Autour des variétés hyperkählériennes:
motifs, supersingularité et automorphismes

Mémoire d'habilitation à diriger des recherches

Présenté par
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Résumé: Dans ce mémoire, on étudie différents aspects des variétés hyperkählériennes: leurs motifs, leur arithmétique et leur dynamique. La première et principale partie concerne leurs anneaux de Chow et leurs motifs de Chow comme objets d'algèbre. On donne plusieurs nouvelles perspectives vers la conjecture de Beauville–Voisin: *via* le produit d'orbifold, *via* des comparaisons aux motifs abéliens et *via* les déformations aux familles universelles. Des extensions ainsi que des nouveaux résultats autour de cette conjecture sont obtenus. La deuxième partie porte sur l'analogue de telles variétés, dites symplectiques irréductibles, sur un corps de caractéristique non nulle: on développe les bases de la théorie des variétés symplectiques irréductibles supersingulières et on propose quelques conjectures que l'on vérifie sur certains espaces de modules des faisceaux. La troisième partie établit quelques propriétés de finitude pour le groupe des automorphismes d'une variété hyperkählérienne.

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Around hyper-Kähler varieties:
motives, supersingularity and automorphisms

Abstract

This memoir studies the motivic, arithmetic and dynamic aspects of the so-called hyper-Kähler varieties. The first and principal part concerns their Chow rings and Chow motives viewed as algebra objects. We provide several new perspectives on the Beauville–Voisin conjecture: *via* orbifold products, *via* comparisons with abelian motives and *via* spreading to the universal family. Meaningful extensions as well as new evidence of this conjecture are obtained. The second part studies analogues of these varieties, called irreducible symplectic varieties, in positive characteristics. We develop the basic theory of supersingular irreducible symplectic varieties, propose a conjectural picture and provide evidence in the case of moduli spaces of sheaves. The third part establishes several basic finiteness results on the automorphism groups of hyper-Kähler manifolds.

Dedicated to my teachers.

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更上一层楼

唐·王之涣·登鹳雀楼

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Overview

This memoir presents part of the research work that I have carried out after my PhD degree, during the period 2014 - 2019. The main object of study is a certain kind of Calabi–Yau varieties called *hyper-Kähler* varieties (also known as *irreducible symplectic* varieties). They are natural generalizations of K3 surfaces. We will touch upon the motivic, arithmetic, and dynamic aspects of these varieties.

A large portion of the content is on the study of algebraic cycles (*i.e.* Chow rings) and Chow motives (as algebra objects) of projective hyper-Kähler varieties, mainly inspired by the Beauville–Voisin conjecture [**Bea07**], [**Voi08**]. We provide several new perspectives on this conjecture: *via* orbifold products, *via* comparisons with abelian motives and *via* spreading to the universal family. Meaningful extensions and closely related conjectures are proposed; new evidence for this conjecture is presented. With a more arithmetic flavour, we investigate the geometry and the motives of the analogous varieties over fields of positive characteristics and especially the supersingular ones, trying to generalize the beautiful theory of supersingular K3 surfaces. Finally, within the context of algebraic dynamic systems, we answer some basic open questions on the finiteness properties of automorphism groups of compact hyper-Kähler manifolds (projective or not).

Let us briefly describe the content of the memoir by chapters. The first two chapters are of introductory nature and have very limited technical details. My goal is that they be reasonably self-contained for a non-specialist and serve as a common base of the subsequent chapters, which then hopefully can be read in an independent manner.

In the first chapter, we recall the basic theory of algebraic cycles and motives as well as the general conjectural picture proposed by Bloch–Beilinson–Murre, with an emphasis on the Frobenius algebra object structure on the motive of smooth projective varieties and the possibility of having a motivic Künneth decomposition that is compatible with this algebra structure (a so-called *multiplicative Chow–Künneth decomposition*). Such a motivic decomposition is not expected to exist in general and was originally studied only for abelian varieties and hyper-Kähler varieties. However, it is interesting to ask in general when such decompositions exist and how to construct them. The Frobenius structure on the motive of a smooth projective variety, or rather its importance, is a fairly recent discovery of Vial and myself [**FV19b**], which I touch upon here only briefly to reflect the most updated possible knowledge and which I wish to pursue in the years to come. We include along the way some of our results obtained in [**FT19**], [**FV19a**], [**FV19b**] and [**FLV19**].

Chapter II specializes Chapter I to cycles and motives of projective hyper-Kähler varieties. The new feature is the *splitting property* conjectured by Beauville [**Bea07**]. We spend the first two sections on the established theory for K3 surfaces and abelian varieties, not only to give historical origins and motivations, but also to present some preliminary results that will be used repeatedly in the subsequent chapters, for instance Beauville–Voisin's fundamental theorem on cycles on projective K3 surfaces [BV04], Beauville–Deninger–Murre's decomposition of Chow rings [Bea86] and motives [DM91] of abelian varieties and O'Sullivan's theory of symmetrically distinguished cycles on abelian varieties [O'S11b]. After that, in the third section, we give an overview of the general conjectural picture on the study of algebraic cycles and motives of projective hyper-Kähler varieties. The main conjectures we are considering here include

- the Beauville splitting conjecture [Bea07],
- the Beauville–Voisin conjecture [Voi08],
- the section property conjecture [FV19a],
- the generalized Franchetta conjecture [FLVS19],
- the multiplicative Chow–Künneth decomposition conjecture [SV16a],
- the distinguished marking conjecture [FV19a],
- the multiplicative Orlov conjecture [FV19b],
- the motivic hyper-Kähler resolution conjecture [FTV19], [FT17], [FN19].

We discuss some relations among them and give some pointers to the subsequent chapters for detailed study.

Chapter III presents my series of works on the so-called *motivic hyper-Kähler resolution conjecture*, mainly based on the papers [FTV19] (joint with Tian and Vial), [FT19] (joint with Tian), [FT17] (joint with Tian), [FN19] (joint with Nguyen) and also incorporating [Fu15]. We develop the orbifold product for the (higher) Chow rings, K-theory and Chow motives, then extend Ruan's crepant resolution conjecture to this motivic level in the hyper-Kähler setting. We establish our conjecture in several cases. The interest is not merely theoretic: this conjecture relates the Chow ring and K-theory of some interesting smooth projective hyper-Kähler varieties to that of their orbifold model, which are much easier to handle. Plenty of applications are given at the end to illustrate its usefulness.

Chapter IV is a tentative theory of distinguished cycles that I developed in [FV19a] with Vial. It is a systematic way of using O'Sullivan's theory of symmetrically distinguished cycles on abelian varieties to study cycles on other smooth projective varieties whose motives are of abelian type. The central conjecture is the section property conjecture and its motivic counterpart the distinguished marking conjecture. We consider them as a promising way to construct the so-called grade-0 cycles $\mathrm{CH}^*(X)_{(0)}$ for a hyper-Kähler variety X. Many examples of varieties having distinguished markings are provided. The feeling is that the potential of the theory is not yet fully exploited due to the unknown algebraicity of the Kuga–Satake construction.

Chapter V investigates algebraic cycles on projective hyper-Kähler varieties that deform to the universal family. The so-called Franchetta property states that those "generically defined" cycles should be easy in the sense that they are determined by their cohomology classes. I survey my joint work [FLVS19] with Laterveer, Vial, and Shen, where we establish some cases of the Franchetta property by using the projective geometry of the total family, an idea originally due to Voisin. It has a close relation to the Beauville–Voisin conjecture: on the one hand, to establish the Franchetta property one sometimes uses some known cases of the Beauville–Voisin conjecture; on the other hand, the Franchetta property, once established, often leads to new evidence for Voisin's refinement [Voi16a] of the Beauville–Voisin conjecture on algebraically coisotropic subvarieties. Moreover, the Franchetta property plays an essential role in establishing some multiplicative Chow–Künneth decompositions. We will illustrate such interactions in the last section.

Chapter VI is based on my joint work [FL18] with Zhiyuan Li. We study irreducible symplectic varieties over a base field of positive characteristic, which are analogues of hyper-Kähler varieties. Guided by the theory of supersingular K3 surfaces, we propose several definitions of supersingularity for these varieties and conjecture that they are all equivalent. We further conjecture that supersingularity for these varieties is characterized geometrically by their unirationality as well as by their rational chain connectedness. We also conjecture that the motive of supersingular irreducible symplectic varieties is as simple as possible: their motive is of Tate type if the odd-degree cohomology vanishes; while in the presence of non-zero odd Betti numbers, their motive is of supersingular abelian type and the analogous section property conjecture is equivalent to the analogue of Beauville's splitting conjecture, which takes a particularly simple form. In the second half of this chapter, we provide evidence for this conjectural picture, mainly for moduli spaces of sheaves on K3 and abelian surfaces.

Chapter VII reports on my collaboration with Andrea Cattaneo [CF19] on some basic finiteness properties of the automorphism groups of compact hyper-Kähler manifolds. We prove that the automorphism group, as well as the group of bimeromorphic transformations, is finitely generated and has only finitely many finite subgroups up to conjugation. As an application in real algebraic geometry, we show that any compact hyper-Kähler manifold has only finitely many real structures up to equivalence.

One recurring phenomenon in the study of hyper-Kähler varieties is that they behave very much like abelian varieties. This philosophy is illustrated throughout this memoir, especially in Chapter IV and Chapter VI.

What's new

Some parts of this memoir that has never appeared in the literature yet.

