

HOCHSCHILD–KOSTANT–ROSENBERG ISOMORPHISM FOR DERIVED DELIGNE–MUMFORD STACKS

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ABSTRACT. We prove a Hochschild–Konstant–Rosenberg theorem for general derived Deligne–Mumford (DM) stacks, extending the results of Arinkin–Căldăraru–Hablicsek in the smooth, global quotient case although with different methods. To formulate our result, we introduce the notion of *orbifold inertia* stack of a derived DM stack; this supplies a finely tuned derived enhancement of the classical inertia stack, which does not coincide with the classical truncation of the free loop space. We show that, in characteristic 0, the shifted tangent bundle of the orbifold inertia stack is equivalent to the free loop space. This yields an explicit HKR equivalence between Hochschild homology and differential forms on the orbifold inertia stack, as algebras. We also construct a stacky filtered circle, leading to a filtration on the Hochschild homology of a derived DM stack whose associated graded complex recovers the de Rham theory of its orbifold inertia stack. This provides a generalization of recent work of Moulinos–Robalo–Toën to the setting of derived DM stacks.

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1. INTRODUCTION

1.1. Historical context. In this article we establish the Hochschild–Konstant–Rosenberg (HKR) equivalence for derived Deligne–Mumford stacks. In their seminal 1962 paper [11] Hochschild, Konstant and Rosenberg proved that, under suitable assumptions, the Hochschild homology of a smooth commutative algebra A is equivalent to its graded algebra of differential forms. Hochschild homology can be defined for any associative algebra A , as the derived tensor product of A with itself as an A -bimodule

$$\mathrm{HH}_*(A) \simeq A \otimes_{A \otimes A^{\mathrm{op}}}^{\mathrm{L}} A$$

The HKR theorem shows in particular that the algebra of differential forms, a subtle geometric invariant, can be computed in terms of a general construction in homological algebra which makes sense in much more general contexts than geometric ones. The HKR theorem is, from this perspective, one of the cornerstones of non-commutative geometry. In the mid-80s Connes and Tsygan discovered that the de Rham differential can also be recovered purely in terms of homological algebra, as a manifestation of a natural S^1 -action on $\mathrm{HH}_*(A)$. This finding opened the

way to many subsequent developments, showing that essentially the whole calculus of differential forms and polyvector fields can be transported to the non-commutative setting.

The HKR theorem can be extended to the global geometric case. The classical formulation of the equivalence in this setting states that if X is a smooth and proper scheme, its Hochschild homology is equivalent to the hyper-cohomology of differential forms

$$(1.1.1) \quad \mathrm{HH}_{-*}(X) \cong \bigoplus_{q-p=*} H^q(X, \Omega_X^p) .$$

The smoothness assumption can be dropped, at the price of working with the exterior powers of the cotangent complex. Within derived algebraic geometry, it is possible to reinterpret Hochschild homology in terms of a basic geometric construction, the free loop space. If X is a derived stack, its free loop space is the mapping stack

$$\mathcal{L}X := \mathbf{Map}(S^1, X) ,$$

and the Hochschild homology of X is equivalent to the algebra of functions on $\mathcal{L}X$:

$$\mathrm{HH}_{-*}(X) \simeq \mathcal{O}(\mathcal{L}X) .$$

As first observed by Ben-Zvi and Nadler in [5], the equivalence (1.1.1) can be understood as the linearization of a more fundamental equivalence of stacks. In characteristic 0, if X is a scheme there is a canonical equivalence

$$(1.1.2) \quad \exp : \mathbb{T}[-1]X \rightarrow \mathcal{L}X ,$$

where $\mathbb{T}[-1]X$ is the shifted tangent bundle of X . It is equally possible to understand the de Rham differential and compare it with the S^1 -action on $\mathcal{L}X$ in this setting, see [27]. The coherent cohomology of the structure sheaf of $\mathbb{T}[-1]X$ computes the right hand side (1.1.1); thus the HKR isomorphism (1.1.1) can be obtained from (1.1.2) by taking global functions. Furthermore, in [5] it was also proven that (1.1.2), despite being well-defined for derived Artin stacks, it is typically *not* an equivalence; rather it is an isomorphism only on formal completions (at the zero section on the source and at constant loops on the target).

1.2. Main goals. The main goal of this paper is to address the question of obtaining a stacky formulation of the HKR theorem for derived Deligne-Mumford (DM) stacks. Giving a map from S^1 to X amounts to choosing a point x of X together with monodromy data at x , i.e. a conjugacy class in the isotropy group of x . If X is DM, the isotropy groups are finite, and the monodromy becomes a discrete parameter. This yields a decomposition of the free loop space $\mathcal{L}X$ into its connected components: the components labeled by non-trivial monodromy data are sometimes referred to as twisted sectors. This is a fundamental new feature for stacks compared to the scheme case, where no twisted sectors appear, and is the main challenge in establishing a HKR theorem for DM stacks. Indeed, because of the contributions from twisted sectors, the Hochschild homology $\mathrm{HH}_*(X)$ can no longer be equivalent to the algebra of differential forms on X .

An HKR theorem for special DM stacks was proven by Arinkin–Căldăraru–Hablicsek in [3]. They restrict themselves to consider underived, smooth, global quotient DM stacks of the form

$$X = [Y/G]$$

where G is a finite group acting on a smooth scheme Y . Under these assumptions, they show that $\mathrm{HH}_*(X)$ can be computed in terms of differential forms on the *inertia stack* of X . Recall that the inertia stack IX is the self-intersection of the diagonal in the category of classical (i.e. underived) stacks,

$$\mathrm{IX} = X \times_{X \times X}^{\mathrm{cl}} X$$

Note that, since X is underived, the inertia is equivalent to the classical truncation of the free loop space

$$(1.2.1) \quad \mathrm{IX} \cong \mathrm{t}_0(\mathcal{L}X) .$$

Additionally, since X is a global quotient, the inertia can be easily computed via the following formula

$$(1.2.2) \quad \mathrm{IX} \cong \bigsqcup_{g \in G/G} [Y^g/C(g)] .$$

Here G/G is the set of conjugacy classes, Y^g denotes the classical locus of points in Y fixed under the action of g , and $C(g)$ is the centralizer of g . The main result of [3] is the construction of a canonical equivalence, for X of the special form $[Y/G]$ as above,

$$\exp : \mathrm{T}[-1]\mathrm{IX} \rightarrow \mathcal{L}X .$$

After linearizing by taking the global sections of the structure sheaf, one obtains the equivalence

$$(1.2.3) \quad \mathrm{HH}_{-*}([Y/G]) \cong \bigoplus_{g \in G/G} \bigoplus_{q-p=*} H^q(Y^g, \Omega_{Y^g}^p)^{C(g)} ,$$

which is a generalization of the global HKR theorem for schemes.

Our main goal is to establish the HKR equivalence for all derived DM stacks. We do not require the stacks to be smooth, or to admit a presentation as global quotients under the action of a finite group. In this respect, our result is new also in the setting of classical DM stacks. Our methods are quite different from the ones in [3], which relied on the construction of a splitting of the derived self-intersection of the diagonal. The extra generality in which we place ourselves makes that approach unviable. In fact, the very definition of the inertia stack of a *derived* DM stack is far from clear. Note that for general derived DM stacks equivalences (1.2.1) and (1.2.2) are no longer available. Indeed, one of the basic properties of the inertia stack is that the stack X itself should appear as one of the connected components of IX . Hence, if X has non-trivial derived structure, the inertia can no longer be defined as the classical truncation of the loop space.

1.3. Statements of the main results. Throughout the paper, we work over a field k of characteristic 0. As we shall explain below, some of our results remain valid also in mixed and positive characteristics. A detailed treatment of our constructions in these more general settings will appear in a forthcoming second version of the present article.

Our first contribution is the definition of a viable replacement of the inertia stack for derived Deligne-Mumford stacks, clearly inspired by the work of Abramovich, Graber and Vistoli [2]. Let C_r be the cyclic group of order r .

Definition 1.1. Let X be a derived DM stack. The *orbifold inertia* stack of X is defined as

$$\mathrm{I}^{\mathrm{DM}}X := \operatorname{colim}_r \mathbf{Map}(\mathrm{BC}_r, X)$$

To confirm that this is a reasonable definition, we check that the stack $\mathrm{I}^{\mathrm{DM}}X$ has several favorable properties. We list them below.

- (1) If X is a classical (i.e. underived) stack, then $\mathrm{I}^{\mathrm{DM}}X$ coincides with the classical inertia stack IX .
- (2) There are isomorphisms of classical truncations:

$$\mathrm{t}_0(\mathrm{I}^{\mathrm{DM}}X) \cong \mathrm{t}_0(\mathcal{L}X) \cong \mathrm{I}(\mathrm{t}_0(X)) .$$

(3) If $X \simeq [Y/G]$ is a derived global quotient DM stack, there is an equivalence

$$(1.3.1) \quad \mathrm{l}^{\mathrm{DM}} X \simeq \bigsqcup_{g \in G/G} [Y^g/C(g)],$$

where Y^g denotes the *genuine* fixed locus of Y under the action of g .

We refer the reader to Definition 5.1 for the notion of genuine fixed locus, a construction first considered by Gabber. Here we shall limit ourselves to remark that, when X is underived, the genuine fixed locus coincides with the classical fixed locus. Hence equivalence (1.3.1) generalizes (1.2.2) to global quotient derived DM stacks.

The following is our main result:

Theorem 1.2 (HKR for derived DM stacks, Theorem 4.9). *For every derived Deligne–Mumford stack X defined over a base ring of characteristic zero, there is a canonical equivalence over X*

$$\widehat{\mathrm{aff}}^* : \mathrm{T}[-1] \mathrm{l}^{\mathrm{DM}} X \longrightarrow \mathcal{L}X.$$

This equivalence is furthermore functorial in X .

Theorem 4.9 recovers the main result of [3] when X is a classical, smooth, global quotient DM stack as in that setting, our map $\widehat{\mathrm{aff}}^*$ coincides with the comparison map exp constructed in *loc. cit.* Taking global sections of the structure sheaves on both sides we obtain the following equivalence, which is a direct generalization of (1.1.1) to derived DM stacks.

Corollary 1.3 (Theorem 4.11, Corollary 4.18). *Let X be a derived Deligne–Mumford stack. Then we have an isomorphism of algebras*

$$(1.3.2) \quad \mathrm{HH}_*(X) \cong H^*(\mathrm{l}^{\mathrm{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathrm{l}^{\mathrm{DM}} X}[1])).$$

where the Hochschild homology $\mathrm{HH}_(X)$ is equipped with its natural algebra structure and $H^*(\mathrm{l}^{\mathrm{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathrm{l}^{\mathrm{DM}} X}[1]))$ is equipped with the natural algebra structure induced from the algebra structure on the symmetric algebra.*

Moreover, the equivalence (1.3.2) intertwines the natural S^1 -algebra structure on

$$\mathcal{O}(\mathcal{L}X) \simeq \mathrm{HH}_*(X)$$

and the natural mixed structure on

$$\mathcal{O}(\mathrm{T}[-1] \mathrm{l}^{\mathrm{DM}} X) \simeq H^*(\mathrm{l}^{\mathrm{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathrm{l}^{\mathrm{DM}} X}[1]))$$

induced by the de Rham differential.

Corollary 1.4 (Corollary 4.14). *Let X be a derived DM stack. There is a canonical isomorphism of graded vector spaces:*

$$\mathrm{HH}^*(X) \simeq \Gamma(\mathrm{T}[-1] \mathrm{l}^{\mathrm{DM}} X, q^!(\mathcal{O}_X)).$$

If X is moreover of finite type and lci, then

$$(1.3.3) \quad \mathrm{HH}^*(X) \simeq \Gamma(\mathrm{l}^{\mathrm{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathrm{l}^{\mathrm{DM}} X}[1]) \otimes i^* \omega_X^\vee[-\dim(X)]).$$

where ω_X is the dualizing sheaf of X , $i: \mathrm{l}^{\mathrm{DM}} X \rightarrow X$ is the canonical morphism.

In Section 5 we work out explicitly some consequences of our results in special settings. First of all, we treat the case of derived DM stacks which admit a presentation as a global quotient of a derived scheme by a finite group G . We prove an explicit formula for the DM inertia in terms of genuine fixed point loci; we have already presented it as equation (1.3.1) above. This implies a decomposition of Hochschild homology parametrized by conjugacy classes, and gives rise to more explicit formulations of the HKR equivalence in that setting.

In Section 5.2 we turn our attention to DM stacks that cannot be presented as global quotients by finite group actions. These examples are particularly relevant for us as they fall beyond the scope of the HKR theorems which are currently available in the literature. As such, they show the reach of our new techniques and findings. We obtain some general statements in the underived case—the derived enhancement of these results will be included in the second version of this article—and give a detailed treatment of two simple but instructive examples: Thurston’s football and teardrop, two stacky projective lines which cannot be obtained as global quotients by finite groups.

In the last section of the paper, we revisit the HKR equivalence from the vantage point of the theory of the filtered circle, a framework which was first proposed by Simpson, and studied in depth in [15]. It is well known that the HKR theorem holds only in characteristic 0; it fails, in general, in mixed or positive characteristic. As proved in [15], however, there is a more fundamental statement, underlying the HKR equivalence, which holds in any characteristic: namely, the existence of the HKR filtration. The Hochschild chain complex $\mathrm{HH}_*(X)$ carries a natural filtration, whose associated graded is

$$\mathrm{R}\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[1]))$$

Following a beautiful observation of Simpson, this filtration can be constructed geometrically.

Recall that a filtered stack, by definition, is a stack X equipped with a map to $[\mathbb{A}^1/\mathbb{G}_m]$. Conceptually, the fiber over $[\mathbb{G}_m/\mathbb{G}_m]$ should be thought of as the stack underlying the filtration, and the fiber over $B\mathbb{G}_m$ as the associated graded object. The *filtered circle* S_{fil}^1 is a stack over $[\mathbb{A}^1/\mathbb{G}_m]$ with fiber pattern

$$\begin{array}{ccccc} \mathbb{D}_{-1} & \longrightarrow & S_{\mathrm{fil}}^1 & \longleftarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ B\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $\mathbb{D}_{-1}$ is the suspension of the spectrum of the dual numbers. Mapping out of S_{fil}^1 to a target derived Artin stack X , we obtain the *filtered loop space* $\mathcal{L}_{\mathrm{fil}}X$. This is a filtered stack whose underlying stack is $\mathcal{L}X$, and whose associated graded stack is the shifted tangent $\mathrm{T}[-1]X$. Under suitable assumptions on X , taking global functions yields the desired HKR filtration on $\mathcal{O}(\mathcal{L}X) \simeq \mathrm{HH}_*(X)$ such that

$$\mathrm{AssGr}(\mathrm{HH}_*(X)) \simeq \mathcal{O}(\mathrm{T}[-1]X) \simeq \Gamma(X, \mathrm{Sym}(\mathbb{L}_X[1]))$$

If X is a char. 0 scheme this filtration collapses, and $\mathrm{HH}_*(X)$ is equivalent to its associated graded. This is precisely the content of the classical HKR theorem. In positive and mixed characteristic, however, the filtration is highly non-trivial, and allows us to interpolate between the Hochschild homology of X and its de Rham theory.

