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CLASSIFICATION OF POLARIZED SYMPLECTIC AUTOMORPHISMS OF FANO VARIETIES OF CUBIC FOURFOLDS

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Abstract. We classify the polarized symplectic automorphisms of Fano varieties of smooth cubic fourfolds (equipped with the Plücker polarization) and study the fixed loci.

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1. Introduction. The purpose of this paper is to classify all polarized symplectic automorphisms of the irreducible holomorphic symplectic projective varieties constructed by Beauville and Donagi [**4**], namely, the Fano varieties of (smooth) cubic fourfolds.

Finite-order symplectic automorphisms of K3 surfaces have been studied in detail by Nikulin in [**18**]. A natural generalization of K3 surfaces to higher dimensions is the notion of *irreducible holomorphic symplectic manifolds* or *hyper-Kähler manifolds* (*cf.* [**2**]), which, by definition, are simply connected compact Kähler manifolds with $H^{2,0}$ generated by a symplectic form (*i.e.* nowhere degenerate holomorphic 2-form). Initiated by Beauville [**1**], some results have been obtained in the study of automorphisms of such manifolds. Let us mention [**3**, **5**, **6**, **7**, **8**].

In [**4**], Beauville and Donagi show that the Fano varieties of lines of smooth cubic fourfolds provide an example of a 20-dimensional family of irreducible holomorphic symplectic projective fourfolds. We propose to classify the polarized *symplectic* automorphisms of this family up to conjugacy. Our classification result is shown in the table below[1]. The following remarks concern this table:

(i) As is remarked in Section 2, such an automorphism comes from a (finite order) automorphism of the cubic fourfold itself. Hence, we express the automorphism in the fourth column as an element f in PGL_6, and our classification is up to conjugacy with respect to the action of PGL_6.

(ii) In the third column, n is the order of f, which is a primary number. The reason why we only listed the automorphisms with primary order is that every finite-order automorphism is a product of commuting automorphisms with primary orders, by the structure of cyclic groups. *cf.* Remark 4.4.

(iii) We give an explicit basis of the family in the fifth column.

[1] Please see the next page.

(iv) In the sixth column, we work out the fixed loci for a generic member. For geometric descriptions of the fixed loci, see Section 5. The numbers of moduli are in the last column.

(v) Some of these families have been discovered and studied before: Family I is described in [**16**]; Family IV-(2) first appeared in [**17**] and is also treated in [**13**]; Family V-(1) is studied in [**8**], where the fixed locus and the number of moduli are calculated. More generally, the classification of prime order automorphisms of cubic fourfolds has been done in [**12**]. In Mongardi's Ph.D. thesis [**15**], he classifies the prime order symplectic automorphisms of hyper-Kähler varieties which are of $K3^{[n]}$-deformation type. To the best of my knowledge, the remaining primary order automorphisms IV-(4), IV-(5), V-(2), V-(3) are new.

THEOREM 1.1 (Classification). *For any smooth cubic fourfold X, let $F(X)$ be its Fano variety of lines, equipped with the Plücker polarization. We classify, in the following list, all families of cubic fourfolds equipped with an automorphism of primary order whose members (resp. general members) are smooth if the family is isotrivial*[2] *(resp. not isotrivial), such that the induced actions on the Fano varieties of lines are symplectic.*

Family	p	$n=p^m$	*automorphism*	*basis for* $\overline{B}$	*fixed loci*	*dimension*
0	*1*	*1*	$f=\mathrm{id}_{\mathbf{P}^5}$	*deg 3 monomials*	$F(X)$	*20*
I	*11*	*11*	$f=\mathrm{diag}(\zeta,\zeta^{-2},$ $\zeta^4,\zeta^3,\zeta^5,1)$ $\zeta=e^{\frac{r}{11}\cdot 2\pi\sqrt{-1}}, 1\le r\le 10$	$x_0^2x_1$ $x_1^2x_2$ $x_2^2x_3$ $x_3^2x_4$ $x_4^2x_0$ x_5^3	*5 pts*	*0*
II	*7*	*7*	$f=\mathrm{diag}(\zeta,\zeta^{-2},\zeta^{-3},\zeta^{-1},$ $\zeta^2,\zeta^3)$ $\zeta=e^{\frac{r}{7}\cdot 2\pi\sqrt{-1}}, 1\le r\le 6$	$x_0^2x_1$ $x_1^2x_2$ $x_2^2x_3$ $x_3^2x_4$ $x_4^2x_5$ $x_5^2x_0$ $x_0x_2x_4$ $x_1x_3x_5$	*9 pts*	*2*

[2]We say a family is isotrivial if it is isomorphic to a constant family after an étale base change. In particular, in an isotrivial family, the cubic fourfolds (equipped with automorphisms) are equivariantly isomorphic.

III	5	5	$f = \mathrm{diag}(\zeta, \zeta^{-2}, \zeta^{-1},$ $\zeta^2, 1, 1)$ $\zeta = e^{\frac{r}{5}\cdot 2\pi\sqrt{-1}}, 1 \leq r \leq 4$	$x_0^2x_1$ $x_1^2x_2$ $x_2^2x_3$ $x_3^2x_0$ $x_4^2x_5$ $x_5^2x_4$ x_5^3 x_4^3 $x_0x_2x_4$ $x_0x_2x_5$ $x_1x_3x_4$ $x_1x_3x_5$	*14 pts*	4
IV-(1)	3	3	$f = \mathrm{diag}(1, 1, 1, 1, \omega, \omega^2)$ $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$	*deg 3 monomials on* $x_0, \dots, x_3$ x_4^3 x_5^3 $x_0x_4x_5$ $x_1x_4x_5$ $x_2x_4x_5$ $x_3x_4x_5$	*27 pts*	8
IV-(2)	3	3	$f = \mathrm{diag}(1, 1, 1, \omega, \omega, \omega)$ $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$	*deg 3 monomials on* x_0, x_1, x_2 *deg 3 monomials on* x_3, x_4, x_5	*an abelian surface*	2
IV-(3)	3	3	$f = \mathrm{diag}(1, 1, \omega, \omega, \omega^2, \omega^2)$ $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$	*deg 3 monomials on* x_0, x_1 *deg 3 monomials on* x_2, x_3 *deg 3 monomials on* x_4, x_5 $x_0x_2x_4$ $x_0x_2x_5$ $x_0x_3x_4$ $x_0x_3x_5$ $x_1x_2x_4$ $x_1x_2x_5$ $x_1x_3x_4$ $x_1x_3x_5$	*27 pts*	8

IV-(4)	3	9	$f = \mathrm{diag}(1, \zeta^{-3}, \zeta^3, \zeta, \zeta^4, \zeta^{-2})$ $\zeta = e^{\frac{r}{9}\cdot 2\pi\sqrt{-1}}, r = 1, 2, 4, 5, 7, 8$	$x_0^2x_1$ $x_1^2x_2$ $x_2^2x_0$ $x_3^2x_4$ $x_4^2x_5$ $x_5^2x_3$	*9 pts*	*0*
IV-(5)	3	9	$f = \mathrm{diag}(1, \zeta^3, \zeta^{-3}, \zeta, \zeta, \zeta^4)$ $\zeta = e^{\frac{r}{9}\cdot 2\pi\sqrt{-1}}, r = 1, 2, 4, 5, 7, 8$	$x_0^2x_1$ $x_1^2x_2$ $x_2^2x_0$ $x_3^2x_4$ $x_3x_4^2$ x_3^3 x_4^3 x_5^3	*9 pts*	*0*
V-(1)	2	2	$f = \mathrm{diag}(1, 1, 1, 1, -1, -1)$	*deg 3 monomials on* $x_0, \ldots, x_3$ $x_0x_5^2, x_1x_5^2, x_2x_5^2, x_3x_5^2$ $x_0x_4^2, x_1x_4^2, x_2x_4^2, x_3x_4^2$ $x_0x_4x_5, x_1x_4x_5,$ $x_2x_4x_5, x_3x_4x_5$	*28 pts & a K3 surface*	*12*
V-(2)	2	4	$f = \mathrm{diag}(1, 1, -1, -1, \sqrt{-1}, -\sqrt{-1})$	$x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$ $x_0\cdot$ *deg 2 monomials on* x_2, x_3 $x_1\cdot$ *deg 2 monomials on* x_2, x_3 $x_2x_4^2$ $x_3x_4^2$ $x_2x_5^2$ $x_3x_5^2$ $x_0x_4x_5$ $x_1x_4x_5$	*16 pts*	*6*

V-(3)	*2*	*8*	$f = \mathrm{diag}(1, -1, \zeta^2, \zeta^{-2}, \zeta, \zeta^3)$ $\zeta = e^{\frac{r}{8}\cdot 2\pi\sqrt{-1}}$, $r = \pm 1 \mod 8$	x_0^3 $x_0x_1^2$ $x_1x_2^2$ $x_1x_3^2$ $x_0x_2x_3$ $x_3x_4^2$ $x_2x_5^2$ $x_1x_4x_5$	*6 pts*	*2*

The structure of this paper is as follows. In Section 2, we set up the basic notation, and show that any polarized automorphism of the Fano variety comes from a finite-order automorphism of the cubic fourfold. Then in Section 3, we reinterpret the assumption of being symplectic into a numerical equation by using Griffiths' theory of residue. Finally we do the classification in Section 4. The basic observation is that the generic smoothness of the family of cubics imposes strong combinatorial constraints.