- Some general facts in §I.3 concerning the multiplicative Chow–Künneth decompositions are known to myself and my collaborators but unfortunately have never been well-documented. We include a rather complete proof for those facts.
  - Lemma I.3.2.
  - Proposition I.3.3. It was Footnote 24 in [FV19a] and we clarify here the proof.
  - Proposition I.3.7. It was scattered in [SV16a] and [FV19a].
  - Remark I.3.10.
  - The whole paragraph on regular surfaces, especially Proposition I.3.14.
- Some interesting new examples of multiplicative Chow–Künneth decompositions, mainly Theorem I.3.17 for cubic fourfolds and Theorem I.3.19 for some Todorov surfaces, worked out recently with Laterveer and Vial, are given in §I.3.3. We plan to make them into a research paper [FLV19] soon.
- The *motivic hyper-Kähler K-equivalence conjecture* II.3.16 and the related discussion in §II.3.4. It puts the motivic hyper-Kähler resolution conjecture II.3.17 into a broader context and reveals its relation with the multiplicative Orlov conjecture II.3.15 via Kawamata's DK-hypothesis conjecture [Kaw18], see III.8.7.

CHAPTER I

Algebraic cycles and motives

I.1. Chow rings

I.1.1. Basic definitions. Fix a field k. For a given irreducible smooth projective variety X of dimension d_X over k and an integer $i \in \mathbf{Z}$, an *algebraic cycle* of codimension i is a formal linear combination of integral subschemes of X of codimension i. The i-th *Chow group* of X, denoted by $\mathrm{CH}^i(X)$, is the group of algebraic cycles of codimension i modulo the rational equivalence relation. Here the group of algebraic cycles rationally equivalent to zero is generated by the difference of two fibers of cycles on $X \times_k \mathbf{P}^1$ which are flat over $\mathbf{P}^1$. The intersection theory endows

$$\mathrm{CH}^*(X) := \bigoplus_{i=0}^{d_X} \mathrm{CH}^i(X)$$

with the structure of a graded commutative ring with unit (given by the fundamental class 1_X), called the *Chow ring* of X. The indexation by dimension $\mathrm{CH}_i(X) := \mathrm{CH}^{d_X - i}(X)$ is sometimes used.

Example I.1.1 (Divisors). Codimension 1 cycles are nothing else but Weil divisors and rational equivalence in this case reduces to the classical linear equivalence, hence $\mathrm{CH}^1(X)$ is the divisor class group. As X is assumed to be smooth, Weil divisors are identified with Cartier divisors and $\mathrm{CH}^1(X)$ is canonically isomorphic to $\mathrm{Pic}(X)$, the group of isomorphism classes of line bundles.

Example I.1.2 (Zero-cycles). Let k be algebraically closed. For any $n \in \mathbf{N}$, let $\mathrm{Sym}^n(X)$ denote the n-th symmetric power of X. One can form the algebraic monoïd $\mathrm{Sym}^\bullet(X) := \coprod_{n \geq 0} \mathrm{Sym}^n(X)$. Then $\mathrm{CH}_0(X)$ can be constructed alternatively as the quotient of the group completion $\mathrm{Sym}^\bullet(X)^+$ by the subgroup generated by (the images of) elements of the form $f(0) - f(\infty)$ for all rational curves $f : \mathbf{P}^1 \to \mathrm{Sym}^n(X)$. A celebrated observation of Mumford [Mum68] is that when $k = \mathbf{C}$ (or rather any universal domain[1]) and $H^0(X, \Omega_X^i) \neq 0$ for some $i > 0$, the infinite procedure involved above cannot be avoided, *i.e.* $\mathrm{CH}_0(X)$ is *infinite dimensional* in the sense that $\mathrm{Sym}^n(X) \times \mathrm{Sym}^n(X) \xrightarrow{-} \mathrm{CH}_0(X)_{\deg 0}$ is not surjective for any n. This contrasts drastically with the case of divisors.

Using Chow varieties in the place of symmetric powers, the same construction carries over for higher dimensional cycles. Similar phenomena of infinite nature also happen for Chow groups $\mathrm{CH}^i(X)$ with $i > 1$, making the Chow ring a fundamentally uncomputable invariant. One way to understand this important invariant is to compare it with more computable invariants of finite nature, for example, the cohomology.

[1] A *universal domain* is an algebraically closed field of infinite transcendental degree over its prime subfield.

I.1.2. Relation to cohomology. For the ease of exposition, let us concentrate on the case $k = \mathbf{C}$, where singular (Betti) cohomology and Hodge theory are at our disposal. The first tool to compare the Chow ring to the cohomology is simply the cycle class map

$$\mathrm{cl} : \mathrm{CH}^*(X) \to H^{2*}(X, \mathbf{Z}).$$

The image of cl is a subgroup of the group of integral Hodge classes, and their the difference, which is expected to be a finite group according to the Hodge conjecture, measures the failure of the integral Hodge conjecture. The kernel of cl is the ideal of homologically trivial cycles (modulo rational equivalence), denoted by $\mathrm{CH}^*(X)_{\mathrm{hom}}$. The second tool is provided by the *Abel–Jacobi map*. More precisely, for any $1 \leq i \leq d_X$, the i-th intermediate jacobian is defined as the following complex torus given by double quotient:

$$J^{2i-1}(X) := F^i H^{2i-1}(X, \mathbf{C}) \backslash H^{2i-1}(X, \mathbf{C}) / H^{2i-1}(X, \mathbf{Z})_{\mathrm{tf}} \simeq F^{d_X - i+1} H^{2d_X - 2i+1}(X, \mathbf{C})^\vee / H_{2d_X - 2i+1}(X, \mathbf{Z})_{\mathrm{tf}},$$

where $F^\bullet$ denotes the Hodge filtration and $H^{2i-1}(X, \mathbf{Z})_{\mathrm{tf}} := H^{2i-1}(X, \mathbf{Z})/H^{2i-1}(X, \mathbf{Z})_{\mathrm{tors}}$ is identified with its image in $H^{2i-1}(X, \mathbf{C})$, similarly for the homology group. Note that we recover two classically constructed abelian varieties associated with a given variety: $\mathrm{Pic}^0(X)$ is $J^1(X)$ and $\mathrm{Alb}(X)$ is $J^{2d_X-1}(X)$. The *Abel–Jacobi map*

$$\mathrm{AJ}^i : \mathrm{CH}^i(X)_{\mathrm{hom}} \to J^{2i-1}(X)(\mathbf{C})$$

sends a homologically trivial algebraic cycle σ to the integration functional over Γ, where Γ is any topological cycle satisfying $\partial\Gamma = \sigma$. The image of AJ^i is conjecturally a countable union of translations, indexed by the *Griffiths group* $\mathrm{Griff}^i(X) := \mathrm{CH}^i(X)_{\mathrm{hom}} / \mathrm{CH}^i(X)_{\mathrm{alg}}$, of the i-th *algebraic intermediate jacobian* $J_a^{2i-1}(X)$, which is by definition the maximal sub-abelian variety in $J^{2i-1}(X)$ and conjecturally consists of the Abel–Jacobi images of the algebraically trivial cycles.

These two tools of comparison from the Chow group to the cohomology can be combined into a single one, namely the *Deligne cycle class map*

$$\mathrm{cl}_D^i : \mathrm{CH}^i(X) \to H_D^{2i}(X, \mathbf{Z}(i)),$$

where the target is the *Deligne cohomology* defined as the hypercohomology $\mathbb{H}^{2i}(X, \mathbf{Z}_D(i))$, where $\mathbf{Z}_D(i)$ is the i-th Deligne complex

$$0 \to \mathbf{Z} \cdot 2\pi\sqrt{-1} \to \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{i-1}.$$

For any $1 \leq i \leq d_X$, $H_D^{2i}(X, \mathbf{Z}(i))$ is the extension of the group of integral Hodge classes of degree $2i$ by the i-th intermediate jacobien $J^{2i-1}(X)$ and cl_D^i incorporates cl^i and AJ^i in a compatible way.

For divisors ($i = 1$), the Abel–Jacobi map is an isomorphism, hence $\mathrm{CH}^1(X) \simeq H_D^2(X, \mathbf{Z}(1))$ is the extension of the Néron–Severi group by $\mathrm{Pic}^0(X)$. However, as observed by Mumford [**Mum68**], already for any surface S with non-vanishing geometric genus, the Albanese map $\mathrm{alb} = \mathrm{AJ}^2 : \mathrm{CH}^2(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)(\mathbf{C})$ has a huge kernel, which is responsible for the infinite dimensionality of $\mathrm{CH}^2(S)$ mentioned in Example I.1.2. An understanding of this Albanese kernel, or more generally other non-trivial Abel–Jacobi kernels, is one of the biggest challenges in the study of algebraic cycles, as is indicated by the difficulty encountered towards the Bloch conjecture and its generalizations.

I.1.3. Some representable Chow groups. Nevertheless, in some geometrically interesting situations, this two-step comparison, or equivalently the Deligne cycle class map, does give a satisfying description of the Chow group. We say the Chow group of algebraically trivial cycles $\mathrm{CH}^i(X)_{\mathrm{alg}}$ is *representable* if the Abel–Jacobi map $\mathrm{AJ}^i : \mathrm{CH}^i(X)_{\mathrm{alg}} \to J_a^{2i-1}(X)$ has finite kernel. Our ad hoc definition slightly differs from the one by Murre [Mur85] in terms of a universal property among regular maps to abelian varieties; but they are conjecturally equivalent.

One important example of representable Chow groups is the following result obtained by Bloch–Srinivas [BS83] and Murre [Mur85] on codimension 2 cycles. They use the technique of decomposition of the diagonal and the Merkurjev–Suslin theorem [MS82].

THEOREM I.1.3 ([BS83], [Mur85]). *If $\mathrm{CH}_0(X)$ is supported*[2] *on an algebraic subset of dimension at most 1, then*

- *The Griffiths group $\mathrm{Griff}^2(X)$ is 0, or equivalently, the algebraic equivalence and homological equivalence coincide for codimension-2 cycles.*
- *$J^3(X) = J_a^3(X)$ is an abelian variety and the Abel–Jacobi map $\mathrm{CH}^2(X)_{\mathrm{hom}} \to J^3(X)(\mathbf{C})$ is an isomorphism.*

For codimension ≥ 3, there seems to be very few interesting examples of representable Chow groups (with non-trivial algebraic intermediate jacobians) worked out in the literature. With the motivation of filling this blank, we are interested in varieties with a non-zero odd degree cohomology of Hodge level 1. Recall that the following are the only hypersurfaces in projective spaces whose middle cohomology has Hodge level 1, hence the algebraic intermediate jacobian is non-trivial:

- cubic threefolds,
- quartic threefolds,
- cubic fivefolds.

