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Special Cubic Four-Folds, K3 Surfaces, and the Franchetta Property

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O'Grady conjectured that the Chow group of 0-cycles of the generic fiber of the universal family over the moduli space of polarized K3 surfaces of genus g is cyclic. This so-called generalized Franchetta conjecture has been solved only for low genera where there is a Mukai model (precisely, when $g \leq 10$ and $g = 12, 13, 16, 18, 20$), by the work of Pavic–Shen–Yin. In this paper, as a non-commutative analogue, we study the Franchetta property for families of special cubic four-folds (in the sense of Hassett) and relate it to O'Grady's conjecture for K3 surfaces. Most notably, by using special cubic four-folds of discriminant 26, we prove O'Grady's generalized Franchetta conjecture for $g = 14$, providing the first evidence beyond Mukai models.

1 Introduction

For an integer $g \geq 2$, let $\mathcal{M}_g$ be the moduli stack of genus g curves and $\pi : \mathcal{C} \to \mathcal{M}_g$ the universal curve. Franchetta conjectured in [21] that the Picard group of the generic fiber of π is free cyclic and generated by the relative canonical bundle ω_π. The conjecture can be equivalently formulated as follows: for any line bundle L on $\mathcal{C}$, the restriction of L to a fiber $C_t := \pi^{-1}(t)$, for any $t \in \mathcal{M}_g$, is a power of the canonical bundle:

$$L|_{C_t} \simeq \omega_{C_t}^{\otimes m}, \text{ for some } m \in \mathbb{Z}.$$

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Franchetta's conjecture was proved by Harer [27] (see also [2] and [38]).

1.1 Franchetta for K3 surfaces

In the end of [46], O'Grady proposed an analogue of Franchetta's conjecture for K3 surfaces. In order to state his conjecture, let us first recall the following seminal result of Beauville and Voisin [8]. Let $\mathrm{CH}^*(-)$ denote the Chow ring.

Theorem 1.1 (Beauville–Voisin). Let S be a projective K3 surface. There exists a canonical 0-cycle $\mathfrak{o}_S \in \mathrm{CH}^2(S)$, defined as the class of any point lying on some rational curve in S, satisfying the following properties:

(i) $\operatorname{Im}\left(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathrm{CH}^2(S)\right) \subset \mathbb{Z}\mathfrak{o}_S$.

(ii) $c_2(T_S) = 24\mathfrak{o}_S$ in $\mathrm{CH}^2(S)$.

We call the canonical 0-cycle $\mathfrak{o}_S$ the *Beauville–Voisin class* of the K3 surface S. The existence of such a canonical class is remarkable, as Mumford proved in [44] that $\mathrm{CH}^2(S)$ is infinite dimensional, in the sense that it cannot be parameterized by a scheme of finite type. The insight of O'Grady is that to generalize Franchetta's conjecture, the Beauville–Voisin class for a K3 surface should play the role of the canonical class for a curve.

Now let us state O'Grady's conjecture in [46, page 717] precisely. Throughout the paper, for an integer $g \geq 2$, we denote by $\mathcal{F}_g$ the moduli stack of primitively polarized K3 surfaces of genus g, that is, a pair (S, H) of a K3 surface S and a primitive ample line bundle H on it with degree $(H^2) = 2g - 2$. Let $\pi : \mathcal{S} \to \mathcal{F}_g$ be the universal family ($\mathcal{S}$ is sometimes denoted by $\mathcal{F}_{g,1}$ in the literature). For any closed point $b \in \mathcal{F}_g$, we denote by S_b the fiber of π over b. Rational Chow groups of algebraic stacks are defined in [54].

Conjecture 1.2 (O'Grady). For any $b \in \mathcal{F}_g$, the Gysin restriction of any cycle $z \in \mathrm{CH}^2(\mathcal{S})_{\mathbb{Q}}$ to the fiber S_b is a multiple of the Beauville–Voisin class, that is,

$$\operatorname{Im}\left(\mathrm{CH}^2(\mathcal{S})_{\mathbb{Q}} \to \mathrm{CH}^2(S_b)_{\mathbb{Q}}\right) = \mathbb{Q}\mathfrak{o}_{S_b}.$$

We will refer to this conjecture as the **generalized Franchetta conjecture**. Note that by the standard argument of "spreading out" (see e.g., [57, Section 1.1.2]), it is equivalent to requiring the same property only for a *very general* point b in $\mathcal{F}_g$.

Conjecture 1.2 is largely open at present. Let us first mention some closely related results:

- Bergeron and Li [10, Theorem 1.2.1] established a cohomological version of the conjecture: for any $z \in \mathrm{CH}^2(\mathcal{S})_{\mathbb{Q}}$, if it is cohomologically trivial on each fiber of π, then its cohomology class $[z]$ vanishes on the preimage of a Zariski open subset of $\mathcal{F}_g$.
- Beauville recently proved in [6] that for any g, there exists a hypersurface in $\mathcal{F}_g$ such that the restricted universal family satisfies the Franchetta property, in the sense of Definition 2.1 below.
- In a series of joint work with Vial [22–24], we formulated and investigated the natural extension of Conjecture 1.2 for higher-dimensional hyper-Kähler varieties, which is proved most notably in the cases of Beauville–Donagi four-folds [7] and Lehn–Lehn–Sorger–van Straten eight-folds [37] associated with the universal family of cubic four-folds.

As for Conjecture 1.2 itself, the only known result so far is the following:

Theorem 1.3 (Pavic–Shen–Yin [47]). Conjecture 1.2 is true for $2 \leq g \leq 10$ and for $g \in \{12, 13, 16, 18, 20\}$.

The values of g appearing in the statement are exactly the ones where a so-called *Mukai model* is available, and indeed, Theorem 1.3 is proven by exploiting the projective geometry of those Mukai models. Here, a Mukai model refers to a description of a general genus g polarized K3 surface as the zero locus of a general section of some globally generated homogeneous vector bundle over a homogeneous variety. Examples are double covers of $\mathbb{P}^2$ ramified along a sextic curve (for $g = 2$), quartic surfaces in $\mathbb{P}^3$ (for $g = 3$), complete intersections of a hyperquadric and a cubic hypersurface in $\mathbb{P}^4$ (for $g = 4$), complete intersections of three hyperquadrics in $\mathbb{P}^5$ (for $g = 5$), complete intersections of three hyperplanes and a hyperquadric with $\mathrm{Gr}(2, 5)$ embedded in $\mathbb{P}^9$ via Plücker (for $g = 6$), and so on. For more details on the geometric constructions, we refer to the original papers of Mukai [39, 40, 42, 43], and also to [47, Section 2] for a summary.

Our main result is the following, which provides the first instance of the generalized Franchetta conjecture 1.2 beyond Mukai models:

Theorem 1.4. Conjecture 1.2 holds for $g = 14$.

What is probably more interesting than the result is our approach to establishing it. Theorem 1.4 is implied by the combination of Theorem 1.5 and Theorem 1.7 below. Let us now give a brief account.

1.2 Franchetta for special cubic four-folds

Special cubic four-folds were first introduced and studied by Hassett [28]. These are cubic four-folds X containing a surface R whose class is not proportional to h^2, the square of the hyperplane class. Special cubic four-folds come in families enumerated by the discriminant d of the sublattice of $H^4(X,\mathbb{Z})$ generated by R and h^2. The moduli space of special cubic four-folds of discriminant d is denoted by $\mathcal{C}_d$, which is non-empty and irreducible when $d \equiv 0, 2 \pmod 6$. For d satisfying an extra numerical condition $(**)$ (see Section 3.2), a special cubic four-fold X of discriminant d has an associated K3 surface S, such that X and S are related Hodge theoretically ([28]), and it turns out there are also strong relations between their derived categories [33] [1] and algebraic cycles (or motives) [14]. All the above is explained in more detail in Section 3.

The proof of Theorem 1.4, which uses special cubic four-folds of discriminant 26, can be summarized as follows. Sending such a cubic four-fold to its (Hodge theoretically) associated K3 surface gives a birational isomorphism between $\mathcal{F}_{14}$ and the moduli space $\mathcal{C}_{26}$. Let U be a common Zariski open subset and denote by $\mathcal{S}$ and $\mathcal{X}$ the universal families of K3 surfaces and of cubic four-folds, respectively. Our proof splits into two parts:

Step 1. Produce a relative correspondence over U between $\mathcal{S}$ and $\mathcal{X}$ and show that Conjecture 1.2 for $\mathcal{S} \to \mathcal{F}_{14}$ is equivalent to the *Franchetta property* (Definition 2.1) for $\mathcal{X} \to \mathcal{C}_{26}$.

Step 2. Establish the Franchetta property for $\mathcal{X} \to \mathcal{C}_{26}$ by using the concrete geometric characterization of such cubic four-folds due to Farkas–Verra [19] as the ones containing certain type of scrolls.

The upshot is that although there is no Mukai model for K3 surfaces of genus 14 at our disposal, we have the following replacement that is almost as good: *a generic K3 surface of genus 14 is a moduli space of Bridgeland-stable objects, with certain isotropic Mukai vector, in the Kuznetsov component of a cubic four-fold that contains a 3-nodal septic rational scroll.*

In this paper, both of the above steps are treated in greater generality. For Step 1, which is accomplished in Section 3.3, we actually give a strong link between the Franchetta properties for special cubic four-folds and for the associated K3 surfaces. Let us state here only the non-technical version. See Theorem 3.4 for a stronger form.