Throughout this paper, we work over the field of complex numbers with a fixed choice of $\sqrt{-1}$.

2. Fano varieties of lines of cubic fourfolds. First of all, let us fix the notation and make some basic constructions. Let V be a 6-dimensional $\mathbf{C}$-vector space, and $\mathbf{P}^5 := \mathbf{P}(V)$ be the corresponding projective space of 1-dimensional subspaces of V. Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold. The following subvariety of the Grassmannian $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$

$$F(X) := \left\{[L] \in \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5) \mid L \subset X\right\}, \tag{1}$$

is called the *Fano variety of lines*[3] of X. It is well-known that $F(X)$ is a 4-dimensional smooth projective variety. Throughout this paper, we always equip $F(X)$ with the polarization $\mathscr{L}$, which is by definition the restriction of the Plücker line bundle on the ambient Grassmannian $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.

Consider the incidence variety (*i.e.* the universal projective line over $F(X)$):

$$P(X) := \{(x, [L]) \in X \times F(X) \mid x \in L\},$$

and then the following natural correspondence:

$$\begin{tikzcd} P(X) \arrow[r, "q"] \arrow[d, "p"'] & X \\ F(X) & \end{tikzcd}$$

we have the following

[3]In the scheme-theoretic language, $F(X)$ is defined to be the zero locus of $s_T \in H^0\left(\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5), \mathrm{Sym}^3 S^\vee\right)$, where S is the universal tautological subbundle on the Grassmannian, and s_T is the section induced by T using the morphism of vector bundles $\mathrm{Sym}^3 H^0(\mathbf{P}^5, \mathscr{O}(1)) \otimes \mathscr{O} \to \mathrm{Sym}^3 S^\vee$ on $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.

THEOREM 2.1 (Beauville-Donagi [4]). *Keeping the above notation,*

(i) *$F(X)$ is a 4-dimensional irreducible holomorphic symplectic projective variety, i.e. $F(X)$ is simply-connected and $H^{2,0}(F(X)) = \mathbf{C} \cdot \omega$ with ω a no-where degenerate holomorphic 2-form.*

(ii) *The correspondence*

$$p_* q^* : H^4(X, \mathbf{Z}) \to H^2(F(X), \mathbf{Z})$$

is an isomorphism of Hodge structures.

By definition, an automorphism ψ of $F(X)$ is called *polarized*, if it preserves the Plücker polarization: $\psi^* \mathscr{L} \simeq \mathscr{L}$. Now, we investigate what it means for an automorphism of $F(X)$ to be polarized.

LEMMA 2.2. *An automorphism ψ of $F(X)$ is polarized if and only if it is induced from an automorphism of the cubic fourfold X itself.*

Proof. See [9, Proposition 4]. □

Define $\mathrm{Aut}(X)$ to be the automorphism group of X, and $\mathrm{Aut}^{pol}(F(X), \mathscr{L})$ or simply $\mathrm{Aut}^{pol}(F(X))$ to be the group of polarized automorphisms of $F(X)$. Then Lemma 2.2 says that the image of the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(F(X))$ is exactly $\mathrm{Aut}^{pol}(F(X))$. This homomorphism of groups is clearly injective (since through each point of X, passes a 1-dimensional family of lines), hence we have

COROLLARY 2.3. *The natural morphism*

$$\mathrm{Aut}(X) \xrightarrow{\simeq} \mathrm{Aut}^{pol}(F(X)),$$

which sends an automorphism f of X to the induced (polarized) automorphism $\hat{f}$ of $F(X)$ is an isomorphism.

REMARK 2.4. This group is a *finite* group. Indeed, since $\mathrm{Pic}(X) = \mathbf{Z} \cdot \mathscr{O}_X(1)$, all its automorphisms come from linear automorphisms of $\mathbf{P}^5$, hence $\mathrm{Aut}(X)$ is a closed subgroup of PGL_6 thus of finite type. On the other hand, $H^0(F(X), T_{F(X)}) = H^{1,0}(F(X)) = 0$, which implies that the group considered is also discrete, therefore finite.

By Corollary 2.3, the classification of polarized symplectic automorphisms of $F(X)$ is equivalent to the classification of automorphisms of cubic fourfolds such that the induced action satisfies the symplectic condition. The first thing to do is to find a reformulation of this *symplectic condition* purely in terms of the action on the cubic fourfold.

3. The symplectic condition. The contents of this section are borrowed from [11, Section 1]. For the sake of completeness, we briefly reproduce it here. Let us keep the notation of the previous section. Suppose the cubic fourfold $X \subset \mathbf{P}^5$ is defined by a polynomial $T \in H^0(\mathbf{P}^5, \mathscr{O}(3)) = \mathrm{Sym}^3 V^\vee$. Let f be an automorphism of X. By Remark 2.4, f is the restriction of a finite-order linear automorphism of $\mathbf{P}^5$ preserving X, still denoted by f. Let $n \in \mathbf{N}^+$ be its order. We can assume without loss of generality that $f : \mathbf{P}^5 \to \mathbf{P}^5$ is given by:

$$f : [x_0 : x_1 : \cdots : x_5] \mapsto [\zeta^{e_0} x_0 : \zeta^{e_1} x_1 : \cdots : \zeta^{e_5} x_5], \tag{2}$$

where $\zeta = e^{\frac{2\pi\sqrt{-1}}{n}}$ is a primitive nth root of unity and $e_i \in \mathbf{Z}/n\mathbf{Z}$ for $i = 0, \ldots, 5$.

It is clear that X is preserved by f if and only if the defining equation T is contained in an eigenspace of $\mathrm{Sym}^3 V^\vee$. More precisely: let the coordinates $x_0, x_1, \ldots, x_5$ of $\mathbf{P}^5$ be a basis of $V^\vee$, then $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha}\in\Lambda}$ is a basis of $\mathrm{Sym}^3 V^\vee = H^0(\mathbf{P}^5, \mathscr{O}(3))$, where $\underline{x}^{\underline{\alpha}}$ denotes $x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_5^{\alpha_5}$. Define

$$\Lambda := \left\{\underline{\alpha} = (\alpha_0, \ldots, \alpha_5) \in \mathbf{N}^5 \mid \alpha_0 + \cdots + \alpha_5 = 3\right\}. \tag{3}$$

We write the eigenspace decomposition of $\mathrm{Sym}^3 V^\vee$:

$$\mathrm{Sym}^3 V^\vee = \bigoplus_{j\in\mathbf{Z}/n\mathbf{Z}} \left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}\right),$$

where for each $j \in \mathbf{Z}/n\mathbf{Z}$, we define the subset of Λ

$$\Lambda_j := \left\{\underline{\alpha} = (\alpha_0, \ldots, \alpha_5) \in \mathbf{N}^5 \mid {}_{e_0\alpha_0+\cdots+e_5\alpha_5=j \mod n}^{\alpha_0+\cdots+\alpha_5=3}\right\}. \tag{4}$$

and the eigenvalue of $\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}$ is thus ζ^j. Therefore, explicitly speaking, we have:

LEMMA 3.1. *A cubic fourfold X is preserved by the f in (2) if and only if there exists a $j \in \mathbf{Z}/n\mathbf{Z}$ such that its defining polynomial $T \in \bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}$.*

Next, we deal with the symplectic condition. Note that Theorem 2.1 (ii) says in particular that

$$p_*q^* : H^{3,1}(X) \xrightarrow{\simeq} H^{2,0}(F(X)),$$

is an isomorphism. If X is equipped with an action f as before, we denote by $\hat{f}$ the induced automorphism of $F(X)$. Since the construction of $F(X)$ as well as the correspondence p_*q^* are both functorial with respect to X, the condition that $\hat{f}$ is *symplectic, i.e.* $\hat{f}^*$ acts on $H^{2,0}(F(X))$ as identity, is equivalent to the condition that f^* acts as identity on $H^{3,1}(X)$. Working this out, we arrive at the congruence equation (5) in the following:

LEMMA 3.2 (Symplectic condition). *Let f be the linear automorphism in (2), and X be a cubic fourfold defined by equation T. Then the followings are equivalent:*

- *f preserves X and the induced action $\hat{f}$ on $F(X)$ is symplectic;*
- *There exists a $j \in \mathbf{Z}/n\mathbf{Z}$ satisfying the equation*

$$e_0 + e_1 + \cdots + e_5 = 2j \mod n, \tag{5}$$

such that the defining polynomial $T \in \bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}$, whereas in (4)

$$\Lambda_j := \left\{\underline{\alpha} = (\alpha_0, \ldots, \alpha_5) \in \mathbf{N}^5 \mid {}_{e_0\alpha_0+\cdots+e_5\alpha_5=j \mod n}^{\alpha_0+\cdots+\alpha_5=3}\right\}.$$

Proof. Firstly, the condition that f preserves X is given in Lemma 3.1. As is remarked before the lemma, $\hat{f}$ is symplectic if and only if f^* acts as identity on $H^{3,1}(X)$. On the other hand, by Griffiths' theory of Hodge structures of hypersurfaces (*cf.* [**20**, Chapter 18]), $H^{3,1}(X)$ is generated by the residue Res $\frac{\Omega}{T^2}$, where $\Omega := \sum_{i=0}^{5}(-1)^i x_i \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_5$ is a generator of $H^0(\mathbf{P}^5, K_{\mathbf{P}^5}(6))$. The map f being defined in (2), we find $f^*\Omega = \zeta^{e_0+\cdots+e_5}\Omega$ and $f^*(T) = \zeta^j T$. Hence the action of f^* on $H^{3,1}(X)$ is multiplication by $\zeta^{e_0+\cdots+e_5-2j}$, from which we obtain the equation (5). □

4. Classification. We now turn to the classification of polarized symplectic automorphisms of primary order of smooth cubic fourfolds. Retaining the notation of Section 3, we define the parameter space

$$\overline{B} := \mathbf{P}\left(\bigoplus_{\underline{\alpha}\in\Lambda_j} \mathbf{C}\cdot \underline{x}^{\underline{\alpha}}\right). \tag{6}$$

Let $B \subset \overline{B}$ be the open subset parameterizing the smooth ones.

In this paper, we are only interested in the *smooth* cubic fourfolds, that is the case when $B \neq \varnothing$, or equivalently, when a general member of $\overline{B}$ is smooth. The easy observation below (see Lemma 4.1) which makes the classification feasible is that this non-emptiness condition imposes strong combinatorial constraints on the defining equations.

LEMMA 4.1. *If a general member in $\overline{B}$ is smooth then for each $i \in \{0, 1, \ldots, 5\}$, there exists $i' \in \{0, 1, \ldots, 5\}$, such that ${x_i}^2 x_{i'} \in \overline{B}$.*

Proof. Suppose on the contrary that, without loss of generality, for $i = 0$, none of the monomials $x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0^2x_5$ are contained in $\overline{B}$, then every equation in this family can be written in the following form:

$$x_0 Q(x_1, \ldots, x_5) + C(x_1, \ldots, x_5),$$

where Q (*resp.* C) is a homogeneous polynomial of degree 2 (*resp.* 3). It is clear that $[1, 0, 0, 0, 0, 0]$ is always a singular point, which is a contradiction. □

Since a finite-order automorphism amounts to the action of a finite cyclic group, which is the product of some finite cyclic groups with order equal to a power of a prime number, we only have to classify automorphisms of *primary* order, that is $n = p^m$ for p a prime number and $m \in \mathbf{N}_+$. To get general results for any order from the classification of primary order case, see Remark 4.4. We thus assume $n = p^m$ in the sequel.

For the convenience of the reader, we summarize all the relevant equations:

$$\begin{cases} e_0 + e_1 + \cdots + e_5 = 2j \mod p^m; \\ \alpha_0 + \cdots + \alpha_5 = 3; \quad \alpha_i \in \mathbf{N}; \\ e_0\alpha_0 + \cdots + e_5\alpha_5 = j \mod p^m; \\ (*)\ \forall i, \exists i' \text{ such that } 2e_i + e_{i'} = j \mod p^m \end{cases} \tag{7}$$

where the last condition $(*)$ comes from Lemma 4.1.

We associate to each solution of (7) a diagram, *i.e.* a finite oriented graph, as follows:

(i) The vertex set is the quotient set of $\{0, \ldots, 5\}$ with respect to the equivalence relation defined by: $i_1 \sim i_2$ if and only if $e_{i_1} = e_{i_2} \mod p^m$.
(ii) For each pair (i, i') satisfying $2e_i + e_{i'} = j \mod p^m$, there is an arrow from i to i'.

The arrows in (*ii*) are well-defined because we have taken into account the equivalence relation in (*i*). It is also obvious that each vertex can have at most one arrow going out. Thanks to condition $(*)$ in (7), we know that each vertex has exactly one arrow going out.

REMARKS 4.2.

(i) If $p \neq 2$, it is easy to see that each vertex has at most one arrow coming in. Since the total going-out degree should coincide with the total coming-in degree, each vertex has exactly one arrow coming-in. As a result, the diagram is in fact a disjoint union of several cycles[4] in this case.
(ii) If $p \neq 3$, then 3 is always invertible modulo $n = p^m$. Let γ be an integer representing the inverse $3^{-1} \mod p^m$. By the change of variables $e_i \rightsquigarrow e_i - \gamma j$, which does not change the automorphism f, we reduce immediately to the case $j = 0$.

Before the detailed case-by-case analysis, let us point out that a cycle in the diagram would have some congruence implications:

LEMMA 4.3.

(i) There cannot be cycles of length 2.
(ii) If $p \neq 3$, there is at most one cycle of length 1.
(iii) If there is a cycle of length l =3, 4, 5 or 6, then p divides $\frac{(-2)^l-1}{3}$.

Proof.

(i) It is because $2e_i + e_{i'} = 2e_{i'} + e_i \mod p^m$ will imply $e_i = e_i' \mod p^m$, contradicting the definition of a cycle.
(ii) A cycle of length 1 means $3e_i = j \mod p^m$, and when $p \neq 3$, e_i is determined by j.
(iii) Without loss of generality, we can assume that the cycle is given by:

$$2e_0 + e_1 = 2e_1 + e_2 = \cdots = 2e_{l-2} + e_{l-1} = 2e_{l-1} + e_0 = j \mod p^m.$$

This system of congruence equations implies that

$$\left((-2)^l - 1\right) e_0 = \frac{(-2)^l - 1}{3} \cdot j \mod p^m. \tag{8}$$

If p does not divide $\frac{(-2)^l-1}{3}$, then by (8), we have $j = 3e_0 \mod p^m$, and hence $e_0 = e_1 = \cdots = e_{l-1}$, contradicting the definition of a cycle.

□

Next, we work out the classification case-by-case. The result is summarized in Theorem 1.1.

[4]Here, we use the terminology 'cycle' in the sense of graph theory: it means a loop in an oriented graph with no arrow repeated. The *length* of a cycle will refer to the number of arrows appearing in it.

Case 0. When $p \geq 13$.
If we have a cycle of length $l \geq 3$, since in Lemma 4.3, $\frac{(-2)^l-1}{3}$ could only be $-3, 5, -11, 21$, all of which are prime to p, this will lead to a contradiction. Therefore, we only have cycles of length 1. As $p \neq 3$, $3^{-1}j \mod p^m$ is well-defined, hence we have $e_0 = e_1 = \cdots = e_5$. As a result, f is the identity action of $\mathbf{P}^5$, which is Family 0 in Theorem 1.1.