In the first two cases, the interesting Chow group is that of 1-cycles, hence is computed by Theorem I.1.3. With Zhiyu Tian, we studied in [FT18] the remaining case of cubic fivefolds, where the interesting Chow group is CH^3 (the other ones can be computed using Theorem I.1.3 and other general results). Here is our main result:

THEOREM I.1.4 (Cubic fivefolds, Fu–Tian [FT18]). *Let X be a smooth cubic hypersurface in $\mathbf{P}^6_{\mathbf{C}}$. Then*

- $\mathrm{Griff}^3(X) = 0$, *that is*, $\mathrm{CH}^3(X)_{\mathrm{hom}} = \mathrm{CH}^3(X)_{\mathrm{alg}}$.
- *The Abel–Jacobi map induces an isomorphism* $\mathrm{AJ}^3 : \mathrm{CH}^3_{\mathrm{hom}} \to J^5(X)(\mathbf{C})$.

In particular, $\mathrm{CH}^3(X)$ is the split extension of $\mathbf{Z}$ by the abelian variety $J^5(X)(\mathbf{C})$.

With a view towards the rationality problem of cubic fivefolds, we also showed in [FT18] that all the unramified cohomology groups vanish:

$$H^i_{\mathrm{ur}}(X, \mathbf{Q}/\mathbf{Z}) = 0 \text{ for all } i > 0.$$

We remark that the thorough study of the unramified cohomology of X, as well as that of the more general $\mathcal{H}$-cohomology, is crucial for the proof of Theorem I.1.4.

REMARK I.1.5. If we enlarge our search range to complete intersections in projective spaces, we have three more types of examples whose middle cohomology has Hodge level 1:

[2]We say that $\mathrm{CH}_0(X)$ is supported on a subscheme Y if $\mathrm{CH}_0(X\backslash Y) = 0$.

- complete intersections of two even-dimensional quadrics,
- complete intersections of three odd-dimensional quadrics,
- complete intersections of a quadric and a cubic in $\mathbf{P}^5$.

The last case is again covered by Theorem I.1.3. For the cases of intersections of two or three quadrics, their Chow groups with rational coefficients were completely computed in [BT16] (see also [Otw99], [Via13b] and [Via13a]); the Chow groups with integral coefficients, which are conjecturally representable, seem worth further exploring.

I.1.4. The Bloch–Beilinson filtration. Despite of the interesting examples of representable ones in the previous subsection, Chow groups are in general not representable. To the best of the author's knowledge, no reasonable structures on the Abel–Jacobi kernels have been established. We nevertheless expect that the Chow group, at least its torsion-free part, can still be understood by the cohomology groups (equipped with Hodge structures). Regarding §I.1.2, it is desirable to continue filtering the Chow group and compare the graded pieces to some Hodge theoretic objects related to cohomology groups of lower degrees. Such a precise relation is conjectured by Bloch and Beilinson as follows, see [Beĭ87], [Blo10] and [Voi03, Conjecture 11.21].

Conjecture I.1.6 (Bloch–Beilinson). *For any complex irreducible smooth projective variety X of dimension d_X and any integer $0 \leq i \leq d_X$, there is a canonical descending filtration $F^\bullet$ on $\mathrm{CH}^i(X)_{\mathbf{Q}}$ with the following properties:*

- *(Functoriality) For any $\Gamma \in \mathrm{CH}^l(X \times Y)_{\mathbf{Q}}$, $\Gamma_* : \mathrm{CH}^i(X)_{\mathbf{Q}} \to \mathrm{CH}^{i+l-d_X}(Y)_{\mathbf{Q}}$ respects the filtration $F^\bullet$.*
- *(Multiplicativity) $F^j \mathrm{CH}^i(X)_{\mathbf{Q}} \cdot F^{j'} \mathrm{CH}^{i'}(X)_{\mathbf{Q}} \subset F^{j+j'} \mathrm{CH}^{i+i'}(X)_{\mathbf{Q}}$.*
- *$F^0 \mathrm{CH}^i(X)_{\mathbf{Q}} = \mathrm{CH}^i(X)_{\mathbf{Q}}$.*
- *$F^1 \mathrm{CH}^i(X)_{\mathbf{Q}} = \mathrm{CH}^i(X)_{\mathrm{hom},\mathbf{Q}}$.*
- *(Separatedness) $F^{i+1} \mathrm{CH}^i(X)_{\mathbf{Q}} = 0$.*
- *(Relation to cohomology) For any $\Gamma \in \mathrm{CH}^l(X\times Y)_{\mathbf{Q}}$, the induced morphism $\mathrm{Gr}^j_F \mathrm{CH}^i(X)_{\mathbf{Q}} \to \mathrm{Gr}^j_F \mathrm{CH}^{i+l-d_X}(Y)_{\mathbf{Q}}$ vanishes if $[\Gamma]_* : H^{2i-j}(X) \to H^{2i-j+2l-2d_X}(Y)$ vanishes on the Hodge components $H^{p,q}(X)$ for all $p, q \in \mathbf{N}$ with $p + q = 2i - j$ and $|p - q| \geq j$.*

Remark I.1.7. One could conjecture further that $F^2 \mathrm{CH}^i(X)_{\mathbf{Q}}$ is given by the kernel of the Abel–Jacobi map.

Remark I.1.8 (Torsion cycles). The above Bloch–Beilinson conjecture completely neglects torsion cycles. As far as the author knows, apart from Roitman's theorem [Roj80] (extended by Milne [Mil82]) on torsion 0-cycles which says that the Albanese map induces an isomorphism between $\mathrm{CH}_0(X)_{\mathrm{tors}}$ and $\mathrm{Alb}(X)(k)_{\mathrm{tors}}$ for any algebraically closed field k, there is no general conjectural expectation on the structure of torsion cycles. Nevertheless, torsion cycles are of course equally (if not more) important. Let us just mention the important observation of Voisin [Voi15b] on the relation between the stable rationality of a rationally connected variety and the vanishing of the torsion[3] cycle $(\Delta_X - x \times X)|_{X\times\eta}$ in $\mathrm{CH}_0(X_{k(X)})$, which leads to a series of exciting progress on the stable Lüroth problem, see for example [Voi16b] for a survey.

[3]The fact that this cycle is of torsion follows from the existence of the decomposition of the diagonal proved by Bloch–Srinivas [BS83].

I.2. Motives

Convention: From now on, all Chow groups are with rational coefficients and we will use the notation CH for $\mathrm{CH}_{\mathbf{Q}}$.

I.2.1. The category of Chow motives. Among morphisms between Chow groups, those induced by correspondences are of the most importance. Due to the lack of the Poincaré duality and the Künneth formula for Chow groups,[4] going from a correspondence $\Gamma \in \mathrm{CH}^*(X \times Y)$ to its induced morphism $\Gamma_* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ loses significant information. To make a systematic use of correspondences, the whole theory becomes more natural in the language of *motives*.

Let us briefly recall the basic definitions following [**And04**]. The category of Chow motives with rational coefficients, denoted by CHM, is a pseudo-abelian[5] rigid symmetric monoidal category, whose objects consist of triples (X, p, n), where X is an irreducible smooth projective variety of dimension d_X over the base field k, $p \in \mathrm{CH}^{d_X}(X \times X)$ is a projector $p \circ p = p$, and $n \in \mathbf{Z}$. A morphism between two Chow motives $M = (X, p, n)$ and $N = (Y, q, m)$ is an element $\gamma \in \mathrm{CH}^{d_X+m-n}(X \times Y)$ such that $\gamma \circ p = q \circ \gamma = \gamma$. The Chow motive of a smooth projective variety X is defined as $\mathfrak{h}(X) := (X, \Delta_X, 0)$, where Δ_X denotes the class of the diagonal inside $X \times X$. We obtain therefore a contra-variant functor

$$\mathfrak{h} : \mathrm{SmProj}^{\mathrm{op}} \to \mathrm{CHM}$$

from the category of smooth projective varieties to the category of Chow motives, where a morphism $f : X \to Y$ is sent to the transposed graph ${}^t\Gamma_f$ as a morphism from $\mathfrak{h}(Y)$ to $\mathfrak{h}(X)$.

The tensor product of two motives is defined in the obvious way by fiber product over the base field, while the dual of $M = (X, p, n)$ is $M^\vee = (X, {}^tp, -n + d_X)$, where ${}^tp$ denotes the transpose of p. Almost tautologically, we have the *motivic Künneth formula*

$$\mathfrak{h}(X \times_k Y) \simeq \mathfrak{h}(X) \otimes \mathfrak{h}(Y)$$

and the *motivic Poincaré duality*

$$\mathfrak{h}(X)^\vee \simeq \mathfrak{h}(X)(d).$$

The commutativity constraints are induced from the natural isomorphisms $X \times_k Y \simeq Y \times_k X$. The tensor unit is given by the *unit motive*, denoted $\mathbb{1} := \mathfrak{h}(\mathrm{Spec}(k))$. We define the Chow group of a motive M by $\mathrm{CH}^i(M) := \mathrm{Hom}(\mathbb{1}(-i), M)$; in particular, we recover the Chow groups of a variety $\mathrm{CH}^i(X) = \mathrm{Hom}(\mathbb{1}(-i), \mathfrak{h}(X))$. The *Tate motive* of weight $-2i$ is the motive $\mathbb{1}(i) := (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)}, i)$. A motive is said to be of *Tate type* if it is isomorphic to a direct sum of Tate motives (of various weights). The notation $M(i) := M \otimes \mathbb{1}(i)$ is often used; hence the motive (X, Δ_X, i) is sometimes denoted by $\mathfrak{h}(X)(i)$.

