(Co-)homological Operations in Chow Theory

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Introduction

Let F be an abelian group, i be a non-negative integer, we have the functor of singular cohomology with coefficient in F:

$$H^i(-, F) : \mathcal{T}op^{op} \to \mathcal{A}b,$$

from the category of topological spaces to the category of abelian groups. Thanks to the representability of singular cohomology by Eilenberg-Maclane spaces [Hat02, Theorem 4.57], we find that the group consisting of the morphisms between two such functors $\mathrm{Hom}_{Fct}(H^i(-, F), H^j(-, F'))$ is $H^j(K(i, F), F')$, where $K(i, F)$ is the Eilenberg-Maclane space representing the singular cohomology functor $H^i(-, F)$, we call such natural transformations *cohomological operations*. In 'generic' cases, there are only some obvious cohomological operations, for example, the zero map, the identity map if $i = j$ and $F = F'$, or more generally, the change of coefficient map if $i = j$ with a morphism of abelian groups $F \to F'$, and a little bit less obviously, Bockstein homomorphisms, etc. In [Ste62], Steenrod constructed, in the case $F = \mathbf{Z}/p\mathbf{Z}$ with p a prime number, some series of cohomological operations which are not at all obvious, called *Steenrod squares* if $p = 2$ and *reduced Steenrod powers* if p is an odd prime number. They measure the obstruction to lift the commutativity of cup-product to the level of co-chain. These cohomological operations give more algebraic structures on cohomology groups, namely the graded $\mathcal{A}(p)$−module structures, where $\mathcal{A}(p)$ is the graded Steenrod algebra.

This extra algebraic structure turns out to be quite important in some circumstances, for instance, the famous counter-example of the integral Hodge conjecture provided by Atiyah and Hirzebruch [AH62], relies on the cohomological operations to prove some torsion classes they constructed cannot be algebraic. It is natural to consider the analogies for algebraic cycles. In [Voe03] Voevodsky constructs the analogous cohomological operations on his motivic cohomology, and hence on the mod p Chow groups. After that, Brosnan [Bro03] also constructs a series of homological operations on the Chow groups in a more elementary way. We need to mention that it remains a question whether Brosnan's cohomological operations coincide with the restriction on the usual Chow groups of Voevodsky's cohomological operations defined for the motivic cohomology groups.

In this note, we want to follow the article of Brosnan [Bro03] to generalize the cohomological operations in the context of Chow groups, which are compatible with the Steenrod's cohomological operations on singular cohomology groups under the cycle class map for complex algebraic varieties.

The first main result is the following:

Theorem 0.1 (=Theorem 3.18) *Let X be a* smooth *n-dimensional algebraic variety over a field k, $A_*(X)$ be the $\mathbf{F}_p$-coefficient Chow group of X, set $A^*(X) = A_{n-*}(X)$. We have a series of operations (Definition 3.26): for any i,*

$$S^i : A^q(X) \to A^{q+i(p-1)}(X)$$

for any non-negative integers q, satisfying:

(a)(Additivity) *Each S^i is additive, i.e. a homomorphism of abelian groups;*

(b)(Functoriality) *For any morphism between* smooth *varieties $f : X \to Y$, and any integers i, q, the operation S^i commutes with the l.c.i.*[1] *pull-back $f^!$.*

$$f^! S^i = S^i f^! : A^q(Y) \to A^{q+i(p-1)}(X)$$

(c)(Range) $S^i : A^q(X) \to A^{q+i(p-1)}(X)$ *is the zero map if $i > q$ or $i < 0$;*

(d)(p-th power) $S^q : A^q(X) \to A^{pq}(X)$ *is the p-th power map, that is, $x \mapsto x^p$, the intersection product* [2]*;*

(e) $S^0 = \mathrm{id}$*;*

(f) $S^0([X]) = [X]$ *and $S^i([X]) = 0$ for any $i \neq 0$;*

(g)(Cycle map) *If the base field $k = \mathbf{C}$, then S^i is compatible with the topological Steenrod operations in the following sense:*
If $p = 2$, then $\mathrm{cl} \circ S^i = \mathrm{Sq}^{2i} \circ \mathrm{cl}$:

$$\begin{array}{ccc} A^q(X) & \xrightarrow{\quad S^i \quad} & A^{q+i}(X) \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^{2q}(X;\mathbf{Z}/2) & \xrightarrow{\mathrm{Sq}^{2i}} & H^{2q+2i}(X;\mathbf{Z}/2) \end{array}$$

where Sq^{2i} *is the Steenrod square, c.f. Theorem 3.6.*

If p is an odd prime, then $\mathrm{cl} \circ S^i = \mathrm{P}^i \circ \mathrm{cl}$:

$$\begin{array}{ccc} A^q(X) & \xrightarrow{\quad S^i \quad} & A^{q+i(p-1)}(X) \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^{2q}(X;\mathbf{Z}/p) & \xrightarrow{\mathrm{P}^{i}} & H^{2q+2i(p-1)}(X;\mathbf{Z}/p) \end{array}$$

where P^i *is the reduced Steenrod power, c.f. Theorem 3.7.*

(h)(Cartan formula) *For any $x \in A^q(X)$, $y \in A^r(X)$, we have*

$$S^k(x \cdot y) = \sum_{i=0}^{k} S^i(x) \cdot S^{k-i}(y)$$

in $A^{q+r+k(p-1)}(X)$, or more neatly, $S^\bullet(x \cdot y) = S^\bullet(x) \cdot S^\bullet(y)$ where $S^\bullet$ is the total Steenrod operation and the product $\cdot$ is the intersection product [3]*;*

[1] f is l.c.i., see the argument before the proof of Proposition 2.6
[2] [Ful98] Chapter 8
[3] [Ful98] Chapter 8

(i)(Adem relations) *For any* $0 < a < pb$, *we have:*

$$S^a \circ S^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} S^{a+b-j} \circ S^j$$

In other words, for a smooth variety, the $\mathbf{F}_p$-coefficient graded Chow group is an graded $\mathscr{S}(p)$-module, and such module structure is natural with respect to the l.c.i pull-backs, where $\mathscr{S}(p)$ is the reduced Steenrod algebra.

As for the property of the operations under push-forwards, we introduce another $\mathscr{S}(p)$-module structure on Chow groups, called *twisted Steenrod operations* defined by

$$U^\bullet(t) := w(T_X, t)^{-1} \circ S^\bullet(t),$$

where T_X is the tangent bundle of X, w is certain characteristic class, and $S^\bullet$ is the total Steenrod operation mentioned above.

Here is the second main result:

Theorem 0.2 (=Theorem 4.5) *Let* $f : X \to Y$ *be a morphism between* smooth *varieties, and we suppose that* f *is* projective in the strong sense, *that is,* f *admits a factorization* $f : X \hookrightarrow Y \times \mathbf{P}^r \to Y$, *where the first map is a closed immersion, and the second one is a trivial projective bundle. Then the operations* U^j *commutes with the proper push-forward* f_**:*

$$f_* \circ U^j = U^j \circ f_* : A_i(X) \to A_{i-(p-1)j}(Y)$$

i.e. the following diagram is commutative:

$$\begin{tikzcd} A_i(X) \arrow[r, "U^j"] \arrow[d, "f_*"'] & A_{i-j(p-1)}(X) \arrow[d, "f_*"] \\ A_i(Y) \arrow[r, "U^j"] & A_{i-j(p-1)}(Y) \end{tikzcd}$$

Thanks to this result, we can extend the twisted Steenrod operations U to any algebraic variety over k, giving an $\mathscr{S}(p)$-module structure on its $\mathbf{F}_p$ coefficient Chow groups (Theorem 4.12). And we will generalize the above result that such $\mathscr{S}(p)$ module structures are natural with respect to any proper push-forward.

The main techniques used to construct the Steenrod operations in Chow theory is parallel to the method used in [Ste62], namely, the equivariant Chow groups in the place of equivariant cohomology groups, and of course some equivariant intersection theory is involved.

This note is organized as follows. We start in §1 with some review of basic notions concerning the group action, for example, freeness, geometric quotients, principal homogenous spaces. We reconstruct the equivariant intersection theory in §2, in the spirit of [EG98] and [Bro03]. In §3, we first summarize the essential properties characterizing the Steenrod operations in algebraic topology, define the reduced Steenrod algebra, then we state the first main result (Theorem 0.1) summarizing the analogous properties of the Steenrod operations on Chow groups that we are going to construct, and introduce the twisted Steenrod operations, prove the second main result (Theorem 0.2), and we will explain how to extend this twisted operations to an arbitrary variety. §4 deals with the explicit construction of the Steenrod operations on Chow groups, including the proof of the first main result (Theorem 0.1). Finally, we give an application of the Steenrod operations in §5, to prove the degree formula following [Mer03].

In this note, the terminology 'variety' means an integral (=irreducible and reduced) separated scheme of finite type defined over a field.

1 Generalities on Group Actions

In this preliminary section, we will talk about some notions concerning the group actions, such as quotients, principal bundles, etc.

Let k be a field, $\overline{k}$ be a fixed algebraic closure of k. Let G be a linear algebraic group over k, and X an algebraic variety defined over k.

1.1 Group Actions, Freeness

The standard reference is [MFK94].

Definition 1.1 (Actions) Let X be an algebraic variety over k, and G be a linear algebraic group over k. An *action* of G on X is a k-morphism:

$$\sigma : G \times X \to X$$

satisfying the following properties:

(a)(Associativity) The following diagram is commutative:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times X \\ \Big\downarrow{\scriptstyle \mu \times \mathrm{id}_X} & & \Big\downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow[\sigma]{} & X \end{array}$$

where $\mu : G \times G \to G$ is the multiplication morphism of the algebraic group G;

(b)(Identity) The composition

$$X \cong \operatorname{Spec} k \times X \xrightarrow{e} G \times X \xrightarrow{\sigma} X$$

equals to $\mathbf{id}_X$,

where $e : \operatorname{Spec} k \to G$ is the morphism of 'the neutral element' of G.

Definition 1.2 (Stabilizer) Let the linear algebraic group G act on an algebraic variety X, $\sigma : G \times X \to X$ be the action, we define:

1. the *(universal) stabilizer* $\mathcal{S}$ is the X-scheme defined by the fibre product:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\omega"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_X"] \\ G \times X \arrow[r, "\Psi=\sigma\times pr_2"] & X \times X \end{tikzcd}$$

   Thus $\mathcal{S}$ is a group scheme over X, *i.e.* a group object in the category $\mathcal{S}ch\,/X$, in fact, since i is a closed immersion, $\mathcal{S}$ is a closed subgroup scheme of $G \times X$ over X.

2. For any point $f \in X(T)$, that is a morphism $f : T \to X$, define $\mathcal{S}(f)$ the *stabilizer of f*, as the fibre of $\mathcal{S}$ at the point f:

$$\begin{tikzcd} \mathcal{S}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[d, "\omega"] \\ T \arrow[r, "f"] & X \end{tikzcd}$$

   In particular, for any geometric point x of X, $x : \mathrm{Spec}(\bar{k}) \to X$, the stabilizer of x, $\mathcal{S}(x)$, is by definition above just the fibre of $\mathcal{S}$ at x, it is a closed subgroup scheme of $G_{\bar{k}}$, in particular, $\mathcal{S}(x)(\bar{k})$ is a subgroup of $G(\bar{k})$.

$$\begin{tikzcd} \mathcal{S}(x) \arrow[r] \arrow[d, "i'"'] & \mathcal{S} \arrow[d, "i"] \\ G_{\bar{k}} \arrow[r] \arrow[d] & G \times X \arrow[d, "pr_2"] \\ \mathrm{Spec}(\bar{k}) \arrow[r, "x"] & X \end{tikzcd}$$

   Thus, roughly speaking, the group scheme $\mathcal{S}$ is a family of subgroups of G parameterized by X, and the subgroup as the fiber over a point x of X is just the stabilizer of x under the action of G.

Definition 1.3 Let the linear algebraic group G act on an algebraic variety X, $\sigma : G \times X \to X$ be the action, and denote

$$\Psi : G \times X \xrightarrow{\sigma\times pr_2} X \times X.$$

We define:

1. the action is called *closed* if the orbit of any geometric point is closed in $X_{\bar{k}}$;

2. the action is called *separated* if the image of $\Psi : G \times X \xrightarrow{\sigma\times pr_2} X \times X$ is closed;

3. the action is called *proper* if $\Psi : G \times X \xrightarrow{\sigma\times pr_2} X \times X$ is proper;

4. the action is called *transitive* if $\Psi : G \times X \xrightarrow{\sigma \times pr_2} X \times X$ is surjective;

5. the action is called *free* or *schematically-free* if $\Psi : G \times X \xrightarrow{\sigma \times pr_2} X \times X$ is a closed immersion;

6. the action is called *geometrically free* if the stabilizer of every geometric point is trivial, that is, for any geometric point of X, say $x : \mathrm{Spec}(\bar{k}) \to X$, the stabilizer $\mathcal{S}(x)$, *i.e.* the fibre of $\mathcal{S}$ on x, is trivial;

7. the action is called *set-theoretically free* if for any geometric point of X, say $x : \mathrm{Spec}(\bar{k}) \to X$, its set-theoretical stabilizer is trivial, that is, the stabilizer of x under the action of group $G(\bar{k})$ on $X(\bar{k})$ is trivial.

Remark 1.4 (Transitivity) Keep the notations as above. The transitivity of the action σ is equivalent to the surjectivity of Ψ by definition, which is further equivalent to the surjectivity of $\Psi(\bar{k})$, on the level of geometric points, thanks to [Gro60, Page 147 Proposition 6.3.10]. However, $\Psi(\bar{k})$ is surjective if and only if the action on the geometric points $\sigma(\bar{k}) : G(\bar{k}) \times X(\bar{k}) \to X(\bar{k})$ is transitive. Therefore, the notion of transitivity defined above coincides with the naive notion of transitivity on the level of geometric points.

Proposition 1.5 (Set-theoretical freeness versus geometrical freeness) *Let G be a linear algebraic group acting on a variety X as above, and $\omega : \mathcal{S} \to X$ be the (universal) stabilizer. Then it admits an identity section $e_X : X \to \mathcal{S}$. Suppose either one of the following is satisfied:*

- $\mathrm{char}(k) = 0$*; or*

- G *is finite étale.*

Then the action is set-theoretically free if and only if it is geometrically free.

Proof. In fact, $\mathcal{S}$ is a group scheme over X, ω is the structure morphism, it admits an identity section $e_X : X \to \mathcal{S}$, $\omega \circ e_X = \mathrm{id}_X$, induced by the neutral element $e : \mathrm{Spec}\, k \to G$. For any geometric point $x : \mathrm{Spec}(\bar{k}) \to X$, the stabilizer of x, denoted by $\mathcal{S}(x)$, is the fibre of $\omega : \mathcal{S} \to X$ over x and the set-theoretical stabilizer of x is just the $\bar{k}$-point of the $\bar{k}$-group scheme $\mathcal{S}(x)$:

$$\mathrm{Stab}_{G(\bar{k})}(x) = \mathcal{S}(x)(\bar{k})$$

Therefore, geometrical freeness of the action, which means $\mathcal{S}(x)$ is trivial for any closed point x, implies $\mathrm{Stab}_{G(\bar{k})}(x)$ is a trivial group by the above displayed formula, thus the action is set-theoretically free.

Conversely, set-theoretical freeness of the action implies $\mathcal{S}(x)(\bar{k})$ is trivial for every closed point x. We notice that under either condition listed in the statement of the proposition, $\mathcal{S}(x)$ is a reduced algebraic group: in characteristic zero, it is a general theorem of Cartier (cf.[DG70, II §6,1.1]), while if G is finite étale, every subgroup of G is obviously reduced. Therefore, $\mathcal{S}(x)$ is reduced with only one closed point, by Nullstellensatz, $\mathcal{S}(x)$ itself is trivial, as wanted. □

Proposition 1.6 *Let G be a linear algebraic group acting on a variety X, suppose either one of the following is satisfied:*

- $\mathrm{char}(k) = 0$*; or*
- *G is finite étale.*

then the action is schematically free if and only if it is proper and set-theoretically free.

We remark that by Proposition 1.5, we have, under the assumptions mentioned in the statement, the equivalence between set-theoretical freeness and geometrical freeness.

Proof. Firstly, we assume that the action is (schematically) free, *i.e.* $\Psi : G \times X \to X \times X$ is a closed immersion (hence proper), then by base change $\omega : \mathcal{S} \to X$ is also a closed immersion (hence proper), and in particular, any fibre of ω is trivial. So we have 'free' implies 'set-theoretically free' and 'proper'.

For the converse, see [EG98, Lemma 8]. We reproduce the proof here, the issue is to prove $\Psi : G \times X \to X \times X$ is a closed immersion. To this end, by assumption, the fibers of Ψ are proper, and at the same time affine (since it is a translation of a closed subgroup of G), thus finite. Therefore Ψ is quasi-finite and proper, and hence finite by Zariski's Main Theorem ([Gro66, Page 45, 8.12.6]). Now we look at an arbitrary geometric fibre of Ψ, say, over the closed point $(x, y) \in X(\bar{k}) \times X(\bar{k})$. This fibre $F_{(x,y)}$ is the *transporter* of y to x, it can also be characterized as the fibre over $\{x\}$ of

$$G_{\bar{k}} \simeq G_{\bar{k}} \underset{\bar{k}}{\times} \{y\} \to X_{\bar{k}},$$

here we write $\{x\}$ and $\{y\}$ for the $\mathrm{Spec}\, \bar{k}$ corresponding to x and y respectively. By definition, $F_{(x,x)}$ is the stabilizer of x, which is the fibre over $\{x\}$ of

$$G_{\bar{k}} \simeq G_{\bar{k}} \underset{\bar{k}}{\times} \{x\} \to X_{\bar{k}}.$$

If $F_{(x,y)}$ is nonempty, then any of its geometric point g defines the right translations τ_g and $\tau_{g^{-1}}$ of $G_{\bar{k}}$, which make the following diagram commutative:

$$\begin{tikzcd} G_{\bar{k}} \simeq G_{\bar{k}} \times_{\bar{k}} \{y\} \arrow[rr, shift left, "\tau_{g^{-1}}"] \arrow[dr] & & G_{\bar{k}} \simeq G_{\bar{k}} \times_{\bar{k}} \{x\} \arrow[ll, shift left, "\tau_g"] \arrow[dl] \\ & X_{\bar{k}} & \end{tikzcd}$$

In particular, the two fibres are isomorphic as varieties over $\bar{k}$. Thanks to Proposition 1.5, the stabilizer $F_{(x,x)}$ is trivial, hence the transporter $F_{(x,y)}$ is also trivial. Now we can conclude that Ψ is a closed embedding, by applying [Gro66, Page 42, 8.11.5]. $\square$

1.2 Quotients

Definition 1.7 (Invariants) Let $G = \operatorname{Spec} H$ be a linear algebraic group over a field k, defined by a commutative Hopf algebra $(H, \cdot, 1, \Delta, \epsilon)$, where $\cdot$ is the (commutative) multiplication, 1 is the unit for the algebra structure, $\Delta : H \to H \otimes_k H$ is the comultiplication, $\epsilon : H \to k$ is the counit map. Let $X = \operatorname{Spec} A$ be an affine algebraic variety over k, where A is a (commutative) reduced finitely generated k-algebra. Suppose we have an action of G over X, defined by the map

$$\rho : A \to H \otimes_k A$$

giving an H-comodule structure on A.

The *invariants* A^G is defined to be the subring of A consisting of the 'cofixed' elements of A by H:

$$A^G := \{a \in A | \rho(a) = 1 \otimes a\}$$

Definition-Proposition 1.8 (Affine case: algebraic quotient) Suppose that G is a linear algebraic group over k, acting on an affine k-variety $X = \operatorname{Spec} A$, ($A$ is a finitely generated reduced k-algebra), the *algebraic quotient* $X /\!\!/ G$ is by definition the affine scheme defined by the invariants:

$$X /\!\!/ G := \operatorname{Spec} A^G$$

We mention that if the algebraic group G is *reductive*, $X /\!\!/ G$ is in fact a k-variety, *i.e.* the invariants A^G is also a reduced finitely generated k-algebra.

Proof. [MFK94] P27, Chapter 1, §2. □

For more about the algebraic quotient in the affine case, see the end of this subsection, Theorem 1.20.

Definition 1.9 (Categorical quotient) Given an action $\sigma : G \times X \to X$, a pair (Y, ϕ) consisting of a variety Y, and a morphism $\phi : X \to Y$ is called a *categorical quotient*, if the following diagram is *cocartesian*:

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd} \tag{1}$$

That is, the above diagram commutes, and Y is the universal push-out, as indicated in the following diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"'] \arrow[d, "pr_2"] & X \arrow[d, "\phi"'] \arrow[ddr, bend left] & \\ X \arrow[r, "\phi"] \arrow[drr, bend right] & Y \arrow[dr, dotted, "\exists!"] & \\ & & \forall Z \end{tikzcd}$$

We say a categorical quotient is *universal*, if it furthermore satisfies the condition below:

- for any $Y' \to Y$, the base change $\phi' : X' = X \times_Y Y' \to Y'$ is always a categorical quotient,

If in the above condition, we only demand the base-change property is satisfied by any *flat* base change $Y' \to Y$, such categorical quotient is called *uniform*.

Remark 1.10 (1) A categorical quotient does not always exist, but if it exists, it is unique up to unique isomorphism by the universal property, so we can call it 'the' categorical quotient is this case;

(2) The algebraic quotient defined in Definition-Proposition 1.8, is actually the categorical quotient *in the category of affine k-varieties.* Indeed, for any affine k-variety $Z = \operatorname{Spec} B$, and any morphism $X \to Z$ invariant under the action of G, that is, a morphism of k-algebra $B \to A$ whose image is contained in the cofixed subring A^G, we have of course the unique factorization $B \to A^G \hookrightarrow A$ corresponding to the unique factorization $X \to Y \to Z$.

Here is a useful result for recognizing categorical quotient.

Proposition 1.11 *Let a linear algebraic group G act on an algebraic variety X, and suppose we have the following commutative diagram:*

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

satisfying:

(i) O_Y *is the subsheaf of invariants of* $\phi_*(O_X)$*;*

(ii) If W is an invariant closed subset of X, then $\phi(W)$ is closed in Y; if $\{W_i\}_{i\in I}$ is a set of invariant closed subsets of X, then:

$$\phi(\cap_{i\in I} W_i) = \cap_{i\in I}\phi(W_i).$$

Then $\phi : X \to Y$ is the categorical quotient, and moreover, ϕ is submersive.

PROOF. See [MFK94] Page 8, Remark (6). □

Mumford defined the following fundamental notion of geometric quotient, c.f: [MFK94, Definition 0.6, Page 4].

Definition 1.12 (Geometric quotients I) A pair (Y, ϕ) consisting of a variety Y, and a morphism $\phi : X \to Y$ is called a *geometric quotient*, and denoted by $Y = X/G$, if we have the diagram (1):

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

which satisfies the following properties:

(1) The diagram commutes;

(2) ϕ is surjective, and the image of $\Psi : G \times X \xrightarrow{(\sigma, pr_2)} X \times X$ is $X \times_Y X$;

(3) ϕ is a submersion, *i.e.* a subset $U \subset Y$ is open, if and only if $\phi^{-1}(U)$ is open is X;

(4) the structure sheaf O_Y is the sub-sheaf of $\phi_* O_X$ consisting of invariant functions. It is equivalent to the exactness of the following sequence: (denote $\psi = \phi \circ \sigma = \phi \circ pr_2$)

$$0 \to O_Y \to \phi_* O_X \xrightarrow{\sigma^\sharp - pr_2{}^\sharp} \psi_*(O_{G\times X}) \tag{2}$$

Remark 1.13 Here is a reformulation of the second condition above. By the diagram (1), we have a natural factorization of Ψ through $X \times_Y X$ as the following diagram shows:

$$\begin{tikzcd} G \times X \arrow[dr, dotted, "\overline{\Psi}"] \arrow[drr, bend left, "\sigma"] \arrow[ddr, bend right, "pr_2"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\phi"] \\ & X \arrow[r, "\phi"] & Y \end{tikzcd}$$

thus this condition is equivalent to $\overline{\Psi}$ is surjective (in the sense of scheme).

We can also define the notion of *universal geometric quotient* and *uniform geometric quotient* in the way of Definition 1.9.

Proposition 1.14 *A geometric quotient is automatically a categorical quotient. Hence the geometric quotient (if it exists) is unique up to unique isomorphism.*

PROOF. [MFK94] Chap 0, §2,Prop0.1, Page 4. $\square$

Proposition 1.15 (Faithfully Flat Descent) *Let a linear algebraic group G act on an algebraic variety X, and $(Y, \phi : X \to Y)$ fitting into a diagram as (1). Let $\alpha : Y' \to Y$ be a morphism, and $\phi' : X' \to Y'$ be the corresponding base change of ϕ. Suppose that α is* faithfully flat and quasi-compact*, then (Y', ϕ') is a geometric quotient of X' implies (Y, ϕ) is a geometric quotient of X. And moreover, if the former one is a universal geometric quotient, then so is the latter.*

Proof. ([MFK94] Page 9, Remark 8) Firstly, we remark that in the definition of geometric quotient, the condition 1, 2 is preserved by any base change and have descent property for surjective (for example, faithfully flat) base change, since, in general, surjectivity is stable by base change and have descent property for surjective base change.
The third condition is not stable by base change in general. But it has the descent property for faithfully flat (in fact, surjective and open suffice) base change under the assumption of surjectivity of ϕ: let U be a subset of Y and assume that $V = \phi^{-1}(U)$ is open in X, we want to show that U is open. Indeed, let $U' = \alpha^{-1}(U)$ be the pre-image of U in Y', and $V' = \phi'^{-1}(U') = \alpha'^{-1}(V)$ is open in X, then by the condition 3 for ϕ', U' is open. Thus $U = \alpha(U')$ is open, since flat morphism is open.
The last condition is preserved by *flat* base change, since the base change of the exact sequence (2)

$$0 \to O'_Y \to \phi_* O'_X \xrightarrow{\sigma'^\sharp - pr_2'^\sharp} \psi_*(O_{G\times X'})$$

is just $(2)\otimes_{O_Y} O_{Y'}$ which is also exact by the flatness of Y' over Y.
And the last condition has descent property for *faithfully flat and quasi-compact* base change, since the sequence $(2)\otimes_{O_Y} O_{Y'}$ is exact if and only if the sequence (2) is. □

We give another equivalent (but sometimes more practical) definition of geometric quotients, following [CTS07] Definition 2.7 :

Definition 1.16 (Geometric quotients II) Let X be an algebraic k-variety equipped with an action of a linear algebraic group G. A *geometric quotient* of X by G is a k-variety Y equipped with a k-morphism $\phi : X \to Y$ such that:

(1') ϕ is constant on G-orbits;

(2') ϕ induces a bijection between $X(\bar{k})/G(\bar{k})$ and $Y(\bar{k})$;

(3') ϕ is open;

(4') For any open subset $V \subset Y$, the natural morphism $O_Y(V) \to O_X(\phi^{-1}(V))^G$ is an isomorphism.

