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DISTINGUISHED CYCLES ON VARIETIES WITH MOTIVE OF ABELIAN TYPE AND THE SECTION PROPERTY

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Abstract

A remarkable result of Peter O'Sullivan asserts that the algebra epimorphism from the rational Chow ring of an abelian variety to its rational Chow ring modulo numerical equivalence admits a (canonical) section. Motivated by Beauville's splitting principle, we formulate a conjectural Section Property which predicts that for smooth projective holomorphic symplectic varieties there exists such a section of algebra whose image contains all the Chern classes of the variety. In this paper, we investigate this property for (not necessarily symplectic) varieties with a Chow motive of abelian type. We introduce the notion of a symmetrically distinguished abelian motive and use it to provide a sufficient condition for a smooth projective variety to admit such a section. We then give a series of examples of varieties for which our theory works. For instance, we prove the existence of such a section for arbitrary products of varieties with Chow groups of finite rank, abelian varieties, hyperelliptic curves, Fermat cubic hypersurfaces, Hilbert schemes of points on an abelian surface or a Kummer surface or a K3 surface with Picard number at least 19, and generalized Kummer varieties. The latter cases provide evidence for the conjectural Section Property and exemplify the mantra that the motives of holomorphic symplectic varieties should behave as the motives of abelian varieties, as algebra objects.

Contents

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Introduction

Let X be a smooth projective variety over a field k. We denote by $\mathrm{CH}(X)$ its Chow ring with rational coefficients, and by $\overline{\mathrm{CH}}(X)$ the quotient of $\mathrm{CH}(X)$ by numerical equivalence of algebraic cycles. The aim of this work is to build upon a recent result of O'Sullivan [38] and give sufficient conditions on a smooth projective variety X for the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ to admit a section that contains the Chern classes of X. This amounts to *lifting* numerical cycle classes to cycle classes in the Chow groups such that the lifted cycles form a subalgebra and the lifting of the Chern classes are the corresponding Chow-theoretic Chern classes.

0.1. Motivation: The motives of holomorphic symplectic varieties. It is an insight of Beauville that the motives of smooth projective holomorphic symplectic varieties should behave in a similar way to the motives of abelian varieties as algebra objects in the category of Chow motives. Following the seminal work [9], Beauville [8] (meta-)conjectured that the conjectural Bloch–Beilinson filtration on the Chow ring of holomorphic symplectic varieties should split. This will subsequently be referred to as the *splitting principle*. That the conjectural Bloch–Beilinson filtration on the Chow ring of abelian varieties should split was established by Beauville [7].

0.1.1. The conjecture of Beauville. A first verifiable consequence of this splitting principle for *simply connected* holomorphic symplectic varieties is the following concrete conjecture, called *weak splitting property*; see [8] for details.

Conjecture (Beauville [8]). *Let X be a simply connected*[1] *smooth projective holomorphic symplectic variety, and denote by $R(X)$ the subalgebra of* $\mathrm{CH}(X)$ *generated by divisors. Then the composition of the following natural maps is injective:*

$$R(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X).$$

This conjecture was checked for K3 surfaces in the seminal work of Beauville and Voisin [9], and in [8] Beauville checked it for Hilbert schemes of length-2 and length-3 subschemes on a K3 surface. The conjecture was later strengthened by Voisin [49], who added the Chern classes of X to the set of generators of $R(X)$ (see also [52]). Since then, the strengthened conjecture has been shown to hold in a number of cases; see [49], [18], [53], [42], [19, §10], and [21].

0.1.2. Multiplicative Chow–Künneth decompositions. Beauville's splitting principle was reformulated in [43] directly on the level of Chow motives, without presupposing the existence of the Bloch–Beilinson filtration. In the case of abelian varieties, Deninger and Murre [16] constructed a canonical Chow–Künneth decomposition of the motive of an abelian variety, lifting to the motivic level the decomposition of Beauville on the level of the Chow ring [7]. It can be checked that the decomposition of Deninger–Murre is compatible with the algebra structure on the Chow motives of abelian varieties; following [43], we say that abelian varieties admit a *multiplicative Chow–Künneth decomposition*. We refer to Section 6 for definitions and properties of (multiplicative) Chow–Künneth decompositions. Similarly, for holomorphic

[1] This condition ensures that $\mathrm{CH}^1(X) \twoheadrightarrow \overline{\mathrm{CH}}^1(X)$ is an isomorphism.

symplectic varieties, the splitting principle suggests the following conjecture, which is case-by-case verifiable.

Conjecture (Multiplicative Chow–Künneth decomposition [43])**.** *A holomorphic symplectic variety X admits a multiplicative Chow–Künneth decomposition with the additional property that the Chern classes $c_i(X)$ belong to* $\mathrm{CH}(X)_{(0)}$.[2]

The decomposition of the small diagonal for K3 surfaces of Beauville–Voisin [9] in fact establishes the existence of a multiplicative Chow–Künneth decomposition for K3 surfaces; see [43, Proposition 8.14]. The existence of a multiplicative Chow–Künneth decomposition was established for the Hilbert scheme of length-2 subschemes on a K3 surface in [43], more generally for the Hilbert scheme of length-n subschemes on a K3 or abelian surface in [48], and for generalized Kummer varieties in [20].

0.1.3. O'Sullivan's theorem. There is another verifiable consequence of Beauville's splitting principle, which will be our main focus here. The Bloch–Beilinson conjectures (or Murre's conjecture (D) [36]) predict that for any smooth projective variety, the composition $\mathrm{CH}^i(X)_{(0)} \hookrightarrow \mathrm{CH}^i(X) \twoheadrightarrow \overline{\mathrm{CH}}^i(X)$ is an isomorphism of $\mathbb{Q}$-vector spaces for all i. In the case where the conjectural Bloch–Beilinson filtration splits, $\mathrm{CH}(X)_{(0)}$ is a $\mathbb{Q}$-subalgebra of $\mathrm{CH}(X)$, and we would therefore expect that $\mathrm{CH}(X)_{(0)}$ provides a section to the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$. In the case of abelian varieties, this was conjectured by Beauville [7]. A breakthrough in that direction is the following result due to O'Sullivan.

Theorem (O'Sullivan [38])**.** *Let A be an abelian variety. Then the $\mathbb{Q}$-algebra epimorphism*

$$\mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$$

admits a section (*as $\mathbb{Q}$-algebras*) *whose image consists of* symmetrically distinguished cycles *in the sense of Definition* 1.7.

See Theorems 1.3 and 1.8 for a more precise version of O'Sullivan's theorem. In particular, O'Sullivan's theorem establishes the following version[3] of Beauville's conjecture for abelian varieties (see [1] and [34] for alternative proofs): *If A is an abelian variety, then the subalgebra of* $\mathrm{CH}(A)$ *generated by symmetric divisors injects into cohomology via the cycle class map.* In this paper, inspired by the work of O'Sullivan [38] on the Chow rings of abelian varieties, we would like to address the following consequence of Beauville's splitting principle.

[2]See (11) for the definition of the grading $\mathrm{CH}(X)_{(*)}$.

[3]This question was asked by Voisin as a more accessible consequence of Beauville's more general conjecture in [7].

Conjecture 1 (Section Property)**.** *Let X be a smooth projective holomorphic symplectic variety. Then the $\mathbb{Q}$-algebra epimorphism*

$$\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$$

admits a section (as $\mathbb{Q}$-algebras) whose image contains the Chern classes of X.

Conjecture 1 implies Beauville's *weak splitting property* conjecture [8], as well as its refinement due to Voisin [49], because $\mathrm{CH}^1(X) \twoheadrightarrow \overline{\mathrm{CH}}^1(X)$ is an isomorphism for smooth projective varieties X with vanishing irregularity. We prove the following result (Propositions 4.1, 4.17, 5.11, 5.12, 5.13 and 5.14) in support of Conjecture 1.

Theorem 1. *Let X be a product of holomorphic symplectic varieties that are birational to either the Hilbert scheme of length-n subschemes on an abelian surface or a Kummer surface or a K3 surface with Picard number ≥ 19, or a generalized Kummer variety. Then Conjecture 1 holds for X.*

Finally, we note that the notion of symmetrically distinguished cycles on an abelian variety A depends on the choice of an origin for A, and in particular that there are at least as many sections to the algebra epimorphism $\mathrm{CH}(A) \to \overline{\mathrm{CH}}(A)$ as the number of rational equivalence classes of points on A. However, in the case of smooth projective irreducible holomorphic symplectic (i.e., hyper-Kähler) varieties, we expect that a section as in Conjecture 1, if it exists, is unique; and we also expect that cycles that are either classes of algebraically co-isotropic subvarieties (see [52]) or restrictions of cycles defined on the *universal family* belong to the image of the section (we refer to [21] for some evidence).

0.2. Distinguished cycles on varieties with motive of abelian type. Although our primary motivation for this work was to establish Theorem 1, we were led to consider the following broader question (see Question 3.6): Suppose X is a smooth projective variety whose Chow motive is isomorphic to a direct summand of the motive of an abelian variety (such varieties are said to have *motive of abelian type*; see Definition 1.1). Are there sufficient conditions on X that ensure that the epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section that is compatible with the intersection product? For that purpose we introduce the notion of *distinguished cycles* on varieties with motive of abelian type; see Definition 3.2. Precisely, distinguished cycles depend a priori on the choice of a *marking*: a marking for a variety X (see Definition 3.1) is an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ of Chow motives, where M is[4] a direct summand of a Chow motive of the form $\oplus_i \mathfrak{h}(A_i)(n_i)$ cut out by an idempotent

[4]Strictly speaking, M should be an object in the category $\mathscr{M}_{sd}^{ab}$ introduced in Definition 2.1.

matrix P of *symmetrically distinguished* cycles, where A_i is an abelian variety and $n_i \in \mathbb{Z}$. Given such a marking, the group of *distinguished cycles* $\mathrm{DCH}_\phi(X)$ consists of the image under P_* of the symmetrically distinguished cycles on each A_i, in the sense of O'Sullivan (see Definition 1.7), transported via the induced isomorphism $\phi_* : \mathrm{CH}(X) \xrightarrow{\simeq} \mathrm{CH}(M)$. The question becomes: What are sufficient conditions on a marking ϕ for $\mathrm{DCH}_\phi(X)$ to be a subalgebra of $\mathrm{CH}(X)$? In Proposition 3.12, we show that it suffices that the following condition holds:

$(\star_{\mathrm{Mult}})$ The small diagonal δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 3}}(X^3)$.

Since it is natural to expect that the Chern classes are distinguished, we will also require that the Chern classes of X are transported to symmetrically distinguished cycles via ϕ, i.e., that the marking ϕ also satisfies the condition

$(\star_{\mathrm{Chern}})$ All Chern classes $c_1(X), c_2(X), \ldots$ belong to $\mathrm{DCH}_\phi(X)$.

These two conditions are gathered in condition $(\star)$ in Definition 3.7, where we also consider the more general situation where X is endowed with the action of a finite group G. The condition $(\star_{\mathrm{Chern}})$ is not only esthetically pleasing, it is also essential to establish that the condition $(\star)$ is stable under natural constructions such as blow-ups (Proposition 4.8).

Therefore in order to prove Theorem 1, it is enough to exhibit a suitable marking for X such that the Chern classes and the small diagonal are distinguished with respect to the (product) markings. If such a suitable marking for X exists, we will say that X satisfies $(\star)$; see Definition 3.7. This condition is strictly stronger than the condition of having motive of abelian type; see Section 7 for examples of varieties with motive of abelian type that do not satisfy $(\star)$ and/or are such that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ does not admit a section. Thus that smooth projective holomorphic symplectic varieties should satisfy the *Section Property* in Conjecture 1 is remarkable. We also want to stress that the original Section Property, i.e., the existence of a section of the algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$, does not behave well enough under basic operations, for instance, products, blow-ups, quotients, etc.; however, the closely related condition $(\star)$ is essentially motivic and behaves much better; see Section 4. In view of Proposition 3.12, one could also be optimistic and go as far as proposing:

Conjecture 2 (Distinguished marking). *A smooth projective holomorphic symplectic variety admits a marking that satisfies* $(\star)$*.*

In particular, this conjecture stipulates that smooth projective holomorphic symplectic varieties have motives of abelian type. Some evidence towards the latter is provided by recent work of Kurnosov–Soldatenkov–Verbitsky [30] on Kuga–Satake constructions.

Although holomorphic symplectic varieties seem to play a central role, we provide many other examples of smooth projective varieties X that satisfy $(\star)$ and hence are such that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section whose image contains the Chern classes of X. The building blocks (see Section 5) are given by abelian varieties (O'Sullivan's theorem), varieties with Chow groups of finite rank (Proposition 5.2), hyperelliptic curves (Corollary 5.4), cubic Fermat hypersurfaces (Proposition 5.7), K3 surfaces with Picard rank ≥ 19 (Proposition 5.12), and generalized Kummer varieties (Proposition 5.14). One can then construct new examples (see Section 4) of varieties satisfying $(\star)$ by taking products (Proposition 4.1), certain projective bundles and blow-ups (Example 4.6, Propositions 4.5 and 4.8, here that the Chern classes are distinguished plays a central role), certain étale or cyclic quotients (Propositions 4.9, 4.11 and 4.12), Hilbert squares and the first two nested Hilbert schemes (Propositions 4.13 and 4.14), Hilbert schemes and nested Hilbert schemes of curves or surfaces satisfying $(\star)$ (Remark 5.6 and Proposition 5.13), and birational transforms of irreducible symplectic varieties (Corollary 4.17). Combining the above-mentioned results, we obtain

Theorem 2. *Let E be the smallest collection of isomorphism classes of smooth projective complex varieties that contains varieties with Chow groups of finite rank (as $\mathbb{Q}$-vector spaces), abelian varieties, hyperelliptic curves, cubic Fermat hypersurfaces, K3 surfaces with Picard rank ≥ 19, and generalized Kummer varieties, and that is stable under the following operations:*

- (i) *if X and Y belong to E, then $X \times Y$ belongs to E;*
- (ii) *if X belongs to E, then $\mathbb{P}(\oplus_i \mathbb{S}_{\lambda_i} T_X)$ belongs to E, where T_X is the tangent bundle of X, the λ_i's are nonincreasing sequences of integers, and $\mathbb{S}_{\lambda_i}$ is the Schur functor associated to λ_i;*
- (iii) *if X belongs to E, then the Hilbert scheme of length-2 subschemes $X^{[2]}$, as well as the nested Hilbert schemes $X^{[1,2]}$ and $X^{[2,3]}$, belong to E;*
- (iv) *if X is a curve or a surface that belongs to E, then for any $n \in \mathbb{N}$, the Hilbert scheme of length-n subschemes $X^{[n]}$, as well as the nested Hilbert schemes $X^{[n,n+1]}$, belong to E;*
- (v) *if one of two birationally equivalent irreducible holomorphic symplectic varieties belongs to E, then so does the other.*

If X is a smooth projective variety whose isomorphism class belongs to E, then X admits a marking that satisfies $(\star)$ so that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section (as $\mathbb{Q}$-algebras) whose image contains the Chern classes of X.

It is further shown in [31] that a certain complete family of Calabi–Yau varieties and certain rigid Calabi–Yau varieties, constructed by Cynk and Hulek, as well as certain varieties constructed by Schreieder, satisfy the condition $(\star)$ so that these varieties can be added to the set E of Theorem 2.

An immediate consequence of Theorem 2 is the following concrete result related to Beauville's weak splitting property and the Beauville–Voisin conjecture (but beyond the hyper-Kähler context).

Corollary 1. *Let X be a smooth projective variety that belongs to the collection E of Theorem 2. Assume that X is regular*[5] *and denote $R(X)$ the $\mathbb{Q}$-subalgebra of $\mathrm{CH}(X)$ generated by divisors and Chern classes. Then the natural composition*

$$R(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$$

is injective.

Note that all smooth projective varieties which we can show satisfy $(\star)$ were already shown to admit a self-dual multiplicative Chow–Künneth decomposition; see [44, Theorem 2], and [20] for the case of generalized Kummer varieties. In fact, condition $(\star)$ implies the existence of a multiplicative Chow–Künneth decomposition (Proposition 6.1). Note also that the structure of Section 4 is similar to the structure of [44, §3]. We refer to Section 6 for more on multiplicative Chow–Künneth decompositions and links to this work. Finally, we note that while the result of Beauville–Voisin [9] shows that the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(S) \twoheadrightarrow \overline{\mathrm{CH}}(S)$ admits a section whose image contains the Chern classes of S, for a K3 surface S, and while it can be shown [48] that the Hilbert scheme of length-n subschemes on a K3 surface has a self-dual multiplicative Chow–Künneth decomposition, we do not know how to show in general that a K3 surface satisfies condition $(\star)$, nor do we know how to show that the Hilbert scheme of length-n subschemes on a K3 surface satisfies the *Section Property* (Conjecture 1). In fact it is even an open problem to show in general that K3 surfaces have motive of abelian type.

Conventions and Notation. We work throughout the paper over an arbitrary algebraically closed field k, except in Sections 4.6, 5.3, 5.4, and 7, where k is assumed to be the field of complex numbers. Chow groups CH^i are always understood to be with rational coefficients. For a smooth projective variety X, we will write $\mathrm{CH}(X)$ for the (graded) rational Chow ring $\bigoplus_i \mathrm{CH}^i(X)$. We

[5] A smooth projective variety X over an algebraically closed field k is called *regular* if its Picard variety is trivial, so that the projection morphism $\mathrm{CH}^1(X) \to \overline{\mathrm{CH}}^1(X)$ is an isomorphism. Note that the *irregularity*, i.e., the dimension of the Picard variety, is always less than or equal to $\dim H^1(X, \mathcal{O}_X)$ and equal to $\dim H^1(X, \mathcal{O}_X)$ when $char(k) = 0$ by Hodge theory.

will denote by $\overline{\mathrm{CH}}^i(X)$ the rational Chow group modulo numerical equivalence and by $\overline{\mathrm{CH}}(X)$ the rational Chow ring modulo numerical equivalence.

An *abelian variety* is always assumed to be connected and with a fixed origin.

1. Symmetrically distinguished cycles

In this section, we review the theory of symmetrically distinguished cycles developed by O'Sullivan in [38] and, with a view towards applications, extend it slightly following the authors' previous work [20] joint with Zhiyu Tian.

1.1. Motives of abelian type. Let $\mathrm{CHM} := \mathrm{CHM}(k)_{\mathbb{Q}}$ and $\mathrm{NumM} := \mathrm{NumM}(k)_{\mathbb{Q}}$ be, respectively, the category of rational Chow motives and that of rational numerical motives over the base field k. By definition, there is a natural (full) projection functor

$$\mathrm{CHM} \to \mathrm{NumM},$$

which sends a Chow motive to the corresponding numerical motive and sends any cycle/correspondence modulo rational equivalence to its class modulo numerical equivalence. A typical object in these two categories is a triple (X, p, n) with X a smooth projective variety over k, $p \in \mathrm{CH}^{\dim X}(X \times X)$ or $\overline{\mathrm{CH}}^{\dim X}(X \times X)$ a projector (i.e., $p \circ p = p$), and $n \in \mathbb{Z}$. See [2] for the basic notions.

Let us introduce the following subcategories of CHM and NumM that will be relevant to our work.

Definition 1.1 (Motives of abelian type)**.** Let $\mathscr{M}^{ab}$ (resp., $\overline{\mathscr{M}^{ab}}$) be the strictly[6] full, thick, and rigid tensor subcategory of CHM (resp., NumM) generated by the motives of abelian varieties. A motive is said to be *of abelian type* if it belongs to $\mathscr{M}^{ab}$; equivalently, if one of its Tate twists is isomorphic to the direct summand of the motive of an abelian variety. We have the restriction of the projection functor:

$$\pi : \mathscr{M}^{ab} \to \overline{\mathscr{M}^{ab}}.$$

Example 1.2. The Chow (resp., numerical) motives of the following algebraic varieties belong to the category $\mathscr{M}^{ab}$ (resp., $\overline{\mathscr{M}^{ab}}$):

(i) projective spaces, Grassmannian varieties, and more generally projective homogeneous varieties under a linear algebraic group and toric varieties;
(ii) smooth projective curves;

[6] A full subcategory is called *strictly full* if it is closed under isomorphisms.