I.2.2. Chow–Künneth decomposition and Murre's conjecture. We review the Bloch–Beilinson conjecture discussed in §I.1.4 from the point of view of decompositions of motives. We fix a Weil cohomology theory $H^*(-)$. For example, if $k = \mathbf{C}$, one can use $H^*(-) = H^*(-, \mathbf{Q})$, the singular cohomology with rational coefficients. The natural extension of the cohomology

[4] Or rather, the lack of good understanding of the natural structures on Chow groups like the Bloch–Beilinson filtration discussed before.

[5] A pseudo-abelian category is an additive category where each projector has an image. This notion goes also under the name of *idempotent completeness*.

theory to the category of rational Chow motives $H^* : \mathrm{CHM} \to \mathrm{Vect}_{\mathbf{Q}}$ is called the *realization functor*.

Definition I.2.1 (Chow–Künneth decomposition). Let X be an irreducible smooth projective variety of dimension d. A *Chow–Künneth* decomposition for X is a direct-sum decomposition

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X)$$

of its rational Chow motive in CHM: such that for any $0 \le i \le 2d$, the realization $H^*(\mathfrak{h}^i(X))$ is $H^i(X)$.

In other words, a Chow–Künneth decomposition is a system of self-correspondences $\left\{\pi^0, \ldots, \pi^{2d}\right\}$ in $\mathrm{CH}^d(X \times X)$ satisfying the following properties:

- (Projectors) $\pi^i \circ \pi^i = \pi^i$ for any i;
- (Orthogonality) $\pi^i \circ \pi^j = 0$ for any $i \neq j$;
- (Completeness) $\pi^0 + \cdots + \pi^{2d} = \Delta_X$;
- $\pi^i_* H^*(X) = H^i(X)$ for any i.

Remark I.2.2 (Duality). Thanks to the motivic Poincaré duality $\mathfrak{h}(X) \simeq \mathfrak{h}(X)^\vee(-d)$, we see that a Chow–Künneth decomposition

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X),$$

naturally admits a *dual decomposition*:

$$\mathfrak{h}(X) = \mathfrak{h}^{2d}(X)^\vee(-d) \oplus \cdots \oplus \mathfrak{h}^0(X)^\vee(-d).$$

In terms of projectors, the dual of a system $\left\{\pi^0, \ldots, \pi^{2d}\right\}$ is $\left\{{}^t\pi^{2d}, \ldots, {}^t\pi^0\right\}$. A Chow–Künneth decomposition is called *self-dual* if $\mathfrak{h}^i(X)^\vee = \mathfrak{h}^{2d-i}(X)(d)$, or equivalently, $\pi^i = {}^t\pi^{2d-i}$, for all $0 \le i \le 2d$.

Note that the existence of an algebraic cycle π^i whose action on $H^*(X)$ has image $H^i(X)$ implies the standard conjecture of Künneth type of degree i (*cf.* [**Kle94**]). A self-dual Chow–Künneth decomposition is conjectured to exist, but not uniquely; the uniqueness only manifests itself as a filtration on Chow groups.

Conjecture I.2.3 (Murre [**Mur93b**]). *Let X be an irreducible smooth projective variety defined over an algebraically closed field k and let d be its dimension. Then*

(A) there exists a Chow–Künneth decomposition $\left\{\pi^0, \ldots, \pi^{2d}\right\}$. Any such decomposition satisfies the following:
(B) For any i, the projectors $\pi^0, \ldots, \pi^{i-1}$ and $\pi^{2i+1}, \ldots, \pi^{2d}$ all act as zero on $\mathrm{CH}^i(X)$.
(C) The filtration on $\mathrm{CH}^i(X)$ defined by $F^0\,\mathrm{CH}^i(X) = \mathrm{CH}^i(X)$ and for any $j > 0$, $F^j\,\mathrm{CH}^i(X) := \ker(\pi^{2i}) \cap \cdots \cap \ker(\pi^{2i-j+1})$, is independent of the choice of the Chow–Künneth decomposition.
(D) $F^1\,\mathrm{CH}^i(X) = \mathrm{CH}^i(X)_{\mathrm{hom}}$.

Jannsen [**Jan94**] showed that the Murre Conjecture I.2.3 is in fact equivalent to the Bloch–Beilinson Conjecture I.1.6, with the filtration in (C) corresponding to the Bloch–Beilinson filtration. We will then speak of Bloch–Beilinson–Murre conjecture.

Although the Bloch–Beilinson–Murre conjecture is wide open, there are some established cases for (parts of) Murre's conjecture. We refer to [**Mur93b**], [**Mur93a**], [**Mur90**], [**Via15**], [**Via13a**] for details.

Remark I.2.4 (π^0 and π^{2d}). In a Chow–Künneth decomposition of a d-dimensional smooth projective variety X, the first and the last projectors are usually taken to be of the form $\pi^0 = z \times 1_X$ and $\pi^{2d} = 1_X \times z'$ respectively, where z, z' are 0-cycles of degree 1 and 1_X is the fundamental class. We point out that if X is Kimura finite-dimensional [Kim05], we must have $\mathfrak{h}^0(X) \simeq \mathbb{1}$ and $\mathfrak{h}(X) \simeq \mathbb{1}(-d)$, therefore π^0 and π^{2d} must be of the above form.

Remark I.2.5 (Product Chow–Künneth decomposition). Given two smooth projective varieties X and Y, by the motivic Künneth formula

$$\mathfrak{h}(X \times Y) \cong \mathfrak{h}(X) \otimes \mathfrak{h}(Y),$$

a Chow–Künneth decomposition of X and a Chow–Künneth decomposition of Y induce naturally a Chow–Künneth decomposition of $X \times Y$ defined as follows: for any i,

$$\mathfrak{h}^i(X \times Y) := \bigoplus_{j=0}^{i} \mathfrak{h}^j(X) \otimes \mathfrak{h}^{i-j}(Y);$$

in terms of projectors:

$$\pi^i_{X\times Y} = \sum_{j=0}^{i} \pi^j_X \otimes \pi^{i-j}_Y.$$

If we use instead

$$\mathfrak{h}(X \times Y) \cong \mathfrak{h}(X)^\vee(-d_X) \otimes \mathfrak{h}(Y),$$

and the dual Chow–Künneth decomposition for X (see Remark I.2.2), then we have another natural Chow–Künneth decomposition for $X \times Y$ given by

$$\mathfrak{h}^i(X \times Y) := \bigoplus_{j=0}^{i} \mathfrak{h}^{2d_X - j}(X)^\vee(-d_X) \otimes \mathfrak{h}^{i-j}(Y);$$

in terms of projectors:

$$\pi^i_{X\times Y} = \sum_{j=0}^{i} {}^t\pi^{2d_X - j}_X \otimes \pi^{i-j}_Y.$$

Of course, these two decompositions coincide if the Chow–Künneth decomposition on X is self-dual.

Murre's conjecture (B) can be viewed as a motivic interpretation of the following well-known principle: there are no non-zero morphisms from small weights to big weights. More precisely, we have:

Proposition I.2.6 (Weight argument). *Let X and Y be two smooth projective varieties. Assuming Murre's conjecture I.2.3 (A) for X and Y, and (B) for $X \times Y$, then for any $i < j - 2l$, we have* $\mathrm{Hom}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y)(l)) = 0$.

Proof. This is formal: we have

$$\mathrm{Hom}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y)(l)) = \mathrm{CH}^{d_X}(\mathfrak{h}^i(X)^\vee(l - d_X) \otimes \mathfrak{h}^j(Y)) \subset \mathrm{CH}^{d_X}(\mathfrak{h}^{2d_X - 2l - i + j}(X \times Y)) = 0,$$

where we use the second Chow–Künneth decomposition on $X \times Y$ in Remark I.2.5. □

I.2.3. Algebra objects. One of the main objectives of this memoir is to understand the *multiplicative* structure of Chow rings of algebraic varieties. This is captured by the algebra structure of the Chow motive of a smooth projective variety. Recall that an *algebra object* in a symmetric monoidal category is an object M together with a unit morphism η from the tensor unit to M and a multiplication morphism $\mu : M \otimes M \to M$ satisfying the usual unit axiom $\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mu \circ (\eta \otimes \mathrm{id})$ and the associativity axiom $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. It is called *commutative* if moreover $\mu = \mu \circ c_{M,M}$ is satisfied, where $c_{M,M}$ is the commutativity constraint of the symmetric monoidal category.

Definition I.2.7 (Algebra structure). Let X be a smooth projective variety and let $\mathfrak{h}(X)$ be its Chow motive. Then $\mathfrak{h}(X)$ admits the following canonical structure of a commutative algebra object in the category CHM of rational Chow motives:

- The unit morphism $\eta : \mathbb{1} \to \mathfrak{h}(X)$ is given by the fundamental class $1_X \in \mathrm{CH}^0(X) = \mathrm{Hom}(\mathbb{1}, \mathfrak{h}(X))$;
- The multiplication $\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ is given by the small diagonal $\delta_X \in \mathrm{CH}^{2d_X}(X^3) = \mathrm{Hom}(\mathfrak{h}(X) \otimes \mathfrak{h}(X), \mathfrak{h}(X))$.