To prove the equivalence of the two definitions of geometric quotients, we need the following lemma [MFK94, Page 6, Remark(4)]:

Lemma 1.17 *If (Y, ϕ) is a geometric quotient* [4] *of X by G , then ϕ is universally open (hence universally submersive* [5]*).*

Proof. Consider the diagram (1):

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

Firstly, one remarks that the action map $\sigma : G \times X \to X$ is universally open. Indeed, σ can be factorized as:

$$\sigma : G \times X \xrightarrow{\mathrm{id} \times \sigma} G \times X \xrightarrow{pr_2} X$$

where the first map is an isomorphism since in the point of view of functor of points, the first map can be written as $(g, x) \mapsto (g, gx)$, which has an obvious inverse $(g, x) \mapsto (g, g^{-1}x)$, and the second projection map is universally open.

Now let $U \subset X$ be an open subset of X. Then $V = pr_2^{-1}(U) = G \times U$ is open. $U' := \sigma(V)$ is open since σ is. And U' is moreover G-invariant. So $U' = \phi^{-1}(\phi(U'))$, and ϕ is submersive by assumption, we get $\phi(U')$ is open. It is obvious that $\phi(U') = \phi(U)$, thus $\phi(U)$ is open. To complete the proof of the lemma, it suffices to remark that the above argument is also valid for a base change of the diagram, since we only used the facts that σ is open, and pr_2 is surjective, and U' is G-invariant. □

Now we can prove the equivalence of the two definitions of geometric quotients, the argument is due to B.Kahn.

Proof. (Equivalence of Definitions 1.12 and Definition 1.16)
In the following proof, we will use (1)(2)(3)(4) to refer to the conditions in Definition 1.12, and (1')(2')(3')(4') for the conditions in Definition 1.16.

The equivalence between (1) and (1'), as well as that of (4) and (4'), are clear. (2) is equivalent to the surjectivities of ϕ and $\overline{\Psi}$ by the Remark 1.13. Recall that the theorem of Fogarty and Hashimoto [Has04] says that Y is of finite type, therefore one can apply Nullstellensatz, to conclude that (2) is equivalent to the surjectivities on the level of closed points:

$$\Psi(\bar{k}) : G(\bar{k}) \times X(\bar{k}) \twoheadrightarrow X(\bar{k}) \times_{Y(\bar{k})} X(\bar{k});$$

and

$$\phi(\bar{k}) : X(\bar{k}) \twoheadrightarrow Y(\bar{k}),$$

hence is equivalent to (2').

[4] In the sense of Mumford, as in Definition 1.12, at present. We will prove that the two definitions are equivalent.

[5] Surjective and open together imply submersive(c.f.[Gro66, 15.7.8])

As for (3) and (3'), assuming first the conditions (1)(2)(3)(4) in Definition 1.12, then by Lemma 1.17, ϕ is universally open. Conversely, assuming the conditions (1')(2')(3')(4') of Definition 1.16. Then we know that ϕ is surjective and open, thus submersive (c.f.[Gro66, 15.7.8]). The proof is complete. □

Here is an important tool to verify geometric quotients:

Proposition 1.18 *Let G be a linear algebraic group, and X an algebraic normal variety with an action $\sigma : G \times X \to X$. Let Y be an irreducible, normal variety with the function field $K(Y)$ of characteristic 0, and let $\phi : X \to Y$ be a dominant morphism fitting in to a commutative diagram (1):*

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{\phi} & Y \end{array}$$

Suppose that for any geometric point $y : \mathrm{Spec}(\bar{k}) \to Y$, the geometric fibre of ϕ over y, contains at most one orbit under the action of $G_{\bar{k}} = G \times_k \bar{k}$ [6]. Then ϕ is universally open, and (Y, ϕ) is a geometric quotient for σ.

Proof. [MFK94] Chapter0, §2, Proposition 0.2, Page 7. □

We remark that by Lemma 1.17, in the above proposition, the universal openness is in fact a consequence of geometric quotient.

In many circumstances, the converse of Proposition 1.18 holds: we have the following criterion of geometric quotient (c.f.[CTS07, Proposition 2.8]):

Proposition 1.19 (Criterion of geometric quotient) *Suppose k is algebraically closed, and of characteristic 0. Let X be a normal algebraic variety over k, equipped with an action of a linear algebraic group G over k. Then a k-morphism $\phi : X \to Y$ is a geometric quotient if and only if the three conditions below hold:*

1. *ϕ is constant on G-orbits;*
2. *Y is the orbit space of the geometric points, that is, ϕ induces a bijection $X(k)/G(k) \to Y(k)$;*
3. *Y is normal.*

Proof. Condition 1 is the same as to say the diagram (1) commutes. Condition 2 is equivalent to there is only one orbit in each geometric fibre. Therefore by Proposition 1.18, $\phi : X \to Y$ is the geometric quotient.

[6]Let $y : \mathrm{Spec}\,\bar{k} \to Y$ be a geometric point, we have a natural action of $G \times_k \bar{k}$ on the geometric fibre $X \times_Y \mathrm{Spec}\,\bar{k}$: $(G \times_k \bar{k}) \times_{\bar{k}} (X \times_Y \bar{k}) = G \times_k X \times_Y \mathrm{Spec}\,\bar{k} \xrightarrow{\sigma \times \mathrm{id}} X \times_Y \mathrm{Spec}\,k$

Conversely, let $\phi : X \to Y$ be a geometric quotient. Condition 1 is trivial. Condition 2 follows from Definition 1.16, (if one wants to see it from Mumford's definition, c.f. the remarks in [MFK94, Page 4 Definition 0.6 (ii)]). We only need to show that Y is normal, under the assumption that X is normal.

The argument to prove the normality here is quite close to the reasoning in the classical case that G is a finite group. Because of the condition (4') in Definition 1.16, we can restrict ourselves in the affine case that $X = \operatorname{Spec} A$, with A a normal domain, and $Y = \operatorname{Spec} B$, where

$$B = A^G = \{a \in A | \rho(a) = 1 \otimes a\}$$

the subdomain of invariants (c.f. Definition 1.7), here we write $\rho : A \to H \otimes A$ for the group action, where H is the commutative Hopf algebra representing G. Set $L = \mathrm{Frac}(A)$ and $K = \mathrm{Frac}(B)$, a subfield of L. Suppose we have an element $b/b' \in K$ ($b' \neq 0$) which is integral over B. When viewed as an element in L, it is *a fortiori* integral over A, therefore $b/b' = a \in A$, by the assumption that A is normal. We apply ρ to the equation $b = ab'$, to get

$$1 \otimes b = \rho(a) \cdot (1 \otimes b') \in H \otimes A,$$

however, we have clearly

$$1 \otimes b = (1 \otimes a) \cdot (1 \otimes b') \in H \otimes A.$$

Remark that $H \otimes A$ is still a domain (G and X are both integral varieties, and so is $G \times X$, since k is algebraically closed), therefore by the above two displayed equations, we get $\rho(a) = 1 \otimes a$, that is, $b/b' = a \in B = A^G$, as wanted. □

Let us return to the affine case:

Theorem 1.20 (Algebraic quotients by reductive groups) *Let X be an affine scheme over k, G be a reductive algebraic group over k acting on X. Then*

(1) The uniform categorical quotient exists: $\phi : X \to Y$, where Y is affine, ϕ is universally submersion;

(2) If X is moreover of finite type over k, then so is the categorical quotient Y;

(3) If $\mathrm{char}(k) = 0$, then the categorical quotient is in fact universal; Moreover, if X is noetherian, then so is the categorical quotient Y;

(4) The categorical quotient $\phi : X \to Y$ is a geometric quotient if and only if the action of G on X is closed. Moreover, if $\mathrm{char}(k) = 0$, it is actually a universal geometric quotient.

PROOF. If $X = \operatorname{Spec} A$, H is the Hopf algebra represents G, and the action $\sigma : G \times X \to X$ corresponds to $\rho : A \to A \otimes_k H$ morphism of k-algebras which gives A a comodule structure. Denote $A^G = \{a \in A | \rho(a) = a \otimes 1\}$ the sub-ring of invariants of A under the action of G. Then in fact $Y = \operatorname{Spec} A^G$. The proof in the case of characteristic 0 uses the Reynolds operator, Proposition 1.11, and three key lemmas. See [MFK94, Page 27 Thm 1.1, Page 30 Amplification 1.3 and Page 194 Thm A.1.1] for details. □

1.3 G-Principal Bundles

Definition 1.21 (G-principal bundles) Let G be a linear algebraic group over k, X be a k-variety equipped with a G-action σ, and a k-morphism $\phi : X \to Y$ fitting into a commutative diagram (1):

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

We will say $X \xrightarrow{\phi} Y$ is a *G-principal bundle*, or a *G-torsor*, if

1. ϕ is *faithfully flat*;
2. the above diagram is *cartesian*, *i.e.* the induced natural map $X \times G \to X \times_Y X$ is an isomorphism.

Proposition 1.22 *If $X \xrightarrow{\phi} Y$ is a G-principal bundle, then (Y, ϕ) is a geometric quotient of X by G, and the action of G on X is (schematically) free. In particular, the above diagram is* cartesian *and* cocartesian.

PROOF. Indeed, the freeness is immediate from the isomorphism $G \times_k X \to X \times_Y X$, and the closed immersion $X \times_Y X \to X \times_k X$.
To check it is an geometric quotient, as in the definition 1.12,
condition 1 is trivial;
condition 2, ϕ is surjective by faithfully flatness and $X \times_k G \to X \times_Y X$ is an isomorphism thus surjective;
condition 3, ϕ is submersive since it is faithfully flat thus open and surjective;
condition 4, the sequence (2) becomes the following, where $\psi := \phi \circ \sigma$

$$0 \to O_Y \to \phi_* O_X \xrightarrow{pr_1{}^\sharp - pr_2{}^\sharp} \psi_*(O_{X \times_Y X}) \simeq \psi_*(O_{G \times X})$$

which is exact since O_X is faithfully flat over O_Y (strict epimorphism)[7] □

[7] Let B be an faithfully flat A-algebra, then the following Amitsur complex is exact[KO74] Page 30, Prop 2.1, where the morphism is the alternative sum of 'face-maps' :

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

The important thing is the converse:

Theorem 1.23 *Given X an algebraic variety over k, and G is a linear algebraic group acting on X. If the action is (schematically) free, and X admits a geometric quotient (Y, ϕ), then $\phi : X \to Y$ is a G-principal bundle.*

For the proof, we refer to [MFK94] Page 16, Proposition 0.9.

Example 1.24 *Let G be an algebraic group over k, and H a closed subgroup of G, then $\pi_1 : G \to G/H$ is an H-principal bundle, for the right H-action on G by right translation; similarly, $\pi_2 : G \to H\backslash G$ is an H-principal bundle, for the left H-action on G by left translation.*

Definition 1.25 One says that a G-principal bundle $\pi : X \to Y$ is *trivial* (or more precisely *trivializable*), if there is a G-equivariant Y-isomorphism $\varphi : X \xrightarrow{\simeq} G \times Y$:

$$\begin{tikzcd}
G \times X \arrow[rr, "\mathrm{id}\times\varphi", "\simeq"'] \arrow[d, "\sigma"'] & & G \times G \times Y \arrow[d, "\mu\times\mathrm{id}"] \\
X \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, "\pi"'] & & G \times Y \arrow[dl, "pr_2"] \\
& Y &
\end{tikzcd}$$

It is worth mentioning the result that a G-principal bundle $\pi : X \to Y$, is trivial if and only if it admits a section $s : Y \to X$, *i.e.* $\pi \circ s = \mathrm{id}_Y$.

Here are two useful tricks to 'trivialize' some principal bundles.

Example 1.26 Let $\pi : X \to X_0$ be a G-principal bundle, then the pull-back

$$\pi' : X \times_{X_0} X \to X$$

is a trivial G-principal bundle.
Indeed, π is faithfully flat implies the same for the base change π', and the diagram (1) is cartesian implies the same for the base changed diagram for $X \times_{X_0} X$. Thus π' is a G-principal bundle. And since it admits a canonical section, namely the diagonal: $\Delta_{X/X_0} : X \to X \times_{X_0} X$, so it is trivial.

Before going into the second trick, let us say something about the natural correspondence between left actions and right actions.

Remark 1.27 (Left and right actions) A left G-action corresponds naturally to a right G-action, and vice versa. Precisely, given a variety X with a left G-action, then we can define the *associated right G-action* on X by $x \cdot g := g^{-1} \cdot x$, for any $x \in X$ and $g \in G$. Here we are using the functors of points, to be more formal, we could define the associated right G-action as the composition:

$$X \times G \xrightarrow{\mathrm{id}\times\iota} X \times G \xrightarrow{\tau} G \times X \xrightarrow{\sigma} X,$$

where ι is the morphism of 'taking the inverse' of the algebraic group G, and τ is simply interchanging the two factors, and σ is the left action as before. And similarly for the construction from a right action to its *associated left action.*

Example 1.28 Let X be an algebraic variety, G be a linear algebraic group acting on X. Consider $X \times G$, it can be equipped with two natural G-actions, we indicate them on the level of functor of points as below, and for the later reference, we will also give the formulation corresponding to the naturally associated right action of G on X, discussed in Remark 1.27.

On one hand, the G-action can be simply the left translation on the second factor:

$$\begin{array}{rcl} G \times (X \times G) & \to & X \times G \\ g_1, (x, g_2) & \mapsto & (x, g_1 g_2); \end{array}$$

On the other hand, we have the diagonal action:

$$\begin{array}{rcl} G \times (X \times G) & \to & X \times G \\ g_1, (x, g_2) & \mapsto & (g_1 x, g_1 g_2) = (x g_1^{-1}, g_1 g_2). \end{array}$$

In fact, $X \times G$ with the above two G-space structures are isomorphic, here is the (G-equivariant) isomorphism as G-spaces:

$$\begin{array}{rcl} X \times G & \xrightarrow{\cong} & X \times G \\ (x, g) & \mapsto & (gx, g) = (xg^{-1}, g), \end{array}$$

where the space on the left hand side is equipped with the action on the second factor by left translation, and the space on the right hand side is equipped with the diagonal action. Since $X \times G$ with the second-factor action is obviously a trivial G-principal bundle with the base space X and the bundle map given by the first projection, in virtue of the isomorphism given above, $X \times G$ with the diagonal action is also a trivializable G-principal bundle, with the base space X, and the bundle map is defined by

$$\begin{array}{rcl} \phi : X \times G & \to & X \\ x, g & \mapsto & g^{-1} x = xg. \end{array}$$

Finally, we want to give a proposition which creates new principal bundles from old ones.

Proposition 1.29 *Suppose that E is the total space of a G-principal bundle, and X is another algebraic variety with a (left) G-action, Assume one of the following conditions:*

1. *X is (quasi-)projective with a linearized G-action [8]; or*

2. *G is connected and X is equivariantly embedded as a closed sub-scheme of a normal variety with G-action; or*

3. *G is special [9],*

[8] [MFK94, Chapter 1, §3]

[9] Definition: A group is *special* if every principal bundle is locally trivial in the Zariski topology. Examples: $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$. Conterexamples: $\mathrm{PGL}_n, \mathrm{SO}_n$, finite groups

then $E \times X$, equipped with the diagonal G-action, is the total space of a G-principal bundle, denoted by $E \times X \to E \times^G X$.

PROOF. [EG98] Page 632, Prop 23. $\square$

Here is the explanation for the notation $E \times^G X$, appeared in the preceding proposition. Under the natural correspondence of left and right action in Remark 1.27, the diagonal action on $E \times X$ is given by:

$$\begin{aligned} G \times (E \times X) &\to E \times X \\ g, (e, x) &\mapsto (ge, gx) = (eg^{-1}, gx), \end{aligned}$$

which explains the notation.

2 Equivariant Intersection Theory

We fix some notations throughout this section: let k be a field, G be a linear algebraic group over k, X be a variety defined over k with a G-action, say $\sigma : G \times X \to X$. And furthermore, we make the following technical assumption:

Assumption 2.1 As in Proposition 1.29, we assume that:

1. X is (quasi-)projective with a linearized G-action; or
2. G is connected and X is equivariantly embedded as a closed sub-scheme of a normal variety with G-action; or
3. G is special.

In this section, we will write $A_j(X)$, $A^j(X)$ or $A^*(X)$ for the usual Chow groups, and the coefficient group(or ring) Λ will be suppressed, for example: $A_j(X)$ means $A_j(X) \otimes \Lambda$. We write $|X|$ for the dimension of X. All flat morphisms are assumed to admit a relative dimension.

2.1 Equivariant Chow Groups

In algebraic topology, given a topological space X with a continuous action of a topological group G, we can define the *equivariant cohomology*:

$$H^i_G(X; \mathbf{Z}) = H^i\left(\frac{X \times \mathrm{EG}}{G}\right),$$

where EG is the universal cover of the classifying space BG, and G acts on the product diagonally.

We intend to extend this notion in the setting of algebraic geometry, the *equivariant Chow group*. To this end, we have to extend the notion of EG to the algebraic setting. Here are three categorical substitutions of EG.

Recall that by *a G-principal bundle*, we mean an algebraic variety equipped with a free G-action, such that the geometric quotient exists as a variety. We write $\mathsf{Rep}(G)$ for the category of k-linear representations of G.

Definition 2.2 (Categorical substitutions of EG) Let G be a linear algebraic group over k, X be an algebraic variety over k with a G-action. We define the following three categories:

1. The category EG:

   Objects: G-invariant open sub-scheme U of a representation $V \in \mathrm{Rep}(G)$ such that U is the total space of a G-principal bundle.

   Morphisms: G-equivariant morphism of varieties.

   We also write EG_r for the full subcategory of EG consisting of the open sub-schemes $U \subset V$ such that the codimension of $S = V - U$ in V is at least r.

2. The category CEG:

   Objects: the smooth total spaces of G-principal bundles;

   Morphisms: G-equivariant morphisms.

3. The category $\mathrm{FSE}_X G$:

   Strictly speaking, it is not a categorical substitution of the topological EG in the algebraic context, but of the topological $EG \times X$ instead, In particular, this is a category depending on X, as the notation suggests.

   Objects: $p_E : E \to X$ a G-equivariant *smooth* morphism with a relative dimension n_E, where E is the total space of a G-principal bundle [10];

   Morphisms: G-equivariant morphisms compatible with the morphisms to X.

We have the following natural functors:

$$\mathrm{EG} \xrightarrow{j} \mathrm{CEG} \xrightarrow{\iota_X} \mathrm{FSE}_X G \tag{3}$$

where j takes an open sub-scheme (with free G-action admitting a geometric quotient) of a representation to itself, and ι_X takes any smooth variety U (with free G-action admitting a geometric quotient) to $p : U \times X \to X$, where G acts on $U \times X$ diagonally, by Assumption 2.1 we made in the beginning of this section and applying Proposition 1.29, we find that $U \times X$ is indeed the total space of a principal bundle, therefore ι_X is well-defined. And finally we define $i_X = \iota_X \circ j : \mathrm{EG} \to \mathrm{FSE}_X G$ to be the composition of functors. We note that j is fully-faithful, while ι_X is only faithful.

Thanks to the principle that modifying a low dimensional subvariety does not effect the higher dimensional Chow group, we intend to use the objects in the above categories to 'approximate' the topological EG. But first of all, we should check that they are non-vacuous, as explained in Totaro [Tot99] Remark 1.4, or Colliot-Thélène and Sansuc [CTS07] Lemma 9.2:

[10] Attention: This G-principal bundle is not supposed to have X as its base. All we need is the total space of a principal bundle.

Proposition 2.3 (Algebraic approximations of EG) *For any given non-negative integer r, EG_r, hence* CEG *and* $\mathrm{FSE}_X G$ *are non-empty.*

PROOF. Take any faithful representation of G: $G \subset \mathrm{GL}_n$, there is a natural embedding: $G \to \mathrm{GL}_{N+n}$ defined by:

$$g \mapsto \left(\begin{array}{c|c} I_N & 0 \\ \hline 0 & g \end{array}\right) \tag{4}$$

where $g \in G$. We denote the image of (4) by $\widetilde{G}$, a subgroup of GL_{N+n} and isomorphic to G.
Let U be the space of $n \times (N+n)$-matrices of rank n, on which there is a natural G action (by multiplication on the left through GL_n). Consider also the projection to the bottom n rows:

$$\pi : \mathrm{GL}_{N+n} \to U$$

it is actually G-equivariant if we let $G \simeq \widetilde{G}$ acts by left multiplication on GL_{N+n}. Define a subgroup of GL_{N+n}:

$$H = \left\{\left(\begin{array}{c|c} \mathrm{GL}_N & * \\ \hline 0 & I \end{array}\right)\right\}$$

It is easy to see that $\mathrm{GL}_{N+n}/H \simeq U$. Since the action of G on U is clearly free, to prove that U is a principal G-bundle, it suffices to show that the geometric quotient U/G exists, equivalently, GL_{N+n}/Γ exists, where Γ is the subgroup of GL_{N+n} generated by H and the image of $\widetilde{G}$. It is easy to verify that $\widetilde{G}$ normalize H, thus $\Gamma = \widetilde{G} \ltimes H$ is the subgroup of GL_{n+N}, consisting of all the matrices of the form:

$$\left(\begin{array}{c|c} \mathrm{GL}_N & * \\ \hline 0 & G \end{array}\right)$$

Γ is clearly a closed subgroup of GL_{N+n}, hence the geometric quotient exists, and U is a G-principal bundle. To complete the proof, we need to check the codimension condition. Since U is an open sub-scheme of the representation V consisting of all the $n \times (n+N)$-matrices, with compliment S. The elements in S are the matrices of rank$< n$, whose codimension is $N+1$, which tends to infinity with N, as wanted. As for the codimension calculation, see the standard elementary lemma below, or one can just use the much more general result [Ful98, lemme A.7.2]. □

Lemma 2.4 *Let M, N, r be positive integers, such that $r \leq \min\{M, N\}$. Then in the vector space of all M by N matrices, the subspace consisting of the matrices of rank $\leq r$ is a closed subscheme of codimension $(M-r)(N-r)$.*

PROOF. Write V for the vector space of all M by N matrices, and X_r for the subspace consisting of the matrices of rank $\leq r$. Since the rank is an upper semi-continuous function on V, X_r is a closed subset of V. It is easy to see that the open subset $U_r := X_r - X_{r-1}$ of X_r, consisting of the matrices of rank r, is homogenous under the action of $\mathrm{GL}_M \times \mathrm{GL}_N$. Indeed, let GL_M act on the left and GL_N act on the right by:

$$\begin{array}{rcl} \mathrm{GL}_M \times \mathrm{GL}_N \times U_r & \to & U_r \\ (P, Q, x) & \mapsto & PxQ^{-1}. \end{array}$$

The two actions are clearly commutative, thus define an action of $\mathrm{GL}_M \times \mathrm{GL}_N$ on U_r. It is elementary that this action is transitive over U_r since it is so on the level of closed points, c.f. Remark 1.4. Therefore U_r is homogenous, hence smooth, and to calculate the codimension of X_r in V, it is enough to calculate the codimension of U_r in V (or in $V - X_{r-1}$) at any its (smooth) point, say, at $x_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix.

Now we deduce the local equations defining U_r (or equivalently X_r) at x_0 by the following argument. Consider an $M \times N$ matrix $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in V, which is *near* x_0 in the sense that x is in the Zariski open subset defined by $\det(A) \neq 0$. Here we have written x by blocs: A is of size $r \times r$, and the sizes of B, C, D follow correspondingly. After elementary operations the rank of x is equals to the rank of

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

Therefore $\mathrm{rank}(x) \leq r$ is equivalent to the system of equations

$$D = CA^{-1}B$$

i.e. these $(M - r)(N - r)$ equations are the local equations defining U_r. Since every entry of D appears in this system of equations exactly once and linearly, it follows that the U_r is locally a complete intersection in $V - X_{r-1}$, of codimension $(M - r)(N - r)$. □

We need an auxiliary equivariant Chow group defined for $\mathrm{FSE}_X G$.

Definition 2.5 (Auxiliary equivariant Chow groups for $\mathrm{FSE}_X G$) Let $p_E : E \to X$ be an object in $\mathrm{FSE}_X G$, where E is a G-principal bundle, p_E is smooth of relative dimension n_E. Define

$$\mathscr{A}_i^G(E) := A_{i+n_E-|G|}(E/G)$$

where E/G is the geometric quotient as a variety.

We note here that given an object U of EG or CEG, in virtue of the natural functors in (3), we have

$$\mathscr{A}_i^G(U \times X) = A_{i+|U|-|G|}(X \times^G U)$$

where $X \times^G U$ means the quotient of the free G-space $X \times U$, see Proposition 1.29. We should remark that the 'bizarre' indexing is derived from the intuition that an element in $\mathscr{A}_i^G(X \times \mathrm{EG})$ is morally an 'i-equivariant cycle in X'.

Proposition 2.6 (Functoriality of $\mathscr{A}_j^G$) *Given a morphism f in* $\mathrm{FSE}_X G$*:*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p_E"'] & & E' \arrow[dl, "p_{E'}"] \\ & X & \end{tikzcd}$$

it induces an l.c.i pull-back $f^* = f_G^! : \mathscr{A}_j^G(E') \to \mathscr{A}_j^G(E)$, which makes $\mathscr{A}_j^G$ into a contravariant functor from $\mathrm{FSE}_X G$ *to the category of abelian groups $\mathcal{A}b$.*

Before we embark for the proof, let us recall some basic facts about the l.c.i pull-back in the classical (non-equivariant) case. We will loosely follow the exposition in [Ful98].

Recall that a morphism of varieties (or more generally, schemes) is called *locally complete intersection*(l.c.i), if it can be factorized into a regular embedding followed by a smooth morphism, and the *codimension* of the l.c.i morphism is the codimension of the regular embedding minus the relative dimension of the smooth map. In particular, it could be negative.