Theorem 1.5. Let d be an integer satisfying the condition $(**)$ (see Section 3.2). Let $g = \frac{d}{2}+1$. If $d \equiv 2 \pmod 6$, then the Franchetta property (Definition 2.1) for the universal family over $\mathcal{C}_d$ is equivalent to the Franchetta property for the universal family over $\mathcal{F}_g$.

In view of Step 2, we are led to ask the following question. As cubic four-folds are considered as non-commutative analogues of K3 surfaces [33], the following can be seen as a non-commutative version of O'Grady's generalized Franchetta conjecture 1.2.

Question 1.6. Let $d > 6$ be an integer $\equiv 0$ or $2 \pmod 6$. Does the universal family $\mathcal{X} \to \mathcal{C}_d$ of special cubic four-folds of discriminant d satisfy the Franchetta property (Definition 2.1)? That is, for any $b \in \mathcal{C}_d$,

$$\operatorname{Im}\left(\mathrm{CH}^3(\mathcal{X})_{\mathbb{Q}} \to \mathrm{CH}^3(X_b)_{\mathbb{Q}}\right) \stackrel{?}{=} \mathbb{Q}h^3. \tag{1}$$

The left-hand side is often denoted by $\mathrm{GDCH}^3_B(X_b)$ in this paper.

We answer this question affirmatively in a few cases:

Theorem 1.7. The Franchetta property (1) holds for the universal family of special cubic four-folds $\mathcal{C}_d$ with discriminant $d = 8, 14, 20, 26, 38$.

Remark 1.8 (Relation with previous results). In Theorem 1.7, the case $d = 8$ provides a new proof of Conjecture 1.2 for $g = 2$; the case $d = 14$ gives a new proof of Conjecture 1.2 for $g = 8$ using Theorem 1.5; the case $d = 20$ has been proven using different methods in our previous joint work with Vial [24, Lemma 6.3]; the case $d = 26$ is the principal case, yielding Theorem 1.4; the case $d = 38$ is proven using Theorem 1.5 and the $g = 20$ case of Conjecture 1.2, demonstrating the flow of information in the reverse direction. Finally, note that the Franchetta property for the universal family over the whole moduli space of cubic four-folds can be easily checked (see however [23, Theorem 2] for stronger and more interesting results).

Potential and limits

Cubic four-folds are instances of the so-called *varieties of K3 type* (see [24]), which means an even-dimensional smooth projective variety X whose Hodge numbers $h^{p,q}(X) = 0$ for all $p \neq q$ except for $h^{m-1,m+1}(X) = h^{m+1,m-1}(X) = 1$ where $2m = \dim(X)$. The terminology is justified by the observation that its middle cohomology group $\mathrm{H}^{2m}(X, \mathbb{Z})$, up to a Tate twist, carries a weight-2 Hodge structure of K3 type. Examples of varieties of K3 type include cubic four-folds, Gushel–Mukai four-folds and six-folds [15, 16, 34, 39], Debarre–Voisin twenty-folds [17], etc.; see [11] for a recent updated list.

We expect that our approach will lead to further progress on Conjecture 1.2: whenever a concrete geometric description is discovered for a family of varieties of K3 type whose generic member has an associated K3 surface that is generic in the moduli space $\mathcal{F}_g$, our argument gives access to the generalized Franchetta conjecture for this g. We view Theorem 1.4 for $g = 14$, as well as our new proofs for $g = 2$ and 8 without using Mukai models (Remark 1.8), merely as the first examples of this approach.

In the past few years, we witnessed a rapid development on the projective geometry of special cubic four-folds [12, 19, 20, 35, 45, 49], Gushel–Mukai four-folds [15, 30], and Debarre–Voisin twenty-folds [9]. These achievements will certainly shed light on the geometry of K3 surfaces in the future.

An initial motivation to construct Mukai models was to prove the *unirationality* of the moduli spaces $\mathcal{F}_g$ for g taking values as in Theorem 1.3. Recent progress in this direction is due to Farkas–Verra [19] (for $g = 14$), Farkas–Verra [20] (for $g = 22$), and Hoff–Staglianò [31] (a new proof for $g = 11$, originally due to Mukai [41]). Both the arguments in [47] and in the present paper require "parameterizing" K3 surfaces by a flag variety, which in practice always takes the form of a unirational parameterization.

However, Gritsenko–Hulek–Sankaran [26] showed that $\mathcal{F}_g$ is of non-negative Kodaira dimension, hence not unirational, for $g \geq 41$ and $g \neq 42, 45, 46, 48$; see similar results for moduli spaces of special cubic four-folds in [45, Proposition 1.3] and for moduli spaces of special Gushel–Mukai four-folds in [48]. Therefore, for a high genus in this range, some entirely new idea is needed to study the generalized Franchetta conjecture.

The paper is organized as follows: in Section 2, we collect some basic facts concerning the Franchetta property. In Section 3, we first recap the theory of special cubic four-folds and their associated K3 surfaces, then we establish the bridge between their Franchetta properties, namely, Theorem 1.5 (or rather its more precise version Theorem 3.4). In the remaining sections, as their titles indicate, we prove Theorem 1.7 case by case and give applications.

Convention:

Throughout the paper, we work over the field of complex numbers $\mathbb{C}$. All Chow groups and Chow motives are with rational coefficients: for any (possibly singular) variety X of dimension d we write $\mathrm{CH}_i(X) = \mathrm{CH}^{d-i}(X)$ for the group of i-dimensional algebraic cycles with $\mathbb{Q}$-coefficients modulo rational equivalence. A *lattice* means a free abelian group of finite rank equipped with a symmetric bilinear pairing.

2 Franchetta Property and Generically Defined Cycles

To tackle with the generalized Franchetta conjecture, we will need to study this property beyond the scope of K3 surfaces:

Definition 2.1 (Franchetta property [22, 23]). Let $\mathcal{X} \to B$ be a smooth projective morphism between complex varieties (or algebraic stacks). For an integer $i \geq 0$, we say that the family $\mathcal{X}/B$ satisfies the *Franchetta property* for codimension-i cycles, if for any $z \in \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ and any $b \in B$, the Gysin restriction $z|_{X_b}$ is rationally equivalent to zero if and only if its cohomology class $[z|_{X_b}] = 0$ in $\mathrm{H}^{2i}(X_b, \mathbb{Q})$. If this holds for all i, we simply say that $\mathcal{X}/B$ has the Franchetta property. Again, by spreading out rational equivalence, it is equivalent to requiring the same property only for very general $b \in B$.

Remark 2.2. Note that there is no implication in either direction between the Franchetta properties for a family $\mathcal{X} \to B$ and for a subfamily $\mathcal{X}_{B'} \to B'$, where B' is a closed subscheme of B (see [23, page 1]). However, if $B' \to B$ is a dominant morphism, the Franchetta property of the base-changed family $\mathcal{X}_{B'} \to B'$ implies the Franchetta property for $\mathcal{X} \to B$ (see [22, Remark 2.6]); in particular, on can freely replace B by a non-empty Zariski open subset.

To study the generalized Franchetta conjecture 1.2, or more generally the Franchetta property (Definition 2.1), it is convenient to introduce the following notion.

Definition 2.3 (Generically defined cycles). let $\pi : \mathcal{X} \to B$ be a smooth projective morphism between complex varieties (or algebraic stacks). Let X be a fiber of π over a closed point. We define the group of *generically defined cycles* on X as the following graded subgroup of $\mathrm{CH}^*(X)$:

$$\mathrm{GDCH}^*_B(X) := \mathrm{Im}\left(\mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(X)\right),$$

where the morphism is the Gysin restriction map.

Using this notation, the Franchetta property (Definition 2.1) for $\mathcal{X}/B$ is equivalent to the injectivity of the cycle class map:

$$\mathrm{GDCH}^*_B(X) \to \mathrm{H}^*(X, \mathbb{Q}),$$

for all (or equivalently, for very general) fibers X.

In [47], a key step is an argument using projective bundles, which is further generalized in [22, 23] into a stratified version. Here we provide the following variant allowing base locus, which is the basic tool in our paper.

Proposition 2.4 (Projective bundle argument: with base locus). Let P be a smooth projective variety and let E be a vector bundle on it. Let $Q \subset P$ be a (possibly singular) closed subvariety. Let

$$B \ \subset \ \bar{B} := \mathbb{P}H^0(P, E \otimes \mathcal{I}_Q)$$

denote the Zariski open subset parameterizing smooth dimensionally transversal sections of E vanishing along Q, and let $\pi : \mathcal{X} \to B$ denote the universal family of zero loci of such sections. Assume that B is not empty, and that the sections in $H^0(P, E \otimes \mathcal{I}_Q)$ globally generate E outside of Q. Then for any fiber X of π, we have

$$\mathrm{GDCH}^*_B(X) = \mathrm{Im}\left(\mathrm{CH}^*(P) \to \mathrm{CH}^*(X)\right) + \mathrm{Im}\left(\mathrm{CH}^*(Q) \to \mathrm{CH}^*(X)\right),$$

where on the right-hand side, the first morphism is the Gysin restriction map and the second morphism is the push-forward via the natural closed immersion.