Case I. When $p = 11$.
Let the order of the automorphism be $n = 11^m$. We can assume $j = 0$ without loss of generality by Remark 4.2(*ii*). As in the previous case, by Lemma 4.3, cycles of length 2, 3, 4 or 6 cannot occur. Thus, the only possible lengths of cycles are 1 and 5. If there is no cycle of length 5, then as before, since $p \neq 3$, all e_i's will be equal and f will be the identity. Let the 5-cycle be

$$2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_3 = 2e_3 + e_4 = 2e_4 + e_0 = j = 0 \mod 11^m.$$

Hence, $e_0 = e_0; e_1 = -2e_0; e_2 = 4e_0; e_3 = -8e_0; e_4 = 16e_0; e_5 = -11e_0 \mod 11^m$ where the last equality comes from the first equation in (7). Moreover as in (8) we have $33e_0 = 0 \mod 11^m$, *i.e.* e_0 is divisible by 11^{m-1}. Therefore, we must have $m = 1$ and $n = 11$:

$$e_0 = e_0; e_1 = -2e_0; e_2 = 4e_0; e_3 = 3e_0; e_4 = 5e_0; e_5 = 0 \mod 11.$$

As a result

$$f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^4, \zeta^3, \zeta^5, 1\right),$$

where $\zeta = e^{\frac{r}{11}\cdot 2\pi\sqrt{-1}}$ and $1 \leq r \leq 10$. Going back to (7), we easily work out all solutions for α_i's, and the corresponding family

$$\overline{B} = \mathbf{P}\left(\operatorname{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_3, x_3^2x_4, x_4^2x_0, x_5^3\rangle\right).$$

It is easy to see that this family is actually isotrivial and in particular the cubic fourfolds in this family are all isomorphic. Hence in order to verify the smoothness of its members, it suffices to verify one, say, $x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 + x_5^3$ by the standard Jacobian criterion, which is quite easy. This is Family I in Theorem 1.1. We would like to mention that this example was discovered in [**16**].

Case II. When $p = 7$.
Let the order be $n = 7^m$. As before, we can assume $j = 0$ by Remark 4.2(*ii*). By Lemma 4.3, cycles of length 2, 3, 4 or 5 cannot occur. Thus, the only possible lengths of cycles are 1 and 6; and except the trivial Family 0, there must be a 6-cycle:

$$2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_3 = 2e_3 + e_4 = 2e_4 + e_5 = 2e_5 + e_0 = 0 \mod 7^m.$$

Hence, $e_0 = e_0; e_1 = -2e_0; e_2 = 4e_0; e_3 = -8e_0; e_4 = 16e_0; e_5 = -32e_0 \mod 7$ and moreover as in (8), $63e_0 = 0 \mod 7^m$ *i.e.* e_0 is divisible by 7^{m-1}. Hence, we must have $m = 1$ and $n = 7$. Therefore

$$e_0 = e_0; e_1 = -2e_0; e_2 = -3e_0; e_3 = -e_0; e_4 = 2e_0; e_5 = 3e_0 \mod 7.$$

One verifies easily that it satisfies $e_0 + \cdots + e_5 = 0$. As a result

$$f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^{-3}, \zeta^{-1}, \zeta^{2}, \zeta^{3}\right),$$

where $\zeta = e^{\frac{r}{7}\cdot 2\pi\sqrt{-1}}$ and $1 \leq r \leq 6$. Going back to (7), we easily work out all solutions for α_i's and the corresponding family

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_3, x_3^2x_4, x_4^2x_5, x_5^2x_0, x_0x_2x_4, x_1x_3x_5\rangle\right).$$

To show that a general member of this family is smooth, we only need to remark that $x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_0$ is smooth. This accomplishes Family II in Theorem 1.1.

Case III. When $p = 5$.
By the remark on orders, we have $m = 1$ or 2. And again, by Remark 4.2(*ii*), we can assume $j = 0$. As before, by Lemma 4.3, cycles of length 2,3,5 or 6 cannot occur. Thus the only possible lengths of cycles are 1 and 4; and except the trivial Family 0, there must be a 4-cycle:

$$2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_3 = 2e_3 + e_0 = 0 \mod 5^m.$$

As in (8), we get $15e_0 = 0 \mod 5^m$, *i.e.* e_0 is divisible by 5^{m-1}. Hence

$$e_0 = e_0; e_1 = -2e_0; e_2 = -e_0; e_3 = 2e_0 \mod 5^m.$$

Since 2-cycle does not exist, for $i = 4, 5$, either e_i takes value in $e_0, \ldots, e_3$, or it is a 1-cycle: $3e_i = 0$, *i.e.* $e_i = 0$. Therefore, in any case, we must have $m = 1$ and $n = 5$.

We write $e_4 = ae_0$ and $e_5 = be_0$ for $a, b \in \mathbf{Z}/5\mathbf{Z}$. Taking into account the first equation in (7), we obtain

$$a + b = 0 \mod 5.$$

As a result,

$$f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^{-1}, \zeta^{2}, \zeta^{a}, \zeta^{-a}\right)$$

where $\zeta = e^{\frac{r}{5}\cdot 2\pi\sqrt{-1}}$ for $1 \leq r \leq 4$ and $a \in \mathbf{Z}/5\mathbf{Z}$. Going back to (7), we work out the solutions for α_i's depending on the value of a:

Subcase III (i). When $a \neq 0$.
We treat the case $a = 1$ first, that is, $p = 5, m = 1, (e_0, \ldots, e_5) = (1, -2, -1, 2, 1, -1)$, and

$$f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^{-1}, \zeta^{2}, \zeta, \zeta^{-1}\right),$$

where $\zeta = e^{\frac{r}{5}\cdot 2\pi\sqrt{-1}}$ for $1 \leq r \leq 4$. Solving α_i's from equation (7), we get the corresponding family

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_3, x_3^2x_0, x_4^2x_1, x_3^2x_4, x_4^2x_3, x_1^2x_5, x_5^2x_3, x_0x_1x_4, x_2x_3x_5\rangle\right).$$

However, there is no smooth cubic fourfolds in this family: in fact each member would have two singular points in the line $(x_0 = x_1 = x_3 = x_4 = 0)$.

When $a = 2$, we have $p = 5, m = 1, (e_0, \dots, e_5) = (1, -2, -1, 2, 2, -2)$, and $\zeta = e^{\frac{r}{5}\cdot 2\pi\sqrt{-1}}$ for $1 \le r \le 4$, therefore $f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^{-1}, \zeta^2, \zeta^2, \zeta^{-2}\right)$.
By the transformation $\zeta \mapsto \zeta^2$, this f is exactly the one when $a = 1$ which is already discussed.
When $a = 3$, by the symmetry, it is the same case as $a = 2$. Similarly, the case $a = 4$ is the same as the case $a = 1$.

Subcase III (ii). When $a = 0$.
$p = 5, m = 1, (e_0, \dots, e_5) = (1, -2, -1, 2, 0, 0)$, and

$$f = \operatorname{diag}\left(\zeta, \zeta^{-2}, \zeta^{-1}, \zeta^2, 1, 1\right),$$

where $\zeta = e^{\frac{r}{5}\cdot 2\pi\sqrt{-1}}$ for $1 \le r \le 4$. Solving α_i's from the equation (7), we obtain the corresponding family

$$\overline{B} = \mathbf{P}\left(\operatorname{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_3, x_3^2x_0, x_4^2x_5, x_5^2x_4, x_5^3, x_4^3, x_0x_2x_4, x_0x_2x_5, x_1x_3x_4, x_1x_3x_5\rangle\right).$$

Moreover, a general cubic fourfold in this family is smooth. Indeed, we give a particular smooth member: $x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_0 + x_4^3 + x_5^3$. This is Family III in Theorem 1.1.

Case IV. When $p = 3$.
Still by Lemma 4.3, we know that cycles of length 2, 4 or 5 cannot occur. Thus, the only possible lengths of cycles are 1, 3 and 6. We first claim that a 6-cycle cannot exist. Suppose on the contrary that the diagram is a 6-cycle:

$$2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_3 = 2e_3 + e_4 = 2e_4 + e_5 = 2e_5 + e_0 = j \mod 3^m,$$

then we have as in (8) that $63e_0 = 21j \mod 3^m$. Thus there exists $r \in \mathbf{Z}/3\mathbf{Z}$, such that $j = 3e_0 + r\cdot 3^{m-1} \mod 3^m$, and

$$e_0 = e_0; e_1 = e_0 + r\cdot 3^{m-1}; e_2 = e_0 - r\cdot 3^{m-1}; e_3 = e_0; e_4 = e_0 + r\cdot 3^{m-1}; e_5 = e_0 - r\cdot 3^{m-1} \mod 3^m.$$

This contradicts the assumption that e_i's are distinct. Therefore, there are only 1-cycles and 3-cycles. A 1-cycle means $3e_i = j \mod 3^m$. On the other hand, a 3-cycle $2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_0 = j \mod 3^m$ would imply $9e_0 = 3j$. In particular, $9e_0 = 9e_1 = \dots = 9e_5 = 3j \mod 3^m$. Without loss of generality, we can demand $e_0 = 0$. As a result, f has the form $f = \operatorname{diag}(1, \zeta^{a_1}, \dots, \zeta^{a_5})$ where $\zeta = e^{\frac{2\pi\sqrt{-1}}{9}}$. In particular, f is of order 3 or 9.

