J. Math. Pures Appl. 130 (2019) 1–35

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

The generalized Franchetta conjecture for some hyper-Kähler varieties

Lie Fu [a,1], Robert Laterveer [b], Charles Vial [c,*],
with an appendix joint with Mingmin Shen [d]

[a] Institut Camille Jordan, Université Claude Bernard Lyon 1, France
[b] Institut de Recherche Mathématique Avancée, CNRS, Université de Strasbourg, France
[c] Fakultät für Mathematik, Universität Bielefeld, Germany
[d] KdV Institute for Mathematics, University of Amsterdam, Netherlands

A R T I C L E I N F O

Article history:
Received 14 September 2018
Available online 24 January 2019

MSC:
14C15
14C17
14C25
14D22
14J10
14J28
14J32

Keywords:
Algebraic cycles and Chow groups
Irreducible holomorphic symplectic varieties
Moduli spaces
K3 surfaces
Cubic fourfolds

A B S T R A C T

The generalized Franchetta conjecture for hyper-Kähler varieties predicts that an algebraic cycle on the universal family of certain polarized hyper-Kähler varieties is fiberwise rationally equivalent to zero if and only if it vanishes in cohomology fiberwise. We establish Franchetta-type results for certain low (Hilbert) powers of low degree K3 surfaces, for the Beauville–Donagi family of Fano varieties of lines on cubic fourfolds and its relative square, and for 0-cycles and codimension-2 cycles for the Lehn–Lehn–Sorger–van Straten family of hyper-Kähler eightfolds. We also draw many consequences in the direction of the Beauville–Voisin conjecture as well as Voisin's refinement involving coisotropic subvarieties. In the appendix, we establish a new relation among tautological cycles on the square of the Fano variety of lines of a smooth cubic fourfold and provide some applications.

R É S U M É

La conjecture généralisée de Franchetta pour les variétés hyper-Kählériennes prédit que la restriction à une fibre d'un cycle algébrique défini sur la famille universelle de certaines variétés hyper-Kählériennes est rationnellement triviale si et seulement si elle est homologiquement triviale. Nous établissons des résultats de type Franchetta pour certaines puissances de bas degré de surfaces K3 de bas degré, pour la famille de Beauville–Donagi des variétés de Fano de droites sur les cubiques lisses de dimension 4 et leur carré relatif, et pour les 0-cycles et les cycles de codimension 2 pour la famille de variétés hyper-Kählériennes de Lehn–Lehn–Sorger–van Straten. Nous en déduisons également de nombreuses conséquences concernant la conjecture de Beauville–Voisin, ainsi que sa généralisation due à Voisin incluant les sous-variétés co-isotropes. Dans l'appendice, nous établissons une nouvelle relation parmi les

* Corresponding author.
E-mail addresses: fu@math.univ-lyon1.fr (L. Fu), robert.laterveer@math.unistra.fr (R. Laterveer), vial@math.uni-bielefeld.de (C. Vial), M.Shen@uva.nl (M. Shen).
[1] Lie Fu is supported by the Agence Nationale de la Recherche through ECOVA (ANR-15-CE40-0002), HodgeFun (ANR-16-CE40-0011), LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program *Investissements d'Avenir* (ANR-11-IDEX-0007), and *Projet Inter-Laboratoire* 2017 by Fédération de Recherche en Mathématiques Rhône-Alpes/Auvergne CNRS 3490.

https://doi.org/10.1016/j.matpur.2019.01.018
0021-7824/

cycles tautologiques sur le carré de la variété de Fano des droites d'une hypersurface cubique lisse de dimension 4 et fournissons des applications.

1. Introduction

The original Franchetta conjecture [14] (proven in [21], see also [29] and [2]) states the following:

Theorem 1.1 *([14], [21], [29], [2]). For an integer $g \geq 2$, let $\mathcal{M}_g$ be the moduli stack of smooth projective curves of genus g, and let $\mathcal{C} \to \mathcal{M}_g$ be the universal curve. Then for any line bundle L on $\mathcal{C}$ and any closed point $b \in \mathcal{M}_g$, the restriction of L to the fiber C_b is a multiple of the canonical bundle of C_b.*

In the case of the universal family of K3 surfaces $\mathcal{S} \to \mathcal{F}_g$, where $\mathcal{F}_g$ is the moduli stack of polarized K3 surfaces of genus g, O'Grady proposed in [37] the following analogue of the Franchetta conjecture. Recall that the Beauville–Voisin class ([7]) of a projective K3 surface S is the degree-1 0-cycle class $\mathfrak{o}_S$ with support any closed point lying on a rational curve of the K3 surface. It enjoys the property that the intersection of any two divisors, as well as the second (Chow-theoretic) Chern class of S, are multiples of $\mathfrak{o}_S$. In the sequel, the Chow groups of stacks are the ones defined in Vistoli [45] (see also Kresch [24]) and are always considered with rational coefficients.

Conjecture 1.2 *(O'Grady [37]). Notation is as above. Then for any algebraic cycle $z \in \mathrm{CH}^2(\mathcal{S})$ and any point $b \in \mathcal{F}_g$, the restriction of z to the fiber K3 surface S_b is a multiple of the Beauville–Voisin class of S_b.*

Using Mukai models, Conjecture 1.2 is verified in [38] for K3 surfaces of genus $g \leq 10$ and $g = 12, 13, 16, 18, 20$. Otherwise, Conjecture 1.2 is still wide open.

The main goal of the paper is to investigate the following higher-dimensional analogue of O'Grady's Conjecture 1.2 concerning projective hyper-Kähler varieties. Recall that a smooth projective variety is called *hyper-Kähler* or *irreducible holomorphic symplectic*, if it is simply connected and $H^{2,0}$ is generated by a nowhere degenerate holomorphic 2-form.

Conjecture 1.3 *(Generalized Franchetta conjecture, cf. [8]). Let $\mathcal{F}$ be the moduli stack of a locally complete family of polarized hyper-Kähler varieties, and let $\mathcal{X} \to \mathcal{F}$ be the universal family. For any $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$, if its restriction to a very general fiber is homologically trivial then its restriction to any fiber is (rationally equivalent to) zero.*

Here, $\mathcal{F}$ and $\mathcal{X}$ are assumed to exist in the category of smooth Deligne–Mumford stacks. If one prefers to avoid stacks, one can add some level structure and obtain a universal family in the category of quasi-projective varieties, *cf.* [8, Section 3.4].

We note that a cycle is homologically trivial when restricted to a very general fiber if and only if it is homologically trivial when restricted to any fiber. Given any smooth family of projective varieties $\mathcal{X} \to \mathcal{F}$ with $\mathcal{F}$ smooth, we will say that $\mathcal{X} \to \mathcal{F}$ satisfies the *Franchetta property* if for any $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$ which is fiberwise homologically trivial, its restriction to any fiber is (rationally equivalent to) zero.

Although it would seem optimistic[2] that Conjecture 1.3 could hold more generally for self-products of hyper-Kähler varieties – *i.e.*, $\mathcal{X} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} \mathcal{X} \to \mathcal{F}$ satisfies the Franchetta property in the sense above – we may nevertheless ask, given a locally complete family $\mathcal{X} \to \mathcal{F}$ of polarized hyper-Kähler varieties, for

[2] When $g \geq 4$, the relative square of the universal curve of genus g does not satisfy the Franchetta property because the degree-0 0-cycle $p_1^*K_C \cdot p_2^*K_C - \deg(K_C)p_1^*K_C \cdot \Delta_C$ is not rationally trivial for C very general of genus $g \geq 4$; see [20].

which integers n does $\mathcal{X}^{n/\mathcal{F}}$ satisfy the Franchetta property. We provide some results in that direction in Theorems 1.4, 1.5, 1.10 and 1.11 below.

Recently, Bergeron and Li [8, Theorem 8.1.1] have proven the cohomological version of the generalized Franchetta Conjecture 1.3 for relative 0-cycles when the second Betti number is sufficiently large, which is an important support in favor of the conjecture, at least for 0-cycles.

Let us also mention that Conjecture 1.3 is closely related to the so-called Beauville–Voisin conjecture and its refinement (see Conjectures 2.3 and 2.4). On the one hand, the proof of some of our main results actually uses some known cases of the Beauville–Voisin conjecture (especially [47]); on the other hand, the generalized Franchetta conjecture implies the part of the Beauville–Voisin conjecture involving only Chern classes and the polarization, see Proposition 2.5.

We outline the main results of the paper, which provide more evidence for the generalized Franchetta conjecture.

1.1. Powers and Hilbert powers of some K3 surfaces

We can establish Franchetta-type results for the relative squares and cubes, as well as the relative Hilbert squares and Hilbert cubes, of the universal family of K3 surfaces which are complete intersections in projective spaces.

Theorem 1.4. *Let $\mathcal{M}$ be the moduli stack of smooth K3 surfaces of genus $g = 3, 4$ or 5, and let $\mathcal{S} \to \mathcal{M}$ be the universal family. Let $\mathcal{X}$ be $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$ or $\mathrm{Hilb}^3_{\mathcal{M}} \mathcal{S}$. For any cycle $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$ and any point $b \in \mathcal{M}$, the restriction of z to the fiber X_b is zero if and only if it is numerically trivial.*

The proof will be given in §4 for squares and Hilbert squares and in §5.2 for the other cases. We note that, thanks to the result of de Cataldo and Migliorini [11], the crucial cases are the self-products $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \mathcal{S}$.

By pushing our techniques further (*cf.* §5.1), we can also treat some other cases of (Hilbert) powers of K3 surfaces:

Theorem 1.5. *The following families satisfy the Franchetta property:*

- (*i*) $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$ *and* $\mathrm{Hilb}^3_{\mathcal{M}} \mathcal{S}$*, where $\mathcal{S} \to \mathcal{M}$ is the universal family of smooth K3 surfaces of genus 2 (double planes).*
- (*ii*) $\mathrm{Hilb}^{r_1}_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathrm{Hilb}^{r_m}_{\mathcal{M}} \mathcal{S}$*, where $\mathcal{S} \to \mathcal{M}$ is the universal family of smooth quartic K3 surfaces and $r_1 + \cdots + r_m \leq 5$.*
- (*iii*) *The relative square and relative Hilbert square of the universal family of K3 surfaces of genera* $6, 7, 8, 9, 10, 12$.

The proof will be given in §5.3, where these results are just special cases of the more general but more technical Theorem 5.8. See also Remark 5.9 which explains that the ranges in Theorems 1.4 and 1.5 above are, at least most of them, already at the limit of our method.

As immediate consequences, we obtain some partial confirmation of Voisin's refinement of the Beauville–Voisin conjecture involving coisotropic subvarieties (Conjecture 2.4):

Corollary 1.6. *Let S be a general K3 surface of genus $g \leq 10$ or 12, and let X be the Hilbert square $X = \mathrm{Hilb}^2(S)$. Let $R^*(X) \subset \mathrm{CH}^*(X)_{\mathbf{Q}}$ denote the $\mathbf{Q}$-subalgebra generated by the polarization class h, the Chern classes c_i, and the Lagrangian surface $T \subset X$ constructed in [22, Proposition 4]. Then $R^*(X)$ injects into cohomology by the cycle class map.*

Corollary 1.7. *Let $S \subset \mathbf{P}^3$ be a quartic K3 surface, and let $X = \mathrm{Hilb}^5 S$, $\mathrm{Hilb}^2 S \times \mathrm{Hilb}^2 S \times S$, $\mathrm{Hilb}^2 S \times S^3$ or $\mathrm{Hilb}^2 S \times \mathrm{Hilb}^3 S$. Let $R^*(X) \subset \mathrm{CH}^*(X)_{\mathbf{Q}}$ denote the $\mathbf{Q}$-subalgebra generated by the polarization class h, the Chern classes c_i, the coisotropic subvarieties E_μ of [50, 4.1 item 1)], the Lagrangian surface $T \subset \mathrm{Hilb}^2 S$ constructed in [22, Proposition 4], and the surface of bitangents $U \subset \mathrm{Hilb}^2 S$. Then $R^*(X)$ injects into cohomology by the cycle class map.*

These two corollaries are proven in §5.3 and also partially extended to products of Hilbert schemes in Corollary 5.10. A similar application to a 19-dimensional family of double EPW sextics is given in §5.5.

Another consequence, whose proof as well as the background is in §5.6, concerns the Bloch conjecture for the anti-symplectic involution on Hilbert squares of quartic surfaces constructed by Beauville [3]:

Corollary 1.8. *Let $X = \mathrm{Hilb}^2 S$ be the Hilbert square of a quartic K3 surface S, and let $\iota\colon X \to X$ be the anti-symplectic involution of Beauville [3]. Then*

$$\begin{aligned} \iota^* = -\mathrm{id}\colon \quad \mathrm{CH}^i(X)_{(2)} &\to \mathrm{CH}^i(X)_{(2)} \quad (i = 2, 4)\ , \\ \iota^* = \mathrm{id}\colon \quad \mathrm{CH}^4(X)_{(j)} &\to \mathrm{CH}^4(X)_{(j)} \quad (j = 0, 4)\ . \end{aligned}$$

(Here, the notation $\mathrm{CH}^*(X)_{(*)}$ refers to the Fourier decomposition of $\mathrm{CH}^*(X)_{\mathbf{Q}}$ constructed by Shen–Vial [40].)

1.2. The Beauville–Donagi family

For the universal family of Fano varieties of lines of cubic fourfolds, which form a locally complete family of projective hyper-Kähler fourfolds of $\mathrm{K3}^{[2]}$-type ([6]), we have the following slightly stronger result than predicted by Conjecture 1.3:

Theorem 1.9. *Let $\mathcal{C}$ be the moduli stack of smooth cubic fourfolds, $\mathcal{X} \to \mathcal{C}$ the universal family and $\mathcal{F} \to \mathcal{C}$ be the universal family of Fano varieties of lines of the fibers of $\mathcal{X}/\mathcal{C}$. Then for any $i \in \mathbf{N}$, any $z \in \mathrm{CH}^i(\mathcal{F})_{\mathbf{Q}}$ and any $b \in \mathcal{C}$, the restriction of z to the fiber F_b is numerically trivial if and only if it is (rationally equivalent to) zero.*[3]

In order to study the next case (Theorem 1.11), we also prove the following analogous result on the relative square of the universal family of Fano varieties of lines:

Theorem 1.10. *Notation is as in Theorem 1.9. Then for $z \in \mathrm{CH}^i(\mathcal{F} \times_{\mathcal{C}} \mathcal{F})_{\mathbf{Q}}$ and any $b \in \mathcal{C}$, the restriction of z to the fiber $F_b \times F_b$ is numerically trivial if and only if it is (rationally equivalent to) zero.*[4]

The proof of Theorem 1.9 (resp. Theorem 1.10) consists of two steps. First we show that cycles that belong to the image of the restriction map $\mathrm{CH}^i(\mathcal{F})_{\mathbf{Q}} \to \mathrm{CH}^i(F_b)_{\mathbf{Q}}$ (*resp.* $\mathrm{CH}^i(\mathcal{F} \times_{\mathcal{C}} \mathcal{F})_{\mathbf{Q}} \to \mathrm{CH}^i(F_b \times F_b)_{\mathbf{Q}}$) are tautological in the sense of Remark 3.3 (*resp.* Definition 6.2). Second we show that relations among tautological cycles modulo numerical equivalence in fact hold modulo rational equivalence. More precisely, we determine completely in terms of generators and relations the rings of tautological cycles for F_b and $F_b \times F_b$. In the case of $F_b \times F_b$, all relations but one had been established in [47] and [40]. The remaining

[3] In fact, we show that the restriction of $\mathrm{CH}^*(\mathcal{F})_{\mathbf{Q}}$ to $\mathrm{CH}^*(F_b)_{\mathbf{Q}}$ is the *tautological* subring, which is defined as the $\mathbf{Q}$-subalgebra generated by the Plücker polarization of F_b and by the Chern classes of F_b, see Remark 3.3.

[4] We actually show that the restriction of $\mathrm{CH}^*(\mathcal{F} \times_{\mathcal{C}} \mathcal{F})_{\mathbf{Q}}$ to $\mathrm{CH}^*(F_b \times F_b)_{\mathbf{Q}}$ is the *tautological* subring, which is defined as the $\mathbf{Q}$-subalgebra generated by the tautological subrings of the two factors together with the classes of the diagonal and the incidence subvariety; see Proposition 6.3.

relation is established in a joint appendix with Mingmin Shen, where we also draw some consequences concerning the multiplicative properties of the Chow motive of F_b.

1.3. The Lehn–Lehn–Sorger–van Straten family

Similarly to the Fano varieties of lines of cubic fourfolds, Lehn–Lehn–Sorger–van Straten (LLSvS) consider in [27] the twisted cubic curves on a cubic fourfold not containing a plane and show that the base of the maximal rationally connected (MRC) quotient of the moduli space of such curves is a hyper-Kähler eightfold. Later Addington and M. Lehn show in [1] that this hyper-Kähler eightfold is of $\mathrm{K3}^{[4]}$-deformation type (*cf.* also [28]). For the universal family of LLSvS hyper-Kähler eightfolds, we have the following result, which confirms the 0-cycle and codimension-2 cases of the generalized Franchetta conjecture.