In the last section of the paper we study a variant of the filtered circle which is adapted to the case of DM stacks. When X is not a scheme, the filtered loop space turns out to be a rather coarse interpolation between the loop space of X and its de Rham theory. To circumvent this issue we design a filtered loop space whose central fiber is $\mathrm{T}[-1]^{\mathrm{DM}}X$. The key ingredient are variants of the filtered circle, which we denote $S_{\mathrm{fil}}^{1,(r)}$, that have fiber pattern

$$\begin{array}{ccccc} [\mathbb{D}_{-1}/\mathbb{C}_r] & \longrightarrow & S_{\mathrm{fil}}^{1,(r)} & \longleftarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ B\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where the quotient $[\mathbb{D}_{-1}/C_r]$ is taken with respect to natural C_r -action. Passing to the (formal) limit as r runs over the natural numbers, we obtain the *stacky filtered circle*, denoted $\widehat{S}_{\text{filt}}^1$, which is a pro-stack over $[\mathbb{A}^1/\mathbb{G}_m]$. If X is a derived DM stack, its *DM filtered loop space* is the filtered stack

$$\mathcal{D}_X^{\text{DM}} := \mathbf{Map}_{/[\mathbb{A}^1/\mathbb{G}_m]}(\widehat{S}_{\text{filt}}^1, X \times [\mathbb{A}^1/\mathbb{G}_m])$$

Theorem 1.5 (Proposition 6.8, Corollary 6.9). *Let X be a DM stack. Then $\mathcal{D}_X^{\text{DM}}$ is a filtered stack with fiber pattern*

$$\begin{array}{ccccc} T[-1]\mathbb{I}^{\text{DM}}X & \longrightarrow & \mathcal{D}_X^{\text{DM}} & \longleftarrow & \mathcal{L}X \\ \downarrow & & \downarrow & & \downarrow \\ B\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \text{Spec}(k) \end{array}$$

Taking global functions yields a filtration on $\text{HH}_*(X)$ such that the associated graded complex is equivalent $\Gamma(\mathbb{I}^{\text{DM}}X, \text{Sym}(\mathbb{L}_{\mathbb{I}^{\text{DM}}X}[1]))$.

We remark that Theorem 1.5 is particularly significant when working in mixed or graded characteristics. In that setting, it should be considered as a generalization of the theory developed in [15], which applies to affine schemes, to the more general setting of DM stacks. In the present iteration of the paper, however, all constructions are implemented in char. 0. The missing details in the mixed and positive characteristic case will appear in the forthcoming second version of the present article.

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Conventions. All the derived Deligne-Mumford stacks considered in this paper are assumed to be 1-stacks.

2. SOME RESULTS ON MAPPING STACKS

In this section we collect some foundational results that we could not locate in the exact form that we needed in the literature. There is an evident overlap of intention with [4, §3.2] and [10, Appendix A], but the results discussed below are slightly more general. We systematically use the theory of tensor products in $\mathbf{Pr}_k^{\text{L}}$, which was not yet fully available at the time [4, 10] were firstly written. We refer the reader to [13, §4.8.1] and to [18, §I.4] for background on this topic.

2.1. Categorical finiteness properties. We introduce the following finiteness conditions on (the ∞ -categories of quasi-coherent complexes of) derived stacks.

Definition 2.1. Let $F \in \mathbf{dSt}_k$ and write $p: F \rightarrow \text{Spec}(k)$ for the structural morphism.

- (1) The derived stack F is called $\otimes$ -universal if for any derived stack $X \in \mathbf{dSt}_k$, the canonical morphism

$$\boxtimes: \text{QCoh}(F) \otimes_k \text{QCoh}(X) \longrightarrow \text{QCoh}(F \times X)$$

is an equivalence.

- (2) The derived stack F is called *categorically quasi-compact* if the functor

$$p_*: \mathrm{QCoh}(F) \longrightarrow \mathrm{Mod}_k$$

commutes with filtered colimits.

- (3) The derived stack F is called *categorically perfect* if the functor

$$p^*: \mathrm{Mod}_k \longrightarrow \mathrm{QCoh}(F)$$

admits a left adjoint p_+ .

Example 2.2. Let $F \in \mathrm{dSt}_k$.

- (1) If $\mathrm{QCoh}(F)$ is dualizable in $\mathbf{Pr}_k^L$, then F is $\otimes$ -universal. Indeed, in this case $\mathrm{QCoh}(F) \otimes_k (-)$ commutes with limits, and therefore both source and target $\boxtimes$ satisfy descent in X and so it is enough to test that it is an equivalence when X is affine. In this case, the claim follows from [4, Proposition 4.13] (applied with $Y = \mathrm{Spec}(k)$).
- (2) Let $K \in \mathbf{Spc}$ be an ∞ -groupoid, seen as a constant derived stack. Then

$$\mathrm{QCoh}(K) \simeq \mathrm{Fun}(K, \mathrm{Mod}_k)$$

is compactly generated, and for every $A \in \mathrm{dCAlg}_k$ we have

$$\mathrm{QCoh}(K) \otimes_k \mathrm{Mod}_A \simeq \mathrm{Fun}(K, \mathrm{Mod}_A),$$

as it follows from [13, Proposition 4.8.1.17]. Thus, K is $\otimes$ -universal. If K is in addition a compact object in $\mathbf{Spc}$, then the constant derived stack K is categorically quasi-compact.

Lemma 2.3. *Assume that F is categorically quasi-compact, that $\mathrm{QCoh}(F) \simeq \mathrm{Ind}(\mathrm{Perf}(F))$ and that $p_*: \mathrm{QCoh}(F) \rightarrow \mathrm{Mod}_k$ preserves perfect complexes. Then F is categorically perfect, and moreover the left adjoint p_+ is characterized by the fact that for any $M \in \mathrm{Perf}(F)$, we have*

$$p_+(M) \simeq (p_*(M^\vee))^\vee.$$

Proof. Under these assumptions, projection formula holds, that is, the natural comparison morphism

$$p_*(M) \otimes N \longrightarrow p_*(M \otimes p^*(N))$$

is an equivalence. The same computation of [19, Proposition 7.11] shows that defining p_+ by the given formula on perfect complexes and taking the left Kan extension along $\mathrm{Perf}(F) \rightarrow \mathrm{Ind}(\mathrm{Perf}(F))$ produces indeed a left adjoint for p^* . See also [14, Proposition 6.4.5.3]. $\square$

Lemma 2.4. *Let $F \in \mathrm{dSt}_k$ be $\otimes$ -universal and categorically quasi-compact. Then for every morphism $f: X \rightarrow Y$ in dSt_k , the square*

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow p_Y^* & & \downarrow p_X^* \\ \mathrm{QCoh}(F \times Y) & \xrightarrow{(\mathrm{id}_F \times f)^*} & \mathrm{QCoh}(F \times X) \end{array}$$

is vertically right adjointable, that is, the natural Beck-Chevalley transformation

$$f^* \circ p_{Y,*} \longrightarrow p_{X,*} \circ (\mathrm{id}_F \times f)^*$$

is an equivalence. If in addition F is categorically perfect, then the above square is also vertically left adjointable.

Proof. Since F is $\otimes$ -universal, we find canonical identifications

$$(\mathrm{id}_F \times f)^* \simeq \mathrm{id}_{\mathrm{QCoh}(F)} \otimes f^*, \quad p_X^* \simeq p^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}, \quad q_X^* \simeq q^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}.$$

Since F is categorically quasi-compact, the adjunction $p^* \dashv p_*$ holds in $\mathbf{Pr}_k^{\mathrm{L}}$. The functoriality of the tensor product in $\mathbf{Pr}_k^{\mathrm{L}}$ guarantees therefore that the adjunctions

$$p^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)} \dashv p_* \otimes \mathrm{id}_{\mathrm{QCoh}(X)} \quad \text{and} \quad p^* \otimes \mathrm{id}_{\mathrm{QCoh}(Y)} \dashv p_* \otimes \mathrm{id}_{\mathrm{QCoh}(Y)}$$

hold. The uniqueness of the adjoint therefore provides two more identifications

$$p_{X,*} \simeq p_* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}, \quad q_{X,*} \simeq q_* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}.$$

At this point, the conclusion is obvious. The same argument shows that if there exists a left adjoint p_+ for p^* , then the adjunction $p_+ \otimes \mathrm{id}_{\mathrm{QCoh}(X)} \dashv p_X^*$ holds as well, and therefore the base change holds as well. $\square$

Notation 2.5. Let

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ & \searrow p \quad \swarrow q & \\ & \mathrm{Spec}(k) & \end{array}$$

be a diagram in $\mathbf{dSt}_k$. The associated $*$ -Beck-Chevalley transformation is the natural transformation

$$q_* \longrightarrow q_* f_* f^* \simeq p_* f^*.$$

If furthermore F and G are categorically perfect, the associated $+$ -Beck-Chevalley transformation is the natural transformation

$$p_+ f^* \longrightarrow p_+ f^* q^* q_+ \simeq p_+ p^* q_+ \longrightarrow q_+.$$

Lemma 2.6. *Let $f: F \rightarrow G$ be a morphism in $\mathbf{dSt}_k$. For every $X \in \mathbf{dSt}_k$, let*

$$p_X: F \times X \longrightarrow X, \quad q_X: G \times X \longrightarrow X$$

be the canonical projections. Assume that:

- (1) *both F and G are $\otimes$ -universal and categorically quasi-compact.*
- (2) *the $*$ -Beck-Chevalley transformation associated to f*

$$q_* \longrightarrow p_* f^*$$

is an equivalence.

Then for every $X \in \mathbf{dSt}_k$, the triangle

$$\begin{array}{ccc} \mathrm{QCoh}(G \times X) & \xrightarrow{(f \times \mathrm{id}_X)^*} & \mathrm{QCoh}(F \times X) \\ & \searrow q_X^* \quad \swarrow p_X^* & \\ & \mathrm{QCoh}(X) & \end{array}$$

is vertically right adjointable. If F and G are moreover categorically perfect, and $+$ -Beck-Chevalley transformation associated to f is an equivalence, then the above triangle is vertically left adjointable as well.

Proof. Assumption (1) provides canonical identifications

$$(f \times \mathrm{id}_X)^* \simeq f^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}, \quad p_X^* \simeq p^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}, \quad q_X^* \simeq q^* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}.$$

As in the proof of Lemma 2.4, we furthermore obtain identifications

$$p_{X,*} \simeq p_* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}, \quad q_{X,*} \simeq q_* \otimes \mathrm{id}_{\mathrm{QCoh}(X)}.$$

Thus, the conclusion follows from (3) and the functoriality of the tensor product in $\mathbf{Pr}_k^L$. The proof of left adjointability follows the same lines. $\square$

The following proof is completely standard, and well-documented in the literature. See [22, Proposition B.3.5] or [16, Proposition 1.4]. The proof is elementary, so we include it for the convenience of the reader:

Proposition 2.7. *Let $F, X \in \mathbf{dSt}_k$ be derived stacks. Assume that F is $\otimes$ -universal and categorically perfect. Assume that X admits a global cotangent complex and is infinitesimally cohesive. Then $\mathbf{Map}(F, X)$ admits a global cotangent complex, given by the formula*

$$\mathbb{L}_{\mathbf{Map}(F, X)} \simeq \pi_+ \mathrm{ev}^*(\mathbb{L}_X) ,$$

where $\mathrm{ev}: X \times \mathbf{Map}(F, X) \rightarrow X$ is the evaluation map and $\pi: X \times \mathbf{Map}(F, X) \rightarrow \mathbf{Map}(F, X)$ is the canonical projection.

Proof. Let $S \in \mathbf{dAff}_k$ be a test derived affine scheme and fix a morphism $S \rightarrow \mathbf{Map}(F, X)$ classifying a morphism $f: S \times F \rightarrow X$. Let $M \in \mathbf{QCoh}(S)_{\geq 0}$ and write $S[M]$ the split square-zero extension determined by M . By definition of mapping stack, the space of liftings

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{Map}(F, X) \\ \downarrow & \nearrow & \\ S[M] & & \end{array}$$

is equivalent to the space of liftings

$$\begin{array}{ccc} S \times F & \xrightarrow{f} & X \\ \downarrow & \nearrow & \\ S[M] \times F & & \end{array} .$$

However, $S[M] \times F \simeq (S \times F)[p_S^*(M)]$, where $p_S: S \times F \rightarrow S$ is the canonical projection. Therefore, the space of such liftings is equivalent to

$$(2.1.1) \quad \mathbf{Map}_{\mathbf{QCoh}(S \times F)}(f^* \mathbb{L}_X, p_S^* M) .$$

Since p^* admits a left adjoint and F is $\otimes$ -universal, the same argument given in Lemma 2.4 implies that p_S^* also admits a left adjoint $p_{S,+}$. This shows that $\mathbf{Map}(F, X)$ admits a cotangent complex at the given point, which is furthermore given by the formula $p_{S,+} f^*(\mathbb{L}_X)$. The base change for the plus pushforward proven in Lemma 2.4 readily implies that $\mathbf{Map}(F, X)$ has a global cotangent complex, given by the claimed formula. $\square$

Variation 2.8. Let $X \in \mathbf{dSt}_k$ be a derived stack and assume that it is infinitesimally cohesive and formally étale. Then for any derived stack F , $\mathbf{Map}(F, X)$ admits a global cotangent complex, given by 0. More generally, if $p: X \rightarrow Y$ is an infinitesimally cohesive and formally étale morphism in $\mathbf{dSt}_k$, then for any derived stack F , the induced morphism $\mathbf{Map}(F, X) \rightarrow \mathbf{Map}(F, Y)$ is again formally étale. The same argument given in Proposition 2.7 supplies an identification of the space of derivations with (2.1.1). The formal étaleness assumption now supplies $f^* \mathbb{L}_p \simeq 0$, so the conclusion is automatic.

Construction 2.9. Let $f: F \rightarrow G$ be a morphism in $\mathbf{dSt}_k$. Assume that F is $\otimes$ -universal and that both F and G are categorically perfect. Fix as well $X \in \mathbf{dSt}_k$ admitting a cotangent complex and consider the induced morphism

$$\phi_f: \mathbf{Map}(G, X) \longrightarrow \mathbf{Map}(F, X) ,$$

which is part of the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbf{Map}(G, X) & \xlongequal{\quad} & \mathbf{Map}(G, X) \\
 & \nearrow \pi_{F,G} & \downarrow \phi_f & & \nearrow \pi_G \\
 F \times \mathbf{Map}(G, X) & \xrightarrow{\quad} & G \times \mathbf{Map}(G, X) & & \\
 \downarrow \text{id}_F \times \phi_f & & \downarrow \text{ev}_G & & \\
 & \nearrow \pi_F & \mathbf{Map}(F, X) & & \\
 F \times \mathbf{Map}(F, X) & \xrightarrow{\quad \text{ev}_F \quad} & X & &
 \end{array}$$

where we set $g := f \times \text{id}_{\mathbf{Map}(G, X)}$. Since F is $\otimes$ -universal and categorically perfect, Lemma 2.4 guarantees that the Beck-Chevalley transformation

$$\phi_f^* \circ \pi_{F,+} \longrightarrow \pi_{F,G,+} \circ (\text{id}_F \times \phi_f)^*$$

is an equivalence. Besides, the top square induces a second Beck-Chevalley transformation

$$(2.1.2) \quad \text{BC}: \pi_{F,G,+} \circ g^* \longrightarrow \pi_{G,+},$$

which, paired with the above equivalence gives rise to a natural transformation

$$\phi_f^* \circ \pi_{F,+} \circ \text{ev}_F^* \longrightarrow \pi_{G,+} \circ \text{ev}_G^*.$$

Evaluating on $\mathbb{L}_X$ gives rise via to the natural comparison map

$$(2.1.3) \quad \phi_f^* \mathbb{L}_{\mathbf{Map}(F, X)} \longrightarrow \mathbb{L}_{\mathbf{Map}(G, X)}.$$

Proposition 2.10. *Let $F, G \in \mathbf{dSt}_k$ be $\otimes$ -universal and categorically perfect derived stacks. Write $p: F \rightarrow \text{Spec}(k)$ and $q: G \rightarrow \text{Spec}(k)$ for the structural morphisms. Let $f: F \rightarrow G$ be a morphism and assume that the $+$ -Beck-Chevalley transformation*

$$p_+ f^* \longrightarrow p_+ f^* q^* q_+ \simeq p_+ p^* q_+ \longrightarrow q_+$$

is an equivalence. Then for every derived stack $X \in \mathbf{dSt}_k$ admitting a cotangent complex, the induced transformation

$$\phi_f: \mathbf{Map}(G, X) \longrightarrow \mathbf{Map}(F, X)$$

is formally étale, that is, it admits a cotangent complex which is moreover zero.