Subcases IV (i). If f is of order 3.
Let $\omega := e^{\frac{2\pi\sqrt{-1}}{3}}$. Then, up to isomorphism, f is one of the following automorphisms:
- $\operatorname{diag}(1, 1, 1, 1, 1, \omega)$: this case does not satisfy condition $(*)$.
- $\operatorname{diag}(1, 1, 1, 1, 1, \omega^2)$: this case does not satisfy condition $(*)$.
- $\operatorname{diag}(1, 1, 1, 1, \omega, \omega^2)$: we find Family IV-(1) in Theorem 1.1. We remark that its general member is indeed smooth because in particular the Fermat cubic fourfold (which is smooth) is contained in this family.
- $\operatorname{diag}(1, 1, 1, 1, \omega, \omega)$: this case does not satisfy condition $(*)$.

- diag$(1, 1, 1, 1, \omega^2, \omega^2)$: this case does not satisfy condition $(*)$.
- diag$(1, 1, 1, \omega, \omega, \omega^2)$: Here we find $\overline{B}$ has a basis:

  $$x_5 \cdot \text{ degree 2 monomials on } x_0, x_1 \text{ and } x_2; x_4x_5^2, x_3x_5^2, x_0x_3x_4, x_1x_3x_4, x_2x_3x_4, x_0x_3^2, x_1x_3^2, x_2x_3^2, x_0x_4^2, x_1x_4^2, x_2x_4^2,$$

  However, any cubic fourfold in this family is singular along a conic curve in the projective plane $(x_3 = x_4 = x_5 = 0)$.
- diag$(1, 1, 1, \omega^2, \omega^2, \omega)$: this is as in the previous case, with ω replaced by ω^2.
- diag$(1, 1, 1, \omega, \omega, \omega)$: By solving (7), we find the following basis for $\overline{B}$:

  $$\text{degree 3 monomials on } x_0, x_1 \text{ and } x_2; \text{degree 3 monomials on } x_3, x_4 \text{ and } x_5.$$

  As the Fermat cubic fourfold is in this family, the general member is also smooth. This is Family IV-(2) in Theorem 1.1. We remark that this family was discovered first in [**17**] and also was studied in [**13**].
- diag$(1, 1, \omega, \omega, \omega^2, \omega^2)$: The basis of $\overline{B}$ is

  $$\text{degree 3 monomials on } x_0 \text{ and } x_1; \text{degree 3 monomials on } x_2 \text{ and } x_3; \text{degree 3 monomials on } x_4 \text{ and } x_5;$$
  $$x_0x_2x_4, x_0x_2x_5, x_0x_3x_4, x_0x_3x_5, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5.$$

  Because $\overline{B}$ contains the Fermat cubic fourfold, its general member is smooth. This is Family IV-(3) in Theorem 1.1.

Subcase IV (ii). If the order $n = 9$ and the diagram consists of two 3-cycles (thus e_i's are distinct, so $m \geq 2$):

$$\begin{cases} 2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_0 = j \\ 2e_3 + e_4 = 2e_4 + e_5 = 2e_5 + e_3 = j \end{cases} \mod 3^m,$$

From which we have $3j = 9e_0 = 9e_3 \mod 3^m$. Hence, there exists $t = \pm 1$ such that $j = 3e_0 + t \cdot 3^{m-1}$ and

$$e_0 = e_0; e_1 = e_0 + t \cdot 3^{m-1}; e_2 = e_0 - t \cdot 3^{m-1}; e_3 = e_0 + r \cdot 3^{m-2};$$
$$e_4 = e_0 + t \cdot 3^{m-1} - 2r \cdot 3^{m-2}; e_5 = e_0 - t \cdot 3^{m-1} + 4r \cdot 3^{m-2} \mod 3^m.$$

where $r \in \mathbf{Z}/9\mathbf{Z}$. Note that $r \neq 0, 3, 6 \mod 9$, since otherwise the e_i's cannot be distinct. By the first equation in (7)

$$t = -r \mod 3.$$

Putting this back into the previous system of equations, we obtain:
$p = 3, m = 2, n = 9, (e_0, \ldots, e_5) = (0, -3, 3, 1, 4, -2), j = -3 \mod 9$, and

$$f = \mathrm{diag}\left(1, \zeta^{-3}, \zeta^3, \zeta, \zeta^4, \zeta^{-2}\right),$$

where $\zeta = e^{\frac{r}{9} \cdot 2\pi\sqrt{-1}}$ for $r \in \{1, 2, 4, 5, 7, 8\}$. Solving α_i's from equation (7), we have the corresponding family

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_0, x_3^2x_4, x_4^2x_5, x_5^2x_3\rangle\right).$$

This family is clearly isotrivial. Since the cubic fourfold $x_0^2x_1 + x_1^2x_2 + x_2^2x_0 + x_3^2x_4 + x_4^2x_5 + x_5^2x_3$ is smooth by Jacobian criterion, so is every cubic fourfold in this family. This is Family IV-(4) in Theorem 1.1.

Subcase IV (iii). If the order $n = 9$ and the diagram contains only one 3-cycle:

$$2e_0 + e_1 = 2e_1 + e_2 = 2e_2 + e_0 = j \mod 9.$$

As before, we can assume $e_0 = 0$, then $e_1 = j$, $e_2 = -j$ and $3j = 0 \mod 9$. In particular, $3|j$. Since $j \neq 0 \mod 9$ (otherwise $e_0 = e_1 = e_2$ is a contradiction), $j = \pm 3$. For $i = 3, 4, 5$, e_i either takes value in $\{e_0, e_1, e_2\}$, or $3e_i = j$.
If $j = 3$, then f has the form

$$f = \mathrm{diag}(1, \zeta^3, \zeta^{-3}, \zeta^a, \zeta^b, \zeta^c),$$

where $a, b, c \in \{0, 3, 6, 1, 4, 7\}$. By the first equation in (7),

$$a + b + c = 6 \mod 9.$$

Thus, either $a, b, c \in \{0, 3, 6\}$, or $a, b, c \in \{1, 4, 7\}$. While the former will make f of order 3, which has been treated in Subcases IV(i). Therefore $a, b, c \in \{1, 4, 7\}$ and $a + b + c = 6$. There are only three possibilities (up to permutations of a, b, c): $(a, b, c) = (1, 1, 4)$ or $(4, 4, 7)$ or $(7, 7, 1)$. However, these three correspond to the following automorphism

$$f = \mathrm{diag}\left(1, \zeta^3, \zeta^{-3}, \zeta, \zeta, \zeta^4\right),$$

Back to (7), we solve the corresponding α_i's to get the following basis for $\overline{B}$:

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle x_0^2x_1, x_1^2x_2, x_2^2x_0, x_3^2x_4, x_4^2x_3, x_3^3, x_4^3, x_5^3\rangle\right).$$

This family is clearly isotrivial. As we have a smooth member $x_0^2x_1 + x_1^2x_2 + x_2^2x_0 + x_3^3 + x_4^3 + x_5^3$ in this family, every cubic fourfold in it is also smooth. This is Family IV-(5) in Theorem 1.1.
If $j = -3$, it reduces to the $j = 3$ case by replacing ζ by ζ^{-1}, thus already included in Family IV-(5) of the theorem.