Theorem 1.11. *Let $\mathcal{C}^\circ$ be the moduli stack of smooth cubic fourfolds not containing a plane and let $\mathcal{Z} \to \mathcal{C}^\circ$ be the universal family of LLSvS hyper-Kähler eightfolds ([27]). Then*

- *(i) for any $b \in \mathcal{C}^\circ$ and for any $\gamma \in \mathrm{CH}^8(\mathcal{Z})$ which is fiber-wise of degree 0, the restriction of γ to the fiber Z_b is (rationally equivalent to) zero.*
- *(ii) for any $b \in \mathcal{C}^\circ$ and for any $\gamma \in \mathrm{CH}^2(\mathcal{Z})_{\mathbf{Q}}$, its restriction to the fiber Z_b is zero if and only if its cohomology class vanishes.*

As a consequence, we deduce a part of the Beauville–Voisin Conjecture 2.3 as well as the refined Conjecture 2.4 for LLSvS eightfolds:

Corollary 1.12. *Given any smooth cubic fourfold X which does not contain a plane, let Z be the LLSvS hyper-Kähler eightfold associated to X. Denote by h the polarization class. Then the classes*

$$h^8, c_2h^6, c_2^2h^4, c_2^3h^2, c_2^4, c_4h^4, c_2c_4h^2, c_2^2c_4, c_6h^2, c_2c_6, c_4^2, c_8 \in \mathrm{CH}_0(Z)_{\mathbf{Q}}$$

are all proportional, where $c_i := c_i(T_Z)$ is the i-th (Chow-theoretic) Chern class of the tangent bundle of Z. We call the generator of degree 1 in this one-dimensional subspace the canonical 0-cycle class *or the* Beauville–Voisin class *of Z, denoted by $\mathfrak{o}_Z$.*

More strongly, let $R^*(Z)$ be the $\mathbf{Q}$-subalgebra generated by the polarization class h, the Chern classes c_i together with the following classes of coisotropic subvarieties of Z:

- *the embedded cubic fourfold $X \subset Z$ ([27]);*
- *the space of twisted cubics contained in a general hyperplane section of X ([42]);*
- *the coisotropic subvarieties of codimension $1, 2, 3, 4$ constructed by Voisin [50, Corollary 4.9];*
- *the fixed locus of the anti-symplectic involution ι of Z ([25]);*
- *the images by ι of all the above subvarieties.*

Then $R^8(Z) = \mathbf{Q} \cdot \mathfrak{o}_Z$.

Conventions. All algebraic varieties are over the field of complex numbers. We work with Chow groups with rational coefficients. For the m-th Hilbert scheme of a surface S, the two notations $S^{[m]}$ and $\mathrm{Hilb}^m(S)$ are used interchangeably and similarly for the relative situation. Chow groups of Deligne–Mumford stacks are the ones defined with rational coefficients by Vistoli [45] (there is a definition with integer coefficients by Kresch [24]).

Acknowledgments. The authors want to thank Nicolas Addington, Zhiyuan Li, Renjie Lyu, Nicolas Ressayre, Qizheng Yin for their interest and helpful comments and discussions. Thanks to the referee for many pertinent suggestions that helped improve the paper.

2. General remarks

2.1. Generic fiber vs. geometric fibers

There is the following slightly different version of the generalized Franchetta conjecture for hyper-Kähler varieties:

Conjecture 2.1. *Let $\mathcal{F}$ be the moduli stack of certain polarized hyper-Kähler varieties and let $\pi : \mathcal{X} \to \mathcal{F}$ be the universal family. Denote by $\mathcal{X}_\eta$ the generic fiber of π, where $\eta = \mathrm{Spec}(\mathbf{C}(\mathcal{F}))$. Then the group $\mathrm{CH}^*(\mathcal{X}_\eta)_{\mathrm{hom}}$ is zero.*

Here homological equivalence is with respect to some classical Weil cohomology; for instance, étale cohomology or de Rham cohomology.

Lemma 2.2. *Conjecture 1.3 and Conjecture 2.1 are equivalent.*

Proof. Let us start by assuming Conjecture 1.3. Using [43, Lemma 2.1], the hypothesis that the restriction of z to the geometric generic fiber is homologically trivial implies that the restriction of z to a very general geometric fiber is also trivial. Now the conclusion of Conjecture 1.3 says that the restriction of z to a very general geometric fiber is (rationally equivalent to) zero. By the standard argument of decomposition of the diagonal ([9], [46], [49]), this implies the existence of a Zariski open dense subset $U \subset \mathcal{F}$, such that $z|_{\mathcal{X}_U}$ is zero. In particular, z_η is rationally equivalent to zero.

For the other direction, since we know that $\mathrm{CH}^*(\mathcal{X}_\eta)_{\mathrm{hom}} = 0$, by restriction we can show Conjecture 1.3 for general fibers. Then a standard specialization argument allows us to conclude for all fibers. □

Thanks to Lemma 2.2, we will focus in this paper on Conjecture 1.3.

2.2. Relation to the Beauville–Voisin conjecture

As is mentioned in the introduction, the generalized Franchetta Conjecture 1.3 is very much related to the following Beauville–Voisin conjecture:

Conjecture 2.3 *(Beauville–Voisin [5], [47]). Let X be a projective hyper-Kähler variety. Let the* Beauville–Voisin subring $\langle c_i(X), Pic(X)\rangle$ *be the* $\mathbf{Q}$*-subalgebra of* $\mathrm{CH}^*(X)$ *generated by line bundles and all (Chow-theoretic) Chern classes of* T_X*. Then the restriction of the cycle class map to the Beauville–Voisin subring is injective. In other words, any polynomial of line bundles and Chern classes of X is homologically equivalent to zero if and only if it is rationally equivalent to zero.*

The original version due to Beauville in [5], under the name of *weak splitting property*, contains only line bundles; the Chern classes of the tangent bundle are introduced by Voisin in [47]. Some active progress towards this conjecture has recently been made: see [5], [47], [16], [51], [39], [18, Theorem 1.14] for the known results and more details. More recently, Voisin [50] proposed the following stronger version of Conjecture 2.3 involving certain types of coisotropic subvarieties. Recall that a subvariety is called *coisotropic* if the tangent space at each regular point of this subvariety is a coisotropic subspace (*i.e.* containing its orthogonal) with respect to the holomorphic symplectic form. We say that a subvariety of codimension i is *strongly coisotropic* if it can be swept out by i-dimensional subvarieties that are constant cycle subvarieties of the ambient hyper-Kähler variety. (Naturally, a strongly coisotropic subvariety is coisotropic.)

Conjecture 2.4 *(Voisin's refinement [50]). Let X be a projective hyper-Kähler variety. Then the restriction of the cycle class map to the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(X)$ generated by line bundles, Chern classes of T_X and strongly coisotropic subvarieties, is injective.*

We would like to point out that the generalized Franchetta conjecture implies the part of the Beauville–Voisin conjecture involving only the Chern classes of the tangent bundle and the polarization class. More generally it actually implies part of the refined Conjecture 2.4 once taking into account strongly coisotropic subvarieties which are defined universally over the moduli space (see Corollaries 1.6, 1.7, 5.11 and 1.12 for examples):

Proposition 2.5. *Let $\mathcal{F}$ be a moduli space of polarized hyper-Kähler varieties. If Conjecture 1.3 holds true for the universal family over $\mathcal{F}$, then for any member X of this family, the cycle class map restricted to the $\mathbf{Q}$-subalgebra generated by the polarization line bundle and the Chern classes of X, is injective.*

More generally, still assuming Conjecture 1.3, for any member X of this family, the cycle class map restricted to the $\mathbf{Q}$-subalgebra generated by the algebraic cycles of X that exist universally over the moduli space, is injective.

Proof. For any member X and any given polynomial in the polarization line bundle and the Chern classes of the tangent bundle $z := P(h, c_i(T_X)) \in \mathrm{CH}^*(X)$ such that the cohomology class of z vanishes, we want to show that $z = 0$. Consider $\gamma := P(h, c_i(T_{\mathcal{X}/\mathcal{F}})) \in \mathrm{CH}^*(\mathcal{X})$. Clearly $\gamma|_X = z$ and hence γ has fiber-wise vanishing cohomology class. Then the generalized Franchetta Conjecture 1.3 says exactly that z is rationally equivalent to zero. The last assertion is more or less tautological. □

2.3. Moduli space vs. parameter space

Remark 2.6. In order to establish the generalized Franchetta conjecture (or more generally the Franchetta property) in some cases, it will be convenient to work over some parameter space which dominates the moduli stack, instead of the moduli stack itself. More precisely, keep the same notation as in Conjecture 1.3 and let $B \to U$ be a surjective morphism from some smooth parameter space B (it will often be denoted by B° in concrete situations) to some smooth Zariski dense open subset U of the moduli stack $\mathcal{F}$. Denote by $\mathcal{Y} \to B$ the pulled-back family of the universal family $\mathcal{X} \to \mathcal{F}$. Then the generalized Franchetta conjecture for $\mathcal{Y} \to B$ implies the generalized Franchetta conjecture for $\mathcal{X} \to \mathcal{F}$ (but not conversely).

$$\begin{array}{ccccc} \mathcal{Y} & \twoheadrightarrow & \mathcal{X}_U & \hookrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow & \square & \downarrow \\ B & \twoheadrightarrow & U & \hookrightarrow & \mathcal{F} \end{array}$$

Indeed, for any $z \in \mathrm{CH}^*(\mathcal{X})$, denote by $z' \in \mathrm{CH}^*(\mathcal{Y})$ its pull-back image in $\mathcal{Y}$. Obviously, the hypothesis that the restriction of z to a very general fiber of $\mathcal{X}/\mathcal{F}$ is homologically trivial implies the same thing for the restriction of z' to the fibers of $\mathcal{Y}/B$. The generalized Franchetta conjecture for $\mathcal{Y}/B$ then implies that z' restricts to zero on each fiber of $\mathcal{Y}/B$. Hence so does z for each fiber of $\mathcal{X}_U \to U$. A specialization argument shows that the same thing holds for each fiber of $\mathcal{X} \to \mathcal{F}$.

3. Fano varieties of lines on cubic fourfolds

In this section, we prove Theorem 1.9, which by Remark 2.6 confirms the generalized Franchetta conjecture for the 20-dimensional locally complete family of polarized hyper-Kähler fourfolds constructed by Beauville–Donagi in [6]. The key idea of the proof is as in [48] and [38]: the universal family has very simple Chow groups.

We start by setting up some notations. Let V be a 6-dimensional vector space and $\mathbf{P}^5 = \mathbf{P}(V)$ be its projectivization. The parameter space of possibly singular cubic fourfolds is given by the following projective space:

$$B := \mathbf{P}\left(H^0(\mathbf{P}^5, \mathcal{O}(3))\right) = \mathbf{P}(\operatorname{Sym}^3 V^\vee) \simeq \mathbf{P}^{55}.$$

Let $B^\circ \subset B$ be the open subset parameterizing smooth cubic fourfolds. We thus have the universal family $\mathcal{X} \to B$ as well as the smooth family $\mathcal{X}^\circ \to B^\circ$ by base-change.

Let $G := \operatorname{Gr}(\mathbf{P}^1, \mathbf{P}^5)(= \operatorname{Gr}(2, 6))$ be the Grassmannian variety parameterizing all projective lines in $\mathbf{P}^5$. Denote by S (*resp.* Q) the tautological rank 2 subbundle (*resp.* rank 4 quotient bundle), fitting into the following short exact sequences of vector bundles over G:

$$0 \to S \to \mathcal{O}_G \otimes V \to Q \to 0.$$

Note that for any equation $f \in \operatorname{Sym}^3 V^\vee$, the above short exact sequence gives a section s_f of the vector bundle $\operatorname{Sym}^3 S^\vee$, whose zero locus $(s_f = 0)$ is exactly the Fano variety of lines of the cubic fourfold defined by f.

Consider the incidence subvariety $\mathcal{F}$ in $B \times G$ defined by

$$\mathcal{F} := \{([f], l) \in B \times G \;\; | \;\; f|_l = 0\},$$

together with the two natural projections:

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\pi}\swarrow & & \searrow^{p} \\ B & & G \end{array}$$

It is easy to see that $\pi : \mathcal{F} \to B$ is the universal Fano variety of lines of fibers of $\mathcal{X}/B$ and that $p : \mathcal{F} \to G$ is a projective bundle whose fiber over a line $l \in G$ parameterizes all (possibly singular) cubic fourfolds containing l.

As in [38, Lemma 2.1], we have the following:

Lemma 3.1. *For any $b \in B$, the following two images of restriction maps are the same:*

$$\operatorname{Im}\left(\operatorname{CH}^*(\mathcal{F}) \to \operatorname{CH}^*(F_b)\right) = \operatorname{Im}\left(\operatorname{CH}^*(G) \to \operatorname{CH}^*(F_b)\right).$$

Proof. The inclusion "$\supseteq$" is trivial (we have the factorization $F_b \hookrightarrow \mathcal{F} \to G$).

Let us show the inverse inclusion. Given any cycle $z \in \operatorname{CH}^*(\mathcal{F})$, we have by the projective bundle formula

$$z = \sum_{k \geq 0} p^*(z_k) \cdot \xi^k,$$

where $z_k \in \operatorname{CH}^*(G)$ and $\xi = c_1\left(\mathcal{O}_p(1)\right)$. As in [38, Lemma 2.1], we easily check that ξ is a linear combination of cycles pulled back from B by π and cycles pulled back from G by p. Hence z is a polynomial of cycles of the form $p^*(\alpha)$ and $\pi^*(\beta)$. The latter type being zero when restricted to any fiber F_b, the restriction of z to F_b is therefore the restriction of some cycle of G. □

Lemma 3.2. *For any* $b \in B^\circ$,

$$\mathrm{Im}\left(\mathrm{CH}^*(G) \to \mathrm{CH}^*(F_b)\right) \subseteq \langle c_i(F_b), \mathrm{Pic}(F_b)\rangle,$$

where the right hand side is the Beauville–Voisin subring *of* $\mathrm{CH}^*(F_b)$ *generated (as a* $\mathbf{Q}$*-algebra) by line bundles and all Chern classes of the tangent bundle of* F_b*.*

Proof. Since $\mathrm{CH}^*(G)$ is generated (as a $\mathbf{Q}$-algebra) by $c_1(S^\vee)$ and $c_2(S^\vee)$, it suffices to show that both of their restrictions to F_b lie in the Beauville–Voisin ring. The first one being a line bundle, it remains to show that $c_2(S^\vee|_{F_b}) \in \langle c_i(F_b), \mathrm{Pic}(F_b)\rangle$. However, using the short exact sequence

$$0 \to T_{F_b} \to T_G|_{F_b} \to \mathrm{Sym}^3 S^\vee|_{F_b} \to 0$$

together with the isomorphism $T_G \simeq S^\vee \otimes Q$, one finds that

$$ch(T_{F_b}) = ch(S^\vee|_{F_b})\,(6 - ch(S|_{F_b})) - ch\left(\mathrm{Sym}^3 S^\vee|_{F_b}\right),$$

and hence $c_2(T_{F_b}) = -ch_2(T_{F_b}) = 5c_1(S^\vee|_{F_b})^2 - 8c_2(S^\vee|_{F_b})$. Therefore $c_2(S^\vee|_{F_b})$ also belongs to the Beauville–Voisin ring.[5] □

We can now easily conclude:

Proof of Theorem 1.9. Let z be an element in $\mathrm{CH}^*(\mathcal{C})$. For any $b \in B^\circ$, thanks to Lemma 3.1, $z|_{F_b}$ is the restriction of some cycle from G, which must lie in the Beauville–Voisin ring $\langle c_i(F_b), \mathrm{Pic}(F_b)\rangle$ by Lemma 3.2. Now the equivalence between homological triviality and rational triviality of $z|_{F_b}$ is a consequence of Voisin's result [47, Theorem 1.4(ii)] saying that the cycle class map restricted to the Beauville–Voisin ring is injective. Finally, numerical equivalence and homological equivalence coincide for Fano varieties of lines of cubic fourfolds by [10]. □

Remark 3.3. In fact, the above proof shows that the restriction of a cycle $z \in \mathrm{CH}^*(\mathcal{C})$ to a fiber Fano variety of lines F is in the so-called *tautological* ring $R^*(F)$, which is the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(F)$, in general smaller than the Beauville–Voisin ring, generated by the Plücker polarization class g and the Chern classes of F. In particular,

- $R^1(F) = \mathbf{Q} \cdot g$;
- $R^2(F) = \mathbf{Q} \cdot g^2 \oplus \mathbf{Q} \cdot c_2$;
- $R^3(F) = \mathbf{Q} \cdot g^3$ (by [47, Lemma 3.5], gc_2 and g^3 are proportional);
- $R^4(F) = \mathbf{Q} \cdot \mathfrak{o}_F$, where $\mathfrak{o}_F$ is the canonical 0-cycle class and c_2^2, c_4, g^4, g^2c_2 are all proportional to it by [47, Lemma 3.2].

4. Hilbert squares of complete intersection K3 surfaces

In this section, we prove Theorem 1.4 for squares and Hilbert squares. There are three families of complete intersection K3 surfaces, namely, quartic surfaces in $\mathbf{P}^3$, complete intersections of quadric and cubic hypersurfaces in $\mathbf{P}^4$ and complete intersections of three quadric hypersurfaces in $\mathbf{P}^5$.

[5] The classes $c_1(S^\vee|_{F_b})$ and $c_2(S^\vee|_{F_b})$ are the classes that Claire Voisin calls g and c respectively in [47].