Proof. It follows from Proposition 2.7 that $\mathbb{L}_{\phi_f}$ is canonically identified with the cofiber of the map (2.1.3). It is therefore enough to prove that said morphism is an equivalence. Tracing the definition given in Construction 2.9 we see that if the Beck-Chevalley transformation (2.1.2) is an equivalence, then the same goes for (2.1.3). Since F and G are $\otimes$ -universal and categorically perfect, the conclusion follows directly from Lemma 2.6. $\square$

2.2. Étale codescent. The second goal of this section is to prove the following codescent property:

Theorem 2.11. *Let G be a connected and underived algebraic group. Let X be a derived Deligne-Mumford stack and let $X_{\text{ét}}$ be the small étale site of X . Then the functor*

$$\mathbf{Map}(\text{BG}, -): X_{\text{ét}} \longrightarrow \mathbf{dSt}_k$$

satisfies étale codescent.

Remark 2.12. The assumption that the source of the mapping stack is of the form BG with G connected cannot be easily weakened. For instance, $\mathbf{Map}(S^1, -)$ only satisfies codescent with respect to *representable* étale covers.

The proof of Theorem 2.11 heavily relies on the following technical result, that is also a key ingredient in the proof of Proposition 4.8 later in the paper:

Lemma 2.13. *Let G be a connected and underived algebraic group. For any underived affine scheme $S \in \text{Aff}_k$ and any derived Deligne-Mumford stack X , the canonical morphism*

$$(2.2.1) \quad \text{Map}(S \times \text{BG}, X) \longrightarrow \text{Map}(S, X)$$

induced by precomposition with $S \rightarrow S \times \text{BG}$, is an equivalence.

Proof. Observe that, since S and $S \times \text{BG}$ are underived, we can replace X by $\text{t}_0(X)$, or, equivalently, assume that X is underived from the very beginning.

Since (2.2.1) is clearly surjective on π_0 , it suffices to argue that it has contractible fibers. Fix therefore $x: S \rightarrow X$ a morphism. The fiber of (2.2.1) at x is described as the fiber of

$$\text{Map}_{\text{St}_{S/}}(S \times \text{BG}, X) \longrightarrow \text{Map}_{\text{St}_{S/}}(S, X) ,$$

or equivalently as fiber of

$$(2.2.2) \quad \text{Map}_{\text{St}_{S//S}}(S \times \text{BG}, S \times X) \longrightarrow \text{Map}_{\text{St}_{S//S}}(S, S \times X) .$$

Following [13, Example 5.2.6.13], we find an equivalence

$$S \times \text{BG} \simeq \text{B}_S(S \times G) \simeq \text{Bar}^{(1)}(S \times G) ,$$

where the bar construction is performed in the ∞ -topos $\text{St}_{/S}$ of *underived* stacks over S . Similarly, S can be seen as the bar construction of the trivial group over S . Combining [13, Notation 5.2.6.11 & Remark 5.2.6.12], we see that

$$\text{Bar}^{(1)}: \text{Mon}_{\mathbb{E}_1}(\text{St}_{S//S}) \longrightarrow \text{St}_{S//S}$$

admits $\text{Cobar}^{(1)}(-) \simeq \Omega^1(-)$ as a right adjoint. Here $\Omega^1(-)$ denotes the based loop space functor in $\text{St}_{S//S}$. We can therefore rewrite the map (2.2.2) as

$$\text{Map}_{\text{Mon}_{\mathbb{E}_1}(\text{St}_{/S})}(S \times G, \Omega^1(S \times X)) \longrightarrow \text{Map}_{\text{Mon}_{\mathbb{E}_1}(\text{St}_{/S})}(S, \Omega^1(S \times X)) ,$$

induced by the unit section $S \rightarrow S \times G$, seen as a morphism of groups. Notice that these mapping spaces are discrete. Thus, in order to conclude the argument it is therefore enough to argue that any morphism of S -groups

$$S \times G \longrightarrow \Omega^1(S \times X)$$

factors through the unit section of $\Omega^1(S \times X)$.

Observe now that

$$\Omega^1(S \times X) := S \times_{S \times X} S \simeq S \times_{S \times X} (\text{IX} \times S) ,$$

and that the unit section of this S -group is the pullback of the diagonal embedding

$$X \times S \longrightarrow \text{IX} \times S .$$

Since X is a Deligne-Mumford stack, the diagonal $X \rightarrow X \times X$ is unramified. This implies that the map $X \rightarrow \text{IX}$ is an open immersion, see [24, Tag 02GE]. It follows that the unit section $S \rightarrow \Omega^1(S \times X)$ is an open immersion. In particular, a morphism $S \times G \rightarrow \Omega^1(S \times X)$ factors through the unit section if and only if it factors topologically. In order to check this latter statement, it is enough to assume that S is the spectrum of a field. In this case, since G is connected by assumption, and the morphism is assumed to be a morphism of groups, it must factor through the connected component of the identity in $\Omega^1(S \times X)$, which is reduced to the unit element itself. $\square$

Proof of Theorem 2.11. Since the functor $\mathbf{Map}(\mathbf{BG}, -)$ commutes with limits, it commutes also with the formation of Čech nerves. In particular, it suffices to show that if $f: U \rightarrow V$ is an étale epimorphism, the same goes for

$$\phi_f: \mathbf{Map}(\mathbf{BG}, U) \longrightarrow \mathbf{Map}(\mathbf{BG}, V) .$$

First, we observe that Variant 2.8 implies that ϕ_f is formally étale. It immediately follows that ϕ_f is an effective epimorphism if and only if its truncation $t_0(\phi_f)$ is. Besides, since G is underived, for any derived stack Y we have a canonical identification

$$t_0 \mathbf{Map}(\mathbf{BG}, Y) \simeq t_0 \mathbf{Map}(\mathbf{BG}, t_0(Y)) .$$

We can therefore assume from the very beginning that X (and hence U and V) is underived.

Fix an affine test scheme $S \in \mathbf{Aff}_k$, and a morphism $S \rightarrow \mathbf{Map}(\mathbf{BG}, V)$, corresponding to a morphism $S \times \mathbf{BG} \rightarrow V$. We have to show that there exists an étale cover $S' \rightarrow S$ such that the restriction

$$S' \times \mathbf{BG} \longrightarrow S \times \mathbf{BG} \longrightarrow V$$

factors through $f: U \rightarrow V$. It follows from Proposition 4.8 that

$$\mathbf{Map}(S \times \mathbf{BG}, V) \longrightarrow \mathbf{Map}(S, V)$$

is an equivalence, and similarly for U in place of V . Thus, the conclusion follows from the fact that $f: U \rightarrow V$ was an effective epimorphism to begin with. $\square$

3. INERTIA FOR DERIVED DELIGNE-MUMFORD STACKS

Let X be a derived Deligne-Mumford stack (see for example [17] for the basic definition). We emphasize that in this work we only consider 1-stacks. We typically think of X as a derived stack, but we will occasionally use its representation as a structured ∞ -topos.

For an underived Deligne-Mumford stack X , it is customary to define the inertia stack of X as

$$\mathbf{IX} := X \times_{X \times X} X ,$$

where the fiber product is taken in the category $\mathbf{St}_k$ of underived stacks. It can equivalently be realized as the (underived) mapping stack from the constant stack associated to $S^1 \in \mathbf{Spc}$:

$$(3.0.1) \quad \mathbf{IX} \cong \mathbf{Map}_{\mathbf{St}_k}(S^1, X) .$$

The very same constructions operated in $\mathbf{dSt}_k$ yields the derived loop stack $\mathcal{L}X$:

$$(3.0.2) \quad \mathcal{L}X \cong \mathbf{Map}(S^1, X) \cong X \times_{X \times X}^d X .$$

To generalize the HKR theorem, one needs to introduce a meaningful construction of the inertia stack.

3.1. Derived orbifold inertia. Our definition is motivated by the work of Abramovich–Graber–Vistoli [1]. For every integer $r > 0$, let $\mathbf{C}_r$ be the cyclic group of order r , seen as a constant group scheme over $\mathrm{Spec}(k)$. When r divides r' , there is a canonical group homomorphism $\mathbf{C}_{r'} \twoheadrightarrow \mathbf{C}_r$, and we set

$$\widehat{\mathbb{Z}} := \varprojlim_r \mathbf{C}_r \in \mathbf{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathrm{Pro}(\mathbf{dSt}_k)) .$$

Passing to the classifying stack, we also set

$$\widehat{S}^1 := \varprojlim_r \mathbf{BC}_r \in \mathrm{Pro}(\mathbf{dSt}_k) .$$

Notice that the canonical homomorphisms $\mathbb{Z} \twoheadrightarrow \mathbf{C}_r$ induce at the level of classifying stacks the map

$$\varpi_r: S^1 \longrightarrow \mathbf{BC}_r ,$$

and all these maps assemble into a morphism

$$(3.1.1) \quad \varpi: S^1 \longrightarrow \widehat{S}^1 .$$

Definition 3.1. Let $X \in \mathbf{dSt}_k$ be a derived stack.

- (1) The r -th orbifold inertia of X is the mapping stack

$$\mathbf{l}^{(r)}X := \mathbf{Map}(\mathbf{BC}_r, X) .$$

- (2) The orbifold inertia of X is the mapping stack

$$\mathbf{l}^{\mathrm{DM}}X := \mathbf{Map}(\widehat{S}^1, X) .$$

Unraveling the definitions, we see that

$$\mathbf{l}^{\mathrm{DM}}X \simeq \operatorname{colim}_r \mathbf{l}^{(r)}X ,$$

where the filtered colimit is computed in $\mathbf{dSt}_k$.

Notation 3.2. The maps ϖ and ϖ_r induce well-defined morphisms

$$\iota_\varpi: \mathbf{l}^{\mathrm{DM}}X \longrightarrow \mathcal{L}X \quad \text{and} \quad \iota_r: \mathbf{l}^{(r)}X \longrightarrow \mathcal{L}X .$$

We also let

$$\pi: \mathcal{L}X \longrightarrow X , \quad \pi^{\mathrm{DM}}: \mathbf{l}^{\mathrm{DM}}X \longrightarrow X , \quad \pi_r: \mathbf{l}^{(r)}X \longrightarrow X$$

be the natural projections.

The main result of this section is the following properties of $\mathbf{l}^{\mathrm{DM}}X$:

Theorem 3.3. *Assume that X is a derived 1-Artin stack locally almost of finite type. Then the following holds.*

- (1) *For every positive integer r , the map*

$$\iota_r: \mathbf{l}^{(r)}X \longrightarrow \mathcal{L}X$$

is a closed immersion. In particular, $\mathbf{l}^{(r)}X$ is a derived Artin stack.

- (2) *If the cotangent complex of X is perfect (resp. has tor-amplitude in $[a, b]$), the same holds for the cotangent complex of $\mathbf{l}^{(r)}X$. In particular, if X is smooth (resp. lci¹), the same holds for $\mathbf{l}^{(r)}X$.*

- (3) *For positive integers $r \mid r'$, the transition map $\mathbf{l}^{(r)}X \rightarrow \mathbf{l}^{(r')}X$ is an open and closed immersion.*

Assume furthermore that X is a derived Deligne-Mumford stack locally almost of finite type. Then:

- (4) *$\mathbf{l}^{\mathrm{DM}}X$ is a derived Deligne-Mumford stack locally almost of finite type and the map*

$$\iota_\varpi: \mathbf{l}^{\mathrm{DM}}X \longrightarrow \mathcal{L}X$$

induces an isomorphism on the truncation: $t_0 \mathbf{l}^{\mathrm{DM}}X \cong t_0 \mathcal{L}X \cong \mathbf{l}(t_0(X))$.

Corollary 3.4. *Let X be a smooth Deligne-Mumford stack locally almost of finite presentation. Then $\mathbf{l}^{\mathrm{DM}}X$ is also smooth. In particular, it is underived and therefore it coincides with*

$$X \times_{X \times X} X ,$$

where the fiber product is computed in underived stacks $\mathbf{St}_k$.

¹In this paper, the terminology *lci* stands for *derived lci*, a.k.a. *quasi-smooth*, which means that the tor-amplitude of the cotangent complex belongs to $[-1, 1]$. Notice that an underived stack X is derived lci if and only if X is lci in the classical sense, see for instance [20, Lemma 2.4] (whose proof works verbatim in the algebraic setting).

Proof. The smoothness follows by combining items (2) and (3) of Theorem 3.3. The fact that it coincides with the usual inertia stack follows from item (4). $\square$

Remark 3.5. Theorem 3.3 recovers the considerations of [2, §3.1] in the derived setting. The proof below might seem weirdly long compared to the arguments given in *loc. cit.* The ultimate reason is that whereas in the classical setting a point of $\mathbb{I}^{(r)}X$ is the datum of a pair (x, g) where x is a point of X and g is an automorphism of x whose order divides r , it is actually challenging to obtain a similar description of $\mathbb{I}^{(r)}X$ in the derived setting. This difficulty can be precisely quantified in terms of a basic computation in homotopy theory, see Proposition 3.12 below.

Remark 3.6. We do not know whether $\mathbb{I}^{\text{DM}}X$ always coincide with $\mathbb{I}X$ whenever X is underived. In other words, $\mathbb{I}^{\text{DM}}X$ *might* have a non-trivial derived structure. Although we find this statement unlikely, we do not have a proof, nor we need it in the rest of the paper.

3.2. Deformation theory of orbifold inertia. We start by some general considerations on the cotangent complex of mapping stacks. The material covered here will be crucial in the proof of Theorem 3.3 (and especially item (4) of that theorem), but it will also be used later on in the proof of the HKR theorem.

For the finiteness conditions appearing in the following lemma, see Section 2.

Lemma 3.7. *For a positive integer r , the stack BC_r is $\otimes$ -universal, categorically quasi-compact and categorically proper.*

Proof. Following [6, Theorem 5.4] (see also [7, Theorem 5.25], we obtain a canonical decomposition

$$(3.2.1) \quad \text{QCoh}(\text{BC}_r) \simeq \coprod_{\zeta \in \mu_r} \text{QCoh}_{\zeta}(\text{BC}_r) ,$$

and moreover we have canonical identifications $\text{QCoh}_{\zeta}(\text{BC}_r) \simeq \text{Mod}_k$. It follows that $\text{QCoh}(\text{BC}_r)$ is compactly generated, and therefore BC_r is $\otimes$ -universal. It is also categorically quasi-compact, as the pushforward is canonically identified with taking the component indexed by $\zeta = 1$ in the decomposition (3.2.1). Finally, this operation preserves perfect complexes, and therefore we deduce that BC_r is categorically perfect as a consequence of Lemma 2.3. $\square$

Lemma 3.8. *Let r and r' be positive integers such that r divides r' , and let $f: \text{BC}_{r'} \rightarrow \text{BC}_r$ be the natural homomorphism. Let $p: \text{BC}_{r'} \rightarrow \text{Spec}(k)$ and $q: \text{BC}_r \rightarrow \text{Spec}(k)$ be the structural morphisms. Then the natural $+$ -Beck-Chevalley transformation*

$$p_+ \circ f^* \longrightarrow q_+$$

is an equivalence.