Now given an l.c.i morphism $f : X \to Y$ of codimension d, which admits a factorization $X \xrightarrow{i} P \xrightarrow{g} Y$ as above. Let s be the relative dimension of g, r be the codimension of i, then $d = r - s$. We define the *l.c.i pull-back* (or *refined Gysin homomorphism* in [Ful98, Chapter 6])

$$f^! = i^! \circ g^* : A_*(Y) \to A_{*-d}(X)$$

where g^* is the flat pull-back: $A_*(Y) \to A_{*+s}(P)$, and $i^! : A_*(P) \to A_{*-r}(X)$ is the *Gysin map* ([Ful98] Chapiter 6). Roughly speaking, the Gysin map associated to a regular embedding is 'intersecting with the zero-section of the normal bundle' [11], which is in turn the inverse of the isomorphic flat pull-back of the vector bundle projection. The remarkable fact is that the l.c.i pull-back $f^!$ is independent of the choice of the factorization, see [Ful98] Proposition 6.6(a).

Proof of the Proposition. Given a morphism in $\mathrm{FSE}_X G$ as in the statement of the proposition, say,

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p_E"'] & & E' \arrow[dl, "p_{E'}"] \\ & X & \end{tikzcd}$$

To justify the name, we firstly exhibit a factorization of f into a regular embedding followed by a smooth morphism, which means, by definition, f is l.c.i.

The construction is simply the usual 'graph map':

$$E \xrightarrow{i=\mathrm{id}\times f} E \times_X E' \xrightarrow{pr_2} E'$$

[11] As for the idea that viewing a cycle of P as a cycle of the normal bundle appropriately, see [Ful98], Chapter 5.

where i is a closed embedding fitting into a diagram:

$$\begin{tikzcd} E \arrow[r, "i=\mathrm{id}\times f"] \arrow[dr, "\mathrm{id}"'] & E \times_X E' \arrow[d, "pr_1"] \\ & E \end{tikzcd}$$

One observes that pr_1 is smooth since it is a base change of $p_{E'}$, and together with the smoothness of id implies that i is a regular embedding. Similarly, thanks to the smoothness of p_E, pr_2 is smooth. Thus $f = pr_2 \circ i$ is l.c.i., of codimension $n_{E'} - n_E$.

Now consider the following square:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ E/G \arrow[r, "f_G"] & E'/G \end{tikzcd}$$

On one hand, E is a G-principal bundle implies that the square is cartesian. on the other hand, the vertical arrows are faithfully flat since E' is a principal bundle. In virtue of the faithfully flat descent property of l.c.i morphisms, We conclude that $f_G : E/G \to E'/G$ is also an l.c.i morphism of codimension $n_{E'} - n_E$. So we can apply the l.c.i. pull-back to f_G, which is recalled before the proof, and obtain: $f_G^! : A_*(E'/G) \to A_{*-n_{E'}+n_E}(E/G)$. Rewriting the two groups according to the indexing in Definition 2.5, we arrive at $f^* = f_G^! : \mathscr{A}_j^G(E') \to \mathscr{A}_j^G(E)$ as wanted. In virtue of [Ful98] Proposition 6.6(c), the functorial property is obvious by construction. The proof is complete. □

Remark 2.7 Note that if f itself is flat, then by faithfully flat descent property of flatness, we know that f_G is also flat, and that the Gysin map $f^* := f_G^!$ coincides with the flat pull-back f_G^* by [Ful98] Proposition 6.6(b).

Now we are ready for the formal definition of equivariant Chow groups.

Definition 2.8 (Equivariant Chow groups I) Let G be a linear algebraic group acting on a variety X, the j-th *equivariant Chow group* of X with respect to the G-action (or G-*equivariant Chow group*, for short), denoted $A_j^G(X)$, is defined by:

$$A_j^G(X) := \varprojlim_{U \in \mathrm{EG}^{op}} \mathscr{A}_j^G(U \times X) \tag{5}$$

In fact, the above projective limit is 'represented', or more precisely, determined by a particular U, as long as the codimension of U is large enough. In other words, we have a more concrete description for $A_j^G(X)$:

Proposition 2.9 (Equivariant Chow groups II) *Fix a nonnegative integer j. Given $U \in \mathrm{EG}_r$ with $r > |X| - j$. Then we have*

$$A_j^G(X) \xrightarrow{\simeq} \mathscr{A}_j^G(X \times U) = A_{j+|U|-|G|}(X \times^G U)$$

In particular, $\mathscr{A}_j^G(X \times U)$ is independent of $U \in \mathrm{EG}_r$ as long as r is large enough.

One notes that there is an evident difference between equivariant Chow groups and usual ones, that is, the equivariant Chow groups can have non-vanishing negative dimensional parts. See Appendix A for an easy example.

The proof relies on two basic observations, which will also be used somewhere else. For later reference, we sum them up into two lemmas:

Lemma 2.10 (Small complement) *Given $E \in \mathrm{FSE}_X G$, and a G-invariant open sub-scheme $E' \subset E$, which is also in $\mathrm{FSE}_X G$, denote $F = E - E'$ the complement. Suppose the codimension of (every irreducible component of) F in E is larger than $|X| - j$. Then the natural pull-back*

$$\mathscr{A}_j^G(E) \xrightarrow{\simeq} \mathscr{A}_j^G(E')$$

is an isomorphism.

In particular, if U' is a G-invariant open sub-scheme of U, with $U, U' \in \mathrm{EG}$ (or CEG), and (every irreducible component of)$S = U - U'$ has codimension in U larger than $|X| - j$, then the natural pull-back $\mathscr{A}_j^G(X \times U) \xrightarrow{\simeq} \mathscr{A}_j^G(X \times U')$ is an isomorphism.

Proof. It follows immediately from the short right exact sequence [Ful98] Proposition 1.8. Explicitly, let n be the relative dimension of $E \to X$ (or $E' \to X$), then E'/G is an open sub-scheme in E/G with complement F/G, thus an exact sequence:

$$A_{j+n-|G|}(F/G) \to A_{j+n-|G|}(E/G) \to A_{j+n-|G|}(E'/G) \to 0$$

Noting that $|F/G| < |E| - (|X| - j) - |G| = n + j - |G|$, we conclude that $A_{j+n-|G|}(E/G) \xrightarrow{\simeq} A_{j+n-|G|}(E'/G)$ is an isomorphism, *i.e.* $\mathscr{A}_j^G(E) \xrightarrow{\simeq} \mathscr{A}_j^G(E')$ as wanted. Applying this to the case $E = X \times U, E' = X \times U'$, $F = X \times S$, then $n = |U|$, $\mathrm{codim}(F, E) = \mathrm{codim}(S, U)$. This completes the proof. □

Lemma 2.11 (Vector bundles) *Given $E \in \mathrm{FSE}_X G$, $V \in \mathrm{Rep}(G)$, then the natural flat pull-back*

$$\mathscr{A}_j^G(E) \xrightarrow{\simeq} \mathscr{A}_j^G(E \times V)$$

is an isomorphism.

Proof. Apply the isomorphism in [Ful98] Theorem 3.3 to the vector bundle $E \times^G V \to E/G$ of rank $\dim(V)$, we obtain an isomorphism:

$$A_{j+n-|G|}(E/G) \xrightarrow{\simeq} A_{j+n-|G|+\dim V}(E \times^G V)$$

The left hand side is just $\mathscr{A}_j^G(E)$ by definition, and noting that the relative dimension of $E \times V$ over X is $n + \dim V$, the right hand side is exactly $\mathscr{A}_j^G(E \times V)$. □

Now we return to the proof of the proposition:

Proof of the Proposition. By assumption, $U \in \mathbf{EG}_r$ is an open subscheme of $V \in \mathrm{Rep}(G)$ with $S = V - U$ of codimension at least r, where $r > |X| - j$. We will show that any element $\alpha \in \mathscr{A}_j^G(X \times U)$ will determine an element in $\mathscr{A}_j^G(X \times U')$ for every $U' \in \mathbf{EG}$ in a compatible way.

For a fixed U', firstly, applying flat pull-back (Remark 2.7) to the morphism $X \times U \times U' \to X \times U$, we get an element $\widetilde{\alpha} \in \mathscr{A}_j^G(X \times U \times U')$. Then we consider the morphism $X \times U \times U' \xrightarrow{j} X \times V \times U' \xrightarrow{pr} X \times U'$. By Lemma 2.10, j induces an isomorphism $\mathscr{A}_j^G(X \times V \times U') \xrightarrow{\simeq} \mathscr{A}_j^G(X \times U \times U')$, and by Lemma 2.11, pr also induces an isomorphism $\mathscr{A}_j^G(X \times U') \xrightarrow{\simeq} \mathscr{A}_j^G(X \times V \times U')$. Thus $\widetilde{\alpha}$ determines an element in $\mathscr{A}_j^G(X \times U')$. The compatibility can be verified formally. We have defined a morphism

$$\phi : \mathscr{A}_j^G(U \times X) \to \varprojlim_{U \in \mathbf{EG}^{op}} \mathscr{A}_j^G(U \times X)$$

and we have an obvious morphism in another direction, namely the projection ψ.

$\psi \circ \phi$ is clearly identity, to complete the proof of Proposition 2.9, we should show that $\phi \circ \psi$ is also identity. While it follows from the uniqueness of the above construction of $\widetilde{\alpha}$ from α. □

For some purpose, the category $\mathbf{EG}$ is not large enough to manipulate the usual geometric constructions/operations, so it is useful to express the equivariant Chow group in terms of larger categories $\mathbf{CEG}$ and $\mathbf{FSE}_X G$, which are more adapted in some circumstance:

Proposition 2.12 (Equivariant Chow group III)

$$A_j^G(X) := \varprojlim_{U \in \mathbf{EG}^{op}} \mathscr{A}_j^G(X \times U) \cong \varprojlim_{U \in \mathbf{CEG}^{op}} \mathscr{A}_j^G(X \times U) \cong \varprojlim_{E \in \mathbf{FSE}_X G^{op}} \mathscr{A}_j^G(E)$$

Proof. Since we have natural inclusions of categories (3), it suffices to verify the following assertion:

For any *compatible* assignment $\{\alpha_{U \times X} \in \mathscr{A}_j^G(U \times X)\}_{U \in \mathbf{EG}}$, that is, for any morphism $U_1 \xrightarrow{f} U_2$, we have $f^*(\alpha_{U_2 \times X}) = \alpha_{U_1 \times X}$, there exists a unique extension of compatible assignment $\{\widetilde{\alpha}_E \in \mathscr{A}_j^G(E)\}_{E \in \mathbf{FSE}_X G}$, *i.e.* $\widetilde{\alpha}_{U \times X} = \alpha_{U \times X}$, and for any morphism in $\mathbf{FSE}_X G$, say, $E_1 \xrightarrow{f} E_2$, we have $f^*(\widetilde{\alpha}_{E_2}) = \widetilde{\alpha}_{E_1}$.

The construction is similar to the one in the proof of Proposition 2.9, we just sketch the proof. Fix the assignment $\alpha_{U \times X}$, for a given $E \in \mathbf{FSE}_X G$, we choose any $U \in \mathbf{EG}_r$,with $r > |X| - j$. Firstly we apply flat pull back to the morphism $U \times E \to U \times X$ to get the element in $\mathscr{A}_j^G(U \times E)$, and then use the isomorphism $\mathscr{A}_j^G(E) \xrightarrow{\simeq} \mathscr{A}_j^G(V \times E) \xrightarrow{\simeq} \mathscr{A}_j^G(U \times E)$ to determine an element in $\mathscr{A}_j^G(E)$, where the two isomorphisms follow from Lemma 2.11 and Lemma 2.10 respectively. The construction is clearly uniquely determined by the compatibility requirement, and it is independent of the particular U

chosen by Proposition 2.9. The compatibility can be verified by straightforward computation. Finally, we should verify $\widetilde{\alpha}_{U\times X} = \alpha_{U\times X}$. But by compatibility, we know that $\widetilde{\alpha}_{U\times X}$ and $\alpha_{U\times X}$ pull back to the same element in $\mathscr{A}^G_j(U \times U \times X)$, but the pull back is an isomorphism since $r > |X| - j$, thus $\widetilde{\alpha}_{U\times X} = \alpha_{U\times X}$. □

Remark 2.13 In the sequel, we will use the three equivalent definitions (Definition 2.8, Proposition 2.9 and Proposition 2.12) of equivariant Chow group freely. In most cases, the more concrete description in Proposition 2.9 is sufficient. While the various characterization by categorical projective limits are sometimes more convenient to prove some formal properties, and allow more constructions.

Remarks 2.14 (Ring structure on A^G in the smooth case) In certain circumstances, the equivariant Chow groups $A^G_*(X)$ has more structures than just abelian groups.

For example, when X is a smooth variety of dimension n with a G-action, we write $A^{i,G}(X) := A^G_{n-i}(X)$. Then the graded group $A^{*,G}(X)$ is in fact a graded ring, we now turn to explain the multiplication structure.

In the proof of Lemma 2.3, we can check that the base space U/G of the G-principal bundle $U \to U/G$ constructed there, is isomorphic to a homogenous space, namely GL_{N+n}/Γ, which is smooth. While the geometric fibres of the flat morphism $X\times^G U \to U/G$ is isomorphic to $X_{\bar{k}}$, which is smooth by assumption. In conclusion, $X \times^G U$ is smooth. Therefore, when U has codimension sufficiently large, we can define the multiplication

$$A^{j,G}(X) \times A^{l,G}(X) \to A^{j+l,G}(X)$$

to be the intersection product of the smooth variety $X \times^G U$ [12]

$$A^j(X \times^G U) \times A^l(X \times^G U) \to A^{j+l}(X \times^G U)$$

through the isomorphisms given in Proposition 2.9.

2.2 Functorialities: Constructions and Formal Properties

In this subsection, we want to talk about some formal functorial properties of equivariant Chow groups, some of which are analogous to the classical cases explained in [Ful98], while others, like restrictions and transfers, are related to the groups acting on the spaces. Here we still keep the technical assumption 2.1, and we will use the more concrete definition of equivariant Chow groups, namely, $A^G_j(X) = \mathscr{A}^G_j(X \times U)$ with $\operatorname{codim} U$ sufficiently large (Proposition 2.9), which leads the constructions and proofs here more geometric, but the drawback is that we should check the independence of the choice of U each time.

We first mention the following preliminary result concerning faithfully flat descent of properties of morphisms. It will be used in this and later subsections repeatedly, and we will simply refer to this result as ‘by faithfully flat descent’.

[12] [Ful98] Chapter 8.

Theorem 2.15 (Faithfully flat descent of properties of morphisms) *Let X, Y, X', Y' be schemes, such that the diagram below is cartesian:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Suppose that g is quasi-compact (which is easily satisfied in our setting) and faithfully flat, then for any of the properties of morphism $\mathscr{P}$ listed below, f has property $\mathscr{P}$ if and only if f' has property $\mathscr{P}$. $\mathscr{P}$ can be:

- *Separated;*
- *Isomorphic;*
- *Open immersion;*
- *Closed immersion;*
- *Proper;*
- *Affine;*
- *Quasi-affine;*
- *Flat;*
- *Flat of relative dimension d;*
- *Finite;*
- *Quasi-finite;*
- *Non-ramified;*
- *Étale;*
- *Smooth.*

PROOF. c.f.[Gro65, Page 29, Proposition 2.7.1]. $\square$

2.2.1 Proper Push-forward and Flat Pull-back

Let X, Y be two G-varieties satisfying Assumption 2.1, and $f : X \to Y$ be a G-equivariant morphism.

If f is proper, we construct a push-forward $f_* : A_j^G(X) \to A_j^G(Y)$ in the following way. For a fixed non-negative integer j, choose any $U \in \mathrm{EG}_r$ of dimension n, with $r > \max\{|X|, |Y|\} - j$. By Proposition 1.29, $X \times U$ and $Y \times U$ are G-principal bundles. The properness of $f \times \mathrm{id}_U : X \times U \to Y \times U$ implies that $(f \times \mathrm{id}_U)_G : X \times^G U \to Y \times^G U$ is also proper by faithfully flat descent.

In virtue of Proposition 2.9, we define the *proper push-forward* $f_* : A_j^G(X) \to A_j^G(Y)$ to be the (classical) proper push-forward ([Ful98] Theorem 1.4) associated to the morphism $X \times^G U \to Y \times^G U$, that is, $(f \times \mathrm{id}_U)_{G*} : A_{j+n-|G|}(X \times^G U) \to A_{j+n-|G|}(Y \times^G U)$

If f is flat of relative dimension d, we want to construct a pull-back $f^* : A_j^G(Y) \to A_{j+d}^G(X)$ in a similar way. For a fixed non-negative integer j, choose any $U \in \mathrm{EG}_r$ of dimension n, with $r > |Y| - j = |X| - (j + d)$. Again, by Proposition 1.29, $X \times U$ and $Y \times U$ are G-principal bundles. The flatness of $f \times \mathrm{id}_U : X \times U \to Y \times U$ implies that $X \times^G U \to Y \times^G U$ is also flat with the same relative dimension d by faithfully flat descent.

Similarly, now we can define the *flat pull-back* $f^* : A_j^G(Y) \to A_{j+d}^G(X)$ to be the (classical) flat pull-back ([Ful98] Theorem 1.7) induced by the flat morphism $(f \times \mathrm{id}_U)_G : X \times^G U \to Y \times^G U$, that is, $(f \times \mathrm{id}_U)_G^* : A_{j+n-|G|}(Y \times^G U) \to A_{j+d+n-|G|}(X \times^G U)$

Lemma 2.16 *The proper push-forward and the flat pull-back constructed above are independent of the choice of U.*

PROOF. We will focus on the proper push-forward case, the proof for flat pull-back is analogous. Fix j, for any two U, U' in EG_r, with $r > \max\{|X|, |Y|\} - j$. Consider the product to relate two constructions:

$$\begin{tikzcd}
X \times U \arrow[r, "f\times \mathrm{id}_U"] & Y \times U \\
X \times U \times U' \arrow[u] \arrow[d] \arrow[r, "f\times \mathrm{id}_U \times \mathrm{id}_{U'}"] & Y \times U \times U' \arrow[u] \arrow[d] \\
X \times U' \arrow[r, "f\times \mathrm{id}_{U'}"] & Y \times U'
\end{tikzcd}$$

By Lemma2.10 and Lemma2.11, the vertical arrows induce isomorphisms between $\mathscr{A}_j^G$, thus after passing to the quotients, the upper and bottom proper push-forwards coincide through the middle one. □

From the construction, it is clear that the proper push-forwards and the flat pull-backs have the usual functorial properties as one expected:

Proposition 2.17 (Functorialities of proper push-forward and flat pull-back) *Let X, Y, Z be G-varieties satisfying Assumption 2.1, $X \xrightarrow{f} Y \xrightarrow{g} Z$ be G-equivariant morphisms.*

1. *If f, g are proper, then*

$$(g \circ f)_* = g_* \circ f_* : A_j^G(X) \to A_j^G(Z)$$

2. *If f, g are flat of relative dimension d and e respectively, then*

$$(g \circ f)^* = f^* \circ g^* : A_j^G(Z) \to A_{j+d+e}^G(X)$$

2.2.2 l.c.i. Pull-back

As the setting in [Ful98] Section 6.6, given a cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{6}$$

where X', Y', X, Y are G-varieties satisfying Assumption 2.1, and all the morphisms are G-equivariant, and the morphism f is assumed to be l.c.i. of codimension c (see the summary before the proof of Proposition 2.6). We want to construct an equivariant pull-back $f^! : A_j^G(Y') \to A_{j-c}^G(X')$.

As above, for a fixed non-negative integer j, choose any $U \in \mathrm{EG}_r$ of dimension n, with $r > |X| - j$. Then we get a cartesian square:

$$\begin{tikzcd} X' \times^G U \arrow[r] \arrow[d] & Y' \times^G U \arrow[d] \\ X \times^G U \arrow[r] & Y \times^G U \end{tikzcd} \tag{7}$$

where the lower horizontal arrow is still l.c.i. of codimension c, by faithfully flat descent. Thanks to Proposition 2.9, we can define the l.c.i. pull-back $f^! : A_j^G(Y') \to A_{j-c}^G(X')$ to be the (classical) l.c.i. pull-back ([Ful98] section6.6) $(f \times \mathrm{id}_U)_G^! : A_{j+n-|G|}(Y' \times^G U) \to A_{j-c+n-|G|}(X' \times^G U)$.

Lemma 2.18 *The above construction of l.c.i. pull-back is independent of the choice of U.*

Proof. As in the case of proper push-forward, for two U, U' of sufficiently large codimension, we consider the cartesian square (7) with U replaced by $U \times U'$, and the morphisms of squares, which are all projections (in particular, they are flat). By the functoriality in [Ful98] Proposition 6.6 (c), we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{A}_j^G(Y' \times U') \arrow[r] \arrow[d] & \mathscr{A}_{j-c}^G(X' \times U') \arrow[d] \\ \mathscr{A}_j^G(Y' \times U \times U') \arrow[r] & \mathscr{A}_{j-c}^G(X' \times U' \times U) \\ \mathscr{A}_j^G(Y' \times U) \arrow[r] \arrow[u] & \mathscr{A}_{j-c}^G(X' \times U) \arrow[u] \end{tikzcd}$$

where the vertical arrows are all flat pull-backs, and the horizontal ones are classical l.c.i. pull-backs ([Ful98] Section 6.6). Applying Lemma 2.10 and Lemma 2.11, we find that the vertical arrows are isomorphisms, which implies that the higher and lower l.c.i.pull-backs coincide through the middle one. $\square$

As is expected, the equivariant l.c.i. pull-back constructed above satisfies the same properties as the classical ones:

Proposition 2.19 (Formal properties of l.c.i. pull-back) *We write only the morphisms, while the spaces involved will be omitted. For the precise formulations and diagrams, c.f. the non-equivariant versions in [Ful98], Chapter 6.*

(a) *If f is both l.c.i. and flat, then the l.c.i.pull-back coincides with the flat-pull back constructed in the preceding subsubsection;*

(b)(Functoriality) *Let f, f' be l.c.i. morphisms, then $(f' \circ f)^! = f^! \circ f'^!$ if the compositions are defined;*

(c)(Naturality) *l.c.i. pull-backs commute with the proper push-forwards and flat pull-backs;*

(d)(Commutativity) *l.c.i pull-backs commute with each other as long as they make sense;*

Proof. By choosing $U \in \mathrm{EG}$ of sufficiently large codimension, taking product of the corresponding diagram with U, then passing to the quotients, all the assumptions remain (being cartesian, properness, flatness, being l.c.i. all have the property of faithfully flat descent) thus these assertions follow immediately from the non-equivariant cases (see [Ful98] Chapter 6). □

2.2.3 Equivariant Chern Classes

Let X be a G-variety satisfying Assumption 2.1 as above. A *G-equivariant vector bundle* $\pi : E \to X$ is by definition a vector bundle such that the bundle space and the base space are equipped with G-actions, and the bundle map p is G-equivariant. Given an equivariant vector bundle, we shall construct the equivariant Chern classes $c_i^G(E)$ operating on the equivariant Chow group $A_j^G(X)$ decreasing the degree by i.

Following the same pattern of the preceding subsections, for any fixed non-negative integer j, take $U \in \mathrm{EG}_r$ of dimension n, with $r > |X| - j + i$, then $E \times U \to X \times U$ is also a G-equivariant vector bundle. By the argument in [EG98] Lemma 2.4.1, $E \times^G U \to X \times^G U$ is a vector bundle. We define the the operation of the i^{th} *G-equivariant Chern class* of E on $A_j^G(X)$, denoted as $c_i^G(E)\cap : A_j^G(X) \to A_{j-i}^G(E)$, to be the usual Chern class operation $c_i(E \times^G U)\cap : A_{j+n-|G|}(X \times^G U) \to A_{j-i+n-|G|}(X \times^G U)$.

As before, we have:

Lemma 2.20 *The above construction is independent of the choice of U.*

Proof. For two choices U and U', consider the cartesian squares consisting of G-principal bundles:

$$\begin{tikzcd} E \times U \arrow[d] & E \times U \times U' \arrow[l] \arrow[r] \arrow[d] & E \times U' \arrow[d] \\ X \times U & X \times U \times U' \arrow[l] \arrow[r] & X \times U' \end{tikzcd}$$

When passing to the quotient, the two squares remain cartesian, that is, the vector bundle $E \times U \times U'/G \to X \times U \times U'/G$ is the pull-back of $E \times^G U \to X \times^G U$, and the pull-back of $E \times^G U' \to X \times^G U'$, too. By the pull-back property of the Chern classes operations ([Ful98] Theorem 3.2(d)), we conclude that the two constructions using U and U' coincide through the construction using $U \times U'$.

□

Now we extend the basic properties of Chern classes to the equivariant version. Recall the definition of *Chern polynomials*, whose coefficients are in fact operations on equivariant Chow groups:

$$c_t^G(E) := \sum_i c_i^G(E) t^i.$$

Proposition 2.21 (Formal properties of equivariant Chern classes) *Let E, E', E'', F be G-equivariant vector bundles over X, then*

(a)(Vanishing) *For all $i > rank(E)$ or $i < 0$, $c_i^G(E) = 0$;*

(b)(Commutativity) *For any $\alpha \in A_l^G(X)$, we have*

$$c_i^G(E) \cap c_j^G(F) \cap \alpha = c_j^G(F) \cap c_i^G(E) \cap \alpha$$

(c)(Projection formula) *If $f : X' \to X$ is a G-equivariant proper morphism, and f^*E is the pull-back along f of E. Then for any $\alpha \in A_j^G(X')$,*

$$f_*(c_i^G(f^*E) \cap \alpha) = c_i^G(E) \cap f_*(\alpha)$$

(d)(Pull-back) *Let $f : X' \to X$ be a G-equivariant flat (or more generally, l.c.i) morphism, and f^*E is the pull-back equivariant vector-bundle. Then for any $\alpha \in A_j^G(X)$,*

$$c_i^G(f^*E) \cap f^*(\alpha) = f^*(c_i^G(E) \cap \alpha)$$

or just simply write: $c_i^G(f^*(E)) = f^*(c_i^G(E))$

(e)(Whitney sum) *For any exact sequence $0 \to E' \to E \to E'' \to 0$ of equivariant vector bundles over X, we have* [13]

$$c_t^G(E) = c_t^G(E') \cdot c_t^G(E'').$$

[13] Here the equality and product are understood as the equality and product of polynomials in t, with the multiplications for coefficients given by compositions of equivariant Chern classes as operations, in virtue of (b)commutativity, the product is well-defined. Of course, in the case that X is smooth, the product is just the intersection product explained in Remark 2.14.