For a smooth projective variety X, the structure of algebra object on its motive $\mathfrak{h}(X) \in$ CHM recovers the intersection product of the Chow ring $\mathrm{CH}^*(X)$. More precisely, for any $\alpha \in \mathrm{CH}^i(X)$, $\beta \in \mathrm{CH}^j(X)$, viewed as morphisms $\alpha : \mathbb{1}(-i) \to \mathfrak{h}(X)$ and $\beta : \mathbb{1}(-j) \to \mathfrak{h}(X)$, their intersection product $\alpha \cdot \beta \in \mathrm{CH}^{i+j}(X)$ is the following composition of morphisms

$$\alpha \cdot \beta : \mathbb{1}(-i-j) = \mathbb{1}(-i) \otimes \mathbb{1}(-j) \xrightarrow{\alpha \otimes \beta} \mathfrak{h}(X) \otimes \mathfrak{h}(X) \xrightarrow{\mu} \mathfrak{h}(X).$$

Similarly, applying the cohomological realization functor, the algebra object structure induces the cup product on cohomology.

I.2.4. Frobenius algebra objects. As observed[6] in our recent joint work with Vial [**FV19b**], the motive of a smooth projective variety carries an additional structure, namely a *Frobenius algebra* structure. Let us first recall the definition.

Definition I.2.8 (Frobenius algebra objects [**FV19b**, §3]). Let $(\mathcal{C}, \otimes, \vee, \mathbb{1})$ be a rigid symmetric monoidal category admitting a $\otimes$-invertible object denoted by $\mathbb{1}(1)$. Let d be an integer. A *degree-d Frobenius algebra object* in $\mathcal{C}$ is the data of an object $M \in \mathcal{C}$ endowed with

- $\eta : \mathbb{1} \to M$, a unit morphism;
- $\mu : M \otimes M \to M$, a multiplication morphism;
- $\lambda : M^\vee \xrightarrow{\sim} M(d)$, an isomorphism, called the *Frobenius structure*;

satisfying the following axioms:

(*i*) (Unit) $\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mu \circ (\eta \otimes \mathrm{id})$;
(*ii*) (Associativity) $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$;
(*iii*) (Frobenius condition) $(\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) = \delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$,

[6] It was probably noticed before, we just could not find it in the literature.

where the *comultiplication* morphism $\delta : M \to M \otimes M(d)$ is defined by dualizing μ via the following commutative diagram

$$\begin{tikzcd} M^\vee \arrow[r, "{}^t\mu"] \arrow[d, "\lambda"', "\simeq"] & M^\vee \otimes M^\vee \arrow[d, "\lambda\otimes\lambda"', "\simeq"] \\ M(d) \arrow[r, "\delta(d)"] & M(d) \otimes M(d). \end{tikzcd}$$

We define also the *counit morphism* $\varepsilon : M \to \mathbb{1}(-d)$ by dualizing η via the following diagram

$$\begin{tikzcd} M^\vee \arrow[r, "{}^t\eta"] \arrow[d, "\simeq"', "\lambda"] & \mathbb{1} \\ M(d). \arrow[ur, "\varepsilon(d)"'] & \end{tikzcd}$$

We remark that ε and δ automatically satisfy the counit and coassociativity axioms.

A Frobenius algebra object M is called *commutative* if the underlying algebra object is commutative: $\mu \circ c_{M,M} = \mu$. Commutativity is equivalent to the cocommutativity of δ.

Remark I.2.9. The classical notion of Frobenius algebras is as follows: it is a finite dimensional k-algebra A together with a non-degenerate pairing $\beta : A \otimes_k A \to k$ such that $\beta(x, yz) = \beta(xy, z)$ for all $x, y, z \in A$. By identifying A and $A^\vee$ using β, one sees that a Frobenius algebra is nothing else but a Frobenius algebra object (of degree 0) in the category of vector spaces. In the case of Frobenius algebra objects of degree 0, the $\otimes$-invertible object $\mathbb{1}(1)$ is not needed in the definition, and it is reduced to the usual notion of Frobenius algebra object in the literature. In this sense, Definition I.2.8 generalizes the existing definition of Frobenius structure by allowing non-zero twists by a $\otimes$-invertible object. We believe that our more flexible notion is necessary and adequate for more sophisticated tensor categories than that of vector spaces, such as the categories of Hodge structures, Galois representations, motives, *etc*.

Remark I.2.10 (Morphisms). Morphisms of Frobenius algebra objects are defined in the natural way, that is, as morphisms $\phi : M \to N$ such that all the natural diagrams involving the structural morphisms are commutative. In particular, all non-zero morphisms between Frobenius algebra objects are in fact invertible. It is easy to show [FV19b, Lemma 2.4] that a morphism $\phi : M \to N$ is a morphism of Frobenius algebra objects if and only if it is an isomorphism and it is *orthogonal* in the sense that $\phi(d)^{-1} = \lambda_M \circ {}^t\phi \circ \lambda_N^{-1}$, or more succinctly, $\phi^{-1} = {}^t\phi$.

Now let us give some natural examples of Frobenius algebra objects.

Example I.2.11 (Cohomology as a graded vector space). Let X be a connected compact orientable (real) manifold of dimension d. Then its cohomology group $H^*(X, \mathbf{Q})$ is naturally a Frobenius algebra object of degree d in the category of $\mathbf{Z}$-graded $\mathbf{Q}$-vector spaces (where morphisms are *degree-preserving* linear maps and the $\otimes$-invertible object is chosen to be $\mathbf{Q}[1]$, the 1-dimensional vector space sitting in degree -1). The unit morphism $\eta : \mathbf{Q} \to H^*(X, \mathbf{Q})$ is given by the fundamental class; the multiplication morphism $\mu : H^*(X, \mathbf{Q}) \otimes H^*(X, \mathbf{Q}) \to H^*(X, \mathbf{Q})$ is the cup product; the Frobenius structure comes from the Poincaré duality

$$\lambda : H^*(X, \mathbf{Q})^\vee \xrightarrow{\sim} H^*(X, \mathbf{Q})[d] = H^*(X, \mathbf{Q}) \otimes \mathbf{Q}[d].$$

The induced comultiplication morphism $\delta : H^*(X, \mathbf{Q}) \to H^*(X, \mathbf{Q}) \otimes H^*(X, \mathbf{Q})[d]$ is the Gysin map for the diagonal embedding $X \hookrightarrow X \times X$; the counit morphism $\varepsilon : H^*(X, \mathbf{Q}) \to \mathbf{Q}[-d]$ is the integration $\int_X$. The Frobenius condition is a classical exercise. Note that $H^*(X, \mathbf{Q})$ is commutative, because the commutativity constraint in the category of graded vector spaces is with the super one.

If instead we consider the cohomology group as merely an ungraded vector space, then it becomes a Frobenius algebra object of degree 0 (*i.e.* in the usual sense); this is one of the main examples in the literature.

Example I.2.12 (Hodge structures). A pure rational Hodge structure is a finite-dimensional $\mathbf{Z}$-graded $\mathbf{Q}$-vector space $H = \bigoplus_{n\in\mathbf{Z}} H^{(n)}$ such that each $H^{(n)}$ is given a Hodge structure of weight n. A morphism between two Hodge structures is required to preserve the weights. The category of pure Hodge structures is naturally a rigid symmetric monoidal category. The $\otimes$-invertible object is chosen to be $\mathbf{Q}(1)$, which is the 1-dimensional vector space $(2\pi i)\cdot\mathbf{Q}$ with Hodge structure purely of type $(-1, -1)$.

Let X be a compact Kähler manifold of (complex) dimension d. Then $H^*(X, \mathbf{Q})$ is naturally a commutative Frobenius algebra object of degree d in the category of pure $\mathbf{Q}$-Hodge structures. The structural morphisms are the same as in Example I.2.11 up to replacing $[d]$ by (d). For instance, the Frobenius structure is $\lambda : H^*(X, \mathbf{Q})^\vee \xrightarrow{\sim} H^*(X, \mathbf{Q})(d)$.

Our main examples of Frobenius algebra objects are the Chow motives of smooth projective varieties. In the rigid symmetric monoidal category of rational Chow motives over a field k, we choose the $\otimes$-invertible object to be the Tate motive $\mathbb{1}(1)$. Then for any smooth projective k-variety X of dimension d, its Chow motive $\mathfrak{h}(X)$ is naturally a commutative Frobenius algebra object of degree d in the category of Chow motives. The structure of algebra objects $\mu : \mathfrak{h}(X)\otimes\mathfrak{h}(X) \longrightarrow \mathfrak{h}(X)$ and $\eta : \mathbb{1} \to \mathfrak{h}(X)$ is already explained in §I.2.3.

The Frobenius structure is defined as the canonical isomorphism

$$\lambda : \mathfrak{h}(X)^\vee \xrightarrow{\sim} \mathfrak{h}(X)(d),$$

called motivic Poincaré duality, given by the class of the diagonal $\Delta_X \in \mathrm{CH}^d(X \times X) = \mathrm{Hom}(\mathfrak{h}(X)^\vee, \mathfrak{h}(X)(d))$. One readily checks that the induced comultiplication morphism

$$\delta : \mathfrak{h}(X) \to \mathfrak{h}(X)\otimes\mathfrak{h}(X)(d)$$

is given by the small diagonal $\delta_X \in \mathrm{CH}^{2d}(X\times X\times X) = \mathrm{Hom}(\mathfrak{h}(X), \mathfrak{h}(X)\otimes\mathfrak{h}(X)(d))$, while the counit morphism

$$\varepsilon : \mathfrak{h}(X) \to \mathbb{1}(-d)$$

is given by the fundamental class.

The following lemma proves that, endowed with these structural morphisms, $\mathfrak{h}(X)$ is indeed a Frobenius algebra object.