Proof. Under the character decomposition (3.2.1) for $\text{QCoh}(\text{BC}_r)$, the functor f^* correspond to the map

$$\prod_{\mu_r} \text{Mod}_k \longrightarrow \prod_{\mu_{r'}} \text{Mod}_k$$

induced by the inclusion $\mu_r \subset \mu_{r'}$. Given $\zeta \in \mu_r$, write k_{ζ} for the element $k \in \text{Mod}_k \simeq \text{QCoh}_{\zeta}(\text{Mod}_k)$. We therefore have $f^*(k_{\zeta}) \simeq k_{\zeta}$, and it suffices at this point to observe that

$$q_+(k_{\zeta}) \simeq q_*(k_{\zeta^{-1}})^{\vee} \simeq \begin{cases} k & \text{if } \zeta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Proposition 3.9. *Let X be a derived Artin stack. Then $\mathbb{I}^{(r)}X$ is infinitesimally cohesive, nilcomplete and admits a cotangent complex. Moreover:*

- (1) if $\mathbb{L}_X$ is perfect of tor-amplitude $[a, b]$, the same holds for $\mathbb{L}_{\mathbb{I}^{(r)}X}$;
- (2) if $r \mid r'$, the induced morphism $\mathbb{I}^{(r)}X \rightarrow \mathbb{I}^{(r')}X$ is formally étale.

Proof. Since X is Artin, it is itself infinitesimally cohesive and nilcomplete. Therefore, the same holds for $\mathbf{Map}(F, X)$ for any derived stack F , in particular for $\mathbb{I}^{(r)}X$. The existence of the cotangent complex follows by combining Lemma 3.7 with Proposition 2.7. Again by Proposition 2.7, the cotangent complex of $\mathbb{I}^{(r)}X$ is computed by the formula

$$\mathbb{L}_{\mathbb{I}^{(r)}X} \cong \pi_+ \mathrm{ev}^*(\mathbb{L}_X).$$

Since π_+ consists of taking the dual of C_r -invariants (and since in characteristic zero this operation is t -exact), we see that point (1) holds. As for statement (2), it follows from the combination of Lemma 3.8 and Proposition 2.10. $\square$

3.3. A computation in homotopy theory. We will reduce the proof of Theorem 3.3 to a computation in homotopy theory. We start by introducing some notations.

Notation 3.10. Let $r \in \mathbb{Z}$ be an integer. We denote by

$$\mathrm{cov}_r: S^1 \rightarrow S^1$$

the standard degree- r covering map, corresponding to the element $r \in \mathbb{Z} \simeq \pi_1(S^1)$. We write

$$\Gamma_r := \mathrm{cofib}(\mathrm{cov}_r) \in \mathbf{Spc}.$$

Notice that the commutative square

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{r} & \mathbb{Z} \\ \downarrow & & \downarrow \\ * & \longrightarrow & C_r \end{array}$$

of groups induces passing to classifying stacks a commutative square

$$\begin{array}{ccc} S^1 & \xrightarrow{\mathrm{cov}_r} & S^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{BC}_r, \end{array}$$

and therefore a canonical comparison map

$$\gamma_r: \Gamma_r \longrightarrow \mathrm{BC}_r.$$

Example 3.11. For $r = 2$, $\Gamma_2 \cong \mathbb{RP}^2$, $\mathrm{BC}_2 \cong \mathbb{RP}^\infty$ and the map γ_2 is the natural inclusion.

Proposition 3.12.

- (1) The space Γ_r admits the bouquet $(S^2)^{\vee r-1}$ as a universal cover.
- (2) The map γ_r exhibits BC_r as the 1-Postnikov truncation of Γ_r .

Proof. We represent S^1 as the complex circle, and we compute Γ_r as the strict pushout of the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{z \mapsto z^r} & S^1 \\ \downarrow & & \downarrow \\ \mathbb{D} & \longrightarrow & \Gamma_r, \end{array}$$

where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. We can alternatively describe Γ_r as follows. Let B_r be the strict pushout

$$\begin{array}{ccc} S^1 \times \mu_r & \longrightarrow & \mathbb{D} \times \mu_r \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & B_r \end{array}$$

There is a canonical action of μ_r on B_r given by

$$\zeta \cdot (z, \zeta') := (\zeta \cdot z, \zeta \zeta') ,$$

and Γ_r coincides with the quotient of B_r by this action. Furthermore, this action is free and properly discontinuous and therefore the quotient map

$$B_r \longrightarrow \Gamma_r$$

is a covering map. Finally, it is easy to see that B_r is geometrically the gluing of r copies of $\mathbb{D}$ along their boundaries and hence homotopically equivalent to the bouquet of $r - 1$ copies of S^2 . This proves statement (1). It also implies that B_r is connected and

$$\pi_n(B_r) \simeq \begin{cases} \mathbb{C}_r & \text{if } n = 1 \\ \pi_n((S^2)^{\vee r-1}) & \text{if } n \geq 2 . \end{cases}$$

Statement (2) follows at once. □

Precomposition with cov_r induces for every $F \in \mathbf{dSt}_k$ a well-defined operation

$$(-)^r : \mathcal{L}F \longrightarrow \mathcal{L}F$$

over F . Write

$$\delta_F : F \longrightarrow \mathcal{L}F$$

for the relative diagonal.

Corollary 3.13. *Assume that F is a 1-Artin derived stack. Then the square*

$$\begin{array}{ccc} \mathbf{l}^{(r)}F & \longrightarrow & \mathcal{L}F \\ \downarrow & & \downarrow (-)^r \\ F & \xrightarrow{\delta_F} & \mathcal{L}F \end{array}$$

is a pullback square on truncations.

Proof. Unraveling the definitions, we see that for every $F \in \mathbf{dSt}_k$, the square

$$\begin{array}{ccc} \mathbf{Map}(\Gamma_r, F) & \longrightarrow & \mathcal{L}F \\ \downarrow & & \downarrow (-)^r \\ F & \xrightarrow{\delta_F} & \mathcal{L}F \end{array}$$

is a pullback square. Since $t_0 : \mathbf{dSt}_k \rightarrow \mathbf{St}_k$ preserves limits, it is therefore enough to argue that the map

$$\mathbf{l}^{(r)}F = \mathbf{Map}(\mathbf{BC}_r, F) \longrightarrow \mathbf{Map}(\Gamma_r, F)$$

becomes an equivalence after applying t_0 . Equivalently, we have to check that for every underived test affine scheme $S \in \mathbf{Aff}_k$, the map

$$\mathbf{Map}(\mathbf{BC}_r, F)(S) \longrightarrow \mathbf{Map}(\Gamma_r, F)(S)$$

is an equivalence. Since both BC_r and Γ_r are the constant stacks, we can identify this map with the map

$$\mathrm{Map}_{\mathbf{Spc}}(\mathrm{BC}_r, F(S)) \longrightarrow \mathrm{Map}_{\mathbf{Spc}}(\Gamma_r, F(S))$$

induced by $\gamma_r: \Gamma_r \rightarrow \mathrm{BC}_r$. Since F is a 1-Artin stack and S is underived, $F(S)$ is a 1-groupoid. Therefore, the conclusion follows directly from the fact that BC_r coincides with the 1-Postnikov truncation of Γ_r , as proven in Proposition 3.12-(2). $\square$

3.4. Proof of Theorem 3.3. We are now ready for the proof of the main theorem of this section. We start by collecting the consequences of the results obtained so far.

Corollary 3.14. *Let X be a 1-Artin stack locally almost of finite presentation and with separated diagonal. Then:*

(1) *for every r the map*

$$\iota_r: \mathbb{I}^{(r)}X \longrightarrow \mathcal{L}X$$

is a closed immersion.

(2) *the derived stack $\mathbb{I}^{(r)}X$ is 1-Artin and locally almost of finite presentation.*

(3) *For any positive integers $r \mid r'$, the induced morphism*

$$\mathbb{I}^{(r)}X \longrightarrow \mathbb{I}^{(r')}X$$

is an open and closed immersion, and therefore $\mathbb{I}^{\mathrm{DM}}X$ is a derived 1-Artin stack locally almost of finite presentation.

Proof. For (1), it is enough to prove that ι_r is a closed immersion after passing to classical truncations. Thanks to Corollary 3.13, it is enough to argue that $\delta_X: X \rightarrow \mathcal{L}X$ is a closed immersion, which immediately follows from the assumption that the diagonal of X is separated. In particular, $t_0(\mathbb{I}^{(r)}X)$ is a 1-Artin stack of finite presentation (in the underived sense). Point (2) follows combining Proposition 3.9 with Lurie’s representability theorem. As for point (3), we already know as a consequence of (1) that the morphism in question is a closed immersion. Proposition 3.9 implies that it is formally étale, and therefore that it must be an open immersion as well. $\square$

We complete the proof of Theorem 3.3 by showing the following:

Proposition 3.15. *Let X be a Deligne-Mumford stack locally almost of finite type. Then the map*

$$\iota_r: \mathbb{I}^{(r)}X \longrightarrow \mathcal{L}X$$

induces an open immersion on truncations. Besides, the morphism

$$\iota_{\varpi}: \mathbb{I}^{\mathrm{DM}}X \longrightarrow \mathcal{L}X$$

is an equivalence on truncations.

Proof. First observe that

$$t_0(\mathcal{L}X) \simeq t_0(\mathcal{L}(t_0(X))) \simeq \mathbb{I}(t_0(X)) .$$

To prove the first statement, we can therefore assume from the very beginning that X is underived. Thanks to Corollary 3.13 it is sufficient to argue that $\delta_X: X \rightarrow t_0(\mathcal{L}X)$ is an open immersion. As already remarked in the proof of Lemma 2.13, since X is a Deligne-Mumford stack, the diagonal $X \rightarrow X \times X$ is unramified and therefore the map $X \rightarrow \mathbb{I}X$ is an open immersion, see [24, Tag 02GE].

As for the second statement, it suffices to argue that ι_{ϖ} is an effective epimorphism. This follows directly since any element of the stabilizers of X has finite order. $\square$

4. HKR ISOMORPHISM FOR DERIVED DELIGNE-MUMFORD STACKS

In this section we prove the HKR theorem for derived Deligne-Mumford stacks. The strategy consists of two steps. First, we consider the natural map

$$S^1 \longrightarrow \mathrm{B}\mathbb{G}_a \times \widehat{S}^1,$$

and prove that when applying the mapping stacks against a derived Deligne-Mumford stack X , it gives rise to an equivalence

$$\mathbf{Map}(\mathrm{B}\mathbb{G}_a \times \widehat{S}^1, X) \xrightarrow{\cong} \mathbf{Map}(S^1, X) \simeq \mathcal{L}X.$$

Then, we conclude by establishing a canonical identification

$$\mathbf{Map}(\mathrm{B}\mathbb{G}_a \times \widehat{S}^1, X) \simeq \mathrm{T}[-1]^{\mathrm{DM}} X,$$

supplied by Proposition 4.8 below.

4.1. Unipotent loops. Let k be a field of characteristic zero.

Proposition 4.1. *The derived stack $\mathrm{B}\mathbb{G}_a$ is perfect in the sense of [4, Definition 3.2]. In particular, the stable ∞ -category $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_a)$ satisfies*

$$\mathrm{QCoh}(\mathrm{B}\mathbb{G}_a) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathrm{B}\mathbb{G}_a)).$$

In other words, perfect complexes on $\mathrm{B}\mathbb{G}_a$ are compact, and they generate the whole $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_a)$.

Proof. This is a particular case of [4, Corollary 3.22]. $\square$

Corollary 4.2. *The derived stack $\mathrm{B}\mathbb{G}_a$ is $\otimes$ -universal and categorically quasi-compact.*

Proof. The $\otimes$ -universality follows from Example 2.2-(1), while categorical quasi-compactness follows from the fact that $\mathcal{O}_{\mathrm{B}\mathbb{G}_a}$ is perfect, and therefore compact in $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_a) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathrm{B}\mathbb{G}_a))$ thanks to Proposition 4.1. $\square$

Recollection 4.3. There is a natural group morphism

$$\mathbb{Z} \longrightarrow \mathbb{G}_a,$$

which induces, after applying the delooping functor B , a morphism

$$(4.1.1) \quad \mathrm{aff}: S^1 \longrightarrow \mathrm{B}\mathbb{G}_a.$$

Write $p: S^1 \rightarrow \mathrm{Spec}(k)$ and $q: \mathrm{B}\mathbb{G}_a \rightarrow \mathrm{Spec}(k)$ for the structural morphisms. Since we are in characteristic zero, [25, Lemma 2.2.5] implies that the natural comparison map

$$q_*(\mathcal{O}_{\mathrm{B}\mathbb{G}_a}) \longrightarrow p_*(\mathcal{O}_{S^1})$$

is an equivalence. In particular, aff exhibits $\mathrm{B}\mathbb{G}_a$ as the affinization of S^1 .

Lemma 4.4. *With the notations of Recollection 4.3, the canonical comparison map*

$$q_*(\mathcal{F}) \longrightarrow p_*(\mathrm{aff}^*(\mathcal{F}))$$

is an equivalence for every eventually coconnective $\mathcal{F} \in \mathrm{QCoh}(\mathrm{B}\mathbb{G}_a)$. In particular, it is an equivalence on $\mathrm{Perf}(\mathrm{B}\mathbb{G}_a)$.

Proof. Write

$$A := \Gamma(\mathrm{B}\mathbb{G}_a, \mathcal{O}_{\mathrm{B}\mathbb{G}_a}).$$

It follows from [25, Lemme 2.2.5] that $\mathrm{B}\mathbb{G}_a \simeq \mathrm{cSpec}(A)$. As in [12, §4.5], we have a natural cocontinuous symmetric monoidal functor

$$\theta: \mathrm{Mod}_A \longrightarrow \mathrm{QCoh}(\mathrm{B}\mathbb{G}_a),$$

Since both $\mathcal{B}\mathbb{G}_a$ and S^1 are categorically quasi-compact (in virtue of Corollary 4.2 and Example 2.2-(2), respectively), it follows that both source and target of the Beck-Chevalley transformation $q_* \rightarrow p_* \circ \text{aff}^*$ commute with arbitrary colimits, since in a stable category commuting with filtered colimits implies commuting with all colimits. Furthermore, since k has characteristic zero, we see that statement is true for $\mathcal{F} := \mathcal{O}_{\mathcal{B}\mathbb{G}_a} \simeq \theta(A)$. Since A is a generator for Mod_A , it follows that the result holds for every $\mathcal{F}$ in the essential image of θ . Recall now from [12, Proposition 4.5.2] that θ is t -exact and that it restricts to an equivalence

$$(\text{Mod}_A)_{\leq n} \simeq \text{QCoh}(\mathcal{B}\mathbb{G}_a)_{\leq n}$$

where we are using homological indexing conventions. This immediately implies the first statement, and the second one follows from the fact that since $\mathcal{B}\mathbb{G}_a$ is smooth and quasi-compact, every perfect complex is eventually coconnective. $\square$

Proposition 4.5. *The derived stack $\mathcal{B}\mathbb{G}_a$ is categorically perfect.*

Proof. We keep writing $q: \mathcal{B}\mathbb{G}_a \rightarrow \text{Spec}(k)$ for the structural morphism. Since $\text{QCoh}(\mathcal{B}\mathbb{G}_a) \simeq \text{Ind}(\text{Perf}(\mathcal{B}\mathbb{G}_a))$ by Proposition 4.1 and since $\mathcal{B}\mathbb{G}_a$ is categorically quasi-compact by Corollary 4.2, Lemma 2.3 shows that it suffices to prove that

$$q_*: \text{QCoh}(\mathcal{B}\mathbb{G}_a) \longrightarrow \text{Mod}_k$$

preserves perfect complexes. Let therefore $\mathcal{F} \in \text{Perf}(\mathcal{B}\mathbb{G}_a)$. By Lemma 4.4, we know that the natural comparison morphism

$$q_*(\mathcal{F}) \longrightarrow p_* \text{aff}^*(\mathcal{F})$$

is an equivalence. It is then enough to observe that $\mathcal{G} := \text{aff}^*(\mathcal{F})$ can be identified with a functor $G: S^1 \rightarrow \text{Perf}_k$, and $p_*(\mathcal{G})$ is identified with the limit of G . Since S^1 is a compact object in $\mathbf{Spc}$, the limit remains perfect, whence the conclusion. $\square$

Remark 4.6. Since $\mathcal{B}\mathbb{G}_a$ is $\otimes$ -universal, the functoriality of the tensor product of ∞ -categories shows that Proposition 4.5 recovers the integration map for $\mathcal{B}\mathbb{G}_a$ of [16, §5.3].