Let us fix some notations. In each of the three cases:

- $\mathbf{P} := \mathbf{P}^3, \mathbf{P}^4$ *resp.* $\mathbf{P}^5$ is the ambient projective space;
- $E := \mathcal{O}_{\mathbf{P}}(4), \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(3),$ *resp.* $\mathcal{O}_{\mathbf{P}}(2)^{\oplus 3}$ is the relevant vector bundle;
- $B := \mathbf{P}H^0(\mathbf{P}, E)$ is the parameter (projective) space and B° is the open subset parameterizing smooth K3 surfaces.
- $\mathcal{S} := \{(x, [s]) \in \mathbf{P} \times B \mid s(x) = 0\}$ is the universal family.

We have therefore the two natural projections, where p is clearly a projective bundle;

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{p} & \mathbf{P} \\ {\scriptstyle \pi}\downarrow & & \\ B & & \end{array} \tag{1}$$

Similarly, the relative square and the open complement of the relative diagonal in it fit into the following diagram

$$\begin{array}{ccc} \mathcal{S} \times_B \mathcal{S} \backslash \Delta_{\mathcal{S}/B} & \xrightarrow{q'} & \mathbf{P} \times \mathbf{P} \backslash \Delta_{\mathbf{P}} \\ {\scriptstyle j}\downarrow & & \downarrow \\ \mathcal{S} \times_B \mathcal{S} & \xrightarrow{q:=(p,p)} & \mathbf{P} \times \mathbf{P} \\ {\scriptstyle \pi_2:=(\pi,\pi)}\downarrow & & \\ B & & \end{array} \tag{2}$$

Note that although q itself is not a projective bundle, its restriction q' is. Let ξ be the first Chern class of $\mathcal{O}_{q'}(1)$. The relative diagonal $\Delta_{\mathcal{S}/B}$ being of codimension 2, ξ extends uniquely to the whole of $\mathcal{S} \times_B \mathcal{S}$, which we still denote by ξ by abuse of notation.

We can show the analogue of Lemma 3.1 in our case[6]:

Proposition 4.1. *For any $b \in B^\circ$, we have:*

$$\operatorname{Im}\left(\mathrm{CH}^*(\mathcal{S} \times_B \mathcal{S}) \to \mathrm{CH}^*(S_b \times S_b)\right) = \operatorname{Im}\left(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b)\right) + \Delta_* \operatorname{Im}\left(\mathrm{CH}^*(\mathbf{P}) \to \mathrm{CH}^*(S_b)\right),$$

where $\Delta : S_b \hookrightarrow S_b \times S_b$ is the diagonal embedding.

Proof. Notation is as in Diagrams (1) and (2). By base-change, it is easy to see that the right-hand side is contained in the left-hand side. Concerning the inverse inclusion, the projective bundle formula gives, for any $z \in \mathrm{CH}^*(\mathcal{S} \times_B \mathcal{S})$,

$$j^*(z) = \sum_{k \geq 0} q'^*(z_k) \cdot \xi^k,$$

for some cycles $z_k \in \mathrm{CH}^*(\mathbf{P} \times \mathbf{P} \backslash \Delta_{\mathbf{P}})$. As in Lemma 3.1, it is easy to see that $\xi = j^* \pi_2^*(h) + q'^*(\alpha)$, where $h = c_1(\mathcal{O}_B(1))$ and $\alpha \in \mathrm{CH}^*(\mathbf{P} \times \mathbf{P} \backslash \Delta_{\mathbf{P}})$. For each k, we denote still by $z_k \in \mathrm{CH}^*(\mathbf{P} \times \mathbf{P})$ its closure and similarly for α. Therefore, we have

[6] Proposition 4.1 will be generalized for the so-called stratified projective bundle in §5.1.

$$z - \sum_k q^*(z_k) \cdot (\pi_2^*(h) - q^*(\alpha))^k \in \operatorname{Ker}(j^*).$$

By the localization sequence, there exists $\gamma \in \mathrm{CH}^*(\mathcal{S})$, such that

$$z - \sum_k q^*(z_k) \cdot (\pi_2^*(h) - q^*(\alpha))^k = \Delta_*(\gamma), \tag{3}$$

where $\Delta : \mathcal{S} \hookrightarrow \mathcal{S} \times_B \mathcal{S}$ is the diagonal embedding.

Since $p : \mathcal{S} \to \mathbf{P}$ is also a projective bundle with $c_1(\mathcal{O}_p(1)) = \pi^*(h)$, we have

$$\gamma = \sum_l p^*(\gamma_l) \cdot \pi^*(h)^l,$$

for some $\gamma_l \in \mathrm{CH}^*(\mathbf{P})$. Substituting this into (3), we get

$$z = \sum_k q^*(z_k) \cdot (\pi_2^*(h) - q^*(\alpha))^k + \sum_l \Delta_*(p^*(\gamma_l) \cdot \pi^*(h)^l). \tag{4}$$

Now for any $b \in B^\circ$, the restriction $z|_{S_b \times S_b}$ is of the desired form simply because the restrictions of $\pi_2^*(h)$ and $p^*(h)$ to the fibers vanish. □

We can now prove the first two parts of Theorem 1.4.

Proof of Theorem 1.4 for relative squares. Keep the same notations as before. Thanks to Proposition 4.1, we only need to show that for any smooth complete intersection K3 surface $S \subset \mathbf{P}$, the cycle class map restricted to

$$\operatorname{Im}(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S \times S)) + \Delta_* \operatorname{Im}(\mathrm{CH}^*(\mathbf{P}) \to \mathrm{CH}^*(S))$$

is injective. Denote $H := c_1(\mathcal{O}_{\mathbf{P}}(1))$ and $h := H|_S$. Since $\mathrm{CH}^*(\mathbf{P} \times \mathbf{P})$ is generated by $\mathrm{pr}_1^*(H)$ and $\mathrm{pr}_2^*(H)$, and $\Delta_*(h) = h \times \mathfrak{o}_S + \mathfrak{o}_S \times h$ (see [7]), it is enough to show that the cycle class map of $S \times S$ restricted to the subalgebra generated by $\mathrm{pr}_1^*(h), \mathrm{pr}_2^*(h)$ and Δ is injective. This is the easiest case of Voisin's [47, Proposition 2.2]. □

Proof of Theorem 1.4 for relative Hilbert squares. Consider the blow-up of $\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ$ along the relative diagonal $\Delta_{\mathcal{S}^\circ/B^\circ}$; the natural involution switching the two factors lifts to the blow-up. It is well-known that the Hilbert square is the quotient of this lifted involution and that

$$\mathrm{CH}^*(\mathrm{Hilb}^2_{B^\circ}(\mathcal{S}^\circ)) \simeq \mathrm{CH}^*(\mathrm{Bl}_\Delta(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ))^{inv} \simeq \mathrm{CH}^*(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ)^{inv} \oplus \mathrm{CH}^{*-1}(\mathcal{S}^\circ),$$

where all isomorphisms are compatible with the restriction to the fibers. Therefore, for any $b \in B^\circ$, the restriction $z|_{S_b^{[2]}}$ of any $z \in \mathrm{CH}^*(\mathrm{Hilb}^2_{B^\circ}(\mathcal{S}^\circ))$ to the fiber over b, viewed as an element in $\mathrm{CH}^*(S_b \times S_b)^{inv} \oplus \mathrm{CH}^{*-1}(S_b)$, lives in $\operatorname{Im}(\mathrm{CH}^*(\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ)^{inv} \to \mathrm{CH}^*(S_b \times S_b)^{inv}) \oplus \operatorname{Im}(\mathrm{CH}^{*-1}(\mathcal{S}^\circ) \to \mathrm{CH}^{*-1}(S_b))$. We can thus conclude thanks to the established cases of the Franchetta property for the relative squares $\mathcal{S}^\circ \times_{B^\circ} \mathcal{S}^\circ$ and for $\mathcal{S}^\circ$. □

5. Some more cases of Hilbert schemes of K3 surfaces

In this section, we push the results and methods of §4 to higher (Hilbert) powers and to K3 surfaces of higher genera. Let us first provide some technical tool for that purpose.

5.1. Stratified projective bundles

As one can observe, Lemma 3.1 and Proposition 4.1 (but also Proposition 6.1 below) share some similarity. The goal of this technical subsection is to summarize these situations.

Definition 5.1 *(Stratified projective bundle).* A projective morphism $q : \mathcal{X} \to Y$ is called a *stratified projective bundle* if there exists a commutative cartesian diagram

$$\begin{array}{ccccccc} \mathcal{X}_r & \hookrightarrow & \cdots \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \mathcal{X}_0 = \mathcal{X} \\ \downarrow q_r & \square & \square & \downarrow q_1 & \square & \downarrow q_0 = q \\ Y_r & \hookrightarrow & \cdots \hookrightarrow & Y_1 & \hookrightarrow & Y_0 = Y \end{array} \tag{5}$$

where all horizontal morphisms are closed immersions, such that for any $0 \le i \le r$, the restriction of q_i

$$q_i' : \mathcal{X}_i \backslash \mathcal{X}_{i+1} \to Y_i \backslash Y_{i+1}$$

is a projective bundle ($\mathcal{X}_{r+1} = Y_{r+1} = \varnothing$). The above diagram is called a *stratification* of q.

Now we can state the following generalization of Lemma 3.1 and Proposition 4.1 (see also Proposition 6.1 for an example).

Proposition 5.2. *Let $q : \mathcal{X} \to Y$ be a stratified projective bundle with a given stratification (5) and $\pi : \mathcal{X} \to B$ be a surjective morphism. Assume moreover that for any $0 \le i \le r$, Y_i is smooth projective, $\mathcal{X}_i$ is flat over a (common) smooth Zariski open subset $B^\circ \subset B$, $\operatorname{codim}_{\mathcal{X}_i}(\mathcal{X}_{i+1}) \ge 2$ and finally that there exists a line bundle on B whose restriction to fibers of the projective bundle q_i' is non-trivial. Then for any $b \in B^\circ$*

$$\operatorname{Im}\left(\mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(X_b)\right) = \sum_{i=0}^{r} \iota_{i*} \operatorname{Im}\left(q_{i,b}^* : \mathrm{CH}^*(Y_i) \to \mathrm{CH}^*(X_{i,b})\right),$$

where X_b (resp. $X_{i,b}$) is the fiber of $\mathcal{X}$ (resp. the Zariski closure of $\mathcal{X}_i \backslash \mathcal{X}_{i+1}$) over b, $\iota_i : X_{i,b} \hookrightarrow X_b$ is the natural inclusion and $q_{i,b}$ is the restriction of q_i to $X_{i,b}$.

Proof. Since the $\mathcal{X}_i$'s are flat over B°, by base-change, the right-hand side is clearly contained in the left-hand side. We use induction on r to prove the other inclusion. For any $z \in \mathrm{CH}^*(\mathcal{X})$, the projective bundle formula shows that

$$j^*(z) = \sum_{k \ge 0} {q_0'}^*(z_k) \cdot \xi^k,$$

for some cycles $z_k \in \mathrm{CH}^*(Y_0 \backslash Y_1)$ where $j : \mathcal{X} \backslash \mathcal{X}_1 \hookrightarrow \mathcal{X}$ is the open immersion and $\xi = c_1(\mathcal{O}_{q_0'}(1))$. By hypothesis, $\xi = j^*\pi^*(h) + {q_0'}^*(\alpha)$, where h is a divisor on B and $\alpha \in \mathrm{CH}^*(Y_0 \backslash Y_1)$. We extend z_k and α to Y_0, keeping the same notation for the classes on Y_0. Therefore

$$z - \sum_k q^*(z_k) \cdot (\pi^*(h) - q^*(\alpha))^k \in \operatorname{Ker}(j^*).$$

By the localization sequence, there exists $\gamma \in \mathrm{CH}^*(\mathcal{X}_1)$, such that

$$z = \sum_k q^*(z_k) \cdot (\pi^*(h) - q^*(\alpha))^k + \iota_*(\gamma), \tag{6}$$

where $\iota : \mathcal{X}_1 \hookrightarrow \mathcal{X}$ is the natural inclusion.

Noting that the restriction of $\pi^*(h)$ to X_b vanishes, we have that

$$z|_{X_b} \in \mathrm{Im}\,(q^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X_b)) + \mathrm{Im}\,(\iota_* : \mathrm{CH}^*(\mathcal{X}_1) \to CH^*(\mathcal{X}))\,|_{X_b},$$

where the second term is $\iota_{1,*}\,\mathrm{Im}\,(\mathrm{CH}^*(\mathcal{X}_1) \to \mathrm{CH}^*(X_{1,b}))$ by flat base-change. Observing that $q_1 : \mathcal{X}_1 \to Y_1$ is again a stratified projective bundle verifying all the conditions, the induction hypothesis allows us to conclude. □

5.2. *Cubes and Hilbert cubes of complete intersection K3 surfaces*

We prove Theorem 1.4 for cubes and Hilbert cubes in this subsection. Notation is as in §4.

The geometry is quite close[7] to the one considered in [15]; in particular, we will study collinear triples in the projective space $\mathbf{P}$. For three points in $\mathbf{P}$ there are four types of relative positions: non-collinear, collinear and distinct, two coincide but not with the third, all coincide. As a result, the evaluation map of the relative cube of the universal family

$$q : \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} \to \mathbf{P} \times \mathbf{P} \times \mathbf{P}$$

is not a projective bundle but is a *stratified* projective bundle (Definition 5.1) with the following stratification:

$$\begin{array}{ccccccccc}
\mathcal{S} = \delta_{\mathcal{S}/B} & \hookrightarrow & \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \hookrightarrow & \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} \cup \mathcal{I} & \hookrightarrow & \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} & \xrightarrow{\pi_3} & B \\
\downarrow{\scriptstyle p} & \square & \downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle q} & & \\
\mathbf{P} = \delta_P & \hookrightarrow & \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \hookrightarrow & J & \hookrightarrow & \mathbf{P} \times \mathbf{P} \times \mathbf{P} & &
\end{array} \tag{7}$$

where in the first row, $\Delta_{i,j} : \mathcal{S} \times_B \mathcal{S} \hookrightarrow \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}$ are the three big (relative) diagonals and $\mathcal{I}$ is the Zariski closure of

$$\mathcal{I}^\circ := \{(x, y, z) \in \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} \ \mid \ x, y, z \text{ collinear and distinct}\}\ ;$$

in the second row, $\Delta_{i,j} : \mathbf{P} \times \mathbf{P} \hookrightarrow \mathbf{P} \times \mathbf{P} \times \mathbf{P}$ are the three big diagonals and

$$J := \{(x, y, z) \in \mathbf{P} \times \mathbf{P} \times \mathbf{P} \ \mid \ x, y, z \text{ collinear}\}\,.$$

Proposition 5.3. *We have for any* $b \in B^\circ$

$$\begin{aligned}
&\mathrm{Im}\,(\mathrm{CH}^*(\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}) \to \mathrm{CH}^*(S_b \times S_b \times S_b)) \\
= {}&\mathrm{Im}\,(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b \times S_b)) \\
&+ \sum_{1 \le i < j \le 3} \Delta_{i,j\,*}\,\mathrm{Im}\,(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b)) \\
&+ \delta_*\,\mathrm{Im}\,(\mathrm{CH}^*(\mathbf{P}) \to \mathrm{CH}^*(S_b))\,,
\end{aligned}$$

[7] In fact, complete intersection K3 surfaces are special cases of the Calabi–Yau complete intersections considered in [15] and so all results in [15] apply.

where $\Delta_{i,j} : S_b^2 \hookrightarrow S_b^3$ are the inclusions of the big diagonals and $\delta : S_b \hookrightarrow S_b^3$ is the inclusion of the small diagonal.