Lemma I.2.13 (Frobenius condition). *Notation is as above. We have an equality of endomorphisms of* $\mathfrak{h}(X)\otimes\mathfrak{h}(X)$*:*

$$(\mathrm{id}\otimes\mu)\circ(\delta\otimes\mathrm{id}) = \delta\circ\mu = (\mu\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta).$$

Proof. We only show $\delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$, the other equality being similar. We have a commutative cartesian diagram without excess intersection:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "\Delta\times\mathrm{id}"] \\ X \times X \arrow[r, "\mathrm{id}\times\Delta"'] & X \times X \times X, \end{tikzcd}$$

where $\Delta : X \to X \times X$ denotes the diagonal embedding. The base-change formula yields

$$(\Delta \times \mathrm{id})^* \circ (\mathrm{id} \times\Delta)_* = \Delta_* \circ \Delta^*$$

on Chow groups, hence also for Chow motives by Manin's identity principle [**And04**, §4.3.1]. Now it suffices to notice that Δ_* is the comultiplication δ and Δ^* is the multiplication μ. □

Remark I.2.14. In general, a tensor functor $F : \mathcal{C} \to \mathcal{C}'$ between two rigid symmetric monoidal categories sends a Frobenius algebra object in $\mathcal{C}$ to such an object in $\mathcal{C}'$. Example I.2.12 is obtained by applying the Betti–Hodge realization functor from the category of Chow motives to that of pure Hodge structures; Example I.2.11 (for Kähler manifolds) is obtained by further applying the forgetful functor ($\mathbf{Q}(1)$ is sent to $\mathbf{Q}[2]$).

To justify introducing the Frobenius structure, we present our following recent joint work with Vial [**FV19b**] on the derived categories of K3 surfaces, which eventually gives a Torelli-type theorem for isogenous K3 surfaces (Theorem I.2.16).

Theorem I.2.15 ([**FV19b**, Theorem 1]). *Let S and S' be twisted derived equivalent K3 surfaces over a field k. Then the Chow motives of S and S' are isomorphic as Frobenius algebra objects, in the category of rational Chow motives over k.*

Concretely, the conclusion of the theorem says that there exists a correspondence $\Gamma \in \mathrm{CH}^2(S \times_k S')$ such that

- Γ is *orthogonal*: $\Gamma \circ {}^t\Gamma = \Delta_{S'}$ and ${}^t\Gamma \circ \Gamma = \Delta_S$.
- Γ is a homomorphism of algebras: $\mu \circ (\Gamma \otimes \Gamma) = \Gamma \circ \mu$. Or equivalently, under the first condition, $(\Gamma \otimes \Gamma \otimes \Gamma)_*(\delta_S) = \delta_{S'}$.

The fact that the motives of S and S' are isomorphic is due to Huybrechts [**Huy18**]. The main motivation to strengthen his result in the above form is the following *motivic global Torelli theorem* for isogenous K3 surfaces.

Theorem I.2.16 ([**FV19b**, Corollary 2]). *Let S and S' be complex projective K3 surfaces. The following statements are equivalent:*

(i) S and S' are isogenous, *that is, there is a Hodge isometry between $H^2(S, \mathbf{Q})$ and $H^2(S', \mathbf{Q})$;*
(ii) $\mathfrak{h}(S)$ and $\mathfrak{h}(S')$ are isomorphic as Frobenius algebra objects.

(*ii*) $\Longrightarrow$ (*i*) is obtained by applying the realization functors. For (*i*) $\Longrightarrow$ (*ii*), we make an essential use of Huybrechts' result [**Huy19**], which says that two projective K3 surfaces S and S' are isogenous if and only if they are twisted derived equivalent in the sense in [**Huy19**], then we apply Theorem I.2.15 to conclude.

To show that the Frobenius condition is necessary, we construct in [**FV19b**, Theorem A.13] an infinite family of pairwise non-isogenous K3 surfaces whose motives are all isomorphic as algebra objects, in particular, their cohomology are isomorphic as Hodge algebras.

The possibility to generalize Theorem I.2.15 to higher dimensions is discussed in §II.3.3 under the name of "the multiplicative Orlov conjecture".

I.3. Multiplicative Chow–Künneth decompositions

I.3.1. Definitions and basic properties. Let X be a smooth projective variety of dimension d. The decomposition

$$H^*(X) = \bigoplus_{i=0}^{2d} H^i(X) \tag{I.1}$$

is obviously compatible with the cup product: for any $0 \le i, j \le 2d$, we have

$$\mathrm{Im}(\smile\colon H^i(X) \otimes H^j(X) \to H^*(X)) \subset H^{i+j}(X).$$

Moving to Chow motives, on the one hand, a Chow–Künneth decomposition of $\mathfrak{h}(X)$, whose existence is predicted by the Murre conjecture (see §I.2.2), lifts (I.1) to the level of Chow motives; on the other hand, we saw in §I.2.3 that there is a natural structure of algebra object on $\mathfrak{h}(X)$ which lifts the cup product. Therefore we can naturally ask whether the above obvious compatibility still holds on the level of motives. More precisely, we have the following notion defined by Shen–Vial [**SV16a**].

Definition I.3.1 (Multiplicative Chow–Künneth (MCK) decomposition). A Chow–Künneth decomposition (Definition I.2.1)

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X)$$

is called *multiplicative*, if for any $0 \le i, j \le 2d$, the restriction of the multiplication $\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ to the direct summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factors through the inclusion of the direct summand $\mathfrak{h}^{i+j}(X)$.

In practice, it is useful to express the above notion in terms of projectors and correspondences.

Lemma I.3.2. *Let X be a smooth projective variety of dimension d. Let $\{\pi^0, \dots, \pi^{2d}\}$ be the system of projectors corresponding to a Chow–Künneth decomposition of X (see Definition I.2.1). Then the following conditions are equivalent:*

(*i*) *The Chow–Künneth decomposition is multiplicative;*
(*ii*) *For any i, j, k such that $i + j \neq k$, we have $\pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = 0$.*
(*iii*) *For any i, j, we have $\pi^{i+j} \circ \delta_X \circ (\pi^i \otimes \pi^j) = \delta_X \circ (\pi^i \otimes \pi^j)$.*
(*iv*) $\delta_X = \sum_{i,j} \pi^{i+j} \circ \delta_X \circ (\pi^i \otimes \pi^j)$.

Here δ_X denotes the small diagonal of X^3, viewed as a correspondence from $X \times X$ to X.

Proof. Noting that μ is induced by δ_X by definition, the equivalence between (*i*) and (*ii*) becomes tautological.
(*ii*) $\Longrightarrow$ (*iv*): by the completeness of the system $\sum_i \pi^i = \Delta_X$, we see that

$$\delta_X = \sum_{i,j,k} \pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = \sum_{k=i+j} \pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) + \sum_{k \neq i+j} \pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = \sum_{i,j} \pi^{i+j} \circ \delta_X \circ (\pi^i \otimes \pi^j).$$

(*iv*) $\Longrightarrow$ (*iii*): it is enough to post-compose both sides of (*iv*) with $\pi^i \otimes \pi^j$ and use the orthogonality between the projectors.
(*iii*) $\Longrightarrow$ (*ii*): it suffices to pre-compose both sides of (*iii*) with π^k and use the orthogonality between the projectors. ∎

As is explained in §I.2.4, the motive $\mathfrak{h}(X)$ of a d-dimensional smooth projective variety X is moreover endowed with a Frobenius algebra object structure. One naturally wonders whether a multiplicative Chow–Künneth decomposition, if it exists, is further compatible with the Frobenius structure $\lambda : \mathfrak{h}(X)^\vee \xrightarrow{\simeq} \mathfrak{h}(X)(d)$. One easily sees that it is the case if the decomposition is self-dual: $\mathfrak{h}^i(X)^\vee \simeq \mathfrak{h}^{2d-i}(X)(d)$ for all i (see Remark I.2.2). It turns out that an MCK decomposition is automatically self-dual:

Proposition I.3.3 (Multiplicativity implies self-duality [**FV19a**, §6 Footnote 24]). *Let X be a smooth projective variety of dimension d admitting a multiplicative Chow–Künneth decomposition $\{\pi^0, \dots, \pi^{2d}\}$. Then it is self-dual, that is, $\pi^i = {}^t\pi^{2d-i}$ for all i.*

Proof. Projecting both sides of (*iv*) in Lemma I.3.2 to the first two factors (or equivalently, pre-composing with the canonical morphism $\varepsilon : \mathfrak{h}(X) \to \mathbb{1}(-d)$ given by the fundamental class), one finds

$$\varepsilon \circ \delta_X = \sum_{i,j} \varepsilon \circ \pi^{i+j} \circ \delta_X \circ (\pi^i \otimes \pi^j).$$

As $\pi^{2d} \in \mathrm{End}(\mathfrak{h}(X))$ is the composition $\mathfrak{h}(X) \xrightarrow{\varepsilon} \mathbb{1}(-d) \hookrightarrow \mathfrak{h}(X)$, we get from the orthogonality between π^{2d} and the other projectors that for any $i+j \neq 2d$, one has $\varepsilon \circ \pi^{i+j} = 0$ and $\varepsilon \circ \pi^{2d} = \varepsilon$. Therefore the equality reduces to the following form

$$\varepsilon \circ \delta_X = \sum_i \varepsilon \circ \delta_X \circ (\pi^{2d-i} \otimes \pi^i).$$

Now noting that $\varepsilon \circ \delta_X$ is the diagonal class $\Delta_X \in \mathrm{CH}^d(X \times X)$, we obtain

$$\Delta_X = \sum_i ({}^t\pi^{2d-i} \otimes \pi^i)_*(\Delta_X),$$

In other words, $\mathrm{id} = \sum_i {}^t\pi^{2d-i} \circ \pi^i$. This allows us to conclude by composing with π^i and ${}^t\pi^{2d-i}$:

$$\pi^i = \left(\sum_j {}^t\pi^{2d-j} \circ \pi^j\right) \circ \pi^i = {}^t\pi^{2d-i} \circ \pi^i = {}^t\pi^{2d-i} \circ \left(\sum_j {}^t\pi^{2d-j} \circ \pi^j\right) = {}^t\pi^{2d-i}.$$