Subcase IV (iv). If $n = 9$ and the diagram has only 1-cycles, *i.e.* for any $0 \leq i \leq 5$,

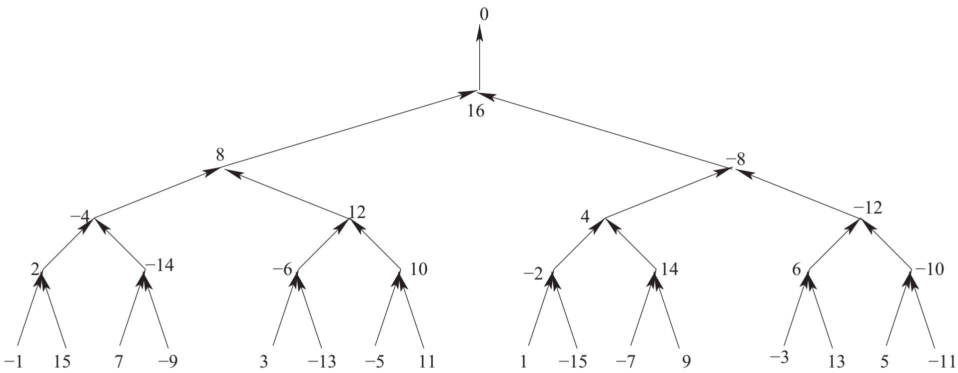
$$3e_i = j \mod 9.$$

In particular, $3|j$. First of all, $j \neq 0$, otherwise, f is of order 3, which is treated in Subcases IV(i).
If $j = 3$. Then $e_i \in \{1, 4, 7\}$ for any i. Taking into account the first equation of (7), we find all the solutions for $(e_0, \ldots, e_5)$, up to permutations:

$$\begin{aligned}(e_0, \ldots, e_5) = {} & (1, 1, 1, 4, 4, 4), (1, 1, 1, 7, 7, 7), (4, 4, 4, 7, 7, 7), (1, 1, 1, 1, 4, 7),\\ & (4, 4, 4, 4, 1, 7), (7, 7, 7, 7, 1, 4), (1, 1, 4, 4, 7, 7)\end{aligned}$$

where the automorphisms in the first line are equal to $\mathrm{diag}(1, 1, 1, \omega, \omega, \omega)$, which has been done in Family IV-(2); the automorphisms in the second line are equal to $\mathrm{diag}(1, 1, 1, 1, \omega, \omega^2)$, which has been done in Family IV-(1); the last automorphism is equal to $\mathrm{diag}(1, 1, \omega, \omega, \omega^2, \omega^2)$, which has been done in Family IV-(3) in Theorem 1.1.

Case V. When $p = 2$.
By Lemma 4.3, we find that the associated diagram has only 1-cycles. The new phenomenon is that the coming-in degree in this case is not necessarily 1. As before, we reduce to the case $j = 0$ by Remark 4.2(*ii*). Then we claim that the order of f divides 32. Indeed, for any 1-cycle $3e_i = 0 \mod 2^m$ implies $e_i = 0 \mod 2^m$. Hence a vertex pointing to a 1-cycle is divisible by 2^{m-1}, and a vertex pointing to a vertex pointing to a 1-cycle is divisible by 2^{m-2}, *etc.* And every vertex arrives at the 1-cycle vertex after at most 5 steps.



Let us put $\mathbf{Z}/32\mathbf{Z}$ into the above complete binary tree. Then our associated diagram clearly is a sub-diagram of this tree, satisfying two properties:

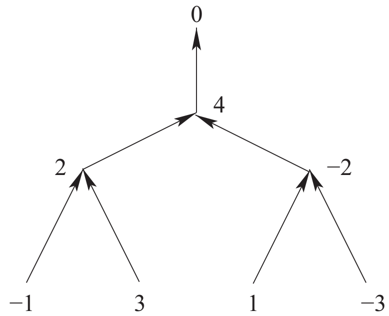
- If a vertex belongs to the diagram then so do its ancestors;
- The sum of vertices (multiplicities counted) is zero modulo 32.

It is immediate that the leaves (vertices on the bottom sixth level) cannot appear in the diagram: since by the parity of their sum, if there are leaves, there are at least two. But we already have five ancestors to include, while we have only six places in total. Next, we remark that the vertices in the fifth level cannot belong to the diagram either: since the sum is divisible by 4, there are at least two vertices from the fifth level if there is any, and they should have the same father (otherwise we need to include at least five ancestors, and it will be out of place). Therefore we only have four possibilities, and it is straightforward to check that none of them has sum zero as demanded.

As a result, we have a further reduction: since only the first four levels can appear, the order of f always divides 8. We can assume now $n = 8$ and $e_i \in \mathbf{Z}/8\mathbf{Z}$. Similarly, we put $\mathbf{Z}/8\mathbf{Z}$ into the following complete binary tree:

Then our diagram is a sub-diagram of this tree which is as before 'ancestor-closed' and its multiplicities-counted sum is zero modulo 8. We easily work out all the possibilities as follows.

- $(e_0, \dots, e_5) = (0, 0, 0, 0, 4, 4)$ or $(0, 0, 4, 4, 4, 4)$. In this case, f is the involution $\mathrm{diag}(1, 1, 1, 1, -1, -1)$ and we reduce to $n = p = 2$ with $(e_0, \dots, e_5) = (0, 0, 0, 0, 1, 1)$. The equation for α_i's becomes $\alpha_4 + \alpha_5 = 0 \mod 2$. This is Family V-(1) in Theorem 1.1, whose generic smoothness is easy to verify: $x_4^2x_0 + x_5^2x_1 + x_4x_5x_2 + x_0^3 + x_1^3 + x_2^3 + x_3^3$ is smooth. This family of cubic fourfolds has been studied in [**8**].



- $(e_0, \dots, e_5) = (0, 0, 0, 4, 2, 2)$ or $(0, 4, 4, 4, -2, -2)$. They both correspond to the following automorphism of order $n = 4$:

$$f = \mathrm{diag}(1, 1, 1, -1, \sqrt{-1}, \sqrt{-1}).$$

We thus reduce to $n = 4$, and $(e_0, \dots, e_5) = (0, 0, 0, 2, 1, 1)$. The equation (7) for α_i's becomes $2\alpha_3 + \alpha_4 + \alpha_5 = 0 \mod 4$. We easily obtain a basis for $\overline{B}$:

$$\text{degree 3 monomials on } x_0, x_1, x_2; x_0x_3^2, x_1x_3^2, x_2x_3^2, x_3x_4x_5, x_3x_4^2, x_3x_5^2.$$

Unfortunately, any cubic fourfold in this family is singular on two points on the line defined by $(x_0 = x_1 = x_2 = x_3 = 0)$.
- $(e_0, \dots, e_5) = (0, 0, 0, 4, -2, -2)$ or $(0, 4, 4, 4, 2, 2)$. They both correspond to the same f, which becomes the automorphism of the previous case if we replace $\sqrt{-1}$ by $-\sqrt{-1}$.
- $(e_0, \dots, e_5) = (0, 4, 2, 2, 2, -2)$ or $(0, 4, 2, -2, -2, -2)$. They both correspond to the following automorphism of order $n = 4$:

$$f = \mathrm{diag}(1, -1, \sqrt{-1}, \sqrt{-1}, \sqrt{-1}, -\sqrt{-1}).$$

We thus reduce to $n = 4$ and $(e_0, \dots, e_5) = (0, 2, 1, 1, 1, -1)$, the equation for α_i's becomes $2\alpha_3 + \alpha_4 - \alpha_5 = 1 \mod 4$. The basis for $\overline{B}$ is:

$$x_4 \cdot \text{degree 2 monomials on } x_0, x_1, x_2; x_3^2x_4, x_0x_3x_5, x_1x_3x_5, x_2x_3x_5, x_5^3, x_4^2x_5.$$

Each cubic fourfold in this family is singular along a conic curve in the plane defined by $(x_3 = x_4 = x_5 = 0)$.
- $(e_0, \dots, e_5) = (0, 0, 4, 4, 2, -2)$. It is the following automorphism of order $n = 4$:

$$f = \mathrm{diag}(1, 1, -1, -1, \sqrt{-1}, -\sqrt{-1}).$$

We then reduce to $n = 4$ and $(e_0, \dots, e_5) = (0, 0, 2, 2, 1, -1)$. The equation for α_i's is $2\alpha_2 + 2\alpha_3 + \alpha_4 - \alpha_5 = 0 \mod 4$. It corresponds Family V-(2) in Theorem 1.1. It is easy to find a smooth member, for example, $x_0^3 + x_1^3 + x_2^2x_0 + x_3^2x_1 + x_4^2x_2 + x_5^2x_3$.

- $(e_0, \dots, e_5) = (0, 4, 4, 2, -1, -1)$ or $(0, 4, 4, 2, 3, 3)$ or $(0, 4, 4, -2, 1, 1)$ or $(0, 4, 4, -2, -3, -3)$. Although they are *different* automorphisms of order $n = 8$, they correspond to four possible choices of the primitive eighth root of unity ζ in the automorphism:

$$f = \mathrm{diag}(1, -1, -1, \zeta^{-2}, \zeta, \zeta),$$

where $\zeta = e^{\frac{r}{8}\cdot 2\pi\sqrt{-1}}$ for $r = \pm 1, \pm 3$. For each choice of r, the equation (7) becomes

$$4\alpha_1 + 4\alpha_2 - 2\alpha_3 + \alpha_4 + \alpha_5 = 0 \mod 8.$$

The solutions form a basis for $\overline{B}$:

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle x_0^3, x_0x_1^2, x_0x_2^2, x_0x_1x_2, x_2x_3^2, x_1x_3^2, x_3x_4^2, x_3x_5^2, x_3x_4x_5\rangle\right).$$

However, any cubic fourfold in this family is singular at two points on the line defined by $(x_0 = x_1 = x_2 = x_3 = 0)$.