Proof. Let $\bar{\mathcal{X}} \to \bar{B}$ denote the universal family of zero loci of sections. The assumption that E is globally generated outside of Q by its sections vanishing along Q implies that the evaluation map $\bar{\mathcal{X}} \to P$ restricts to a projective bundle over the open subset $P \setminus Q$. Reasoning with the projective bundle formula as in [47, Lemma 1.1] or [23, Proposition 2.6], this readily gives that

$$\mathrm{Im}\big(\mathrm{CH}^*(\bar{\mathcal{X}} \setminus (Q \times \bar{B})) \to \mathrm{CH}^*(X \setminus Q)\big) = \mathrm{Im}\big(\mathrm{CH}^*(P \setminus Q) \to \mathrm{CH}^*(X \setminus Q)\big).$$

By the localization exact sequence for Chow groups, this implies that

$$\mathrm{GDCH}^*_B(X) \ \subset \ \mathrm{Im}\big(\mathrm{CH}^*(P) \to \mathrm{CH}^*(X)\big) + \mathrm{Im}\big(\mathrm{CH}^*(Q) \to \mathrm{CH}^*(X)\big).$$

The converse inclusion is obvious. ■

The following easy observation abstracts a basic setup that will be repeatedly used in our proof of Theorem 1.7. In practice, P is some incidence variety in $B \times T$, which dominates B.

Lemma 2.5. Let P, B, T be varieties and let $p\colon P \to B$ and $q\colon P \to T$ be morphisms. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. For a point $b \in B$ lying in the image of p, let $X := X_b$ be the fiber of π over b. Then

$$\mathrm{GDCH}^*_B(X) \subset \bigcap_{t \in q(p^{-1}(b))} \mathrm{GDCH}^*_{q^{-1}(t)}(X),$$

where on the right-hand side, X is viewed as a fiber in the base change to P (or rather to $q^{-1}(t)$) of the family $\mathcal{X}/B$.

Proof. For any $(b, t) \in B \times T$ such that $p^{-1}(b) \cap q^{-1}(t) \neq \emptyset$, we have, by restricting p, a morphism $q^{-1}(t) \to B$ whose image contains b. Therefore, $\mathrm{GDCH}^*_B(X) \subset \mathrm{GDCH}^*_{q^{-1}(t)}(X)$, where $X = X_b$. One can conclude by letting t run through $q(p^{-1}(b))$. ■

3 Special Cubic Four-Folds and Associated K3 Surfaces

Let X be a cubic four-fold, that is, a smooth hypersurface of degree 3 in $\mathbb{P}^5$. Its middle cohomology group $\mathrm{H}^4(X, \mathbb{Z})$ equipped with the intersection pairing is naturally a unimodular lattice abstractly isometric to $\mathbf{I}_{21,2}$ and, up to a Tate twist, it also carries a weight-2 Hodge structure of K3 type with Hodge numbers $(1, 21, 1)$. Denote by $h := c_1(\mathcal{O}_X(1)) \in \mathrm{H}^2(X, \mathbb{Z})$ the hyperplane section class. The h-primitive cohomology group

$$\mathrm{H}^4(X, \mathbb{Z})_0 = \{h^2\}^\perp$$

is a Hodge structure of K3 type with Hodge numbers $(1, 20, 1)$, and as a lattice is isometric to the following *cubic lattice*:

$$\Gamma := E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2,$$

where E_8 is the unique positive definite unimodular even lattice of rank 8, U is the hyperbolic plane, and A_2 is the lattice with intersection form $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. We can fix embeddings without loss of generality (such embeddings are unique up to isometries of $\mathbf{I}_{21,2}$):

$$h^2 \in \mathbf{I}_{21,2} \quad \text{and} \quad \Gamma = \{h^2\}^\perp \subset \mathbf{I}_{21,2}.$$

The moduli space of cubic four-folds is denoted by $\mathcal{C} := |\mathcal{O}_{\mathbb{P}^5}(3)|_{\mathrm{sm}}/\,\mathrm{PGL}_6$. The local period domain

$$\Omega(\Gamma) := \left\{\omega \in \mathbb{P}(\Gamma \otimes \mathbb{C}) \mid \omega^2 = 0,\ \omega \cdot \overline{\omega} < 0\right\} \tag{2}$$

is equipped with a natural action of the group

$$\widetilde{O}(\Gamma) := \{g \in O(\Gamma) \mid \bar{g}|_{A_\Gamma} = \mathrm{id}_{A_\Gamma}\} = \{\tilde{g} \in O(\mathbf{I}_{21,2}) \mid \tilde{g}(h^2) = h^2\},$$

where $A_\Gamma = \Gamma^\vee/\Gamma$ is the discriminant group of Γ. The corresponding quotient is called the *global period domain*

$$\mathcal{D} := \Omega(\Gamma)/\widetilde{O}(\Gamma),$$

which is a normal and quasi-projective variety by [3]. Sending a cubic four-fold X to its period $H^{3,1}(X)$, we get the period map $\mathcal{C} \to \mathcal{D}$, which is shown to be an open immersion by Voisin [55].

3.1 Special cubics

Denote the subgroup of integral Hodge classes by

$$\mathrm{H}^{2,2}(X,\mathbb{Z}) := \mathrm{H}^{2,2}(X) \cap \mathrm{H}^4(X,\mathbb{Z}),$$

which is also the subgroup of algebraic classes, thanks to the integral Hodge conjecture proved by Voisin [56].

For a very general cubic four-fold, $\mathrm{H}^{2,2}(X,\mathbb{Z}) = \mathbb{Z}h^2$. Following [28], a cubic four-fold X is called *special*, if $\mathrm{H}^{2,2}(X,\mathbb{Z})$ is of rank at least two. More precisely, a *marked* cubic four-fold is a (special) cubic four-fold together with a primitive embedding of lattices $K \hookrightarrow \mathrm{H}^{2,2}(X,\mathbb{Z})$ from a rank-2 lattice K such that the image contains h^2. A *labelled* cubic four-fold is a cubic four-fold together with a primitive rank-2 sublattice $K \subset \mathrm{H}^{2,2}(X,\mathbb{Z})$ containing h^2. Such an embedding (resp. a sublattice) is called a *marking* (resp. a *labelling*), and the determinant of the intersection matrix of K is called the *discriminant* of the (marked or labelled) special cubic four-fold. It turns out ([28, Proposition 3.2.4]) that the lattice K, as well as its embedding into $\mathbf{I}_{21,2}$, is determined by d, up to isometries of $\mathbf{I}_{21,2}$ preserving the class h^2. Hence it is conventional to denote K by K_d, and we can fix without loss of generality the embeddings

$$h^2 \in K_d \subset \mathbf{I}_{21,2},$$

$$\Gamma_d := K_d^\perp \subset \Gamma.$$

By Hassett [28, Theorem 1.0.1]), for a positive integer d, there exists a special cubic four-fold of discriminant d if and only if

$$(*) \quad d > 6 \quad \text{and} \quad d \equiv 0 \text{ or } 2 \pmod 6;$$

moreover, for such an integer d, the locus of special cubic four-folds of discriminant d is an irreducible divisor in the moduli space $\mathcal{C}$, denoted by $\mathcal{C}_d$.

The period domains of labelled and marked cubic four-folds of discriminant d are

$$\mathcal{D}_d^{\mathrm{lab}} = \Omega(\Gamma_d)/\widetilde{O}(\Gamma, K_d), \tag{3}$$

$$\mathcal{D}_d^{\mathrm{mar}} = \Omega(\Gamma_d)/\widetilde{O}(\Gamma_d), \tag{4}$$

where $\Omega(\Gamma_d)$ is defined similarly as in (2), and $\widetilde{O}(\Gamma, K_d)$ (resp. $\widetilde{O}(\Gamma_d)$) is the subgroup of elements of $\widetilde{O}(\Gamma)$ that preserves (resp. acts trivially on) the sublattice K_d.

Define the moduli spaces of marked and labelled cubic four-folds of discriminant d respectively as $\mathcal{C}_d^{\mathrm{mar}} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_d^{\mathrm{mar}}$ and $\mathcal{C}_d^{\mathrm{lab}} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_d^{\mathrm{lab}}$, which are normal quasi-projective varieties. There are natural morphisms

$$\begin{tikzcd}
\mathcal{C}_d^{\mathrm{mar}} \arrow[r, two heads] \arrow[d] & \mathcal{C}_d^{\mathrm{lab}} \arrow[r, two heads] \arrow[d] & \mathcal{C}_d \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "\mathcal{P}"] \\
\mathcal{D}_d^{\mathrm{mar}} \arrow[r, two heads] & \mathcal{D}_d^{\mathrm{lab}} \arrow[r, two heads] & \mathcal{D}_d \arrow[r, hook] & \mathcal{D}
\end{tikzcd}$$

where the vertical period maps are open immersions, the middle horizonal morphisms are normalizations, while the left horizontal ones are isomorphisms if $d \equiv 2 \pmod 6$ and are finite of degree 2 if $d \equiv 0 \pmod 6$ (see [28, Proposition 5.2.1]).