PROOF. By the same argument in the proof of Proposition 2.19, this proposition is derived from the classical case ([Ful98]Theorem 3.2) immediately. □

Now we turn to the useful equivariant version of the *excess intersection formula*:

Theorem 2.22 (Excess intersection formula: equivariant version) *Let X, X', X'', Y, Y', Y'' be G-equivariant varieties, all the morphisms of the following diagram are G-equivariant, and the two squares are both cartesian.*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "q"'] & Y'' \arrow[d, "p"] \\ X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

where i and i' are regular embeddings of codimension d and d' respectively, with normal bundles $N = N_{X/Y}$ and $N' = N_{X'/Y'}$ respectively. There is canonical embedding of bundles $N' \hookrightarrow g^*N$, the excess normal bundle *$E = g^*N/N'$ is a vector bundle of rank $e = d - d'$. Then*

$$i^! = c_e^G(q^*E) \cap i'^! : A_*(Y'') \to A_{*-d}(X'')$$

where $i^!, i'^!$ are equivariant l.c.i. pull-backs, and c_e^G is the equivariant Chern class. This theorem is also valid with i and i' being l.c.i.

PROOF. After a routine translation, it follows immediately from the classical excess intersection formula [Ful98] Theorem 6.3, and Theorem 6.6(c). □

Remark 2.23 *The excess intersection formula is also valid with i and i' being l.c.i.*

Finally, let us calculate an example which will be used later:

Example 2.24 Let R be the standard representation of $\mathbf{Z}/p$. [14] If we view R as an $\mathbf{Z}/p$-equivariant vector bundle of rank $p-1$ over a point, then the top equivariant Chern class

$$c_{p-1}^{\mathbf{Z}/p}(R) = -l^{p-1} =: \eta$$

in $A_{1-p}^{\mathbf{Z}/p}(\mathrm{pt}) = \mathbf{F}_p \cdot l^{p-1}$, (see Appendix A for the calculation of the $\mathbf{Z}/p$-equivariant Chow groups of a point).

[14] That is, the quotient of the regular representation by the one-dimensional invariant subrepresentation generated by the sum of the base elements. Note that R is $(p-1)$-dimensional.

(Indication of the proof: Fix a p-th root of unity ζ, the representation R splits into one-dimensional representations as $R = \oplus_{j=1}^{p-1} L_j$, where the action of $1 \bmod p \in \mathbf{Z}/p$ on L_j is multiplying by ζ^j. Therefore

$$
\begin{aligned}
c_{p-1}^{\mathbf{Z}/p}(R) &= \prod_{j=1}^{p-1} c_1^{\mathbf{Z}/p}(L_j) \\
&= \prod_{j=1}^{p-1} c_1^{\mathbf{Z}/p}(L_1^{\otimes j}) \\
&= \prod_{j=1}^{p-1} j c_1^{\mathbf{Z}/p}(L_1) \\
&= (p-1)! \cdot (c_1^{\mathbf{Z}/p}(L_1))^{p-1} \\
&= -l^{p-1}
\end{aligned}
$$

the last equality is by Wilson's theorem and the fact that the $(p-1)$-th power of any element in $(\mathbf{Z}/p)^*$ is 1.)

Now we turn to the constructions involving the change of the group.

2.2.4 Restrictions and Transfers

Let X be a variety with an action of a linear algebraic group G satisfying Assumption 2.1 as above. Let $H \hookrightarrow G$ be a closed subgroup of codimension d. Recall that G/H is always quasi-projective. In the context concerning transfers, we will further assume that *the quotient G/H is proper hence projective* (of dimension d), which means that the subgroup H is parabolic. The action of H on X is always assumed to be the one induced from the action of G, *i.e.* $H \times X \hookrightarrow G \times X \xrightarrow{\sigma} X$. We want to construct the morphisms analogous to the restrictions and transfers for equivaraint cohomology:

$$\mathrm{Res}_H^G : A_j^G(X) \to A_j^H(X)$$

$$\mathrm{Tr}_H^G : A_j^H(X) \to A_{j+d}^G(X)$$

First of all, we need a lemma which gives the *fundamental fibrations* relating the G-principal bundles to H-principal bundles. To illustrate the idea, let us compare the case of Lie groups: if G is a Lie group acting freely on a manifold E, giving a G-principal bundle [15] $E \to E/G$, then for any closed subgroup H, H is also a Lie group acting on E freely, thus giving an H-principal bundle $E \to E/H$. Moreover, we have the *fundamental fibration*

$$G/H \hookrightarrow E/H \to E/G,$$

which means that E/H is a fibre bundle over E/G, with fibre type G/H. In fact, E/H is the fibre bundle associated to the G-principal bundle $E \to E/G$ with G/H the (non-linear) representation of G:

$$E/H = E \times^G G/H.$$

[15] Here is the notion of principal bundles in differential geometry, c.f. [KN96]

c.f. [KN96] Chapter 1, Section 5 for this standard procedure in differential geometry.

Now we give the following lemma recovering certain part of the above assertions in the setting of algebraic geometry.

Lemma 2.25 (Fundamental fibrations) *Let $\pi : E \to B$ be a G-principal bundle with base space $B = E/G$, H be a closed subgroup of G, acting on E via G. Then*

1. *The geometric quotient E/H exists, and $E/H = E \times^G G/H$;*

2. *$\phi : E \to E/H$ is an H-principal bundle;*

3. *We have a canonical map $\phi : E/H \to E/G$, which is faithfully flat of relative dimension $|G/H| = |G| - |H| = d$.*

4. *If H is a parabolic subgroup of G (i.e. G/H is projective), then ϕ is proper.*

Proof. The argument here is due to B.Kahn. Consider the right square of the following diagram:

$$\begin{tikzcd}
E \times G \arrow[r, "\simeq"] \arrow[d, "pr_1"'] & E \times G \arrow[r, "\gamma"] \arrow[d, "\alpha"] & E \times G/H \arrow[d, "\beta"] \\
E \arrow[r, "\simeq"] & E \times^G G \arrow[r, "\pi"] & E \times^G G/H
\end{tikzcd}$$

In this diagram:

- The middle $E \times G$ is equipped with the diagonal G-action, and a right H-action defined by right translation on the second factor G, note that the two actions are commutative;

- Since $p : G \to G/H$ is an H-principal bundle (Example 1.24), so is $\gamma = \mathrm{id} \times p : E \times G \to E \times G/H$, and $E \times G/H$ has a diagonal G-action inherited from $E \times G$;

- Using the trick of Example 1.28, we know that the geometric quotient $\frac{E \times G}{G} = E \times^G G$ exists, and $\alpha : E \times G \to E \times^G G$ is a G-principal bundle. Moreover, this G-principal bundle is isomorphic to the trivial G-principal bundle $pr_1 : E \times G \to E$ (Here $E \times G$ has the G-action of left translation on the second factor), hence the left square of the above diagram with the isomorphisms as indicated;

- Since E is a G-principal bundle, and G/H is quasi-projective, whose G-action can be lineared, then thanks to Proposition 1.29, $E \times G/H$ is the total space of a G-principal bundle, *i.e.* the right vertical arrow, $\beta : E \times G/H \to E \times^G G/H$ is a G-principal bundle;

- Since γ is G-equivariant map between two G-principal bundles, we get a morphism between their base spaces: $\pi : E \times^G G \to E \times^G G/H$, which makes the right square of the diagram *cartisian*. Now by faithfully flat descent (Proposition 1.15), and the fact that γ is an H-principal bundle, we conclude that the bottom right arrow π is also an H-principal bundle.

- Taking the isomorphisms of the left square into account, we find that the composition of the two morphisms of the bottom line, still denoted π, is an H-principal bundle. While the transport of the H-action back onto E is the natural right action of H on E associated to the natural left H-action defined via G. Therefore, the geometric quotient E/H exists, and $E/H = E \times^G G/H$, this proves 1, and moreover, $\pi : E \to E/H$ is an H-principal bundle, this proves 2.

As for the third and fourth assertions in the lemma, consider the natural map ('the fundamental fibration'):

$$\phi : E/H = E \times^G G/H \xrightarrow{\overline{pr_1}} E/G,$$

when pulling back to E by the bundle map $E \to E/G$, the above displayed morphism becomes:

$$\widetilde{\phi} : E \times_{E/G} E/H = (E \times_{E/G} E) \times^G G/H = E \times G \times^G G/H = E \times G/H \to E$$

which is simply the first projection, thus $\widetilde{\phi}$ is faithfully flat of relative dimension $|G/H| = |G| - |H| = d$, and when G/H is projective, hence proper, so is $\widetilde{\phi}$. Now we can deduce for ϕ the properties wanted, because the bundle map $E \to E/G$ is faithfully flat, and the properties of morphism considered, namely faithfully flatness of a given relative dimension, properness, have the permanence property with respect to faithfully flat morphisms that is, they can be tested after any faithfully flat base change. □

From the above lemma, we deduce directly the following corollaries.

Corollary 2.26 *1. There are natural functors* $\mathrm{EG}_r \to \mathrm{EH}_r$, $\mathrm{CEG} \to \mathrm{CEH}$, $\mathrm{FSE}_X G \to \mathrm{FSE}_X H$, *given by restricting the action of* G *to* H.

2. For any $E \in \mathrm{FSE}_X G$, *the natural morphism* $\phi : E/H \to E/G$ *is faithfully flat of relative dimension* $|G/H| = |G| - |H| = d$;

3. If H *is a parabolic subgroup of* G, *then for any* $E \in \mathrm{FSE}_X G$, *the natural morphism* $\phi : E/H \to E/G$ *is proper.*

Now we are ready for the constructions of restrictions and transfers.

Take any $U \in \mathrm{EG}_r$ of dimension n, with r large enough. By Corollary 2.26 (1), we can also view U as an object in EH_r. Apply Corollary 2.26(2) to $E = X \times U \in \mathrm{FSE}_X G$, we find that the natural map $X \times^H U \xrightarrow{\phi} X \times^G U$ is faithfully flat of relative dimension $|G/H| = |G| - |H| = d$. So we can define the *restriction map*

$$\mathrm{Res}^G_H : A^G_j(X) \to A^H_j(X)$$

to be the flat pull-back $\phi^* : A_{j+n-|G|}(X \times^G U) \to A_{j+n-|H|}(X \times^H U)$.

If moreover H is a parabolic subgroup of G, by Corollary 2.26(3), the natural map $X \times^H U \xrightarrow{\phi} X \times^G U$ is proper. Therefore we can define the *transfer map*

$$\mathrm{Tr}^G_H : A^H_j(X) \to A^G_{j+d}(X)$$

to be the proper push-forward $\phi_* : A_{j+n-|H|}(X \times^H U) \to A_{j+n-|H|}(X \times^G U)$, noting that $j + n - |H| = j + d + n - |G|$.

By exactly the same method as before, we have:

Lemma 2.27 *The above constructions are independent of the choice of U.*

Here we have the expected properties as in the equivariant cohomology case:

Proposition 2.28 (Formal properties of restriction and transfer) *Let G be an algebraic group, H be a (closed) subgroup of G of codimension d, and L be a (closed) subgroup of H of codimension e. Let X, X', Y, Y' be G-varieties as above, with the G-equivariant cartesian square below:*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We assume that $f : X \to Y$ is l.c.i. of codimension c.

(a)(Functoriality)

$$\mathrm{Res}^H_L \circ \mathrm{Res}^G_H = \mathrm{Res}^G_L : A^G_j(X) \to A^L_j(X);$$

If furthermore H is a parabolic subgroup of G, and L is a parabolic subgroup of H, then:

$$\mathrm{Tr}^G_H \circ \mathrm{Tr}^H_L = \mathrm{Tr}^G_L : A^L_j(X) \to A^G_{j+d+e}(X).$$

(b)(Degree map) *If we assume further that G/H is (constant) finite of degree N, i.e. $[G : H] = N$, then $\mathrm{Tr}^G_H \circ \mathrm{Res}^G_H$ is the map of multiply by N on $A^G_j(X)$*

(c)(l.c.i. Pull-backs)

$$f^!_H \circ \mathrm{Res}^G_H = \mathrm{Res}^G_H \circ f^!_G : A^G_j(Y') \to A^H_{j-c}(X');$$

If furthermore H is a parabolic subgroup of G, then:

$$f^!_G \circ \mathrm{Tr}^G_H = \mathrm{Tr}^G_H \circ f^!_H : A^H_j(Y') \to A^G_{j-c+d}(X')$$

where the subscripts of f is used to indicate the group involved when we do equivaraint l.c.i. pull-backs.

(d)(Restriction and transfer of Chern classes) *If E is a G-equivariant vector bundle over X, then*

$$\mathrm{Res}_H^G \circ c_i^G(E)\cap = c_i^H(E) \cap \circ \mathrm{Res}_H^G : A_j^G(X) \to A_{j-i}^H(X);$$

If furthermore H is a parabolic subgroup of G, then:

$$\mathrm{Tr}_H^G \circ c_i^H(E)\cap = c_i^G(E) \cap \circ \mathrm{Tr}_H^G : A_j^H(X) \to A_{k-i+d}^G(X)$$

Proof. Fix a $U \in \mathrm{EG}_r$ of dimension n, with r sufficiently large.

The two equalities in (a) follow from the functoriality of flat pull-back and proper push-forward respectively, applied to the composition of the two 'fundamental fibrations':

$$\frac{X \times U}{L} \to \frac{X \times U}{H} \to \frac{X \times U}{G}$$

For (b), by Corollary 2.26(2)(3), with $E = X \times U$, we have [16]: $X \times^H U \xrightarrow{\phi} X \times^G U$ is faithfully flat, finite of degree $[G : H] = N$, which implies $\phi_* \circ \phi^*$ is the map of multiply by N by [Ful98] Example 1.7.4. This concludes (b) by the constructions of Res_H^G and Tr_H^G.

For the first assertion of (c), we need the following cartesian diagram:

$$\begin{tikzcd}
\frac{X'\times U}{H} \arrow[r] \arrow[d, "\phi_{X'}"'] & \frac{Y'\times U}{H} \arrow[d, "\phi_{Y'}"] \\
\frac{X'\times U}{G} \arrow[r] \arrow[d] & \frac{Y'\times U}{G} \arrow[d] \\
\frac{X\times U}{G} \arrow[r, "f_G"'] & \frac{Y\times U}{G}
\end{tikzcd}$$

where the upper vertical arrows are the natural maps, the 'fundamental fibrations', explained in Lemma 2.25 and Corollary 2.26, and the lower horizontal arrow f_G is l.c.i. since f is. And [Ful98] Theorem 6.2(b)(or rather Proposition 6.6(c)) tells us that:

$$\phi_{X'}^* \circ f_G^! = f_G^! \circ \phi_{Y'}^* : A_*(Y' \times^G U) \to A_*(X' \times^H U)$$

By the definition of Res_H^G, we only need to show in the following diagram that

$$f_G^! = f_H^! : A_*(Y' \times^H U) \to A_*(X' \times^H U)$$

$$\begin{tikzcd}
\frac{X'\times U}{H} \arrow[r] \arrow[d] & \frac{Y'\times U}{H} \arrow[d] \\
\frac{X\times U}{H} \arrow[r, "f_H"] \arrow[d] & \frac{Y\times U}{H} \arrow[d] \\
\frac{X\times U}{G} \arrow[r, "f_G"'] & \frac{Y\times U}{G}
\end{tikzcd}$$

[16] In fact, to get the following information of the degree, we need to trace back to the last steps of the proof of Lemma 2.25.

But this follows from [Ful98] Theorem 6.2(c) (or rather Proposition 6.6(c)). The proof of the second assertion of (c) is the same, replacing [Ful98] Proposition6.2(b) by 6.2(c) applied to proper push-forward instead of flat pull-back.

For the first assertion of (d), we consider the following cartesian diagram:

$$\begin{tikzcd} E \times^H U \arrow[r] \arrow[d] & E \times^G U \arrow[d] \\ X \times^H U \arrow[r, "\phi"] & X \times^G U \end{tikzcd}$$

i.e. it is a pull-back of vector bundle: $\phi^*(E \times^G U) = E \times^H U$. By the pull-back property of Chern classes [Ful98]Theorem 3.2(d), we have

$$\phi^* \circ c_i(E \times^G U) \cap = c_i(E \times^H U) \cap \circ \phi^*$$

which is exactly what we want by the definition of equivariant Chern classes and restriction map. For the second assertion of (d), we use the projection formula of Chern classes [Ful98] Theorem 3.2(c) instead of the pull-back property above, we get

$$\phi_* \circ c_i(E \times^H U) \cap = c_i(E \times^G U) \cap \circ \phi_*$$

as wanted. □

2.3 Equivariant Cycle Class Map

We write $Z_i(X)$ for the group consisting of i-dimensional cycles of X. In the usual (non-equivariant) case, we have the cycle class map

$$Z_i(X) \xrightarrow{\mathrm{cl}} A_i(X)$$

In this section, we want to define the equivariant analogy, $Z_i(X)^G \xrightarrow{\mathrm{cl}^G} A_i^G(X)$, in certain circumstances.

Throughout this subsection, X is an algebraic variety, G is a constant finite group acting on X satisfying Assumption 2.1. Define the group of *G-invariant cycles*:

$$Z_i(X)^G := \{z \in Z_i(X) | g \cdot z = z, \forall g \in G\}$$

Now we turn to the construction, fix a non-negative integer i.

Construction 2.29 (Cycle class map) As usual, we choose $U \in \mathrm{EG}_r$ of dimension n, with r large enough. Then by Proposition 1.29, $X \times U$ is a G-principal bundle, and $A_i^G(X) \simeq A_{i+n}(X \times^G U)$. We define the *(equivariant)cycle class map* $\mathrm{cl}^G : Z_i(X)^G \to A_i^G(X)$ to be the composition:

$$\begin{tikzcd} Z_i(X)^G \arrow[r, "\mathrm{cl}^G"] \arrow[d] & A_i^G(X) = A_{i+n}(X \times^G U) \\ Z_{i+n}(X \times U)^G \arrow[r, "\simeq"] & Z_{i+n}(X \times^G U) \arrow[u] \end{tikzcd}$$

where for any $\alpha \in Z_i(X)^G$, a G-invariant cycle of X, the left vertical arrow takes it to a G-invariant cycle of $X\times U$, namely $\alpha\times[U]$; and then the bottom horizontal arrow, which is the inverse of the isomorphic pull-back $Z_{i+n-|G|}(X \times^G U) \to Z_{i+n}(X \times U)^G$, takes it to a cycle of $X \times^G U$, which gives rise to an element of $A_i^G(X)$, *i.e.* an equivariant cycle of X. If there is no risk of confusion, we will write cl instead of cl^G.

It is obvious that the cycle class map such defined is a homomorphism, and we have the following routine lemma:

Lemma 2.30 *The above construction is independent of the choice of U.*

To illustrate the meaning of the equivariant cycle class map and for later reference, we prove the following proposition.

Proposition 2.31 *Let X be a G-space, where G is a constant finite group. then for every $\alpha \in Z_i(X)$, we have $\mathrm{Tr}_1^G(\mathrm{cl}(\alpha)) = \mathrm{cl}^G(\sum_{g\in G} g \cdot \alpha)$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
Z_i(X) \arrow[r, "\mathrm{cl}"] \arrow[d, "\sum_{g\in G} g\cdot"'] & A_i(X) \arrow[d, "\mathrm{Tr}_1^G"] \\
Z_i(X)^G \arrow[r, "\mathrm{cl}^G"] & A_i^G(X)
\end{tikzcd}$$

Proof. Choose $U \in \mathrm{EG}_r$ of dimension n with r sufficiently large, fix $\alpha \in Z_i(X)$. Consider the covering space (fiber bundle with discrete fiber) $X\times U \xrightarrow{\phi} X\times^G U$. Since $\sum_{g\in G}(g \cdot \alpha) \times [U] = \sum_{g\in G} g \cdot (\alpha \times [U]) = \phi^*\phi_*(\alpha \times [U])$, by the definition of cl^G,

$$\mathrm{cl}^G(\sum_{g\in G} g \cdot \alpha) = \phi_*(\alpha \times [U])$$

which is $\mathrm{Tr}_1^G(\mathrm{cl}(\alpha))$ by the definition of transfer. The proof is complete. □

3 Steenrod Operations on Smooth Varieties

In this section, we begin to talk about the Steenrod operations over cohomology groups and Chow groups, with the coefficient field being $\mathbf{F}_p$.

In the first subsection, the Steenrod operations in the context of algebraic topology are discussed, the Steenrod algebra is defined. And then in the second subsection, we introduce the reduced Steenrod algebra, which is better adapted to the current setting. In the third subsection, the formal properties of Steenrod operations on Chow groups for smooth varieties are summarized. Finally, in the fourth subsection, we give the explicit construction of Steenrod operations and also the proofs of the results in the third subsection.

Throughout the rest of the note, all the Chow groups and cohomology groups unless otherwise stated are assumed to be with coefficients $\mathbf{F}_p$.

3.1 Steenrod Operations in Algebraic Topology

We will talk about the origin of our construction in algebraic topology, which is developed by N.E.Steenrod [Ste62].

First of all, let us give some general ideas of cohomological operations:

Definition 3.1 (Cohomological Operations) Let F, F' be abelian groups. Consider the cohomology functors $H^*(-;F)$, $H^*(-;F')$. A *cohomological operation* is a natural transformation of functors $H^i(-;F) \to H^j(-;F')$

Lemma 3.2 *The group of all cohomological operations from $H^i(-;F)$ to $H^j(-;F')$ is isomorphic to the group $H^j(K(F,i);F')$, i.e.*

$$\mathrm{Hom}_{fct}(H^i(-;F), H^j(-;F')) \cong H^j(K(F,i);F')$$

where $K(F,i)$ is the Eilenberg-Maclane space.

Proof. Recall that the cohomology functor is representable by the spectrum of Eilenberg-Maclane space, that is, $H^i(-,F) = [-, K(F,i)]$, where $[\ ,\ ]$ denotes the **Hom** (bi-)functor in the pointed homotopy category. So we deduce the result formally as following:

$$\begin{array}{rcll} \mathrm{Hom}(H^i(-;F), H^j(-;F')) = & \mathrm{Hom}([-,K(F,i)],[-,K(F',j)]) & \text{(Representability)} \\ = & [K(F,i), K(F',j)] & \text{(Yoneda Lemma)} \\ = & H^j(K(F,i);F') & \text{(Representability)} \end{array}$$

□

Remark 3.3 We can extend the preceding definition and lemma to any generalized cohomological theory, replacing the Eilenberg-Maclane spectrum by another one.

Let us focus on the singular cohomolgy case. We notice that the Eilenberg-Maclane space $K(F,i)$ can be constructed with cells of dimension at least i, so $H^j(K(F,i);F') = 0$ for $0 < j < i$ [17], which means that any nontrivial cohomological operations cannot decrease the degree.

Here are some examples:

Example 3.4 **(a)(Change of coefficients)** If $F \to F'$ is a homomorphism of abelian groups, we have $H^i(X,F) \to H^i(X,F')$, the change of coefficients map, which is obviously a cohomological operation;

(b)(Multiplication) If the coefficient group F is moreover a ring, then thanks to the ring structure of the cohomology, we have that $H^i(X,F) \to H^{2i}(X,F)$ which takes any element to its square, is a cohomological operation;

[17] This can also be derived from the theorem of Hurewicz.

(c)(Bockstein homomorphism) If the coefficient group F is taken to be $\mathbf{Z}/p$, where p is prime, we have the Bockstein homomorphism $H^i(X; \mathbf{Z}/p) \to H^{i+1}(X; \mathbf{Z}/p)$ which is the connecting map in the long exact sequence associated to the short exact sequence of the coefficient groups: $0 \to \mathbf{Z}/p \xrightarrow{\cdot p} \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$;

(d)(Steenrod squares) If the coefficient ring $F = \mathbf{Z}/2$, then Steenrod has defined a series of cohomological operations: $\mathrm{Sq}^i : H^q(X, \mathbf{Z}/2) \to H^{q+i}(X, \mathbf{Z}/2)$, called *Steenrod squares*, see the discussion below for the details;

(e)(Steenrod's reduced powers) If the coefficient ring $F = \mathbf{Z}/p$, where p is an odd prime number, then Steenrod has defined a series of cohomological operations: $\mathrm{P}^i : H^q(X, \mathbf{Z}/p) \to H^{q+2i(p-1)}(X, \mathbf{Z}/p)$, called *reduced powers*, see the discussion below for the details;

We are mainly interested in the last three operations in the above collection of examples. There are several reasons for this, one is that we want to take account of compositions and relations of such operations, the convenient way is to make it into an algebra of 'operations' acting on the same cohomology ring, so we want the coefficient group to be fixed, so (a) is excluded. Another reason is that we want our operations to be *stable*, which means 'commuting with suspension', while (b) does not satisfy such property. On the other hand, the Steenrod squares when in coefficient $\mathbf{Z}/2$ and the reduced powers when in coefficient $\mathbf{Z}/p$ are stable, and the Bockstein homomorphism is 'anti-stable'. (See the three theorems below).

Now we sum up the basic properties of the three cohomological operations into three theorems, for the proofs we refer to the book of Steenrod[Ste62].