Proof. It is straight-forward to check that (7) indeed stratifies q into projective bundles and that the codimension of $\mathcal{I}$ in $\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}$ is $\dim(\mathbf{P}) - 1$ (*cf.* [15, Lemma 1.2]), which is ≥ 2. Moreover, it is clear that $\pi_3^* \mathcal{O}_B(1)$ restricts to the relative ample tautological line bundle on fibers of all projective bundles. All assumptions of Proposition 5.2 being satisfied, it implies that for any $b \in B^\circ$

$$\begin{aligned}
&\operatorname{Im}\left(\mathrm{CH}^*(\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}) \to \mathrm{CH}^*(S_b \times S_b \times S_b)\right) \\
&= \operatorname{Im}\left(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b \times S_b)\right) + \iota_* \operatorname{Im}\left(\mathrm{CH}^*(J) \to \mathrm{CH}^*(I_b)\right) \\
&\quad + \sum_{1 \leq i < j \leq 3} \Delta_{i,j*} \operatorname{Im}\left(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b)\right) + \delta_* \operatorname{Im}\left(\mathrm{CH}^*(\mathbf{P}) \to \mathrm{CH}^*(S_b)\right),
\end{aligned}$$

where $\iota : I_b \hookrightarrow S_b^3$ is the inclusion of the Zariski closure of the locus of collinear and distinct triples. We only have to show that the second term on the right-hand side is redundant. Indeed, for any $b \in B^\circ$, consider the cartesian square

$$\begin{array}{ccc}
\Delta_{12} \cup \Delta_{13} \cup \Delta_{23} \cup I_b & \hookrightarrow & S_b \times S_b \times S_b \\
\downarrow & \square & \downarrow \\
J & \hookrightarrow & \mathbf{P} \times \mathbf{P} \times \mathbf{P}
\end{array}$$

Here the intersection is transversal along $I_b \backslash \cup \Delta_{i,j}$ (without excess intersection) and $\operatorname{codim}_{S_b^3} I_b = \operatorname{codim}_{\mathbf{P}^{\times 3}}(J) = \dim P - 1$, while along $\Delta_{i,j}$ the intersection has excess dimension $\dim \mathbf{P} - 3$ (*cf.* [15, Lemma 1.2]) with excess normal bundle $\frac{\mathrm{pr}_1^*(E|_{S_b})}{\mathcal{O}(1) \boxtimes \mathcal{O}(-1)}$ (*cf.* [15, Lemma 1.5]).[8] The excess intersection class on $\Delta_{i,j} = S_b \times S_b$ is therefore a polynomial in h_1 and h_2 with $h_i := \mathrm{pr}_i^*(c_1(\mathcal{O}(1)|_{S_b}))$, hence is the pull-back of an element in $\mathrm{CH}^*(\mathbf{P} \times \mathbf{P})$. As a result, by the excess intersection formula (*cf.* [19, §6.3]) applied to the above cartesian square, any element in the second term $\iota_* \operatorname{Im}(\mathrm{CH}^*(J) \to \mathrm{CH}^*(I_b))$, up to an element in the third term $\sum_{1 \leq i < j \leq 3} \Delta_{i,j*} \operatorname{Im}(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b))$, is an element in the first term $\operatorname{Im}(\mathrm{CH}^*(\mathbf{P} \times \mathbf{P} \times \mathbf{P}) \to \mathrm{CH}^*(S_b \times S_b \times S_b))$, thus is redundant. □

We are now ready to prove the remaining cases of Theorem 1.4:

Proof of Theorem 1.4 for relative cubes. Denote by $h = c_1(\mathcal{O}_{\mathbf{P}}(1)|_{S_b})$ and $h_i := \mathrm{pr}_i^*(h)$. Thanks to Proposition 5.3, for any $z \in \mathrm{CH}^*(\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S})$ and any $b \in B^\circ$, the restriction $z|_{S_b \times S_b \times S_b}$ is a polynomial in $h_1, h_2, h_3, \Delta_{12}, \Delta_{13}, \Delta_{23}$ (and $\delta = \Delta_{12}\Delta_{23}$). We can conclude by the $m = 3$ case of Voisin's [47, Proposition 2.2], where the essential point is the decomposition of the small diagonal δ due to Beauville–Voisin [7, Proposition 3.2]. □

Proof of Theorem 1.4 for relative Hilbert cubes. To simplify the notation, we denote $S^{[m]} := \mathrm{Hilb}^m(S)$ and similarly $\mathcal{S}^{[m]/B} := \mathrm{Hilb}^m_B \mathcal{S}$. Let us first recall the result of de Cataldo–Migliorini [11] in the special case of Chow groups of Hilbert cubes of surfaces: for any surface S, denote by $\rho : S^{[3]} \to S^{(3)}$ the Hilbert–Chow morphism which sends a 0-dimensional subscheme to its support 0-cycle. We have the incidence subvarieties

[8] So there is no excess intersection in the case of quartic surfaces.

$$U := \left\{(z, x_1, x_2, x_3) \in S^{[3]} \times S^3 \;\mid\; \rho(z) = x_1 + x_2 + x_3\right\} ;$$
$$V := \left\{(z, x_1, x_2) \in S^{[3]} \times S^2 \;\mid\; \rho(z) = 2x_1 + x_2\right\} ;$$
$$W := \left\{(z, x) \in S^{[3]} \times S \;\mid\; \rho(z) = 3x\right\} ;$$

and the main result of [11] says that together they induce an injective morphism

$$(U_*, V_*, W_*) : \mathrm{CH}^*(S^{[3]}) \hookrightarrow \mathrm{CH}^*(S^3) \oplus \mathrm{CH}^*(S^2) \oplus \mathrm{CH}^*(S).$$

Note that the above correspondences have natural family counterparts, denoted by $\mathcal{U}, \mathcal{V}, \mathcal{W}$.

Let $z \in \mathrm{CH}^*(\mathcal{S}^{[3]/B})$ be such that the cohomology class of $z|_{S_b^{[3]}}$ vanishes. By the above injectivity, it is enough to show that for any $b \in B^\circ$, $U_*\left(z|_{S_b^{[3]}}\right)$, $V_*\left(z|_{S_b^{[3]}}\right)$ and $W_*\left(z|_{S_b^{[3]}}\right)$ are zero. To this end, observe that $U_*\left(z|_{S_b^{[3]}}\right) = \mathcal{U}_*(z)|_{S_b^3}$ is the restriction of a cycle of the total family $\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}$ with trivial cohomology class, hence is zero by the relative cube case of Theorem 1.4 just proven. Similarly, the vanishing of $V_*\left(z|_{S_b^{[3]}}\right)$ and $W_*\left(z|_{S_b^{[3]}}\right)$ follow from the relative square case proven in §4 and [38] respectively.

Finally, the proof of the case of $\mathcal{S} \times_B \mathcal{S}^{[2]/B}$ is similar (in fact, easier) by using the motivic decomposition for Hilbert squares. □

5.3. Beyond complete intersection K3 surfaces

The techniques we utilized above in order to prove Theorem 1.4 for (Hilbert) squares and cubes of complete intersection K3 surfaces can also be employed to attack the generalized Franchetta Conjecture 1.3 for families of K3 surfaces for which Mukai models are available. In this subsection, we give a sufficient condition for the Franchetta property to hold for Hilbert schemes of K3 surfaces in a certain range. It is convenient to introduce the following notion:

Definition 5.4 *(Tautological ring).* Let (S, H) be a polarized K3 surface and $r \in \mathbf{N}$. Denote $h := c_1(H) \in \mathrm{CH}^1(S)$. The *tautological ring* $R^*(S^r)$ is the subring of the (rational) Chow ring $\mathrm{CH}^*(S^r)$ generated by the big diagonals $\Delta_{i,j}$ $(1 \le i < j \le r)$, the polarization classes $h_i := \mathrm{pr}_i^*(h)$ and the Beauville–Voisin classes $\mathfrak{o}_i := \mathrm{pr}_i^*(\mathfrak{o}_S)$ $(1 \le i \le r)$.

Remark 5.5. Using [7, Proposition 2.6], we see that the tautological rings of different powers of a K3 surface are closed under push-forwards and pull-backs along all kinds of (partial) diagonal inclusions.

Recall that for a natural number g, we say that a *Mukai model* for K3 surfaces of genus g exists, if there exist an ambient homogeneous space $G = G_g$ (often a Grassmannian) and a globally generated homogeneous vector bundle $E = E_g$ on G such that the zero locus of a general section of E gives a general K3 surface of genus g. For the available constructions of Mukai models and the corresponding G and E, we refer to [38] as well as the original sources [30], [31], [32], [33]. Accordingly, we have a universal family

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\;p\;} & G \\ {\scriptstyle\pi}\big\downarrow & & \\ B = H^0(G, E) & & \end{array}$$

and we denote $B^\circ \subset B$ the locus parameterizing smooth K3 surfaces of genus g.

The crucial condition for our techniques to work is the following:

Definition 5.6. For an $r \in \mathbf{N}^*$, we say that the Mukai model (G, E) satisfies the condition $(\star_r)$ if

$(\star_r)$: for any $x_1, \cdots, x_r$ distinct points of G, the following evaluation map is surjective

$$H^0(G, E) \to \bigoplus_{i=1}^{r} E_{x_i}.$$

Or equivalently, $H^0(G, E \otimes I_{x_1} \otimes \cdots \otimes I_{x_r})$ is of codimension $r \cdot \mathrm{rank}(E)$ in $H^0(G, E)$.

Clearly, $(\star_r)$ implies $(\star_k)$ for all $k < r$.

Proposition 5.7. *The notation is as above. Fix a genus g for which a Mukai model exists for K3 surfaces of genus g and fix such a Mukai model which satisfies condition $(\star_r)$. Assume that*

$$\mathrm{Im}\left(\mathrm{CH}^*(\mathcal{S}) \to \mathrm{CH}^*(S_b)\right) = R^*(S_b),$$

for any $b \in B^\circ$. Then

$$\mathrm{Im}\left(\mathrm{CH}^*(\mathcal{S}^{r/B}) \to \mathrm{CH}^*(S_b^r)\right) = R^*(S_b^r),$$

for any $b \in B^\circ$.

Proof. The proof is to rephrase every step of §5.2 in the general setting. We proceed by induction on r. Consider the evaluation map $q : \mathcal{S}^{r/B} \to G^r$, which is a stratified projective bundle (Definition 5.1) with the stratification on G^r given by the different types of incidence relations for r points of G:

$$\begin{array}{ccccccccc}
\mathcal{X}_n = \mathcal{S} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \mathcal{X}_0 = \mathcal{S}^{r/B} & \longrightarrow & B \\
\downarrow{\scriptstyle q_n = p} & \square & & \square & \downarrow{\scriptstyle q_1} & \square & \downarrow{\scriptstyle q_0 = q} & & \\
Y_n = G & \hookrightarrow & \cdots & \hookrightarrow & Y_1 & \hookrightarrow & Y_0 = G^r & &
\end{array} \tag{8}$$

By Proposition 5.2, for any $b \in B^\circ$,

$$\mathrm{Im}\left(\mathrm{CH}^*(\mathcal{S}^{r/B}) \to \mathrm{CH}^*(S_b^r)\right) = \sum_{i=0}^{n} \iota_{i,*}\, \mathrm{Im}\left(\mathrm{CH}^*(Y_i) \to \mathrm{CH}^*(\mathcal{X}'_{ib})\right), \tag{9}$$

where $\mathcal{X}'_i$ is the Zariski closure of $\mathcal{X}_i \backslash \mathcal{X}_{i+1}$. Let us show that each term of (9) is in the tautological ring $R^*(S_b)$ by ascending order for $0 \le i \le n$:

- If $i = 0$, since the Chow ring of G satisfies the Künneth formula, we only need to show that
$$\mathrm{Im}\left(\mathrm{CH}^*(G) \to \mathrm{CH}^*(S_b)\right) \subset R^*(S_b),$$
which is true by assumption.
- If a general point of Y_i is parameterizing r points of G where at least two of them coincide, then the contribution of the i-th term of (9) factors through $R^*(S_b^{r-1})$ (via the diagonal push-forward) by the induction hypothesis, hence is contained in $R^*(S_b^r)$ (Remark 5.5).
- If a general point of Y_i is parameterizing r different points of G, then the hypothesis $(\star_r)$ means precisely that any r different points of G impose independent conditions on B, each of codimension $\mathrm{rank}(E)$.

Therefore, $\mathcal{X}_i'$, the Zariski closure of $\mathcal{X}_i \backslash \mathcal{X}_{i+1}$, has codimension in $\mathcal{X}_{i-1}$ equal to $\operatorname{codim}_{Y_{i-1}}(Y_i)$. The excess intersection formula ([19, §6.3]) applied to the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_i = \mathcal{X}_{i+1} \cup \mathcal{X}_i' & \hookrightarrow & \mathcal{X}_{i-1} \\ \downarrow & \square & \downarrow \\ Y_i & \hookrightarrow & Y_{i-1} \end{array}$$

tells us that modulo the $(i+1)$-th term of (9), the contribution of the i-th term is contained in the $(i-1)$-th term. □

Theorem 5.8. *Fix a genus g for which a Mukai model exists for K3 surfaces of genus g, and fix such a Mukai model. Assume that*

- *(i) the Mukai model satisfies the condition $(\star_r)$;*
- *(ii) Conjecture 1.2 is true for the universal family $\mathcal{S} \to B$ of K3 surfaces of genus g;*
- *(iii) the cycle class map restricted to the tautological ring $R^*(S^r)$ is injective for the very general K3 surface S of genus g.*

Then the Franchetta property holds for $\mathcal{S}^{[r_1]/B} \times_B \cdots \times_B \mathcal{S}^{[r_m]/B}$, for any $r_1, \cdots, r_m$ whose sum is $\leq r$.

Proof. The case of relative powers $\mathcal{S}^{k/B}$, for any $k \leq r$, is a direct consequence of Proposition 5.7 and the hypothesis on the injectivity of the cycle class map on the tautological ring. The other cases reduce to the cases of $\mathcal{S}^{k/B}$ for all $1 \leq k \leq r$ by making use of de Cataldo–Migliorini's result [11] for Chow motives of Hilbert schemes of surfaces. □

We apply Theorem 5.8 to some Mukai models to get concrete unconditional results:

Proof of Theorem 1.5. Assumption (ii) is proven for $g \in \{2, \ldots, 10\} \cup \{12\}$ in [38]. Assumption (iii) is taken care of for $r \leq 43$ by Voisin's [47, Proposition 2.2]. It remains to check assumption (i) of Theorem 5.8; we proceed by a case-by-case analysis of the positivity of the homogeneous bundle in the Mukai model. See Mukai's series of papers [30], [31], [32], [33] for more information on the geometry of these models.

- K3 surfaces of genus $g = 2$ are[9] smooth degree 6 hypersurfaces in the weighted projective space $\mathbf{P} := \mathbf{P}(1,1,1,3)$. The Mukai model for this family is thus $(G, E) = (\mathbf{P}, \mathcal{O}(6))$. Note that the K3 surfaces in this family all avoid the singular point $O := [0,0,0,1]$. Let us check the condition $(\star_3)$, *i.e.*, that the evaluation map

$$H^0(\mathbf{P}, \mathcal{O}(6)) \to \bigoplus_{i=1}^{3} \mathbf{C}_{x_i}$$

is surjective for distinct $x_1, x_2, x_3 \neq O$, where $\mathbf{C}_x$ denotes the fiber of $\mathcal{O}(6)$ at x. It is easy to see that $\mathbf{P}(1,1,1,3)$ is isomorphic to the projective cone over the third Veronese embedding of $\mathbf{P}^2$ (*cf.* [13]) and O is the vertex. By upper-semicontinuity, it is enough to treat the most degenerate case for three distinct points of $\mathbf{P}\backslash\{O\}$, which is when they lie in the same ruling of the projective cone. In this case, as the restriction of $\mathcal{O}(6)$ to the ruling is $\mathcal{O}(2)$, the condition $(\star_3)$ follows from the surjections:

[9] Equivalently, these K3 surfaces are also double covers of $\mathbf{P}^2$ ramified along smooth sextic curves.

$$H^0(\mathbf{P}, \mathcal{O}(6)) \twoheadrightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2)) \twoheadrightarrow \bigoplus_{i=1}^{3} \mathbf{C}_{x_i},$$

where $\mathbf{P}^1$ is the ruling which contains x_i's.

- For quartic surfaces ($g = 3$), let us first show that $(\mathbf{P}^3, \mathcal{O}(4))$ satisfies $(\star_5)$, *i.e.*, that the evaluation map

$$H^0(\mathbf{P}^3, \mathcal{O}(4)) \to \bigoplus_{i=1}^{5} \mathbf{C}_{x_i}$$

is surjective for distinct x_i's. Again, it is enough to treat the most degenerate cases, namely:

  – when $x_1, \cdots, x_5$ are collinear, then this follows from the surjectivity of the restriction and the evaluation

$$H^0(\mathbf{P}^3, \mathcal{O}(4)) \twoheadrightarrow H^0(\mathbf{P}^1, \mathcal{O}(4)) \twoheadrightarrow \bigoplus_{i=1}^{5} \mathbf{C}_{x_i},$$

  where $\mathbf{P}^1$ is the line containing these points.

  – when $x_1, \cdots, x_5$ are in a conic C. Then the Koszul resolution provides an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-3) \to \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_C \to 0,$$

  which allows us to see that the restriction map $H^0(\mathbf{P}^3, \mathcal{O}(4)) \to H^0(C, \mathcal{O}_C(8))$ is surjective. Since $H^0(C, \mathcal{O}_C(8)) \to \bigoplus_{i=1}^{5} \mathbf{C}_{x_i}$ is clearly surjective, we are done.

  The condition $(\star_5)$ is proven.

- For $g = 6$, the Mukai model is $(G, E) = (\mathrm{Gr}(2, 5), \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2))$, where $\mathcal{O}(1)$ is the Plücker line bundle. It is clear that the condition $(\star_2)$ is equivalent to the surjectivity of

$$H^0(G, \mathcal{O}(1)) \to \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2}$$

for any two distinct points $x_1, x_2 \in G$. This last condition follows from the very ampleness of the Plücker line bundle $\mathcal{O}(1)$.