4.2. Shifted tangents. Let X be a derived Artin stack. For every integer $n \in \mathbb{Z}$ we set

$$\mathbb{T}[n]X := \text{Spec}_X(\text{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[-n])) .$$

Since

$$\begin{array}{ccc} \mathbb{L}_X[-n] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}_X[-n+1] \end{array}$$

is a pushout in $\text{QCoh}(X)$, we see that

$$\begin{array}{ccc} \mathbb{T}[n-1]X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{T}[n]X \end{array}$$

is a pullback square in $\mathbf{dSt}_k$, where both maps $X \rightarrow \mathbb{T}[n]X$ are the inclusion of the zero section.

Consider the ordinary split square-zero extension

$$k[\varepsilon] := k \oplus k\varepsilon ,$$

with $\varepsilon^2 = 0$. We set

$$\mathbb{D}_0 := \text{Spec}(k[\varepsilon]) .$$

For every derived Artin stack X , we have a canonical equivalence

$$\mathbf{Map}(\mathbb{D}_0, X) \simeq \mathbf{T}X ,$$

see for instance [26, Proposition 1.4.1.6]. Define $\mathbb{D}_{-1}$ as the following pushout in $\mathbf{dSt}_k$:

$$\begin{array}{ccc} \mathbb{D}_0 & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathbb{D}_{-1} . \end{array}$$

Remark 4.7. In fact, it follows from [16, Proposition 5.11] that there is a canonical identification $\mathbb{D}_{-1} \simeq \mathbf{B}\widehat{\mathbb{G}}_a$.

It immediately follows from this discussion that

$$\mathbf{Map}(\mathbb{D}_{-1}, X) \simeq X \times_{\mathbf{T}X} X \simeq \mathbf{T}[-1]X .$$

The natural inclusion $\mathbb{D}_0 \rightarrow \mathbb{G}_a$ induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longleftarrow & \mathbb{D}_0 & \longrightarrow & \mathrm{Spec}(k) \\ \parallel & & \downarrow & & \parallel \\ \mathrm{Spec}(k) & \longleftarrow & \mathbb{G}_a & \longrightarrow & \mathrm{Spec}(k) , \end{array}$$

and therefore a canonical comparison map

$$u: \mathbb{D}_{-1} \longrightarrow \mathbf{B}\mathbb{G}_a .$$

In particular, for every derived stack X , we obtain a natural comparison morphism

$$u^*: \mathbf{Map}(\mathbf{B}\mathbb{G}_a, X) \longrightarrow \mathbf{Map}(\mathbb{D}_{-1}, X) \simeq \mathbf{T}[-1]X .$$

Proposition 4.8. *For X a derived Deligne-Mumford stack, the morphism u^* is an equivalence.*

Proof. It is shown in [16, Proposition 5.22] that the map u^* is formally étale. It suffices therefore to argue that u^* is an isomorphism on truncations: this will imply that $\mathbf{Map}(\mathbf{B}\mathbb{G}_a, X)$ is geometric, and that u^* is an equivalence. For this, it is enough to argue that for any underived test scheme $S \in \mathbf{Aff}_k$, the morphism induced by precomposition with $S \rightarrow S \times \mathbf{B}\mathbb{G}_a$

$$(4.2.1) \quad \mathrm{Map}(S \times \mathbf{B}\mathbb{G}_a, X) \longrightarrow \mathrm{Map}(S, X)$$

is an equivalence. Since $\mathbb{G}_a$ is connected, this follows from Lemma 2.13. $\square$

4.3. The HKR equivalence for derived Deligne-Mumford stacks. Consider the natural morphism

$$\widehat{\mathrm{aff}} := (\mathrm{aff}, \varpi): S^1 \longrightarrow \mathbf{B}\mathbb{G}_a \times \widehat{S}^1 .$$

See (3.1.1) and (4.1.1) for the definitions of ϖ and aff . For any derived Deligne-Mumford stack X , precomposition with $\widehat{\mathrm{aff}}$ induces a natural transformation

$$\mathbf{Map}(\mathbf{B}\mathbb{G}_a \times \widehat{S}^1, X) \longrightarrow \mathbf{Map}(S^1, X) = \mathcal{L}X .$$

We can furthermore rewrite

$$\mathbf{Map}(\mathbf{B}\mathbb{G}_a \times \widehat{S}^1, X) \simeq \mathbf{Map}(\mathbf{B}\mathbb{G}_a, \mathbf{Map}(\widehat{S}^1, X)) \simeq \mathbf{Map}(\mathbf{B}\mathbb{G}_a, \mathbf{l}^{\mathrm{DM}}X) .$$

Since $\mathbf{l}^{\mathrm{DM}}X$ is again a derived Deligne-Mumford stack by Theorem 3.3-(5), Proposition 4.8 provides an equivalence:

$$\mathbf{Map}(\mathbf{B}\mathbb{G}_a, \mathbf{l}^{\mathrm{DM}}X) \simeq \mathbf{T}[-1]\mathbf{l}^{\mathrm{DM}}X .$$

The goal of this section is to prove the following result.

Theorem 4.9 (HKR for derived DM stacks). *The comparison map*

$$\widehat{\text{aff}}^* : \mathbb{T}[-1]^{\text{DM}} X \longrightarrow \mathcal{L}X$$

is an equivalence over X .

We denote by aff_r the natural morphism $(\text{aff}, \varpi_r) : S^1 \longrightarrow \text{BG}_a \times \text{BC}_r$.

Proposition 4.10. *Fix a positive integer $r > 0$ and write $p : S^1 \rightarrow \text{Spec}(k)$ and $q_r : \text{BG}_a \times \text{BC}_r \rightarrow \text{Spec}(k)$ for the structural morphisms. The $+$ -Beck-Chevalley transformation*

$$p_+ \text{aff}_r^* \longrightarrow q_{r,+} .$$

induced by $\text{aff}_r : S^1 \longrightarrow \text{BG}_a \times \text{BC}_r$ is an equivalence.

Proof. Observe that both source and target of the $+$ -Beck-Chevalley transformation commute with colimits. In addition

$$\text{QCoh}(\text{BG}_a \times \text{BC}_r) \simeq \text{QCoh}(\text{BG}_a) \otimes \text{QCoh}(\text{BC}_r) ,$$

and since we are in characteristic zero, both BG_a and BC_r are perfect stacks. In particular, it suffices to prove that for $\mathcal{F} \in \text{Perf}(\text{BG}_a)$ and $\mathcal{G} \in \text{Perf}(\text{BC}_r)$, the $+$ -Beck-Chevalley transformation is an equivalence on $\mathcal{F} \boxtimes \mathcal{G}$. Fix $\mathcal{G} \in \text{Perf}(\text{BC}_r)$. Reasoning as in the proof of Lemma 4.4, with the help of the functor θ , we see that it is enough to argue that

$$p_+ \text{aff}_r^*(\mathcal{O}_{\text{BG}_a} \boxtimes \mathcal{G}) \longrightarrow q_{r,+}(\mathcal{O}_{\text{BG}_a} \boxtimes \mathcal{G})$$

is an equivalence. Now, [6, Theorem 5.4] (see also [7, Theorem 5.25] for a proof in the ∞ -categorical setting) supplies a canonical identification

$$\text{Perf}(\text{BC}_r) \simeq \prod_{\zeta \in \mu_r} \text{Perf}_\zeta(\text{BC}_r) ,$$

where $\text{Perf}_\zeta(\text{BC}_r)$ denotes the full subcategory of ζ -homogeneous objects. We have further identifications $\text{Perf}_\zeta(\text{BC}_r) \simeq \text{Perf}_k$, and we write k_ζ for k seen as an element in $\text{Perf}_\zeta(\text{BC}_r)$. Notice that $\mathcal{O}_{\text{BG}_a} \boxtimes k_\zeta$ is sent by aff_r^* to the local system having k as stalk and monodromy given by multiplication by ζ . We still denote this local system by k_ζ . Hence $p_+ \text{aff}_r^*(\mathcal{O}_{\text{BG}_a} \boxtimes k_\zeta) \simeq p_+(k_\zeta) = p_*(k_{\zeta^{-1}})^\vee$. In particular, $p_+ \text{aff}_r^*(\mathcal{O}_{\text{BG}_a} \boxtimes k_\zeta)$ is the colimit

$$\begin{array}{ccc} k \oplus k & \xrightarrow{(1, -\zeta)} & k \\ \downarrow (1, -1) & & \downarrow \\ k & \longrightarrow & p_+ \text{aff}_r^*(\mathcal{O}_{\text{BG}_a} \boxtimes k_\zeta) . \end{array}$$

From here, we immediately obtain the identification

$$p_+ \text{aff}_r^*(\mathcal{O}_{\text{BG}_a} \boxtimes k_\zeta) \simeq \text{cofib}(1 - \zeta : k \rightarrow k) ,$$

This implies that it is zero whenever $\zeta \neq 1$. We are therefore reduced to check the statement for $k_1 \simeq \mathcal{O}_{\text{BC}_r}$, and in this case the statement is obvious. $\square$

We are now ready for to prove our main result.

Proof of Theorem 4.9. Combining Proposition 4.10 and Proposition 2.10, we deduce that $\widehat{\text{aff}}^*$ is formally étale. As a consequence of Theorem 3.3, both source and target are derived Deligne-Mumford stacks, and therefore it is sufficient to prove that $\widehat{\text{aff}}^*$ induces an equivalence on truncations. This is guaranteed by Theorem 3.3-(4). $\square$

4.4. Multiplicative HKR isomorphism for Hochschild homology. Recall that for a derived Deligne-Mumford stack X , its i -th *Hochschild homology* is defined as

$$(4.4.1) \quad \mathrm{HH}_i(X) := H^{-i}(X \times X, \Delta_* \mathcal{O}_X \otimes^{\mathbb{L}} \Delta_* \mathcal{O}_X),$$

where $\Delta: X \rightarrow X \times X$ is the diagonal map.

Taking the direct sum, we get a graded vector space:

$$(4.4.2) \quad \mathrm{HH}_*(X) = \bigoplus_i \mathrm{HH}_i(X).$$

Since the Hochschild homology is functorial and satisfies the Künneth formula, $\mathrm{HH}_*(X)$ admits a natural graded algebra structure.

Note that by the projection formula and the base-change formula,

$$\begin{aligned} \mathrm{HH}_{-*}(X) & \simeq H^*(X, \Delta^* \Delta_* \mathcal{O}_X) \\ & \simeq H^*(X, p_* \mathcal{O}_{\mathcal{L}X}) \\ & \simeq H^*(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}), \end{aligned}$$

where $p: \mathcal{L}X \rightarrow X$ is the natural morphism. This interpretation of Hochschild homology using loop stack also makes the algebra structure transparent.

Theorem 4.11. *Let X be a derived Deligne–Mumford stack. Then we have an isomorphism of algebras*

$$(4.4.3) \quad \mathrm{HH}_{-*}(X) \cong H^*(\mathrm{I}^{\mathrm{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathrm{I}^{\mathrm{DM}}X}[1])).$$

where the Hochschild homology $\mathrm{HH}_{-*}(X)$ is equipped with its natural graded algebra structure and $H^*(\mathrm{I}^{\mathrm{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathrm{I}^{\mathrm{DM}}X}[1]))$ is equipped with the natural graded algebra structure induced from the algebra structure on the symmetric algebra.

Proof. Thanks to Theorem 4.9, we have an isomorphism of derived stacks over X :

$$(4.4.4) \quad \mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X \xrightarrow{\sim} \mathcal{L}X.$$

Therefore, by taking (derived) functions, we have an isomorphism of commutative algebras:

$$(4.4.5) \quad H^*(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \cong H^*(\mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X, \mathcal{O}_{\mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X}).$$

We conclude by noting that $\pi_*(\mathcal{O}_{\mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X}) \simeq \mathrm{Sym}(\mathbb{L}_{\mathrm{I}^{\mathrm{DM}}X}[1])$ as algebras, where $\pi: \mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X \rightarrow \mathrm{I}^{\mathrm{DM}}X$ is the natural projection. $\square$

4.5. HKR isomorphism for Hochschild cohomology. Theorem 4.9 also allows to obtain a statement for Hochschild cohomology. To formulate it let us denote by

$$p: \mathcal{L}X \longrightarrow X, \quad q: \mathrm{T}[-1]\mathrm{I}^{\mathrm{DM}}X \longrightarrow X$$

the natural projections.

Notation 4.12. Let $f: Y \rightarrow X$ be a morphism. We denote by

$$\mathrm{Coh}^b(Y/X)$$

the full subcategory of $\mathrm{QCoh}(Y)$ spanned by almost perfect complexes on Y having finite tor-amplitude relative to X .

Example 4.13. Assume that X is a smooth Deligne-Mumford stack and that $f: Y \rightarrow X$ is representable by quasi-compact algebraic spaces. Then unraveling the definitions we see that $\mathrm{Coh}^b(Y/X)$ canonically coincide with $\mathrm{Coh}^b(Y)$.

The morphisms p and q induce well-defined morphisms

$$p_*: \operatorname{Ind}(\operatorname{Coh}^b(\mathcal{L}X/X)) \longrightarrow \operatorname{QCoh}(X), \quad q_*: \operatorname{Ind}(\operatorname{Coh}^b(\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X/X)) \longrightarrow \operatorname{QCoh}(X).$$

By construction, both these functors commute with (filtered and hence all) colimits. Furthermore, since p and q are proper, the definition of $\operatorname{Coh}^b(-/X)$ implies that p_* and q_* preserve compact objects. It follows that they both admit continuous right adjoints that we denote

$$p^!: \operatorname{QCoh}(X) \longrightarrow \operatorname{Ind}(\operatorname{Coh}^b(\mathcal{L}X/X)), \quad q^!: \operatorname{QCoh}(X) \longrightarrow \operatorname{Ind}(\operatorname{Coh}^b(\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X/X)).$$

In general, we set

$$\mathcal{H}^*(X) := p_*p^!(\mathcal{O}_X),$$

and we refer to it as the *Hochschild cohomology sheaf* of X , whose (derived) global sections recover the *Hochschild cohomology* of X :

$$\operatorname{HH}^*(X) := \Gamma(X, \mathcal{H}^*(X)) = \Gamma(\mathcal{L}X, p^!\mathcal{O}_X).$$

Corollary 4.14. *Let X be a derived DM stack. There is a canonical isomorphism*

$$\mathcal{H}^*(X) \simeq q_*q^!(\mathcal{O}_X)$$

in $\operatorname{QCoh}(X)$, and a canonical isomorphism

$$\operatorname{HH}^*(X) \simeq \Gamma(\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X, q^!(\mathcal{O}_X))$$

in Mod_k . If X is moreover of finite type and lci, then

$$(4.5.1) \quad \operatorname{HH}^*(X) \simeq \Gamma(\mathbb{I}^{\operatorname{DM}}X, \operatorname{Sym}(\mathbb{L}_{\mathbb{I}^{\operatorname{DM}}X}[1]) \otimes i^*\omega_X^\vee[-\dim(X)]).$$

where ω_X is the dualizing sheaf of X , $i: \mathbb{I}^{\operatorname{DM}}X \rightarrow X$ is the canonical morphism, and $\dim(X)$ is the dimension of X (a locally constant $\mathbb{Z}$ -valued function).