(iii) Kummer K3 surfaces; K3 surfaces with Picard numbers at least 19 as well as their (nested) Hilbert schemes;
(iv) abelian torsors;
(v) Hilbert schemes of abelian surfaces;
(vi) generalized Kummer varieties;
(vii) Fermat hypersurfaces;
(viii) projective bundles over and products of the examples above.

As far as the authors know, all examples of motives that have been proven to be (Kimura) finite dimensional ([26]) belong[7] to the category $\mathscr{M}^{ab}$.

Let us state the following result of [38], which is built upon [26] and [4]:

Theorem 1.3 (O'Sullivan [38, Theorem 5.5.3])**.** *The projection $\otimes$-functor $\pi : \mathscr{M}^{ab} \to \overline{\mathscr{M}^{ab}}$ has a right-inverse T, which is unique up to a unique tensor isomorphism above the identity.*

Remark 1.4. See Theorem 1.8, together with Remark 2.5, for a down-to-earth understanding of Theorem 1.3.

Remark 1.5. The existence of the right-inverse $\otimes$-functor T is ensured by a general result of André–Kahn [4] concerning the so-called Wedderburn categories, and such a section is unique only up to a *nonunique* tensor conjugacy. The Hopf algebra structure on the motive of an abelian variety, given by the diagonal embedding and the group structure (in particular, the (-1)-involution), allows O'Sullivan to make the section T unique up to a *unique* tensor conjugacy above the identity.

Remark 1.6. The section T in Theorem 1.3 cannot be defined uniquely. Indeed, let B be a torsor under an abelian variety A of dimension g. Obviously A and B have isomorphic Chow motives. If a canonical section T were constructed for morphisms between $\mathbb{1}(-g)$ and $\mathfrak{h}(B)$, then we would have a canonical 1-dimensional subspace $\mathrm{DCH}_0(B)$ inside the infinite-dimensional space $\mathrm{CH}_0(B)$, hence a canonical degree-one 0-cycle of B. However, as the origin of B is not fixed, there is neither a privileged point nor a privileged nontrivial 0-cycle.

1.2. Symmetrically distinguished cycles on abelian varieties.

O'Sullivan defines the following concrete notion of symmetrically distinguished cycles on an abelian variety A, and shows (Theorem 1.8) that these provide a section to

$$\mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$$

that is compatible with the intersection product.

[7]When k has characteristic zero, there are many varieties whose motive is not in $\mathscr{M}^{ab}$, while conjecturally all varieties have finite-dimensional motive.

Definition 1.7 (Symmetrically distinguished cycles on abelian varieties [38])**.** Let A be an abelian variety, and let $\alpha \in \mathrm{CH}(A)$. For each integer $m \geq 0$, denote by $V_m(\alpha)$ the $\mathbb{Q}$-vector subspace of $\mathrm{CH}(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \times \alpha^{r_2} \times \cdots \times \alpha^{r_n}),$$

where $n \leq m$, $r_j \geq 0$ are integers, and where $p : A^n \to A^m$ is a closed immersion each component $A^n \to A$ of which is either a projection or the composite of a projection with the involution $[-1] : A \to A$. Then α is *symmetrically distinguished* if for every m the restriction of the projection $\mathrm{CH}(A^m) \to \overline{\mathrm{CH}}(A^m)$ to $V_m(\alpha)$ is injective. The subgroup of symmetrically distinguished cycles is denoted by $\mathrm{DCH}(A)$.

Here is the main result of O'Sullivan [38], which is the most important ingredient that we use throughout this paper.

Theorem 1.8 (O'Sullivan [38, Theorem 6.2.5])**.** *Let A be an abelian variety. Then the symmetrically distinguished cycles in* $\mathrm{CH}(A)$ *form a graded* $\mathbb{Q}$*-subalgebra* $\mathrm{DCH}(A)$ *that contains symmetric divisors and that is stable under pull-backs and push-forwards along homomorphisms of abelian varieties. Moreover, the composition*

$$\mathrm{DCH}(A) \hookrightarrow \mathrm{CH}(A) \twoheadrightarrow \overline{\mathrm{CH}}(A)$$

is an isomorphism of $\mathbb{Q}$*-algebras.*

Remark 1.9. Given an abelian variety A, thanks to Theorem 1.8, it is easy to see by looking at the eigenvalues of multiplication-by-m endomorphisms ($m \in \mathbb{Z}$) that $\mathrm{DCH}(A)$ is a subalgebra of $\mathrm{CH}(A)_{(0)}$, where $\mathrm{CH}(A)_{(*)}$ refers to Beauville's decomposition[8] [7]. Moreover, the inclusion $\mathrm{DCH}^i(A) \subseteq \mathrm{CH}^i(A)_{(0)}$ is an equality for $i \leq 1$ as well as for $i \geq \dim A - 1$ by the Fourier transform [5]. Beauville's conjectures on abelian varieties in [7] would imply that the subalgebra $\mathrm{DCH}(A)$ is equal to the direct summand $\mathrm{CH}(A)_{(0)}$. In this sense, O'Sullivan's work [38] can be viewed as a step towards Beauville's conjectures.

1.3. . . . on abelian torsors with torsion structures. For later use, we give a minor extension of O'Sullivan's theory. The main idea appeared in our previous work [20]: to treat the Chow motives of some algebraic varieties like Hilbert schemes of abelian surfaces and generalized Kummer varieties, it is inevitable to deal with "disconnected abelian varieties" where there is no natural choice for the origins on the components, whence the notion of symmetrically distinguished cycles a priori fails. However, a simple but crucial

[8]Beauville's decomposition coincides with the decomposition induced, as in (11), by the Chow–Künneth decomposition of Deninger–Murre [16].

observation made in [20] is that we have a canonical notion of torsion points on these components.

Definition 1.10 (Abelian torsors with torsion structure [20])**.** An *abelian torsor with torsion structure*, or an *a.t.t.s.* for short, is a pair (X, Q_X) where X is a connected smooth projective variety and Q_X is a subset of closed points of X such that there exists an isomorphism, as algebraic varieties, $f : X \xrightarrow{\simeq} A$ from X to an abelian variety A which induces a bijection between Q_X and $\mathrm{Tor}(A)$, the set of all torsion points of A. A choice of such an isomorphism f is called a *marking*. A morphism of a.t.t.s.'s $(X, Q_X) \to (Y, Q_Y)$ consists of a morphism of algebraic varieties $f : X \to Y$ such that $f(Q_X) \subseteq Q_Y$.

This notion of a.t.t.s. sits in between the notion of abelian variety (with fixed origin) and that of abelian torsor (without origin).

Definition 1.11 (Symmetrically distinguished cycles on a.t.t.s.'s)**.** Given an a.t.t.s. (X, Q_X), an algebraic cycle $\gamma \in \mathrm{CH}(X)$ is called *symmetrically distinguished* if, for a marking $f : X \xrightarrow{\simeq} A$, the cycle $f_*(\gamma) \in \mathrm{CH}(A)$ is symmetrically distinguished in the sense of O'Sullivan (Definition 1.7). By [20, Lemma 6.7], this definition is independent of the choice of marking. An algebraic cycle on a disjoint union of a.t.t.s.'s is symmetrically distinguished if it is so on each component. We denote by $\mathrm{DCH}(X)$ the subspace of symmetrically distinguished cycles.

We have the following generalization of Theorem 1.8; see [20, Proposition 6.9]. Its proof uses the fact that torsion points on an abelian variety are all rationally equivalent (with $\mathbb{Q}$-coefficients).

Theorem 1.12. *Let (X, Q_X) be an a.t.t.s. Then the symmetrically distinguished cycles in $\mathrm{CH}(X)$ form a graded $\mathbb{Q}$-subalgebra that is stable under pull-backs and push-forwards along morphisms of a.t.t.s.'s. Moreover, the composition $\mathrm{DCH}(X) \hookrightarrow \mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ is an isomorphism of $\mathbb{Q}$-algebras.*

We refer to [20, §6.2] for more properties of symmetrically distinguished cycles on a.t.t.s.'s.

2. Symmetrically distinguished abelian motives

To make a more flexible use of O'Sullivan's Theorem 1.8 in the language of motives, we introduce the following category $\mathscr{M}_{sd}^{ab}$. We refer to Remarks 2.5 and 2.7 for some motivation.

Definition 2.1 (The category $\mathscr{M}_{sd}^{ab}$)**.** The category of *symmetrically distinguished abelian motives*, denoted $\mathscr{M}_{sd}^{ab}$, is defined as follows:

(i) An object consists of the data of
 - a positive integer $r \in \mathbb{N}^*$;

- a length-r sequence of abelian varieties (thus with fixed origins) $A_1, \dots, A_r$;
- a length-r sequence of integers $n_1, \dots, n_r \in \mathbb{Z}$;
- an $(r \times r)$-matrix $P := (p_{i,j})_{1 \le i,j \le r}$ with

$$p_{i,j} \in \mathrm{DCH}^{\dim A_i + n_j - n_i}(A_i \times A_j)$$

a *symmetrically distinguished* cycle (Definition 1.7) such that $P \circ P = P$; that is, for all $1 \le i, j \le r$, we have

$$\sum_{k=1}^{r} p_{k,j} \circ p_{i,k} = p_{i,j} \text{ in } \mathrm{CH}^{\dim A_i + n_j - n_i}(A_i \times A_j).$$

Such an object is denoted in what follows by a triple

$$(A_1 \sqcup \cdots \sqcup A_r, P = (p_{i,j}), (n_1, \dots, n_r)).$$

(ii) The group of morphisms from $(A_1 \sqcup \cdots \sqcup A_r, P = (p_{i,j}), (n_1, \dots, n_r))$ to another object $(B_1 \sqcup \cdots \sqcup B_s, Q = (q_{i,j}), (m_1, \dots, m_s))$ is defined to be the subgroup of

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \mathrm{CH}^{\dim A_i + m_j - n_i}(A_i \times B_j)$$

(whose elements are viewed as an $(s \times r)$-matrix) given by

$$Q \circ \left(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \mathrm{CH}^{\dim A_i + m_j - n_i}(A_i \times B_j) \right) \circ P,$$

where the multiplication law is the one between matrices.

(iii) The composition is defined as usual by composition of correspondences.

(iv) The category $\mathscr{M}_{sd}^{ab}$ is an additive category where the direct sum is given by

$$\begin{aligned} &\left(\bigsqcup_{i=1}^{r} A_i, P, (n_1, \dots, n_r) \right) \oplus \left(\bigsqcup_{j=1}^{s} B_j, Q, (m_1, \dots, m_s) \right) \\ = &\left(\bigsqcup_{i=1}^{r} A_i \sqcup \bigsqcup_{j=1}^{s} B_j, P \oplus Q := \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, (n_1, \dots, n_r, m_1, \dots, m_s) \right). \end{aligned}$$

(v) The category $\mathscr{M}_{sd}^{ab}$ is a symmetric monoïdal category where the tensor product is defined by

$$\left(\bigsqcup_{i=1}^{r} A_i, P, (n_1, \ldots, n_r)\right) \otimes \left(\bigsqcup_{j=1}^{s} B_j, Q, (m_1, \ldots, m_s)\right)$$
$$= \left(\bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{s} A_i \times B_j, P \otimes Q, (n_i m_j; 1 \leq i \leq r, 1 \leq j \leq s)\right),$$

where $P \otimes Q$ is the Kronecker product of two matrices.

In particular, for any $m \in \mathbb{Z}$, the mth *Tate twist*, i.e., the tensor product with the Tate object $\mathbb{1}(m) := (\operatorname{Spec} k, \operatorname{Spec} k, m)$ sends $(A_1 \sqcup \cdots \sqcup A_r, P, (n_1, \ldots, n_r))$ to $(A_1 \sqcup \cdots \sqcup A_r, P, (n_1 + m, \ldots, n_r + m))$. All Tate objects are $\otimes$-invertible.

(vi) The category $\mathscr{M}_{sd}^{ab}$ is rigid; the dual of $(A_1 \sqcup \cdots \sqcup A_r, P = (p_{i,j}), (n_1, \ldots, n_r))$ is given by $(A_1 \sqcup \cdots \sqcup A_r, {}^tP := ({}^tp_{j,i}), (d_1 - n_1, \ldots, d_r - n_r))$, where $d_k = \dim A_k$ and the (i,j)th entry of ${}^tP$ is ${}^tp_{j,i} \in \mathrm{CH}^{d_i + (d_j - n_j) - (d_i - n_i)}(A_i \times A_j)$, the transpose of $p_{j,i} \in \mathrm{CH}^{d_j + n_i - n_j}(A_j \times A_i)$.

In a similar way, one can define the rigid symmetric monoïdal additive category $\mathscr{M}_{sd}^{atts}$ by replacing in the above definition abelian varieties (thus with origin fixed) by abelian torsors with torsion structure (thus with only the subset of torsion points fixed; cf. Section 1.3). With the notion and basic properties of symmetrically distinguished cycles extended to the case of abelian torsors with torsion structure in Section 1.3, all the above constructions go through. It is important to point out that[9] $\mathscr{M}_{sd}^{ab}$ and $\mathscr{M}_{sd}^{atts}$ are *not* subcategories of CHM since in the definition of motives, one uses varieties instead of *pointed* varieties or varieties with additional structures. See, however, Lemma 2.2 below.

There are natural fully faithful additive tensor functors

$$F : \mathscr{M}_{sd}^{ab} \to \mathscr{M}^{ab} \quad \text{and} \quad F' : \mathscr{M}_{sd}^{atts} \to \mathscr{M}^{ab},$$

which send an object $(A_1 \sqcup \cdots \sqcup A_r, P = (p_{i,j}), (n_1, \ldots, n_r))$ to the Chow motive $\operatorname{Im}(P : \bigoplus_{i=1}^r \mathfrak{h}(A_i)(n_i) \to \bigoplus_{i=1}^r \mathfrak{h}(A_i)(n_i))$. Here we use the facts that CHM is pseudo-abelian and that P induces an idempotent endomorphism of $\bigoplus_{i=1}^r \mathfrak{h}(A_i)(n_i)$ by construction.

For any object M in $\mathscr{M}_{sd}^{ab}$ or $\mathscr{M}_{sd}^{atts}$ and any $i \in \mathbb{Z}$, the ith *Chow group* $\mathrm{CH}^i(M)$ is defined to be $\mathrm{CH}^i(F(M))$, which is nothing but

$$\operatorname{Hom}_{\mathscr{M}_{sd}^{ab}}((\operatorname{Spec} k, \operatorname{Spec} k, -i), M).$$

[9]We thank Peter O'Sullivan for reminding us of this subtle point.

Despite the technical construction of the categories $\mathscr{M}_{sd}^{ab}$ and $\mathscr{M}_{sd}^{atts}$, they are, after all, not so different from the category $\mathscr{M}^{ab}$ of abelian motives (Definition 1.1).

Lemma 2.2 (Relation with Chow motives of abelian type)**.** *The functors $F : \mathscr{M}_{sd}^{ab} \to \mathscr{M}^{ab}$ and $F' : \mathscr{M}_{sd}^{atts} \to \mathscr{M}^{ab}$ are equivalences of categories.*

Proof. These two functors are fully faithful by definition, and we only have to show that they are essentially surjective. Consider an object in CHM isomorphic to (A, p, n) with A a g-dimensional abelian torsor, $p \in \mathrm{CH}^g(A \times A)$ a projector, and $n \in \mathbb{Z}$. First we choose an origin for A so that the symmetric distinguishedness makes sense in the rest of the proof. Using the existence of symmetrically distinguished cycles in each numerical cycle class (Theorem 1.8), one can find a symmetrically distinguished element $q \in \mathrm{DCH}^g(A\times A)$ such that q is numerically equivalent to p. As p is a projector, we know that $q \circ q$ is numerically equivalent to q. However, as $q \circ q$ and q are both symmetrically distinguished, they must be equal by the uniqueness of symmetrically distinguished lifting in Theorem 1.8; i.e., q is a projector. Therefore (A, p, n) is isomorphic, in CHM, to (A, q, n), which is in the image of the functor F. Finally, since F factorizes through F', F' is also essentially surjective. $\square$

Now we extend the notion of symmetrical distinguishedness from cycles on abelian varieties (Definition 1.7) to morphisms in the category $\mathscr{M}_{sd}^{ab}$ (and $\mathscr{M}_{sd}^{atts}$).

Definition 2.3 (Symmetrically distinguished morphisms in $\mathscr{M}_{sd}^{ab}$)**.** Given two objects in $\mathscr{M}_{sd}^{ab}$, say, $M := (A_1 \sqcup \cdots \sqcup A_r, P = (p_{i,j}), (n_1, \ldots, n_r))$ and $N := (B_1 \sqcup \cdots \sqcup B_s, Q = (q_{i,j}), (m_1, \ldots, m_s))$, the subspace of *symmetrically distinguished morphisms* from M to N, denoted by $\mathrm{DHom}(M, N)$, is defined to be

$$\mathrm{DHom}(M,N) := Q \circ \bigoplus_{i,j} \mathrm{DCH}^{\dim A_i + m_j - n_i}(A_i \times B_j) \circ P \subseteq \mathrm{Hom}(M,N).$$

Similarly, one can define symmetrically distinguished morphisms in $\mathscr{M}_{sd}^{atts}$. Here $\mathrm{DCH}(A_i \times B_j)$ is in the sense of Definition 1.7 or 1.11.

In particular, for any object M in $\mathscr{M}_{sd}^{ab}$ (or $\mathscr{M}_{sd}^{atts}$) and any integer i, $\mathrm{DHom}(\mathbb{1}(-i), M)$ is a canonical subgroup of $\mathrm{CH}^i(M) = \mathrm{CH}^i(F(M))$. We denote[10]

$$\mathrm{DCH}^i(M) := \mathrm{DHom}(\mathbb{1}(-i), M)$$

and call its elements *symmetrically distinguished cycles* of M.

[10]Beware that our notation slightly conflicts with the notation of [8], where $\mathrm{DCH}^*(X)$ stands for the subalgebra generated by divisors, which is denoted by $R(X)$ in the present paper.

We collect some basic properties of symmetrically distinguished morphisms in the following lemma. Recall that $\pi : \mathscr{M}^{ab} \to \overline{\mathscr{M}^{ab}}$ is the natural projection functor (Definition 1.1).

Lemma 2.4 (Relation with numerical motives of abelian type)**.** *In $\mathscr{M}^{ab}_{sd}$,*

(i) *the composition and the tensor product of two symmetrically distinguished morphisms is again symmetrically distinguished. Hence we have a tensor subcategory $\left(\mathscr{M}^{ab}_{sd},\ s.d.\,morphisms\right)$.*

(ii) *For any two objects $M, N \in \mathscr{M}^{ab}_{sd}$, the functor $\pi \circ F$ induces an isomorphism:*

$$\mathrm{DHom}(M,N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{NumM}}(\pi(F(M)), \pi(F(N))).$$

In particular, for any object $M \in \mathscr{M}^{ab}_{sd}$, the composition of the natural map $\mathrm{DCH}^i(M) \hookrightarrow \mathrm{CH}^i(M) \twoheadrightarrow \overline{\mathrm{CH}}^i(M) := \overline{\mathrm{CH}}^i(F(M))$ is an isomorphism.

(iii) *The composed functor $\pi \circ F : \mathscr{M}^{ab}_{sd} \to \overline{\mathscr{M}^{ab}}$ induces an equivalence of categories*

$$\overline{F} : \left(\mathscr{M}^{ab}_{sd},\ s.d.\,morphisms\right) \xrightarrow{\simeq} \overline{\mathscr{M}^{ab}}.$$

Similar properties also hold for the category $\mathscr{M}^{atts}_{sd}$.