- For $g = 7$, the Mukai model is $(G, E) = \left(\mathrm{OGr}(5, 10), U^{\oplus 8}\right)$, where $\mathrm{OGr}(5, 10)$ is the orthogonal Grassmannian parameterizing isotropic subspaces of dimension 5 in a vector space of dimension 10 equipped with a non-degenerate quadratic form and U is a line bundle corresponding to the half spinor representation. The proof is similar to the previous case: one uses the very ampleness of U.
- For $g = 8$, the Mukai model is $(G, E) = \left(\mathrm{Gr}(2, 6), \mathcal{O}(1)^{\oplus 6}\right)$, where $\mathcal{O}(1)$ is the Plücker line bundle. The proof goes as for $g = 6$ by the very ampleness of the Plücker line bundle.
- For $g = 9$, the Mukai model is $(G, E) = \left(\mathrm{LGr}(3, 6), \mathcal{O}(1)^{\oplus 4}\right)$, where $\mathrm{LGr}(3, 6)$ is the symplectic Grassmannian parameterizing Lagrangian subspaces in a 6-dimensional vector space equipped with a symplectic form and $\mathcal{O}(1)$ is the restriction of the Plücker line bundle of $\mathrm{Gr}(3, 6)$. The proof goes as before: one uses the very ampleness of $\mathcal{O}(1)$.
- For $g = 10$, the Mukai model is $(G, E) = (G_2/P, \mathcal{O}(1)^{\oplus 3})$, where G is the 5-dimensional quotient of the simply-connected semi-simple algebraic group of type G_2 by a maximal parabolic subgroup P and $\mathcal{O}(1)$ is the line bundle associated to the adjoint representation of G_2; in other words, $G = G_2/P \hookrightarrow \mathbf{P}(\mathfrak{g}_2^\vee)$. Again, we can conclude by the very ampleness of $\mathcal{O}(1)$.
- For $g = 12$, we use a slight variant of the above argument. Indeed, the general K3 surface of genus 12 can be constructed as an anti-canonical section in a smooth prime Fano threefold X of genus 12 (*cf.* [4], [22, Section 3.1]). The Fano threefold X has very ample anti-canonical bundle, and $H^3(X, \mathbf{Q}) = 0$

([23, Corollary 4.3.5]) so that X has trivial Chow groups[10] (this Fano threefold X is the variety denoted by $X_{22} \subset \mathbf{P}^{13}$ in [23, Propositions 4.1.11 and 4.1.12]; actually X is an intersection of quadrics). We now consider a variant of Theorem 5.8, replacing G by X and E by $-K_X$. The very ampleness of $-K_X$ ensures that condition $(\star_2)$ holds. As X has trivial Chow groups, there is a Chow–Künneth formula for products of X, and so one is reduced to the statement for the K3 surface S_b, which is [38]. □

Remark 5.9 *(Limit of our method).* Given a Mukai model (G, E),

- the global generation of E corresponds to condition $(\star_1)$, which essentially explains the reason why one can prove the generalized Franchetta conjecture for K3 surfaces with a Mukai model in [38].
- For K3 surfaces of genus 2, $G = \mathbf{P}(1,1,1,3)$ and $E = \mathcal{O}(6)$, the condition $(\star_4)$ is not satisfied: it is violated by three distinct points lying on the same ruling, away from the singular point.
- For the quartic K3 surfaces, $G = \mathbf{P}^3$ and $E = \mathcal{O}(4)$, the condition $(\star_6)$ is not satisfied: it is violated by six collinear distinct points. Similarly, for the other two families of complete intersection K3 surfaces (genus 4 and 5), $(\star_4)$ is violated by four collinear distinct points.
- For K3 surfaces of genus 6 and 8, whose Mukai model is $(G, E) = (\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2))$ and $(\mathrm{Gr}(2,6), \mathcal{O}(1)^{\oplus 6})$ respectively, the condition $(\star_3)$ is not satisfied. Indeed, it is equivalent to the surjectivity of $H^0(G, \mathcal{O}(1)) \to \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2} \oplus \mathbf{C}_{x_3}$, which is violated by three distinct collinear points of G.
- For K3 surfaces of genus 13 and 20, the Mukai models are respectively

$$(G, E) = \left(\mathrm{Gr}(3,7), (\wedge^2 S^\vee)^{\oplus 2} \oplus \wedge^3 Q\right) \text{ and } \left(\mathrm{Gr}(4,9), (\wedge^2 S^\vee)^{\oplus 3}\right).$$

  where S is the tautological subbundle and Q is the tautological quotient bundle. We claim that none of them verifies the condition $(\star_2)$. For example, in the genus 13 case, the condition $(\star_2)$ is equivalent to the surjectivities of the following two evaluation maps

$$H^0(G, \wedge^2 S^\vee) \to \wedge^2 S_x^\vee \oplus \wedge^2 S_y^\vee,$$
$$H^0(G, \wedge^2 Q) \to \wedge^2 Q_x \oplus \wedge^2 Q_y,$$

  for any $x \neq y \in G$, which, by Bott theorem, amount to say that for any two different 3-dimensional subspaces W_1, W_2 in a 7-dimensional vector space V, the natural maps

$$\wedge^2 V^\vee \to \wedge^2(W_1^\vee) \oplus \wedge^2(W_2^\vee)$$
$$\wedge^2 V \to \wedge^2(V/W_1) \oplus \wedge^2(V/W_2)$$

  are surjective. It is not true when $\dim W_1 \cap W_2 \geq 2$. The case of genus 20 is similar.
- For K3 surfaces of genus 18, the Mukai model is $(G, E) = (\mathrm{OGr}(3,9), U^{\oplus 5})$, where U is the rank 2 vector bundle associated to the representation V associated to the fourth dominant weight $\omega_4 = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4)$, of the (semi-simple part of the) maximal parabolic group P. We claim that $(\star_2)$ does not hold, *i.e.*, there exist two different points $x, y \in G$ such that $H^0(G, U) \to U_x \oplus U_y$ is not surjective.[11] Let $x = P/P$ and $y = wP/P$ where $w = s_{\alpha_3}$, as an element in the Weyl group W, is the reflection with respect to the third simple root. Clearly, w does not belong to the Weyl group of P, which is generated by $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_4}$. A direct computation shows that the representation $H^0(G, U)$ has multiplicity one for all weights. On the other hand, ω_4 is a common weight for V and its conjugate by w (since $w.\omega_4 = \omega_4$). Hence $H^0(G, U) \to U_x \oplus U_y$ cannot be surjective.

[10] Following Voisin [48], we say a smooth projective variety has *trivial Chow groups* if the cycle map $\mathrm{cl}^i : \mathrm{CH}^i(X)_{\mathbf{Q}} \to H^{2i}(X, \mathbf{Q})$ is injective for any i.

[11] We thank Nicolas Ressayre for his kind help on the proof.

- If one wants to follow the same strategy of this paper to establish the Franchetta property for (Hilbert) powers beyond the range stated in Theorem 1.4 and Theorem 1.5, one has to deal with some essentially new universal cycles, which may not belong to the tautological ring, or rather, the tautological ring should be enlarged to include some more incidence classes from projective geometry than just the polarization class.

5.4. Applications towards the Beauville–Voisin conjecture

Let us now turn to the consequences of our results in the direction of the Beauville–Voisin conjecture (and its refined version Conjecture 2.4):

Proof of Corollaries 1.6 and 1.7. The strongly coisotropic subvarieties E_μ, and the Lagrangian surfaces T and U, can all be defined over (suitable relative powers of) the universal family, and so these are just special cases of Proposition 2.5, combined with Theorems 1.4 and 1.5. □

One can also prove a version of Corollaries 1.6 and 1.7 for product varieties of arbitrarily high dimension, but the statement is now restricted to 0-cycles and 1-cycles:

Corollary 5.10. *Let X be a product*

$$X = X_1 \times X_2 \times \cdots \times X_s, \quad \dim X = 2m,$$

where X_j is a Hilbert scheme $S^{[r]}$ with S a K3 surface. Let $\widetilde{R}^*(X) \subset \mathrm{CH}^*(X)$ denote[12] *the $\mathbf{Q}$-subalgebra generated by (pullbacks of) divisors on X_j, the Chern classes $c_i(T_{X_j})$, plus the following coisotropic subvarieties:*

- *the strongly coisotropic subvarieties E_μ of [50, 4.1 item 1)];*
- *the Lagrangian surfaces $T \subset X_j$ constructed in [22, Proposition 4] (if $X_j = S^{[2]}$ and S is of genus $g \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$);*
- *the surface of bitangents $U \subset X_j$ (if $X_j = S^{[2]}$ and S is a quartic K3 surface).*

Then $\widetilde{R}^{2m}(X)$ and $\widetilde{R}^{2m-1}(X)$ inject into cohomology via the cycle class map.

Proof of Corollary 5.10. This uses the fact that the X_j have a *multiplicative Chow–Künneth decomposition* $\{\pi^k_{X_j}\}$, in the sense of [40, Chapter 8], [41]; see also Appendix A.2. This induces a bigrading of the Chow ring of X_j, given by

$$\mathrm{CH}^i(X_j)_{(k)} := (\pi^{2i-k}_{X_j})_* \mathrm{CH}^i(X_j).$$

It is readily seen that the projectors $\pi^k_{X_j}$ are universally defined (*i.e.*, they exist as a relative cycle for the family $\mathcal{X}^\circ_j \times_{B^\circ} \mathcal{X}^\circ_j$). Theorem 5.8 applied to the relative cycle $\mathcal{T} - (\pi^2_{\mathcal{X}_j})_*(\mathcal{T})$ (where we use the formalism of relative correspondences as in [35, Section 8.1]), thus implies that

$$T \in \mathrm{CH}^2(X_j)_{(0)}.$$

Similarly, we find that $U \in \mathrm{CH}^2(X_j)_{(0)}$. The fact that E_μ belongs to $\mathrm{CH}^*(X_j)_{(0)}$ is true for Hilbert schemes of arbitrary K3 surfaces, *cf.* [50, Lemma 4.3].

[12] In this paper, the notation $R^*(X)$ is reserved for the *tautological ring* of a power of K3 surface, see Definition 5.4.

The product X also has a multiplicative Chow–Künneth decomposition, and hence there is a bigrading of the Chow ring $\mathrm{CH}^*(X)$ by [40, Theorem 8.6]. Since divisors and Chern classes of X_j are also in $\mathrm{CH}(X_j)_{(0)}$, and pullback under any projection $X \to X_j$ preserves the bigrading [41, Corollary 1.6], we see that there is an inclusion

$$\widetilde{R}^*(X) \ \subset \mathrm{CH}^*(X)_{(0)}.$$

The corollary now follows, since it is known that $\mathrm{CH}^i(X)_{(0)}$ injects into cohomology for $i \geq \dim(X) - 1$, see [44, Introduction]. □

5.5. Double EPW sextics

The interested reader will have no trouble finding further applications in the flavor of Corollaries 1.6 and 1.7. For instance, consider the Hilbert square $X = S^{[2]}$, where S is a general K3 surface of genus 6. As shown by O'Grady [36, Section 4], X is isomorphic to a small resolution X_A^ϵ of a singular double EPW sextic X_A (notation is as in [36]). Let $\epsilon\colon X \to X_A$ denote the small resolution, and let $f_A\colon X_A \to Y_A$ denote the double cover to the associated EPW sextic Y_A. The surface S being general corresponds to the fact that the Lagrangian vector space A is general (in the precise sense given in [36, §4]) in the divisor $\Delta \subset \mathrm{LGr}(\wedge^3 V)$ studied in [36]. This construction produces Lagrangian surfaces in X: the surface

$$P := \epsilon^{-1}(\mathrm{Sing}(X_A))$$

(which is isomorphic to $\mathbf{P}^2$ since X_A has only one singular point), and the surface

$$\mathrm{Fix} := \epsilon^{-1}(\mathrm{Fix}(\iota)),$$

where $\mathrm{Fix}(\iota)$ denotes the fixed point locus of the (anti-symplectic) covering involution ι of X_A.

These Lagrangian surfaces are easily seen to be universally defined. Indeed, as shown in [36], there is a stratification

$$Y_A[3] \subset Y_A[2] \subset Y_A[1] = Y_A$$

of the EPW sextic Y_A. Here the surface $Y_A[2]$ is the singular locus of Y_A and the point $Y_A[3]$ is the unique singularity of $Y_A[2]$. One has

$$\mathrm{Fix} = (f_A \circ \epsilon)^{-1}(Y_A[2]) \ \text{ and } \ P = (f_A \circ \epsilon)^{-1}(Y_A[3]).$$

On the other hand (as explained in [36, Section 3]), there exist family versions $\mathcal{Y}[i]$ of the subvarieties $Y_A[i]$ over the base Δ. One can perform a base change

$$\begin{array}{ccc} \overline{\mathcal{X}}_{B^\circ} & \to & \overline{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{Y}_{B^\circ} & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ B^\circ & \to & \Delta \end{array}$$

where $B^\circ \subset B$ is an open such that the rational map $B \dashrightarrow \Delta$ of [36, Section 4] is defined, and $\overline{\mathcal{X}}$ is the tautological family of singular double EPW sextics over Δ. One obtains relative versions of P and of Fix by pulling back $\mathcal{Y}[i]$ under the birational morphism $\mathcal{X}_{B^\circ} \to \overline{\mathcal{X}}_{B^\circ}$.

Thus, applying Theorem 1.5 one obtains the following:

Corollary 5.11. *Let $X = S^{[2]}$, where S is a general K3 surface of genus 6. The $\mathbf{Q}$-subalgebra*

$$< D_1, D_2, c_i(T_X), P, Fix, T > \quad \subset \ \mathrm{CH}^*(X)$$

injects into cohomology via the cycle class map. (Here D_1, D_2 are two divisors generating the Picard group of X, and T is the Lagrangian surface of [22, Proposition 4].)

5.6. An application to Bloch's conjecture

Given a quartic K3 surface S, Beauville [3] constructed an interesting involution ι on $X := S^{[2]}$, which, generically, sends $\{x_1, x_2\}$ to $\{x_3, x_4\}$, where $x_1, \ldots, x_4$ are the four intersection points of the line $\overline{x_1, x_2}$ with S. The involution ι is anti-symplectic. According to the generalized Bloch conjecture (*cf.* [46, §11.2]), which roughly says that CH_0 is "controlled" by the holomorphic forms, the action of ι on $\mathrm{CH}_0(X)$ should be the identity on $\mathrm{Gr}^0_F \mathrm{CH}_0(X)$ and on $\mathrm{Gr}^4_F \mathrm{CH}_0(X)$ (just as on $H^0(X)$ and $H^{4,0}(X)$) and should be $-\mathrm{id}$ on $\mathrm{Gr}^2_F \mathrm{CH}_0(X)$ (just as on $H^{2,0}(X)$), where $F^\cdot$ is the conjectural Bloch–Beilinson filtration. On the other hand, as conjectured in [5] by Beauville and worked out by Shen–Vial in [40] in the case of Hilbert squares of K3 surfaces, we have a canonical splitting of this filtration for X, giving a direct sum decomposition:

$$\mathrm{CH}^4(X) = \mathrm{CH}^4(X)_{(0)} \oplus \mathrm{CH}^4(X)_{(2)} \oplus \mathrm{CH}^4(X)_{(4)}.$$

Hence the action of ι on the three summands should be id, $-\mathrm{id}$ and id, respectively. Our results allow us to confirm this expectation.

Proof of Corollary 1.8. Let $\mathcal{S}^\circ \to B^\circ$ be the universal family of smooth quartic K3 surfaces and $\mathcal{X}^\circ \to B^\circ$ be the relative Hilbert square. As noted above, the bigrading $\mathrm{CH}^*(X)_{(*)}$ is induced by a self-dual multiplicative Chow–Künneth decomposition $\{\pi^k_X\}$ that is universally defined. The anti-symplectic involution ι can also be defined on the level of the universal family; let us denote $\Gamma_\iota \in \mathrm{CH}^4(\mathcal{X}^\circ \times_{B^\circ} \mathcal{X}^\circ)$ the graph of the involution $\iota\colon \mathcal{X}^\circ \to \mathcal{X}^\circ$.

The relative correspondence

$$\pi^i_{\mathcal{X}} \circ \Gamma_\iota \circ \pi^j_{\mathcal{X}} \quad \in \ \mathrm{CH}^4(\mathcal{X}^\circ \times_{B^\circ} \mathcal{X}^\circ)$$

is fiberwise homologically trivial for $i \neq j$. Theorem 1.5 (*ii*) for $\mathrm{Hilb}^2_B \mathcal{S} \times_B \mathrm{Hilb}^2_B \mathcal{S}$ implies that

$$\left(\pi^i_{\mathcal{X}} \circ \Gamma_\iota \circ \pi^j_{\mathcal{X}}\right)|_{X_b \times X_b} = 0 \quad \text{in } \mathrm{CH}^4(X_b \times X_b), \quad \forall i \neq j \ \ \forall b \in B^\circ \ , \tag{10}$$

i.e., Γ_{ι_b} belongs to $\mathrm{CH}^4(X_b \times X_b)_{(0)}$, and thus ι_b preserves the bigrading $\mathrm{CH}^*(X_b)_{(*)}$.

Next, the fact that ι_b is anti-symplectic means that for any $b \in B^\circ$ there exists a divisor $D_b \subset X_b$, and a cycle γ_b supported on $D_b \times D_b$, such that

$$\left((\Delta_{\mathcal{X}} + \Gamma_\iota) \circ \pi^2_{\mathcal{X}}\right)|_{X_b \times X_b} = \gamma_b \quad \text{in } H^8(X_b \times X_b).$$

Using a Hilbert schemes argument as in [48, Proposition 3.7], the D_b and γ_b can be spread out, *i.e.*, there exist a divisor $\mathcal{D} \subset \mathcal{X}$ and a relative cycle γ supported on $\mathcal{D} \times_{B^\circ} \mathcal{D}$ such that

$$\left((\Delta_{\mathcal{X}} + \Gamma_\iota) \circ \pi^2_{\mathcal{X}} - \gamma\right)|_{X_b \times X_b} = 0 \quad \text{in } H^8(X_b \times X_b) \quad \forall b \in B^\circ.$$

Applying Theorem 1.5 once more, we find that

$$\Big((\Delta_{\mathcal{X}} + \Gamma_\iota) \circ \pi^2_{\mathcal{X}} - \gamma\Big)|_{X_b \times X_b} = 0 \quad \text{in } \mathrm{CH}^4(X_b \times X_b) \quad \forall b \in B^\circ. \tag{11}$$

For general $b \in B^\circ$, the restriction $\gamma|_{X_b\times X_b}$ will be supported on (divisor)×(divisor), and so $\gamma|_{X_b\times X_b}$ will act as 0 on $\mathrm{CH}^2(X_b)_{(2)}$. It follows that

$$(\iota_b)^* = -\,\mathrm{id}\colon \quad \mathrm{CH}^2(X_b)_{(2)} \ \to\ \mathrm{CH}^2(X_b)_{(2)} \quad \text{for general } b \in B^\circ.$$

To extend this to *all* $b \in B^\circ$, one notes that the above construction can be done with a divisor $\mathcal{D} \subset \mathcal{X}$ in general position with respect to X_b.