Theorem 3.5 (Bockstein homomorphisms) *Fix a prime number p. For any pair of topological spaces (X, A), we have $\beta : H^q(X, A; \mathbf{Z}/p) \to H^{q+1}(X, A; \mathbf{Z}/p)$ for any non-negative integer q, satisfying:*

(a)(Functoriality) *β is functorial with respect to the pair of spaces, more precisely, for any morphism f between pairs of topological spaces, write f^* for the induced morphism on cohomological rings, then*

$$f^*\beta = \beta f^*$$

thus β is a cohomological operation;

(b) $\beta^2 = 0$*;*

(c) *β **anti-commutes** with the connecting morphism δ in the long exact sequence associated to a pair of spaces:*

$$\beta\delta = -\delta\beta$$

(d)(Anti-stable) *We denote the suspension isomorphism as $\Sigma : H^q(X; \mathbf{Z}/p) \to H^{q+1}(\Sigma X; \mathbf{Z}/p)$, then we have*

$$\Sigma\beta = -\beta\Sigma$$

(e)(Cartan formula) *For any* $x \in H^q(X, A; \mathbf{Z}/p)$ *and* $y \in H^r(X, B; \mathbf{Z}/p)$,

$$\beta(x \cup y) = \beta(x) \cup y + (-1)^q x \cup \beta(y)$$

Theorem 3.6 ($\mathbf{Z}/2$ coefficient, Steenrod Squares) *For any pair of topological spaces* (X, A)*, and for any integer* i*, we have an operation*

$$\mathrm{Sq}^i : H^q(X, A; \mathbf{Z}/2) \to H^{q+i}(X, A; \mathbf{Z}/2)$$

for any non-negative integer q*, satisfying:*

(a)(Functoriality) Sq^i *is functorial with respect to the pair of spaces, more precisely, for any morphism* f *between pairs of topological spaces, write* f^* *for the induced morphism on cohomological rings, then*

$$f^* \mathrm{Sq}^i = \mathrm{Sq}^i f^*$$

i.e. Sq^i *is a cohomological operation;*

(b)(Naturality) Sq^i *commutes with the connecting morphism in the long exact sequence associated to a pair of spaces;*

(c)(Range) $\mathrm{Sq}^i : H^q(X, A; \mathbf{Z}/2) \to H^{q+i}(X, A; \mathbf{Z}/2)$ *is zero when* $i > q$ *or* $i < 0$*;*

(d)(Square) $\mathrm{Sq}^i : H^i(X, A; \mathbf{Z}/2) \to H^{2i}(X, A; \mathbf{Z}/2)$ *is the square map, i.e. it takes any element* x *to* $x^2 = x \cup x$*;*

(e) $\mathrm{Sq}^0 = \mathrm{id}$*;* $\mathrm{Sq}^1 = \beta$ *the Bockstein homomorphism;*

(f)(Cartan formula) *For any element* $x \in H^q(X, A; \mathbf{Z}/2)$ *and* $y \in H^r(X, B; \mathbf{Z}/2)$*, we have*

$$\mathrm{Sq}^k(x \cup y) = \sum_{i=0}^{k} \mathrm{Sq}^i(x) \cup \mathrm{Sq}^{k-i}(y)$$

in $H^{q+r+k}(X, A \cup B; \mathbf{Z}/2)$*;*

(g)(Stability) *We write the suspension isomorphism as* $\Sigma : H^q(X; \mathbf{Z}/2) \to H^{q+1}(\Sigma X; \mathbf{Z}/2)$*, then we have*

$$\Sigma \mathrm{Sq}^i = \mathrm{Sq}^i \Sigma$$

(h)(Adem's relation) *For any* $0 < a < 2b$*, we have*

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j \tag{8}$$

Theorem 3.7 ($\mathbf{Z}/p$ coefficient, Steenrod reduced powers) *For any pair of topological spaces* (X, A)*, and for any integer* i*, we have*

$$\mathrm{P}^i : H^q(X, A; \mathbf{Z}/p) \to H^{q+2i(p-1)}(X, A; \mathbf{Z}/p)$$

for any non-negative integer q*, satisfying:*

(a)(Functoriality) P^i *is functorial with respect to the pair of spaces, more precisely, for any morphism f between pairs of topological spaces, write f^* for the induced morphism on cohomological rings, then*

$$f^* \mathrm{P}^i = \mathrm{P}^i f^*$$

i.e. P^i *is a cohomological operation;*

(b)(Naturality) P^i *commutes with the connecting morphism in the long exact sequence associated to a pair of spaces;*

(c)(Range) $\mathrm{P}^i : H^q(X, A; \mathbf{Z}/p) \to H^{q+2i(p-1)}(X, A; \mathbf{Z}/p)$ *is zero when* $2i > q$ *or* $i < 0$*;*

(d)($p^{\mathbf{th}}$-power) $\mathrm{P}^q : H^{2q}(X, A; \mathbf{Z}/p) \to H^{2pq}(X, A; \mathbf{Z}/p)$ *is the p^{th}-power map, i.e. it takes any element x to x^p;*

(e) $\mathrm{P}^0 = \mathrm{id}$*;*

(f)(Cartan formula) *For any elements $x \in H^q(X, A; \mathbf{Z}/p)$ and $y \in H^r(X, B; \mathbf{Z}/p)$, we have*

$$\mathrm{P}^k(x \cup y) = \sum_{i=0}^{k} \mathrm{P}^i(x) \cup \mathrm{P}^{k-i}(y)$$

in $H^{q+r+2k(p-1)}(X, A \cup B; \mathbf{Z}/p)$;

(g)(Stability) *We write the suspension isomorphism as $\Sigma : H^q(X; \mathbf{Z}/p) \to H^{q+1}(\Sigma X; \mathbf{Z}/p)$, then we have*

$$\Sigma \mathrm{P}^i = \mathrm{P}^i \Sigma$$

(h)(Adem's relations) *For any $0 < a < pb$, we have*

$$\mathrm{P}^a \mathrm{P}^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \mathrm{P}^{a+b-j} \mathrm{P}^j \tag{9}$$

For any $a \leq b$ we have

$$\mathrm{P}^a \beta \mathrm{P}^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta \mathrm{P}^{a+b-j} \mathrm{P}^j + \sum_{j=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \mathrm{P}^{a+b-j} \beta \mathrm{P}^j \tag{10}$$

Notice: If one takes $p = 2$ in (9), one gets (8), in other words, they are the same.

To better summarize the above constructions by Steenrod, we introduce the notion of Steenrod algebra:

Definition 3.8 (Steenrod algebra $\mathscr{A}$) For any prime number, we define a Steenrod algebra.

$\mathcal{A}(2)$**:** The *Steenrod algebra* for the prime 2, denoted $\mathcal{A}(2)$, is by definition the graded associative algebra of generators $\{\mathrm{Sq}^i\}_{i\geq 0}$ and β, subject to the Adem relations (8):

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2\rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j$$

and

$$\mathrm{Sq}^1 = \beta; \mathrm{Sq}^0 = 1$$

The grading of $\mathcal{A}(2)$ is determined by $\deg(\mathrm{Sq}^i) = i$;

$\mathcal{A}(p)$**:** The *Steenrod algebra* for the odd prime p, denoted $\mathcal{A}(p)$, is by definition the graded associative algebra of generators $\{\mathrm{P}^i\}_{i\geq 0}$ and β, and subject to the Adem relations (9) and (10):

$$\mathrm{P}^a\,\mathrm{P}^b = \sum_{j=0}^{\lfloor a/p\rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \mathrm{P}^{a+b-j}\,\mathrm{P}^j$$

$$\mathrm{P}^a\,\beta\,\mathrm{P}^b = \sum_{j=0}^{\lfloor a/p\rfloor} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta\,\mathrm{P}^{a+b-j}\,\mathrm{P}^j + \sum_{j=0}^{\lfloor (a-1)/p\rfloor} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \mathrm{P}^{a+b-j}\,\beta\,\mathrm{P}^j$$

and

$$\mathrm{P}^0 = 1$$

The grading of $\mathcal{A}(p)$ is determined by $\deg(\beta) = 1$ and $\deg(\mathrm{P}^i) = 2i(p-1)$.

Now we get a neat formulation of part of the preceding theorems:

Theorem 3.9 ($\mathcal{A}(p)$-modules) *Fix a prime number p. For any pair of topological spaces (X, A), the graded ring of cohomology $H^*(X, A; \mathbf{Z}/p)$ is an $\mathcal{A}(p)$-module, and the operation of $\mathcal{A}(p)$ on $H^*(X, A; \mathbf{Z}/p)$ is functorial with respect to the pair. In a more formal language: $H^*(-; \mathbf{Z}/p)$ is a contravariant functor from the category of pairs of topological spaces to $\mathcal{R}\mathit{ings} \cap_{\mathcal{A}(p)} \mathcal{M}\mathit{od}$, the category of rings with a structure of $\mathcal{A}(p)$ module.*

The above theorem provides us a lot of algebraic constraints on the morphisms between topological spaces other than the ones provided by the ring structure.

Remark 3.10 In fact, $\mathcal{A}(p)$ is moreover a cocommutative graded Hopf algebra. For the discussions on the Hopf algebra structure, preferable basis, and the structure of its dual Hopf algebra, we refer to the original paper of Milnor [Mil58], and the book of Steenrod [Ste62].

3.2 Reduced Steenrod Algebra

We now intend to extend the notion of Steenrod operations to the algebraic-geometric setting, that is, analogous operations on Chow groups. Historically, the question is posed by Fulton in [Ful98] Example 19.1.18, where he gives a short proof of a theorem of Kawai that the topological Steenrod operation preserves algebraic classes. Voevodsky constructed reduced power operations on motivic cohomology (which is isomorphic to higher Chow groups) in [Voe03] to prove the Milnor Conjecture, while the construction given there is very different from the methods used by Steenrod. Following the idea in the book of Steenrod [Ste62], Brosnan made a construction of Steenrod operations for the (usual) Chow groups in [Bro03]. But there is no proof at the moment that these coincide with Voevodsky's constructions. The content of this note is to present Brosnan's way of construct the Steenrod operations.

Suppose that we have an algebraic variety of dimension n defined over $\mathbf{C}$, if we have found such an operation $S : A_*(X) \to A_*(X)$, then a reasonable requirement is that S should commute with the cycle map:

$$\begin{array}{ccc} A_*(X) & \xrightarrow{S} & A_*(X) \\ \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\ H^*(X;\mathbf{Z}/p) & \xrightarrow{S} & H^*(X;\mathbf{Z}/p) \end{array}$$

where the operation S on the bottom arrow is the corresponding topological Steenrod operation. Recall that the cycle map is, c.f. [Ful98, Chapter 19]:

$$\mathrm{cl} : A_k(X) \to H_{2k}^{BM}(X;\mathbf{Z}/p) \simeq H^{2(n-k)}(X;\mathbf{Z}/p)$$

can only maps to the even-dimensional part, so the Bockstein homomorphism and the Steenrod squares (when $p = 2$) of odd-degree do not extend to the algebraic setting [18]. Such consideration leads us to the notion of *reduced Steenrod algebra*.

To begin with, we introduce an affine group scheme G, or equivalently a Hopf algebra H, whose dual will be the reduced Steenrod algebra we want. The treatment here follows [Mer03].

The construction can be divided into two steps, define the algebra structure on H and set $G = \mathrm{Spec}(H)$, and then define the coalgebra structure by specifying the group law of G. But anyway, we will give the explicit formula for the comultiplication of H.

Construction 3.11 (Group scheme G and Hopf algebra H) Consider the polynomial ring $H = \mathbf{F}_p[\mathbf{b}] = \mathbf{F}_p[b_1, b_2, ...]$ in infinitely many variables as a graded $\mathbf{F}_p$-algebra with $\deg(b_i) = p^i - 1$. Set the affine scheme $G = \mathrm{Spec}(H)$.

To define the coalgebra structure of H, we define the group scheme structure on G as below. For any commutative $\mathbf{F}_p$-algebra A, the set of A-points of G, namely $G(A) = \mathrm{Hom}_{\mathbf{F}_p-\mathrm{alg}}(H, A)$, can be identified with the set of sequences in

[18] However, they do have generalizations for higher Chow groups, c.f. [Voe03]

A, which can be further identified with the set of all formal power series of the following form, *i.e.* with degrees powers of p and coefficients in A,

$$t + a_1 t^p + a_2 t^{p^2} + \cdots \in A[[t]]$$

The multiplication of $G(A)$ is the composition of two formal power series, *i.e.* for two $f', f'' \in A[[t]]$ of the above form, we define the product $*$ as

$$(f' * f'')(t) = f''(f'(t))$$

the result is clearly of the above type. We set the unit element of $G(A)$ to be $f(t) = t$, then it is easy to see that every element has an inverse. In short, $G(A)$ is a group.

It is easy to verify the functorial properties of this construction, that is, given any morphism of $\mathbf{F}_p$-algebras, say $A \to B$, then the map it induces by composition $G(A) \to G(B)$, is a homomorphism of groups.

In conclusion, this gives G a affine group scheme structure, or equivalently, a Hopf algebra structure on H.

Remarks 3.12 **(a)(Comultiplication of H)** It is useful to write down the comultiplication of H explicitly. For this, we identify $\mathbf{F}_p[\mathbf{b}] \otimes \mathbf{F}_p[\mathbf{b}]$ with $\mathbf{F}_p[\mathbf{b}', \mathbf{b}'']$, by sending $b_i \otimes 1$ to b'_i, $1 \otimes b_i$ to b''_i. Now by rephrasing the definition of G, we get the formula of the comultiplication of H:

$$\begin{array}{rcl} \Delta : \mathbf{F}_p[\mathbf{b}] & \to & \mathbf{F}_p[\mathbf{b}', \mathbf{b}''] \\ b_k & \mapsto & \displaystyle\sum_{\substack{i+j=k \\ i,j \geq 0}} (b'_i)^{p^j} b''_j \end{array}$$

where $b_0 = 1$

(b)(Grading) Recall that the degree of b_i is by definition $p^i - 1$, one notes that the comultiplication of H preserves the degree: every term of $\Delta(b_k)$ has degree $(p^i - 1) \cdot p^j + (p^j - 1) = p^{i+j} - 1 = p^k - 1$, which is the degree of b_k itself. Thus H is a *graded* Hopf algebra.

Let us write the grading of H as: $H = \oplus_{m \geq 0} H_m$, where H_m is the homogenous part of H of degree m. For any multi-index of finite support[19] $I = (i_1, i_2, \cdots)$, we denote $\lambda_I \mathbf{b}^I := \lambda_I b_1^{i_1} b_2^{i_2} \cdots$, and we define the *norm* of $I = (i_1, i_2, \cdots)$ to be

$$|I| := \sum_k (p^k - 1) i_k \tag{11}$$

by assumption, the sum is finite. Then

$$H_m = \left\{ \sum_{|I|=m} \lambda_I \mathbf{b}^I | \lambda_I \in \mathbf{F}_p \right\}$$

And moreover, it is elementary that H_m is finite dimensional, since there are only finitely many multi-indices I of a fixed norm.

[19] *i.e.* only finitely many of them are non-zero

Before we define the reduced Steenrod algebra, let us recall the notion of the dual of a graded Hopf algebra. *A priori*, the dual vector space H^* of an infinitely dimensional Hoph algebra H only has the structure of an algebra, but not a coalgebra in general. The reason is that the multiplication of H does not necessarily give rise to a comultiplication for H^* because the inclusion $H^* \otimes H^* \subsetneq (H \otimes H)^*$ is strict when H is infinitely dimensional.

However, in the case of a graded Hopf algebra with finitely dimensional graded pieces, like the H constructed above, the graded dual can be well defined.

Definition 3.13 (Dual of graded Hopf algebra) Let $H = \oplus_{m\geq 0} H_m$ be a graded Hopf algebra *with finitely dimensional graded pieces* (like the Hopf algebra constructed above, so we abuse the notation). We can define the *dual* of H, denoted by $H^\vee$, to be $H^\vee = \oplus_{m\geq 0} H_m^*$, where H_m^* is the dual vector space of H_m. The multiplication and comultiplication of $H^\vee$ are defined by restricting to the homogenous parts, which is finite dimensional. More precisely, take the multiplication of $H^\vee$ as example, we write a homogenous part of the comultiplication of H:

$$H_k \xrightarrow{\Delta} \bigoplus_{i+j=k} H_i \otimes H_j$$

taking the vector space dual (noting that they are all finite dimensional),

$$\bigoplus_{i+j=k} H_i^* \otimes H_j^* \xrightarrow{\Delta^*} H_k^*$$

which gives the good multiplication for two homogenous elements of total degree k. The comultiplication of $H^\vee$ is defined analogously.

We now turn to the notion of reduced Steenrod algebra.

Definition 3.14 (Reduced Steenrod algebra) Let H be the graded Hopf algebra defined in Construction 3.11 (for the grading, see Remarks 3.12(b)). Define the *reduced Steenrod algebra* $\mathscr{S}$ to be its graded dual Hopf algebra in the sense of Definition 3.13, *i.e.* $\mathscr{S} = H^\vee$. In case we want to specify the prime number p that we fixed in the beginning, we would write $\mathscr{S}(p)$ to avoid possible confusions.

We write

$$H = \bigoplus_{m\geq 0} \bigoplus_{|I|=m} \mathbf{F}_p \cdot \mathbf{b}^I$$

i.e. $\{\mathbf{b}^I\}_I$ forms a basis of H, where I ranges over all multi-indices. By definition, the reduced Steenrod algebra

$$\mathscr{S} = \bigoplus_{m\geq 0} \bigoplus_{|I|=m} \mathbf{F}_p \cdot s^I$$

where $\{s^I\}$ is the basis of $\mathscr{S}$ dual to $\{\mathbf{b}^I\}$.

The multiplication of $\mathscr{S}$, denoted by $\circ$, is determined by the comultiplication $\Delta : H \to H \otimes H$, in the way of Definition 3.13. More precisely, let I, J be two multi-indices, and recall the definition of the norm of a multi-index in (11), then

$$s^I \circ s^J = \sum_{|K|=|I|+|J|} a_K s^K$$

where the sum is taken over all such multi-indices K (only finitely many), and the coefficient a_K equals to the coefficient of $\mathbf{b}'^I \mathbf{b}''^J$ in $\Delta(\mathbf{b}^K)$, which is determined by the formula in Remarks 3.12(a).

Now we give a theorem to justify the name of $\mathscr{S}$. This theorem is due to Milnor [Mil58]. The essential ingredients of the proof can be found in the book of Switzer [Swi75] Chapter 18., which are purely algebraic and elementary, but somehow tedious.

Theorem 3.15 (Generators and relations of $\mathscr{S}$) *Set $s^{(i,0,0\cdots)} =: s^i$ in $\mathscr{S}$, then $s^0 = 1$ and*

(a)(Generators) *$\{s^i\}_{i\geq 1}$ is a system of generators of $\mathscr{S}$;*

(b)(Relations) *The relations of the system $\{s^i\}_{i\geq 1}$ is generated by the Adem relations (9) [20]:*

$$s^a \circ s^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} s^{a+b-j} \circ s^j \tag{12}$$

Corollary 3.16 (Reduced Steenrod algebra) *The reduced Steenrod algebra is the Steenrod algebra modulo the two-sided ideal generated by the Bockstein element:*

$$\mathscr{S}(p) = \mathcal{A}(p)/ <\beta>$$

The isomorphism is given by $s^i \mapsto \mathrm{Sq}^{2i}$ in the $\mathbf{Z}/2$ case, and $s^i \mapsto \mathrm{P}^i$ in the $\mathbf{Z}/p$ case ($p > 2$).

PROOF. It is trivial from the theorem and the definition of Steenrod algebras, Definition 3.8. □

Remark 3.17 ($\mathscr{S}(p)$-modules) In the sequel, we will encounter some $\mathscr{S}(p)$-*modules*, that is, representations of the associative algebra $\mathscr{S}(p)$. Recall that in Definition 3.14, we have a basis for $\mathscr{S}$, namely s^I, where I ranges over all multi-indices. We will view these elements as 'operations' over the modules, and the two kinds of notations s^I and s^i will be used freely, with the identification $S^i = S^{(i,0,\ldots)}$.

[20]When $p = 2$, it is (8), see the notice following Theorem 3.7.

3.3 Steenrod Operations on Chow Groups: Smooth Case

From now on, all the Chow groups (usual or equivariant) are assumed to be of coefficient $\mathbf{F}_p$, that is: $A_*(X) = \mathrm{CH}_*(X) \otimes \mathbf{Z}/p$, where p is a prime number. And the characteristic of the base field is assumed to be different from p. As explained in the beginning of the preceding subsection, the 'best' thing we can expect to generalize is a $\mathscr{S}(p)$-module structure on $A_*(X)$, where $\mathscr{S}(p)$ is the reduced Steenrod algebra defined in the preceding subsection.

Fortunately, the 'best' thing happens, Brosnan [Bro03] constructed the operations S^i, and proved Adem's relations. Through his constructions, he also showed that some formal properties can also be generalized to the algebraic setting.

In this subsection, we want to state the results of Brosnan, take them as axioms, and give some simple applications, while the detailed construction and the verification of the properties will be worked out in the next section.

Now here comes the main theorem:

Theorem 3.18 (Formal properties of Steenrod operations: smooth case) *Let X be a* smooth *n-dimensional algebraic variety over k with* $\mathrm{char}(k) \neq p$, *$A_*(X)$ be the $\mathbf{F}_p$-coefficient Chow group of X, set $A^*(X) = A_{n-*}(X)$. For any integer i, we have an operation constructed in the next subsection Definition 3.26:*

$$S^i : A^q(X) \to A^{q+i(p-1)}(X)$$

for any non-negative integers q, satisfying:

(a)(Additivity) *Each S^i is additive, i.e. a homomorphism of abelian groups;*

(b)(Functoriality) *For any morphism between* smooth *varieties $f : X \to Y$, and any integers i, q, the operation S^i commutes with the l.c.i.*[21] *pull-back $f^!$.*

$$f^! S^i = S^i f^! : A^q(Y) \to A^{q+i(p-1)}(X)$$

(c)(Range) $S^i : A^q(X) \to A^{q+i(p-1)}(X)$ *is the zero map if $i > q$ or $i < 0$;*

(d)(p-th power) $S^q : A^q(X) \to A^{pq}(X)$ *is the p-th power map, that is, $x \mapsto x^p$, the intersection product* [22]*;*

(e) $S^0 = \mathrm{id}$*;*

(f) $S^0([X]) = [X]$ *and* $S^i([X]) = 0$ *for any $i \neq 0$;*

(g)(Cycle map) *If the base field $k = \mathbf{C}$, then S^i is compatible with the topological Steenrod operations in the following sense:*

[21] f is l.c.i., see the argument before the proof of Proposition 2.6

[22] [Ful98] Chapter 8

If $p = 2$, *then* $\mathrm{cl} \circ S^i = \mathrm{Sq}^{2i} \circ \mathrm{cl}$*:*

$$\begin{tikzcd}
A^q(X) \arrow[r, "S^i"] \arrow[d, "\mathrm{cl}"'] & A^{q+i}(X) \arrow[d, "\mathrm{cl}"] \\
H^{2q}(X;\mathbf{Z}/2) \arrow[r, "\mathrm{Sq}^{2i}"] & H^{2q+2i}(X;\mathbf{Z}/2)
\end{tikzcd}$$

where Sq^{2i} *is the Steenrod square, c.f. Theorem 3.6.*
If p *is an odd prime number, then* $\mathrm{cl} \circ S^i = \mathrm{P}^i \circ \mathrm{cl}$*:*

$$\begin{tikzcd}
A^q(X) \arrow[r, "S^i"] \arrow[d, "\mathrm{cl}"'] & A^{q+i(p-1)}(X) \arrow[d, "\mathrm{cl}"] \\
H^{2q}(X;\mathbf{Z}/p) \arrow[r, "\mathrm{P}^i"] & H^{2q+2i(p-1)}(X;\mathbf{Z}/p)
\end{tikzcd}$$

where P^i *is the reduced Steenrod power, c.f. Theorem 3.7.*

(h)(Cartan formula) *For any* $x \in A^q(X)$, $y \in A^r(X)$, *we have*

$$S^k(x \cdot y) = \sum_{i=0}^{k} S^i(x) \cdot S^{k-i}(y)$$

in $A^{q+r+k(p-1)}(X)$, *or more neatly,* $S^\bullet(x \cdot y) = S^\bullet(x) \cdot S^\bullet(y)$ *where the product* $\cdot$ *is the intersection product* [23],

(i)(Adem relations) *For any* $0 < a < pb$, *we have (12):*

$$S^a \circ S^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} S^{a+b-j} \circ S^j$$

Proof. The explicit construction is the content of the next subsection, and the detailed proof is also given at the end of it. However, we give the reference of the proofs here. All the proofs in the next subsection are reproduced from Brosnan [Bro03]: (a):Theorem 8.2; (b): Remark 8.12; (c): Corollary 9.6; (d): Proposition 9.4(iv); (e): Corollary 9.6; (f):Proposition 9.4(ii); (g): Corollary 9.12; (h): Theorem 9.3; (i):Theorem 11.3. □

Combining (a),(b),(c),(e),(i) of the above theorem, we get a partial generalization of Theorem 3.9:

Theorem 3.19 ($\mathscr{S}(p)$-module structure I: smooth case) *Fix a prime number* p. *Denote* $A^*(-)$ *the functor of the* $\mathbf{F}_p$*-coefficient Chow ring. For any* smooth *variety* X *over* k *with* $\mathrm{char}(k) \neq p$, *the Chow ring* $A^*(X)$ *is a graded* $\mathscr{S}(p)$*-module, and the operation of* $\mathscr{S}(p)$ *on* $A^*(X)$ *is natural with respect to the functorial pull-backs.*
In a more formal language: $A^*(-) : \mathcal{S}m_k^{op} \to \mathcal{R}ings \cap_{\mathscr{S}(p)} \mathcal{M}od$ *is a contravariant functor from the category of smooth varieties to the category of rings with a structure of* $\mathscr{S}(p)$*-module.*

[23][Ful98] Chapter 8

Remark 3.20 In the rest of the note, we will also consider the the action of other basis elements of $\mathscr{S}$ over the Chow ring of a smooth variety, denoted as

$$S^I : A^j(X) \to A^{j+|I|}(X)$$

for any multi-index I. We have $S^i = S^{(i,0,\dots)}$, see Remark 3.17.

3.4 The Construction and Proofs: Smooth Case

In this subsection, we will give the detailed construction of the Steenrod operations on Chow groups of a smooth variety, as promised in the preceding subsection. The proofs of the 'axioms' listed in Theorem 3.18 will be given at the end of this subsection.

3.4.1 p-th Power Map

In this subsubsection, let X be a (not necessarily smooth) variety, p be a positive integer. We will write X^p for $\underbrace{X \times X \times \cdots \times X}_{p}$, the product of p copies of X. Let G be any subgroup of $\mathfrak{S}_p$, the symmetric group of p letters. We will define a morphism $P : A_i(X) \to A^G_{pi}(X^p)$, roughly speaking, it sends a cycle of X to the 'quotient' by the group of its p-th exterior power. Now we turn to the details:

The p-th exterior power defines a map:

$$\begin{aligned} {}^{\times p} : Z_i(X) \to \quad & Z_{pi}(X^p) \\ \alpha \mapsto \quad & \underbrace{\alpha \times \cdots \times \alpha}_{p} \end{aligned}$$

The image is obviously G-invariant, by abuse of notation, we have the morphism ${}^{\times p} : Z_i(X) \to Z_{pi}(X^p)^G$. And then we recall the equivariant cycle class map defined in Subsection 2.3 (Construction 2.29): $\mathrm{cl}^G : Z_{pi}(X^p)^G \to A^G_{pi}(X^p)$

Proposition 3.21 (p-th power) *The composition*

$$Z_i(X) \xrightarrow{\times p} Z_{pi}(X^p)^G \xrightarrow{\mathrm{cl}^G} A^G_{pi}(X^p)$$

passes to rational equivalence, i.e. it factors through $A_i(X)$, thus define the required morphism P below:

$$\begin{tikzcd} Z_i(X) \arrow[r, "\times p"] \arrow[d, "\mathrm{cl}"'] & Z_{pi}(X^p)^G \arrow[d, "\mathrm{cl}^G"] \\ A_i(X) \arrow[r, "P"] & A^G_{pi}(X^p) \end{tikzcd}$$

We call the bottom arrow $P : A_i(X) \to A^G_{pi}(X^p)$ the *p-th power map*[24].