Proof. (i) is a consequence of Theorem 1.8, which implies that symmetrically distinguished cycles on abelian varieties are closed under tensor product and that symmetrically distinguished correspondences between abelian varieties are closed under composition.

For (ii), let $M := (\bigsqcup_{i=1}^{r} A_i, P, (n_1, \ldots, n_r))$ and $N := (\bigsqcup_{j=1}^{s} B_j, Q, (m_1, \ldots, m_s))$. Then, on the one hand, we have by definition

$$\mathrm{DHom}(M,N) := Q \circ \bigoplus_{i,j} \mathrm{DCH}^{\dim A_i + m_j - n_i}(A_i \times B_j) \circ P,$$

and, on the other hand,

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{NumM}}\left(\pi(F(M)), \pi(F(N))\right) \\ &\quad = \mathrm{Hom}_{\mathrm{NumM}}\left(\overline{P}\left(\oplus\overline{\mathfrak{h}}(A_i)(n_i)\right), \overline{Q}\left(\oplus\overline{\mathfrak{h}}(B_j)(m_j)\right)\right) \\ &\quad = \overline{Q} \circ \bigoplus_{i,j} \overline{\mathrm{CH}}^{\dim A_i + m_j - n_i}(A_i \times B_j) \circ \overline{P}. \end{aligned}$$

By Theorem 1.8, for any $1 \le i \le r, 1 \le j \le s$, the natural map induced by $\pi \circ F$,

$$\mathrm{DCH}^{\dim A_i + m_j - n_i}(A_i \times B_j) \xrightarrow{\simeq} \overline{\mathrm{CH}}^{\dim A_i + m_j - n_i}(A_i \times B_j),$$

is an isomorphism. Now the fact that P and Q are matrices of symmetrically distinguished cycles allows us to conclude.

For (iii), the full faithfulness is the content of (ii), while the essential surjectivity follows from that of F (Lemma 2.2) and π.

The same argument also works for the category $\mathscr{M}_{sd}^{atts}$ by using Theorem 1.12 in place of Theorem 1.8. $\square$

Remark 2.5. Lemmas 2.2 and 2.4 are contrasting: on the one hand, the whole category $\mathscr{M}_{sd}^{ab}$ is equivalent to $\mathscr{M}^{ab}$, the category of abelian Chow motives; on the other hand, the subcategory with same objects as $\mathscr{M}_{sd}^{ab}$ and with symmetrically distinguished morphisms (Definition 2.3) is equivalent to $\overline{\mathscr{M}^{ab}}$, the category of abelian numerical motives. Thus $\mathscr{M}_{sd}^{ab}$ fulfills our purpose exactly to make a bridge between $\mathscr{M}^{ab}$ and $\overline{\mathscr{M}^{ab}}$. More precisely, we have the commutative diagram

$$\begin{tikzcd} \mathscr{M}_{sd}^{ab} \arrow[r, "F", "\simeq"'] & \mathscr{M}^{ab} \arrow[d, "\pi"] \\ (\mathscr{M}_{sd}^{ab},\ \text{s.d. morph}) \arrow[u, hook] \arrow[r, "\simeq", "\overline{F}"'] & \overline{\mathscr{M}^{ab}} \end{tikzcd}$$

which gives an explicit way to understand O'Sullivan's categorical Theorem 1.3 via his more down-to-earth Theorem 1.8. Namely, we no longer deal with the right-inverse tensor functor T, whose existence is proven in a somehow abstract way and whose uniqueness is up to a tensor conjugacy, but instead we have, via the equivalences F and $\overline{F}$, a concrete subcategory of symmetrically distinguished morphisms inside $\mathscr{M}_{sd}^{ab}$, which plays the role of the section functor T. We think the construction and basic properties of $\mathscr{M}_{sd}^{ab}$ and its subcategory of symmetrically distinguished morphisms would have independent interest in the future study of algebraic cycles on abelian varieties or, more generally, varieties with motives of abelian type.

Finally, let us note the following simple consequence of Lemma 2.4(iii), which will be crucial when dealing with quotients (or, more generally, generically finite surjective morphisms) in Section 4.4.

Lemma 2.6. *The category $(\mathscr{M}_{sd}^{ab},$ s.d. morphisms$)$, with objects as in $\mathscr{M}_{sd}^{ab}$ but with morphisms restricted to symmetrically distinguished morphisms, is pseudo-abelian.*

Proof. This follows from the equivalence of categories in Lemma 2.4(iii) and the fact that $\overline{\mathscr{M}^{ab}}$ is pseudo-abelian. $\square$

Remark 2.7. In fact, the category $(\mathscr{M}_{sd}^{ab},$ s.d. morphisms$)$ is the pseudo-abelian additive envelop of the category $Corr_{sd}^{ab}$ of symmetrically distinguished correspondences between abelian varieties: more precisely, an object of $Corr_{sd}^{ab}$ is a couple (A, n) with A an abelian variety (with fixed origin) and $n \in \mathbb{Z}$, and morphisms between two objects $(A, n), (B, m)$ are given by

$$\mathrm{Hom}((A,n),(B,m)) := \mathrm{DCH}^{\dim A+m-n}(A \times B);$$

the composition is the usual one for correspondences.

3. Distinguished cycles

3.1. Definitions and basic properties. Here are the key notions of this paper.

Definition 3.1 (Marking)**.** Let X be smooth projective variety such that its Chow motive $\mathfrak{h}(X)$ belongs to $\mathscr{M}^{ab}$. A *marking* for X consists of an object $M \in \mathscr{M}^{ab}_{sd}$ together with an isomorphism

$$\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M) \quad \text{in CHM},$$

where $F : \mathscr{M}^{ab}_{sd} \xrightarrow{\simeq} \mathscr{M}^{ab}$ is the equivalence in Definition 2.1.

By Lemma 2.2, a marking for a smooth projective variety X with motive of abelian type always exists. In practice, starting from Section 4, we will abusively ignore the difference between $\mathscr{M}^{ab}_{sd}$ and its image in $\mathscr{M}^{ab}$ by F and write a marking as an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ for $M \in \mathscr{M}^{ab}_{sd}$.

Definition 3.2 (Distinguished cycles)**.** Let X be a smooth projective variety such that its Chow motive $\mathfrak{h}(X)$ belongs to $\mathscr{M}^{ab}$. Given a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ with $M \in \mathscr{M}^{ab}_{sd}$, we define the subgroup of *distinguished cycles* of codimension i of X, denoted by $\mathrm{DCH}^i_\phi(X)$, or sometimes $\mathrm{DCH}^i(X)$ if ϕ is clear from the context, to be the preimage of $\mathrm{DCH}^i(M)$ (see Definition 2.3) via the induced isomorphism $\phi_* : \mathrm{CH}^i(X) \xrightarrow{\simeq} \mathrm{CH}^i(M)$.

Almost by construction, we have the following.

Lemma 3.3. *For any smooth projective variety X such that $\mathfrak{h}(X) \in \mathscr{M}^{ab}$ and any marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ with $M \in \mathscr{M}^{ab}_{sd}$, the composition*

$$\mathrm{DCH}^i_\phi(X) \hookrightarrow \mathrm{CH}^i(X) \twoheadrightarrow \overline{\mathrm{CH}}^i(X)$$

is an isomorphism. In other words, ϕ provides a section (as graded vector spaces) of the natural projection $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$.

Proof. In the commutative diagram

$$\begin{tikzcd}
\mathrm{DCH}^i_\phi(X) \arrow[r, hook] \arrow[d, "\phi_*", "\simeq"'] & \mathrm{CH}^i(X) \arrow[r, two heads] \arrow[d, "\phi_*", "\simeq"'] & \overline{\mathrm{CH}}^i(X) \arrow[d, "\overline{\phi}_*", "\simeq"'] \\
\mathrm{DCH}^i(M) \arrow[r, hook] & \mathrm{CH}^i(M) \arrow[r, two heads] & \overline{\mathrm{CH}}^i(M)
\end{tikzcd}$$

the composition of the bottom line is an isomorphism by Lemma 2.4. Therefore the composition of the top line is also an isomorphism, hence $\mathrm{DCH}^i_\phi(X)$ provides a section. $\square$

Remark 3.4 (Fundamental class)**.** Given a smooth projective variety X, its fundamental class $\mathbb{1}_X$ is always distinguished for any choice of marking. Indeed, we can assume that X is connected, thus $\mathrm{CH}^0(X) = \mathbb{Q}\cdot\mathbb{1}_X$, and Lemma 3.3 ensures that $\mathbb{1}_X$ is distinguished.

Distinguished cycles behave well with respect to tensor products and projections.

Proposition 3.5 (Tensor products and projections)**.** *Let X, Y be two smooth projective varieties with motive of abelian type, endowed with markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} F(N)$. Then*

$$\phi\otimes\psi : \mathfrak{h}(X\times Y) \xrightarrow{\simeq} F(M\otimes N)$$

provides a marking for $X\times Y$, and the exterior product $\mathrm{CH}^i(X)\times\mathrm{CH}^j(Y) \xrightarrow{\otimes} \mathrm{CH}^{i+j}(X\times Y)$ respects distinguished cycles:

$$\mathrm{DCH}^i_\phi(X)\times\mathrm{DCH}^j_\psi(Y) \xrightarrow{\otimes} \mathrm{DCH}^{i+j}_{\phi\otimes\psi}(X\times Y).$$

Moreover, denoting $p : X\times Y \to X$ the natural projection, we have

$$p_*\mathrm{DCH}^i_{\phi\otimes\psi}(X\times Y)\subseteq\mathrm{DCH}^{i-\dim Y}_\phi(X) \text{ and } p^*\mathrm{DCH}^i_\phi(X)\subseteq\mathrm{DCH}^i_{\phi\otimes\psi}(X\times Y),$$

and similarly for the natural projection, $q : X\times Y\to Y$.

Proof. That $\phi\otimes\psi$ provides a marking for $X\times Y$ such that the exterior product respects distinguished cycles follows directly from Lemma 2.4(i), which says that the tensor product of two symmetrically distinguished morphisms is symmetrically distinguished. To see that push-forwards and pull-backs along projections respect distinguished cycles, it is enough, by Lemma 2.4(i), to see that $\mathrm{id}_M\otimes f : M\otimes N \to M\otimes\mathbb{1}(-d)$ is a symmetrically distinguished morphism (Definition 2.3), where $d := \dim Y$ and $f : N \to \mathbb{1}(-d)$ is induced by the morphism $\mathfrak{h}(Y)\to\mathbb{1}(-d)$ determined by the fundamental class of Y. By Lemma 2.4(i), we only have to see that f is a symmetrically distinguished morphism, which is explained in Remark 3.4. $\square$

3.2. The main questions and the key condition ($\star$).

Question 3.6. Here are the most important properties of the distinguished cycles that we are going to investigate:

- When does $\bigoplus_i \mathrm{DCH}^i_\phi(X)$ form a (graded) $\mathbb{Q}$-subalgebra of $\mathrm{CH}(X)$?
- When do the Chern classes of X belong to $\bigoplus_i \mathrm{DCH}^i_\phi(X)$?

To this end, let us introduce the following condition for smooth projective varieties whose Chow motive is of abelian type.

Definition 3.7. We say that a smooth projective variety X with $\mathfrak{h}(X) \in \mathscr{M}^{ab}$ satisfies condition $(\star)$ if

There exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ (with $M \in \mathscr{M}^{ab}_{sd}$) such that

$(\star_{\mathrm{Mult}})$ **(Multiplicativity)** the small diagonal δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 3}}(X^3)$; that is, under the induced isomorphism $\phi_*^{\otimes 3} : \mathrm{CH}(X^3) \xrightarrow{\simeq} \mathrm{CH}(M^{\otimes 3})$, the image of δ_X is symmetrically distinguished, i.e., in $\mathrm{DCH}(M^{\otimes 3})$;

$(\star_{\mathrm{Chern}})$ **(Chern classes)** all Chern classes of T_X belong to $\mathrm{DCH}_\phi(X)$.

More generally, if X is a smooth projective variety equipped with the action of a finite group G, we say that (X, G) satisfies $(\star)$ if there exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ that satisfies, in addition to $(\star_{\mathrm{Mult}})$ and $(\star_{\mathrm{Chern}})$ above:

$(\star_G)$ **(G-invariance)** the graph g_X of $g : X \to X$ belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$ for any $g \in G$.

We will see in Corollary 3.16 that condition $(\star_{\mathrm{Mult}})$ implies that the *top* Chern class of T_X is distinguished.

Lemma 3.8 (Diagonal)**.** *Notation is as before.*

(i) *Condition* $(\star_{\mathrm{Mult}})$ *implies that the diagonal* Δ_X *belongs to* $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$.

(ii) *The condition that* Δ_X *is distinguished is equivalent to saying that the isomorphism* $\sigma : M \xrightarrow{\simeq} M^\vee(-d_X)$*, given by the commutativity of the following diagram,*[11] *is symmetrically distinguished in the sense of Definition* 2.3*, where the top morphism is the Poincaré duality in* CHM (*induced by* Δ_X)*:*

$$\begin{tikzcd} \mathfrak{h}(X) \arrow[r, "PD_X", "\simeq"'] \arrow[d, "\phi", "\simeq"'] & \mathfrak{h}(X)^\vee(-d_X) \\ F(M) \arrow[r, "\simeq", "F(\sigma)"'] & F(M^\vee(-d_X)) \arrow[u, "\phi^\vee(-d_X)"', "\simeq"] \end{tikzcd} \tag{1}$$

Proof. Statement (ii) is tautological, and statement (i) follows from Proposition 3.5 together with the observation that Δ_X is the push-forward of δ_X along the projection $\mathrm{pr}_{1,2} : X \times X \times X \to X \times X$. $\square$

Lemma 3.9 (Equivalent formulation of $(\star_{\mathrm{Mult}})$)**.** *Let* $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ *be a marking as above, and let* d_X *be the dimension of* X*. The condition* $(\star_{\mathrm{Mult}})$ *is equivalent to saying that the morphism* $\mu : M^{\otimes 2} \to M$*, determined by the commutativity of the following diagram,*[12] *is a symmetrically*

[11] Recall that F is an equivalence (Lemma 2.2), so $F(\sigma)$ determines σ.
[12] Recall that F is an equivalence (Lemma 2.2), so $F(\mu)$ determines μ.

distinguished morphism, where the top morphism is the intersection product in CHM *induced by the small diagonal:*

(2)
$$\begin{tikzcd} \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}"', "\simeq"] & \mathfrak{h}(X) \arrow[d, "\simeq"', "\phi"] \\ F(M^{\otimes 2}) \arrow[r, "F(\mu)"'] & F(M) \end{tikzcd}$$

Proof. First we claim[13] that the condition that μ is symmetrically distinguished implies that σ in diagram (1) is symmetrically distinguished (or equivalently, Δ_X is distinguished by Lemma 3.8(ii)). Indeed, consider the commutative diagram

$$\begin{tikzcd} \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}"', "\simeq"] & \mathfrak{h}(X) \arrow[r, "\mathbb{1}_X"] \arrow[d, "\simeq"', "\phi"] & \mathbb{1}(-d_X) \arrow[d, equal] \\ F(M^{\otimes 2}) \arrow[r, "F(\mu)"'] & F(M) \arrow[r, "F(\nu)"'] & F(\mathbb{1}(-d_X)) \end{tikzcd}$$

where the left square is (2), the top right morphism is induced by the fundamental class of X, and ν is the morphism determined by the commutativity of the right square. By Remark 3.4, ν is a symmetrically distinguished morphism. Now the outer square of the previous diagram gives the right square in the following diagram:

$$\begin{tikzcd} \mathfrak{h}(X) \arrow[r, "\eta_{\mathfrak{h}(X)}\otimes \mathrm{id}_{\mathfrak{h}(X)}"] \arrow[d, "\phi"', "\simeq"] & \mathfrak{h}(X)^\vee \otimes \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\mathrm{id}_{h(X)^\vee}\otimes(\mathbb{1}_X\circ\delta_X)"] \arrow[d, "(\phi^\vee)^{-1}\otimes\phi^{\otimes 2}"', "\simeq"] & \mathfrak{h}(X)^\vee(-d_X) \\ F(M) \arrow[r, "F(\eta_M\otimes \mathrm{id}_M)"'] & F(M^\vee \otimes M^{\otimes 2}) \arrow[r, "F(\mathrm{id}_{M^\vee}\otimes(\nu\circ\mu))"'] & F(M^\vee)(-d_X) \arrow[u, "\phi^\vee(-d_X)", "\simeq"'] \end{tikzcd}$$

where in the left square, $\eta_M : \mathbb{1} \to M^\vee \otimes M$ is the unit of the duality for M, and similarly for $\mathfrak{h}(X)$. Therefore, by definition, the isomorphism σ in diagram (1) is given by

$$\sigma = (\mathrm{id}_{M^\vee} \otimes(\nu \circ \mu)) \circ (\eta_M \otimes \mathrm{id}_M) .$$

As μ, ν, and η_M are all symmetrically distinguished morphisms, so is σ by Lemma 2.4(i).

Now let us show the equivalence between ($\star_{\mathrm{Mult}}$) and the symmetric distinguishedness of μ. Thanks to the above claim and to Lemma 3.8, for both directions of implication one can suppose that σ is symmetrically distinguished.

[13]We thank Peter O'Sullivan for mentioning this to us.

Thus the following isomorphism, induced by composing with $\sigma^{\otimes 2} \otimes \mathrm{id}_M$, preserves the symmetrically distinguished elements:

$$\begin{aligned}\mathrm{CH}^{2d_X}(M^{\otimes 3}) = \mathrm{Hom}(\mathbb{1}, M(d_X)^{\otimes 2} \otimes M) &\xrightarrow{\simeq} \mathrm{Hom}(\mathbb{1}, (M^\vee)^{\otimes 2} \otimes M) \\ &= \mathrm{Hom}(M^{\otimes 2}, M).\end{aligned}$$

We can conclude by observing that this isomorphism sends $\phi_*^{\otimes 3}(\delta_X)$ to μ. $\square$

Let us also mention the following convenient sufficient condition for $(\star_G)$.

Lemma 3.10 (G-invariant marking)**.** *Let X be a smooth projective variety endowed with an action of a finite group G. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ be a marking as above. If Δ_X is distinguished and if for any $g \in G$, we have $\phi \circ g = F(\overline{g}) \circ \phi$ for some symmetrically distinguished cycle $\overline{g}$, then ϕ satisfies $(\star_G)$, where $g : \mathfrak{h}(X) \to \mathfrak{h}(X)$ is the automorphism induced by g.*

Proof. For any $g \in G$, consider the composition

$$\mathbb{1}(-\dim X) \xrightarrow{\Delta_X} \mathfrak{h}(X) \otimes \mathfrak{h}(X) \xrightarrow{\mathrm{id} \otimes g} \mathfrak{h}(X) \otimes \mathfrak{h}(X) \xrightarrow{\phi \otimes \phi} M \otimes M.$$

We obtain that $(\phi \otimes \phi)_* \Gamma_g = (\phi \otimes \phi) \circ (\mathrm{id} \otimes g) \circ \Delta_X = F(\mathrm{id} \otimes \overline{g}) \circ (\phi \otimes \phi) \circ \Delta_X$, where the latter term is symmetrically distinguished from the assumption on Δ_X. This means exactly that the graph Γ_g is distinguished. $\square$

Remark 3.11 (Another formulation)**.** The following interpretation of the condition $(\star_{\mathrm{Mult}})$ using the section T in Theorem 1.3 was kindly suggested to us by Peter O'Sullivan.[14] For an algebraic variety X with $\mathfrak{h}(X) \in \mathscr{M}^{ab}$, the existence of a marking satisfying $(\star_{\mathrm{Mult}})$ is equivalent to the existence of an isomorphism of algebra objects

$$\varphi : \mathfrak{h}(X) \xrightarrow{\simeq} T(\overline{\mathfrak{h}}(X)).$$

As such an isomorphism induces a section of the epimorphism $\mathrm{CH}(X) \to \overline{\mathrm{CH}}(X)$. The condition $(\star_{\mathrm{Chern}})$ can be translated into saying that the Chern classes belong to the image of the section. Similarly, in the presence of a G-action, the condition $(\star_G)$ can be spelled out by its graphs.