The statement for $\mathrm{CH}^4(X_b)_{(2)}$ follows upon taking the transpose of relation (11), and using the relation (10). The remaining statements are proven similarly. □

Remark 5.12. Corollary 1.8 was proven in a more convoluted way in [26].

6. Lehn–Lehn–Sorger–van Straten hyper-Kähler eightfolds

In this section we first show Theorem 1.10 and then deduce from it Theorem 1.11.

Keep the same notation as in §3. We still have a correspondence:

$$\begin{array}{ccc} & \mathcal{F}\times_B\mathcal{F} & \\ {}^{\pi_2:=(\pi,\pi)}\swarrow & & \searrow^{q:=(p,p)} \\ B & & G\times G \end{array}$$

However the problem is that q is no longer a projective bundle: the fiber of q over a pair of lines (l,l') is the subspace of cubic fourfolds containing both l and l', whose dimension depends therefore on the relative position of (l,l'). To adapt the same strategy to this case, we use similar techniques as in [48], [17] by studying the various strata of the morphism q. There are three possible relative positions between two projective lines in $\mathbf{P}^5$: identical, intersecting but not identical, not intersecting.

On the one hand, for a (general) cubic fourfold X with Fano variety of lines F, let

$$I := \{(l,l') \in F\times F \ \mid\ l\cap l' \neq \varnothing\}$$

be the 6-dimensional incidence subvariety of $F\times F$. The incidence subvariety I has two natural projections to F with fiber over $l\in F$ the surface S_l parameterizing lines inside X meeting l. Similarly, we consider the family version of this incidence subvariety inside $\mathcal{F}\times_B\mathcal{F}$:

$$\mathcal{I} := \{(b,l,l') \in \mathcal{F}\times_B\mathcal{F} \ \mid\ l\cap l'\neq\varnothing\} = \{(b,l,l')\in B\times G\times G \ \mid\ l,l'\subset X_b \ \ ; l\cap l'\neq\varnothing\}\,.$$

On the other hand, we define $J := \{(l,l')\in G\times G \ \mid\ l\cap l'\neq\varnothing\}$ to be the incidence subvariety of $G\times G$.

These incidence subvarieties, together with the diagonals, give the stratification:

$$\begin{array}{ccccccc} \mathcal{F} = \Delta_{\mathcal{F}/B} & \hookrightarrow & \mathcal{I} & \hookrightarrow & \mathcal{F}\times_B\mathcal{F} & \xrightarrow{\pi_2} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q|_{\mathcal{I}}} & & \downarrow{\scriptstyle q} & & \\ G = \Delta_G & \hookrightarrow & J & \hookrightarrow & G\times G & & \end{array}$$

where q is a projective bundle outside of $\mathcal{I}$ and $q|_{\mathcal{I}}$ is also a projective bundle outside of $\Delta_{\mathcal{F}}$; in other words, q is a *stratified projective bundle* in the sense of Definition 5.1.

Let B° be the Zariski open subset of B parameterizing smooth cubic fourfolds. Applying Proposition 5.2 to q, we have the following analogue of Lemma 3.1 and Proposition 4.1 in our case:

Proposition 6.1. *For any $b \in B^\circ$, we have*

$$\begin{aligned} &\operatorname{Im}\left(\mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}^*(F_b \times F_b)\right) \\ = &\operatorname{Im}\left(\mathrm{CH}^*(G \times G) \to \mathrm{CH}^*(F_b \times F_b)\right) + i_* \operatorname{Im}\left(\mathrm{CH}^*(J) \to \mathrm{CH}^*(I_b)\right) + \Delta_* \operatorname{Im}\left(\mathrm{CH}^*(G) \to \mathrm{CH}^*(F_b)\right), \end{aligned}$$

where $i : I_b \hookrightarrow F_b \times F_b$ and $\Delta : F_b \hookrightarrow F_b \times F_b$ are the inclusions.

As the incidence subvariety J is singular along the smaller stratum Δ_G, it is more convenient to work with a natural resolution of singularities. To this end, we define

$$\begin{aligned} \widetilde{\mathcal{I}} &:= \left\{(b, x, l, l') \in B \times \mathbf{P}^5 \times G \times G \;\mid\; l, l' \subset X_b \;;\; x \in l \cap l'\right\} ; \\ \widetilde{J} &:= \left\{(x, l, l') \in \mathbf{P}^5 \times G \times G \;\mid\; x \in l \cap l'\right\} ; \\ \mathcal{P} &:= \left\{(b, x, l) \in B \times \mathbf{P}^5 \times G \;\mid\; l \subset X_b \;;\; x \in l\right\} ; \\ Q &:= \left\{(x, l) \in \mathbf{P}^5 \times G \;\mid\; x \in l\right\}, \end{aligned}$$

where $\widetilde{\mathcal{I}}$ (*resp.* $\widetilde{J}$) admits a natural birational morphism to $\mathcal{I}$ (*resp.* J), which contracts $\mathcal{P}$ (*resp.* Q) to $\mathcal{F}$ (*resp.* G). We summarize the situation in the following diagram whose squares are all cartesian:

$$\begin{array}{ccccccccc} \mathcal{F} & \twoheadleftarrow & \mathcal{P} & \hookrightarrow & \widetilde{\mathcal{I}} & \longrightarrow & \mathcal{I} & \hookrightarrow & \mathcal{F} \times_B \mathcal{F} \\ \downarrow p & \square & \downarrow q'|_{\mathcal{P}} & \square & \downarrow q' & \square & \downarrow q|_{\mathcal{I}} & \square & \downarrow q \\ G & \twoheadleftarrow & Q & \hookrightarrow & \widetilde{J} & \longrightarrow & J & \hookrightarrow & G \times G \end{array}$$

Recall that $G = \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$, S is the tautological rank-2 subbundle, $g := c_1(S^\vee|_F) \in \mathrm{CH}^1(F)$ is the Plücker polarization class, and $c := c_2(S^\vee|_F) \in \mathrm{CH}^2(F)$. We computed in Lemma 3.2 that $c_2(F) = 5g^2 - 8c$. In $\mathrm{CH}^*(F \times F)$, $g_i := \mathrm{pr}_i^*(g)$ and $c_i := \mathrm{pr}_i^*(c)$ for $i = 1, 2$.

Definition 6.2 *(Tautological ring of $F \times F$).* Let X be a smooth cubic fourfold and F be its Fano variety of lines. We define the *tautological ring* of $F \times F$, denoted by $R^*(F \times F)$, to be the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(F \times F)$ generated by the classes $c_1, c_2, g_1, g_2, \Delta, I$, where Δ and I are the classes in $\mathrm{CH}^*(F \times F)$ of the diagonal Δ_F and the incidence subvariety I respectively.

Proposition 6.3. *For any point $b \in B^\circ$, we have*

$$\operatorname{Im}\left(\mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}^*(F_b \times F_b)\right) = R^*(F_b \times F_b).$$

Proof. To simplify the notation, let us leave out the subscript b. Thanks to Proposition 6.1, we only need to deal with the following three cases:

- For $\operatorname{Im}\left(\mathrm{CH}^*(G \times G) \to \mathrm{CH}^*(F \times F)\right)$, it is enough to observe that $\mathrm{CH}^*(G \times G)$ satisfies the Künneth formula (since the cycle class map $\mathrm{CH}^*(G \times G) \to H^*(G \times G, \mathbf{Q})$ is an isomorphism).

- For $i_* \operatorname{Im}(\mathrm{CH}^*(J) \to \mathrm{CH}^*(I))$, consider

$$\widetilde{I} := \{(x, l, l') \in X \times F \times F \ \mid \ x \in l \cap l'\} \quad \text{and}$$
$$\widetilde{J} := \left\{(x, l, l') \in \mathbf{P}^5 \times G \times G \ \mid \ x \in l \cap l'\right\}$$

fitting into the diagram

$$\begin{array}{ccccc}
\widetilde{I} & \xrightarrow{\tau'} & I & \overset{i}{\hookrightarrow} & F \times F \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\widetilde{J} & \xrightarrow{\tau} & J & \overset{j}{\hookrightarrow} & G \times G \\
\downarrow{\scriptstyle \pi} & & & & \\
\mathbf{P}^5 & & & &
\end{array}$$

Denote by $\widetilde{i} = \tau' \circ i$ and $\widetilde{j} = \tau \circ j$. Then any cycle in J can be written as $\tau_*(\alpha)$ for some $\alpha \in \mathrm{CH}^*(\widetilde{J})$. Observe that $\widetilde{J}$ is a $\mathbf{P}^4 \times \mathbf{P}^4$-bundle over $\mathbf{P}^5$ such that the two relative $\mathcal{O}(1)$ on the fibers are given by $\widetilde{j}^*(g_1)$ and $\widetilde{j}^*(g_2)$, respectively. Therefore α is a linear combination of cycles of the form $\pi^*(h^k)\widetilde{j}^*(g_1^l g_2^m)$ where $k, l, m \in \mathbf{N}$ and $h = \mathcal{O}_{\mathbf{P}^5}(1)$. We have

$$\begin{aligned}
& i_*(\tau_*(\pi^*(h^k)\widetilde{j}^*(g_1^l g_2^m))|_I) \\
&= i_* \circ \tau'_* \left(\pi^*(h^k)\widetilde{j}^*(g_1^l g_2^m)|_{\widetilde{I}}\right) \\
&= \widetilde{i}_* \left(\pi^*(h^k)|_{\widetilde{I}} \cdot \widetilde{i}^*(g_1^l g_2^m)\right) \\
&= g_1^l g_2^m \cdot i_*(\tau_* \pi^*(h^k)|_I) \\
&= g_1^l g_2^m \cdot \Gamma_{h^k},
\end{aligned}$$

where Γ_{h^k}, defined in [40, Appendix A], is the cycle of $F \times F$ represented by the subvariety

$$\{(l, l') \in F \times F \ \mid \ \exists x \in H_1 \cap \cdots \cap H_k \text{ such that } x \in l \cap l'\},$$

where $H_1, \cdots, H_k$ are k general hyperplanes in $\mathbf{P}^5$. It is proven in [40, Appendix A] that when $k \geq 1$, Γ_{h^k} is actually a polynomial in c_1, c_2, g_1, g_2, while $\Gamma_{h^0} = I$.

- For $\Delta_* \operatorname{Im}(\mathrm{CH}^*(G) \to \mathrm{CH}^*(F))$, let us remark that for any $\alpha \in \mathrm{CH}^*(F)$, we have $\Delta_*(\alpha) = \Delta \cdot \mathrm{pr}_1^*(\alpha)$. Thus it suffices to recall that $\operatorname{Im}(\mathrm{CH}^*(G) \to \mathrm{CH}^*(F))$ is generated by g and c. □

Consequently, in order to prove Theorem 1.10, we need to study the injectivity of the cycle class map restricted to the tautological ring $R^*(F \times F)$.

Proposition 6.4. *Let X be a smooth cubic fourfold and let F be its Fano variety of lines. Then the cycle class map restricted to the tautological ring $R^*(F \times F)$ is injective.*

Proof. It suffices to show the proposition for general cubic fourfolds, in which case

$$\mathrm{cl} : R^*(F \times F) \to Hdg^{2*}(F \times F)_{\mathbf{Q}}$$

is surjective. Let us show it is injective.

First it is not hard to count the dimensions of the spaces of Hodge classes:

i	0	1	2	3	4	5	6	7	8
$\dim Hdg^{2i}$	1	2	6	8	12	8	6	2	1

It is enough to show that the $R^i(F \times F)$ have the same dimensions.

The following relations in $R^*(F \times F)$ are at our disposal.

(i) $g_1 \cdot \Delta = g_2 \cdot \Delta\,; c_1 \cdot \Delta = c_2 \cdot \Delta$.

(ii) For $i = 1, 2$, we have $12 g_i c_i = 5 g_i^3\,; 4\,c_i^2 = g_i^4$.

(iii) Voisin's relation [47][13]:

$$I^2 = 2\Delta + I \cdot (g_1^2 + g_1 g_2 + g_2^2) + \Gamma_2(g_1, g_2, c_1, c_2),$$

where Γ_2 is a polynomial of weighted degree 4.

(iv) In [40, Proposition 17.5], one finds

$$\Delta \cdot I = 6c_1\Delta - 3g_1^2\Delta.$$

(v) In [40, Lemma 17.6], there is a polynomial P of weighted degree 4 such that

$$c_1 \cdot I = P(g_1, g_2, c_1, c_2)\,;$$
$$c_2 \cdot I = P(g_2, g_1, c_2, c_1).$$

Using these relations, we easily get for each degree a list of generators (as vector-spaces):

- $R^0 = \langle \mathbb{1} \rangle$;
- $R^1 = \langle g_1, g_2 \rangle$;
- $R^2 = \langle g_1^2, g_1 g_2, g_2^2, c_1, c_2, I \rangle$;
- $R^3 = \langle g_1^3, g_1^2 g_2, g_1 g_2^2, g_2^3, g_1 c_2, g_2 c_1, g_1 I, g_2 I \rangle$;
- $R^4 = \langle g_1^4, g_1^3 g_2, g_1^2 g_2^2, g_1 g_2^3, g_2^4, g_1^2 c_2, g_2^2 c_1, c_1 c_2, g_1^2 I, g_2^2 I, g_1 g_2 I, \Delta \rangle$;
- $R^5 = \langle g_1^4 g_2, g_1^3 g_2^2, g_1^2 g_2^3, g_1 g_2^4, g_1^3 c_2, g_2^3 c_1, g_1^2 g_2 I, g_1 g_2^2 I, g_1 \Delta \rangle$;
- $R^6 = \langle g_1^4 g_2^2, g_1^3 g_2^3, g_1^2 g_2^4, g_1^4 c_2, g_2^4 c_1, g_1^2 g_2^2 I, g_1^2 \Delta \rangle$;
- $R^7 = \langle g_1^4 g_2^3, g_1^3 g_2^4 \rangle$;
- $R^8 = \langle g_1^4 g_2^4 \rangle$.

Observe that we have the same number of generators as the dimension of Hdg^{2i} for $i \neq 5$ or 6. Therefore the cycle class map $R^i(F \times F) \to H^{2i}(F \times F, \mathbf{Q})$ is injective for $i = 0, 1, 2, 3, 4, 7, 8$.

(vi) As for $i = 5$ (*resp.* $i = 6$), we use the following (new) tautological relation established in the Appendix Theorem A.1:

$$6\Delta_*(g) + g_1 g_2 (g_1 + g_2) \cdot I = Q(g_1, g_2, c_1, c_2),$$

where Q is a polynomial.

Therefore the generator $g_1 \Delta = \Delta_*(g)$ (*resp.* $g_1^2 \Delta$) is redundant, hence $R^i(F \times F) \to H^{2i}(F \times F)$ is also injective in these two degrees. □

[13] The coefficients are made precise by [40, Proposition 17.4].

Remark 6.5. As a manifestation of the same principle as in §5.3, the extra difficulty encountered here (excess dimension of I, the new tautological relation *etc.*) can be traced back to the lack of positivity of the vector bundle $E = \mathrm{Sym}^3 S^\vee$ on $G = \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5)$, namely it satisfies only $(\star_1)$ but not $(\star_2)$, where S is the tautological subbundle on G.

We can now easily conclude the proof of Theorem 1.10:

Proof of Theorem 1.10. As the standard conjecture is proven for F_b in [10] (this can also be seen more elementarily by noting that the incidence correspondence I induces an isomorphism from $H^6(F_b, \mathbf{Q})$ to $H^2(F_b, \mathbf{Q})$), numerical equivalence coincides with homological equivalence on powers of F_b. Since the moduli stack $\mathcal{C}$ is dominated by the parameter space B° of smooth cubic fourfolds, by Remark 2.6, we only need to show the conclusion for the family $\mathcal{F}^\circ \times_{B^\circ} \mathcal{F}^\circ \to B^\circ$. Since any cycle of $\mathcal{F}^\circ \times_{B^\circ} \mathcal{F}^\circ$ is the restriction of a cycle of $\mathcal{F} \times_B \mathcal{F}$, it is enough to show that for any $b \in B^\circ$, the restriction of a cycle $\gamma \in \mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F})$ to $F_b \times F_b$ is zero if and only if it is homologically trivial, which is proven by combining Proposition 6.3 and Proposition 6.4. □

With Theorem 1.10 proven, we proceed to study the 0-cycles and codimension-2 cycles of the LLSvS hyper-Kähler eightfolds. The key input is Voisin's degree 6 dominant rational map [50, Proposition 4.8]

$$F \times F \dashrightarrow Z.$$

Let $B^{\circ\circ}$ be the Zariski open subset of B parameterizing smooth cubic fourfolds not containing a plane. Consider the family version of Voisin's construction (over $B^{\circ\circ}$): $\psi : \mathcal{F}^{\circ\circ} \times_{B^{\circ\circ}} \mathcal{F}^{\circ\circ} \dashrightarrow \mathcal{Z}$.