[24] In Brosnan[Bro03], he called this map *the fundamental operation.*

Proof. Let $Z_i(X|\mathbf{P}^1)$ denote the group consisting of the families of i-dimensional cycles parameterized by $\mathbf{P}^1$, that is, $(i+1)$-dimensional cycles of $X \times \mathbf{P}^1$ with components dominating $\mathbf{P}^1$. Consider the map

$$\begin{aligned} Z_i(X|\mathbf{P}^1) &\to Z_i(X) \\ \alpha &\mapsto \alpha(0) - \alpha(\infty) \end{aligned}$$

where α is an $(i+1)$-dimensional cycle of $X \times \mathbf{P}^1$ dominating $\mathbf{P}^1$, and $\alpha(t)$ is the fiber over $t \in \mathbf{P}^1$: $\alpha(t) = i_t^*(\alpha)$ (See [Ful98] Chapter 1). Therefore, the first row of the diagram below is exact:

$$\begin{tikzcd} Z_i(X|\mathbf{P}^1) \arrow[r] \arrow[d, "\times_{\mathbf{P}^1} p"'] & Z_i(X) \arrow[r, "\mathrm{cl}"] \arrow[d, "\times p"'] & A_i(X) \arrow[r] \arrow[d, dashed, "P"'] & 0 \\ Z_{pi}(X^p|\mathbf{P}^1)^G \arrow[r] & Z_{pi}(X^p)^G \arrow[r, "\mathrm{cl}^G"] & A^G_{pi}(X^p) & \end{tikzcd}$$

The left vertical arrow is the fiber-wise p-th exterior power, as indicated by the fiber product notation. It is also obvious that the second row is a complex, *i.e.* the composition is 0. By a routine diagram chasing, we find the dashed vertical arrow is well-defined, which is exactly the morphism we are looking for. □

Here are some formal functorial properties of the p-th power map:

Proposition 3.22 (Functoriality of the p-th power) *The p-th power map commutes with proper push-forwards, l.c.i pull-backs, and Chern classes.*

More precisely, let $f : X \to Y$ be a morphism of varieties, G a subgroup of $\mathfrak{S}_p$. Write $f^p := \underbrace{f \times f \times \cdots \times f}_{p} : X^p \to Y^p$ the morphism induced by f. We write a subscript under P to indicate the space involved in the p-th power map. Then:

(a)(Push-forward) *If f is* proper*, then the following diagram commutes:*

$$\begin{tikzcd} A_i(X) \arrow[r, "P_X"] \arrow[d, "f_*"'] & A^G_{pi}(X^p) \arrow[d, "f^p_*"] \\ A_i(Y) \arrow[r, "P_Y"] & A^G_{pi}(Y^p) \end{tikzcd}$$

where the left vertical arrow is the usual proper push-forward, while the right one is the equivariant proper push-forward.

(b)(Pull-back) *If f is* l.c.i. *of codimension c, then the following diagram commutes:*

$$\begin{tikzcd} A_i(Y) \arrow[r, "P_Y"] \arrow[d, "f^!"'] & A^G_{pi}(Y^p) \arrow[d, "f^{p!}"] \\ A_{i-c}(X) \arrow[r, "P_X"] & A^G_{p(i-c)}(X^p) \end{tikzcd}$$

where the left vertical arrow is the usual l.c.i. pull-back, while the right one is the equivariant l.c.i. pull-back.

(c)(Chern classes) *If $E \to X$ is a vector bundle of rank r, $E^p \to X^p$ is the G-equivariant vector bundle. Then the following diagram commutes:*

$$\begin{tikzcd} A_i(X) \arrow[r, "P"] \arrow[d, "c_j(E)\cap"'] & A^G_{pi}(X^p) \arrow[d, "c^G_{pj}(E^p)\cap"] \\ A_{i-j}(X) \arrow[r, "P"] & A^G_{p(i-j)}(X^p) \end{tikzcd}$$

where the left vertical arrow is the usual Chern class action, while the right one is the equivariant Chern class action.

Proof. The proofs are just straightforward verifications, which we will omit here. □

3.4.2 Basic Construction and Proofs

In this subsubsection, let X be a smooth variety of dimension n over k, p be a fixed prime number, with $\mathrm{char}(k) \neq p$. To define an $\mathscr{S}(p)$-module structure on $A_*(X)$, by the description of the reduced Steenrod algebra Theorem 3.15, it suffices to construct a series of operations S^i satisfying the Adem relations (12).

Definition 3.23 (Construction of D_X) Firstly, applying the p-th power map of the preceding subsection to the case $G = \mathbf{Z}/p$, we have:

$$P : A_i(X) \to A^{\mathbf{Z}/p}_{pi}(X^p)$$

Secondly, consider the diagonal inclusion $\delta : X \hookrightarrow X^p$, since X is smooth, δ is a regular embedding of codimension $n(p-1)$, where n is the dimension of X. Let the finite group $\mathbf{Z}/p$ act on X trivially, and on X^p by permutation, we find that δ is indeed equivariant. Applying the construction of equivariant l.c.i pull-back in 2.2.2, we get:

$$\delta^! : A^{\mathbf{Z}/p}_{pi}(X^p) \to A^{\mathbf{Z}/p}_{pi-(p-1)n}(X)$$

And finally, according to the result in Appendix A, Theorem A.2, we write

$$A^{\mathbf{Z}/p}_{pi-(p-1)n}(X) = \bigoplus_{j\geq 0} A_{pi-(p-1)n+j}(X) \cdot l^j$$

We denote the composition of the three maps

$$D_X : A_i(X) \to A^{\mathbf{Z}/p}_{pi-(p-1)n}(X) = \bigoplus_{j\geq 0} A_{pi-(p-1)n+j}(X) \cdot l^j$$

Sometimes when we are dealing with only one space, the subscript could be omitted.

There are two basic lemmas:

Lemma 3.24 (Additivity) *The D_X constructed above is additive, i.e. a homomorphism of groups.*

PROOF. Let us keep the above notation. Given $x, y \in A_i(X)$ two cycles of X. Then

$$\begin{aligned} D_X(x+y) &= \delta^!(P(x+y)) \\ &= \delta^! \circ \mathrm{cl}^{\mathbf{Z}/p}((x+y)^{\times p}) \\ &= \delta^! \circ \mathrm{cl}^{\mathbf{Z}/p}(x^{\times p}+y^{\times p}) + \delta^! \circ \mathrm{cl}^{\mathbf{Z}/p}(\sum_{g\in \mathbf{Z}/p} g\cdot z) \end{aligned}$$

where z is a cycle in X^p. The reason for the last equality is that apart from $x^{\times p}+y^{\times p}$, the rest terms can be regrouped into the form $\sum_{g\in\mathbf{Z}/p} g\cdot z \in Z_{pi}(X^p)^{\mathbf{Z}/p}$, but Proposition 2.31 implies that $\mathrm{cl}^{\mathbf{Z}/p}(\sum_{g\in\mathbf{Z}/p} g\cdot z) = \mathrm{Tr}_1^{\mathbf{Z}/p} \circ \mathrm{cl}(z)$ in $A^{\mathbf{Z}/p}_{pi}(X^p)$. Thus

$$\begin{aligned} D_X(x+y) &= \delta^! \circ \mathrm{cl}^{\mathbf{Z}/p}(x^{\times p}+y^{\times p}) + \delta^! \circ \mathrm{Tr}_1^{\mathbf{Z}/p} \circ \mathrm{cl}(z) \\ &= D_X(x) + D_X(y) + \delta^! \circ \mathrm{Tr}_1^{\mathbf{Z}/p} \circ \mathrm{cl}(z) \\ &= D_X(x) + D_X(y) + \mathrm{Tr}_1^{\mathbf{Z}/p} \circ \delta^! \circ \mathrm{cl}(z) \end{aligned}$$

The last equality follows from Proposition 2.28(c)[25]. So it remains to show that $\mathrm{Tr}_1^{\mathbf{Z}/p} = 0 : A_j(X) \to A_j^{\mathbf{Z}/p}(X)$. To this end, recall that (Proposition 2.28(b)) the composition $A_j^{\mathbf{Z}/p}(X) \xrightarrow{\mathrm{Res}_1^{\mathbf{Z}/p}} A_j(X) \xrightarrow{\mathrm{Tr}_1^{\mathbf{Z}/p}} A_i^{\mathbf{Z}/p}(X)$ is multiplication by p, *i.e.* 0. However, since the $\mathbf{Z}/p$-action on X is trivial, so the restriction map can be read as the push-forward of the projection map:

$$pr_{1*} : A_{j+|B|}(X\times B) \to A_j(X),$$

where $B = \frac{E}{\mathbf{Z}/p}$ is the quotient of $E \in E(\mathbf{Z}/p)_r$ with r sufficiently large. The above displayed morphism is clearly surjective, hence $\mathrm{Tr}_1^{\mathbf{Z}/p} = 0 : A_j(X) \to A_j^{\mathbf{Z}/p}(X)$. □

Lemma 3.25 *We keep the notation: $D_X : A_i(X) \to \bigoplus_{j\geq 0} A_{pi-(p-1)n+j}(X)\cdot l^j$. Then for any $x \in A_i(X)$, $D_X(x)$ has its j-th component 0, for every j not divisible by $(p-1)$.*

In other words, we can write

$$D_X : A_i(X) \to \bigoplus_{j\geq 0} A_{pi-(p-1)(n-j)}(X)\cdot l^{j(p-1)}$$

[25] The two $\delta^!$'s are different, the first one is $\mathbf{Z}/p$-equivariant l.c.i.pull-back, the second one is usual l.c.i pull-back

Proof. Let N be the normalizer of $\mathbf{Z}/p$ in $\mathfrak{S}_p$, then $N \simeq \mathbf{Z}/p \rtimes \mathrm{Aut}(\mathbf{Z}/p)$, while $\mathrm{Aut}(\mathbf{Z}/p) \simeq (\mathbf{Z}/p)^* \simeq \mathbf{Z}/(p-1)$, Consider the action of $\mathrm{Aut}(\mathbf{Z}/p) \simeq \mathbf{Z}/(p-1)$ on $A^{\mathbf{Z}/p}_{pi-(p-1)n}(X) = \bigoplus_{j\geq 0} A_{pi-(p-1)n+j}(X) \cdot l^j$. Examining the calculation in Appendix A, we find the action is just

$$\begin{aligned} \mathrm{Aut}(\mathbf{Z}/p) \simeq (\mathbf{Z}/p)^* \times \mathbf{F}_p[l] \rightarrow &\quad \mathbf{F}_p[l] \\ (k, l^j) \mapsto &\quad k^j l^j \end{aligned}$$

The action on l^j is trivial if and only if $(p-1)$ divides j.

On the other hand, since the restriction map commutes with l.c.i. pull-back, we have the following diagram:

$$\begin{tikzcd} A_i(X) \arrow[d, "P"'] \arrow[dr, "P"] & \\ A^N_{pi}(X^p) \arrow[r, "\mathrm{Res}^N_{\mathbf{Z}/p}"] \arrow[d, "\delta^!"] & A^{\mathbf{Z}/p}_{pi}(X^p) \arrow[d, "\delta^!"] \\ A^N_{pi-(p-1)n}(X) \arrow[r, "\mathrm{Res}^N_{\mathbf{Z}/p}"] & A^{\mathbf{Z}/p}_{pi-(p-1)n}(X) \end{tikzcd}$$

The action of $\mathrm{Aut}(\mathbf{Z}/p) \simeq \mathbf{Z}/(p-1)$ on $A^{\mathbf{Z}/p}_{pi-(p-1)n}(X)$ is determined by the restriction map. By a general argument in [Bro03, Theorem 8.3] using [Bro03, Proposition 3.4]: 'Inner automorphisms act trivially on equivariant cohomology', this action must be trivial. Therefore all the coefficient of $D_X(x)$ in front of l^j with j not divisible by $(p-1)$, is zero. For the details, see [Bro03]. □

Now we turn to the definition of the Steenrod operations:

Definition 3.26 (Steenrod operations (smooth)) Given a smooth algebraic variety X over k with $\mathrm{char}(k) \neq p$. Keep the notation as above, and denote $\eta := -l^{p-1}$ as in Example 2.24. Thanks to Lemma 3.25, we can write

$$D_X : A_i(X) \rightarrow \bigoplus_{j\geq 0} A_{pi-(p-1)(n-j)}(X) \cdot \eta$$

For a $x \in A_i(X)$, write $D_X(x) = \sum_{j\geq 0}(-1)^j b_j \cdot l^{j(p-1)} = \sum_{j\geq 0} b_j \eta^j$, where $b_j \in A_{pi-(p-1)(n-j)}(X)$. We define the *Steenrod operations* $S^j : A_i(X) \rightarrow A_{i-(p-1)j}(X)$, by a reindexing as following:

$$S^j(x) := b_{n-i-j} \in A_{i-(p-1)j}(X) \tag{13}$$

Remark 3.27 To better keep track of degrees, we also introduce the power series notation following Brosnan[Bro03]8.4 and Definition 8.5: set $\eta = -l^{p-1}$, for $x \in A_i(X)$, let $D_X(x,t) \in A_*(X)[[t]]$ be the power series such that $D_X(x,\eta) = D_X(x)$. And then, define $S(x)(t) = t^{n-i} D_X(x, 1/t)$. So by definition,

$$S(x)(t) = \sum_{j\geq 0} S^j \cdot t^j$$

Now we turn to the verification of the various properties of the Steenrod operations listed in §3.3 Theorem 3.18.

PROOF. (of **Theorem 3.18**)

(a) follows immediately from Lemma 3.24;

(b) is trivial from the construction: since every step of the construction of D is functorial, and the reindexing (13) only concerns the codimension of cycles, which is preserved by the l.c.i. pull-backs, every S^j is functorial with respect to the pull-backs;

For (d), set $i = n - q$, fix $x \in A^q(X) = A_i(X)$, then $S^q(x)$ is the constant coefficient of $D_X(x,t)$, which turns out to be the image of $D_X(x) \in A_*^{\mathbf{Z}/p}X$ under the map $\mathrm{Res}_1^{\mathbf{Z}/p} : A_*^{\mathbf{Z}/p}(X) \to A_*(X)$. Thanks to Proposition 2.28(c), the right square of the following diagram is commutative:

$$\begin{tikzcd} A_i(X) \arrow[r, "P"] \arrow[d, equal] & A_{pi}^{\mathbf{Z}/p}(X^p) \arrow[r, "\delta^!_{\mathbf{Z}/p}"] \arrow[d, "\mathrm{Res}_1^{\mathbf{Z}/p}"'] & A_{pi-n(p-1)}^{\mathbf{Z}/p}(X) \arrow[d, "\mathrm{Res}_1^{\mathbf{Z}/p}"] \\ A_i(X) \arrow[r, "\times p"] & A_{pi}(X^p) \arrow[r, "\delta^!_1"] & A_{pi-n(p-1)}(X) \end{tikzcd}$$

the left square is obviously commutative, and the composition of the top line is just the map D_X. Consequently,

$$\begin{aligned} S^q(x) &= \mathrm{Res}_1^{\mathbf{Z}/p} \circ D_X(x) \\ &= \mathrm{Res}_1^{\mathbf{Z}/p} \circ \delta^!_{\mathbf{Z}/p} \circ P(x) \\ &= \delta^!_1(x^{\times p}) \\ &= x^p \end{aligned}$$

For (f), we plunge into the above construction, recalling the definition of equivariant Chow group, we arrive at a regular embedding:

$$\frac{X \times U}{\mathbf{Z}/p} = X \times \frac{U}{\mathbf{Z}/p} \hookrightarrow \frac{X \times \cdots \times X \times U}{\mathbf{Z}/p}$$

where $U \in \mathrm{E}(\mathbf{Z}/p)_r$ with r large enough. And the p-th power of the cycle $[X]$ is the whole ambient space of the right hand side, thus its pull-back is the whole space of the left hand side, namely $X \times \frac{U}{\mathbf{Z}/p}$, according to the fact that the Gysin map of a regular embedding sends the cycle of the ambient space to the cycle of the subspace. Then the result follows.

We prove (c) and (e) together, it is trivial from the reindex (13) that S^i is zero for $i > q$. As for the $i \leq 0$ case, by the linearity (a), we can suppose $x = [Y]$, where Y is a subvariety of X. Denote $Z = Y_{\mathrm{sing}}$ the singular part of Y, and set $U = X - Z$, the open immersion being $j : U \hookrightarrow X$. Applying the functoriality to the open immersion $j : U \hookrightarrow X$, we get $S^i_U([Y - Z]) = j^*(S^i_X([Y]))$ [26]. Noting that for dimension greater than $|Z|$, j^* is an isomorphism, so if we can prove

[26] the spaces on which we do Steenrod operations are indicated by the subscripts.

(c) and (e) for U and $Y - Z$, then we can prove them for X and Y, since for $i \leq 0$, $S^i([Y])$ has dimension at least $|Y|$, which is larger than $|Z|$.
Therefore, we can assume that X and Y are both smooth, hence the embedding is regular. Here we want to use the Step 1 of Theorem 4.5, whose proof will be independent of (c) and (e) of Theorem 3.18 here, we have $U^\bullet_Y([Y]) = U^\bullet_X([Y])$, by the definition of $U^\bullet$, that is:

$$S^\bullet_Y([Y], t) \cap w(T_Y, t)^{-1} = S^\bullet_X([Y], t) \cap w(T_X, t)^{-1}$$

for the definition and several elementary properties used of the characteristic class w, see the beginning of the next section. Since $S^\bullet_Y([Y]) = [Y]$ by (f), and $w(T_Y, t)^{-1} \cdot w(T_X|_Y, t) = w(N_{X/Y}, t)$ by Whitney sum formula (c.f.Remarks 4.2(4)), it is equivalent to

$$[Y] \cap w(N_{X/Y}, t) = S^\bullet_X([Y], t)$$

from which we conclude $S^0([Y]) = [Y]$ and $S^j([Y]) = 0$ for any $j < 0$, hence (c) and (e).

For (g), we just mention that the construction in the algebraic context given above is parallel to the construction in [Ste62] Chapter VII, every step has corresponding cycle map to bridge two constructions.

For (h), we consider the following version of Cartan formula (h'):

(**h'**)(**Cartan formula'**) *For any $x \in A^q(X)$, $y \in A^r(Y)$, we have*

$$S^k(x \times y) = \sum_{i=0}^{k} S^i(x) \times S^{k-i}(y)$$

in $A^{q+r+k(p-1)}(X \times Y)$, or more neatly, $S^\bullet(x \times y) = S^\bullet(x) \times S^\bullet(y)$, where the product is the exterior product;

(h) is an immediate consequence of (h') and the functoriality (b) applying to the diagonal map $\delta : X \hookrightarrow X \times X$:

$$S^k(x \cdot y) = S^k(\delta^!(x \times y)) = \delta^!(S^k(x \times y)) = \delta^!(\sum_{i=0}^{k} S^i(x) \times S^{k-i}(y)) = \sum_{i=0}^{k} S^i(x) \cdot S^{k-i}(y)$$

As for (h'), note that $S^\bullet(x \times y) = S^\bullet(x) \times S^\bullet(y)$ is equivalent to $D_{X \times Y}(x \times y) = D_X(x) \times D_Y(y)$, which can be proved using the formal properties of P and l.c.i. pull-backs:

$$\begin{aligned} D_{X \times Y}(x \times y) &= (\delta_X \times \delta_Y)^!(P_{X \times Y}(x \times y)) \\ &= \delta_X^!(P_X(x)) \times \delta_Y^!(P_Y(y)) \\ &= D_X(x) \times D_Y(y) \end{aligned}$$

Finally, for (i), we refer to the book of Steenrod [Ste62] Chapter VIII, or Brosnan's paper [Bro03]§11. □

4 Twisted Steenrod Operations on Arbitrary Varieties

In this section we intend to extend the Steenrod operations to arbitrary varieties. However, the operations defined for smooth varieties in the preceding section (Theorem 3.19) do not generalize naturally. The reason is that they do not behave well enough under push-forwards (see the details below), while our basic strategy is to use a birational push-forward method. Therefore, we will introduce in the first subsection another slightly different $\mathscr{S}$-module structure on the Chow groups of smooth varieties, namely the *twisted Steenrod operations* U, which commute with projective push-forward, and we will show that this kind of twisted operations extends in to arbitrary varieties by two methods: in the second subsection, we present the approach using resolution of singularities, while in the third subsection, we will use a smooth embedding. Recall that we always assume that the characteristic of the base field is different from p, and all the Chow groups $A_*(-)$ are of coefficient $\mathbf{F}_p$.

4.1 Twisted Steenrod Operations: Smooth case

First of all, one should notice an important fact: the Steenrod operations do not commute with projective (or proper) push-forwards. Fortunately, as in the case of Grothendieck-Riemann-Roch theorem, the failure is measured by certain characteristic class of the 'difference' of the tangent bundles.

We introduce this characteristic class first, by an abuse of the *splitting principle*:

Definition 4.1 (Characteristic class w) Let E be a vector bundle of rank r over a variety X, $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the Chern roots of E. We define the characteristic class w by

$$w(E, t) = \prod_{i=1}^{r}(1 + \lambda_i^{p-1}t).$$

As usual, the indeterminate t is to better keep track of the degree, and we index the classes such that $w(E, t) = \sum_{j=0}^{r} w_j(E)t^j$. Clearly, the degree of w_j is $(p-1)j$, that is, viewing the characteristic classes as operations on the $\mathbf{F}_p$ coefficient Chow groups, as in the language of [Ful98], then we have:

$$w_j(E) : A_*(X) \to A_{*-(p-1)j}(X)$$

We sometimes use the *total w class*: $w(E) = w(E, 1) = \prod_{i=1}^{r}(1 + \lambda_i^{p-1})$

Remarks 4.2 We have:

1. w_j has degree $(p-1)j$, *i.e.* viewed as an operation, it will decrease the dimension of cycles by $(p-1)j$;

2. $w_0(E) = 1$;

3. **(Whitney sum)** For a exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, we have $w(E,t) = w(E',t) \cdot w(E'',t)$;

4. $w_r(E) = c_r(E)^{p-1}$, where c_r is the top Chern class, and the product is the composition of characteristic class operations, and in particular for smooth base varieties, it is just the intersection product.

For later reference, we compute an example where characteristic class w appears naturally.

Example 4.3 Let E be a vector bundle of rank r on a variety X, both equipped with trivial $\mathbf{Z}/p$ actions. Recall some notation in Example 2.24, R is the standard representation of $\mathbf{Z}/p$, and denote $R_X = R \otimes_k \mathcal{O}_X$ to be the trivial vector bundle on X of fiber R, equipped with the natural $\mathbf{Z}/p$-action. Then the $\mathbf{Z}/p$-equivariant top Chern class of $R_X \otimes E$ is

$$c^{\mathbf{Z}/p}_{r(p-1)}(R_X \otimes E) = \eta^r w(E, 1/\eta)$$

where η is by definition (Example 2.24) the top equivariant Chern class of R viewed as a vector bundle on a point, $c^{\mathbf{Z}/p}_{p-1}(R) =: \eta = -l^{p-1}$.

(Indication of the proof: the Chern roots of R_X are given in Example 2.24: $l, 2l, \ldots, (p-1)l$, where $l = c^{\mathbf{Z}/p}_1(L_{jX})$, for some $1 \le j \le p-1$; the Chern roots of E are $\lambda_1, \lambda_2, \ldots, \lambda_r$, then the Chern roots of $R_X \otimes E$ are $\{jl + \lambda_i\}_{1 \le j \le p-1, 1 \le i \le r}$. One deduces that

$$\begin{aligned} c^{\mathbf{Z}/p}_{r(p-1)}(R_X \otimes E) &= \prod_{j=1}^{p-1}\prod_{i=1}^{r}(jl + \lambda_i) \\ &= \prod_{i=1}^{r}(\lambda_i^{p-1} - l^{p-1}) \\ &= \prod_{i=1}^{r}(\lambda_i^{p-1} + \eta) \\ &= \eta^r w(E, 1/\eta) \end{aligned}$$

as wanted.)

Now we show that the difference of the w-classes of the tangent bundles measures the non-commutativity of the push-forward and the Steenrod operations.

Definition 4.4 (Twisted Steenrod operations I: smooth case) Let X be a smooth variety over k, with $\mathsf{char}(k) \neq p$, we define another kind of operations by

$$U^\bullet(t) := w(T_X, t)^{-1} \circ S^\bullet(t)$$

where $U^\bullet(t) = \sum_j U^j t^j$, $S^\bullet(t) = \sum_j S^j t^j$, and $w(T_X, t)^{-1}$ is the inverse power series[27] of the w-class of the tangent bundle of X. Then by the first point

[27]Since $w_0 = 1$, it is always invertible as power series.

of Remarks 4.2, the operation U^j, like S^j and w_j, decreases the dimension by $j(p-1)$. We sometimes write a subscript under U to indicate the space on which we do these new operations. We call this operation U, the *twisted Steenrod operation.*

Theorem 4.5 (Push-forwards and twisted Steenrod operations I) *Let $f : X \to Y$ be a morphism between* smooth *varieties over k with* $\mathrm{char}(k) \neq p$, *and we suppose that f is* projective in the strong sense, *that is, f admits a factorization $f : X \hookrightarrow Y \times \mathbf{P}^r \to Y$, where the first map is a closed immersion, and the second one is a trivial projective bundle. Then the twisted Steenrod operations U^j commutes with the proper push-forward f_*:*

$$f_* \circ U^j = U^j \circ f_* : A_i(X) \to A_{i-(p-1)j}(Y)$$

i.e. the following diagram is commutative:

$$\begin{tikzcd} A_i(X) \arrow[r, "U^j"] \arrow[d, "f_*"'] & A_{i-j(p-1)}(X) \arrow[d, "f_*"] \\ A_i(Y) \arrow[r, "U^j"] & A_{i-j(p-1)}(Y) \end{tikzcd}$$

Proof. Suppose the dimension of X and Y are n and m respectively. Since f admits a factorization as stated in the theorem, we can prove the theorem in two separate cases: f is a closed immersion, or f is the projection $Y \times \mathbf{P} \to Y$.