3.2 Associated K3

Given a marked special cubic four-fold $(X, K_d \hookrightarrow \mathrm{H}^{2,2}(X, \mathbb{Z}))$ of discriminant d, we say a polarized K3 surface (S, H) is *Hodge-theoretically associated* to X if there exists a Hodge isometry:

$$\mathrm{H}^4(X, \mathbb{Z}) \supset K_d^{\perp} \xrightarrow{\simeq} H^{\perp}(-1) \subset \mathrm{H}^2(S, \mathbb{Z})(-1),$$

where (-1) is the Tate twist and changes the sign of the quadratic form. Note that by comparing the discriminant, we have $\deg(H^2) = d$, that is, the K3 surface is of genus $g = \frac{d}{2} + 1$.

Hassett [28, Theorem 5.1.3] showed that a special cubic four-fold of discriminant d has a Hodge-theoretically associated K3 surface if and only if

$(**)$ d satisfies $(*)$ and $d/2$ is not divisible by 9 or any prime number $p \equiv -1 \pmod 3$.

Such d's are called *admissible*, and the first few values are 14, 26, 38, 42, 62, 74, 78, etc.

On the other hand, following [33], let

$$\mathrm{Ku}(X) := \left\{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{RHom}(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2\right\}$$

be the *Kuznetsov component* of (the bounded derived category of coherent sheaves of) X, which is a 2-Calabi–Yau category. One says that an algebraic K3 surface S is *homologically associated* to X if there is an equivalence of triangulated categories:

$$\mathrm{Ku}(X) \simeq \mathrm{D}^{\mathrm{b}}(S).$$

Both notions of Hodge-theoretically and homologically associated K3 surfaces are very much motivated by the rationality problem of cubic four-folds, a topic that we do not treat in this paper. However, what is important to us is the following relation between these two notions.

Theorem 3.1 (Addington–Thomas [1, Theorem 1.1], [4, Corollary 1.7]). Let d be an integer satisfying $(*)$. Let $X \in \mathcal{C}_d$, a special cubic four-fold of discriminant d. The following conditions are equivalent:

(i) The integer d is admissible, that is, it satisfies the condition $(**)$.
(ii) X has a homologically associated K3 surface: $\mathrm{Ku}(X) \simeq \mathrm{D}^{\mathrm{b}}(S)$ for some projective K3 surface S.

The arguments in [1] and [4] actually show that assuming (i), there is a polarized K3 surface of degree d homologically associated to X.

Essentially by taking the characteristic classes of the Fourier–Mukai kernel in (ii), Bülles [14] established the following relation between the motive of a special cubic four-fold and the motive of its associated K3 surface. This can be seen as a motivic lifting of the result of Addington and Thomas [1, Theorem 1.2]. See also [25, Theorem 3] for a different proof resulting in a stronger version taking into account the quadratic space structure.

Theorem 3.2 (Bülles [14, Theorem 0.4]). Given a special cubic four-fold $X \in \mathcal{C}_d$ with d satisfying $(**)$, there exist a polarized K3 surface (S, H) of degree d and an isomorphism in the category of rational Chow motives CHM:

$$\mathfrak{h}(X) \simeq \mathfrak{h}(S)(-1) \oplus \mathbb{1} \oplus \mathbb{1}(-2) \oplus \mathbb{1}(-4). \tag{5}$$

In particular, there is an algebraic cycle $Z \in \mathrm{CH}^3(X \times S)$, which induces an isomorphism of rational Chow groups:

$$\mathrm{CH}_1(X)_{\mathrm{hom}} \simeq \mathrm{CH}_0(S)_{\mathrm{hom}}. \tag{6}$$

Remark 3.3. The original proof in [14] shows that the cycle Z can be chosen to be the codimension-3 component of the Mukai vector of the Fourier–Mukai kernel $\mathcal{E}$ inducing the equivalence between $\mathrm{Ku}(X)$ and $\mathrm{D^b}(S)$:

$$Z = v_3(\mathcal{E}).$$

The proof in [25] actually shows that $Z' := v_3(\mathcal{E}^R)$ gives the inverse of the isomorphism (6), where $\mathcal{E}^R := \mathcal{E}^\vee \otimes p_X^*\omega_X[4]$ is the Fourier–Mukai kernel of the right adjoint.

3.3 Linking two Franchetta properties

The main purpose of this section is the following result, which for an admissible d, transforms the generalized Franchetta conjecture 1.2 for K3 surfaces of degree d into the Franchetta property for *marked* special cubic four-folds of discriminant d. Denote by $\mathcal{C}_{d,1}^{\mathrm{mar}} \to \mathcal{C}_d^{\mathrm{mar}}$ the universal family of cubic four-folds and by $\mathcal{F}_{g,1} \to \mathcal{F}_g$ the universal family of K3 surfaces.

Theorem 3.4. Let d be an integer satisfying the condition $(**)$. Let $g = \frac{d}{2} + 1$. The Franchetta property for codimension-2 cycles for $\mathcal{F}_{g,1} \to \mathcal{F}_g$ is equivalent to the Franchetta property for codimension 3-cycles for $\mathcal{C}_{d,1}^{\mathrm{mar}} \to \mathcal{C}_d^{\mathrm{mar}}$.

Remark 3.5. Theorem 1.5 is a consequence of Theorem 3.4, since when $d \equiv 2 \pmod 6$, $\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d$ is the normalization map, hence does not affect the Franchetta property. On the other hand, if $d \equiv 0 \pmod 6$, then $\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d$ is of degree 2. Hence by Remark 2.2, the Franchetta property for $\mathcal{F}_g$ implies the Franchetta property for $\mathcal{C}_d$.

To prove Theorem 3.4, it is crucial to adapt Addington–Thomas' Theorem 3.1 and Bülles' Theorem 3.2 into their family version. Let $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ be the *K3 lattice*. Let $\Lambda_d := E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d)$ be the abstract lattice underlying the second primitive cohomology of a polarized K3 surface of degree d. Hassett's condition $(**)$ mentioned above is equivalent to the existence of an isometry, up to a sign, between the lattices Γ_d and Λ_d.

For an integer d satisfying $(**)$, upon fixing an isometry $\epsilon : \Gamma_d \xrightarrow{\simeq} \Lambda_d(-1)$, we have an induced isomorphism between the period domain of marked special cubic four-folds of discriminant d and the period domain of polarized K3 surfaces of degree d:

$$\mathcal{D}_d^{\mathrm{mar}} = \Omega(\Gamma_d)/\widetilde{O}(\Gamma_d) \xrightarrow{\simeq} \mathcal{N}_d = \Omega(\Lambda_d)/\widetilde{O}(\Lambda_d),$$

which gives rise to a birational isomorphism (depending on the choice of ϵ) between the moduli space of marked cubic four-folds of discriminant d and the moduli space of polarized K3 surfaces of genus $g := \frac{d}{2} + 1$:

$$\phi \colon \mathcal{C}_d^{\mathrm{mar}} \overset{\simeq}{\dashrightarrow} \mathcal{F}_g. \tag{7}$$

The rational map ϕ sends a marked cubic four-fold to its Hodge-theoretically associated polarized K3 surface.

Let $\mathcal{F}_g^\circ$ be a Zariski open subset of $\mathcal{F}_g$ where ϕ is an isomorphism. The restrictions over $\mathcal{F}_g^\circ$ of the universal families $\mathcal{C}_{d,1}^{\mathrm{mar}}$ and $\mathcal{F}_{g,1}$ are denoted by $\mathcal{X} \to \mathcal{F}_g^\circ$ and $\mathcal{S} \to \mathcal{F}_g^\circ$ respectively.

For a cubic four-fold X, Addington–Thomas [1,Definition 2.2] equipped the topological K-theory of the Kuznetsov component $\mathrm{Ku}(X)$ with a lattice structure via the Euler pairing, abstractly isometric to the Mukai lattice $\widetilde{\Lambda} := E_8^{\oplus 2} \oplus U^{\oplus 4}$, and a natural weight-2 Hodge structure of K3 type via the Mukai-vector map:

$$v \colon \mathrm{K}_{\mathrm{top}}(\mathrm{Ku}(X)) \hookrightarrow \mathrm{H}^*(X, \mathbb{Q}).$$

The resulting *Mukai lattice* of $\mathrm{Ku}(X)$, denoted by $\widetilde{\mathrm{H}}(\mathrm{Ku}(X), \mathbb{Z})$, always contains the $A_2(-1)$-lattice $\langle \lambda_1, \lambda_2 \rangle$, where λ_i is the class of $p(\mathcal{O}_{\mathrm{line}}(i))$, and $p \colon \mathrm{D^b}(X) \to \mathrm{Ku}(X)$ is the left adjoint of the inclusion functor $\mathrm{Ku}(X) \hookrightarrow \mathrm{D^b}(X)$. Identifying $\widetilde{\mathrm{H}}(\mathrm{Ku}(X), \mathbb{Z})$ with its image via v, the Mukai vectors λ_i are given explicitly as follows, denoted by the same notation:

$$\lambda_1 = 3 + \frac{5}{4}h - \frac{7}{32}h^2 - \frac{77}{384}h^3 + \frac{41}{2048}h^4;$$
$$\lambda_2 = -3 - \frac{1}{4}h + \frac{15}{32}h^2 + \frac{1}{384}h^3 - \frac{153}{2048}h^4.$$

Now for the family $\mathcal{X} \to \mathcal{F}_g^\circ$, we have the local system of Mukai lattices over $\mathcal{F}_g^\circ$.

$$\mathbb{H} := \left\{\widetilde{\mathrm{H}}(\mathrm{Ku}(X_t), \mathbb{Z})\right\}_{t\in\mathcal{F}_g^\circ}$$

Lemma 3.6. There exist sections $\mathbf{v}, \mathbf{v}', \mathbf{w}$ of the local system $\mathbb{H}$ such that they are fiberwise algebraic and satisfy $\mathbf{v}^2 = 0$, $\mathbf{v}\cdot\mathbf{v}' = 1$, $\mathbf{v}\cdot\mathbf{w} = 0$, and $\mathbf{w}^2 = -d$. (By definition, a section of a local system is flat and global (i.e., monodromy invariant).