- $(e_0, \dots, e_5) = (0, 0, 4, 2, -1, 3)$ or $(0, 0, 4, -2, 1, -3)$. They are *different* automorphisms of order $n = 8$. In fact, the four possible choices of the primitive eighth root of unity ζ collapse into two cases. The automorphism:

$$f = \mathrm{diag}(1, 1, -1, \zeta^2, \zeta^{-1}, \zeta^3),$$

where $\zeta = e^{\frac{r}{8}\cdot 2\pi\sqrt{-1}}$ for $r = \pm 1$ (here $r = \pm 3$ will give the same two automorphisms). In this case, (7) gives

$$4\alpha_2 + 2\alpha_3 - \alpha_4 + 3\alpha_5 = 0 \mod 8.$$

We easily resolve it to obtain

$$\overline{B} = \mathbf{P}\left(\mathrm{Span}\langle \text{degree 3 monomials on } x_0 \text{ and } x_1, x_0x_2^2, x_1x_2^2, x_2x_3^2, x_3x_4^2, x_3x_5^2\rangle\right).$$

But each member in this family is singular at least at two points of the line defined by $(x_0 = x_1 = x_2 = x_3 = 0)$.

- $(e_0, \dots, e_5) = (0, 4, 2, -2, 1, 3)$ or $(0, 4, 2, -2, -1, -3)$. As in the previous case, although they are *different* automorphisms of order $n = 8$, each corresponds to two possible choices of the primitive eighth root of unity ζ. The automorphism is

$$f = \mathrm{diag}(1, -1, \zeta^2, \zeta^{-2}, \zeta, \zeta^3),$$

where $\zeta = e^{\frac{r}{8}\cdot 2\pi\sqrt{-1}}$ for $r = \pm 1$ (here $r = \pm 3$ will give the same two automorphisms). In this case, we get the Family V-(3) in Theorem 1.1. To verify the smoothness of the general member we claim that the cubic fourfold defined by the following equation is smooth:

$$T := x_0^3 + x_0x_1^2 + x_1x_2^2 + x_1x_3^2 + x_0x_2x_3 + x_3x_4^2 + x_2x_5^2.$$

Indeed, it is straightforward to check that the system of equations $\frac{\partial T}{\partial x_0} = \frac{\partial T}{\partial x_1} = \cdots = \frac{\partial T}{\partial x_5} = 0$ has no non-zero solutions.

The classification is now complete and the result is summarized in Theorem 1.1.

REMARKS 4.4. We have some explanations to make concerning the usage of our list.

- In the fifth column of the table in Theorem 1.1, we give a basis for the *compactified* parameter space $\overline{B}$, which contains of course singular members. To pick out the smooth ones (*i.e.* to determine the *non-empty* open dense subset B), we have to apply the usual method of Jacobian criterion.
- Strictly speaking, the moduli space of cubic fourfolds is the geometric quotient

$$M := \mathbf{P}\left(H^0(\mathbf{P}^5, \mathscr{O}(3))\right) // \mathrm{PGL}_6,$$

and each $\overline{B}$ we have given in the theorem is a sub-projective space of $\mathbf{P}\left(H^0(\mathbf{P}^5, \mathscr{O}(3))\right)$, whose image in M is (a component of) the 'moduli space' of cubic fourfolds admitting a 'symplectic' automorphism of certain primary order.
- For an automorphism f of a given order n, say $n = 2^{r_2}3^{r_3}5^{r_5}7^{r_7}11^{r_{11}}$, where $r_2 = 0, 1, 2$ or 3; $r_3 = 0{,}1$ or 2 and $r_5, r_7, r_{11} = 0$ or 1. Then $f = f_2f_3f_5f_7f_{11}$ where f_p is an automorphisms of order p^{r_p} commuting with each other. Thus, they can be diagonalised simultaneously. Therefore to classify automorphisms of a given order, it suffices to intersect the corresponding $\overline{B}$'s in the list, after *independent* scaling and permutation of coordinates, inside the complete linear system $\mathbf{P}\left(H^0(\mathbf{P}^5, \mathscr{O}(3))\right)$. Of course it may end up with an empty family or a family consisting of only singular members.

EXAMPLE 4.5. We investigate the example of Fermat cubic fourfold $X = (x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0)$. We know that (*cf.* [**19**, **14**]) its automorphism group is $\mathrm{Aut}(X) = (\mathbf{Z}/3\mathbf{Z})^5 \rtimes \mathfrak{S}_6$, which is generated by multiplications by third roots of unity on coordinates and permutations of coordinates. Using Griffiths' residue description of Hodge structure as in the proof of Lemma 3.2, we find that

$$\mathrm{Aut}^{\mathrm{pol,symp}}(F(X)) = \left\{f \in \mathrm{Aut}(X) \mid f^*|_{H^{3,1}(X)} = \mathrm{id}\right\} = (\mathbf{Z}/3\mathbf{Z})^4 \rtimes \mathfrak{A}_6,$$

where each element has the form:

$$[x_0, x_1, x_2, x_3, x_4, x_5] \mapsto [x_{\sigma(0)}, \omega^{i_1}x_{\sigma(1)}, \ldots, \omega^{i_5}x_{\sigma(5)}],$$

where $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$, $i_1, \ldots, i_5 = 0, 1$ or 2 with sum $i_1 + \cdots + i_5$ divisible by 3, and $\sigma \in \mathfrak{A}_6$ is a permutation of $\{0, 1, \ldots, 5\}$ with even sign.

Then X is

- not in Family I, II, IV-(4), IV-(5) or V-(3) simply because X does not admit automorphisms of order 11, 7, 9 or 8;
- in Family III, since $[x_0, x_1, x_2, x_3, x_4, x_5] \mapsto [x_1, x_2, x_3, x_4, x_0, x_5]$ is an order 5 automorphism which induces a symplectic automorphism on its Fano variety of lines. The eigenvalues of the corresponding permutation matrix are $1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4$, thus it is exactly the automorphism in the list (up to a linear automorphism of $\mathbf{P}^5$).
- in IV-(1), IV-(2), IV-(3) obviously;
- in V-(1) and V-(2), because $[x_0, x_1, x_2, x_3, x_4, x_5] \mapsto [x_1, x_2, x_3, x_0, x_5, x_4]$ is an order 4 automorphism inducing a symplectic automorphism of its Fano variety. The eigenvalues are $1, 1, -1, -1, \sqrt{-1}, -\sqrt{-1}$, therefore the automorphism is the one given in V-(2) (up to a linear automorphism of $\mathbf{P}^5$).

5. Fixed loci. We calculate the fixed loci of a generic member for each example in the list of Theorem 1.1. Firstly, we make several general remarks concerning the fixed loci:

REMARK 5.1. For a smooth variety, the fixed locus of any automorphism of finite order is a (not necessarily connected) smooth subvariety. For a proof, *cf.* [**10**, Lemma 4.1]. If furthermore the variety is symplectic and the finite-order automorphism preserves the symplectic form, then the components of the fixed locus are *symplectic* subvarieties. Indeed, for a given fixed point, the automorphism acts also on the tangent space at this fixed point, preserving the symplectic form, where the tangent space of the component of the fixed locus passing through this point is exactly the fixed subspace. However, since the fixed subspace is orthogonal to the other eigenspaces with respect to the symplectic form, it must be a symplectic subspace. As a consequence, the fixed locus is a (smooth) symplectic subvariety.

According to the above remark, in the case of this paper, the fixed loci must be disjoint unions of (isolated) points, K3 surfaces and abelian surfaces, and we will see that all three types do occur in the list in Theorem 1.1.

We now describe the fixed loci in our classification. For a cubic fourfold X with an action f, we denote $\hat{f}$ the induced action on $F(X)$. Then the fixed points of $\hat{f}$ in $F(X)$ are the lines contained in X which are preserved by f. Since any automorphism of $\mathbf{P}^1$ admits two (not necessarily distinct) fixed points, it suffices to check for each line joining two fixed points of f in X whether it is contained in X. In the following, we choose to give an outline only for new or interesting families in our list which have not been treated before, while referring to the literature for the complete result.