This formulation of $(\star)$ has the obvious advantage of being both natural and intrinsic. However, to work out examples, which is the main objective of this paper, as well as to prove theorems in practice (Sections 4 and 5), we find it more convenient to stick to Definition 3.7 together with its interpretation given in Lemma 3.9.

The motivation to study condition $(\star)$ is the following.

[14]The condition that $\mathfrak{h}(X) \in \mathscr{M}^{ab}$ corresponds to the condition $X \in \mathscr{V}^0$ in [38, §6.3], and for such X, the existence of a marking satisfying $(\star_{\mathrm{Mult}})$ corresponds to the condition $X \in \mathscr{V}^{00}$ in [38, §6.3].

Proposition 3.12 (Subalgebra)**.** *Let X be a smooth projective variety with motive of abelian type. If X satisfies condition $(\star_{\mathrm{Mult}})$, then there is a section, as* graded algebras*, for the natural surjective morphism* $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$*. If, moreover, $(\star_{\mathrm{Chern}})$ is satisfied, then all Chern classes of X are in the image of this section.*

In other words, under $(\star)$, we have a graded $\mathbb{Q}$-subalgebra $\mathrm{DCH}(X)$ *of the Chow ring* $\mathrm{CH}(X)$*, which contains all the Chern classes of X and is mapped isomorphically to* $\overline{\mathrm{CH}}(X)$*. We call elements of* $\mathrm{DCH}(X)$ distinguished cycles *of X.*

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ be a marking, where $M \in \mathscr{M}^{ab}_{sd}$. If ϕ satisfies $(\star)$, then we define $\mathrm{DCH}(X) := \mathrm{DCH}_\phi(X)$ as in Definition 3.2, and this provides a section to the epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ as graded vector spaces by Lemma 3.3. To show that it provides a section as *algebras*, one has to show that $\mathrm{DCH}_\phi(X)$ is closed under the intersection product of X (the unit $\mathbb{1}_X$ is automatically distinguished by Remark 3.4). Let $\alpha \in \mathrm{DCH}^i_\phi(X)$, and let $\beta \in \mathrm{DCH}^j_\phi(X)$. Then by definition the morphisms $\phi \circ \alpha : \mathbb{1}(-i) \to F(M)$ and $\phi \circ \beta : \mathbb{1}(-j) \to F(M)$ determine symmetrically distinguished morphisms. By Lemma 2.4(i), $(\phi^{\otimes 2}) \circ (\alpha \otimes \beta) = (\phi \circ \alpha) \otimes (\phi \circ \beta) : \mathbb{1}(-i-j) \to F(M^{\otimes 2})$ also determines a symmetrically distinguished morphism:

$$\begin{tikzcd}
\mathbb{1}(-i-j) \arrow[r, "\alpha\otimes\beta"] \arrow[dr] & \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}", "\simeq"'] & \mathfrak{h}(X) \arrow[d, "\phi", "\simeq"'] \\
 & F(M^{\otimes 2}) \arrow[r, "F(\mu)"] & F(M)
\end{tikzcd}$$

Condition $(\star)$ implies that μ, which is determined by the above commutative diagram, is a symmetrically distinguished morphism. Therefore, the composition $\phi \circ \delta_X \circ (\alpha \otimes \beta)$ in the above diagram determines a symmetrically distinguished morphism, which means that $\alpha \cdot \beta = \delta_{X,*}(\alpha \otimes \beta)$ is in $\mathrm{DCH}_\phi(X)$. The assertion concerning Chern classes is tautological. □

We deduce that condition $(\star_{\mathrm{Mult}})$ actually already implies all the analogous statements for all sorts of diagonals on higher powers (note the analogy with [43, Proposition 8.7(*iii*)] in the context of self-dual multiplicative Chow–Künneth decompositions).

Corollary 3.13 (Other diagonals)**.** *Let X be a smooth projective variety with $\mathfrak{h}(X) \in \mathscr{M}^{ab}$. If X satisfies condition $(\star_{\mathrm{Mult}})$, then all the classes of the partial diagonals*[15] *in a self-product of X are distinguished.*

[15] A *partial diagonal* of a self-product X^n is a subvariety of the form $\{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for all } i \sim j\}$ for an equivalence relation $\sim$ on $\{1, \dots, n\}$.

Proof. Let us fix a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ satisfying condition $(\star_{\mathrm{Mult}})$ and write DCH for $\mathrm{DCH}_{\phi^{\otimes ?}}$. Observe that any partial diagonal can be written as the intersection product of several big diagonals.[16] By Proposition 3.12, we only have to show that any big diagonal of a self-product is distinguished. However, a big diagonal is the exterior product of the distinguished class $\Delta_X \in \mathrm{DCH}(X \times X)$ (by Lemma 3.8) with copies of the fundamental class $\mathbb{1}_X \in \mathrm{DCH}(X)$ (see Remark 3.4), and is henceforth distinguished, thanks to Proposition 3.5. $\square$

3.3. Distinguished morphisms and distinguished correspondences.

Definition 3.14 (Distinguished morphisms and distinguished correspondences)**.** Let X and Y be two smooth projective varieties equipped, respectively, with markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} F(N)$ with $M, N \in \mathscr{M}_{sd}^{ab}$. A correspondence $\Gamma \in \mathrm{CH}(X \times Y)$ is said to be *distinguished* if it is distinguished with respect to the product marking on $X \times Y$, i.e., $\Gamma \in \mathrm{DCH}_{\phi\otimes\psi}(X \times Y)$, or equivalently the morphism $(\phi \otimes \psi)(\Gamma) : M \to N$ is symmetrically distinguished in the sense of Definition 2.3. A morphism $f : X \to Y$ is said to be *distinguished* if its graph belongs to $\mathrm{DCH}_{\phi\otimes\psi}(X \times Y)$.

The notion of distinguished morphisms and distinguished correspondences is only really relevant in the case where the markings satisfy condition $(\star_{\mathrm{Mult}})$.

Proposition 3.15. *Let X, Y, and Z be smooth projective varieties equipped with markings that satisfy $(\star_{\mathrm{Mult}})$, and let $\Gamma \in \mathrm{DCH}(X \times Y)$ and $\Gamma' \in \mathrm{DCH}(Y \times Z)$ be distinguished correspondences. Then:*

(i) $\Gamma_*\mathrm{DCH}(X) \subseteq \mathrm{DCH}(Y)$ *and* $\Gamma^*\mathrm{DCH}(Y) \subseteq \mathrm{DCH}(X)$*;*
(ii) $\Gamma' \circ \Gamma \in \mathrm{DCH}(X \times Z)$.

Proof. This is a direct consequence of Propositions 3.5 and 3.12. $\square$

Corollary 3.16 (Top Chern class)**.** *Let X be an n-dimensional smooth projective variety equipped with a marking satisfying $(\star_{\mathrm{Mult}})$. Then the top Chern class of X is distinguished, i.e., $c_n(T_X) \in \mathrm{DCH}_0(X)$.*

In particular, for a smooth projective curve, $(\star_{\mathrm{Chern}})$ is implied by $(\star_{\mathrm{Mult}})$.

Proof. Observe that the small diagonal δ_X, viewed as a correspondence between X and $X \times X$, is distinguished by hypothesis and it transforms Δ_X to $c_n(X)$:

$$\delta_X^*(\Delta_X) = c_n(X).$$

Under the hypothesis $(\star_{\mathrm{Mult}})$, we know that $\Delta_X \in \mathrm{DCH}(X \times X)$ by Lemma 3.8. Hence Proposition 3.15(i) yields that the top Chern class $c_n(X)$ is distinguished.

[16] A *big diagonal* of a self-product X^n is a subvariety of the form $\{(x_1, \dots, x_n) \in X^n \mid x_i = x_j\}$ for some $1 \leq i \neq j \leq n$.

As for the case of curves, it suffices to recall, moreover, that the fundamental class is automatically distinguished by Remark 3.4. □

4. Operations preserving condition $(\star)$

In this section, we provide some standard operations on varieties that preserve $(\star)$. From now on, we systematically omit the functor $F : \mathscr{M}^{ab}_{sd} \to \mathscr{M}^{ab}$, which is an equivalence of categories (Lemma 2.2), in the notation of a marking.

4.1. Product varieties. Given two smooth projective varieties X and Y with markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$, their product will always be understood to be endowed with the marking

$$\phi \otimes \psi : \mathfrak{h}(X \times_k Y) \cong \mathfrak{h}(X) \otimes \mathfrak{h}(Y) \xrightarrow{\simeq} M \otimes N,$$

which we will refer to as the *product marking*. If X and Y are endowed with the action of a finite group G, then $X \times Y$ is endowed with the natural diagonal action of G. Our condition $(\star)$ (see Definition 3.7) behaves well with respect to products as seen below.

Proposition 4.1 (Products)**.** *Assume X and Y are two smooth projective varieties satisfying condition $(\star)$. Then the natural marking on the product $X \times Y$ satisfies $(\star)$ and has the additional property that the graphs of the two natural projections are distinguished.*

If in addition X and Y are equipped with the action of a finite group G and the respective markings satisfy $(\star_G)$, then the product marking on $X \times Y$ satisfies $(\star_G)$.

Proof. By assumption, there are markings $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ satisfying $(\star)$. The assertion $(\star_{\mathrm{Mult}})$ (resp., $(\star_G)$) follows from Proposition 3.5 applied to X and Y replaced by X^3 and Y^3 (resp., X^2 and Y^2). Indeed, $\delta_{X\times Y} = \delta_X \otimes \delta_Y$ (resp., $g_{X\times Y} = g_X \otimes g_Y$).

The assertion $(\star_{\mathrm{Chern}})$ concerning the Chern classes follows directly from the formula

$$c_i(X \times Y) = \sum_{j=0}^{i} c_j(X) \otimes c_{i-j}(Y)$$

and Proposition 3.5.

Finally, as the diagonal $\Delta_X \in \mathrm{CH}(X \times X)$ and fundamental class $\mathbb{1}_Y$ of Y are distinguished (Lemma 3.8, Remark 3.4), Proposition 3.5 tells us that the graph of the projection $X \times Y \to X$, which is equal to $\Delta_X \otimes \mathbb{1}_Y \in \mathrm{CH}(X \times X \times Y)$, is distinguished. The proof is similar for the other projection $X \times Y \to Y$. □

Remark 4.2 (Permutations)**.** Suppose X has a marking that satisfies $(\star)$. Then any permutation of the factors of X^n defines a distinguished correspondence in $\mathrm{DCH}(X^{2n})$ for the product marking by Corollary 3.13.

Remark 4.3. Assume X and Y are two smooth projective varieties endowed with the action of the finite groups G and H, respectively. The product $G \times H$ acts naturally on the product $X \times Y$. Suppose X and Y satisfy $(\star_G)$ and $(\star_H)$, respectively. Then the same arguments as above show that the product marking on $X \times Y$ satisfies $(\star_{G\times H})$.

4.2. Projective bundles. We show in this subsection that the condition $(\star)$ is stable by forming projective bundles as long as the Chern classes of the vector bundle are distinguished.

Let X be a smooth projective variety of dimension d, and let E be a vector bundle over X of rank $(r+1)$. Let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle.[17] Let ξ be the first Chern class of $\mathcal{O}_\pi(1)$.

Recall the *projective bundle formula* (see [2, §4.3.2]):

$$b : \bigoplus_{k=0}^{r} \mathfrak{h}(X)(-k) \xrightarrow{\simeq} \mathfrak{h}(\mathbb{P}E), \tag{3}$$

which is given factorwise by $\xi^k \cdot \pi^* : \mathfrak{h}(X)(-k) \to \mathfrak{h}(\mathbb{P}E)$ for $0 \le k \le r$.

The following lemma[18] computes the small diagonal for $\mathbb{P}E$. A piece of notation is convenient: for an element $\omega \in \mathrm{CH}^k(X)$, viewed as a morphism $\mathbb{1} \to \mathfrak{h}(X)(k)$, we will talk about the morphism *multiplication by* ω, denoted by $\cdot\omega : \mathfrak{h}(X) \to \mathfrak{h}(X)(k)$, which is by definition the following composition:

$$\mathfrak{h}(X) \xrightarrow{\mathrm{id}\otimes\omega} \mathfrak{h}(X) \otimes \mathfrak{h}(X)(k) \xrightarrow{\delta_X(k)} \mathfrak{h}(X)(k).$$

With a marking being fixed, if ω belongs to $\mathrm{DCH}(X)$ and X satisfies $(\star_{\mathrm{Mult}})$, then by Proposition 3.15 multiplication by ω is a distinguished morphism.

Lemma 4.4 (Small diagonal of projective bundles)**.** *Notation is as above. The intersection product*

$$\delta_{\mathbb{P}E} : \mathfrak{h}(\mathbb{P}E) \otimes \mathfrak{h}(\mathbb{P}E) \to \mathfrak{h}(\mathbb{P}E)$$

induces, via (3)*, a morphism* $\left(\bigoplus_{k=0}^{r} \mathfrak{h}(X)(-k)\right)^{\otimes 2} \to \bigoplus_{m=0}^{r} \mathfrak{h}(X)(-m)$ *such that for any* $0 \le k, l, m \le r$*, the morphism*

$$\mathfrak{h}(X)(-k) \otimes \mathfrak{h}(X)(-l) \to \mathfrak{h}(X)(-m)$$

is described as:

- *If* $m > k + l$ *or* $m > r$*, it is the zero map.*

[17] The $\mathbb{P}$ we are using here is the space of 1-dimensional subspaces, thus different from Grothendieck's convention.

[18] This should be known, but the authors could not find a proper reference.

- *If* $m = k+l \leq r$*, it is induced by the intersection product of* X*, namely,* δ_X*.*
- *If* $k+l \leq r$ *and* $m \neq k+l$*, it is the zero map.*
- *If* $m \leq r < k+l$*, then it is the composition*

$$\mathfrak{h}(X)(-k) \otimes \mathfrak{h}(X)(-l) \xrightarrow{\delta_X(-k-l)} \mathfrak{h}(X)(-k-l) \xrightarrow{\cdot\omega} \mathfrak{h}(X)(-m),$$

where the second morphism is the multiplication by the following characteristic class (*with* s *being the Segre class*)[19]

$$\omega := \sum_{t=0}^{r-m} c_t(E) s_{k+l-m-t}(E) \in \mathrm{CH}^{k+l-m}(X).$$

Proof. By Manin's identity principle ([2, §4.3.1]), we only have to prove the lemma for Chow groups. Let us first compute the inverse b^{-1} of the isomorphism in the projective bundle formula

$$b : \bigoplus_{k=0}^{r} \mathrm{CH}^{*-k}(X) \xrightarrow{\simeq} \mathrm{CH}^*(\mathbb{P}E).$$

Assume $\gamma \in \mathrm{CH}^*(\mathbb{P}E)$ is the image of $(z_0, z_1, \ldots, z_r) \in \bigoplus_{k=0}^{r} \mathrm{CH}^{*-k}(X)$, i.e.,

$$\gamma = \sum_{k=0}^{r} \pi^*(z_k) \cdot \xi^k.$$

For any $t \geq 0$, $\pi_*(\gamma \cdot \xi^t) = \sum_{k=0}^{r} \pi_*(\pi^*(z_k) \cdot \xi^{k+t}) = \sum_{k=0}^{r} z_k \cdot s_{k+t-r}(E)$. Since the total Segre class is the inverse of the total Chern class, we have for any $0 \leq k \leq r$,

$$z_k = \sum_{t=0}^{r-k} c_t(E) \cdot \pi_*(\gamma \cdot \xi^{r-k-t}).$$

This gives b^{-1}. Now let us go back to the product formula. We have to compute the composition $b^{-1} \circ (b \otimes b)$ whose (k, l, m)th component for any $0 \leq k, l, m \leq r$ is the composition

$$\mathrm{CH}(X) \otimes \mathrm{CH}(X) \xrightarrow{(\xi^k \cdot \pi^*, \xi^l \cdot \pi^*)} \mathrm{CH}(\mathbb{P}E) \otimes \mathrm{CH}(\mathbb{P}E) \xrightarrow{\cdot} \mathrm{CH}(\mathbb{P}E) \xrightarrow{b_m^{-1}} \mathrm{CH}(X),$$

where the last morphism is $\sum_{t=0}^{r-m} c_t(E) \cdot \pi_*(\bullet \cdot \xi^{r-m-t})$ by the formula for b^{-1}. Now, for any $z, z' \in \mathrm{CH}(X)$, the mth component of $\pi^*(z) \cdot \xi^k \cdot \pi^*(z') \cdot \xi^l = \pi^*(z \cdot z') \cdot \xi^{k+l}$ is $\sum_{t=0}^{r-m} c_t(E) \cdot \pi_*(\pi^*(z \cdot z') \cdot \xi^{k+l} \cdot \xi^{r-m-t}) = z \cdot z' \cdot (\sum_{t=0}^{r-m} c_t(E) s_{k+l-m-t}(E))$. We can easily conclude in all cases. □

[19] The total Segre class is by definition the inverse of the total Chern class; cf. [22, Chapter 3].

Proposition 4.5 (($\star$) and projective bundles)**.** *Let X be a smooth projective variety, and let E be a vector bundle over X of rank $(r+1)$. Let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle. If we have a marking for X satisfying ($\star$) such that all Chern classes of E are distinguished, then $\mathbb{P}E$ has a natural marking such that $\mathbb{P}E$ satisfies ($\star$) and such that the projection $\pi : \mathbb{P}E \to X$ is distinguished.*

If in addition X is equipped with the action of a finite group G such that E is G-equivariant and such that the marking of X satisfies ($\star_G$), then the natural marking of $\mathbb{P}E$ satisfies ($\star_G$).

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be a marking that satisfies ($\star$) and is such that $c_k(E) \in \mathrm{DCH}(X)$. Using the projective bundle formula (3), we obtain a marking for $\mathbb{P}E$:

$$\lambda : \mathfrak{h}(\mathbb{P}E) \xrightarrow{\simeq} \bigoplus_{k=0}^{r} M(-k).$$

Let us show that λ satisfies ($\star$).

For ($\star_{\mathrm{Mult}}$), one uses the interpretation of ($\star_{\mathrm{Mult}}$) given in Lemma 3.9. Since δ_X as well as the Chern classes and Segre classes of E are distinguished, the condition ($\star_{\mathrm{Mult}}$) follows from Lemma 4.4.

For ($\star_{\mathrm{Chern}}$), we first claim that for any k, the cycle $\pi^*(\alpha) \cdot \xi^k$ is distinguished if $\alpha \in \mathrm{CH}(X)$ is so. For $k \leq r$, this is by definition, while for $k > r$, we use the equality $\xi^{r+1} + \pi^*(c_1(E))\xi^r + \cdots + \pi^*(c_{r+1}(E)) = 0$ and the distinguishedness of the Chern classes of E to reduce to the treated cases. Now from the short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \pi^*(E) \otimes \mathcal{O}_\pi(1) \to T_{\mathbb{P}E/X} \to 0,$$

$$0 \to T_{\mathbb{P}E/X} \to T_{\mathbb{P}E} \to \pi^* T_X \to 0,$$

we see that all the Chern characters of $\mathbb{P}E$ are linear combinations of terms of the form $\pi^*(\alpha) \cdot \xi^k$, where α is a polynomial of Chern classes of X and of E. By assumption α is distinguished; hence so are the Chern characters of $\mathbb{P}E$. With ($\star_{\mathrm{Mult}}$) being proven for $\mathbb{P}E$, we know that $\mathrm{DCH}(\mathbb{P}E)$ is a subalgebra by Proposition 3.12. We are then done because Chern classes are polynomials of Chern characters.