Proof. This is essentially [1, Theorem 3.1]. Indeed, by the definition of $\mathcal{D}_d^{\mathrm{mar}}$ in (4), we see that the monodromy invariant subspace of $\mathrm{H}^*(X, \mathbb{Q})$ contains $\langle 1, h, h^2, h^3, h^4\rangle + K_d$, whose inverse image by the Mukai-vector map v, denoted by L_d, is the saturation of the lattice $\langle\lambda_1, \lambda_2\rangle \oplus \mathbb{Z}v_d$, where v_d is the generator of the orthogonal complement of h^2 in K_d:

$$K_d = \overline{\mathbb{Z}h^2 \oplus \mathbb{Z}v_d}, \quad L_d = \overline{\langle\lambda_1, \lambda_2\rangle \oplus \mathbb{Z}v_d}.$$

All classes in L_d are fiberwise Hodge, hence algebraic. By construction, L_d is a rank 3 primitive sublattice in $\widetilde{\Lambda}$ of discriminant d such that

$$\Gamma \supset K_d^\perp =: \Gamma_d = L_d^\perp \subset \widetilde{\Lambda}.$$

By [1, Theorem 3.1, (1) $\Rightarrow$ (2)], or more directly, by [32, Lemma 1.10, Remark 1.11], there is an isomorphism

$$L_d \xrightarrow{\simeq} U \oplus \mathbb{Z}(-d).$$

One can then take $\mathbf{v}$, $\mathbf{v}'$ to be the standard basis of U and $\mathbf{w}$ to be the generator of $\mathbb{Z}(-d)$. ■

Example 3.7. Let us give the explicit formulas of the vectors in the cases $g = 14$ and 22.

- When $g = 14$, or equivalently $d = 26$, the monodromy invariant part of the local system $\mathbb{H}$ contains the lattice generated by λ_1, λ_2 and an extra class τ, with the intersection form

$$\begin{array}{c|ccc} & \lambda_1 & \lambda_2 & \tau \\ \hline \lambda_1 & -2 & 1 & 0 \\ \lambda_2 & 1 & -2 & 1 \\ \tau & 0 & 1 & 8 \end{array} \tag{8}$$

  Then we take $\mathbf{v} = \lambda_1 + 3\lambda_2 + \tau$, $\mathbf{v}' = \lambda_1$ and $\mathbf{w} = 11\lambda_1 + 22\lambda_2 + 7\tau$.
- When $g = 22$, or equivalently $d = 42$, the monodromy invariant part of the local system $\mathbb{H}$ contains the lattice generated by λ_1, λ_2 and an extra class $\tau = v_{42}$, with the intersection form

$$\begin{array}{c|ccc} & \lambda_1 & \lambda_2 & \tau \\ \hline \lambda_1 & -2 & 1 & 0 \\ \lambda_2 & 1 & -2 & 0 \\ \tau & 0 & 0 & 14 \end{array} \tag{9}$$

  Then we take $\mathbf{v} = \lambda_1 + 3\lambda_2 + \tau$, $\mathbf{v}' = \lambda_1$ and $\mathbf{w} = 14\lambda_1 + 28\lambda_2 + 9\tau$.

Now we can extend Addington–Thomas' result Theorem 3.1 into the following family version.

Proposition 3.8. Let d be an integer satisfying $(**)$ and $g = \frac{d}{2} + 1$. Let $\mathcal{X}$ and $\mathcal{S}$ be the family of cubic four-folds and K3 surfaces over $\mathcal{F}_g^\circ$ as above. Up to replacing $\mathcal{F}_g^\circ$ by a non-empty Zariski open subset, there exists a relative Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X} \times_{\mathcal{F}_g^\circ} \mathcal{S})$ such that for any $t \in \mathcal{F}_g^\circ$, the Fourier–Mukai transform with kernel $\mathcal{E}_t \in \mathrm{D}^{\mathrm{b}}(X_t \times S_t)$ induces an equivalence $\mathrm{Ku}(X_t) \xrightarrow{\simeq} \mathrm{D}^{\mathrm{b}}(S_t)$.

Proof. A distinguished connected component of the (numerical) stability manifold of cubic four-folds is constructed in [5]. Let $\mathbf{v}, \mathbf{v}', \mathbf{w}$ be as in Lemma 3.6. By [4, Theorem 29.4], for a $\mathbf{v}$-generic stability condition $\underline{\sigma}$ on $\mathcal{X}$ over $\mathcal{F}_g^\circ$, there is a relative moduli space $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ of Bridgeland stable objects in $\mathrm{Ku}(\mathcal{X}/\mathcal{F}_g^\circ)$ with Mukai vector $\mathbf{v}$, which is (up to shrinking $\mathcal{F}_g^\circ$) a relative projective K3 surface over $\mathcal{F}_g^\circ$. By the existence of the vector $\mathbf{v}'$ with $\mathbf{v} \cdot \mathbf{v}' = 1$, this moduli space $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ is fine. The existence of the vector $\mathbf{w}$ with $\mathbf{w}^2 = d$ implies that $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$ admits a relative polarization over $\mathcal{F}_g^\circ$ of degree d. We can therefore identify $\mathcal{S}$ with $\mathcal{M}_{\underline{\sigma}}(\mathbf{v})$. Let $\mathcal{E}$ be the universal sheaf. Then the corresponding Fourier–Mukai transform is an equivalence by [4, Lemma 33.2]. ■

We deduce the following family version of Bülles' result.

Corollary 3.9. Let the notation be as before. Up to shrinking $\mathcal{F}_g^\circ$, there exist cycles $Z \in \mathrm{CH}^3(\mathcal{X} \times_{\mathcal{F}_g^\circ} \mathcal{S})$ and $Z' \in \mathrm{CH}^3(\mathcal{S} \times_{\mathcal{F}_g^\circ} \mathcal{X})$, with the property that for any $t \in \mathcal{F}_g^\circ$, the cycles $Z_t, Z'_t \in \mathrm{CH}^3(X_t \times S_t)$ induce mutually inverse isomorphisms

$$\mathrm{CH}_1(X_t)_{\mathrm{hom}} \simeq \mathrm{CH}_0(S_t)_{\mathrm{hom}}.$$

Proof. Let $\mathcal{E}$ be as in Proposition 3.8 and let $\mathcal{E}^R$ be the relative Fourier–Mukai kernel of the right adjoint. Then Theorem 3.2 and Remark 3.3 show that $Z := v_3(\mathcal{E})$ and $Z' := v_3(\mathcal{E}^R)$ induce fiberwise inverse isomorphisms between $\mathrm{CH}_1(X_t)_{\mathrm{hom}}$ and $\mathrm{CH}_0(S_t)_{\mathrm{hom}}$. ■

Remark 3.10. By applying the argument (Manin's identity principle) as in the proof of [14, Theorem 0.4] to the relative Fourier–Mukai kernel $\mathcal{E}$ as well as its right adjoint, we can also show that there is an isomorphism between $\mathfrak{h}(\mathcal{X})$ and $\mathfrak{h}(\mathcal{S})(-1) \oplus \mathbb{1} \oplus \mathbb{1}(-2) \oplus \mathbb{1}(-4)$, as relative Chow motives over $\mathcal{F}_g^\circ$.

Proof. of Theorem 3.4 (hence Theorem 1.5) Given a point $t \in \mathcal{F}_g^\circ$, consider the following commutative diagram where the vertical arrows are Gysin restriction maps:

$$\begin{tikzcd} \mathrm{CH}^3(\mathcal{X}) \arrow[r, "Z_*"] \arrow[d, "r_1"] & \mathrm{CH}^2(\mathcal{S}) \arrow[d, "r_2"] \\ \mathrm{CH}^3(X_t) \arrow[r, "Z_{t,*}"] & \mathrm{CH}^2(S_t). \end{tikzcd} \tag{10}$$

Assume first the Franchetta property for $\mathcal{F}_{g,1}/\mathcal{F}_g$. For any $\alpha \in \mathrm{Im}(r_1) \cap \mathrm{CH}^3(X_t)_{\mathrm{hom}}$, the above diagram shows that $Z_{t,*}(\alpha) \in \mathrm{CH}^2(S_t)_{\mathrm{hom}} \cap \mathrm{Im}(r_2)$, hence is zero by assumption. By Corollary 3.9, $Z_{t,*}$ is an isomorphism, thus $\alpha = 0 \in \mathrm{CH}^3(X_t)$, that is, the Franchetta property is satisfied for $\mathcal{C}_{d,1}^{\mathrm{mar}}/\mathcal{C}_d^{\mathrm{mar}}$.

