613–632 (2002)
16. Bridgeland, T., King, A., Reid, M.: The McKay correspondence as an equivalence of derived categories. *J. Am. Math. Soc.* **14**(3), 535–554 (2001)
17. Campana, F.: Orbifolds à première classe de Chern nulle. In: *The Fano Conference*, pp. 339–351. Univ. Torino, Turin (2004)
18. Candelas, P., de la Ossa, X., Green, P., Parkes, L.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B* **359**(1), 21–74 (1991)
19. Chambert-Loir, A., Nicaise, J., Sebag, J.: *Motivic integration*, volume 325 of *Progress in Mathematics*. Birkhäuser/Springer, New York (2018)
20. Chang, H.-L., Guo, S., Li, J.: Polynomial structure of Gromov–Witten potential of quintic 3-folds. *Ann. Math. (2)* **194**(3), 585–645 (2021)
21. Chang, H.-L., Guo, S., Li, J., Li, W.-P.: BCOV’s Feynman rule of quintic 3-folds. preprint, [arXiv:1810.00394](https://arxiv.org/abs/1810.00394) (2018)
22. Chang, H.-L., Guo, S., Li, J., Li, W.-P.: The theory of N-mixed-spin-P fields. *Geom. Topol.* **25**(2), 775–811 (2021)
23. Chang, H.-L., Li, J., Li, W.-P., Liu, C.-C.M.: *Mixed-Spin-P fields of Fermat quintic polynomials*. Cambridge J. Math., Volume 7, Number 3 (2019)
24. Chang, H.-L., Li, J., Li, W.-P., Liu, C.-C.M.: An effective theory of GW and FJRW invariants of quintic Calabi–Yau manifolds. *J. Differ. Geom.* **120**(2), 251–306 (2022)
25. Chen, Q., Janda, F., Ruan, Y.: The logarithmic gauged linear sigma model. *Invent. Math.* **225**, 1077–1154 (2021)
26. Craw, A.: An introduction to motivic integration. In: *Strings and geometry*, vol. 3 of *Clay Math. Proc.*, pp. 203–225. American Mathematical Society, Providence, RI (2004)
27. Căldăraru, A.: Non-birational Calabi–Yau threefolds that are derived equivalent. *Internat. J. Math.* **18**(5), 491–504 (2007)
28. Dai, X., Yoshikawa, K.-I.: Analytic torsion for log-Enriques surfaces and Borchers product. *Forum Math. Sigma*, 10:Paper No. e77, 54 (2022)
29. Denef, J., Loeser, F.: Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.* **135**(1), 201–232 (1999)
30. Denef, J., Loeser, F.: Motivic integration, quotient singularities and the McKay correspondence. *Compos. Math.* **131**(3), 267–290 (2002)
31. Eriksson, D., Freixas i Montplet, G., Mourougane, C.: BCOV invariants of Calabi–Yau manifolds and degenerations of Hodge structures. *Duke Math. J.* **170**(3), 379–454 (2021)
32. Eriksson, D., Freixas i Montplet, G., Mourougane, C.: On genus one mirror symmetry in higher dimensions and the BCOV conjectures. *Forum Math. Pi*, 10:Paper No. e19, 53 (2022)
33. Fang, H., Lu, Z.: Generalized Hodge metrics and BCOV torsion on Calabi–Yau moduli. *J. Reine Angew. Math.* **588**, 49–69 (2005)
34. Fang, H., Lu, Z., Yoshikawa, K.-I.: Analytic torsion for Calabi–Yau threefolds. *J. Differ. Geom.* **80**(2), 175–259 (2008)
35. Flenner, H., Kosarew, S.: On locally trivial deformations. *Publ. Res. Inst. Math. Sci.* **23**(4), 627–665 (1987)
36. Flenner, H., Zaidenberg, M.: Log-canonical forms and log canonical singularities. *Math. Nachr.* **254**(255), 107–125 (2003)

37. Fujiki, A.: On primitively symplectic compact Kähler V -manifolds of dimension four. In: Classification of algebraic and analytic manifolds (Katata, 1982), volume 39 of *Progr. Math.*, pp. 71–250. Birkhäuser Boston, Boston, MA (1983)
38. Givental, A.B.: Equivariant Gromov–Witten invariants. *Int. Math. Res. Notices* **13**, 613–663 (1996)