The distinguishedness of (the graph of) the projection $\pi : \mathbb{P}(E) \to X$ is obvious: via the markings ϕ and λ, it is equivalent to saying that the inclusion of the first summand

$$M \hookrightarrow M \oplus M(-1) \oplus \cdots \oplus M(-r)$$

is a symmetrically distinguished morphism.

Finally, assume that X is equipped with the action of a finite group G such that E is G-equivariant. Note that with the induced action of G on $\mathbb{P}E$, we have that π is G-equivariant and we have that $(g_{\mathbb{P}E})_*\xi = \xi$ (since G preserves $\mathcal{O}_\pi(1)$). Thus the action of G commutes with b and $b^\vee$. Since we are assuming that the marking ϕ of X satisfies $(\star_G)$, we find that the marking λ satisfies $(\star_G)$. □

Example 4.6. If X is a smooth projective variety with a marking that satisfies $(\star)$, then natural examples of vector bundles with distinguished Chern classes are given by the tangent bundle T_X as well as other vector bundles obtained from it by performing duals, tensor products, and direct sums. More generally, one may consider direct sums of vector bundles of the form $\mathbb{S}_\lambda T_X$, where λ is a nonincreasing sequence of integers and $\mathbb{S}_\lambda$ is the associated Schur functor. By Proposition 4.5, the projective bundle associated to any such vector bundle has a marking that satisfies $(\star)$.

4.3. Blow-ups. We will show in this subsection that condition $(\star)$ in Definition 3.7 passes to a blow-up in the expected way.

We fix the following notation throughout this subsection. Let X be a smooth projective variety of dimension d, let $i : Y \hookrightarrow X$ be a closed immersion of a smooth subvariety of codimension c, and let $\mathscr{N} := \mathscr{N}_{Y/X}$ be the normal bundle. Let $\widetilde{X}$ be the blow-up of X along Y, and let E be the exceptional divisor, which is identified with $\mathbb{P}(\mathscr{N})$. Denote by ξ the first Chern class of $\mathcal{O}_p(1) = \mathscr{N}^\vee_{E/\widetilde{X}}$. The names of some relevant morphisms are in the following cartesian diagram:

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\tau"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd} \tag{4}$$

Recall the blow-up formula (see [2, §4.3.2])

$$b : \mathfrak{h}(X) \oplus \bigoplus_{k=1}^{c-1} \mathfrak{h}(Y)(-k) \xrightarrow{\simeq} \mathfrak{h}(\widetilde{X}), \tag{5}$$

which is given by:

- $\tau^* : \mathfrak{h}(X) \to \mathfrak{h}(\widetilde{X})$;
- for any $1 \leq k \leq c-1$, $j_*(\xi^{k-1} \cdot p^*(-)) : \mathfrak{h}(Y)(-k) \to \mathfrak{h}(\widetilde{X})$.

The following lemma[20] computes the small diagonal of $\widetilde{X}^3$.

Lemma 4.7 (Small diagonal of blow-ups)**.** *The intersection product*

$$\delta_{\widetilde{X}} : \mathfrak{h}(\widetilde{X}) \otimes \mathfrak{h}(\widetilde{X}) \to \mathfrak{h}(\widetilde{X})$$

is described via the isomorphism (5) *as follows:*

[20] This should be known, but the authors could not find a proper reference.

- *$\mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ is the intersection product (induced by δ_X).*
- *For any $1 \le k \le c-1$, $\mathfrak{h}(X) \otimes \mathfrak{h}(Y)(-k) \to \mathfrak{h}(Y)(-k)$ is the composition*

  $$\mathfrak{h}(X) \otimes \mathfrak{h}(Y)(-k) \xrightarrow{i^* \otimes \mathrm{id}} \mathfrak{h}(Y) \otimes \mathfrak{h}(Y)(-k) \xrightarrow{\delta_Y(-k)} \mathfrak{h}(Y)(-k).$$
- *For any $1 \le k, l \le c-1$,*

  $$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \to \mathfrak{h}(X)$$

  is the composition

$$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \xrightarrow{\delta_Y(-k-l)} \mathfrak{h}(Y)(-k-l) \xrightarrow{-\cdot s_{k+l-c}(\mathscr{N})} \mathfrak{h}(Y)(-c) \xrightarrow{i_*} \mathfrak{h}(X),$$

  where in second morphism, s stands for the Segre class.
- *For any $1 \le k, l, m \le c-1$,*

  $$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \to \mathfrak{h}(Y)(-m)$$

  is as follows:
  - *if $m \ge c$ or $m > k+l$, it is the zero map;*
  - *if $m = k+l \le c-1$, then it is induced by $-\delta_Y$;*
  - *if $m \ne k+l \le c-1$, then it is the zero map;*
  - *if $m \le c-1 < k+l$, it is the composition*

  $$\mathfrak{h}(Y)(-k) \otimes \mathfrak{h}(Y)(-l) \xrightarrow{\delta_Y(-k-l)} \mathfrak{h}(Y)(-k-l) \xrightarrow{\cdot\omega} \mathfrak{h}(Y)(-m),$$

    where the second morphism is the multiplication by the following characteristic class with s standing for the Segre class:

  $$\omega := -\sum_{t=1}^{c-m} s_{k+l-m-t+1}(\mathscr{N}) \cdot c_{t-1}(\mathscr{N}) \in \mathrm{CH}^{k+l-m}(Y).$$

Proof. We only have to prove the lemma for Chow groups thanks to Manin's identity principle ([2, §4.3.1]). As in Lemma 4.4, we compute the inverse of

$$b : \mathrm{CH}^*(X) \oplus \bigoplus_{k=1}^{c-1} \mathrm{CH}^{*-k}(Y) \to \mathrm{CH}^*(\widetilde{X}).$$

Assume $\gamma = \tau^*(z_0) + \sum_{k=1}^{c-1} j_*(p^*(z_k) \cdot \xi^{k-1})$, where $z_0 \in \mathrm{CH}(X)$ and $z_k \in \mathrm{CH}(Y)$ for all $1 \le k \le c-1$. Then b^{-1} is given by $z_0 = \tau_*(\gamma)$, and for all $1 \le k \le c-1$,

$$z_k = -\sum_{t=1}^{c-k} p_*(j^*(\gamma) \cdot \xi^{c-k-t}) \cdot c_{t-1}(\mathscr{N}).$$

Now concerning intersection products, we have to compute $b^{-1} \circ (b \otimes b)$. We only give the computation of the (k, l, m)th component when $1 \le k, l, m \le c-1$

and leave the other cases to the reader. Let $z, z' \in \mathrm{CH}(Y)$. Then the mth component of the product

$$j_*(p^*(z)\cdot\xi^{k-1})\cdot j_*(p^*(z')\cdot\xi^{l-1}) = j_*\left(p^*(z)\cdot\xi^{k-1}\cdot j^*(j_*(p^*(z')\cdot\xi^{l-1}))\right)$$
$$= -j_*\left(p^*(z\cdot z')\cdot\xi^{k+l-1}\right)$$

is

$$\begin{aligned} & \sum_{t=1}^{c-m} p_*\left(j^*j_*\left(p^*(z\cdot z')\cdot\xi^{k+l-1}\right)\cdot\xi^{c-m-t}\right)\cdot c_{t-1}(\mathscr{N}) \\ = \quad & -\sum_{t=1}^{c-m} p_*\left(p^*(z\cdot z')\cdot\xi^{k+l+c-m-t}\right)\cdot c_{t-1}(\mathscr{N}) \\ = \quad & -\sum_{t=1}^{c-m} z\cdot z'\cdot s_{k+l-m-t+1}(\mathscr{N})\cdot c_{t-1}(\mathscr{N}). \end{aligned}$$

Then all cases follow easily. □

Proposition 4.8 (($\star$) and blow-ups)**.** *Let X be a smooth projective variety, and let $i: Y \hookrightarrow X$ be a smooth closed subvariety. If we have markings satisfying condition ($\star$) for X and Y such that the inclusion morphism $i: Y \hookrightarrow X$ is distinguished (Definition 3.14), then $\widetilde{X}$, the blow-up of X along Y has a natural marking that satisfies ($\star$) and is such that the morphisms in diagram (4) are all distinguished.*[21]

If in addition X is equipped with the action of a finite group G such that $G\cdot Y = Y$ and such that the markings of X and Y satisfy ($\star_G$), then the natural marking of $\widetilde{X}$ also satisfies ($\star_G$).

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ and $\psi : \mathfrak{h}(Y) \xrightarrow{\simeq} N$ be markings satisfying ($\star$). Using the blow-up formula (5), ϕ and ψ induce a marking for $\widetilde{X}$:

$$\lambda : \mathfrak{h}(\widetilde{X}) \xrightarrow{\simeq} M \oplus \bigoplus_{k=1}^{c-1} N(-k), \tag{6}$$

which we will show to satisfy ($\star$).

Using the short exact sequence $0 \to T_Y \to T_X|_Y \to \mathscr{N} \to 0$, we see that the Chern classes of $\mathscr{N}$ can be expressed as polynomials of Chern classes of Y and Chern classes of X restricted to Y, which are all in $\mathrm{DCH}(Y)$ by hypothesis ($\star_{\mathrm{Chern}}$) for X and Y. Since $\mathrm{DCH}(Y)$ is a subalgebra (Proposition 3.12), all Chern classes of $\mathscr{N}$ are distinguished on Y. The condition ($\star_{\mathrm{Mult}}$) then follows from Lemma 4.7 (together with Proposition 3.15), since all Segre and Chern classes as well as the morphisms $i^* : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ and $i_* : \mathfrak{h}(Y) \to \mathfrak{h}(X)(c)$,

[21]The exceptional divisor E is endowed with the natural marking of Proposition 4.5 by its projective bundle structure over Y.

and the intersection products $\delta_X : \mathfrak{h}(X)^{\otimes 2} \to \mathfrak{h}(X)$ and $\delta_Y : \mathfrak{h}(Y)^{\otimes 2} \to \mathfrak{h}(Y)$ are all distinguished by assumption.

That the morphisms in diagram (4) are all distinguished in the sense of Definition 3.14 is straightforward: the inclusion morphism $i : Y \hookrightarrow X$ is distinguished by assumption; the projective bundle $p : E \to Y$ is distinguished thanks to Proposition 4.5; the distinguishedness of of τ is equivalent to saying that (via the markings ϕ and λ) the inclusion of the first summand $M \hookrightarrow M \oplus \bigoplus_{k=1}^{c-1} N(-k)$ is symmetrically distinguished, which is obvious; finally, one checks easily that via the natural markings, the morphism $j^* : \mathfrak{h}(\widetilde{X}) \to \mathfrak{h}(E)$ corresponds to the morphism

$$(i^*, -\mathrm{id}, \ldots, -\mathrm{id}) : M \oplus N(-1) \oplus \cdots \oplus N(-c+1) \to N \oplus N(-1) \oplus \cdots \oplus N(-c+1),$$

which is obviously symmetrically distinguished.

Now for $(\star_{\mathrm{Chern}})$, we use the formula for Chern classes of a blow-up given in [22, Theorem 15.4]. Given the distinguishedness of the Chern classes of T_X, T_Y, and $\mathscr{N}$, we only have to show that for any $\alpha \in \mathrm{DCH}(Y)$ and $k \in \mathbb{N}$, the class $j_*(p^*(\alpha) \cdot \xi^k) \in \mathrm{CH}(\widetilde{X})$ is distinguished. But that is immediate, because each of j, p, α, and $\xi = -j^* j_*(1)$ is distinguished by the above.

Finally, assume that X is equipped with the action of a finite group G such that $G \cdot Y = Y$. Note that with the induced action of G on E and $\widetilde{X}$, we have that the morphisms in diagram (4) are G-equivariant. Thus the action of G commutes with b and $b^\vee$. Since we are assuming that the markings of X and Y satisfy $(\star_G)$, we find that the marking λ satisfies $(\star_G)$. □

4.4. Generically finite morphism. In this subsection, we show that the condition $(\star)$ passes from the source variety of a surjective and generically finite morphism to the target variety under natural assumptions.

Proposition 4.9. *Let $\pi : X \to Y$ be a generically finite and surjective morphism between smooth projective varieties. If X has a marking satisfying $(\star_{\mathrm{Mult}})$ and such that the cycle ${}^t\Gamma_\pi \circ \Gamma_\pi$ is distinguished in $\mathrm{CH}(X \times X)$, then Y has a natural marking that satisfies $(\star_{\mathrm{Mult}})$ and is such that the graph of π is distinguished.*

Proof. Let d be the degree of π, and let n be the dimension of X and Y. Let $M \in \mathscr{M}_{sd}^{ab}$, and let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be a marking satisfying $(\star_{\mathrm{Mult}})$. The graph of π and its transpose induce, respectively,

$$\pi_* := \Gamma_\pi : \mathfrak{h}(X) \to \mathfrak{h}(Y),$$

$$\pi^* := {}^t\Gamma_\pi : \mathfrak{h}(Y) \to \mathfrak{h}(X)$$

such that $\pi_* \circ \pi^* = d \cdot \Delta_Y \in \mathrm{End}(\mathfrak{h}(Y))$. Therefore $\frac{1}{d}{}^t\Gamma_\pi \circ \Gamma_\pi = \frac{1}{d}\pi^* \circ \pi_* \in \mathrm{End}(\mathfrak{h}(X))$ is a projector and π^* induces an isomorphism of Chow motives

$$\mathfrak{h}(Y) \xrightarrow{\simeq} \left(X, \frac{1}{d}{}^t\Gamma_\pi \circ \Gamma_\pi, 0\right).$$

Consider the projector

$$q := \phi \circ \frac{1}{d}{}^t\Gamma_\pi \circ \Gamma_\pi \circ \phi^{-1}$$

in $\mathrm{End}(M)$. Since ${}^t\Gamma_\pi \circ \Gamma_\pi$ is distinguished by assumption, q is a symmetrically distinguished idempotent endomorphism of M. By Lemma 2.6, we have a canonical image

$$N := \mathrm{Im}\,(q : M \to M)\,,$$

with $N \in \mathscr{M}_{sd}^{ab}$ and such that the projection $p : M \twoheadrightarrow N$ and the inclusion $i : N \hookrightarrow M$ are symmetrically distinguished morphisms in $\mathscr{M}_{sd}^{ab}$.

By definition, we have $p \circ i = \mathrm{id}$ and $i \circ p = q = \phi \circ \frac{1}{d}\pi^* \circ \pi_* \circ \phi^{-1}$. Therefore the composition

$$\lambda := p \circ \phi \circ \pi^* : \mathfrak{h}(Y) \to N$$

is an isomorphism with inverse $\lambda^{-1} = \frac{1}{d}\pi_* \circ \phi^{-1} \circ i$. Note that λ is nothing else but the following composition of isomorphisms:

$$\mathfrak{h}(Y) \xrightarrow{\simeq} \left(X, \frac{1}{d}{}^t\Gamma_\pi \circ \Gamma_\pi, 0\right) \xrightarrow{p \circ \phi} N.$$

We now show that the marking for Y provided by the isomorphism λ satisfies ($\star_{\mathrm{Mult}}$). We consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{h}(Y) \otimes \mathfrak{h}(Y) \arrow[r, "\delta_Y"] \arrow[d, hook, "\pi^* \otimes \pi^*"] & \mathfrak{h}(Y) \arrow[d, hook, "\pi^*"] \arrow[ddd, bend left=60, "\lambda"] \\
\mathfrak{h}(X) \otimes \mathfrak{h}(X) \arrow[r, "\delta_X"] \arrow[d, "\simeq"', "\phi \otimes \phi"] & \mathfrak{h}(X) \arrow[d, "\simeq"', "\phi"] \\
M \otimes M \arrow[r, "\mu_X"] & M \arrow[d, two heads, "p"] \\
N \otimes N \arrow[r, "\mu_Y"] \arrow[u, hook, "i \otimes i"] & N
\end{tikzcd}$$

where $\mu_Y := p \circ \mu_X \circ (i \otimes i)$ is clearly symmetrically distinguished as μ_X, $i \otimes i$, and p are so. By Lemma 3.9, it suffices to check that $\mu_Y \circ (\lambda \otimes \lambda) = \lambda \circ \delta_Y$.

This is straightforward:

$$\begin{aligned}\mu_Y \circ (\lambda \otimes \lambda) &= p \circ \mu_X \circ (i \otimes i) \circ (\lambda \otimes \lambda)\\ &= p \circ \mu_X \circ (\phi \otimes \phi) \circ (\pi^* \otimes \pi^*)\\ &= p \circ \phi \circ \pi^* \circ \delta_Y\\ &= \lambda \circ \delta_Y,\end{aligned}$$

where the second equality uses $i \circ \lambda = i \circ p \circ \phi \circ \pi^* = q \circ \phi \circ \pi^* = \phi \circ \pi^*$ and the third equality uses the commutativity of the previous diagram.

That the graph of $\pi : X \to Y$ is distinguished is equivalent to the condition that the natural inclusion $N \hookrightarrow M$, or equivalently $p : M \twoheadrightarrow N$, is a symmetrically distinguished morphism. $\square$

Remark 4.10 (($\star$) and semismall morphisms)**.** When $\pi : X \to Y$ is semismall (cf. Section 5.5.1), then the condition on the cycle ${}^t\Gamma_\pi \circ \Gamma_\pi$ in Proposition 4.9 is equivalent to the more explicit condition that the class of $X \times_Y X$ in $\mathrm{CH}_n(X \times X)$ is distinguished.

Proposition 4.11 (($\star$) and étale covers)**.** *Notation and assumptions are as in Proposition* 4.9*. If, moreover, π is étale and the marking for X satisfies* ($\star_{\mathrm{Chern}}$)*, then the natural marking for Y also satisfies* ($\star_{\mathrm{Chern}}$)*.*

Proof. Let d be the degree of π. For any $i \in \mathbb{N}$, $c_i(Y) = \frac{1}{d}\pi_*\pi^*(c_i(Y)) = \frac{1}{d}\pi_* c_i(X)$ is distinguished since $c_i(X)$ is distinguished and π is a distinguished morphism. $\square$

Proposition 4.12 (($\star$) and finite group quotients)**.** *Let X be a smooth projective variety endowed with an action of a finite group G such that the quotient $Y := X/G$ is smooth. If there is a marking for (X, G) satisfying* ($\star_{\mathrm{Mult}}$) *and* ($\star_G$)*, then Y has a natural marking that satisfies* ($\star_{\mathrm{Mult}}$) *and is such that the quotient morphism $\pi : X \to Y$ is distinguished.*

Moreover, if $\pi : X \to Y$ is étale or a cyclic covering along a divisor D such that $D \in \mathrm{DCH}(X)$ and if the marking for X satisfies ($\star_{\mathrm{Chern}}$)*, then the natural marking for Y also satisfies* ($\star_{\mathrm{Chern}}$)*.*

Proof. The assertions concerning ($\star_{\mathrm{Mult}}$) and the distinguishedness of π follow from Proposition 4.9. Indeed, by Remark 4.10, in order to apply Proposition 4.9, it suffices to check that the class of $X \times_Y X$ is distinguished. In the present situation of finite group quotient, $X \times_Y X$ is nothing but $\sum_g \Gamma_g$, which is distinguished in $\mathrm{CH}(X \times X)$ by ($\star_G$).