Similarly, by using Z' in Corollary 3.9, one can show that the Franchetta property for $\mathcal{C}_{d,1}^{\mathrm{mar}}/\mathcal{C}_d^{\mathrm{mar}}$ implies that for $\mathcal{F}_{g,1}/\mathcal{F}_g$. ■

Proof of Theorem 1.7 for $d = 14, 38$. Since 14 and 38 are both $\equiv 2 \pmod 6$, Theorem 1.5 applies. Therefore, the Franchetta property for the universal family over $\mathcal{C}_{14}$ and $\mathcal{C}_{38}$ are equivalent to Conjecture 1.2 for $g = 8$ and 20, respectively, which are proved in [47]. ■

4 Franchetta for $\mathcal{C}_8$ and $\mathcal{F}_2$

In this section, we first show Theorem 1.7 for $d = 8$, and then deduce from it a new proof of Conjecture 1.2 for $g = 2$. The key is the geometric characterization of special cubic four-folds of discriminant 8: those are exactly the ones containing a plane [28, Section 4.1.1].

Consider the following varieties.

$$B := \left\{X \subset \mathbb{P}^5 \mid X \text{ is a cubic four-fold containing a plane}\right\}.$$

$$P := \{(R,X) \mid X \text{ is a cubic four-fold}, R \text{ is a plane contained in } X\}.$$

We have natural morphisms $p\colon P \to B$ and $q\colon P \to \mathrm{Gr}(\mathbb{P}^2,\mathbb{P}^5)$ sending a couple (R,X) to X and R respectively. By construction, p,q are surjective, and the fiber of q over a point $[R] \in \mathrm{Gr}(\mathbb{P}^2,\mathbb{P}^5)$ parametrizes all the cubic four-folds containing the plane R, which is a Zariski open subset of $\mathbb{P}H^0(\mathbb{P}^5,\mathcal{I}_R \otimes \mathcal{O}(3)) \simeq \mathbb{P}^{45}$.

Proof of Theorem 1.7 for $d = 8$. As there is a dominant morphism $B \to \mathcal{C}_8$, it suffices to show the Franchetta property for the universal family of cubic four-folds $\mathcal{X} \to B$. For any $b \in B$, denote by X_b the corresponding special cubic four-fold of discriminant 8, and let R be any plane contained in X_b (for generic b, there is only one plane). It is obvious (or one uses Lemma 2.5) that

$$\mathrm{GDCH}^3_B(X_b) \subset \mathrm{GDCH}^3_{B_R}(X_b),$$

where $B_R \subset B$ is the subfamily of cubic four-folds containing the plane R.

By Proposition 2.4,

$$\mathrm{GDCH}^3_{B_R}(X_b) = \mathrm{Im}\big(\mathrm{CH}^3(\mathbb{P}^5) \to \mathrm{CH}^3(X_b)\big) + \mathrm{Im}\big(\mathrm{CH}^1(R) \to \mathrm{CH}^3(X_b)\big) = \mathbb{Q}h^3 + \mathbb{Q}l,$$

where h is the hyperplane section class and l is the class of a line in R. However, l and h^3 are proportional. Indeed, denoting by $i\colon R \hookrightarrow X$ and $\iota\colon X \hookrightarrow \mathbb{P}^5$ the natural closed immersions, we have

$$h^3 = \iota^*(\iota_*(R)) = R \cdot c_1(\mathcal{N}_{X/\mathbb{P}^5}) = R \cdot 3h = 3i_*i^*(h) = 3l. \tag{11}$$

Therefore, $\mathrm{GDCH}^3_B(X_b) = \mathbb{Q}h^3$. ■

As an application, we provide a proof of the generalized Franchetta conjecture 1.2 for $g = 2$, which is different from the one in [47] using Mukai models.

Proof of Conjecture 1.2 for $g = 2$. A generic cubic four-fold X in $\mathcal{C}_8$ contains only one plane, denoted by R. Projecting from R endows the blow-up $X' := \mathrm{Bl}_R X$ with a quadric fibration structure $\pi: X' \to \mathbb{P}^2$, where the base $\mathbb{P}^2$ parameterizes all $\mathbb{P}^3$'s containing R, and the fibers of π are exactly the quadric surfaces that are residual intersections (to R) of the corresponding $\mathbb{P}^3$ with X. The Stein factorization of the relative Hilbert scheme of lines of π is as follows:

$$\mathrm{Hilb}^{\mathrm{line}}(X'/\mathbb{P}^2) \to S \to \mathbb{P}^2,$$

where the first map is a $\mathbb{P}^1$-fibration and the second map is a double cover. The surface S is the associated (twisted) K3 surface (see [55, §1], [33]). We identify $\mathrm{CH}_0(\mathrm{Hilb}^{\mathrm{line}}(X'/\mathbb{P}^2))$ and $\mathrm{CH}_0(S)$. Note that there is a natural map i: $\mathrm{Hilb}^{\mathrm{line}}(X'/\mathbb{P}^2) \to F(X)$, providing a uniruled divisor in the Fano variety of lines. By [51, Example 1.5], the following composition is an isomorphism:

$$\mathrm{CH}_0(S) \simeq \mathrm{CH}_0(\mathrm{Hilb}^{\mathrm{line}}(X'/\mathbb{P}^2)) \to \mathrm{CH}_0(F(X)) \to \mathrm{CH}_1(X),$$

where the first map is induced by i and the second map is induced by the incidence variety $\{(l, x) \in F(X) \times X \mid x \in l\}$.

It is clear from the above construction that the isomorphism between $\mathrm{CH}_0(S)$ and $\mathrm{CH}_1(X)$ can be defined generically over the moduli space $\mathcal{C}_8$, which admits a dominant map to $\mathcal{F}_2$. Therefore, the Franchetta property for the universal family of cubic four-folds over $\mathcal{C}_8$, which is just proved previously, implies the generalized Franchetta conjecture 1.2 for $\mathcal{F}_2$. ■

Remark 4.1 (Twisted K3 surfaces). Recently, Brakkee [13] constructed and studied moduli spaces of twisted polarized K3 surfaces, as well as their relations with special cubic four-folds. In particular, the following is shown ([13, page 1475]): let g be a positive integer such that $d = 2g-2$ satisfies $(**)$ and $d \equiv 2 \pmod 6$, then for any r not divisible by 3, there exists a birational isomorphism between the moduli space of special cubic four-folds $\mathcal{C}_{dr^2}$ and the moduli space $\mathcal{F}_g[r]$ of order-r twisted K3 surfaces of genus g. Note that forgetting the Brauer class gives rise to a natural surjective map $\mathcal{F}_g[r] \to \mathcal{F}_g$. The same argument as in Section 3, in particular Theorem 3.4, can be adapted to the twisted case to show that *for d and g as before, the Franchetta property for the universal family over $\mathcal{C}_{dr^2}$ is equivalent to the Franchetta property for the universal family of K3 surfaces over $\mathcal{F}_g[r]$, hence implies the generalized Franchetta conjecture 1.2 for g.* Our new proof of Conjecture 1.2 for $g = 2$ given above is the special case where $g = r = d = 2$.

5 Franchetta for $\mathcal{C}_{20}$

The $d = 20$ case of Theorem 1.7 is already proved in [24, Lemma 6.3] using the so-called Küchle four-folds of type c7. In this section, we give an alternative proof, which is very similar to the case $d = 8$ treated in Section 4. The geometric input is Hassett's result [28, Section 4.1.4] that special cubic four-folds of discriminant 20 are characterized generically as the ones containing a Veronese surface, that is, the image of the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ via the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. Similarly, as in Section 4, set

$$B := \left\{X \subset \mathbb{P}^5 \mid X \text{ is a cubic four-fold containing a Veronese surface}\right\},$$

$$T := \left\{R \subset \mathbb{P}^5 \mid R \text{ is a Veronese surface}\right\},$$

$$P := \{(R, X) \mid X \text{ is a cubic four-fold}, R \text{ is a Veronese surface contained in } X\},$$

together with natural surjective morphisms $p\colon P \to B$ and $q\colon P \to T$.

Proof of Theorem 1.7 for $d = 20$. Let $\mathcal{X} \to B$ be the universal family. For any $b \in B$, denote the fiber by X_b and take a Veronese surface $R \subset X_b$. Let $B_R \subset B$ be the subvariety parametrizing cubic four-folds containing R, then

$$\mathrm{GDCH}^3_B(X_b) \subset \mathrm{GDCH}^3_{B_R}(X_b).$$

As R is cut out by quadrics, for any point $x \in \mathbb{P}^5 \backslash R$, there exists a cubic four-fold containing R and avoiding x. Hence Proposition 2.4 applies and gives that

$$\mathrm{GDCH}^3_{B_R}(X_b) = \mathrm{Im}\big(\mathrm{CH}^3(\mathbb{P}^5) \to \mathrm{CH}^3(X_b)\big) + \mathrm{Im}\big(\mathrm{CH}^1(R) \to \mathrm{CH}^3(X_b)\big) = \mathbb{Q}h^3 + \mathbb{Q}l,$$

where l is a line in R (so a conic in $\mathbb{P}^5$). A similar computation as in (11) gives that $3l = 2h^3$. Therefore, $\mathrm{GDCH}^3_B(X_b) = \mathbb{Q}h^3$. ■

6 Franchetta for $\mathcal{C}_{26}$ and $\mathcal{F}_{14}$

In this section, we establish Conjecture 1.2 in the case $g = 14$. Thanks to Theorem 3.4 (or Theorem 1.5), it is equivalent to proving Theorem 1.7 for $d = 26$. The key ingredient in our argument is the following geometric characterization of such special cubic fourfolds generically as the ones containing rational normal scrolls of degree 7 with 3 nodes. In the sequel, we often simply call such scrolls *3-nodal and septic*.