□

I.3.2. Multiplicative splitting of the Bloch–Beilinson filtration. A multiplicative Chow–Künneth decomposition $\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \cdots \oplus \mathfrak{h}^{2d}(X)$ naturally gives rise to a (finite) *multiplicative* bigrading on the Chow ring $\mathrm{CH}^*(X) = \bigoplus_{i,s} \mathrm{CH}^i(X)_{(s)}$, where

$$\mathrm{CH}^i(X)_{(s)} := \mathrm{CH}^i\left(\mathfrak{h}^{2i-s}(X)\right) := \mathrm{Hom}\left(\mathbb{1}(-i), \mathfrak{h}^{2i-s}(X)\right). \tag{I.2}$$

We call the new indexation by s the *grade* of a cycle. Here the multiplicativity means that $\mathrm{CH}^i(X)_{(s)} \cdot \mathrm{CH}^{i'}(X)_{(s')} \subset \mathrm{CH}^{i+i'}(X)_{(s+s')}$, which clearly follows from the multiplicativity of the Chow–Künneth decomposition. The new grading is chosen so that, via Murre's conjecture I.2.3 (C), we have

$$\mathrm{CH}^i(X)_{(s)} \simeq \mathrm{Gr}^s_F \mathrm{CH}^i(X),$$

where $F^\cdot$ is also the expected Bloch–Beilinson filtration. In other words, a multiplicative Chow–Künneth decomposition induces a *multiplicative splitting* of the Bloch–Beilinson filtration on the Chow ring.

In the presence of a multiplicative Chow–Künneth decomposition, the Bloch–Beilinson–Murre conjecture I.1.6 and I.2.3 takes the following form:[7]

Conjecture I.3.4 (Bloch–Beilinson–Murre conjecture with MCK decomposition). *Let X be a smooth projective variety equipped with a multiplicative Chow–Künneth decomposition and the associated bigrading* (I.2) *on* $\mathrm{CH}^*(X)$. *Then*

(*i*) $\mathrm{CH}^i(X)_{(s)} = 0$ *for any* $s < 0$.

(*ii*) *The cycle class map restricted to the subalgebra of grade-0 cycles, i.e.* $\mathrm{CH}^*(X)_{(0)}$, *is injective.*

(*iii*) $\mathrm{CH}^i(X)_{(s)} = 0$ *for any* $s > i$.

(*iii′*) $\mathrm{CH}^i(X)_{(s)} = 0$ *if* $H^{p,q}(X) = 0$ *for all* $p, q \in \mathbf{N}$ *such that* $p + q = 2i - s$ *and* $|p - q| \geq s$.

Here (*i*) + (*iii*) and (*ii*) correspond to Murre's conjecture I.2.3 (B) and (D) respectively. (*iii′*) is obviously stronger than (*iii*) and is implied by the last condition in the Bloch–Beilinson conjecture I.1.6.

Such a multiplicative bigrading of the Chow ring satisfying the conditions (*i*) and (*ii*) in Conjecture I.3.4 was first conjectured for abelian varieties by Beauville [**Bea86**] (the bigrading is established there), then proved for K3 surfaces by Beauville–Voisin [**BV04**] and finally conjectured for hyper-Kähler varieties by Beauville [**Bea07**] under the name of *splitting property* (with evidence in some cases). This direction will be discussed in detail in Chapter II.

I.3.3. To be or not to be. Although a Chow–Künneth decomposition is conjectured to exist for all smooth projective varieties, we will see that a multiplicative one is expected to exist only in quite restrictive situations.

The original and main motivation comes from the study of Chow rings of varieties with trivial canonical bundle. This topic will be developed in greater details in the subsequent chapters. Here let us just give a spoiler: a canonical multiplicative Chow–Künneth decomposition was constructed for abelian varieties [**Bea86**], [**DM91**], [**Kün94**], K3 surfaces [**BV04**] (interpreted by [**SV16a**]) and some (conjecturally all) hyper-Kähler varieties [**SV16a**], [**Via17**], [**FTV19**]. However, there are counter-examples of (strict) Calabi–Yau varieties that do not admit multiplicative Chow–Künneth decompositions [**Bea07**, Example 1.7], at least assuming Murre's conjecture I.2.3.

Removing the Calabi–Yau condition, we can ask the following general question.

Question I.3.5. When does a smooth projective variety admit a multiplicative Chow–Künneth decomposition and if exists, when is it unique?

For the time being, the answer to this natural question is not clear beyond the Calabi–Yau case. We will record in the sequel some experiments the author carried out with his collaborators [**FV19a**], [**FLV19**].

We first mention that in [**SV16b**], Shen and Vial showed that the notion of multiplicative Chow–Künneth decomposition is robust enough to allow many standard procedures to produce new examples out of old ones. More precisely, we have the following.

Proposition I.3.6 ([**SV16b**]). *Let X and Y be smooth projective varieties admitting an MCK decomposition.*

[7] Of course, any Chow–Künneth decomposition already induces a splitting of the conjectural Bloch–Beilinson filtration on the Chow groups by the same formula (I.2), but we are not interested in this general setting because the splitting is highly non-canonical and is not compatible with the intersection product.

- *(Product) The product $X \times Y$ has a naturally induced MCK decomposition: for all k, $\pi^k_{X\times Y} := \sum_{i=0}^k \pi^i_X \otimes \pi^{k-i}_Y$.*
- *(Projective bundle) If E is a vector bundle on X whose Chern classes are all of grade 0 (§I.3.2), then $\mathbf{P}(E)$ has a natural MCK decomposition.*
- *(Blow-up) If Y is a subvariety of X such that the Chern classes of the normal bundle are of grade 0 and the graph of the embedding, as an element in $\mathrm{CH}(X \times Y)$, is of grade[8] 0, then the blow-up of X along Y admits a natural MCK decomposition.*
- *(Quotient) If a finite group G acts on X in such a way that the graphs of the automorphisms, as elements in $\mathrm{CH}(X \times X)$, are of grade 0, then the quotient X/G admits a natural MCK decomposition.*

Note that a variety whose motive is of Tate type (for example, toric varieties, homogeneous variety) admits a canonical multiplicative Chow–Künneth decomposition.

To answer Question I.3.5, on the one hand, it now makes sense to search for more "indecomposable" or "minimal" varieties that admit multiplicative Chow–Künneth decompositions to feed the machine in Proposition I.3.6; on the other hand, we need to find counter-examples and obstructions to the existence of multiplicative Chow–Künneth decompositions. Let us now present some first experiments we are doing on both aspects, mainly on varieties with non-trivial canonical bundle, leaving the K-trivial case to the next chapters.

Curves. As mentioned before, the projective line (a special case of homogeneous varieties) and elliptic curves (special cases of abelian varieties) admit canonical multiplicative Chow–Künneth decompositions. Let C be a smooth projective curve of genus $g \geq 2$. As it is Kimura finite-dimensional [**Kim05**], by Remark I.2.4 and Proposition I.3.3, a multiplicative Chow–Künneth decomposition for C must take the form:

$$\pi^0 = z \times 1_C, \pi^1 = \Delta_C - z \times 1_C - 1_C \times z, \pi^2 = 1_C \times z, \tag{I.3}$$

where z is a 0-cycle of degree 1 and 1_C is the fundamental class. Given such a 0-cycle z, there is the natural embedding $\iota : C \to J(C)$, which sends a point $p \in C$ to $\mathcal{O}_C(p - z)$. Denote $[C] := \iota_*(1_C) \in \mathrm{CH}_1(J(C))$.

Proposition I.3.7 (MCK for curves). *Notation is as above ($g \geq 2$). Let z be a 0-cycle of degree 1 on C. Then the following conditions are equivalent:*

(*i*) *The Chow–Künneth decomposition* (I.3) *determined by z is multiplicative.*
(*ii*) *The modified small diagonal $\Gamma_3(C, z) = 0$ in $\mathrm{CH}^2(C^3)$, where*

$$\Gamma_3(C,z) := \delta_C - p_{12}^*(\Delta_C)p_3^*(z) - p_{23}^*(\Delta_C)p_1^*(z) - p_{13}^*(\Delta_C)p_2^*(z) + p_1^*(z)p_2^*(z) + p_1^*(z)p_3^*(z) + p_2^*(z)p_3^*(z).$$

(*iii*) *The class $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}$.*

In particular, if it exists, an MCK decomposition for C is unique and is given by (I.3) *with $z = \frac{1}{2g-2}K_C$. In this case, Conjecture I.3.4 is satisfied.*

Proof. The equivalence between (*i*) and (*ii*) follows from a direct computation using Lemma I.3.2 (*iv*) (see [**SV16a**, Proposition 8.14] for example).
(*ii*) $\Longrightarrow$ (*iii*) was proved in [**FV19a**, Proposition 7.1] using an idea from [**BV04**, Proposition 3.2].

---

[8] Here, $X \times Y$ is endowed with the natural product MCK decomposition and hence with a bigrading on its Chow ring (Remark I.3.2).

Let $f : C^3 \to J(C)$ be the composition of the embedding $\iota^3 : C^3 \to J(C)^3$ followed by the summation on $J(C)$. We have

$$f_*(\Gamma_3(C,z)) = [3]_*[C] - 3[2]_*[C] + 3[C] = 0 \quad \text{in } \mathrm{CH}_1(J(C)).$$

Use the Beauville decomposition [**Bea86**] of $\mathrm{CH}_1(J(C))$, we see that $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}$.
(*iii*) $\Longrightarrow$ (*i*) is implied by [**FV19a**, Propositions 5.3 and 6.1].
For the uniqueness, let (*ii*) act on Δ_C, we get $c_1(T_C) = (2-2g)z$. Hence z is determined by the curve.
Finally, the Bloch–Beilinson–Murre conjecture is easy to check: $\mathrm{CH}^0(C) = \mathrm{CH}^0(\mathfrak{h}^0(C)) = \mathbf{Q}\cdot 1_C$; $\mathrm{CH}^1(C)_{(1)} = \mathrm{Pic}^0(C) = J(C)$ hence it vanishes if and only if $h^{1,0}(C) = g(C) = 0$. □

The next two examples illustrates that Proposition I.3.7 leads to both existence and non-existence results on MCK decompositions for curves:

Example I.3.8 (Curves with MCK decompositions). If C is hyperelliptic, take z to be a Weierstrass point. Then, by [**Tav14**, Proposition 2.1], the class $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}$. By Proposition I.3.7, C has a multiplicative Chow–Künneth decomposition.