As for the condition ($\star_{\mathrm{Chern}}$), the étale case is treated in Proposition 4.11. Suppose $\pi : X \to Y$ is a degree d cyclic covering branched along a divisor D such that $D \in \mathrm{DCH}(X)$. In order to show that the natural marking on Y satisfies ($\star_{\mathrm{Chern}}$), it suffices to show by the projection formula that $\pi^* \mathrm{ch}(T_Y)$

is distinguished. We have a short exact sequence

$$0 \longrightarrow T_X \longrightarrow \pi^* T_Y \longrightarrow O_D(dD) \longrightarrow 0.$$

Since X satisfies ($\star_{\text{Chern}}$), it is enough to show that $\mathrm{ch}(O_D(dD))$ belongs to $\mathrm{DCH}(X)$. Now $O_D(dD)$ fits into the short exact sequence

$$0 \longrightarrow O_X((d-1)D) \longrightarrow O_X(dD) \longrightarrow O_D(dD) \longrightarrow 0.$$

Since the class of the divisor D is assumed to belong to the $\mathbb{Q}$-subalgebra $\mathrm{DCH}(X)$, we find that indeed $\mathrm{ch}(O_D(dD)) = \mathrm{ch}(O_X(dD)) - \mathrm{ch}(O_X((d-1)D))$ belongs to $\mathrm{DCH}(X)$, which concludes the proof. ☐

4.5. Hilbert squares and nested Hilbert schemes.

Proposition 4.13 (Hilbert squares)**.** *Assume X is a smooth projective variety with a marking that satisfies* ($\star$)*. Then $X^{[2]}$ has a natural marking that satisfies* ($\star$) *and is such that the universal family $\{(x,z) : x \in \mathrm{Supp}(z)\} \subseteq X \times X^{[2]}$ is distinguished (with respect to the product marking).*

Proof. The product $X \times X$ is naturally endowed with the action of $G := \mathbb{Z}/2$ that switches the factors, and the locus of fixed points is the diagonal, which is isomorphic to X. By Remark 4.2, the product marking on $X \times X$ satisfies ($\star_G$). Therefore, we may apply Proposition 4.8 to obtain a marking on the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal that satisfies ($\star$) and ($\star_G$). Now $X^{[2]}$ is the quotient of the latter blow-up by the cyclic action of $\mathbb{Z}/2$. Thus Proposition 4.12 provides a marking for $X^{[2]}$ that satisfies ($\star$).

Finally, we show that the universal family $Y := \{(x,z) : x \in \mathrm{Supp}(z)\}$ is distinguished. First note that Y is isomorphic to $\widetilde{X \times X}$ so that Y is endowed with the natural marking coming from that of X. In order to conclude, we only need to show that the graph Γ of the inclusion morphism $i : Y \hookrightarrow X \times X^{[2]}$ is distinguished. This is clear because the components $Y \to X$ and $Y \to X^{[2]}$ of i, which consist of the composition $\widetilde{X \times X} \to X \times X \to X$ and the quotient morphism $\widetilde{X \times X} \to X^{[2]}$, are distinguished. ☐

Recall that by a result of Cheah [13], for a smooth projective variety X of dimension ≥ 3, the only smooth nested Hilbert schemes of finite length subschemes on X are $X^{[2]}$, $X^{[3]}$, $X^{[1,2]}$, and $X^{[2,3]}$. By the same method, we have the following proposition.

Proposition 4.14 (Nested Hilbert schemes)**.** *The assumption is as in Proposition 4.13. Then $X^{[1,2]}$ and $X^{[2,3]}$ have natural markings satisfying* ($\star$) *and are such that the classes of the universal subschemes are distinguished.*

Proof. It is clear that $X^{[1,2]}$ is isomorphic to $\widetilde{X \times X}$, the blow-up of $X \times X$ along the diagonal, hence satisfies ($\star$) by Proposition 4.8. Similarly, $X^{[2,3]}$ is isomorphic to the blow-up of $X \times X^{[2]}$ along the universal subscheme Y. As is mentioned in the proof of the previous proposition, Y is isomorphic to

$X^{[1,2]}$ and hence to $\widetilde{X \times X}$; thus it has a marking satisfying ($\star$). As $X^{[2]}$ is endowed with the marking in Proposition 4.13, $X \times X^{[2]}$ is endowed with the product marking satisfying ($\star$) by Proposition 4.1. Moreover, the Chern classes of the normal bundle of Y in $X \times X^{[2]}$ are distinguished since they are polynomials of the Chern classes of T_Y, of T_X pulled-back to $Y = \widetilde{X \times X}$ via the first projection and of $T_{X^{[2]}}$ pulled-back to Y via the $\mathbb{Z}/2$ quotient map (cf. the computation in [44, Theorem 6.1]), which are all distinguished by Propositions 4.8 and 4.12. Again by Proposition 4.8, $X^{[2,3]}$ has a marking satisfying ($\star$). The assertions about the universal subschemes follow from Corollary 3.13. $\square$

Remark 4.15 (Hilbert cubes)**.** An argument similar to the above combined with the explicit description of the Hilbert cube $X^{[3]}$ in [44] shows that $X^{[3]}$ satisfies ($\star$) once X does. Indeed, $X^{[3]}$ is constructed from X^3 in five steps (cf. [44] or [17]): the first three are successive blow-ups of X^3, each time along a center satisfying ($\star$) with normal bundle having distinguished Chern classes; the fourth step is a quotient map by a distinguished cyclic $\mathbb{Z}/3$-action; the final step is a blow-down of divisor with distinguished normal bundle to a center satisfying ($\star$). Thus using Propositions 4.1, 4.5, 4.8, 4.12, and Corollary 3.13 repeatedly in the first four steps, and using in the final step the analogue of the technical [44, Lemma 6.4] (with $\mathrm{CH}(-)_{(0)}$ replaced by $\mathrm{DCH}(-)$), one can obtain a marking of $X^{[3]}$ satisfying ($\star$). The details are left to the interested reader.

4.6. Birational transforms for hyper-Kähler varieties. Using Huybrechts's fundamental result [24] on deformation equivalence between birational hyper-Kähler varieties, Rieß [41] shows that the Chow rings of birational hyper-Kähler varieties are isomorphic. Actually her proof yields the following more precise result.

Theorem 4.16 (Rieß [41, §3.3 and Lemma 4.4])**.** *Let X and Y be d-dimensional irreducible holomorphic symplectic varieties. If they are birational, then there exists a correspondence $Z \in \mathrm{CH}_d(X \times Y)$ such that*

(i) $(Z \times Z)_* : \mathrm{CH}_d(X \times X) \to \mathrm{CH}_d(Y \times Y)$ *sends* Δ_X *to* Δ_Y*;*
(ii) $(Z \times Z \times Z)_* : \mathrm{CH}_d(X \times X \times X) \to \mathrm{CH}_d(Y \times Y \times Y)$ *sends* δ_X *to* δ_Y*;*
(iii) $Z_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ *sends* $c_i(X)$ *to* $c_i(Y)$ *for any* $i \in \mathbb{N}$*;*
(iv) Z *induces an isomorphism of algebra objects* $\mathfrak{h}(X) \to \mathfrak{h}(Y)$ *in* CHM *with inverse given by* ${}^tZ$*.*

In particular, Z induces an isomorphism between their Chow rings (*resp., cohomology rings*)*.*

Corollary 4.17. *Let X and Y be d-dimensional irreducible holomorphic symplectic varieties that are birationally equivalent. If X has a marking that satisfies $(\star)$, then so does Y.*

Proof. Let $Z \in \mathrm{CH}_d(X \times Y) = \mathrm{Hom}(\mathfrak{h}(X), \mathfrak{h}(Y))$ be the correspondence in Theorem 4.16. Let $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ be a marking satisfying $(\star)$. Then we consider the marking $\psi = \phi \circ Z^* : \mathfrak{h}(Y) \xrightarrow{\simeq} M$. The fact that ψ satisfies the conditions $(\star_{\mathrm{Mult}})$ and $(\star_{\mathrm{Chern}})$ follows from Theorem 4.16(ii), (iii), respectively, together with the corresponding property of ϕ. $\square$

5. Examples of varieties satisfying condition $(\star)$

We provide in this section some examples of varieties satisfying condition $(\star)$. Together with the operations in Section 4, we obtain even more examples. Thanks to Proposition 3.12, the rational Chow ring of each of them possesses a subalgebra consisting of distinguished cycles, which is mapped isomorphically to the numerical Chow ring and contains all Chern classes of the variety.

5.1. Easy examples. First of all, as $(\star)$ is certainly a property preserved by isomorphisms of algebraic varieties, we have by O'Sullivan's Theorem 1.8 the following lemma.

Lemma 5.1. *Any abelian torsor, that is, a variety isomorphic to an abelian variety, satisfies $(\star)$.*

Another set of examples generalizes the projective spaces.

Proposition 5.2. *Let X be a smooth projective variety over a field k, and let Ω be a universal domain containing k. Assume that X satisfies at least one of the following conditions:*

1. *$X \simeq G/P$ is a homogeneous variety, where G is a linear algebraic group and P is a parabolic subgroup.*
2. *X is a toric variety.*
3. *The bounded derived category $D^b_{coh}(X)$ has a full exceptional collection.*
4. *The cycle class map $\mathrm{CH}^*(X_\Omega) \to H^*(X_\Omega, \mathbb{Q}_\ell)$ is injective for some prime $\ell \neq \mathrm{char}\, k$.*
5. *The Chow group $\mathrm{CH}^*(X_\Omega)$ is a finite-dimensional $\mathbb{Q}$-vector space.*

Then X satisfies $(\star)$.

Proof. Actually any of these conditions ensures that the Chow motive of X is of Lefschetz–Tate type:

$$\mathfrak{h}(X) \simeq \bigoplus_i \mathbb{1}(a_i),$$

with $a_i \in \mathbb{Z}$. It is well known for (1) and (2), while for (3) it is established in [10] and [33]. For (4), it is the main result of [32] (see also [50, §2.2] for a recent account), and for (5), it is proven in [27], [47]. □

5.2. Curves. Recall that the smooth projective curves of genus 0 and 1 are covered in §5.1. We consider in this subsection curves of higher genera.

Let C be a smooth projective curve with genus $g \geq 2$. Its Jacobian variety JC is a principally polarized abelian variety of dimension g with origin denoted by O and theta divisor denoted by $\Theta \in \mathrm{CH}^1(JC)$, which is always assumed to be symmetric. By choosing a base point $z \in C$, we have the Abel–Jacobi embedding:

$$\begin{array}{rcl} \iota_z : C & \hookrightarrow & JC, \\ x & \mapsto & \mathcal{O}_C(x-z). \end{array}$$

Associated to z, there is also the motivic decomposition of C:

$$\mathfrak{h}(C) = \mathfrak{h}^0(C) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C),$$

where $\mathfrak{h}^0(C) := (C, z \times C, 0) \simeq \mathbb{1}$, $\mathfrak{h}^2(C) := (C, C \times z, 0) \simeq \mathbb{1}(-1)$, and $\mathfrak{h}^1(C) := (C, \Delta_C - z \times C - C \times z, 0)$.

Proposition 5.3. *Let C be a smooth projective curve of genus $g \geq 2$. If there exists a point $z \in C$ such that $\iota_z(C) \in \mathrm{CH}_1(JC)$ is symmetrically distinguished,*[22] *then C satisfies condition $(\star)$.*

Proof. Let us fix z and simply write $\iota := \iota_z$ and $C := \iota_z(C)$. Assume that $C \in \mathrm{CH}_1(JC)$ is symmetrically distinguished. Since the 1-cycles C and $\frac{1}{(g-1)!}\Theta^{g-1}$ are numerically equivalent and symmetrically distinguished, they are actually equal (i.e., rationally equivalent), thanks to Theorem 1.8.

Deninger and Murre construct in [16] a canonical motivic decomposition

$$\mathfrak{h}(JC) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(JC).$$

Let $\pi^i \in \mathrm{CH}^g(JC \times JC)$ be the projector corresponding to $\mathfrak{h}^i(JC)$. For example, $\pi^0 = [O] \times JC$ and $\pi^{2g} = JC \times [O]$. See [29] for the explicit formulae of the other projectors π^i. One important feature, easily seen from Theorem 1.8, is that they are all symmetrically distinguished.

We claim that $\Gamma_\iota =: \iota_* : \mathfrak{h}(C) \to \mathfrak{h}(JC)(g-1)$ induces isomorphisms:

- $\mathfrak{h}^2(C) \xrightarrow{\simeq} \mathfrak{h}^{2g}(JC)(g-1) := (JC, JC \times [O], g-1)$;
- $\mathfrak{h}^1(C) \xrightarrow{\simeq} \mathfrak{h}^{2g-1}(JC)(g-1) := (JC, \pi^{2g-1}, g-1)$;

[22] By Remark 1.9, this condition is equivalent to $\iota_z(C) \in \mathrm{CH}_1(JC)_{(0)}$.

- $\mathfrak{h}^0(C) \xrightarrow{\simeq} L^{g-1}\mathfrak{h}^0(JC)(g-1) := (JC, \frac{1}{g!}\Theta \times \Theta^{g-1}, g-1)$; the latter is a direct summand of $\mathfrak{h}^{2g-2}(JC)(g-1)$ in the Lefschetz decomposition constructed by Künnemann in [28],

where L is the Lefschetz operator (see [28]). Indeed, all these morphisms are in the Kimura category $\mathscr{M}^{ab}$ (see [26]). The functor $\mathscr{M}^{ab} \to \overline{\mathscr{M}^{ab}}$ is therefore conservative (cf. [3, Corollary 3.16]). One easily checks that these morphisms are isomorphisms modulo homological, thus a fortiori numerical, equivalence.

Putting them together, we have a marking for C:

$$\phi := \iota_* : \mathfrak{h}(C) \xrightarrow{\simeq} M := (JC, JC \times [O] + \pi^{2g-1} + \frac{1}{g!}\Theta \times \Theta^{g-1}, g-1).$$

Let us show $(\star_{\mathrm{Mult}})$: since the inclusion of the direct summand M into $\mathfrak{h}(JC)$ is clearly symmetrically distinguished, to show that $\phi_*^{\otimes 3}(\delta_C)$ is symmetrically distinguished, it suffices to show that $\iota_*^3 : \mathrm{CH}_1(C^3) \to \mathrm{CH}_1(JC^3)$ sends the small diagonal δ_C to a symmetrically distinguished cycle of $JC \times JC \times JC$. However, by the following commutative diagram

$$\begin{tikzcd} C \arrow[r, hook, "\delta_C"] \arrow[d, hook, "\iota"'] & C \times C \times C \arrow[d, hook, "\iota^3"] \\ JC \arrow[r, hook, "\delta_{JC}"] & JC \times JC \times JC \end{tikzcd}$$

we have that $\iota_*^3(\delta_C) = \delta_{JC,*}(\iota(C))$ is symmetrically distinguished by the assumption and Theorem 1.8.

The condition $(\star_{\mathrm{Chern}})$ on Chern classes follows from $(\star_{\mathrm{Mult}})$ since C is a curve (Corollary 3.16). □

Corollary 5.4. *All hyperelliptic curves satisfy condition* $(\star)$.

Proof. For a hyperelliptic curve C, choose any Weierstrass point to define the Abel–Jacobi embedding. Then the involution $[-1]$ on JC preserves C and acts on C by the hyperelliptic involution. By [46, Proposition 2.1], in the Beauville decomposition of $\mathrm{CH}^{g-1}(JC)$, the class of C belongs to $\mathrm{CH}^{g-1}(JC)_{(0)}$. On the other hand, $\mathrm{CH}^{g-1}(JC)_{(0)}$ is the Fourier transform [7] of $\mathrm{CH}^1(JC)_{(0)}$ which maps isomorphically to $\overline{\mathrm{CH}}^1(JC)$. Therefore, the natural cycle class map $\mathrm{CH}^{g-1}(JC)_{(0)} \to \overline{\mathrm{CH}}^{g-1}(JC)$ is also an isomorphism. Consequently, all cycles in $\mathrm{CH}^{g-1}(JC)_{(0)}$, in particular the class of C, are symmetrically distinguished. One can now conclude by invoking Proposition 5.3. □

Remark 5.5. The case of hyperelliptic curves is mentioned in [38, §6.3].

Remark 5.6 (Hilbert schemes of a hyperelliptic curve)**.** Recall that the Hilbert scheme of length-n subschemes on a smooth curve C is nothing but the nth symmetric power $C^{(n)}$ of C. Now if C satisfies $(\star_{\mathrm{Mult}})$, then by

Proposition 4.1, C^n satisfies ($\star_{\mathrm{Mult}}$), and by Proposition 4.12, $C^{(n)}$ satisfies ($\star_{\mathrm{Mult}}$). By Corollary 3.16, C also satisfies ($\star_{\mathrm{Chern}}$), and the same computation as in [43, p. 95] shows that $C^{(n)}$ satisfies ($\star_{\mathrm{Chern}}$). Therefore, it follows from Corollary 5.4 that the Hilbert schemes of a hyperelliptic curve satisfy ($\star$).

5.3. Fermat hypersurfaces. An important class of (higher-dimensional) varieties whose motive is known to be of abelian type is provided by the Fermat hypersurfaces, by using the inductive structure discovered by Shioda–Katsura [45]. Note that Proposition 5.2 implies that smooth quadric hypersurfaces satisfy ($\star$) since their Chow groups are finite-dimensional vector spaces.

In the sequel of this subsection, we fix a degree $d \geq 3$ and, for any $r \in \mathbb{N}$, we let X_r denote the Fermat hypersurface of degree d in $\mathbb{P}^{r+1}$:

$$X_r := \{(x_0, \ldots, x_{r+1}) \mid x_0^d + \cdots + x_{r+1}^d = 0\} \subset \mathbb{P}^{r+1}.$$

Recall the inductive structure (cf. [45, Theorem 1]): let ϵ be a (fixed) dth root of -1 and let ζ be a (fixed) dth root of unity. For any $r, s \in \mathbb{N}$, we have the following commutative diagram:

(7)
$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[r] \arrow[d, "\beta"] \arrow[drr, "\psi"] & Z/\mu_d \arrow[d, "\tau"] \arrow[dr, phantom, "\square"] & X_{r-1} \times \mathbb{P}^s \coprod \mathbb{P}^r \times X_{s-1} \arrow[l, hook'] \arrow[d] \\
X_{r-1} \times X_{s-1} \arrow[r, hook, "i_r \times i_s"] & X_r \times X_s \arrow[r, dashed, "\varphi"] & X_{r+s} & X_{r-1} \coprod X_{s-1} \arrow[l, hook']
\end{tikzcd}$$

where $i_r : X_{r-1} \hookrightarrow X_r$ is the embedding given by $(x_0, \ldots, x_r) \mapsto (x_0, \ldots, x_r, 0)$; $\varphi : ((x_0, \ldots, x_{r+1}), (y_0, \ldots, y_{s+1})) \mapsto (y_{s+1}x_0, \ldots, y_{s+1}x_r, \epsilon x_{r+1}y_0, \ldots, \epsilon x_{r+1}y_s)$; β and τ are blow-ups; the action of μ_d on the blow-up Z is induced by its action on X_r and X_s given by $(x_0, \ldots, x_{r+1}) \mapsto (x_0, \ldots, x_r, \zeta x_{r+1})$ and $(y_0, \ldots, y_{s+1}) \mapsto (y_0, \ldots, y_s, \zeta y_{s+1})$, respectively.

The main result of this subsection is the following.

Proposition 5.7 (Fermat cubics)**.** *If $d = 3$, then there exist, for all $r \in \mathbb{N}$, a marking $\phi_r : \mathfrak{h}(X_r) \xrightarrow{\simeq} M_r$, for the cubic Fermat hypersurface X_r, such that:*

(i) *the embedding $i_r : X_{r-1} \hookrightarrow X_r$ is distinguished* (*Definition* 3.14)*;*
(ii) *the action of μ_d on X_r is distinguished, i.e., ϕ_r satisfies* ($\star_{\mu_d}$)*;*
(iii) *ϕ_r satisfies the condition* ($\star$) *of Definition* 3.7.

In particular, all Fermat cubic hypersurfaces satisfy condition ($\star$).

Proof. We proceed by induction on r. For $r = 1$, $X_1 = \{x_0^3 + x_1^3 + x_2^3 = 0\}$ is a cubic curve in $\mathbb{P}^2$; by fixing an origin, it becomes an elliptic curve. We fix $(-1, 1, 0)$ as its origin. Trivially, X_1 satisfies ($\star$) (§5.1). The embedding $X_0 \hookrightarrow X_1$ is given by three points $(-1, 1, 0), (-\zeta, 1, 0), (-\zeta^2, 1, 0)$, which are of

3-torsion,[23] therefore distinguished. As for the action of μ_d, which is given by $(x_0, x_1, x_2) \mapsto (x_0, x_1, \zeta x_2)$, it is clearly an automorphism of abelian variety, hence also distinguished.