39. Givental, A.B.: A mirror theorem for toric complete intersections. In: *Topological Field Theory, Primitive Forms and Related Topics* (Kyoto, 1996), vol. 160 of *Progr. Math.*, pp. 141–175. Birkhäuser Boston, MA (1998)
40. Guo, S., Janda, F., Ruan, Y.: A mirror theorem for genus two Gromov–Witten invariant of quintic 3-fold. preprint, [arXiv: 1709.07392](https://arxiv.org/abs/1709.07392) (2017)
41. Guo, S., Janda, F., Ruan, Y.: Structure of higher genus Gromov–Witten invariants of quintic 3-folds. preprint, [arXiv: 1812.11908](https://arxiv.org/abs/1812.11908), (2018)
42. Kato, K.: Heights of motives. *Proc. Jpn. Acad. Ser. A Math. Sci.* **90**(3), 49–53 (2014)
43. Knudsen, F.F., Mumford, D.: The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”. *Math. Scand.* **39**(1), 19–55 (1976)
44. Kollár, J., Mori, S.: *Birational geometry of algebraic varieties*, vol. 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
45. Kontsevich, M.: *Motivic Integration*. Lecture at Orsay (1995)
46. Lian, B.H., Liu, K., Yau, S.-T.: Mirror principle. I. *Asian J. Math.* **1**(4), 729–763 (1997)
47. Ma, X.: Orbifolds and analytic torsions. *Trans. Am. Math. Soc.* **357**(6), 2205–2233 (2005)
48. Ma, X.: Orbifold submersion and analytic torsions. In: *Arithmetic L-functions and Differential Geometric Methods*, vol. 338 of *Progr. Math.*, pp. 141–177. Birkhäuser/Springer, Cham (2021)
49. Maillot, V., RöSSLer, D.: On the birational invariance of the BCOV torsion of Calabi–Yau threefolds. *Commun. Math. Phys.* **311**(2), 301–316 (2012)
50. Morrison, D.R.: Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. *J. Am. Math. Soc.* **6**(1), 223–247 (1993)
51. Quillen, D.: Determinants of Cauchy–Riemann operators on Riemann surfaces. *Funct. Anal. Appl.* **19**, 31–34 (1985)
52. Ray, D.B., Singer, I.M.: Analytic torsion for complex manifolds. *Ann. Math.* **2**(98), 154–177 (1973)
53. Sakai, F.: Kodaira dimensions of complements of divisors. In: *Complex Analysis and Algebraic Geometry*, pp. 239–257 (1977)
54. Satake, I.: On a generalization of the notion of manifold. *Proc. Natl. Acad. Sci. USA* **42**, 359–363 (1956)
55. Satake, I.: The Gauss–Bonnet theorem for V -manifolds. *J. Math. Soc. Jpn.* **9**, 464–492 (1957)
56. Uehara, H.: A counterexample of the birational Torelli problem via Fourier–Mukai transforms. *J. Algebraic Geom.* **21**(1), 77–96 (2012)
57. Włodarczyk, J.: Toroidal varieties and the weak factorization theorem. *Invent. Math.* **154**(2), 223–331 (2003)
58. Yasuda, T.: Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.* **140**(2), 396–422 (2004)
59. Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Commun. Pure Appl. Math.* **31**(3), 339–411 (1978)
60. Yoshikawa, K.-I.: Analytic torsion and an invariant of Calabi–Yau threefold. In: *Differential geometry and physics*, vol. 10 of *Nankai Tracts Math.*, pp. 480–489. World Sci. Publ., Hackensack, NJ (2006)
61. Zhang, Y.: BCOV invariant and blow-up. To appear in *Compositio Mathematica*. [arxiv:2003.03805](https://arxiv.org/abs/2003.03805)
62. Zhang, Y.: An extension of BCOV invariant. *International Mathematics Research Notices*, rnaa265 (2020)
63. Zinger, A.: Standard versus reduced genus-one Gromov–Witten invariants. *Geom. Topol.* **12**(2), 1203–1241 (2008)
64. Zinger, A.: The reduced genus 1 Gromov–Witten invariants of Calabi–Yau hypersurfaces. *J. Am. Math. Soc.* **22**(3), 691–737 (2009)

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