Proof of Theorem 1.11. Take a resolution of indeterminacies:

$$\begin{array}{ccc} \mathcal{F}^{\circ\circ}\,\widetilde{\times_{B^{\circ\circ}}}\,\mathcal{F}^{\circ\circ} & & \\ \downarrow \tau & \overset{f}{\searrow} & \\ \mathcal{F}^{\circ\circ}\times_{B^{\circ\circ}}\mathcal{F}^{\circ\circ} & \underset{\psi}{\dashrightarrow} & \mathcal{Z}. \end{array}$$

For (i), let $\gamma \in \mathrm{CH}^8(\mathcal{Z})$ be a relative 0-cycle whose degree on fibers is zero. Then, for any $b \in B^{\circ\circ}$,

$$(\tau_* f^*(\gamma))\,|_{F_b \times F_b} = \tau_{b*}\left(f^*(\gamma)|_{\widetilde{F_b \times F_b}}\right) = \tau_{b*} f_b^*\left(\gamma|_{Z_b}\right).$$

Thus $\tau_* f^*(\gamma)$ is a relative 0-cycle of fiber degree zero on $\mathcal{F}^{\circ\circ} \times_{B^{\circ\circ}} \mathcal{F}^{\circ\circ}$ and by Theorem 1.10, we know that

$$\tau_{b*} f_b^*\left(\gamma|_{Z_b}\right) = 0 \text{ in } \mathrm{CH}^8(F_b \times F_b).$$

For $b \in B^{\circ\circ}$ general, τ_b is birational hence induces an isomorphism on CH_0, hence $f_b^*\left(\gamma|_{Z_b}\right) = 0$. Moreover, since f_b is generically finite of degree 6 (still under the assumption that b is general), we have

$$\gamma|_{Z_b} = \frac{1}{6} f_{b*} f_b^*(\gamma|_{Z_b}) = 0.$$

A specialization argument shows that $\gamma|_{Z_b} = 0$ for all $b \in B^{\circ\circ}$.

As for (ii), *i.e.*, codimension-2 cycles: since $H^3(Z_b, \mathbf{Q}) = H^3(F_b \times F_b) = 0$, any cycle in $\mathrm{CH}^2(Z_b)$ or in $\mathrm{CH}^2(F_b \times F_b)$ is homologically trivial if and only if its Abel–Jacobi invariant vanishes. Now the same proof as in (i) works because the Abel–Jacobi kernel for codimension-2 cycles CH^2_{AJ}, just as CH_0, is a birational invariant (for smooth projective varieties), hence

$$\tau_{b*} : \mathrm{CH}^2(\widetilde{F_b \times F_b})_{hom} \to \mathrm{CH}^2(F_b \times F_b)_{hom}$$

is an isomorphism. □

Proof of Corollary 1.12. In view of Theorem 1.11, this is just a special case of Proposition 2.5. □

Remark 6.6. As above, let Y_b be a smooth cubic fourfold not containing a plane, and Z_b the associated LLSvS eightfold. Our argument to prove Theorem 1.11 breaks down for $\mathrm{CH}^j(Z_b)$ with $2 < j < 8$, because Voisin's map is not a morphism. It is known [34], [12] that the indeterminacy locus of Voisin's map is the incidence subvariety $I \subset F_b \times F_b$, and a resolution of indeterminacy is obtained by blowing up I. To extend Theorem 1.11 to the full Chow ring $\mathrm{CH}^*(Z_b)$, it remains to prove analogues of Propositions 6.3 and 6.4 for $\mathcal{I}$, the family of incidence varieties.

Appendix A. On a new tautological relation

Let X be a smooth cubic fourfold and F be its Fano variety of lines, which is a hyper-Kähler fourfold by [6]. In this appendix, we establish a new relation (Theorem A.1), up to rational equivalence, among 3-dimensional tautological cycle classes of $F \times F$. Some interesting applications of this tautological relation are also discussed. We try to keep the appendix as self-contained as possible.

Throughout this appendix, let us fix the following notation:

- $\mathbf{P}^5$ is the ambient space and X is a smooth cubic hypersurface in it.
- $h := c_1(\mathcal{O}_{\mathbf{P}^5}(1))$; $h|_X$ is still denoted by h.
- $G := \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^5) \simeq \mathrm{Gr}(2, 6)$ is the Grassmannian of projective lines in $\mathbf{P}^5$.
- $F := F(X)$ is the Fano variety of lines of X.
- S is the tautological subbundle on G.
- $g := c_1(S^\vee)$ is the Plücker polarization class; $g|_F$ is still denoted by g.
- $c := c_2(S)$; $c|_F$ is still denoted by c.
- $h_i := \mathrm{pr}_i^*(h)$, $g_i := \mathrm{pr}_i^*(g)$ and $c_i := \mathrm{pr}_i^*(c)$ where pr_i is the i-th projection.
- If $P := \mathbf{P}(S|_F)$ denotes the incidence variety in $F \times X$, then the natural projection $p : P \to F$ is the universal projective line and $q : P \to X$ is the evaluation map.
- $I \subset F \times F$ is the incidence subvariety parameterizing pairs of intersecting lines contained in X.
- $\widetilde{I} := P \times_X P$. Note that I is its image in $F \times F$ via the natural projection.

The main result of this appendix is the following.

Theorem A.1. *There exists a polynomial Q (of weighted degree 5) such that the following equality holds in $\mathrm{CH}^5(F \times F)$:*

$$6\Delta_*(g) + g_1 g_2(g_1 + g_2) \cdot I = Q(g_1, g_2, c_1, c_2), \tag{12}$$

where $\Delta : F \hookrightarrow F \times F$ is the diagonal embedding.

Remark A.2. The polynomial Q is not unique. A cohomological computation shows that

$$Q(g_1, g_2, c_1, c_2) = \frac{1}{4}(g_1^4 g_2 + g_1 g_2^4) + \frac{7}{12}(g_1^3 g_2^2 + g_1^2 g_2^3)$$

is one possible choice of Q.

A.1. Proof of the tautological relation

We have the following diagram

$$\begin{array}{ccc} \widetilde{I} & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_X} \\ P \times P & \xrightarrow{(q,q)} & X \times X \\ {\scriptstyle (p,p)}\big\downarrow & & \\ F \times F & & \end{array} \tag{13}$$

Let us first introduce some natural cycles on $F \times F$. For any $i \in \mathbf{N}$, define

$$\Gamma_{h^i} := (p,p)_*(q,q)^*(\Delta_{X*}(h^i)) \in \mathrm{CH}^{i+2}(F \times F).$$

Note that Γ_{h^0} is nothing but the incidence correspondence I. Geometrically, Γ_{h^i} is represented by the locus of pairs of lines contained in X intersecting at a point which lies on the intersection of i general hyperplane sections of X.

Lemma A.3. *For any $i > 0$, the cycle Γ_{h^i} is a polynomial of g_1, g_2, c_1, c_2. Precisely,*

$$\begin{aligned} \Gamma_h &= \frac{1}{18}(g_1^3 + 6g_1^2 g_2 + 6g_1 g_2^2 + g_2^3 - 6g_1 c_2 - 6g_2 c_1)\,; \\ \Gamma_{h^2} &= \frac{1}{18}(g_1^3 g_2 + 6g_1^2 g_2^2 + g_1 g_2^3 - 6g_1^2 c_2 - 6g_2^2 c_1 + 6c_1 c_2)\,; \\ \Gamma_{h^3} &= \frac{1}{18}(g_1^3 g_2^2 + g_1^2 g_2^3 - g_1^3 c_2 - g_2^3 c_1)\,; \\ \Gamma_{h^4} &= \frac{1}{108} g_1^3 g_2^3. \end{aligned}$$

Proof. A slightly more complicated (but equivalent) form of the first two formulas is proven in [40, Proposition A.6]. For the convenience of the reader, we give a complete proof here. The excess intersection formula [19, §6.3] applied to the following cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}^5 \\ {\scriptstyle \Delta_X}\big\downarrow & \square & \big\downarrow{\scriptstyle \Delta_{\mathbf{P}}} \\ X \times X & \longrightarrow & \mathbf{P}^5 \times \mathbf{P}^5 \end{array}$$

yields that, for any $i \in \mathbf{N}$, we have in $\mathrm{CH}^*(X \times X)$

$$3\Delta_{X*}(h^{i+1}) = \Delta_{\mathbf{P}*}(h^i)|_{X \times X}.$$

From $\Delta_{\mathbf{P}*}(h^i) = h_1^5 h_2^i + \cdots + h_1^i h_2^5$, we obtain

$$\Delta_{X*}(h^i) = \frac{1}{3}\left(h_1^4 h_2^i + \cdots + h_1^i h_2^4\right).$$

Therefore

$$\begin{aligned}\Gamma_{h^i} &= (p,p)_*(q,q)^*(\Delta_{X,*}(h^i)) \\ &= \frac{1}{3}(p,p)_*(q,q)^*\left(h_1^4 h_2^i + \cdots + h_1^i h_2^4\right) \\ &= \frac{1}{3}\left(f_4 \times f_i + \cdots + f_i \times f_4\right),\end{aligned}$$

where $f_j := p_* q^*(h^j)$ and where $\times$ is the exterior product $\mathrm{pr}_1^*(-) \cdot \mathrm{pr}_2^*(-)$. All the formulas in the statement then follow from the facts that $f_1 = 1$, $f_2 = g$, $f_3 = g^2 - c$ and $f_4 = \frac{1}{6}g^3$ (*cf.* [40, Lemma A.4], [47, Lemma 3.2] and [47, Lemma 3.5]). □

Define $I_0 := I \backslash \Delta_F$ to be the subvariety of $F \times F$ parameterizing pairs of distinct intersecting lines in X. We then have a natural morphism

$$q_0 : I_0 \to X$$

which sends two lines to their intersection point.

Lemma A.4. *The inclusion $I_0 \hookrightarrow F \times F \backslash \Delta_F$ is a local complete intersection and the Chern classes of the normal bundle $N := N_{I_0/F \times F \backslash \Delta_F}$ are given by*

$$\begin{aligned}&c_1(N) = (g_1 + g_2)|_{I_0} - q_0^*(h);\\ &c_2(N) = (g_1^2 + g_1 g_2 + g_2^2)|_{I_0} - 3(g_1 + g_2)|_{I_0} \cdot q_0^*(h) + 6q_0^*(h^2).\end{aligned}$$

Proof. Note that $\widetilde{I} \subset P \times P$ is a local complete intersection (since $\widetilde{I} \subset P \times P$ is obtained from the local complete intersection $\Delta_X \subset X \times X$ via base change) and that $\widetilde{I} \backslash \Delta_P \subset P \times P$ is a section of $P \times P \to F \times F$ over I_0. We apply [19, B.7.5] and see that $I_0 \subset F \times F \backslash \Delta_F$ is a local complete intersection. Using the section $\widetilde{I} \backslash \Delta_P$, we view I_0 as a subvariety of $P \times P$. Then we get the following short exact sequence

$$0 \to pr_1^* T_{P/F} \oplus pr_2^* T_{P/F} \to N_{I_0/P \times P} \to N_{I_0/F \times F} \to 0 .$$

Note that by construction, we have

$$N_{I_0/P \times P} = q_0^* T_X.$$

The Chern classes of $N_{I_0/F \times F}$ are computed as follows:

$$\begin{aligned}c(N) &= \frac{q_0^* c(T_X)}{pr_1^* c(T_{P/F}) \cdot pr_2^* c(T_{P/F})} \\ &= \frac{(1+h)^6}{(1+3h)(1+2q_0^* h - g_1|_{I_0})(1+2q_0^* h - g_2|_{I_0})}.\end{aligned}$$

The lemma follows from the expansion of the above equation. □

Remark A.5. The previous lemma implies that

$$I^2|_{F\times F\backslash\Delta_F} = I\cdot(g_1^2+g_1g_2+g_2^2) - 3(g_1+g_2)\Gamma_h + 6\Gamma_{h^2}.$$

Thus by Lemma A.3 there exists $\alpha\in\mathbf{Q}$ and a polynomial Γ_2 such that in $\mathrm{CH}^4(F\times F)$ we have

$$I^2 = \alpha\cdot\Delta_F + I\cdot(g_1^2+g_1g_2+g_2^2) + \Gamma_2(g_1,g_2,c_1,c_2),$$

for some $\alpha\in\mathbf{Q}$. This was proven by Voisin [47]. In fact, $\alpha=2$, as is computed in [40, Proposition 17.4].

Proof of Theorem A.1. Let us first prove the theorem for a general cubic fourfold X. Fix three general hyperplane sections H_1,H_2,H_3 of X. For $i=1,2,3$, let

$$Z_i := \{(l,l')\in F\times F \;\mid\; l\cap l'\cap H_1\cap\cdots\cap H_i\neq\varnothing\}.$$

On the one hand, as mentioned before, the class of Z_i in $\mathrm{CH}^{2+i}(F\times F)$ is equal to Γ_{h^i}; on the other hand, denoting $Z_i^o := Z_i\backslash\Delta_F$ the complement of the diagonal in Z_i, the class of Z_i^o in $\mathrm{CH}^i(I_0)$ is equal to $q_0^*(h^i)$ by definition. This yields the diagram

$$\begin{array}{ccccc} Z_3^o\subset Z_2^o\subset Z_1^o & \hookrightarrow & I_0 & \overset{\iota}{\hookrightarrow} & F\times F\backslash\Delta_F \\ & & \downarrow{\scriptstyle q_0} & & \\ & & X & & \end{array}$$

Denoting N the normal bundle of ι, we obtain

$$\begin{aligned} I\cdot\Gamma_h|_{F\times F\backslash\Delta_F} &= I_0\cdot\iota_*q_0^*(h) \\ &= \iota_*\left(q_0^*(h)\cdot c_2(N)\right) \\ &= \iota_*\left((g_1^2+g_1g_2+g_2^2)|_{I_0}\cdot q_0^*(h) - 3(g_1+g_2)|_{I_0}\cdot q_0^*(h^2) + 6q_0^*(h^3)\right) \\ &= \left((g_1^2+g_1g_2+g_2^2)\cdot Z_1 - 3(g_1+g_2)\cdot Z_2 + 6Z_3\right)|_{F\times F\backslash\Delta_F} \\ &= \left((g_1^2+g_1g_2+g_2^2)\cdot\Gamma_h - 3(g_1+g_2)\cdot\Gamma_{h^2} + 6\Gamma_{h^3}\right)|_{F\times F\backslash\Delta_F}, \end{aligned}$$

where the third equality uses Lemma A.4. By Lemma A.3, there exists a polynomial P_1 such that

$$I\cdot\Gamma_h|_{F\times F\backslash\Delta_F} = P_1(g_1,g_2,c_1,c_2)|_{F\times F\backslash\Delta_F}.$$

Here, more precisely, one can compute by Lemma A.3 and the relation $12gc=5g^3$ that

$$P_1(g_1,g_2,c_1,c_2) = \frac{5}{12}\left(g_1^4g_2 + 4g_1^3g_2^2 + 4g_1^2g_2^3 + g_1g_2^4 - 3g_1^3c_2 - 3g_2^3c_1\right).$$

By the localization short exact sequence of Chow groups, there exists an element $D\in\mathrm{CH}^1(F)$ such that in $\mathrm{CH}^5(F\times F)$ we have

$$I\cdot\Gamma_h + \Delta_*(D) = P_1(g_1,g_2,c_1,c_2).$$

Since X is assumed (for now) to be general, $\mathrm{CH}^1(F)$ is generated by g, hence $D=\lambda g$ for some $\lambda\in\mathbf{Q}$. This yields that in $\mathrm{CH}^5(F\times F)$ we have

$$I \cdot \Gamma_h + \lambda \Delta_*(g) = P_1(g_1, g_2, c_1, c_2). \tag{14}$$

However, we know that $I \cdot c_1$, $I \cdot c_2$, $I \cdot g_1^3$ and $I \cdot g_2^3$ are polynomials in g_1, g_2, c_1, c_2 by [40, Lemma 17.6] (*cf.* the known relations collected in the proof of Proposition 6.4). The first formula in Lemma A.3 then yields that

$$I \cdot \Gamma_h = \frac{1}{3} I \cdot \left(g_1^2 g_2 + g_1 g_2^2\right) + P_2(g_1, g_2, c_1, c_2) \tag{15}$$

for some polynomial P_2.

Putting (14) and (15) together, we know that there exists a polynomial Q such that the following equality holds in $\mathrm{CH}^5(F \times F)$:

$$3\lambda \cdot \Delta_*(g) + I \cdot \left(g_1^2 g_2 + g_1 g_2^2\right) = Q(g_1, g_2, c_1, c_2).$$

By considering the action of both sides on the cohomology, we easily see that $\lambda = 2$ and that

$$Q(g_1, g_2, c_1, c_2) = \frac{1}{4}(g_1^4 g_2 + g_1 g_2^4) + \frac{7}{12}(g_1^3 g_2^2 + g_1^2 g_2^3).$$

Therefore the desired relation is proven for a general cubic fourfold. As all the cycles appearing are universally defined in the universal Fano variety of lines, a specialization argument shows that this relation must also hold for any smooth cubic fourfold. □

A.2. Some applications to the Fourier decomposition of F

Our aim is to use Theorem 1.10, which is based on Theorem A.1, to complement the results of [40] concerning the multiplicative structure of the Chow motive of the Fano variety of lines on a smooth cubic fourfold.