Let $P_0 := [1, 0, 0, 0, 0, 0]$, $P_1 := [0, 1, 0, 0, 0, 0], \ldots, P_5 := [0, 0, 0, 0, 0, 1]$. Denote by $\overline{PQ}$ the line joining two points P and Q. We have explicit descriptions of the fixed loci:

Family I: the five fixed points correspond to $\overline{P_0P_2}$,$\overline{P_0P_3}$, $\overline{P_1P_3}$, $\overline{P_1P_4}$, $\overline{P_2P_4}$.

Family II: the nine fixed points are given by the following nine lines: $\overline{P_0P_2}$, $\overline{P_0P_3}$, $\overline{P_0P_4}$, $\overline{P_1P_3}$, $\overline{P_1P_4}$, $\overline{P_1P_5}$, $\overline{P_2P_4}$, $\overline{P_2P_5}$, $\overline{P_3P_5}$.

Family III: let $C(x_4, x_5) + R(x_0, \ldots, x_5)$ be the defining equation of a cubic fourfold in this family, where C is a homogeneous polynomial of degree 3, and R is a polynomial with the degrees of x_4 and x_5 at most 1. Then the 14 fixed points correspond to $\overline{P_0P_2}$, $\overline{P_1P_3}$ and $\overline{P_i, [0, 0, 0, 0, x_4, x_5]}$ for $0 \le i \le 3$ and $[x_4, x_5]$ being solutions of C.

Family IV-(1): let $C(x_0, \ldots, x_3) + R$ be the defining equation of a cubic fourfold in this family, where C is of degree 3 and each term of R contains x_4 or x_5. The fixed locus corresponds to the 27 lines contained in the cubic surface defined by C.

Family IV-(2): let $C_1(x_0, x_1, x_2) + C_2(x_3, x_4, x_5)$ be the defining equation of a cubic fourfold in this family, where C_1, C_2 are of degree 3. Let E_1, E_2 be the elliptic curve defined by C_1 and C_2 respectively, then the fixed locus is isomorphic to the abelian surface $E_1 \times E_2$. See [**17**], [**13**].

Family IV-(3): let $C_1(x_0, x_1) + C_2(x_2, x_3) + C_3(x_4, x_5) + R$ be the defining equation of a cubic fourfold in this family, where C_i are of degree 3 while each term of R is square-free. Then the fixed locus corresponds to the 27 lines $\overline{Q_{ik}Q_{jl}}$ for $0 \le i < j \le 3$ and $k, l = 1, 2, 3$, where Q_{i1}, Q_{i2}, Q_{i3} are the three points satisfying the equation C_i.

Family IV-(4): the nine fixed points correspond to the following nine lines: $\overline{P_0P_3}$, $\overline{P_0P_4}$, $\overline{P_0P_5}$, $\overline{P_1P_3}$, $\overline{P_1P_4}$, $\overline{P_1P_5}$, $\overline{P_2P_3}$, $\overline{P_2P_4}$, $\overline{P_2P_5}$.

Family IV-(5): let $C(x_3, x_4) + a_0x_0^2x_1 + a_1x_1^2x_2 + a_2x_2^2x_0 + a_5x_5^3$ be the defining equation of a cubic fourfold in this family, where C is of degree 3. Let Q_1, Q_2, Q_3 be the three points on the line $\overline{P_3P_4}$ satisfying C. Then the fixed locus in $F(X)$ corresponds to the 9 lines: $\overline{P_iQ_j}$ for $i = 0, 1, 2$ and $j = 1, 2, 3$.

Family V-(1): the fixed locus is a K3 surface, see [**8**].

Family V-(2): the fixed point set of f, viewed as an automorphism of $\mathbf{P}^5$, consists of the disjoint union of $\overline{P_0P_1}$, $\overline{P_2P_3}$ and P_4, P_5. The lines $\overline{P_2P_3}$ and $\overline{P_4P_5}$ are contained in X; there are three points $Q_1, Q_2, Q_3 \in \overline{P_0P_1}$ such that $\overline{Q_iP_j}$ is contained in X, for $i = 1, 2, 3$ and $j = 4, 5$; there are two points $Q_4, Q_5 \in \overline{P_2P_3}$ such that $\overline{Q_4P_4}$ and $\overline{Q_5P_5}$ are contained in X; finally for each $Q_i \in \overline{P_0P_1}$, $1 \leq i \leq 3$, there exist two points on $\overline{P_2P_3}$ such that the joining line is contained in X. Thus, $\hat{f}$ has altogether $2 + 3 \times 2 + 2 + 3 \times 2 = 16$ isolated fixed points.
Family V-(3): the fixed points are given by the six lines: $\overline{P_1P_4}$, $\overline{P_1P_5}$, $\overline{P_2P_3}$, $\overline{P_2P_4}$, $\overline{P_3P_5}$, $\overline{P_4P_5}$.

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REFERENCES

1. A. Beauville, Some remarks on Kähler manifolds with $c\_1 = 0$, *Classification of algebraic and analytic manifolds (Katata, 1982)*, Progr. Math., vol. 39 (Birkhäuser Boston, Boston, MA, 1983), 1–26. MR 728605 (86c:32031).

2. A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, *J. Differ. Geom.* **18**(4) (1983), 755–782 (1984). MR 730926 (86c:32030).

3. A. Beauville, Antisymplectic involutions of holomorphic symplectic manifolds, *J. Topol.* **4**(2) (2011), 300–304. MR 2805992 (2012e:53083).

4. A. Beauville and R. Donagi, La variété des droites d'une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **301**(14) (1985), 703–706. MR 818549 (87c:14047)

5. S. Boissière, Automorphismes naturels de l'espace de Douady de points sur une surface, *Canad. J. Math.* **64**(1) (2012), 3–23. MR 2932167

6. S. Boissière, M. Nieper-Wißkirchen and A. Sarti, Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties, *J. Math. Pures Appl.* **95**(5) (2011), 553–563. MR 2786223 (2012g:14070).

7. S. Boissière and A. Sarti, A note on automorphisms and birational transformations of holomorphic symplectic manifolds, *Proc. Amer. Math. Soc.* **140**(12) (2012), 4053–4062. MR 2957195

8. C. Camere, Symplectic involutions of holomorphic symplectic four-folds, *Bull. Lond. Math. Soc.* **44**(4) (2012), 687–702. MR 2967237.

9. F. Charles, A remark on the Torelli theorem for cubic fourfolds, preprint, arXiv:1209.4509.

10. P. Donovan, The Lefschetz-Riemann-Roch formula, *Bull. Soc. Math. France* **97** (1969), 257–273. MR 0263834 (41 #8433).

11. L. Fu, On the action of symplectic automorphisms on the CH_0-groups of some hyper-Kähler fourfolds, *Mathematische Zeitschrift*, DOI: 10.1007/s00209-015-1424-9. (2015).

12. V. González-Aguilera and A. Liendo, Automorphisms of prime order of smooth cubic n-folds, *Arch. Math. (Basel)* **97**(1) (2011), 25–37. MR 2820585 (2012g:14071).

13. K. Kawatani, On the birational geometry for irreducible symplectic 4-folds related to the Fano schemes of lines, preprint, arXiv:0906.0654.

14. A. Kontogeorgis, Automorphisms of Fermat-like varieties, *Manuscr. Math.* **107**(2) (2002), 187–205. MR 1894739 (2003a:14043).

15. G. Mongardi, Automorphisms of hyperkähler manifolds, Ph.D. thesis (University of RomaTRE, 2013).

16. G. Mongardi, On symplectic automorphisms of hyper-Kähler fourfolds of $K3^{[2]}$ type, *Michigan Math. J.* **62**(3) (2013), 537–550. MR 3102529.

17. Y. Namikawa, Deformation theory of singular symplectic n-folds, *Math. Ann.* **319**(3) (2001), 597–623. MR 1819886 (2002b:32025).

18. V. V. Nikulin, Finite groups of automorphisms of Kählerian $K3$ surfaces, *Uspehi Mat. Nauk.* **31** (2(188)) (1976), 223–224. MR 0409904.

19. T. Shioda, Arithmetic and geometry of Fermat curves, *Algebraic Geometry Seminar (Singapore, 1987)*, (World Sci. Publishing, Singapore, 1988), 95–102. MR 966448 (90a:11037).

20. C. Voisin, *Hodge theory and complex algebraic geometry. II*, Cambridge Studies in Advanced Mathematics, vol. 77 (Cambridge University Press, Cambridge, 2003), Translated from the French by Leila Schneps. MR 1997577 (2005c:32024b).