Step 1. The case when f is a closed immersion of smooth varieties.
In this case, $n < m$, the codimension is $d = m - n$. Moreover, since X and Y are smooth, f is a regular embedding. We have the cartesian diagram:

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[d, "f"'] & X^p \arrow[d, "f^p"] \\ Y \arrow[r, "\delta_Y"] & Y^p \end{tikzcd}$$

where all morphisms are viewed as $\mathbf{Z}/p$ morphisms, the excess normal bundle is $E = f^*(T_Y \otimes R_Y)/(T_X \otimes R_X) = (f^*T_Y/T_X) \otimes R_X = N_{X/Y} \otimes R_X$, as $\mathbf{Z}/p$-equivariant vector bundle, which is of rank $e = (p-1)d$. Then by the equivariant excess intersection formula, Theorem 2.22,

$$\delta_Y^! = c_e^{\mathbf{Z}/p}(E) \cap \circ \delta_X^! : A_*^{\mathbf{Z}/p}(X^p) \to A_{*-(p-1)m}^{\mathbf{Z}/p}(X) \tag{14}$$

On the other hand, we have the following commutative diagram, in virtue of the naturality of l.c.i pull-backs (Proposition 2.19(c))

$$\begin{tikzcd} A_i(X) \arrow[r, "P_X"] \arrow[d, "f_*"'] & A_{pi}^{\mathbf{Z}/p}(X^p) \arrow[r, "\delta_Y^!"] \arrow[d, "f_*^p"'] & A_{pi-(p-1)m}^{\mathbf{Z}/p}(X) \arrow[d, "f_*"] \\ A_i(Y) \arrow[r, "P_Y"] & A_{pi}^{\mathbf{Z}/p}(Y^p) \arrow[r, "\delta_Y^!"] & A_{pi-(p-1)m}^{\mathbf{Z}/p}(Y) \end{tikzcd}$$

where the operation P is the p^{th} power map defined in Subsection 3.4.1.

Now in the above diagram, by the construction in the beginning of Subsection 3.4.2, the composition of the bottom line is just D_Y, and the composition of the top line is $c_e^{\mathbf{Z}/p}(E) \circ D_X$ by (14), Therefore,

$$D_Y \circ f_* = f_* \circ c_e^{\mathbf{Z}/p}(E) \circ D_X$$

By the calculation in the above Example 4.3,

$$\begin{aligned} c_e^{\mathbf{Z}/p}(E) &= c_e^{\mathbf{Z}/p}(N_{X/Y} \otimes R_X) \\ &= \eta^d w(N_{X/Y}, 1/\eta) \end{aligned}$$

For any given $x \in A_i(X)$, to prove the theorem $f_* \circ U^\bullet(x,t) = U^\bullet(f_*(x),t)$, we only need to show it with $t = 1/\eta$. By the explicit construction of S_X from D_X in Definition 3.26, we recall that $D_X(x,\eta) = D_X(x)$, $S_X(x,t) = t^{n-i}D_X(x,1/t)$, and $U^\bullet(t) := w(T_X,t)^{-1} \circ S^\bullet(t)$, we deduce:

$$\begin{aligned} f_* \circ U^\bullet(x,t)|_{t=1/\eta} &= f_*(w(T_X,t)^{-1} \cdot S_X^\bullet(x,t))|_{t=1/\eta} \\ &= f_*(w(f^*T_Y,t)^{-1} \cdot w(N_{X/Y},t)) \cdot t^{n-i}D_X(x,1/t)|_{t=1/\eta} \\ &= w(T_Y,t)^{-1} \cdot f_*(w(N_{X/Y},t)) \cdot t^{n-i}D_X(x,1/t))|_{t=1/\eta} \\ &= w(T_Y,t)^{-1}|_{t=1/\eta} \cdot f_*(w(N_{X/Y},1/\eta) \cdot \eta^{i-n}D_X(x)) \\ &= w(T_Y,t)^{-1}|_{t=1/\eta} \cdot f_*(c_e^{\mathbf{Z}/p}(E) \cdot \eta^{i-m}D_X(x)) \\ &= w(T_Y,t)^{-1}|_{t=1/\eta} \cdot D_Y(f_*(x)) \cdot \eta^{i-m} \\ &= w(T_Y,t)^{-1} \cdot D_Y(f_*(x),1/t)t^{m-i}|_{t=1/\eta} \\ &= w(T_Y,t)^{-1} \cdot S_Y^\bullet(f_*(x),t)|_{t=1/\eta} \\ &= U^\bullet(f_*(x),t)|_{t=1/\eta} \end{aligned}$$

The closed immersion case is proved.

Step 2. The case when $X = Y \times \mathbf{P}^r \xrightarrow{f=pr_1} Y$ is a trivial projective bundle over a smooth variety.
In this case $n-m=r$. We will use the following version of Cartan formula (h'):

(**h'**)(**Cartan formula'**) *For any $x \in A^q(X)$, $y \in A^r(Y)$, we have*

$$S^k(x \times y) = \sum_{i=0}^{k} S^i(x) \times S^{k-i}(y)$$

in $A^{q+r+k(p-1)}(X \times Y)$, or more neatly, $S^\bullet(x \times y) = S^\bullet(x) \times S^\bullet(y)$, where the product is the exterior product;

(h') follows immediately from the Cartan formula in the main theorem 3.18(h) by considering the product space and using the simple fact that

$$(x \times [Y]) \cdot ([X] \times y) = x \times y$$

in $A(X \times Y)$.

For a given $\alpha \in A_i(Y \times \mathbf{P}^r) = \oplus_j A_j(Y) \otimes A_{i-j}(\mathbf{P}^r)$, write

$$\alpha = \sum_j \beta_j \times [\mathbf{P}^{i-j}]$$

according to the decomposition. Then $f_*(\alpha) = \beta_i$. On the other hand,

$$\begin{aligned} f_* U^\bullet(\alpha, t) &= f_*(\sum_j U^\bullet(\beta_j \times [\mathbf{P}^{i-j}], t)) \\ &= \sum_j f_*(U^\bullet_Y(\beta_j, t) \times U^\bullet_{\mathbf{P}^r}([\mathbf{P}^{i-j}], t)) \\ &= \sum_j f_*(U^\bullet_Y(\beta_j, t) \times (U^\bullet_{\mathbf{P}^r}([\mathbf{P}^{i-j}], t))_0) \end{aligned} \tag{15}$$

by the Cartan formula. However, $(U^\bullet_{\mathbf{P}^r}([\mathbf{P}^a], t))_0 = (U^\bullet_{\mathbf{P}^a}([\mathbf{P}^a], t))_0$ by the closed immersion case, and

$$\begin{aligned} U^\bullet_{\mathbf{P}^a}([\mathbf{P}^a], t) &= w(T_{\mathbf{P}^a}, t)^{-1} \cdot S^\bullet([\mathbf{P}^a], t) \\ &= \frac{w(O_{\mathbf{P}^a}, t)}{w(O_{\mathbf{P}^a}(1), t)^{a+1}} \cdot S^\bullet([\mathbf{P}^a], t) \\ &= (1 + th^{p-1})^{-(a+1)} \cdot [\mathbf{P}^a] \end{aligned}$$

the first equality is by the definition of the operation U; the second equality follows from the Euler sequence of projective spaces; while the third one is Theorem 3.18(f). Consequently, $(U^\bullet_{\mathbf{P}^r}([\mathbf{P}^a], t))_0$ does not vanish only if a is divisible by $(p-1)$, say $a = (p-1)q$, then

$$(U^\bullet_{\mathbf{P}^r}([\mathbf{P}^a], t))_0 = \binom{-q(p-1)-1}{q} \cdot [\mathbf{P}^0] = (-1)^q \binom{pq}{q} [\mathbf{P}^0] = \begin{cases} 0 \bmod p, & \text{if } q \neq 0 \\ 1 \bmod p, & \text{if } q = 0 \end{cases}$$

Combine this with (15), we get

$$f_* U^\bullet(\alpha, t) = f_*(U^\bullet_Y(\beta_i, t) \times [\mathbf{P}^0] = U^\bullet(\beta_i, t) = U^\bullet(f_*(\alpha), t)$$

which completes the proof for the trivial projective bundle case, hence the theorem. □

Remark 4.6 Note that the assumption in the preceding theorem is satisfied in a quite common situation: when $f : X \to Y$ is a *projective* morphism between smooth quasi-projective varieties. Here *projective* is in the usual sense: namely, f is proper (of finite type), and X admits an f-ample line bundle. We remark that if f is projective and Y is quasi-projective, then X is automatically quasi-projective. In fact it is well-known that the pull-back of a sufficient large multiple of an ample line bundle on Y tensors with an relative ample line bundle will get an ample line bundle on X.

4.2 Extending the Twisted Steenrod Operations to Arbitrary Varieties I: via Resolution of Singularities

Theorem 4.5 in the preceding subsection permits us to extend the operations U to varieties which are not necessarily smooth. In this subsection we will invoke the resolution of singularities in this construction, therefore we have to restrict ourselves in characteristic 0, while in the next subsection we will give another approach without using the resolution of singularities to overcome this defect to extend the operations to any characteristic except p.

Construction 4.7 (Twisted Steenrod operations II: general case) Given a variety X (not necessarily smooth) defined over a field k of characteristic 0, we want to define the operations $U^j : A_i(X) \to A_{i-(p-1)j}(X)$, such that they coincide the ones constructed above for smooth varieties, c.f. Definition 4.4. By linearity, it suffices to define $U^\bullet([V])$ for every i-dimensional subvariety V of X. To this end, for a fixed i-dimensional subvariety V of X, take the resolution of singularities $\mu : W \to V$, then W is smooth, μ is birational, projective in the strong sense, consisting of successive blow-up of smooth center in the singular locus of X. We abuse the notation to write $\mu : W \to X$. Then by the preceding construction for smooth varieties, we have the operations $U^j_W : A_i(W) \to A_{i-(p-1)j}(W)$. We define $U^j_X([V])$ to be the image of $U^j_W([W])$ under the proper push-forward $\mu_* : A_{i-(p-1)j}(W) \to A_{i-(p-1)j}(X)$, *i.e.*

$$U^\bullet_X([V], t) = \mu_*(U^\bullet_W([W], t)) = \mu_*([W] \cap w(T_W, t)^{-1})$$

We need to check that such construction is well-defined, see the following two lemmas.

Lemma 4.8 *The above construction is independent of the resolution of singularities chosen.*

Proof. Suppose we are given two resolutions of singularities $\mu' : W' \to V$ and $\mu : W \to V$, then they are dominated by a third one $\mu'' : W'' \to V$

$$\begin{tikzcd} & W'' \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\mu''"] & \\ W \arrow[dr, "\mu"'] & & W' \arrow[dl, "\mu'"] \\ & V & \end{tikzcd}$$

In fact, we can get W'' as the resolution of singularities of the fiber product $W \times_V W'$, thus we can assume that p, q are projective in the strong sense, birational. Thanks to Theorem 4.5, $p_*(U^j([W''])) = U^j(p_*[W'']) = U^j([W])$ and similarly $q_*(U^j([W''])) = U^j([W'])$, hence

$$\mu_*(U^j([W])) = \mu_* p_*(U^j([W''])) = \mu'_* q_*(U^j([W''])) = \mu'_*(U^j([W']))$$

i.e. the two definitions coincide. □

Lemma 4.9 *The construction 4.7 above passes to the rational equivalence classes.*

Proof. Keep the notations of Construction 4.7: let X be an algebraic variety over k of characteristic 0, which is not necessarily smooth. Since the subgroup $\mathcal{Z}_i(X)_{rat}$ consisting of the cycles rational equivalent to 0 is generated by the cycles of the form $[f^{-1}(0)] - [f^{-1}(1)]$, in the following kind of diagram with W an $(i+1)$ dimensional subvariety of $X \times \mathbf{P}^1$, and f flat (or equivalently, dominant) [28]:

$$\begin{tikzcd} W \arrow[rr, hook, "\iota"] \arrow[dr, "f"'] & & X \times \mathbf{P}^1 \arrow[dl, "pr_2"] \\ & \mathbf{P}^1 & \end{tikzcd}$$

therefore it is enough to show that $U^j([f^{-1}(0)]) - U^j([f^{-1}(1)])$ is rational equivalent to 0 for any fixed W as above.

Consider the two fibres involved: $f^{-1}(0)$ and $f^{-1}(1)$, they are Cartier divisors of W. In general, $f^{-1}(0)$, $f^{-1}(1)$ and W are not necessarily smooth. To reduce to the smooth case, we will make use of the theorem of (log-)resolution of singularities, see for example [Laz04, Page 241, Theorem 4.1.3] for a version enough for our purpose. Apply the theorem of (log-)resolution of singularities to $(W, f^{-1}(0), f^{-1}(1))$, we get a projective and birational morphism $\mu : W' \to W$ via a sequence of blow-ups along smooth centers supported in the singular loci of $f^{-1}(0)$, $f^{-1}(1)$ and W, such that W' is smooth, and except(μ) the exceptional locus of μ, together with $\mu^*(f^{-1}(0)) + \mu^*(f^{-1}(1))$ the pull-backs (as divisors) of the two fibres is a divisor with SNC support, in particular, the preimage in W' of any of these two fibres consists of a resolution of singularities and some exceptional components.

Now consider the following diagram:

$$\begin{tikzcd} W' \arrow[r, "\mu"] \arrow[dr, "f'"'] & W \arrow[r, hook, "\iota"] \arrow[d, "f"] & X \times \mathbf{P}^1 \arrow[r, "pr_1"] \arrow[dl, "pr_2"] & X \\ & \mathbf{P}^1 & & \end{tikzcd}$$

For $s = 0$ or 1 in $\mathbf{P}^1$, by the construction above, the fibre $f'^{-1}(s)$ over s of the composition $f' = f \circ \mu$ consists of two parts: one part is a resolution of singularities of the fibre $f^{-1}(s)$, which is a smooth variety birational to $f^{-1}(s)$; and the other part consists of some smooth exceptional components (with simple normal crossings). We denote the first part by $f^{-1}(s)^{\sim}$, since that is the birational transform of $f^{-1}(s)$ via μ^{-1}, and we denote the second part by E_s. We remark that with Theorem 4.5 applied to the inclusion $\alpha : f^{-1}(s)^{\sim} \hookrightarrow W'$, we find that, for $s = 0$ or 1,

$$\begin{aligned} U^j_X([f^{-1}(s)]) &= pr_{1*}\iota_*\mu_*\alpha_* U^j_{f^{-1}(s)^{\sim}}([f^{-1}(s)^{\sim}]) && \text{(by the above construction)} \\ &= pr_{1*}\iota_*\mu_* U^j_{W'}([f^{-1}(s)^{\sim}]) && \text{(by Theorem 4.5)} \end{aligned}$$

[28] c.f. [Ful98, Propostion 1.6]

thus,

$$\begin{aligned}
U_X^j([f^{-1}(0)]) &= pr_{1*}\iota_*\mu_* U_{W'}^j([f^{-1}(0)^\sim]) && \text{(the preceding remark)}\\
&= pr_{1*}\iota_*\mu_*(U_{W'}^j([f^{-1}(0)^\sim]) + E_0) && \text{(the push-forward of exceptional cycles are 0)}\\
&= pr_{1*}\iota_*\mu_*(U_{W'}^j([f'^{-1}(0)])) && \text{(since } [f'^{-1}(0)] = [f^{-1}(0)^\sim] + E_0)
\end{aligned}$$

And we have a similar formula for $U_X^j([f^{-1}(1)])$. Note that $U_{W'}^j([f'^{-1}(0)]) = U_{W'}^j([f'^{-1}(1)])$, since W' is smooth and the twisted Steenrod operations on W' preserve rational equivalence by definition, and that pr_{1*}, ι_*, μ_* also preserve rational equivalence. Therefore we have $U_X^j([f^{-1}(0)]) = U_X^j([f^{-1}(1)])$, as wanted. □

Now we prove that the twisted Steenrod operations commute with proper push-forwards.

Theorem 4.10 (Push-forwards and twisted Steenrod operations II) *Let $f : X \to Y$ be a* proper *morphism between varieties over k of characteristic 0* [29]*, let U_X, U_Y be the twisted Steenrod operations on the $\mathbf{F}_p$ coefficient Chow groups of X, Y respectively. Then the following diagram commutes:*

$$\begin{tikzcd}
A_i(X) \arrow[r, "U_X^j"] \arrow[d, "f_*"'] & A_{i-(p-1)j}(X) \arrow[d, "f_*"] \\
A_i(Y) \arrow[r, "U_Y^j"] & A_{i-(p-1)j}(Y)
\end{tikzcd}$$

It is a generalization of Theorem 4.5 to any proper morphism between two arbitrary varieties.

Proof. We view the proof as the continuation of the proof of Theorem 4.5.

Step 3. The case when $f : X \hookrightarrow Y$ is a closed immersion, without assuming the smoothness of X and Y. In fact, it is obvious by the definition of the twisted Steenrod operation 4.7.

Step 4. The case when $f : X = Y \times \mathbf{P}^r \to Y$ is a trivial projective bundle, without the assumption that X and Y are smooth.
For this case, we first notice that $A_*(X) = A_*(Y) \otimes A_*(\mathbf{P}^r)$, thus by linearity, we can focus on a element in $A_*(X)$ of the form $[V]\times[\mathbf{P}^j]$, where V is a subvariety of Y, and $[\mathbf{P}^j]$ is a sub-linear space of $\mathbf{P}^r$. Now take any resolution of singularities of V, say $\mu : V' \to V$, we can assume that μ is birational and projective in the strong sense. Make the base change:

$$\begin{tikzcd}
W' = V' \times \mathbf{P}^r \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "f'"'] & X = Y \times \mathbf{P}^r \arrow[d, "f"] \\
V' \arrow[r, "\mu"'] & Y
\end{tikzcd}$$

[29]This assumption can be replaced by $\mathrm{char}(k) \neq p$, see the next subsection.

Now the result follows from the defintion in 4.7.

Step 5. The case when $f : X \to Y$ is projective in the strong sense, *i.e.* admits a factorization:

$$X \hookrightarrow Y \times \mathbf{P}^r \twoheadrightarrow Y$$

In this case, we conclude immediately from Step 3 and Step 4.

Step 6. The case when $f : X \to Y$ is projective (in the usual sense of Grothendieck [30]), with Y quasi-projective[31]. We will see this implies f is projective in the strong sense.

In this case, since f is projective, f can be factorized as:

$$f : X \hookrightarrow \mathbf{P}(E) \to Y$$

where the first map is a closed immersion, and the second one is a projective bundle, E is a vector bundle over Y. Since twisting E with a line bundle on Y does not change the associated projective bundle $\mathbf{P}(E)$, we can replace E by $E \otimes A^{\otimes N}$, where A is an ample line bundle on Y, and choose N sufficiently large to make E generated by its global sections:

$$H^0(Y, E) \otimes_k O_Y \to E \to 0$$

Taking the associated projective bundle, we obtain

$$\mathbf{P}(E) \hookrightarrow Y \times \mathbf{P}(V) \twoheadrightarrow Y$$

where V is the vector space $H^0(Y, E)$. We note that in the above displayed line, the first map is a closed immersion, while the second one is a trivial projective bundle. Composed with $X \hookrightarrow \mathbf{P}(E)$, we find f is projective in the strong sense:

$$X \hookrightarrow Y \times \mathbf{P}(V) \twoheadrightarrow Y$$

Therefore, we arrive at the situation of Step 5 above.

Step 7. The case when $f : X \to Y$ is proper, and Y quasi-projective.

In this case, we apply Chow's lemma to have the following diagram:

$$\begin{tikzcd} X' \arrow[rr, "\mu"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

where f' is projective, hence also μ, and $\mu : X' \to X$ is a Chow envelope, which is birational surjective and projective in the strong sense. Apply Step 3 and Step 6 to μ and f' respectively, we know the push-forwards and the Steenrod operations commute for μ and for f', hence commute also for f since μ_* is surjective.

[30] That is, f is proper of finite type and X admits an f-ample line bundle, see Remark 4.6.

[31] Then X is automatically quasi-projective, see Remark 4.6.

Step 8. The general case when $f : X \to Y$ is proper in general.
In this case, we apply the Chow' Lemma to Y to get the following diagram:

$$\begin{tikzcd} X' \arrow[r, "\mu'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\mu"'] & Y \end{tikzcd}$$

where μ, μ' are Chow envelopes which are birational, projective in the strong sense, Y' is quasi-projective, and f' is still proper. Since μ'_* is surjective , therefore applying Step 5 and Step 7, we know that the push-forwards and the Steenrod operations commute for μ, μ' and for f', which implies they also commute for f. This completes the proof. □

In fact, like the Steenrod operations $S^\bullet$ provides an $\mathscr{S}$-module structure on $A_*(X)$ for any smooth variety X, the twisted Steenrod operations $U^\bullet$ also give an $\mathscr{S}$-module structure on $A_*(X)$ for any variety X [32], for this, we need to verify the Adem relations:

Theorem 4.11 (Adem relations for $U^\bullet$: characteristic 0) *Let X be any variety over k of characteristic 0* [33]*, $U^\bullet$ be the operations on $A_*(X)$ defined above. Then they satisfy Adem's relations (12): For any $0 < a < pb$,*

$$U^a \circ U^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} U^{a+b-j} \circ U^j$$

Proof. In [Mer03], Merkurjev gives a conceptual proof for this fact. He extends the notion of Steenrod operations such that for every finite support multi-index I, there is a corresponding operation S^I, where $S^{(i,0,0,\cdots)} = S^i$. He also extends the Chern classes to multi-indexed Chern classes such that $C_{(i,0,0,\cdots)}(E) = w(E,t)$, and we set $U^\bullet := C_\bullet \cdot S^\bullet$. Then $U^{(i,0,0,\cdots)} = U^i$. He gives an easy criterion to check when a series of actions s^I on an abelian group gives an $\mathscr{S}$-module structure. Through this, he can easily prove that: as long as $\{S^I\}_I$ gives an $\mathscr{S}$-module structure, which is equivalent to the Adem relation for the S^i's, we have $\{U^I\}_I$ also gives an $\mathscr{S}$-module structure, which is equivalent to the Adem relation for the U^i's. For the details, see [Mer03]. □

Consequently, combining the above two theorems, we have the following:

Theorem 4.12 ($\mathscr{S}$-module structure II: general case) *Let X be any variety over k of characteristic 0* [34]*, $U^\bullet$ be the twisted Steenrod operations on $A_*(X)$. Then the graded Chow group $A_*(X)$ is a graded $\mathscr{S}$-module, defined by*

[32] Of course, the two $\mathscr{S}$-module structures are different when X is smooth, they differ by a *twist* of the w class of the tangent bundle.

[33] This assumption can be replaced by $\mathrm{char}(k) \neq p$, see the next subsection.

[34] This assumption can be replaced by $\mathrm{char}(k) \neq p$, see the next subsection.

U, and the operations of $\mathscr{S}$ on $A_*(X)$ commute with the functorial proper push-forwards.
In a more formal language: $A_*(-) : (\mathcal{V}ar_k, \text{proper morphisms}) \to {}_{\mathscr{S}}\mathcal{M}od^{\mathbf{N}}$ is a (covariant) functor from the category of varieties over k with proper morphisms to the category of graded (left) $\mathscr{S}$-module.

Remark 4.13 We will also consider the the action of other basis elements of $\mathscr{S}$ over the graded Chow group of a variety, denoted as, for any multi-index I:

$$U^I : A_j(X) \to A_{j-|I|}(X)$$

with $U^i = U^{(i,0,\dots)}$. See Remark 3.17. By the preceding theorem, these operations commute with proper push-forwards.

4.3 Extending the Twisted Steenrod Operations to Arbitrary Varieties II: via Embeddings

In the preceding subsection, we have constructed the twisted Steenrod operations on the $\mathbf{F}_p$ coefficient Chow groups of an arbitrary variety over a field of characteristic 0. There are two main ingredients in that approach, namely Theorem 4.5 and resolution of singularities.

In this subsection, we want to sketch another approach to the construction of the twisted Steenrod operations, in which Theorem 4.5, or rather the excess intersection formula, remains an important ingredient, while resolution of singularities is no longer needed, hence our construction and properties are valid in any characteristic not equal to p. We will follow the line in Brosnan [Bro03]. We will assume that $\mathrm{char}(k) \neq p$ in this subsection.

The strategy here is to do everything we have done in subsection 3.4.2 in a more general way. Explicitly, we proceed in several steps as follows:

Step 1.

We first deal with the varieties which admit embeddings into smooth varieties, say X is a variety over k of dimension n, and $X \hookrightarrow W$ is a closed immersion with W smooth of dimension m. We have the following cartesian diagram:

$$\begin{tikzcd} X \arrow[r, hook, "\delta_X"] \arrow[d, hook] & X^p \arrow[d, hook] \\ W \arrow[r, hook, "\delta_W"] & W^p \end{tikzcd}$$

Since W is smooth, the bottom arrow is a regular closed embedding. Similar to the construction in subsection 3.4.2, we take the $\mathbf{Z}/p$-equivariant Gysin pull-back:

$$\delta_W^! : A_{pi}^{\mathbf{Z}/p}(X^p) \to A_{pi-(p-1)m}^{\mathbf{Z}/p}(X)$$

composed with the p-th power map defined in subsection 3.4.1:

$$P : A_i(X) \to A_{pi}^{\mathbf{Z}/p}(X^p)$$

we get

$$D_X^W : A_i(X) \to A^{\mathbf{Z}/p}_{pi-(p-1)m}(X) = \bigoplus_{j\geq 0} A_{pi-(p-1)m+j}(X)\cdot l^j$$

where the equality is from Appendix A.

We can prove the analogies of Lemma 3.24 (additivity) and Lemma 3.25 for D_X^W:

Lemma 4.14 (=Theorem 8.2 in [Bro03]) *D_X^W is a group homomorphism.*

Lemma 4.15 (=Theorem 8.3 in [Bro03]) *All terms of $D_X^W(\alpha)$ with degree not divisible by $(p-1)$ are 0.*

We refer to the paper for the proofs.

By the preceding lemmas, we can define the following operation as in Definition 3.26 (or rather the remark after it): for any $\alpha \in A_i(X)$

$$S^{W,\bullet}(\alpha)(t) := t^{m-i} D_X^W(\alpha, 1/t)$$

By the same argument, we can prove the following 'main' theorem as in Theorem 3.18:

Theorem 4.16 (Analogous to the main theorem 3.18) *Let X be an algebraic variety over k with $\mathrm{char}(k) \neq p$, of dimension n, $X \hookrightarrow W$ is a closed embedding into a smooth variety of dimension m. A_* be the $\mathbf{F}_p$-coefficient Chow group. The operation $S^{W,\bullet}$ constructed above satisfy: for any i,*

$$S^{W,i} : A_j(X) \to A_{j-i(p-1)}(X)$$

for any non-negative integers j, and:

(a)(Additivity) *Each $S^{W,i}$ is additive, i.e. a homomorphism of groups;*

(b)(Functoriality) *For any inclusion of subvarieties of W, $f : X \hookrightarrow Y$, and any i, q, the operation S^i commutes with the restriction map:*

$$f^! S^{W,i} = S^{W,i} f^! : A_j(Y) \to A_{j-i(p-1)}(X)$$

(c)(Range) *$S^{W,i} : A_j(X) \to A^{j-i(p-1)}(X)$ is the zero map if $i > m-j$ or $i < 0$;*

(d) *$S^{W,0} = \mathrm{id}$;*

(e) *$S^{W,\bullet}([X]) = [X] \cap w(N_{X/W})$ if X is smooth too.*

(f)(Cartan formula) *For any $x \in A_*(X)$, $y \in A_*(Y)$, and two embeddings $X \hookrightarrow W$, $Y \hookrightarrow U$, then we have*

$$S^{W\times U,\bullet}(x \times y) = S^{W,\bullet}(x) \times S^{U,\bullet}(y)$$

where the product is the exterior product;

(g)(Adem relations) *For any $0 < a < pb$, we have (12):*

$$S^{W,a} \circ S^{W,b} = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} S^{W,a+b-j} \circ S^{W,j}$$

Step 2.