A.2.1. An explicit Chow–Künneth decomposition for F

Recall that a Chow–Künneth decomposition for a smooth projective variety X of dimension d is a decomposition of the diagonal $\Delta_X \in \mathrm{CH}^d(X \times X)$ into a sum $\Delta_X = \pi_X^0 + \cdots + \pi_X^{2d}$ of mutually orthogonal idempotent correspondences $\pi_X^i \in \mathrm{CH}^d(X \times X)$ whose action in cohomology is given by $(\pi_X^i)_* H^*(X, \mathbf{Q}) = H^i(X, \mathbf{Q})$. It is a conjecture of Murre that all smooth projective varieties should admit a Chow–Künneth decomposition. In [40], it is shown that the Fano variety of lines on a smooth cubic fourfold admits a Chow–Künneth decomposition; see especially [40, Theorem 3.3]. Such a decomposition is obtained by modifying the following correspondences in $\mathrm{CH}^4(F \times F)$:

$$\pi_F^0 = \frac{1}{23 \cdot 25} l_1^2, \quad \pi_F^2 = \frac{1}{25} L \cdot l_1, \quad \pi_F^4 = \frac{1}{2}(L^2 - \frac{1}{25} l_1 \cdot l_2), \quad \pi_F^6 = \frac{1}{25} L \cdot l_2, \quad \pi_F^8 = \frac{1}{23 \cdot 25} l_2^2. \tag{16}$$

Here, $L := \frac{1}{3}(g_1^2 + \frac{3}{2} g_1 g_2 + g_2^2 - c_1 - c_2) - I \in \mathrm{CH}^2(F \times F)$ is a (and in fact "the", by Proposition 6.4) tautological cycle representing the Beauville–Bogomolov form; see [40, Proposition 19.1]. The cycle $l \in \mathrm{CH}^2(F)$ is the restriction of L to the diagonal, and, as before, a subscript i indicates the pull-back along the projection $F \times F \to F$ to the i-th factor.

As was expected from [40, Conjecture 3], these correspondences already define a Chow–Künneth decomposition:

Proposition A.6. *The correspondences in* (16) *define a Chow–Künneth decomposition of* F.

Proof. The correspondences π_F^{2i} of (16) are cycles on $F \times F$ that belong to the image of the restriction map $\mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}^*(F \times F)$, and they define a Künneth decomposition of the diagonal in cohomology by [40, Corollary 1.7]. (Here $\mathcal{F} \to B$ is the universal Fano variety of lines as defined in §3). It follows readily from Theorem 1.10 that they define a Chow–Künneth decomposition. □

A.2.2. A new multiplicativity statement

Using the Chow–Künneth decomposition (16) given by Proposition A.6, we can define, for all integers i and j,

$$\mathrm{CH}^i(F)_{(j)} := (\pi_F^{2i-j})_* \, \mathrm{CH}^i(F) \ .$$

Concretely, we have (*cf.* [40])

$$\begin{aligned}
\mathrm{CH}^4(F) &= \mathrm{CH}^4(F)_{(0)} \oplus \mathrm{CH}^4(F)_{(2)} \oplus \mathrm{CH}^4(F)_{(4)} \\
\mathrm{CH}^3(F) &= \mathrm{CH}^3(F)_{(0)} \oplus \mathrm{CH}^3(F)_{(2)} \\
\mathrm{CH}^2(F) &= \mathrm{CH}^2(F)_{(0)} \oplus \mathrm{CH}^2(F)_{(2)} \\
\mathrm{CH}^1(F) &= \mathrm{CH}^1(F)_{(0)} \\
\mathrm{CH}^0(F) &= \mathrm{CH}^0(F)_{(0)}.
\end{aligned}$$

In [40], it was proven that for the Fano variety of lines on a *very general* cubic fourfold, the decomposition $\mathrm{CH}^i(F)_{(j)}$ defines a bigrading on the Chow ring $\mathrm{CH}^*(F)$, in the sense that for all integers i, i', j, j' we have

$$\mathrm{CH}^i(F)_{(j)} \cdot \mathrm{CH}^{i'}(F)_{(j')} \subseteq \mathrm{CH}^{i+i'}(F)_{(j+j')}.$$

In the case of the Fano variety of lines on a non-very general cubic fourfold, the following two relations could not be established (see [40, Remark 22.9]):

$$\mathrm{CH}^1(F) \cdot \mathrm{CH}^2(F)_{(0)} \subseteq \mathrm{CH}^3(F)_{(0)} \, ; \tag{17}$$

$$\mathrm{CH}^2(F)_{(0)} \cdot \mathrm{CH}^2(F)_{(0)} \subseteq \mathrm{CH}^4(F)_{(0)} = \mathbf{Q} \cdot \mathfrak{o}_F. \tag{18}$$

Using Theorem 1.10, which is based on the new relation (12), we can now prove one of the missing two inclusions:

Proposition A.7. *Let F be the Fano variety of lines on a smooth cubic fourfold. Then*

$$\mathrm{CH}^1(F) \cdot \mathrm{CH}^2(F)_{(0)} = \mathrm{CH}^3(F)_{(0)}.$$

Proof. We first show that $\mathrm{CH}^3(F)_{(0)} \subseteq \mathrm{CH}^1(F) \cdot \mathrm{CH}^2(F)_{(0)}$. On the one hand, the cycle class map $\mathrm{CH}^3(F)_{(0)} \to H^6_{alg}(F, \mathbf{Q})$ is an isomorphism; on the other hand, the hard Lefschetz isomorphism implies that $H^6_{alg}(F, \mathbf{Q})$ is generated by $g^2 \cdot H^2_{alg}(F, \mathbf{Q}) = g^2 \cdot \mathrm{CH}^1(F)$. Hence $\mathrm{CH}^3(F)_{(0)}$ is generated by intersections of three divisors, which is contained in $\mathrm{CH}^1(F) \cdot \mathrm{CH}^2(F)_{(0)}$ since we know that $\mathrm{CH}^1(F) \cdot \mathrm{CH}^1(F) \subseteq \mathrm{CH}^2(F)_{(0)}$.

For the reverse inclusion, which is (17), by [40, Proposition 22.7], we only need to show that if α is a cycle in $\mathrm{CH}^2(F)_{(0)}$, then $g \cdot \alpha$ belongs to $\mathrm{CH}^3(F)_{(0)}$. To this end, we consider the correspondence

$$\Gamma := \pi_F^4 \circ \Gamma_\iota \circ {}^t\Gamma_\iota \circ \pi_F^4 \quad \in \mathrm{CH}^5(F \times F) \ ,$$

where $\iota \colon H \hookrightarrow F$ denotes the inclusion of a hyperplane with respect to the Plücker embedding. Clearly, Γ is homologically trivial. But Γ is universally defined, and so Theorem 1.10 implies that Γ is rationally trivial. The action of Γ on $\mathrm{CH}^2(F)_{(0)}$ is the same as

$$\mathrm{CH}^2(F)_{(0)} \xrightarrow{\cdot g} \mathrm{CH}^3(F) \to \mathrm{CH}^3(F)_{(2)}$$

(where the second arrow is projection on a direct summand), and so we are done. □

With notations as in §6, it seems that the final missing inclusion (18) cannot be obtained from considering the subring $\mathrm{Im}\,(\mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}^*(F_b \times F_b))$. Rather, a streamlined proof of all inclusions $\mathrm{CH}^i(F)_{(j)} \cdot \mathrm{CH}^{i'}(F)_{(j')} \subseteq \mathrm{CH}^{i+i'}(F)_{(j+j')}$ would follow from establishing that the Chow–Künneth decomposition (16) is *multiplicative* in the sense of [40, §8], meaning that

$$\pi_F^k \circ \delta_F \circ (\pi_F^i \otimes \pi_F^j) = 0 \quad \text{in } \mathrm{CH}^8(F \times F \times F), \quad \text{for all } k \neq i + j,$$

where δ_F denotes the class of the small diagonal in $F \times F \times F$ viewed as a correspondence from $F \times F$ to F. This in turn would follow from establishing the Franchetta property for the relative cube of the universal Fano variety of lines, *i.e.* from showing that

$$\mathrm{Im}\,(\mathrm{CH}^*(\mathcal{F} \times_B \mathcal{F} \times_B \mathcal{F}) \to \mathrm{CH}^*(F_b \times F_b \times F_b))$$

injects into cohomology by the cycle class map for all b. An approach would consist in first showing that this subring consists of "tautological cycles" and then in establishing enough "tautological relations", as was done in Propositions 6.3 and 6.4 in the case of the relative square.

References

[1] Nicolas Addington, Manfred Lehn, On the symplectic eightfold associated to a pfaffian cubic fourfold, J. Reine Angew. Math. 731 (2017) 129–137.
[2] Enrico Arbarello, Maurizio Cornalba, The Picard groups of the moduli spaces of curves, Topology 26 (2) (1987) 153–171.
[3] Arnaud Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in: Classification of Algebraic and Analytic Manifolds, Katata, 1982, Birkhäuser Boston, Boston, 1983.
[4] Arnaud Beauville, Fano threefolds and K3 surfaces, in: The Fano Conference, University Torino, Turin, 2004.
[5] Arnaud Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic Cycles and Motives, vol. 2, in: London Math. Soc. Lecture Note Ser., vol. 344, Cambridge Univ. Press, Cambridge, 2007, pp. 38–53.
[6] Arnaud Beauville, Ron Donagi, La variété des droites d'une hypersurface cubique de dimension 4, C. R. Acad. Sci., Sér. 1 Math. 301 (14) (1985) 703–706.
[7] Arnaud Beauville, Claire Voisin, On the Chow ring of a K3 surface, J. Algebraic Geom. 13 (3) (2004) 417–426.
[8] Nicolas Bergeron, Zhiyuan Li, Tautological classes on moduli space of hyperkähler manifolds, Preprint, arXiv:1703.04733, 2017.
[9] Spencer Bloch, Vasudevan Srinivas, Remarks on correspondences and algebraic cycles, Am. J. Math. 105 (5) (1983) 1235–1253.
[10] François Charles, Eyal Markman, The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces, Compos. Math. 149 (3) (2013) 481–494.
[11] Mark Andrea de Cataldo, Luca Migliorini, The Chow groups and the motive of the Hilbert scheme of points on a surface, J. Algebra 251 (2) (2002) 824–848.
[12] Huachen Chen, The Voisin map via families of extensions, Preprint, arXiv:1806.05771, 2018.
[13] Igor Dolgachev, Weighted projective varieties, in: Group Actions and Vector Fields, in: Lecture Notes in Math., vol. 956, 1982, pp. 34–71.
[14] A. Franchetta, Sulle serie lineari razionalmente determinate sulla curva a moduli generali di dato genere, Matematiche, Catania 9 (1954) 126–147.
[15] Lie Fu, Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi–Yau complete intersections, Adv. Math. 244 (2013) 894–924.
[16] Lie Fu, Beauville–Voisin conjecture for generalized Kummer varieties, Int. Math. Res. Not. IMRN (12) (2015) 3878–3898.
[17] Lie Fu, On the action of symplectic automorphisms on the CH_0-groups of some hyper-Kähler fourfolds, Math. Z. 280 (1–2) (2015) 307–334.
[18] Lie Fu, Zhiyu Tian, Motivic hyperkähler resolution conjecture: II. Hilbert schemes of K3 surfaces, Preprint, 2017.
[19] William Fulton, Intersection Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 2, Springer-Verlag, 1998.
[20] Mark Green, Phillip Griffiths, An interesting 0-cycle, Duke Math. J. 119 (2) (2003) 261–313.
[21] John Harer, The second homology group of the mapping class group of an orientable surface, Invent. Math. 72 (2) (1983) 221–239.
[22] Atanas Iliev, Laurent Manivel, Prime Fano threefolds and integrable systems, Math. Ann. 339 (4) (2007) 937–955.

[23] Vasilii Iskovskikh, Yuri Prokhorov, Fano varieties, in: A.N. Parsin, I.R. Shafarevich (Eds.), Algebraic Geometry V, in: Encyclopaedia of Math. Sci., vol. 47, Springer, 1998.
[24] Andrew Kresch, Cycle groups for Artin stacks, Invent. Math. 138 (3) (1999) 495–536.
[25] Martí Lahoz, Manfred Lehn, Emanuele Macrì, Paolo Stellari, Generalized twisted cubics on a cubic fourfold as a moduli space of stable objects, J. Math. Pures Appl. 114 (2018) 85–117.
[26] Robert Laterveer, Bloch's conjecture for certain hyperkähler fourfolds, Pure Appl. Math. Q. 13 (4) (2018) 639–692.
[27] Christian Lehn, Manfred Lehn, Christoph Sorger, Duco van Straten, Twisted cubics on cubic fourfolds, J. Reine Angew. Math. 731 (2017) 87–128.
[28] Christian Lehn, Twisted cubics on singular cubic fourfolds – on Starr's fibration, Math. Z. 290 (1–2) (2018) 379–388.
[29] Nicole Mestrano, Conjecture de Franchetta forte, Invent. Math. 87 (2) (1987) 365–376.
[30] Shigeru Mukai, Curves, K3 surfaces and Fano 3-folds of genus = 10, in: H. Hijikata, H. Hironaka, M. Maruyama, H. Matsumura, M. Miyanishi, T. Oda, K. Ueno (Eds.), Algebraic Geometry and Commutative Algebra, vol. I, 1988, pp. 357–377.
[31] Shigeru Mukai, Polarized K3 surfaces of genus 18 and 20, in: G. Ellingsrud, C. Peskine, G. Sacchiero, S.A. Strömme (Eds.), Complex Projective Geometry, Trieste, 1989/Bergen, 1989, in: London Mathematical Society, Lecture Note Series, vol. 179, Cambridge University Press, Cambridge, 1992, pp. 264–276.
[32] Shigeru Mukai, Polarized K3 surfaces of genus thirteen, in: S. Mukai, Y. Miyaoka, S. Mori, A. Moriwaki, I. Nakamura (Eds.), Moduli Spaces and Arithmetic Geometry, in: Advanced Studies in Pure Mathematics, vol. 45, The Mathematical Society of Japan, Tokyo, 2006, pp. 315–326.
[33] Shigeru Mukai, K3 surfaces of genus sixteen, Preprint, available at http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1743.pdf, 2012. (Accessed 4 July 2016).
[34] Giosuè Emanuele Muratore, The indeterminacy locus of the Voisin map, Preprint, arXiv:1711.06218.
[35] Jacob Murre, Jan Nagel, Chris Peters, Lectures on the Theory of Pure Motives, University Lecture Series, vol. 61, Amer. Math. Soc., Providence, 2013.
[36] Kieran G. O'Grady, Double covers of EPW-sextics, Mich. Math. J. 62 (2013) 143–184.
[37] Kieran G. O'Grady, Moduli of sheaves and the Chow group of K3 surfaces, J. Math. Pures Appl. (9) 100 (5) (2013) 701–718.
[38] Nebojsa Pavic, Junliang Shen, Qizheng Yin, On O'Grady's generalized Franchetta conjecture, Int. Math. Res. Not. IMRN 2016 (2016) 1–13.
[39] Ulrike Rieß, On Beauville's conjectural weak splitting property, Int. Math. Res. Not. IMRN (20) (2016) 6133–6150.
[40] Mingmin Shen, Charles Vial, The Fourier transform for certain hyperkähler fourfolds, Mem. Am. Math. Soc. 240 (1139) (2016), vii+163.
[41] Mingmin Shen, Charles Vial, On the motive of the Hilbert cube $X^{[3]}$, Forum Math. Sigma 4 (2016) e30.
[42] Evgeny Shinder, Andrey Soldatenkov, On the geometry of the Lehn–Lehn–Sorger–van Straten eightfold, Kyoto J. Math. 57 (4) (2017) 789–806.
[43] Charles Vial, Algebraic cycles and fibrations, Doc. Math. 18 (2013) 1521–1553.
[44] Charles Vial, On the motive of some hyperkähler varieties, J. Reine Angew. Math. 725 (2017) 235–247.
[45] Angelo Vistoli, Intersection theory on algebraic stacks and on their moduli spaces, Invent. Math. 97 (3) (1989) 613–670.
[46] Claire Voisin, Hodge Theory and Complex Algebraic Geometry. II, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2003.
[47] Claire Voisin, On the Chow ring of certain algebraic hyper-Kähler manifolds, Pure Appl. Math. Q. 4 (3, part 2) (2008) 613–649.
[48] Claire Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, Ann. Sci. Éc. Norm. Supér. (4) 46 (3) (2013) 449–475.
[49] Claire Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Annals of Mathematics Studies, vol. 187, Princeton University Press, Princeton, NJ, 2014.
[50] Claire Voisin, Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties, in: K3 Surfaces and Their Moduli, in: Progr. Math., vol. 315, Birkhäuser/Springer, 2016, pp. 365–399.
[51] Qizheng Yin, Finite-dimensionality and cycles on powers of $K3$ surfaces, Comment. Math. Helv. 90 (2) (2015) 503–511.