Theorem 6.1 (Farkas–Verra [19]). A generic member $X \in \mathcal{C}_{26}$ contains a 2-dimensional family of 3-nodal septic scrolls, parameterized by a non-empty Zariski open subset of the Hodge-theoretically associated genus 14 K3 surface of X. Conversely, given a 3-nodal septic scroll $R \subset \mathbb{P}^5$, a cubic four-fold containing R is special of discriminant 26.

Another key ingredient is on the defining equations of these scrolls:

Lemma 6.2 (Russo–Staglianò [50]). Let $R \subset \mathbb{P}^5$ be a generic 3-nodal septic scroll. Then R is cut out by cubic equations.

Proof. This has been checked in [50, Section 7], cf. item *(ii)* in Table 1 of loc. cit. Let us provide some details of the checking procedure using Macaulay2, which was kindly communicated to us by Michael Hoff. As in [19, pages 7–8], let $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, y_3]$ be the homogeneous coordinate ring of $\mathbb{P}^8$ and let $R' \subset \mathbb{P}^8$ be the smooth septic scroll defined as the locus where the following matrix is of rank ≤ 1, that is, the ideal of R' is generated by the 2×2 minors of the matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 \end{pmatrix}.$$

The (5-dimensional) secant variety $\mathrm{Sec}(R') \subset \mathbb{P}^8$ is defined as the locus where the following matrix is of rank ≤ 2, that is, the ideal of $\mathrm{Sec}(R')$ is generated by the 3×3 minors of the matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & y_0 & y_1 \\ x_1 & x_2 & x_3 & y_1 & y_2 \\ x_2 & x_3 & x_4 & y_2 & y_3 \end{pmatrix}.$$

Then we choose the following three points on $\mathrm{Sec}(R')\backslash R'$:

$$P_1 = [1:0:0:0:0:1:0:0:1],$$
$$P_2 = [1:1:1:1:1:1:2:2:2],$$
$$P_3 = [1:2:4:8:8:8:16:32:64].$$

The linear projection from $\mathbb{P}^8 \dashrightarrow \mathbb{P}^5$ from the projective plane spanned by P_1, P_2, P_3 sends R' to a 3-nodal septic scroll $R_0 \subset \mathbb{P}^5$, whose graded Betti diagram is as follows (computed by Macaulay2):

$$\begin{array}{ccccc} 1 & - & - & - & - \\ - & - & - & - & - \\ - & 13 & 24 & 15 & 3 \end{array} \tag{12}$$

This means that the minimal free resolution of the homogeneous coordinate ring of R_0 is as follows,

$$0 \to S(-6)^{\oplus 3} \to S(-5)^{\oplus 15} \to S(-4)^{\oplus 24} \to S(-3)^{\oplus 13} \to S \to S_{R_0} \to 0,$$

where S is the homogeneous coordinate ring of $\mathbb{P}^5$.

By upper semi-continuity, the entries of the Betti diagram for a generic 3-nodal septic scroll $R \subset \mathbb{P}^5$ are less than or equal to the ones in the above diagram (12). However, since the alternating sum of entries of each diagonal (i.e., line of 45°) in the Betti diagram is determined by the Hilbert polynomial (see [18, Corollary 1.10]), the purity of (12) (i.e., there is at most one non-zero entry in each diagonal) implies that the numbers in (12) are the minimal possible ones. Therefore, any generic 3-nodal septic scroll R has the same Betti diagram (12).

In other words, the minimal free resolution of the homogeneous coordinate ring of a generic 3-nodal septic scroll R takes the following form:

$$0 \to S(-6)^{\oplus 3} \to S(-5)^{\oplus 15} \to S(-4)^{\oplus 24} \to S(-3)^{\oplus 13} \to S \to S_R \to 0.$$

In particular, one sees that the ideal of R is generated by 13 cubics (with 24 linear relations). ■

Let us now consider the following parameter spaces:

$$T := \{R \subset \mathbb{P}^5 \mid R \text{ is a 3-nodal septic scroll}\}.$$

$$B := \{X \subset \mathbb{P}^5 \mid X \text{ is a cubic four-fold containing a 3-nodal septic scroll}\}.$$

$$P := \{R \subset X \subset \mathbb{P}^5 \mid X \text{ is a cubic four-fold}, R \in T\}.$$

We emphasize that in the above definitions, we do not quotient out by automorphisms, hence the spaces are some open subsets of certain Hilbert schemes in $\mathbb{P}^5$.

Then we have natural morphisms in the following diagram.

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ B & & T. \end{tikzcd} \tag{13}$$

By Theorem 6.1 (combined with [19, Proposition 3.4]), we have the following.

Lemma 6.3. In the above diagram.

(i) The natural map $B \to \mathcal{C}_{26}$ is dominant.

(ii) The morphism p is surjective. Its general fibers are Zariski open subsets of K3 surfaces.

(iii) The morphism q is surjective. Its general fibers are Zariski open subsets of $\mathbb{P}^{12}$.

Let $\pi : \mathcal{X} \to B$ be the universal family of cubic four-folds over B.

Proposition 6.4. For any $b \in B$, let X_b be the fiber of π over b. Then

$$\mathrm{GDCH}^3_B(X_b) \subset \bigcap_{t \in q(p^{-1}(b))} \left(\mathbb{Q}h^3 + \mathbb{Q}\ell_t\right),$$

where ℓ_t is the class in $\mathrm{CH}^3(X_b)$ of the ruling of R_t, and R_t is the scroll parameterized by $t \in T$.

Proof. Applying Lemma 2.5 to the diagram (13), we have that for any $b \in B$,

$$\mathrm{GDCH}^3_B(X_b) \subset \bigcap_{t \in q(p^{-1}(b))} \mathrm{GDCH}^3_{q^{-1}(t)}(X_b).$$

However, $q^{-1}(t)$ parameterizes all cubic four-folds containing R_t, which is an open subset of $\mathbb{P}\mathrm{H}^0(\mathbb{P}^5, \mathcal{O}(3) \otimes \mathcal{I}_{R_t}) \simeq \mathbb{P}^{12}$. Lemma 6.2 guarantees that for any point outside of R_t, there is a cubic four-fold containing R_t but not this point. Therefore, Proposition 2.4 implies that

$$\mathrm{GDCH}^3_{q^{-1}(t)}(X_b) = \mathrm{Im}\big(\mathrm{CH}^3(\mathbb{P}^5) \to \mathrm{CH}^3(X_b)\big) + \mathrm{Im}\big(\mathrm{CH}^1(R_t) \to \mathrm{CH}^3(X_b)\big). \tag{14}$$

On the right-hand side of (14), the first term is obviously $\mathbb{Q}h^3$. As for the second term, since there is a surjection $\mathbb{F}_1 \to R_t$ (see [19, Section 3]), where $\mathbb{F}_1 = \mathrm{Bl}_o \mathbb{P}^2$ is the first Hirzebruch surface, the group $\mathrm{CH}^1(R_t)$ is at most 2-dimensional, generated by the restriction $h|_{R_t}$ and the class ℓ_t of the ruling of the scroll. The class $h|_{R_t}$, when pushed-forward to X_b, is $h \cdot R_t$. To conclude, it suffices to show that $h \cdot R_t \in \mathrm{CH}^3(X_b)$ is proportional to h^3. To this end, let $\iota \colon X_b \to \mathbb{P}^5$ be the natural inclusion. Then we have

$$3h \cdot R_t = \iota^* \iota_*(R_t).$$

Since $\iota_*(R_t) = 7H^3 \in \mathrm{CH}^3(\mathbb{P}^5)$, where H is the hyperplane class of $\mathbb{P}^5$, we obtain that $3h \cdot R_t = 7h^3$. The proof is complete. ■

Now we are ready to prove the main results, Theorem 1.4, or equivalently, Theorem 1.7 for $d = 26$.

Proof of Theorem 1.7 for $d = 26$. Since there is a dominant morphism $B \to \mathcal{C}_{26}$, it is enough to show the Franchetta property for codimension-3 cycles for the universal family of special cubic four-folds $\pi \colon \mathcal{X} \to B$. Thanks to Proposition 6.4, it suffices to show that for a general cubic four-fold X of discriminant 26, there exists a 3-nodal septic scroll $R \subset X$, such that the class of the ruling ℓ of R, viewed as an element in $\mathrm{CH}^3(X)$, is proportional to h^3.

Let S be the K3 surface that is Hodge-theoretically associated to X. By [19] (see Theorem 6.1), there is a dense open subset $S_0 \subset S$ parameterizing the 3-nodal septic scrolls contained in X. Choose a constant cycle curve C intersecting S_0, which is possible because constant cycle curves are Zariski dense in S (see e.g., [58, Lemma 2.3]). For any $t \in C \cap S_0$, let R_t be the corresponding scroll in X. Since all rulings of R_t are parameterized by a rational curve T_t, we can view T_t as a rational curve in the Fano variety of lines $F(X)$. Therefore, we have well-defined (i.e., independent of $t \in C$) cycle classes $L := L_t \in \mathrm{CH}_0(F(X))$ and $\ell := P_*(L) \in \mathrm{CH}^3(X)$, where $P \subset F(X) \times X$ is the incidence subvariety (i.e., the universal projective line).