Assuming the assertions (i)–(iii) for $r \leq n$, let us establish them for $r = n+1$. We set in the sequel $s = 1$ in diagram (7) and also $\epsilon = -1$. By the induction hypothesis and the fact that distinguished morphisms are stable under products, the embedding $X_{n-1} \times X_0 \hookrightarrow X_n \times X_1$ is distinguished. Therefore Z satisfies $(\star)$ by Proposition 4.8. Again by the induction hypothesis, the action of μ_d on $X_n \times X_1$ is distinguished with distinguished ramification locus, which implies by Proposition 4.12 that Z/μ_d satisfies $(\star)$. We now claim that the marking on X_{n+1} defined via τ satisfies $(\star)$. We thank the referee for providing the following argument. For $(\star_{\mathrm{Mult}})$ it is enough by Proposition 4.9 to show that

$$ {}^t\Gamma_\tau \circ \Gamma_\tau = (\tau \times \tau)^*(\Delta_{X_{n+1}}) $$

is distinguished. The exceptional divisors for τ are E_0, E_1, E_2, E_3 with

$$ E_0 = X_{n-1} \times \mathbb{P}^1 = X_{n-1} \times (X_1/\mu_3) $$

and E_i for $i > 0$ a component $\mathbb{P}^n$ of $\mathbb{P}^n \times X_0$. We have

$$ (\tau \times \tau)^*(\Delta_{X_{n+1}}) = \Delta_{Z/\mu_3} + \alpha_0 + \alpha, $$

where α_0 is the push-forward along $E_0 \times E_0 \to (Z/\mu_3) \times (Z/\mu_3)$ of

$$ E_0 \times_{X_{n-1}} E_0 = \Delta_{X_{n-1}} \times \mathbb{P}^1 \times \mathbb{P}^1, $$

and α is supported on $\coprod_{i>0} E_i \times E_i$. Both Δ_{Z/μ_3} and α_0 are distinguished, and α is distinguished because for $i > 0$ every cycle on $E_i \times E_i = \mathbb{P}^n \times \mathbb{P}^n$ is distinguished. Finally $(\star_{\mathrm{Chern}})$ follows from [22, Theorem 15.4]. In particular, (iii) for $r = n+1$ is proven.

[23] In fact, the nine 3-torsion points of the Fermat elliptic curve are exactly its intersection with the coordinate axes $(x_0 = 0), (x_1 = 0)$, and $(x_2 = 0)$. Indeed, these nine points lie on 12 lines. Each line contains three of these points and each point lies on four lines. Now use the fact that the sum of the three points in the intersection of any line with the elliptic curve is the hyperplane section class. We easily deduce that 3 times any of the nine points is the hyperplane section class. Hence they are all 3-torsion points if any one of them is fixed as the origin.

For (i), we have the following commutative diagram, where i is the embedding determined by the point $(1, 0, -\zeta) \in X_1$:

$$\begin{tikzcd} Z \arrow[d, "\beta"] \arrow[dr, "\psi"] & \\ X_n \times X_1 \arrow[r, dashed, "\varphi"] & X_{n+1} \\ X_n \arrow[u, hook, "i"] \arrow[ur, hook, "i_{n+1}"'] & \end{tikzcd}$$

Since $(1, 0, -\zeta)$ is a torsion point of X_1, i^* is distinguished. Therefore, with ψ and β being distinguished by construction, $i_{n+1}^* = i_{n+1}^* \circ \psi_* \circ \psi^* = i^* \circ \beta_* \circ \psi^*$ is also distinguished.

Finally for (ii), the action of μ_d on X_{n+1} comes, via the diagram (7), from the action of μ_d on X_1 which is given by $(y_0, y_1, y_2) \mapsto (y_0, \zeta y_1, y_2)$. It is clearly an automorphism of abelian variety, hence is distinguished. $\square$

So far, we are not able to determine whether other Fermat hypersurfaces satisfy $(\star)$, but we would like to make the following conjecture.

Conjecture 5.8. *The Fermat hypersurfaces which are Calabi–Yau or Fano, i.e., $d \le r + 2$, satisfy condition $(\star)$.*

Remark 5.9. The conclusion of Conjecture 5.8 cannot hold in general for Fermat hypersurfaces of general type; cf. Proposition 7.4 (together with Proposition 6.1) below for counterexamples in the case of Fermat curves starting from degree 4.

Remark 5.10. It is interesting to notice that for $d = 4$, we know that the quartic Fermat surface satisfies $(\star)$ for a different reason: it is a Kummer surface (cf. [25, Chapter 14, Example 3.18]), and Proposition 5.11 applies. One could therefore show Conjecture 5.8 for $d = 4$ by a similar induction argument as in Proposition 5.7 once we know the case of the Fermat quartic threefold (and some natural compatibilities with the Fermat quartic surface).

5.4. K3 surfaces with large Picard number. While K3 surfaces are expected to have motive of abelian type via the Kuga–Satake construction, this has only been established in scattered cases. This includes Kummer surfaces, and [40, Theorem 2] K3 surfaces with Picard number ≥ 19.

5.4.1. Kummer surfaces. By definition the Kummer surface $K_1(A)$ attached to the abelian surface A is the fiber over 0 of the morphism $A^{[2]} \to A^{(2)} \to A$, which is the composition of the sum morphism $A^{(2)} \to A$ with the Hilbert–Chow morphism $A^{[2]} \to A^{(2)}$.

Proposition 5.11. *A Kummer surface admits a marking that satisfies $(\star)$.*

Proof. The Kummer surface $K_1(A)$ has the following alternative description: the $[-1]$-involution on A induces an involution, denoted ι, on the blow-up $\widetilde{A}$ of A along its subgroup of 2-torsion points, and $K_1(A)$ is the $\mathbb{Z}/2$-quotient of $\widetilde{A}$ for that action. By Proposition 4.8, $(\widetilde{A}, \mathbb{Z}/2)$ has a marking that satisfies $(\star)$. We can then conclude from Proposition 4.12 that $K_1(A)$ has a marking that satisfies $(\star)$. □

Later on (cf. Proposition 5.14), we will generalize Proposition 5.11 by establishing that generalized Kummer varieties admit a marking that satisfies $(\star)$.

5.4.2. K3 surfaces with Picard number ≥ 19. Such K3 surfaces admit [35] a Nikulin involution (that is, a symplectic involution) with quotient birationally equivalent to a Kummer surface.

Proposition 5.12. *A K3 surface with Picard number* ≥ 19 *admits a marking that satisfies* $(\star)$.

Proof. Let X be a K3 surface with a Nikulin involution; by [37, §5] X has eight isolated fixed points, which we denote $Q_1, \ldots, Q_8$. Let $\pi : X \to X/\iota$ be the quotient morphism; X/ι has ordinary double points at the points $P_i := \pi(Q_i)$ so that if $f : Y \to X/\iota$ denotes the minimal resolution, then the exceptional divisors of f are smooth rational (-2)-curves $C_i := f^{-1}(P_i)$.

Let X now be a K3 surface with Picard number ≥ 19. According to [35, Corollary 6.4], X admits a Shioda–Inose structure, meaning that X admits a Nikulin involution ι such that Y is a Kummer surface and such that $f^*\pi_*$ induces a Hodge isometry $T_X(2) \simeq T_Y$, where T_X refers to the transcendental lattice of X. The latter was upgraded to an isomorphism of Chow motives by Pedrini [40, Theorem 2]. Precisely, given S a K3 surface, let us denote o_S as the Beauville–Voisin zero-cycle; cf. [9]. We fix a basis $\{D_j\}$ of $\mathrm{CH}^1(S)$, and denote $\{D_j^\vee\}$ as the dual basis with respect to the intersection product. We then define the idempotent correspondences $\pi_S^0 := o_S \times S$, $\pi_S^4 := S \times o_S$, $\pi_S^{2,alg} := \sum_j D_j^\vee \times D_j$, and $\pi_S^{2,tr} := \Delta_S - \pi_S^0 - \pi_S^4 - \pi_S^{2,alg}$. The motive $\mathfrak{h}^{alg}(S) := (S, \pi_S^0 + \pi_S^{2,alg} + \pi_S^4)$ is the algebraic motive of S (it is isomorphic to a direct sum of Lefschetz–Tate motives), and the motive $\mathfrak{t}^2(S) := (S, \pi_S^{2,tr})$ is the transcendental motive of S. Pedrini [40] showed that $f^*\pi_*$ induces an isomorphism of motives $\mathfrak{t}^2(X) \simeq \mathfrak{t}^2(Y)$ (with inverse $\frac{1}{2}\pi^* f_*$).

We fix a marking for the Kummer surface Y that satisfies $(\star)$; such a marking does exist by Proposition 5.11. Since $\mathrm{DCH}^1(Y) = \mathrm{CH}^1(Y)$, we have that the classes of the smooth rational curves C_i are distinguished, and we also have that the projectors $\pi_Y^0, \pi_Y^4, \pi_Y^{2,alg}$, and $\pi_Y^{2,tr}$ are distinguished. Then we claim that the marking given by the decomposition $\mathfrak{h}(X) \simeq \pi^* f_* \mathfrak{t}^2(Y) \oplus \mathfrak{h}^{alg}(X)$ satisfies $(\star)$. That it satisfies $(\star_{\mathrm{Chern}})$ is obvious since $c_1(X) = 0$ and since by [9], $c_2(X)$ is a multiple of o_X and hence is mapped to zero in

$\mathrm{CH}^2(\mathfrak{t}^2(Y))$. By refined intersection [22], the cycle $(f,f,f)^*(\pi,\pi,\pi)_*\delta_X$ is supported on $(f,f,f)^{-1}(\pi,\pi,\pi)(\delta_X) = \delta_Y \cup \bigcup_i C_i \times C_i \times C_i$. Since C_i is a smooth rational curve, we have that $\mathrm{CH}_2(C_i \times C_i \times C_i)$ admits $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$, and $C_i \times C_i \times c_i$ as a basis, where c_i is any point on C_i. The cycle $(f,f,f)^*(\pi,\pi,\pi)_*\delta_X$ is therefore a linear combination of δ_Y and, for $1 \le i \le 8$, of $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$ and $C_i \times C_i \times c_i$. By [9], the class of c_i in $\mathrm{CH}^2(Y)$ is the Beauville–Voisin zero-cycle o_Y; thus $c_i \in \mathrm{DCH}^2(Y)$. The cycles $c_i \times C_i \times C_i$, $C_i \times c_i \times C_i$ and $C_i \times C_i \times c_i$ therefore belong to $\mathrm{DCH}(Y \times Y \times Y)$ by Proposition 3.5. Since δ_Y is distinguished, this establishes $(\star_{\mathrm{Mult}})$, i.e., that δ_X is distinguished. $\square$

5.5. (Nested) Hilbert schemes of surfaces, generalized Kummer varieties. In this subsection, we produce a series of varieties satisfying $(\star)$. The first series of examples is given by the Hilbert schemes and (two-step) nested Hilbert schemes of points on a surface that satisfies $(\star)$, e.g., an abelian surface, a Kummer surface (Proposition 5.11), a K3 surface with Picard rank ≥ 19 (Proposition 5.12), or the product of two hyperelliptic curves (Corollary 5.4). Note that by a result of Cheah [13] the only nested Hilbert schemes of a smooth surface S that are smooth are the Hilbert schemes $S^{[n]}$ and the nested Hilbert schemes $S^{[n,n+1]}$ for $n \in \mathbb{N}$.

Proposition 5.13. *Let S be a smooth projective surface that satisfies $(\star)$. Then, for any $n \in \mathbb{N}$, the Hilbert scheme of length-n subschemes on S, denoted $S^{[n]}$, and the nested Hilbert scheme $S^{[n,n+1]}$, satisfy condition $(\star)$.*

The second series of example is built from an abelian surface A: the associated Kummer K3 surface as well as its higher-dimensional generalizations. Recall that the nth *generalized Kummer* variety (see [6]) is the symplectic resolution of the quotient $A_0^{n+1}/\mathfrak{S}_{n+1}$, where A_0^{n+1} is the abelian variety $\ker\big(+ : A^{n+1} \to A\big)$, upon which the symmetric group acts naturally by permutations.

Proposition 5.14. *For any $n \in \mathbb{N}$, the generalized Kummer variety $K_n(A)$ associated to an abelian surface A satisfies condition $(\star)$.*

The proofs of Propositions 5.13 and 5.14 will be given concomitantly in full in §5.5.2. Note that the case of Kummer surfaces (which are the generalized Kummer varieties of dimension 2) was already treated in Proposition 5.11. We start by recalling some results of de Cataldo and Migliorini [15] concerning the motives of Hilbert schemes of surfaces, or more generally that of a semismall resolution.

5.5.1. The motive of semismall resolutions. Recall that a morphism $f : Y \to X$ is called *semismall* if for all integer $k \ge 0$, the codimension of the locus $\big\{x \in X : \dim f^{-1}(x) \ge k\big\}$ is at least $2k$. In particular, f is generically finite. In [15], assuming $f : Y \to X$ is a semismall resolution with Y smooth and projective, de Cataldo and Migliorini computed the Chow motive of Y in terms of the Chow motives of projective compactifications of *relevant strata* of f provided these are finite group quotients of smooth varieties; we refer to [15] for a precise statement. In our case of interest, this has the following consequence. Suppose S is a smooth projective surface and suppose A is an abelian surface. Let us make some standard construction and fix the notation.

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{|\lambda|}) = (1^{a_1} \cdots n^{a_n})$ of a positive integer n where $a_i = \#\{j : 1 \leq j \leq n; \lambda_j = i\}$ and where $|\lambda| := a_1 + \cdots + a_n$ denotes the length of λ, we define $\mathfrak{S}_\lambda := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}$. We define S^λ to be $S^{|\lambda|}$, equipped with the natural action of $\mathfrak{S}_\lambda$ and with the natural morphism to $S^{(n)}$ by sending $(x_1, \ldots, x_{|\lambda|})$ to $\sum_{j=1}^{|\lambda|} \lambda_j [x_j]$. We denote the quotient

$$S^{(\lambda)} := S^\lambda / \mathfrak{S}_\lambda \simeq S^{(a_1)} \times \cdots \times S^{(a_n)}$$

and we define the incidence correspondence

$$\Gamma^\lambda := (S^{[n]} \times_{S^{(n)}} S^\lambda)_{\mathrm{red}} \subset S^{[n]} \times S^\lambda.$$

The correspondence $\Gamma^{(\lambda)} \subset S^{[n]} \times S^{(\lambda)}$ is then the quotient $\Gamma^\lambda / \mathfrak{S}_\lambda$. Similarly, the correspondence $\Gamma_1^{(\lambda,j)} \subset S^{[n,n+1]} \times S^{(\lambda)} \times S$ is defined to be the incidence subvariety

$$\Gamma_1^{(\lambda,j)} := \left\{ (\xi \subset \xi', z, x) \mid (\xi, z) \in \Gamma^{(\lambda)}\,; x = \xi'/\xi \text{ has multiplicity} \geq j \text{ in } \xi \right\}.$$

For an integer $a \geq 0$, the motive of the quotient $S^{(a)}$ is thought of as the direct summand of the motive of S^a with respect to the idempotent $\frac{1}{a!} \sum_{\sigma \in \mathfrak{S}_a} \sigma$. When $S = A$ is an abelian surface, this idempotent is symmetrically distinguished, while in the case when S is a smooth projective surface satisfying $(\star)$ it is also distinguished (see Remark 4.2). In the case $S = A$ is an abelian surface, taking the fiber over 0 of the sum map $A^n \to A$ and of the sum map composed with the Hilbert–Chow morphism $A^{[n]} \to A^{(n)} \to A$, we define likewise $A_0^\lambda, A_0^{(\lambda)}, \Gamma_0^\lambda$, and $\Gamma_0^{(\lambda)}$.

Then the strata associated to the semismall resolutions

$$S^{[n]} \to S^{(n)}, \quad K_{n-1}(A) \to A_0^{(n)}, \quad \text{and} \quad S^{[n,n+1]} \to S^{(n)} \times S$$

are indexed by the set $\mathscr{P}(n)$ of partitions of n in the first two cases and $\coprod_{\lambda \in \mathscr{P}(n)} I_\lambda$ with $I_\lambda = \{0\} \coprod \{j \mid a_j \neq 0\}$ in the last case; and we have morphisms (in fact, isomorphisms by Theorem 5.15 below) of Chow motives

$$\Gamma := \bigoplus_{\lambda \in \mathscr{P}(n)} \Gamma^{(\lambda)} \ \mathfrak{h}(S^{[n]}) \longrightarrow \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(S^{(\lambda)})(|\lambda| - n), \tag{8}$$

$$\Gamma_0 := \bigoplus_{\lambda \in \mathscr{P}(n)} \Gamma_0^{(\lambda)} \ \mathfrak{h}(K_{n-1}(A)) \longrightarrow \bigoplus_{\lambda \in \mathscr{P}(n)} \mathfrak{h}(A_0^{(\lambda)})(|\lambda| - n), \tag{9}$$

$$(10)\quad \Gamma_1 := \bigoplus_{\lambda\in\mathscr{P}(n)}\bigoplus_{j\in I_\lambda}\Gamma_1^{(\lambda,j)}\ \mathfrak{h}(S^{[n,n+1]}) \longrightarrow \bigoplus_{\lambda\in\mathscr{P}(n)}\bigoplus_{j\in I_\lambda}\mathfrak{h}(S^{(\lambda)}\times S)(|\lambda|-n-\delta_{0,j}),$$

where $\delta_{0,j}$ is 0 if $j=0$ and is 1 if $j\neq 0$.

Theorem 5.15 (de Cataldo and Migliorini)**.** *The morphisms of Chow motives Γ, Γ_0, and Γ_1 are isomorphisms with inverses given, respectively, by*

$$\Gamma' := \sum_{\lambda\in\mathscr{P}(n)}\frac{1}{m_\lambda}{}^t\Gamma^{(\lambda)},\quad \Gamma'_0 := \sum_{\lambda\in\mathscr{P}(n)}\frac{1}{m_\lambda}{}^t\Gamma_0^{(\lambda)}$$

and

$$\Gamma'_1 := \sum_{\lambda\in\mathscr{P}(n)}\sum_{j\in I_\lambda}\frac{1}{m_{\lambda,j}}{}^t\Gamma_1^{(\lambda,j)},$$

where the superscript "t" indicates transposition, and also where $m_\lambda := (-1)^{n-|\lambda|}\prod_{i=1}^{|\lambda|}\lambda_i$ and $m_{\lambda,j} := (-1)^{n-|\lambda|}a_j\prod_{i=1}^{|\lambda|}\lambda_i$ are nonzero constants, where $a_j = 1$ if $j=0$ and $a_j = \#\{i : 1\leq i\leq n\,;\lambda_i = j\}$ if $j\neq 0$.