Proof. Since p_* and q_* are canonically identified by Theorem 4.9, we immediately obtain the first assertion. Applying Γ , it yields the second assertion. Assume now that X is a lci derived DM stack of finite type. By Theorem 3.3, $\mathbb{I}^{\operatorname{DM}}X = \mathbb{I}^{(r)}X$ for some r , which is also lci. We decompose the morphism q into the composition

$$q: \mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X \xrightarrow{\pi} \mathbb{I}^{\operatorname{DM}}X \xrightarrow{i} X.$$

The standard fiber sequence of cotangent complexes for the map π shows that the dualizing sheaf $\omega_{\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X}$ is trivial. As a result, the relative dualizing sheaf ω_q is isomorphic to $q^*\omega_X^\vee$ and $\dim(q) = -\dim(X)$. Therefore,

$$q^!\mathcal{O}_X \simeq \omega_q[\dim(q)] \simeq q^*\omega_X^\vee[-\dim(X)]$$

We conclude by the following computation:

$$\begin{aligned} \operatorname{HH}^*(X) &\simeq \Gamma(\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X, q^!(\mathcal{O}_X)) \\ &\simeq \Gamma(\mathbb{T}[-1]\mathbb{I}^{\operatorname{DM}}X, q^*\omega_X^\vee[-\dim(X)]) \\ &\simeq \Gamma(\mathbb{I}^{\operatorname{DM}}X, \pi_*\pi^*i^*\omega_X^\vee[-\dim(X)]) \\ &\simeq \Gamma(\mathbb{I}^{\operatorname{DM}}X, \operatorname{Sym}(\mathbb{L}_{\mathbb{I}^{\operatorname{DM}}X}[1]) \otimes i^*\omega_X^\vee[-\dim(X)]), \end{aligned}$$

where the last step uses the projection formula. $\square$

In the classical case, Corollary 4.14 can be reformulated in a much more concrete way, generalizing [3, Corollary 1.17] to the case where X is not necessarily a global quotient by a finite group. The following consequence is one instance:

Corollary 4.15. *Assume that X is a smooth (hence underived) DM stack. Consider the connected-component decomposition of $\mathbf{l}^{\mathrm{DM}}X = \mathbf{l}X$:*

$$\mathbf{l}X = \bigsqcup_{i \in I} Z_i,$$

where I is the set of connected components of $\mathbf{l}X$. Let $c_i = \dim(X) - \dim(Z_i)$, and let $\omega_{Z_i/X}$ be the relative dualizing sheaf of the natural map $Z_i \rightarrow X$. Then we have an isomorphism

$$\mathrm{HH}^*(X) \simeq \bigoplus_{i \in I} \bigoplus_{p+q=*} H^{p-c_i}(Z_i, \bigwedge^q \mathbb{T}_{Z_i} \otimes \omega_{Z_i/X}).$$

Proof. We apply (4.5.1) and use the relative canonical bundle formula $\omega_X^\vee|_{Z_i} \cong \omega_{Z_i}^\vee \otimes \omega_{Z_i/X}$ and the canonical isomorphism $\Omega_{Z_i}^p \otimes \omega_{Z_i}^\vee \cong \bigwedge^{\dim(Z_i)-p} \mathbb{T}_{Z_i}$. $\square$

4.6. The de Rham Differential. We now discuss the compatibility between our HKR theorem Theorem 4.9 and the de Rham differential. Let us first recall the theory of S^1 -algebras and mixed algebras and their relation to the HKR isomorphism.

Recollection 4.16. The main result of [27] shows that any choice of formality isomorphism ϕ of $\mathbf{C}^*(S^1; k)$ as a *Hopf* algebra provides an equivalence A_ϕ fitting in the following commutative diagram

$$\begin{array}{ccc} S^1\text{-}\mathbf{CAlg}_k & \xrightarrow[\sim]{A_\phi} & \varepsilon\text{-}\mathbf{CAlg}_k \\ & \searrow U_{S^1} & \swarrow U_\varepsilon \\ & \mathbf{CAlg}_k & \end{array}$$

where U_{S^1} and U_ε are the natural forgetful functors. Furthermore, the left adjoint functors L_{S^1} and L_ε (of U_{S^1} and U_ε respectively) admit the following explicit description: for any $A \in \mathbf{CAlg}_k$,

- $U_{S^1} L_{S^1}(A) \simeq S^1 \otimes A \simeq A \otimes_{A \otimes_k A} A$, so $L_{S^1}(A)$ is canonically identified with the Hochschild homology complex of A (considered only up to *quasi-isomorphism*) with the free S^1 -action.
- $U_\varepsilon L_\varepsilon(A) \simeq \mathrm{DR}(A) := \mathrm{Sym}_A(\mathbb{L}_{A/k}[1])$, so $L_\varepsilon(A)$ is identified with the derived de Rham algebra of A , with mixed structure given by the de Rham differential.

Recollection 4.17. If $\mathcal{Y}$ is any ∞ -topos, we can apply $(-) \otimes \mathcal{Y}$ to the above diagram to obtain an equivalence

$$\begin{array}{ccc} S^1\text{-}\mathbf{CAlg}_k(\mathcal{Y}) & \xrightarrow[\sim]{A_\phi} & \varepsilon\text{-}\mathbf{CAlg}_k(\mathcal{Y}) \\ & \searrow U_{S^1} & \swarrow U_\varepsilon \\ & \mathbf{CAlg}_k(\mathcal{Y}) & \end{array}$$

where now $\mathbf{CAlg}_k(\mathcal{Y})$ and its variants denote the ∞ -categories of sheaves with values in $\mathbf{CAlg}_k$ (or in its variants).

We fix a derived Deligne-Mumford stack X , and we write

$$\mathbf{l}\mathcal{X} := (\mathbf{l}^{\mathrm{DM}}X)_{\mathrm{\acute{e}t}}$$

for the small étale site of $\mathbf{l}^{\mathrm{DM}}X$. Since the étale topos is insensitive to the derived structure, we see that this is equally the étale topos of $\mathcal{L}X$ and of $\mathbb{T}[-1]\mathbf{l}^{\mathrm{DM}}X$, as well as of the *classical* inertia stack of $t_0(X)$. In particular, we can represent $\mathcal{L}X$ and $\mathbb{T}[-1]\mathbf{l}^{\mathrm{DM}}X$ in the language of structured ∞ -topoi (see for instance [17]) as

$$(\mathbf{l}\mathcal{X}, \mathcal{O}_{\mathcal{L}X}), \quad (\mathbf{l}\mathcal{X}, \mathcal{O}_{\mathbb{T}[-1]\mathbf{l}^{\mathrm{DM}}X}),$$

respectively.

The structure sheaf $\mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X}$ defines an object in $\mathrm{CAlg}_k(\mathcal{I}\mathcal{X})$, and *by definition* it is canonically identified with

$$U_\varepsilon L_\varepsilon(\mathcal{O}_{|\mathrm{DM}_X}) = \mathrm{DR}(\mathcal{O}_{|\mathrm{DM}_X}) .$$

Notice that the HKR isomorphism provided by Theorem 4.9 coincides with identity on classical truncations; it follows that the underlying geometric morphism of ∞ -topoi is the identity. In the language of structured ∞ -topoi it thus gives rise to a canonical identification

$$(4.6.1) \quad \mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X} \simeq \mathcal{O}_{\mathcal{L}_X}$$

inside $\mathrm{CAlg}_k(\mathcal{I}\mathcal{X})$. At the same time, the equivalence of Recollection 4.17 supplies an equivalence (depending on the choice of the formality ϕ)

$$\mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X} \simeq A_\phi(S^1 \otimes_k \mathcal{O}_{|\mathrm{DM}_X}) .$$

The canonical map

$$\mathcal{O}_{|\mathrm{DM}_X} \longrightarrow \mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X}$$

in $\mathrm{CAlg}_k(\mathcal{I}\mathcal{X})$ combined with the universal property of $S^1 \otimes_k \mathcal{O}_{|\mathrm{DM}_X}$ and the canonical S^1 -action on $\mathcal{O}_{\mathcal{L}_X}$ allows to obtain a canonical S^1 -equivariant map

$$(4.6.2) \quad S^1 \otimes_k \mathcal{O}_{|\mathrm{DM}_X} \longrightarrow \mathcal{O}_{\mathcal{L}_X}$$

in $S^1\text{-}\mathrm{CAlg}_k(\mathcal{I}\mathcal{X})$. By construction, under the equivalence of Recollection 4.17 and the identification (4.6.1), this corresponds to the identity of $\mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X}$ as an object in $\varepsilon\text{-}\mathrm{CAlg}_k(\mathcal{I}\mathcal{X})$. In particular, the map (4.6.2) becomes an equivalence after applying the forgetful functor U_{S^1} . Since this functor is conservative, it follows that (4.6.2) is an equivalence itself.

We can summarize this discussion as follows:

Corollary 4.18. *It is possible to promote the equivalence (4.6.1) induced by Theorem 4.9 in such a way that the natural S^1 -algebra structure on $\mathcal{O}_{\mathcal{L}_X}$ corresponds, via the equivalence of categories of Recollection 4.17, to the natural mixed structure on $\mathcal{O}_{\mathbb{T}[-1]|\mathrm{DM}_X}$.*

5. EXAMPLES

5.1. Global quotient. We compute the orbifold inertia of the global quotient of a derived scheme by a finite group, in terms of the fixed loci of this action. The derived structure leads to some subtleties in the definition of fixed locus. Let us first clarify this.

Definition 5.1 (Genuine fixed loci). Let Y be a derived stack equipped with an action of an algebraic group G . Hence we have a structural morphism $[Y/G] \rightarrow \mathrm{BG}$. The *genuine fixed locus* of the G -action on Y is defined as the following section stack.

$$Y^G := \mathbf{Sect}_{\mathrm{BG}}([Y/G]) := \mathrm{Spec}(k) \times_{\mathbf{Map}(\mathrm{BG}, \mathrm{BG})} \mathbf{Map}(\mathrm{BG}, [Y/G]) .$$

Let g be a finite-order automorphism of a derived stack Y . Let $\langle g \rangle$ be the finite cyclic group generated by g . Then the *genuine fixed locus* of g on Y , denote by Y^g , is defined to be genuine fixed locus of the action of $\langle g \rangle$ on Y :

$$Y^g := Y^{\langle g \rangle} = \mathbf{Sect}_{\mathrm{B}\langle g \rangle}([Y/\langle g \rangle]) .$$

Remark 5.2 (Residual action by centralizer). Let the notation be as in Definition 5.1. For any $g \in G$, let $Z(g) := Z_G(g)$ be the centralizer of g in G . Then the genuine fixed locus Y^g admits a canonical residual action of $Z(g)$. Indeed, the action of $Z(g)$ on Y clearly descends to an action on $[Y/\langle g \rangle]$ that respects the structural morphism $[Y/\langle g \rangle] \rightarrow \mathrm{B}\langle g \rangle$. Hence $\mathbf{Map}(\mathrm{B}\langle g \rangle, [Y/\langle g \rangle])$ admits a $Z(g)$ -action that respects the structural morphism to $\mathbf{Map}(\mathrm{B}\langle g \rangle, \mathrm{B}\langle g \rangle)$. By definition of Y^g as the section stack, we get a $Z(g)$ -action on it.

Remark 5.3 (Conjugation action). Let G be a finite group acting on a derived stack Y . For any $g, h \in G$, it is clear from the definition that we have a canonical isomorphism

$$(5.1.1) \quad h.: Y^g \xrightarrow{\sim} Y^{hgh^{-1}},$$

which is given by $x \mapsto h.x$ on the level of functor of points. These isomorphisms assemble into the so-called *conjugation* action of G on $\bigsqcup_{g \in G} Y^g$, where G acts on the indexing set by conjugation.

Proposition 5.4. *Let Y be a derived scheme equipped with an action of a finite group G . Then we have isomorphisms*

$$\mathrm{I}^{\mathrm{DM}}[Y/G] \simeq \bigsqcup_{[g] \in G/G} [Y^g/Z(g)] \simeq \left[\left(\bigsqcup_{g \in G} Y^g \right) / G \right],$$

where G/G denotes the set of conjugacy classes of G , and in the last stack, the G -action is the conjugation action of Remark 5.3.

Proof. The second isomorphism is clear, let us prove the first isomorphism. Let $g \in G$ be an element of order r . We have pull-back diagrams

$$(5.1.2) \quad \begin{array}{ccccc} Y^g & \longrightarrow & \mathbf{Map}(\mathbf{B}\langle g \rangle, [Y/\langle g \rangle]) & \longrightarrow & \mathbf{Map}(\mathbf{B}\langle g \rangle, [Y/Z(g)]) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{id}} & \mathbf{Map}(\mathbf{B}\langle g \rangle, \mathbf{B}\langle g \rangle) & \longrightarrow & \mathbf{Map}(\mathbf{B}\langle g \rangle, \mathbf{B}Z(g)) \end{array}$$

The left square is a pull-back square is by definition. The right square is a pull-back square since it is obtained by applying $\mathbf{Map}(\mathbf{B}\langle g \rangle, -)$ to the following pull-back square:

$$(5.1.3) \quad \begin{array}{ccc} [Y/\langle g \rangle] & \longrightarrow & [Y/Z(g)] \\ \downarrow & & \downarrow \\ \mathbf{B}\langle g \rangle & \longrightarrow & \mathbf{B}Z(g) \end{array}$$

Now since the cotangent complex of $\mathbf{Map}(\mathbf{B}\langle g \rangle, \mathbf{B}Z(g))$ is trivial, the bottom horizontal map

$$\mathrm{Spec}(k) \rightarrow \mathbf{Map}(\mathbf{B}\langle g \rangle, \mathbf{B}Z(g))$$

is étale. Therefore the upper horizontal map $Y^g \rightarrow \mathbf{Map}(\mathbf{B}\langle g \rangle, [Y/Z(g)])$ is also étale.

Now note that the top map factors through the quotient $[Y^g/Z(g)]$. This follows from the fact that $\mathrm{pt} \rightarrow \mathbf{Map}(\mathbf{B}\langle g \rangle, \mathbf{B}Z(g))$ factors through $\mathbf{B}Z(g)$. Hence

$$[Y^g/Z(g)] \rightarrow \mathbf{Map}(\mathbf{B}\langle g \rangle, [Y/Z(g)]) \rightarrow \mathrm{I}^{(r)}[Y/G]$$

is also étale, where $r = |g|$. In order to conclude the proof we only have to show that the above morphisms induce an isomorphism between the truncation of $\mathrm{I}^{\mathrm{DM}}[Y/G]$ and $\bigsqcup_{[g] \in G/G} [\mathrm{t}_0(Y)^g/Z(g)]$. This follows from the following computation

$$\mathrm{t}_0(\mathrm{I}^{\mathrm{DM}}[Y/G]) \cong [\mathrm{t}_0 Y/G] \times_{[\mathrm{t}_0 Y/G] \times [\mathrm{t}_0 Y/G]} [\mathrm{t}_0 Y/G] \cong [\mathrm{t}_0 Y/G] \cong \bigsqcup_{[g] \in G/G} [\mathrm{t}_0(Y)^g/Z(g)],$$

where we used that the fiber products commute with truncation functor t_0 . $\square$

The following corollary recovers and generalizes [3, Corollary 1.7].

Corollary 5.5. *Let Y be a derived scheme equipped with an action of a finite group G . Then*

(1) We have an isomorphism of algebras

$$\begin{aligned} \mathrm{HH}_{-*}([Y/G]) &\simeq \left(\bigoplus_{g \in G} \bigoplus_{q-p=*} H^q(Y^g, \Omega_{Y^g}^p) \right)^G \\ &\simeq \bigoplus_{[g] \in G/G} \bigoplus_{q-p=*} H^q(Y^g, \Omega_{Y^g}^p)^{Z(g)}. \end{aligned}$$

where $\Omega^p := \bigwedge^p \mathbb{L}$ stands for the p -th term of the derived de Rham complex.

(2) If moreover Y is lci, we have an isomorphism of vector spaces:

$$\mathrm{HH}^*([Y/G]) \simeq \bigoplus_{[g] \in G/G} \bigoplus_{q+p=*} H^q(Y^g, \bigwedge^p \mathbb{T}_{Y^g} \otimes \det(\mathbb{N}_g)[-c_g])^{Z(g)},$$

where $\mathbb{N}_g$ is the normal bundle $\mathbb{N}_{Y^g/Y}$ and c_g is its rank; in other words, $\mathbb{N}_g[-1]$ denotes the relative tangent complex of the canonical morphism $Y^g \rightarrow Y$.