To rule out the dependence of the construction on the choice of the embedding space W, we can prove the following lemma, which is essentially in the same spirit of Theorem 4.5, and their proofs rely on the same basic fact, namely the excess intersection formula Theorem 2.22. More precisely:

Lemma 4.17 (=Proposition 6.3 in [Bro03]) *Let $j_i : X \to W_i$ be two embeddings of X into two smooth varieties. Then*

$$c_{top}^{\mathbf{Z}/p}(R \otimes T_{W_2}|_X) \cap \delta_{W_1}^! = c_{top}^{\mathbf{Z}/p}(R \otimes T_{W_1}|_X) \cap \delta_{W_2}^!$$

where c_{top} means ' top Chern class', and R is the standard representation of $\mathbf{Z}/p$.

For the proof of the lemma, see Brosnan [Bro03].

Corollary 4.18 (=Proposition 8.10 in [Bro03]) *If W_1 and W_2 are two smooth varieties containing X, then*

$$S^{W_1,\bullet}(\alpha) \cap w(T_{W_2}) = S^{W_2,\bullet}(\alpha) \cap w(T_{W_1})$$

Proof. It follows immediately from the proceding lemma and the calculation of the equivariant top Chern classes of the form $c_{\mathrm{top}}^{\mathbf{Z}/p}(R \otimes E)$ with E a vector bundle over X, see Example 4.3. □

Now we have the definition:

Definition 4.19 (=Definition 8.13 in [Bro03]) *For a variety X embedded in any smooth variety W, we define the twisted Steenrod operation:*

$$U_X^\bullet(\alpha, t) = S^{W,\bullet}(\alpha, t) \cap w(T_W, t)^{-1}$$

From the preceding corollary, it is a well-defined operation, *i.e.* independent of the choice of embedding.

Step 3.

Extend the operations U to arbitrary varieties.
By Step 2, we can deal with all quasi-projective varieties, for an arbitrary variety X, by Chow's lemma, we can take its *Chow envelope*:

$$\mu : X' \to X$$

where X' is quasi-projective, and μ is birational, projective. Then we can define the twisted Steenrod operations on X by:

$$U_X^\bullet(\alpha) = \mu_* U_{X'}^\bullet(\alpha')$$

where α' is a cycle of X' such that $\mu_*(\alpha') = \alpha$.

We should verify that the above definition is independent of the choice of α' and X', we refer to the paper [Bro03] §10.

Step 4.

To see such defined operations coincide with the ones constructed in Construction 4.7 in characteristic 0, it suffices to verify that the operations constructed above also commute with the proper push-forward.

Theorem 4.20 (=Proposition 10.3 in [Bro03]= Theorem 4.10) *If $f : X \to Y$ is a proper morphism, then $f_* U^\bullet = U^\bullet f_*$*

For the proof, we refer to [Bro03], Proposition 10.3.

Step 5.

The Adem relations Theorem 4.11 can be extended to any characteristic different from p:

Theorem 4.21 (Adem relations for $U^\bullet$: any characteristic) *Let X be any algebraic variety over k with $\mathrm{char}(k) \neq p$, $U^\bullet$ be the operations on $A_*(X)$ defined above. Then they satisfy Adem's relations (12): For any $0 < a < pb$,*

$$U^a \circ U^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} U^{a+b-j} \circ U^j$$

And consequently, Theorem 4.12 can be formulated with the characteristic 0 assumption replaced by $\mathrm{char}(k) \neq p$.
For the proof of the above theorem, see [Bro03] §11.

5 An Application: the Degree Formula

In this section, we want to give an example where we use the Steenrod operations to solve some geometric problems. More concretely, we will talk about the degree formula and some of its interesting properties. The exposition here will more or less follow the article [Mer03].

We fix the notations and settings first. Let X be a variety over a field k, of dimension d, with the structure morphism $q : X \to \mathrm{Spec}\, k$. We also fix a prime number $p \neq \mathrm{char}(k)$, and write $A_i(X) = \mathrm{CH}_i(X) \otimes \mathbf{Z}/p$, where CH_* is the $\mathbf{Z}$-coefficient Chow group. Recall that for a closed point $x \in X$, its *degree* is defined to be the integer:

$$\deg(x) := [\kappa(x) : k]$$

where $\kappa(x)$ is the residual field of x, finite dimensional over k.

First of all, we introduce an integer associated to a variety, concerning the degrees of its rational points:

Definition 5.1 Given a variety X over k, we set

$$n_X := \gcd \deg(x)$$

where the greatest common divisor is taken over all closed points of X. It is easy to see another equivalent definition of this number:

$$n_X = \gcd[L : k]$$

where L is taken over all finite field extensions of k such that $X(L) \neq \emptyset$.

Now we will make a definition through the following argument.

The proper push-forward property of twisted Steenrod operations in Theorem 4.10 (or Remark 4.13) yields the commutative diagram: for any multi-index I with $|I| = d$

$$\begin{tikzcd} A_d(X) \arrow[r, "U^I"] \arrow[d, "q_*"'] & A_0(X) \arrow[d, "q_*"] \\ A_d(\operatorname{Spec} k) = 0 \arrow[r, "U^I"'] & A_0(\operatorname{Spec} k) = \mathbf{Z}/p \end{tikzcd}$$

For the action of the Steenrod operations of the form U^I, see Remark 3.17, and Remark 4.13. Consider $U^I([X]) \in A_0(X)$, take one of its representatives in $\mathrm{CH}_0(X)$, that is, an element $u_X^I \in \mathrm{CH}_0(X)$ whose modulo-p class in $A_0(X)$ is $U^I([X])$. However, $A_d(\operatorname{Spec} k) = 0$ implies that $q_*(U^I([X])) = 0$, which means $\deg(u_X^I)$ is an integer divisible by p, thanks to the following commutative diagram:

$$\begin{tikzcd} \mathrm{CH}_0(X) \arrow[r] \arrow[d, "q_*=\deg"'] & A_0(X) \arrow[d, "q_*"] \\ \mathrm{CH}_0(\operatorname{Spec} k) = \mathbf{Z} \arrow[r] & A_0(\operatorname{Spec} k) = \mathbf{Z}/p \end{tikzcd}$$

If we change the representative u_X^I, then it differs by a zero-cycle with coefficients divisible by p, consequently its degree differs by a multiple of pn_X, by Definition 5.1. As a result, the integer $\frac{1}{p}\deg(u_X^I)$ is independent of the choice of the representative up to a multiple of n_X. Thus the following congruence class is well-defined:

Definition 5.2 Keep the notations as above, we define a class in $\mathbf{Z}/n_X\mathbf{Z}$, for any multi-index I with $|I| = \dim X$:

$$I^p(X) := \frac{1}{p}\deg(u_X^I) \mod n_X$$

Since $\deg(u_X^I)$ is always a multiple of n_X, it is clear that

$$p \cdot I^p(X) = 0 \in \mathbf{Z}/n_X\mathbf{Z}$$

Theorem 5.3 (Degree formula) *Let X, Y be varieties over k with $\operatorname{char}(k) \neq p$, $f : X \to Y$ be a k-morphism. Denote $d = \dim(X)$, n_X and n_Y as in Definition 5.1. Then:*

- *n_Y divides n_X:*

$$n_Y | n_X$$

- *For any multi-index I of norm $|I| = d$, and any prime $p \neq \mathrm{char}(k)$, we have the* degree formula*:*

$$I^p(X) = \deg(f) \cdot I^p(Y) \in \mathbf{Z}/n_Y\mathbf{Z}$$

In the second point, the left hand side is understood to pass from $\mathbf{Z}/n_X\mathbf{Z}$ to $\mathbf{Z}/n_Y\mathbf{Z}$ thanks to the first point.

PROOF. For the first assertion, since n_Y divides the degree of every closed point of Y, thus for any closed point of X, say x, denoting $y = f(x)$, we have

$$\deg(x) = [\kappa(x) : k] = [\kappa(x) : \kappa(y)] \cdot [\kappa(y) : k]$$

is divisible by n_Y. Consequently, n_Y divides the greatest common divisor of the degrees of the closed points of X, which is, by definition, n_X.

For the second assertion, the degree formula, we apply Theorem 4.10 or rather Remark 4.13 to obtain the commutative diagram:

$$\begin{tikzcd} A_d(X) \arrow[r, "U^I_X"] \arrow[d, "f_*"'] & A_0(X) \arrow[d, "f_*"'] \\ A_d(Y) \arrow[r, "U^I_Y"'] & A_0(Y) \end{tikzcd}$$

which yields:

$$f_*(U^I([X])) = U^I(f_*([X])) = \deg(f) \cdot U^I([Y])$$

Take $u^I_X \in \mathrm{CH}_0(X)$ and $u^I_Y \in \mathrm{CH}_0(Y)$ representatives of $U^I_X \in A_0(X)$ and $U^I_Y \in A_0(Y)$ respectively as in Definition 5.2, then the last displayed equality implies:

$$f_*(u^I_X) \in \deg(f)u^I_Y + p\,\mathrm{CH}_0(Y) \tag{16}$$

Let q_X, q_Y be the structure morphisms of X and Y to $\mathrm{Spec}\, k$ respectively. Then we have the commutative diagram.

$$\begin{tikzcd} \mathrm{CH}_0(X) \arrow[rr, "f_*"] \arrow[dr, "q_{X*}=\deg"'] & & \mathrm{CH}_0(Y) \arrow[dl, "q_{Y*}=\deg"] \\ & \mathrm{CH}_0(\mathrm{Spec}\, k) = \mathbf{Z} & \end{tikzcd}$$

Applying q_{Y*} to (16), we find

$$\deg(u^I_X) = q_{X*}(u^I_X) = q_{Y*}f_*(u^I_X) \in \deg(f)\deg(u^I_Y) + pn_Y\mathbf{Z}$$

then dividing by p, we get:

$$\frac{1}{p}\deg(u^I_X) \in \deg(f) \cdot \frac{1}{p}\deg(u^I_Y) + n_Y\mathbf{Z}$$

i.e. $I^p(X) = \deg(f)I^p(Y) \bmod n_Y$ as wanted. □

Now we turn to some applications of the degree formula.

Definition 5.4 (I^p-rigidity) Let X be a variety over k of dimension d. Fix a prime number $p \neq \mathrm{char}(k)$. For a multi-index I with $|I| = d$. We say that X is *I^p-rigid*, if $I^p(X) \neq 0$ in $\mathbf{Z}/n_X\mathbf{Z}$.

Proposition 5.5 *Given X a d-dimensional variety over k, $p \neq \mathrm{char}(k)$, I a multi-index with norm $|I| = d$. Let v_p be the p-adic valuation. Then:*

1. $v_p(n_X) \leq v_p(\deg(u_X^I))$;
2. *X is I^p-rigid if and only if $v_p(n_X) = v_p(\deg(u_X^I))$, and in this case, they are strictly positive.*

Proof. The first assertion is obvious, since n_X divides the degree of any 0-cycle. The second assertion also follows readily from the definition: since X is I^p-rigid, $\frac{1}{p}\deg(u_X^I)$ is not divisible by n_X, while as in the first assertion, $\deg(u_X^I)$ is divisible by n_X. These imply that n_X is a multiple of p, and $\deg(u_X^I)$, as a multiple of n_X, has no more p-powers than n_X, in other words, $v_p(n_X) = v_p(\deg(u_X^I)) > 0$ as wanted; the converse is obvious. □

The following theorem is the main result of this section, recall that for a correspondence $\alpha : X \vdash Y$, the *multiplicity* of α is defined to be the degree of α when projects to X.

Theorem 5.6 *Let X and Y be varieties over a field k. Fix a prime $p \neq \mathrm{char}(k)$. Let I be a multi-index such that $|I| = d := \dim(X) > 0$. Suppose that:*

1. *there is a correspondence $\alpha : X \vdash Y$ of multiplicity not divisible by p;*
2. *X is I^p-rigid;*
3. $v_p(n_X) \leq v_p(n_Y)$.

Then:

1. $\dim(X) \leq \dim(Y)$;
2. *if* $\dim(X) = \dim(Y)$, *then*
   - *there is a correspondence $\beta : Y \vdash X$ of multiplicity not divisible by p;*
   - *Y is I^p-rigid;*
   - $v_p(n_X) = v_p(n_Y)$.

Proof. We can assume that α is a *prime* correspondence, that is, an irreducible subvariety Z of $X \times Y$. Denote the projections to two factors $\pi_1 : Z \to X$, $\pi_2 : Z \to Y$.

If $\dim(X) \geq \dim(Y)$, set $m := \dim(X) - \dim(Y)$. Consider $Y' := Y \times \mathbf{P}^m_k$, and embeds Y into Y' by $Y \simeq Y \times \{\mathrm{pt}\} \subset Y'$, where pt is a rational point of $\mathbf{P}^m_k$, then $n_Y = n_{Y'}$. Now Z can be viewed as a subvariety of $X \times Y'$. We also denote the two projections $\pi_1 : Z \to X$, $\pi_2' : Z \to Y'$.

By the degree formula (Theorem 5.3),

$$I^p(Z) = \deg(\pi_1) I^p(X) \in \mathbf{Z}/n_X\mathbf{Z}$$

By assumption 2, X is I^p-rigid, we have $I^p(X) \neq 0 \in \mathbf{Z}/n_X$, but $pI^p(X) = 0 \in \mathbf{Z}/n_X$, *i.e.* $I^p(X)$ lacks exactly one power of p to be divisible by n_X, however, $\deg(\pi_1)$ is not divisible by p by assumption 1, which implies that

$$I^p(Z) \neq 0 \in \mathbf{Z}/n_X\mathbf{Z} \tag{17}$$

still lacking one power of p.

Applying again the degree formula to the projection $\pi_2' : Z \to Y'$, we obtain:

$$I^p(Z) = \deg(\pi_2') I^p(Y') \in \mathbf{Z}/n_Y' = \mathbf{Z}/n_Y \tag{18}$$

(17) combined with the third assumption $v_p(n_X) \leq v_p(n_Y)$, implies that

$$I^p(Z) \neq 0 \in \mathbf{Z}/n_Y\mathbf{Z}$$

lacking at least one power of p. Combining this with (18) we find that

$$I^p(Y') \neq 0 \in \mathbf{Z}/n_Y \tag{19}$$

However $pI^p(Y') = 0$ in $\mathbf{Z}/n_Y$ implies that $\deg(\pi_2')$ is not divisible by p, in particular, it is non-zero, *i.e.* π_2' is dominating. As a consequence, $m = 0$ (*i.e.* $\dim(X) = \dim(Y)$) and $Y = Y'$, $\pi_2' = \pi_2$. Thus (19) means exactly that Y is I^p-rigid as wanted. We just take the correspondence $\beta : Y \vdash X$ to be defined by Z, the fact that p does not divide $\deg(\pi_2)$ means the multiplicity of β is not divisible by p as wanted. Finally, $pI^p(Y') = 0$ in $\mathbf{Z}/n_Y$ implies that $I^p(Y')$ and hence $I^p(Z)$ lacks exactly one power of p to be divisible by n_Y, as the same situation for n_X in (17), as a result, $v_p(n_X) = v_p(n_Y)$. This completes the proof. □

By the preceding theorem, the I^p-rigidity relates to the *p-compressibility*:

Definition 5.7 (p-compressibility) *Given a variety X over a field k is called* p-compressible, *if there is a rational map $X \dashrightarrow Y$ to another variety over k such that* $\dim(Y) < \dim(X)$ *and* $v_p(n_Y) \geq v_p(n_X)$.

Since a rational map can be viewed as correspondence of multiplicity 1, the theorem implies the following:

Corollary 5.8 (Rigidity to incompressibility) *Let X be a variety of dimension d over a field k, p be a prime which not equals to the characteristic of k, I be a multi-index of norm $|I| = d$. Suppose that X is I^p-rigid, then X is not p-compressible.*

Finally, as a complement, we give the birational invariance of n_X and $I^p(X)$ for smooth projective varieties.

Remark 5.9 (Birational nature of n_X and $I^p(X)$) *Let k be a field, and p be a prime not equals to the characteristic of k. Let I be a multi-index with norm $|I| = d$. Then:*

1. *the integer n_X is a birational invariant for smooth projective varieties X over k of dimension d;*
2. *the congruence class $I^p(X) \in \mathbf{Z}/n_X\mathbf{Z}$ is birational invariant for smooth projective varieties X over k of dimension d.*

PROOF. For the first point, we define an integer n_L for any finite generated field extension $L \supset k$ as following: let Ω be the set of all valuations v of L such that $v|_k = 0$, and that the residue field $\kappa(v)$ is finite over k. Define $n_L := \gcd_{v\in\Omega}[\kappa(v) : k]$. Now let $L = k(X)$ be the function field of X. On one hand, since X is smooth, a classical argument gives a valuation $v \in \Omega$, with the same residue field as $\kappa(x)$. Therefore n_X is the greatest common divisor taken over a smaller rang than n_L does, thus n_L divides n_X. Now we recall the 'classical' argument: since the local ring $\mathcal{O}_x$ of every (closed) point $x \in X$ is regular, thus the maximal ideal $\mathfrak{m}_x$ is generated by a regular sequence, which gives a chain of d places, now by [ZS75, Theorem 37, Page 106], we have a composition of d discrete valuation of rank 1, which is a valuation with the residue field $\kappa(x)$.

On the other hand, for a fixed valuation $v \in \Omega$, we have the valuation ring $\mathcal{O} = \{a|v(a) \geq 0\} \subset L$ and its maximal ideal $\mathfrak{m} = \{a|v(a) > 0\} \subset \mathcal{O}$, by the valuative criterion of properness, the natural morphism $\operatorname{Spec} L \to X$ factorizes uniquely through certain $\operatorname{Spec} \mathcal{O} \to X$ since X is assumed to be projective. This map gives a closed point $x \in X$ by the image of the closed point defined by $\mathfrak{m}$ in $\operatorname{Spec} \mathcal{O}$. Therefore, we have a tower of finite extensions: $k \subset \kappa(x) \subset \kappa(v)$, thus $[\kappa(x) : k]$ divides $[\kappa(v) : k]$, hence n_X divides $[\kappa(v) : k]$. Therefore n_X divides n_L. In conclusion, $n_L = n_X$ for smooth variety X, which implies n_X is a birational invariant for smooth varieties.

For the second point, let X, Y be birational equivalent smooth varieties, with function field L. The birational map gives a correspondence of degree 1 between them, which is a subvariety of $X \times Y$, called Z. Then Z has birational morphisms to X and to Y given by the projections. Applying the degree formula to the two projections, we find that $I^p(Z) = I^p(X)$ in $\mathbf{Z}/n_X = \mathbf{Z}/n_L$ and $I^p(Z) = I^p(Y)$ in $\mathbf{Z}/n_Y = \mathbf{Z}/n_L$. As a result, $I^p(X) = I^p(Y)$ in $\mathbf{Z}/n_L$, as wanted. □

Appendix

A Chow Theory of Cyclic Groups

In this appendix, we want to calculate $A_i^{\mathbf{Z}/p}(\mathrm{pt})$, the $\mathbf{Z}/p$-equivariant Chow groups of a point[35], a result which has already been used in our construction. We will constrained in the case that k contains the p^{th} roots of unity, in particular, when $\mathsf{char}(k) \neq p$. For the general case, see the treatment of Brosnan [Bro03] §7.

Write $G = \mathbf{Z}/p$, by the definition of the equivariant Chow groups, we take a $U \in \mathrm{EG}_r$ of dimension n, with r sufficiently large, then:

$$A_i^{\mathbf{Z}/p}(\mathrm{pt}) = A_{i+n}(\frac{U}{\mathbf{Z}/p})$$

from which we find that only for $i \leq 0$, it is non-zero. In spirit, recall that U is an approximation of the EG in algebraic topology, thus $\frac{U}{\mathbf{Z}/p}$ is an approximation of the BG, the classifying space of G, in algebraic topology. So the issue is to calculate the Chow theory of the classifying space of $\mathbf{Z}/p$.

To this end, we need the following construction of approximations of EG and BG. Fix a primitive p-th root of unity $\zeta \in k$. Write $\mathbf{A}_k^n$ to be the affine n-space, with a $\mathbf{Z}/p$-action being multiply by the roots of unity:

$$\begin{aligned} \mathbf{Z}/p \times \mathbf{A}_k^n &\to \mathbf{A}_k^n \\ (j \pmod p, z) &\mapsto \zeta^j \cdot z \end{aligned}$$

From now on, we will use $\mathbf{Z}/p$ and $\mu_p(k)$ interchangeably.

For the approximation of EG, it is quite natural to take $U \in \mathrm{EG}_{r+1}$ to be $\mathbf{A}_k^{r+1} - \{0\}$. It is obvious that U is a $\mathbf{Z}/p$-principal bundle. And then we get the 'approximation' of BG: set $B = \frac{U}{\mathbf{Z}/p}$. Now by Proposition 2.9,

$$A_i^{\mathbf{Z}/p}(\mathrm{pt}) = A_{i+r+1}(B)$$

for any $i \geq -r$.

To compute $A_*(B)$, we note that there is a *fundamental fibration* (c.f. Lemma 2.25) over the projective space:

$$\frac{\mathbf{G_m}(k)}{\mu_p(k)} \hookrightarrow B = \frac{\mathbf{A}_k^{r+1} - \{0\}}{\mu_p(k)} \to \mathbf{P}_k^r = \frac{\mathbf{A}_k^{r+1} - \{0\}}{\mathbf{G_m}(k)} \tag{20}$$

where $\frac{\mathbf{G_m}(k)}{\mu_p(k)} = \frac{k^*}{\mathbf{Z}/p}$ denotes the fibre type. Thanks to the isomorphism

$$\frac{k^*}{\mathbf{Z}/p} \xrightarrow{\simeq} k^* \tag{21}$$

which takes z to z^p, we can view the fiber bundle (20) as a k^*-bundle, and let (21) determine the coordinate on the fibers. Now let us figure out the

[35] By a *point*, denoted pt, we means $\mathrm{Spec}(k)$

'transition functions'. Note that $\mathbf{A}_k^{r+1} - \{0\}$ is the complement of the zero section of Serre's tautological line bundle $\mathcal{O}_{\mathbf{P}^r}(-1)$. We cover the projective space $\mathbf{P}_k^r$ by $(r+1)$ standard charts, the transition functions of the k^*-bundle $\mathbf{A}_k^{r+1} - \{0\}$ are of the form $\left(\frac{z_i}{z_j}\right)$, the transition functions for the $\frac{k^*}{\mathbf{Z}/p}$-bundle $B \to \mathbf{P}^r$ in (20) is in the same form. Then when one uses the coordinate of fiber as (21), the transition functions of $B \to \mathbf{P}^r$ as a k^*-bundle is of the form $\left(\frac{z_i}{z_j}\right)^p$, which is the transition functions of the line bundle $\mathcal{O}_{\mathbf{P}^r}(-p)$. So we conclude that $B \to \mathbf{P}^r$ is the complement of the zero section of the line bundle $\mathcal{O}_{\mathbf{P}^r}(-p)$ on $\mathbf{P}^r$.

By the exact sequence in [Ful98] Proposition 1.8, we have the short exact sequence:

$$A_*(\mathbf{P}^r) \xrightarrow{i_*} A_*(\mathcal{O}_{\mathbf{P}^r}(-p)) \to A_*(B) \to 0$$

where we write the zero section simple as $\mathbf{P}^r$, and the inclusion is denoted $i : \mathbf{P}^r \hookrightarrow \mathcal{O}_{\mathbf{P}^r}(-p)$. We know that the flat pull-back of a vector bundle projection is an isomorphism: $A_{*-1}(\mathbf{P}^{r-1}) \to A_*(\mathcal{O}_{\mathbf{P}^r}(-p))$, the inverse isomorphism is the Gysin map: $i^* : A_*(\mathcal{O}_{\mathbf{P}^r}(-p)) \to A_{*-1}(\mathbf{P}^{r-1})$. Through this identification, we have:

$$\begin{tikzcd}
A_*(\mathbf{P}^r) \arrow[r, "i_*"] \arrow[dr, "c_1(\mathcal{O}(-p))\cap"'] & A_*(\mathcal{O}_{\mathbf{P}^r}(-p)) \arrow[r] \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\simeq"] & A_*(B) \arrow[r] & 0 \\
& A_{*-1}(\mathbf{P}^r) \arrow[u, bend left, "\pi^*"] & &
\end{tikzcd}$$

Since we are in the $\operatorname{mod} p$ case, the morphism $c_1(\mathcal{O}(-p))\cap$ is 0, from which we get that $A_*(B) = \mathbf{Z}$, and the generator is the hyperplane section, which implies $A_i^{\mathbf{Z}/p}(\mathrm{pt}) = A_{i+r+1}(B) = \mathbf{Z}$ for $i \geq -r$, thus $A_i^{\mathbf{Z}/p}(\mathrm{pt}) = \mathbf{Z}$ for any $i \leq 0$.

Taking the multiplication relation into account, and changing into codimensional notation, we conclude the following theorem:

Theorem A.1 (Chow groups of the classifying space of a cyclic group)

$$A^{\mathbf{Z}/p,*}(\mathrm{pt}) = \mathbf{F}_p[l]$$

where l is of degree 1 (i.e. codimension 1), which is the hyperplane section generator.

Similarly, we have:

Theorem A.2 *Let X be a variety with trivial $\mathbf{Z}/p$ action, then its equivariant Chow groups are:*

$$A_*^{\mathbf{Z}/p}(X) = A_*(X) \otimes_{\mathbf{F}_p} \mathbf{F}_p[l]$$

where l is of degree 1, i.e. of codimension 1.

In particular, $A_i^{\mathbf{Z}/p}(X) = \oplus_{j \geq 0} A_{i+j}(X) l^j$.

Proof. We keep the notation in the preceding argument. Since the action on X is trivial, we have $\frac{X \times U}{\mathbf{Z}/p} = X \times B$. One notes that B admits a cell-decomposition, thus $A_*(X \times B) = A_*(X) \otimes A_*(B)$, by [Ful98] Example 1.10.2. Combining these with the preceding theorem, we have the required result. $\square$

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