We claim that *the class* $L \in \mathrm{CH}_0(F(X))$ *has a 2-dimensional rational orbit*. Indeed, by [28], for $d = 26$, there is an isomorphism

$$\varphi \colon S^{[2]} \xrightarrow{\simeq} F(X)$$

between the Hilbert square of S and $F(X)$. Since C is a constant cycle curve in S, we have the following constant cycle surface in $S^{[2]}$:

$$W := \{z \in S^{[2]} \mid \mathrm{supp}(z) = \{t\}, t \in C\},$$

whose image under φ gives rise to a constant cycle surface in $F(X)$. To prove the claim, we only need to see that the points of this constant cycle surface $\varphi(W)$ represent the class $L \in \mathrm{CH}_0(F(X))$. To this end, let $\rho \colon S^{[2]} \to S^{(2)}$ be the Hilbert–Chow morphism, then by the construction of [19], for any $t \in S$, the septic rational curve $T_t \subset F(X)$ parameterizing the rulings of R_t is exactly $\varphi(\rho^{-1}(t))$, where t is viewed as a point of the diagonal $\Delta_S \subset S^{(2)}$. Hence the class of points on $\varphi(W)$ is L. The claim is proved. In other words, $L \in \mathsf{S}_2\mathrm{CH}_0(F(X))$, where $\mathsf{S}_\bullet$ refers to Voisin's orbit filtration on 0-cycles [59].

However, thanks to Voisin's result [59, Proposition 4.5] (or [59, Theorem 2.5]), we know that $\mathsf{S}_2\mathrm{CH}_0(F(X))$ is one-dimensional, generated by g^4, where g is the Plücker polarization class of $F(X)$. Hence $L \in \mathbb{Q}g^4$ in $\mathrm{CH}_0(F(X))$.

Since the incidence subvariety P in $F(X) \times X$ induces a morphism

$$P_* \colon \mathrm{CH}_0(F(X)) \to \mathrm{CH}^3(X),$$

which sends g^4 to $36h^3$ (see e.g., [52, Lemma A.4]) and $P_*(L) = \ell$ by construction, one can conclude that $\ell \in \mathbb{Q}h^3$. In other words, the ruling class ℓ_t is a proportional h^3 for any $t \in C \cap S_0$. The proof is complete. ∎

7 Franchetta for $\mathcal{C}_{14}$ and $\mathcal{F}_8$

The argument in Section 6 can also be applied to give a new proof of Conjecture 1.2 for $g = 8$, or equivalently (by Theorem 3.4), the Franchetta property for the universal family of special cubic four-folds over $\mathcal{C}_{14}$.

Recall that a generic cubic four-fold in $\mathcal{C}_{14}$ is a *Pfaffian* cubic (see Beauville–Donagi [7]), namely, a 4-dimensional smooth linear section of the Pfaffian cubic hypersurface

$$\mathrm{Pf} := \left\{ \phi \in \mathbb{P}(\bigwedge^2 V) \mid \phi \wedge \phi \wedge \phi = 0 \right\},$$

where V is a 6-dimensional vector space. The associated K3 surface S is the dual 2-dimensional linear section of $\mathrm{Gr}(2, V^\vee) \subset \mathbb{P}(\bigwedge^2 V^\vee)$. The key ingredient is the following characterization of cubic four-folds in $\mathcal{C}_{14}$ by smooth rational normal quartic scrolls, simply called *quartic scrolls* in the sequel, in analogy with Theorem 6.1.

Theorem 7.1 (Hassett [28, 4.1.3], Beauville–Donagi [7, Section 2], Tregub [53]). Let $\mathcal{C}_{14}$ be the moduli space of special cubic four-folds with discriminant 14.

(i) A generic member X in $\mathcal{C}_{14}$ is Pfaffian and contains a quartic scroll and conversely, a cubic four-fold containing a quartic scroll is in $\mathcal{C}_{14}$.

(ii) A Pfaffian cubic four-fold X contains a two-dimensional family of quartic scrolls parameterized by the associated K3 surface S. Moreover, there is a natural isomorphism $S^{[2]} \simeq F(X)$.

Another geometric fact we need is the following, see, for example, [29, §1.4].

Lemma 7.2. A quartic scroll in $\mathbb{P}^5$ is cut out by quadric equations.

Proof. In fact, a quartic scroll in $\mathbb{P}^5$ can be defined by the 2×2 minors of the matrix

$$\begin{pmatrix} u & v & x & y \\ v & w & y & z \end{pmatrix} \tag{15}$$

where $[u : v : w : x : y : z]$ are the homogeneous coordinates of $\mathbb{P}^5$. ■

Proof of Theorem 1.7 for $d = 14$. Consider

$$B := \{X \subset \mathbb{P}^5 \mid X \text{ is a Pfaffian cubic four-fold}\};$$
$$T := \{R \subset \mathbb{P}^5 \mid R \text{ is a quartic scroll}\};$$
$$P := \{R \subset X \subset \mathbb{P}^5 \mid X \text{ is a cubic four-fold}, R \in T\},$$

together with natural morphisms $p\colon P \to B$ and $q\colon P \to T$, which are surjective by Theorem 7.1 (i). Since $B \to \mathcal{C}_{14}$ is dominant, it suffices to show the Franchetta property for the universal family of cubic four-folds $\mathcal{X} \to B$.

Similarly to Proposition 6.4, we first show that

$$\mathrm{GDCH}^3_B(X_b) \subset \bigcap_{t \in q(p^{-1}(b))} \left(\mathbb{Q}h^3 + \mathbb{Q}\ell_t\right), \tag{16}$$

where $\ell_t \in \mathrm{CH}^3(X_b)$ of the class of a ruling of the scroll R_t, for any $t \in T$. Indeed, Lemma 2.5 yields that for any $b \in B$,

$$\mathrm{GDCH}^3_B(X_b) \subset \bigcap_{t \in q(p^{-1}(b))} \mathrm{GDCH}^3_{q^{-1}(t)}(X_b);$$

while for any $t \in q(p^{-1}(b))$, Lemma 7.2 allows us to apply Proposition 2.4 to obtain that

$$\mathrm{GDCH}^3_{q^{-1}(t)}(X_b) = \mathrm{Im}\left(\mathrm{CH}^3(\mathbb{P}^5) \to \mathrm{CH}^3(X_b)\right) + \mathrm{Im}\left(\mathrm{CH}^1(R_t) \to \mathrm{CH}^3(X_b)\right).$$

On the right-hand side, the first term gives $\mathbb{Q}h^3$, and the second term is generated by the push-forward of $h|_{R_t}$ and ℓ_t, since R_t is a rational ruled surface. A similar computation as in Proposition 6.4 yields that the push-forward of $h|_{R_t}$ is $\frac{4}{3}h^3$ in $\mathrm{CH}_1(X_b)$. The equality (16) is proved.

It remains to show that for any Pfaffian cubic four-fold X, there exists a quartic scroll $R \subset X$, such that the class of the ruling ℓ of R, viewed in $\mathrm{CH}^3(X)$, is proportional to h^3. The argument is as in the proof in Section 6 for the $d = 26$ case of Theorem 1.7. Let S be the associated K3 surface. Choose a (sufficiently generic) constant cycle curve C in S, then the rulings of the scrolls parametrized by $t \in C$ (see Theorem 7.1 (ii)) all represent the same classes $L \in \mathrm{CH}_0(F(X))$ and $\ell \in \mathrm{CH}^3(X)$.

The constant cycle curve C gives rise to a constant cycle surface in $S^{[2]}$:

$$W := \{z \in S^{[2]} \mid \mathrm{supp}(z) = \{t\}, t \in C\}.$$

Using the isomorphism $\varphi\colon S^{[2]} \simeq F(X)$ (Theorem 7.1 (*ii*)), we obtain a constant cycle surface in $F(X)$. One can check from the explicit construction of the isomorphism φ given in [7, Proposition 5] that the rational curve in $F(X)$ corresponding to the family of rulings of the scroll R_t parameterized by $t \in S$, is exactly the image under φ of $\rho^{-1}(t) \simeq \mathbb{P}^1$, where t is viewed as a point in $\Delta_S \subset S^{(2)}$ and $\rho\colon S^{[2]} \to S^{(2)}$ is the Hilbert–Chow morphism. It follows that for any point $w \in W$, the 0-cycle class $w \in \mathrm{CH}_0(S^{[2]})$ (which does not depend on w as W is a constant cycle surface) maps via φ to $L \in \mathrm{CH}_0(F(X))$. Therefore, the class L has 2-dimensional rational orbit, hence must be a multiple of g^4 by Voisin [59, Proposition 4.5]. By [52, Lemma A.4], we conclude that $\ell = P_*(L)$ is a multiple of h^3, as desired. ■

Corollary 7.3. Conjecture 1.2 holds for $g = 8$.

Proof. By Theorem 1.5, it follows from the Franchetta property for special cubic four-folds in $\mathcal{C}_{14}$, which has just been proved.

(We remark that instead of appealing to the general result Theorem 1.5, the second author has established in [36,Corollary 4.4] directly the link between the CH_1 of a Pfaffian cubic four-fold and the CH_0 of the associated K3 surface, which is generically defined. This avoids the use of techniques from derived categories.) ■

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