Proof. We apply Theorem 4.11 and Corollary 4.14 (or Corollary 4.15) to $X = [Y/G]$ and combine with Proposition 5.4. $\square$

In addition to Definition 5.1, the following notion of fixed locus is also frequently used in derived algebraic geometry :

Definition 5.6 (Derived fixed loci). Let Y be a derived stack and g an automorphism of Y . We define the *derived fixed locus* $Y^{\mathbb{R}g}$ via the pull-back diagram

$$(5.1.4) \quad \begin{array}{ccc} Y^{\mathbb{R}g} & \longrightarrow & Y \\ \downarrow & & \downarrow \Delta \\ Y & \xrightarrow{\Delta_g} & Y \times Y \end{array}$$

where Δ_g is the g -twisted diagonal map (i.e. the graph of the automorphism of Y given by g), which is given by the following formula on functor of points:

$$(5.1.5) \quad \Delta_g : Y \rightarrow Y \times Y, \quad \Delta_g(y) := (y, gy).$$

The following lemma clarifies the relation between the genuine fixed loci (Definition 5.1) and the derived fixed loci (Definition 5.6) :

Lemma 5.7. *Let Y be a derived scheme equipped with a finite-order automorphism g . Then the genuine fixed locus Y^g is a derived scheme, and we have*

$$\mathbb{T}[-1]Y^g \simeq \mathcal{L}Y^g \simeq Y^{\mathbb{R}g}.$$

Proof. We give here a proof using our main result Theorem 4.9. A direct proof is certainly desired. Let r be the order of g , and let $G = \langle g \rangle$ be the cyclic group generated by g acting naturally on Y . By Theorem 4.9 and Proposition 5.4, we have

$$\mathcal{L}[Y/G] \simeq \mathbb{T}[-1]^{\mathrm{DM}}[Y/G] \simeq \mathbb{T}[-1] \bigsqcup_{k=1}^r [Y^{g^k}/G] \simeq \bigsqcup_{k=1}^r [\mathbb{T}[-1]Y^{g^k}/G]$$

On the other hand,

$$\mathcal{L}[Y/G] \simeq [Y/G] \times_{[Y/G] \times [Y/G]} [Y/G] \simeq \bigsqcup_{k=1}^r [Y^{\mathbb{R}g^k}/G].$$

Since the HKR isomorphism induces identity on the classical truncation, we obtain an equivalence

$$(5.1.6) \quad [\mathbb{T}[-1]Y^g/G] \simeq [Y^{\mathbb{R}g}/G].$$

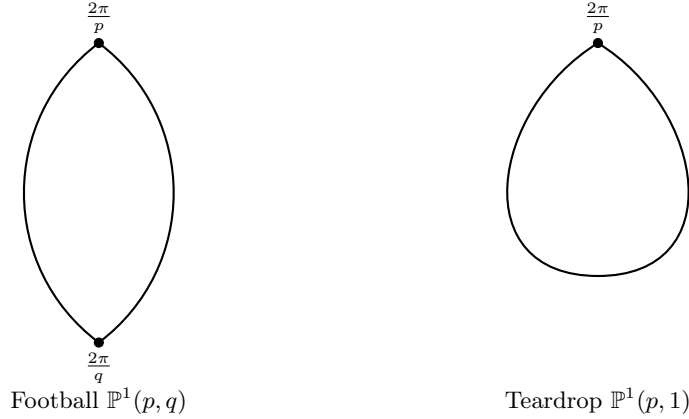
As G is generated by g , hence the G -action on Y^g is trivial. It follows that $\mathbb{T}[-1]Y^g \simeq Y^{\mathbb{R}g}$.

Since $Y^{\mathbb{R}g}$ is a derived scheme by construction, $\mathbb{T}[-1]Y^g$ is also a derived scheme. Therefore, Y^g , as a closed subscheme of $\mathbb{T}[-1]Y^g$, must be a derived scheme. Finally, applying Theorem 4.9 to Y^g , one yields $\mathbb{T}[-1]Y^g \simeq \mathcal{L}Y^g$. $\square$

5.2. Beyond global quotients. Already in the category of underived classical stacks, most DM stacks are not of the form $[Y/G]$ with G a finite group acting on an algebraic space Y . By [21, Proposition 6], any complex variety with quotient singularities (i.e. *orbifold* in the classical sense as in [23]) admits a canonical DM stack structure with only non-trivial stabilizers in codimension ≥ 2 .

To illustrate the usefulness of the generality of Theorem 4.11, let us compute the Hochschild homology of some DM stacks that are not accessible with the previously known results like [3].

Example 5.8 (Thurston's football and teardrop). Given two positive integers p, q that are coprime to each other, let $X = \mathbb{P}^1(p, q) = \text{Proj}(k[x, y])$ be the weighted projective line with $\deg(x) = p, \deg(y) = q$. The underlying orbifold is often referred to as *Thurston's football* (or *teardrop* when $q = 1$); see the pictures below. It is well-known that X is *not* of the form of a global quotient of a scheme by a finite group.



By Corollary 3.4,

$$(5.2.1) \quad {}^{\text{I}}\text{DM}X \simeq {}^{\text{I}}X \simeq X \sqcup (\text{I}(\text{BC}_p \sqcup \text{BC}_q)) \simeq X \sqcup ([\mathbb{C}_p/\mathbb{C}_p] \sqcup [\mathbb{C}_q/\mathbb{C}_q]) \simeq X \sqcup \bigsqcup_{p-1} \text{BC}_p \sqcup \bigsqcup_{q-1} \text{BC}_q$$

Consequently,

$$\text{HH}_0(X) = H^0(X, \mathcal{O}_X) \oplus H^1(X, \mathbb{L}_X) \oplus H^0(\text{BC}_p, \mathcal{O}_{\text{BC}_p})^{p-1} \oplus H^0(\text{BC}_q, \mathcal{O}_{\text{BC}_q})^{q-1} = k^{p+q},$$

and $\text{HH}_i(X) = 0$ for any $i \neq 0$.

Although most DM stacks are not global quotients by finite groups, in practice, by working with quotient stacks by linear algebraic groups, we can already deal with a fairly large class of DM stacks. For instance, the weighted projective line $\mathbb{P}^1(p, q)$ is equivalent to the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ where the 2-dimensional representation of $\mathbb{G}_m$ is of weight p, q . More generally, in [9, Theorem 2.18], it is shown that any smooth DM stack with trivial generic stabilizer is a quotient stack of the form $[Y/G]$ with G a linear algebraic group acting on an algebraic space Y . Motivated by this consideration, we compute the inertia stack as well as the Hochschild homology of smooth quotient stacks by linear algebraic groups.

Proposition 5.9. *Let Y be a finite-type smooth (hence underived) scheme equipped with a faithful action of a linear algebraic group G . Assume that the quotient stack $[Y/G]$ is a separated Deligne-Mumford stack. Then there are only finitely many conjugacy classes in G with non-empty fixed loci; we denote this finite set by $\mathcal{C}$. Then we have an isomorphism*

$$\mathrm{I}^{\mathrm{DM}}[Y/G] \simeq \bigsqcup_{[g] \in \mathcal{C}} [Y^g/Z(g)],$$

where $Z(g)$ is the centralizer of g in G .

Proof. Since $[Y/G]$ is separated DM, any $[g] \in \mathcal{C}$ must be of finite order, and the universal stabilizer (a.k.a. the inertia scheme)

$$\mathrm{I}_G(Y) := Y \times_{Y \times Y} (Y \times G)$$

is finite over Y , where the fiber product is underived. By [8, Lemma 2.1] the set $\mathcal{C}$ is finite and consists of conjugacy classes of semi-simple elements.

By assumption, X is a smooth underived DM stack, hence Corollary 3.4 implies that

$$\mathrm{I}^{\mathrm{DM}}X \simeq \mathrm{I}X \simeq [\mathrm{I}_G(Y)/G].$$

For any $[g] \in \mathcal{C}$, since g is semi-simple, the elements in its conjugacy class form a closed subvariety of G , denoted by $C_G(g) := \{hgh^{-1} \mid h \in G\}$.

For any $[g] \in \mathcal{C}$, there is a natural morphism

$$(5.2.2) \quad [Y^g/Z(g)] \rightarrow [\mathrm{I}_G(Y)/G].$$

We claim that it is étale. Indeed, by the smoothness of Y , Y^g is a smooth subscheme of Y , and for any $y \in Y^g$, the tangent space $\mathbb{T}_{y,Y^g}$ is canonical identified with the subspace of g -invariants $(\mathbb{T}_{y,Y})^g$. Therefore the tangent space of y in $[Y^g/Z(g)]$ is given by the two-term complex

$$(5.2.3) \quad \mathbb{T}_{y,[Y^g/Z(g)]} \simeq [\mathrm{Lie}(Z(g)) \rightarrow (\mathbb{T}_{y,Y})^g].$$

Similarly,

$$(5.2.4) \quad \mathbb{T}_{(g,y),[\mathrm{I}_G Y/G]} \simeq [\mathrm{Lie}(G) \rightarrow \mathbb{T}_{(g,y),\mathrm{I}_G Y}].$$

By definition, we have a disjoint decomposition of $\mathrm{I}_G(Y)$ by looking at the projection to G :

$$\mathrm{I}_G(Y) = \bigsqcup_{[g] \in \mathcal{C}} \mathrm{I}_{[g]}Y,$$

where $\mathrm{I}_{[g]}Y = \{(hgh^{-1}, y) \mid h \in G, y \in Y^{hgh^{-1}}\}$. By construction, $\mathrm{I}_{[g]}Y \rightarrow C_G(g)$ is a smooth fibration with fibers isomorphic to Y^g . Hence $\mathbb{T}_{y,\mathrm{I}_G Y} = \mathbb{T}_{y,\mathrm{I}_{[g]}Y}$ fits into the bottom short exact sequence in the following diagram:

$$(5.2.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Lie}(Z(g)) & \longrightarrow & \mathrm{Lie}(G) & \longrightarrow & \mathbb{T}_{C_G(g)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & (\mathbb{T}_{y,Y})^g & \longrightarrow & \mathbb{T}_{(g,y),\mathrm{I}_G Y} & \longrightarrow & \mathbb{T}_{C_G(g)} \longrightarrow 0 \end{array}$$

The top short exact sequence follows from the fact that as a variety, $C_G(g) \simeq G/Z(g)$, where $Z(g)$ is the centralizer of g . Putting (5.2.3), (5.2.4) and (5.2.5) together, we conclude that (5.2.2) is an étale morphism for any $[g] \in \mathcal{C}$.

When $[g]$ runs through the finite set $\mathcal{C}$, we get an étale morphism

$$(5.2.6) \quad \bigsqcup_{[g] \in \mathcal{C}} [Y^g/Z(g)] \rightarrow [\mathrm{I}_G(Y)/G] \simeq \mathrm{I}X.$$

To check that it is an isomorphism, it suffices to check that the following morphism is an isomorphism

$$\bigsqcup_{[g] \in \mathcal{C}} G \times^{Z(g)} Y^g \rightarrow \mathbf{l}_G(Y)$$

$$(h, y) \mapsto (hgh^{-1}, hy)$$

where $Z(g)$ acts on $G \times Y^g$ via $t.(h, y) := (ht^{-1}, ty)$. This morphism is étale since it is the base-change of (5.2.6), and moreover it induces a bijection on geometric points, therefore, it is an isomorphism. $\square$

Corollary 5.10. *Let Y be a finite-type smooth scheme equipped with a faithful action of a linear algebraic group G . Assume that the quotient stack $[Y/G]$ is a separated Deligne-Mumford stack. Then*

$$\mathrm{HH}_{-*}([Y/G]) \simeq \bigoplus_{[g] \in \mathcal{C}} \bigoplus_{q-p=*} H^q([Y^g/Z(g)], \Omega_{[Y^g/Z(g)]}^p).$$

Proof. It follows from the combination of Theorem 4.11 and Proposition 5.9. $\square$

6. THE CANONICAL HKR DEFORMATION FOR DELIGNE-MUMFORD STACKS

In [15], Moulinos, Robalo and Toën constructed the so-called *filtered circle*, which is a deformation S_{filt}^1 over the stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$ with generic fiber S^1 . In characteristic zero, the central fiber of their construction is canonically identified with $\mathbb{D}_{-1} \simeq \widehat{\mathbf{B}\mathbb{G}_a}$, where $\mathbb{D}_{-1}$ is the suspension of $\mathbb{D}_0 = \mathrm{Spec} k[\epsilon]/(\epsilon^2)$. As a result, for any derived stack X , one can form

$$\mathcal{D}_X := \mathbf{Map}_{/\Theta}(S_{\mathrm{filt}}^1, X \times \Theta) \in \mathbf{dSt}_{/\Theta}.$$

Formal properties of the mapping stack imply that the generic fiber of $\mathcal{D}_X$ is identified with $\mathcal{L}X$, whereas its central fiber is identified with $\mathbf{T}[-1]X$.

In light of our main result, Theorem 4.9, it is desirable to modify the above construction to produce a deformation of $\mathcal{L}X$ to $\mathbf{T}[-1]^{\mathrm{DM}}X$ over Θ , at least when X is a derived Deligne-Mumford stack. With this goal in mind, we propose the following construction:

Definition 6.1. For any positive integer n , we define the following k -algebra

$$A_n := k[x_1, \dots, x_n]/I_n$$

where the ideal I_n is defined as

$$I_n = (x_i x_j; 1 \leq i < j \leq n) = \bigcap_{i=1}^n (x_1, \dots, \widehat{x_i}, \dots, x_n).$$

We consider $\mathrm{Spec}(A_n)$ as a scheme over $\mathbb{A}^1 = \mathrm{Spec} k[t]$, where the structural map

$$\pi_n: \mathrm{Spec}(A_n) \rightarrow \mathbb{A}^1$$

is induced by the algebra homomorphism

$$k[t] \rightarrow A_n$$

$$t \mapsto x_1 + \dots + x_n,$$

which is clearly $\mathbb{G}_m$ -equivariant with respect to the scaling action.

We note that geometrically, $\mathrm{Spec}(A_n)$ is the union of the coordinate axis in $\mathbb{A}^n$. The fiber of π_n over $t \neq 0$ consists of n distinct reduced points $\{(t, 0, \dots, 0), \dots, (0, \dots, 0, t)\}$, and its fiber over $t = 0$ is the spectrum of $A_n/(x_1 + \dots + x_n) \simeq k[x_1, \dots, x_n]/\langle \mathfrak{m}^2, x_1 + \dots + x_n \rangle$ with $\mathfrak{m} = (x_1, \dots, x_n)$; this is a length- n subscheme of $\mathbb{A}^n$ supported at the origin.

Definition 6.2 (Filtered circle with cyclic action). For any positive integer n , we construct a filtered circle with an action of C_n , the cyclic group of order n . We define $\bar{S}_{\text{filt}}^{1,(n)}$ as the colimit of the following diagram in the category $\mathbf{dSt}_{\mathbb{A}^1}$:

$$(6.0.1) \quad \begin{array}{ccc} & & \text{Spec } A_{n-1} \\ & \nearrow f_1 & \\ \text{Spec } A_n & \longrightarrow & \vdots \\ & \searrow f_n & \\ & & \text{Spec } A_{n-1} \end{array}$$

where there are n copies of $\text{Spec } A_{n-1}$ on the right, and for each $1 \leq s \leq n$, the morphism f_s from $\text{Spec } A_n$ to the s -th copy of $\text{Spec } A_{n-1}$ is induced by the following $k[t]$ -algebra homomorphism (using the same notation):

$$\begin{aligned} f_s : A_{n-1} &\rightarrow A_n \\ x_1 &\mapsto x_s + x_{s+1} \\ x_i &\mapsto x_{s+i} \quad \text{for } i = 2, \dots, n-1, \end{aligned}$$

where the index of the variables x_i are considered modulo n .