Proof. The proof that the morphism (8) is an isomorphism with inverse given by Γ' can be found in [14] (or [15]), the proof that the morphism (9) is an isomorphism with inverse given by Γ'_0 can be found in [20, Corollary 6.3], and the proof that the morphism (10) is an isomorphism in [15, Theorem 3.3.1], while the fact that its inverse is given by Γ'_1 follows from the proof of [15, Theorem 2.3.8]. □

5.5.2. Proof of Propositions 5.13 and 5.14. The argument is based on Voisin's universally defined cycle theorem on self-products of surfaces [51, Theorem 5.12]. Let us write X for either (i) the Hilbert scheme of length-n subschemes on a surface S satisfying ($\star$) (Proposition 5.13), (ii) the nth nested Hilbert scheme of a surface S satisfying ($\star$) (Proposition 5.13), or (iii) a generalized Kummer variety $K_n(A)$ (Proposition 5.14). We are going to show that the markings given by (8), (9), and (10) satisfy ($\star$). For that purpose, we have to show that the class of the small diagonal δ_X (resp., the Chern classes of X) are mapped in cases (i) and (ii) to a distinguished cycle on self-products of S under the correspondences $\Gamma\otimes\Gamma\otimes\Gamma$ and $\Gamma_1\otimes\Gamma_1\otimes\Gamma_1$ (resp., Γ and Γ_1), where Γ (resp., Γ_1) is the isomorphism (8) (resp., (10)), and in case (iii) to a symmetrically distinguished cycle on an a.t.t.s. under the correspondence $\Gamma_0\otimes\Gamma_0\otimes\Gamma_0$ (resp., Γ_0), where Γ_0 is the isomorphism (9).

In cases (i) and (ii), one argues as in [48, §3.2] or as in [20, Proposition 5.7]. The main idea is that, thanks to Voisin's theorem [51, Theorem 5.12], $\Gamma_* c_i(X)$ and $(\Gamma\otimes\Gamma\otimes\Gamma)_*\delta_X$ (resp., $\Gamma_{1,*}c_i(X)$ and $(\Gamma_1\otimes\Gamma_1\otimes\Gamma_1)_*\delta_X$) are cycles that are polynomials in pull-backs along projections of Chern classes of S and the diagonal Δ_S. Since S is assumed to satisfy $(\star)$, diagonals and Chern classes are distinguished, and hence the above cycles are all distinguished.

In case (iii), this is achieved for the small diagonal by arguing as in the proof of [20, Proposition 6.12] and for the Chern classes as in the proof of [20, Proposition 7.13]. A key point to establish $(\star_{\mathrm{Mult}})$ is that the small diagonal $\delta_{K_n(A)}$ is the restriction of the small diagonal $\delta_{A^{[n+1]}}$ under the 3-fold product of the inclusion $K_{n-1}(A) \to A^{[n]}$. The proof of $(\star_{\mathrm{Chern}})$ is similar once one has observed that the Chern classes $c_i(K_{n-1}(A))$ are the restrictions of the Chern classes $c_i(A^{[n]})$. One cannot invoke Voisin's theorem directly here, and one has to utilize the commutativity of the following diagram, whose squares are all cartesian and without excess intersections:

$$\begin{tikzcd}
(A^{[n]})^3 & \Gamma^\lambda \times \Gamma^\mu \times \Gamma^\nu \arrow[l, "p''"'] \arrow[r, "q''"] & A^\lambda \times A^\mu \times A^\nu \\
(A^{[n]})^{3/A} \arrow[u, hook] \arrow[ur, phantom, "\square"] & \Gamma^\lambda \times_A \Gamma^\mu \times_A \Gamma^\nu \arrow[l, "p'"'] \arrow[r, "q'"] \arrow[u, hook] \arrow[ur, phantom, "\square"] & A^\lambda \times_A A^\mu \times_A A^\nu \arrow[u, hook, "j"'] \\
K_{n-1}(A)^3 \arrow[u, hook] \arrow[ur, phantom, "\square"] & \Gamma_0^\lambda \times \Gamma_0^\mu \times \Gamma_0^\nu \arrow[l, "p"'] \arrow[r, "q"] \arrow[u, hook] \arrow[ur, phantom, "\square"] & A_0^\lambda \times A_0^\mu \times A_0^\nu \arrow[u, hook, "i"']
\end{tikzcd}$$

Here λ, μ, ν are partitions of n; all fiber products in the second row are over A; the second row is the base change by the inclusion of the small diagonal $A \hookrightarrow A^3$ of the first row; the third row is the base change by $\{O_A\} \hookrightarrow A$ of the second row.

We need to show that $(\Gamma_0^\lambda \times \Gamma_0^\mu \times \Gamma_0^\nu)_*(\delta_{K_{n-1}(A)}) = q_*p^*(\delta_{K_{n-1}(A)})$ is symmetrically distinguished on the a.t.t.s. $A_0^\lambda \times A_0^\mu \times A_0^\nu$ for all partitions λ, μ, ν of n.

As in the proof of [20, Proposition 6.12], we have, thanks to [20, Lemma 6.6], that $A^\lambda \times_A A^\mu \times_A A^\nu$ and $A_0^\lambda \times A_0^\mu \times A_0^\nu$ are naturally disjoint unions of a.t.t.s.'s and the inclusions i and j are morphisms of a.t.t.s.'s on each component in the sense of Definition 1.10.

Denote $\delta_{A^{[n]}/A}$ the small diagonal inside the relative fiber product $(A^{[n]})^{3/A}$. Now by functorialities and the base change formula (cf. [22, Theorem 6.2]), we have

$$j_* \circ q'_* \circ p'^*(\delta_{A^{[n]}/A}) = q''_* \circ p''^*(\delta_{A^{[n]}}),$$

which is a polynomial of big diagonals of $A^{|\lambda|+|\mu|+|\nu|}$ by Voisin's result [51, Proposition 5.6], thus symmetrically distinguished in particular. By [20, Lemma 6.10], $q'_* \circ p'^*(\delta_{A^{[n]}/A})$ is symmetrically distinguished on each component of $A^\lambda \times_A A^\mu \times_A A^\nu$. Again by functorialities and the base change formula, we have

$$q_* \circ p^*(\delta_{K_{n-1}(A)}) = i^* \circ q'_* \circ p'^*(\delta_{A^{[n]}/A}).$$

Since i is a morphism of an a.t.t.s. on each component, one concludes that $q_* \circ p^*(\delta_{K_{n-1}(A)})$ is symmetrically distinguished on each component, which concludes the proof. □

Remark 5.16 (Self-dual multiplicative Chow–Künneth decomposition for nested Hilbert schemes)**.** The arguments of the proof of Proposition 5.13 can be used to show that if a smooth projective surface S has a self-dual multiplicative Chow–Künneth decomposition (see Section 6 for the definition), then so do the nested Hilbert schemes $S^{[n,n+1]}$. Thus one may add the operation of taking nested Hilbert schemes of surfaces to [44, Theorem 2].

6. Link with multiplicative Chow–Künneth decompositions

A *Chow–Künneth decomposition* on a smooth projective variety X of dimension d is a set $\{\pi_X^i : 0 \leq i \leq 2d\}$ of mutually orthogonal idempotent correspondences in $X \times X$ that add up to Δ_X and whose cohomology classes in $H^{2d}(X \times X)$ are the components of the diagonal in $H^{2d-i}(X) \otimes H^i(X)$ for the Künneth decomposition. The notion of Chow–Künneth decomposition was introduced by Murre, who conjectured that all smooth projective varieties should admit such a decomposition [36]. Murre's conjecture is intimately linked to the conjectures of Beilinson and Bloch; cf. [2].

The notion of *multiplicative* Chow–Künneth (MCK) decomposition was introduced in [43] and further studied in [20], [44], [48], and [19]. A Chow–Künneth decomposition $\{\pi_X^i : 0 \leq i \leq 2d\}$ on a smooth projective variety X of dimension d induces a bigrading decomposition of the Chow groups of self-powers of X via the formula

$$\mathrm{CH}^i(X^n)_{(j)} := (\pi_{X^n}^{2i-j})_* \mathrm{CH}^i(X^n), \tag{11}$$

where by definition X^n is endowed with the product Chow–Künneth decomposition

$$\pi_{X^n}^k := \sum_{k_1+\cdots+k_n=k} \pi_X^{k_1} \otimes \cdots \otimes \pi_X^{k_n}.$$

A Chow–Künneth decomposition $\{\pi_X^i : 0 \leq i \leq 2d\}$ is *multiplicative* if δ_X belongs to $\mathrm{CH}^{2d}(X \times X \times X)_{(0)}$. As pointed out by the referee, this multiplicative condition implies[24] that the diagonal Δ_X belongs to $\mathrm{CH}^d(X \times X)_{(0)}$, or, equivalently, that the Chow–Künneth decomposition $\{\pi_X^i : 0 \leq i \leq 2d\}$ is *self-dual*, meaning that $\pi_X^i = {}^t\pi_X^{2d-i}$ for all i. (In particular, the above remark makes it possible to simplify some of the arguments of [44, §3].) The existence of a multiplicative Chow–Künneth decomposition for X ensures that $\mathrm{CH}^*(X)_{(0)}$ defines a graded subalgebra of $\mathrm{CH}^*(X)$. Finally, a natural condition that appeared in [44] is that the Chern classes of X belongs to $\mathrm{CH}^*(X)_{(0)}$. As is apparent from the above and from the previous sections, the theory for DCH^* is in every way similar to that of $\mathrm{CH}^*(-)_{(0)}$ (compare with [44]).

According to Murre's conjecture (D), for any choice of a Chow–Künneth decomposition $\{\pi_X^i : 0 \leq i \leq 2d\}$, we should have that the restriction of the projection morphism $\mathrm{CH}^*(X) \to \overline{\mathrm{CH}}^*(X)$ to $\mathrm{CH}^*(X)_{(0)}$ is an isomorphism; see [36]. Thus conjecturally the existence of a self-dual multiplicative Chow–Künneth decomposition for X provides a splitting to the algebra homomorphism $\mathrm{CH}^*(X) \to \overline{\mathrm{CH}}^*(X)$, in the same that a marking that satisfies $(\star)$ does.

Proposition 6.1 ($(\star)$ and MCK decomposition)**.** *Let X be a smooth projective variety with a marking ϕ that satisfies $(\star_{\mathrm{Mult}})$. Then X has a self-dual multiplicative Chow–Künneth decomposition with the property that $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$. Moreover, equality holds if Murre's conjecture (D) in* [36] *is true.*

Proof. The proof of Proposition 4.1 shows that if X and Y are two smooth projective varieties each endowed with markings satisfying $(\star_{\mathrm{Mult}})$, then the product marking on $X \times Y$ also satisfies $(\star_{\mathrm{Mult}})$. Moreover, the graphs of the projection morphisms are distinguished for the product markings. Therefore, the composition of distinguished correspondences is distinguished.

Let A be an abelian variety, and let $p \in \mathrm{DCH}(A \times A)$ be a symmetrically distinguished projector. The Deninger–Murre Chow–Künneth projectors π_A^i in [16] of A are symmetrically distinguished. Since the Chow–Künneth projectors are central modulo homological equivalence, we see that $p \circ \pi_A^i = \pi_A^i \circ p \in \mathrm{CH}^*(A \times A)$ and in particular that these provide distinguished Chow–Künneth projectors for (A, p).

[24] Indeed, if a is the structural morphism of X, we have $a_* \circ \pi^{2d}_{X\,*} = a_*$, so that projecting $\delta_X = (\pi^{4d}_{X^3})_*\delta_X$ onto $X \times X$ gives $\Delta_X = (\pi^{2d}_{X^2})_*\Delta_X$. From the latter, it follows that $\pi_X^i = (\Delta_X \otimes \pi_X^i)_*\Delta_X = (\pi_X^{2d-i} \otimes \pi_X^i)_*\Delta_X = (\pi_X^{2d-i} \otimes \Delta_X)_*\Delta_X = {}^t\pi_X^{2d-i}$; and conversely from $\pi_X^i = {}^t\pi_X^{2d-i}$ for all i, it follows that $\Delta_X = \sum_i \pi_X^i \circ \pi_X^i = \sum_i ({}^t\pi_X^i \otimes \pi_X^i)_*\Delta_X = \sum_i (\pi_X^{2d-i} \otimes \pi_X^i)_*\Delta_X = (\pi^{2d}_{X^2})_*\Delta_X$.

It follows that, assuming X has a marking ϕ that satisfies ($\star_{\mathrm{Mult}}$), X admits a distinguished Chow–Künneth decomposition. We conclude that X has a self-dual multiplicative Chow–Künneth decomposition by noting that since a Künneth decomposition is always self-dual and multiplicative, any distinguished Chow–Künneth decomposition is self-dual and multiplicative.

Finally, the inclusion $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$ is due to the following three facts: the product Chow–Künneth decomposition $\{\pi^i_{X^n}\}$ is distinguished, the cycle $(\pi^i_{X^n})_*\alpha$ is homologically trivial (and hence numerically trivial) for all $\alpha \in \mathrm{CH}^j(X^n)$ and all $i \neq 2j$, and $(\pi^i_{X^n})_*\alpha$ is distinguished if α is as well. Murre's conjecture (D) for X^n stipulates that $\mathrm{CH}^i(X^n)_{(0)}$ should inject in cohomology via the cycle class map, and in particular that the surjective quotient morphism $\mathrm{CH}^i(X^n) \to \overline{\mathrm{CH}}^*(X^n)$ is an isomorphism when restricted to $\mathrm{CH}^i(X^N)_{(0)}$. Since the quotient morphism is surjective when restricted to $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n)$, Murre's conjecture implies $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) = \mathrm{CH}^*(X^n)_{(0)}$. $\square$

7. Varieties with motive of abelian type that do not satisfy ($\star$)

The previous sections raise the question of determining a natural class of varieties which satisfy condition ($\star$) of Definition 3.7 or, more weakly, the *Section Property*. Beyond the case of hyper-Kähler varieties, which we expect to satisfy the Section Property, the answer is unfortunately not clear to us at this stage. To give some hint, in this section we provide some examples of varieties with motive of abelian type (i.e., in $\mathscr{M}^{ab}$) which fail to satisfy ($\star$) and/or the Section Property.

7.1. The Ceresa cycle and condition ($\star$). Let C be a smooth projective curve. In this section we give a necessary condition on the Ceresa cycle of C for C to admit a marking that satisfies ($\star$). In fact, we give a necessary condition on the Ceresa cycle of C for C to admit a self-dual multiplicative Chow–Künneth decomposition; see Proposition 6.1.

Fix a zero-cycle α of degree 1 on C, and denote $\iota : C \to J(C)$ as the Abel–Jacobi map which maps a point $c \in C$ to the divisor class $[c] - \alpha$. We denote $[C]$ as the class of the image of C under ι. Denote $[k] : J(C) \to J(C)$ as the multiplication-by-k homomorphism. The *Ceresa cycle* is then the one-cycle $[C] - [-1]_*[C]$; it is numerically trivial, and its class modulo algebraic equivalence does not depend on the choice of the degree 1 zero-cycle α.

Proposition 7.1. *Let C be a smooth projective curve. If C has a self-dual multiplicative Chow–Künneth decomposition, then the Ceresa cycle is algebraically trivial.*

Proof. Since a smooth projective curve has finite-dimensional motive in the sense of Kimura [26], any idempotent that is homologically equivalent to the Künneth projector on $H^0(C)$ is rationally equivalent to $\alpha \times C$ for some zero-cycle α of degree 1. Thus if C has a self-dual multiplicative Chow–Künneth decomposition, it must be of the form $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ for some zero-cycle α of degree 1. According to [43, Proposition 8.14] this decomposition is multiplicative if and only if the modified diagonal cycle

$$\begin{aligned}\mathfrak{z} := \delta_C - \{(x,x,\alpha)\} - \{(x,\alpha,x)\} - \{(\alpha,x,x)\} + \{(x,\alpha,\alpha)\}\\ + \{(\alpha,x,\alpha)\} + \{(\alpha,\alpha,x)\}\end{aligned}$$

is zero in $\mathrm{CH}_1(C\times C\times C)$. Now we argue as in the proof of [9, Proposition 3.2]. Let $\iota : C \to J(C)$ be the Abel–Jacobi map which maps a point $c \in C$ to the divisor class $[c] - \alpha$, and let $\iota^3 : C^3 \to J(C)$ be the map deduced from ι by summation. We have

$$(\iota^3)_*(\mathfrak{z}) = [3]_*[C] - 3[2]_*[C] + 3[C] = 0 \quad \text{in } \mathrm{CH}_1(J(C)).$$

According to the Beauville decomposition [7], we have

$$\mathrm{CH}_1(J(C)) = \mathrm{CH}_1(J(C))_{(0)} \oplus \cdots \oplus \mathrm{CH}_1(J(C))_{(g-1)},$$

where g is the dimension of $J(C)$, and where $[k]_*$ acts on $\mathrm{CH}_1(J(C))_{(s)}$ by multiplication by k^{2+s}. Since $3^{2+s} - 3 \cdot 2^{2+s} + 3 > 0$ for $s > 0$, we find that $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}$. In particular, taking $k = -1$, we see that $[C] - [-1]_*[C] = 0$ in $\mathrm{CH}_1(J(C))$, and hence that the Ceresa cycle is algebraically trivial. □

7.2. A very general curve of genus > 2 does not satisfy $(\star)$. Although motives of curves are of abelian type, they do not necessarily have a marking that satisfies $(\star)$.

Proposition 7.2. *Let C be a curve, and let α be a degree 1 zero-cycle on C. If C is very general of genus > 2, then the self-dual Chow–Künneth decomposition $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ is not multiplicative, and C does not satisfy $(\star)$.*

Proof. Ceresa [12] proves that the Ceresa cycle of a very general curve of genus > 2 is not algebraically trivial. The proposition then follows from Proposition 7.1 (together with Proposition 6.1). □

Remark 7.3. This example involving the Ceresa cycle is mentioned in [38, §6.3].

7.3. The Fermat quartic curve does not satisfy $(\star)$.

Proposition 7.4. *Let C be a Fermat curve of degree d with $d \geq 4$, and let α be a zero-cycle of degree one on C. If $d \leq 1000$, then the self-dual Chow–Künneth decomposition $\pi^0_C := \alpha \times C$, $\pi^2_C := C \times \alpha$, $\pi^1_C := \Delta_C - \pi^0_C - \pi^2_C$ is not multiplicative, and C does not satisfy $(\star)$.*

Proof. B. Harris [23] and S. Bloch [11] prove that the Ceresa cycle of quartic Fermat curves is algebraically nontrivial, and Otsubo [39] proves that the Ceresa cycle of Fermat curves of degree $4 \leq d \leq 1000$ is not algebraically trivial. We can now apply Proposition 7.1 (together with Proposition 6.1). $\square$

7.4. Varieties with motive of abelian type that do not admit a section. By considering a K3 surface of Picard rank ≥ 19, the following proposition provides a simple example of a variety X whose motive is of abelian type but for which the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ does not admit a section. In particular, by Proposition 3.12, such a variety X does not satisfy $(\star)$.

Proposition 7.5. *Let S be a complex K3 surface, and let P be a point of S not representing the Beauville–Voisin zero-cycle. Denote by $\widetilde{S}$ the blow-up of S along P. Then the $\mathbb{Q}$-algebra epimorphism $\mathrm{CH}(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}(\widetilde{S})$ does not admit a section.*

Proof. The theorem of Beauville–Voisin [9] asserts that $\mathrm{Im}(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \to \mathrm{CH}^2(S))$ has rank one and is spanned by the class of any point lying on a rational curve on S. Such a class is called the Beauville–Voisin zero-cycle. Since $\dim_{\mathbb{Q}} \mathrm{CH}^2(S) = \infty$, there exists a point P on S whose class is not rationally equivalent to the Beauville–Voisin zero-cycle. It is then straightforward to check that $\mathrm{Im}(\mathrm{CH}^1(\widetilde{S}) \otimes \mathrm{CH}^1(\widetilde{S}) \to \mathrm{CH}^2(\widetilde{S}))$ has rank 2 and is spanned by the class of P and the Beauville–Voisin zero-cycle. Since $\mathrm{CH}^1(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}^1(\widetilde{S})$ is an isomorphism, if $\mathrm{CH}(\widetilde{S}) \twoheadrightarrow \overline{\mathrm{CH}}(\widetilde{S})$ had a section, then $\mathrm{Im}(\mathrm{CH}^1(\widetilde{S}) \otimes \mathrm{CH}^1(\widetilde{S}) \to \mathrm{CH}^2(\widetilde{S}))$ would have rank 1 (equal to $\mathrm{rk}\, \overline{\mathrm{CH}}^2(\widetilde{S})$). This is a contradiction. $\square$

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