We endow a C_n -action on the Diagram 6.0.1 given by the cyclic permutation of the indices of the variables in A_n and the cyclic permutation of the maps f_i along with their targets $\text{Spec}(A_{n-1})$. One can easily check that this is a diagram automorphism, and hence it gives rise to a C_n -action on the colimit $\bar{S}_{\text{filt}}^{1,(n)}$. This C_n -action respects the morphism π_n to $\mathbb{A}^1$.

Remark 6.3 (Compatibility). For any positive integers m and n with m dividing n , the natural surjective group homomorphism $g: C_n \twoheadrightarrow C_m$ induces a g -equivariant map of stacks over $\mathbb{A}^1$

$$\bar{\phi}_n^m : \bar{S}_{\text{filt}}^{1,(m)} \rightarrow \bar{S}_{\text{filt}}^{1,(n)}$$

given by the algebra homomorphism

$$\begin{aligned} A_n &\rightarrow A_m \\ x_i &\mapsto x_{g(i)}; \end{aligned}$$

and for any $1 \leq s \leq n$, the morphism $f_s: \text{Spec } A_n \rightarrow \text{Spec } A_{n-1}$ is sent to the map $f_{g(s)}: \text{Spec } A_m \rightarrow \text{Spec } A_{m-1}$.

Lemma 6.4 (Generic fiber). For any $t \neq 0$ in $\mathbb{A}^1$, the fiber F_t^n of $\bar{S}_{\text{filt}}^{1,(n)} \rightarrow \mathbb{A}^1$ over t is homotopy equivalent to S^1 and the induced C_n -action is given by rotation.

Proof. Over $t \neq 0$, the fiber of $\text{Spec } A_n \rightarrow \mathbb{A}^1$ consists of n reduced points. Therefore the fiber F_t^n is the homotopy colimit of the following diagram:

$$(6.0.2) \quad \begin{array}{ccc} & & n-1 \text{ pts} \\ & \nearrow f_{1,t} & \\ n \text{ pts} & \longrightarrow & \vdots \\ & \searrow f_{n,t} & \\ & & n-1 \text{ pts} \end{array}$$

By replacing each map by a cofibration, one sees that the homotopy colimit is S^1 . Following the computation of homotopy colimit, it is straightforward to see that the C_n -action is the standard rotation action. $\square$

Lemma 6.5 (Central fiber). *The central fiber F_0^n of $\bar{S}_{\text{filt}}^{1,(n)} \rightarrow \mathbb{A}^1$ over $t = 0$ is given by $\mathbb{D}_{-1}$.*

Proof. We work in the ∞ -category of pointed spaces $\mathbf{Spc}_*$. We write

$$X_n = \{x_1, \dots, x_n\}$$

for the discrete pointed space consisting of n -points, x_1 being the marked point. For $1 \leq i \leq n$ we let $X_{n,i}$ be the discrete pointed space obtained from X_n identifying x_i with x_{i+1} (with the convention that $x_{n+1} := x_1$), and denote by

$$\varphi_i: X_n \longrightarrow X_{n,i}$$

the canonical morphisms. Notice that there are non-canonical identifications $X_{n,i} \simeq X_{n-1}$. Furthermore, it is straightforward to verify that the colimit in $\mathbf{Spc}_*$ of the diagram

$$\begin{array}{ccc} & & X_{n,1} \\ & \nearrow \varphi_1 & \\ X_n & \longrightarrow & \vdots \\ & \searrow \varphi_n & \\ & & X_{n,n} \end{array}$$

is given by S^1 .

Recall now from [13, Proposition 4.8.2.11] that $\mathbf{Spc}_*$ is the tensor unit of the canonical symmetric monoidal ∞ -category of pointed presentable ∞ -categories. In particular, there is a canonical action

$$\wedge: \mathbf{Spc}_* \otimes \mathbf{dSt}_{\mathbb{A}^1//\mathbb{A}^1} \longrightarrow \mathbf{dSt}_{\mathbb{A}^1//\mathbb{A}^1},$$

with the property that $S^0 \wedge -$ acts as the identity of $\mathbf{dSt}_{\mathbb{A}^1//\mathbb{A}^1}$ and that $\wedge$ commutes with colimits in both variables. Writing

$$X_n \simeq \overbrace{S^0 \vee \dots \vee S^0}^{n-1}$$

we immediately see that

$$X_n \wedge \text{Spec}(A_2) \simeq \text{Spec}(A_n),$$

and that the maps φ_i induce the maps f_i . Since colimits in $\mathbf{dSt}_{/\mathbb{A}^1}$ are universal, and since the central fiber of $\text{Spec}(A_2)$ is given by $\mathbb{D}_0$, we conclude that the central fiber of $\bar{S}_{\text{filt}}^{1,(n)} \rightarrow \mathbb{A}^1$ is computed by the smash product

$$S^1 \wedge \mathbb{D}_0 \simeq \mathbb{D}_{-1},$$

where the smash denotes the action of $\mathbf{Spc}_*$ on $\mathbf{dSt}_{\text{Spec}(k)//\text{Spec}(k)}$. The proof is therefore complete. $\square$

Lemma 6.6. *The affinization of the central fibre F_0^n is given by BG_a with trivial C_n -action.*

Proof. F_0^n is given by the colimit of the following diagram

$$(6.0.3) \quad \begin{array}{ccc} & & \text{Spec } B_{n-1} \\ & \nearrow f_1 & \\ \text{Spec } B_n & \longrightarrow & \vdots \\ & \searrow f_n & \\ & & \text{Spec } B_{n-1} \end{array}$$

where $B_n = A_n / \langle \sum_{i=1}^n x_i \rangle \simeq k[x_1, \dots, x_n] / \langle \mathfrak{m}^2, x_1 + \dots + x_n \rangle$, where $\mathfrak{m} = (x_1, \dots, x_n)$, and the definition of the maps f_s is as in Definition 6.1. As we are interested in the affinization of the central fiber, it is sufficient to compute the homotopy limit of the diagram in the category of algebras

$$(6.0.4) \quad \begin{array}{ccc} & & B_{n-1} \\ & \nwarrow f_1 & \\ B_n & \longleftarrow & \vdots \\ & \nwarrow f_n & \\ & & B_{n-1} \end{array}$$

The affinization of the central fiber will be equivalent to the coaffine stack defined as the cospectrum of the homotopy limit of Diagram (6.0.4).

Let us compute this homotopy limit in the category of positively graded cdga's (where we are using cohomological conventions). Since the forgetful functor from the category of cdga's to the category of complexes preserve homotopy limits, in order to calculate the homotopy limit, we can first place ourselves in the category of positively graded complexes. The short exact sequence

$$0 \rightarrow B_{n-1} \xrightarrow{f_i} B_n \rightarrow \text{Coker}(f_i) \rightarrow 0$$

allows us to replace B_{n-1} by the two-term complex $[B_n \rightarrow \text{Coker}(f_i)]$. Accordingly, we can replace the morphism $f_i : B_{n-1} \rightarrow B_n$ by the following morphism of complexes:

$$\begin{array}{ccc} B_n & \longrightarrow & \text{Coker}(f_i) \\ \downarrow \text{id} & & \downarrow \\ B_n & \longrightarrow & 0 \end{array}$$

This is a fibrant replacement, as it is a degree-wise surjective. Now the homotopy limit of Diagram (6.0.4), after these fibrant replacements, is computed degree-wise as the classical limit, and is given by the following two-term complex

$$(6.0.5) \quad L := [B_n \rightarrow \bigoplus_{i=1}^n \text{Coker}(f_i)].$$

Note that for any i , $\text{Coker}(f_i)$ is the 1-dimensional k -vector space generated by x_n . Identifying each $\text{Coker}(f_i)$ with k in this way, the map in the complex (6.0.5) is given as follows:

$$a_0 + a_1 x_1 + \dots + a_n x_n \mapsto (a_1 - a_2, a_2 - a_3, \dots, a_n - a_1).$$

Moreover, the C_n -action is given by the cyclic permutation of the summands $\text{Coker}(f_i)$. In other words, L is isomorphic to the complex

$$\begin{aligned} k^{n+1}/k \cdot (0, 1, 1, \dots, 1) &\rightarrow k^n \\ (a_0, a_1, \dots, a_n) &\mapsto (a_1 - a_2, a_2 - a_3, \dots, a_n - a_1). \end{aligned}$$

with C_n -action permuting cyclically $a_1, \dots, a_n$.

As a complex, L is equivalent to $k \oplus k[-1]$, and hence as k -cdga, L must be equivalent to $k[\eta]$ with $|\eta| = 1$. Now taking into account the C_n -action: from the C_n -equivariant retraction from $k^{n+1}/k \cdot (0, 1, 1, \dots, 1)$ to $k \cdot (1, 0, \dots, 0)$, L is isomorphic to $k \oplus [k^n/(1, 1, \dots, 1) \rightarrow k^n]$, where the map in $k^n/(1, 1, \dots, 1) \rightarrow k^n$ is $(a_1, \dots, a_n) \mapsto (a_1 - a_2, a_2 - a_3, \dots, a_n - a_1)$, whose cokernel is the sum map $k^n \rightarrow k$. When the characteristic of k is coprime to n , we have an C_n -equivariant section of the sum map given as

$$\begin{aligned} k &\rightarrow k^n \\ 1 &\mapsto \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right), \end{aligned}$$

hence L is isomorphic to $k[\eta]$ with trivial C_n -action when $\text{char}(k)$ is coprime to n .

It follows that the affinization of the colimit of (6.0.3) is the cospectrum of the cdga $k[\eta]$, with η in degree 1, and with the trivial C_n -action when $\text{char}(k) \nmid n$. In characteristic 0, this is equivalent to $\text{B}\mathbb{G}_a$ with trivial C_n -action, as we wanted to show. $\square$

Construction 6.7. Note that the map

$$\bar{S}_{\text{filt}}^{1,(n)} \rightarrow \mathbb{A}^1$$

is equivariant with respect to the C_n -action on the source we have just described, and the trivial C_n -action on the target. Passing to the C_n -quotient we obtain a map

$$\tilde{S}_{\text{filt}}^{1,(n)} := [\bar{S}_{\text{filt}}^{1,(n)} / C_n] \rightarrow \mathbb{A}^1$$

which is $\mathbb{G}_m$ -equivariant with respect to the natural rescaling action.

We define the n -th *orbifold filtered circle* to be the quotient of $\tilde{S}_{\text{filt}}^{1,(n)}$ by this action of $\mathbb{G}_m$: we set

$$S_{\text{filt}}^{1,(n)} := [\tilde{S}_{\text{filt}}^{1,(n)} / \mathbb{G}_m] \rightarrow [\mathbb{A}^1 / \mathbb{G}_m] = \Theta$$

For any integers m dividing n , let $g: C_n \rightarrow C_m$ be the natural surjective homomorphism, the g -equivariant transition maps

$$\overline{\phi_n^m}: \bar{S}_{\text{filt}}^{1,(m)} \rightarrow \bar{S}_{\text{filt}}^{1,(n)}$$

restrict to the degree- n/m cover $S^1 \rightarrow S^1$ on the generic fiber, and to the identity map on the central fiber. They are equivariant with respect to the C_n -action, and the rescaling action by $\mathbb{G}_m$. Thus they descend to transition maps over Θ

$$\phi_n^m: S_{\text{filt}}^{1,(m)} \rightarrow S_{\text{filt}}^{1,(n)}$$

We define the *orbifold filtered circle*

$$\widehat{S}_{\text{filt}}^1 := \varinjlim_n S_{\text{filt}}^{1,(n)} \in \text{Pro}(\text{dSt}/\Theta).$$

Given a derived stack X , we define the *Deligne-Mumford filtered loop space* as

$$\mathcal{D}_X^{\text{DM}} := \mathbf{Map}_{/\Theta}(\widehat{S}_{\text{filt}}^1, X \times \Theta).$$

Proposition 6.8. *Let X be a derived stack.*

(1) *The generic fiber of $\mathcal{D}_X^{\text{DM}}$ is canonically identified with*

$$\mathbf{Map}(S^1/C_n, X) \simeq \mathcal{L}X.$$

(2) Assume furthermore that X is a derived Deligne-Mumford stack. Then the central fibre of $\mathcal{D}_X^{\text{DM}}$ is canonically identified with

$$\mathbf{Map}(\mathbf{B}\mathbb{G}_a \times \widehat{S}^1, X) \simeq \mathbf{T}[-1]^{\text{DM}} X .$$

Proof. The generic fibre of $\mathcal{D}_X^{\text{DM}}$ is canonically identified with $\mathbf{Map}(S^1/C_n, X)$ by definition. The further identification with $\mathcal{L}X$ follows from the canonical isomorphism $S^1/C_n \simeq S^1$ induced by the n -th power map $S^1 \rightarrow S^1$.

We now prove item (2). First, we observe that the central fiber of $\mathcal{D}_X^{\text{DM}}$ is canonically identified with

$$\text{colim}_n \mathbf{Map}(\mathbb{D}_{-1}/C_n, X) .$$

The canonical map $\mathbb{D}_{-1} \rightarrow \text{Aff}(\mathbb{D}_{-1})$ induces a comparison map

$$(6.0.6) \quad \mathbf{Map}(\text{Aff}(\mathbb{D}_{-1})/C_n, X) \longrightarrow \mathbf{Map}(\mathbb{D}_{-1}/C_n, X) ,$$

functorial in n . Furthermore, Lemma 6.6 implies that

$$\text{Aff}(\mathbb{D}_{-1})/C_n \simeq \text{Aff}(\mathbb{D}_{-1}) \times \mathbf{B}C_n \simeq \mathbf{B}\mathbb{G}_a \times \mathbf{B}C_n .$$

It is therefore enough to argue that (6.0.6) is an equivalence whenever X is a derived Deligne-Mumford stack. Since both $\mathbb{D}_{-1}/C_n$ and $\text{Aff}(\mathbb{D}_{-1})/C_n$ are colimits of the simplicial diagrams encoding the C_n action, one readily reduces to check that the canonical map

$$\mathbf{Map}(\text{Aff}(\mathbb{D}_{-1}) \times C_n^{\times m}, X) \longrightarrow \mathbf{Map}(\mathbb{D}_{-1} \times C_n^{\times m}, X)$$

is an equivalence for every $m \geq 0$. Since each $C_n^{\times m}$ is a disjoint union of finitely many points, one further reduces to check that the canonical map

$$(6.0.7) \quad \mathbf{Map}(\text{Aff}(\mathbb{D}_{-1}), X) \longrightarrow \mathbf{Map}(\mathbb{D}_{-1}, X)$$

is an equivalence. Notice that this latter statement is true for X affine, by the universal property of the affinization. On the other hand, since we are in characteristic zero,

$$\text{Aff}(\mathbb{D}_{-1}) \simeq \mathbf{B}\mathbb{G}_a .$$

Since $\mathbb{G}_a$ is connected, Theorem 2.11 implies that the source of (6.0.7) satisfies étale codescent. The same holds for the target, and therefore the conclusion follows. $\square$

Corollary 6.9. *Let X be a derived Deligne-Mumford stack. Then $\text{HH}_*(X)$ is equipped with a natural filtration whose associated graded complex is equivalent to $\Gamma(\mathbf{l}^{\text{DM}} X, \text{Sym}(\mathbb{L}_{\mathbf{l}^{\text{DM}} X}[1]))$.*

Proof. By Proposition 6.8, we have a diagram of fiber-products

$$\begin{array}{ccccc} T[-1]^{\text{DM}} X & \longrightarrow & \mathcal{D}_X^{\text{DM}} & \longleftarrow & \mathcal{L}X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \text{Spec}(k) = [\mathbb{G}_m/\mathbb{G}_m] \end{array}$$

As explained in [15, Remark 2.2.8], to conclude we need to show that both squares satisfy base change. Following [15], a sufficient condition for this to hold is that X admits a flat hypercover by affines with flat transition maps. Since X is Deligne-Mumford this condition is satisfied, and this concludes the proof. $\square$

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