Example I.3.9 (Curves without MCK decompositions [**FV19a**, §7]). As was pointed out before, any MCK decomposition for a curve C is determined by a 0-cycle of degree 1. If C admits an MCK decomposition, then Proposition I.3.7 (*iii*) implies in particular that the *Ceresa cycle* $[C] - [-1]_*[C] \in \mathrm{CH}_1(J(C))$ vanishes. Note that modulo algebraic equivalence, the class $[C]$, as well as the class of the Ceresa cycle, is independent of the choice of z.

Ceresa [**Cer83**] proves that the Ceresa cycle of a very general curve of genus > 2 is not algebraically trivial. As to more explicit examples, Otsubo [**Ots12**] proves that the Ceresa cycle of Fermat curves of degree $4 \leq d \leq 1000$ is not algebraically trivial. Therefore, very general curves of genus > 2 and Fermat curves of degree $4 \leq d \leq 1000$ do not admit any multiplicative Chow–Künneth decomposition.

Remark I.3.10 (MCK modulo algebraic equivalence). If instead of the rational equivalence relation, we work with algebraic equivalence, the analogue of Proposition I.3.7 still holds and the choice of the 0-cycle z becomes irrelevant. More precisely, given a smooth projective curve C, the following conditions are equivalent:

(*i*) C admits a multiplicative Chow–Künneth decomposition modulo the algebraic equivalence.[9]
(*ii*) The modified small diagonal $\Gamma_3(C, \mathrm{pt})$ is algebraically trivial.
(*iii*) The class $[C]$ belongs to $\mathrm{CH}_1(J(C))_{(0)}/\,\mathrm{alg}$.

Of course, (*iii*) implies as before the following condition
(*iv*) the Ceresa cycle $[C] - [-1]_*[C]$ is algebraically trivial.
Now the point of this remark is that (*iv*) is actually equivalent to (*iii*), hence also to (*i*) or (*ii*). Indeed, in the Beauville decomposition

$$\mathrm{CH}_1(J(C)) = \bigoplus_{s=0}^{g-1} \mathrm{CH}_1(J(C))_{(s)},$$

denote by $C_{(s)}$ the grade-s component of the class $[C]$. It is well-defined modulo algebraic equivalence. Then (*iv*) implies that $C_{(1)} = 0$. By Marini's result [**Mar08**, Corollary 26], we have $C_{(s)} = 0$ for all $s > 0$, that is, $[C] \in \mathrm{CH}_1(J(C))_{(0)}/\,\mathrm{alg}$.

[9] Again, it must be of the form (I.3).

In conclusion, the vanishing of the Ceresa cycle characterizes the multiplicativity of the canonical Chow–Künneth decomposition of the curve modulo algebraic equivalence.

Regular surfaces. Let S be a smooth projective surface with $H^1(S, \mathcal{O}_S) = 0$, *i.e.* it is regular. Then for any 0-cycle z of degree 1 on S, we have a self-dual Chow–Künneth decomposition

$$\pi^0 = z \times 1_S, \pi^4 = 1_S \times z, \pi^2 = \Delta_S - \pi^0 - \pi^4, \pi^1 = \pi^3 = 0. \tag{I.4}$$

Assuming the Kimura finite dimensionality conjecture [**Kim05**], any self-dual decomposition should be of this form (Remark I.2.4). Similarly to Proposition I.3.7, we have the following result.

Proposition I.3.11 (MCK for regular surfaces). *Let S be a regular smooth projective surface. Let z be a 0-cycle of degree 1 on S. Then the following conditions are equivalent:*

(*i*) *The Chow–Künneth decomposition* (I.4) *is multiplicative.*

(*ii*) *The modified small diagonal $\Gamma_3(S, z) \in \mathrm{CH}^4(S^3)$ vanishes, where*

$$\Gamma_3(S,z) := \delta_S - p_{12}^*(\Delta_S)p_3^*(z) - p_{23}^*(\Delta_S)p_1^*(z) - p_{13}^*(\Delta_S)p_2^*(z) + p_1^*(z)p_2^*(z) + p_1^*(z)p_3^*(z) + p_2^*(z)p_3^*(z).$$

Moreover, they imply the following two properties:

(*iii*) $\mathrm{Im}\left(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathrm{CH}^2(S)\right) = \mathbf{Q} \cdot z.$

(*iv*) $c_2(T_S) = \chi_{\mathrm{top}}(S) \cdot z$*, where χ_{top} is the topological Euler characteristic.*

In particular, if it exists, an MCK decomposition of the form (I.4) *for S is unique.*

Finally, Conjecture I.3.4 (*i*)*,* (*ii*)*,* (*iii*) *are satisfied and* (*iii′*) *is the content of the Bloch conjecture.*[10]

Proof. The equivalence between (*i*) and (*ii*) is a direct computation. The implication from them to (*iii*) and (*iv*) is proved as in [**BV04**]: let both sides of (*ii*) act on the exterior product of two divisors to obtain (*iii*) and on Δ_S to get (*iv*). The uniqueness of z, hence the MCK decomposition, follows from (*iii*) by intersecting any two ample divisors. Concerning the Bloch–Beilinson–Murre conjecture, we only need to observe that $\mathrm{CH}^2(S)_{(2)}$ is isomorphic to the Albanese kernel. ◻

Example I.3.12 (Regular elliptic surfaces). One important example of regular surfaces that have multiplicative Chow–Künneth decompositions is the case of projective K3 surfaces, proved by Beauville–Voisin [**BV04**]. They establish the vanishing of the modified small diagonal by using the fact that a K3 surface is covered by a pencil of elliptic curves. The same argument actually works for regular elliptic surfaces (necessarily over a projective line).

Remark I.3.13 (MCK on the image). Let S be a smooth projective regular surface admitting a multiplicative Chow–Künneth decomposition (I.4). If $f : S \to S'$ is a surjective morphism to another smooth projective surface. Then S' must be regular and it admits a multiplicative Chow–Künneth decomposition. Indeed, by Proposition I.3.11, we have a degree-one 0-cycle z on S such that $\Gamma_3(S, z) = 0$ in $\mathrm{CH}^4(S^3)$. It is easy to check that $(f, f, f)_*(\Gamma_3(S, z)) = \deg(f)\Gamma_3(S', f_*(z))$. Again by Proposition I.3.11, S' has an MCK decomposition. This remark works equally for curves.

[10] The Bloch conjecture (*cf.* [**Blo10**]) says that an algebraic surface S has trivial Albanese kernel $\ker\left(\mathrm{CH}^2(S)_{\deg 0} \xrightarrow{\mathrm{alb}} \mathrm{Alb}(S)\right)$ if $h^{2,0}(S) = 0$. This conjecture is known for surfaces of Kodaira dimension at most 1 [**BKL76**], for Kimura finite-dimensional surfaces [**Kim05**] (for instance those dominated by a product of curves, see [**BCGP12**] for examples) and for a few surfaces of general type [**Voi92**], [**Voi14**].

Proposition I.3.11 (*iii*) and (*iv*) give obstructions to the existence of multiplicative Chow–Künneth decompositions for regular surfaces and we can moreover use them to see the difference between (*iii*) and MCK decompositions.

Proposition I.3.14 (MCK decomposition *vs.* degeneration of intersection product). *For any $d \geq 7$, a very general smooth surface of degree d in $\mathbf{P}^3$ does not admit any multiplicative Chow–Künneth decomposition of the form* (I.4). *However, since a very general surface has Picard number 1, the conditions* (*iii*) *and* (*iv*) *in Proposition I.3.11 are obviously satisfied with $z = \frac{1}{d}c_1(\mathcal{O}(1))^2$.*

Proof. Generalizing the author's example of octic surface in [**Fu13**, §1.4], O'Grady [**O'G16**] constructed for each d a smooth surface S of degree d in $\mathbf{P}^3$ with

$$\dim \operatorname{Im}\left(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathrm{CH}^2(S)\right) \geq \left[\frac{d-1}{3}\right].$$

So by Proposition I.3.11 (*iii*) such an S does not have an MCK decomposition of the form (I.4) when d is at least 7. Again by Proposition I.3.11, the modified small diagonal $\Gamma_3(S, \frac{1}{d}c_1(\mathcal{O}(1))^2)$ is non-zero. However, since the cycle $\Gamma_3(S, \frac{1}{d}c_1(\mathcal{O}(1))^2)$ is defined universally for all smooth degree-d surfaces, its non-vanishing on one member, namely S, implies that it is non-trivial for a very general member by an argument using Hilbert schemes. □

Some Fano examples. One interesting example of multiplicative Chow–Künneth decomposition for Fano varieties is the following.

Theorem I.3.15 ([**FV19a**, Proposition 5.7]). *Any Fermat cubic hypersurface admits a multiplicative Chow–Künneth decomposition.*

The proof is obtained by applying the theory of distinguished cycles on abelian motives developed in [**FV19a**] (see Chapter IV for more details) to the inductive structure of Fermat hypersurfaces discovered by Shioda–Katsura [**SK79**].

Remark I.3.16. We conjecture more generally in [**FV19a**, Conjecture 5.8] that all Fano or Calabi–Yau Fermat hypersurfaces admit multiplicative Chow–Künneth decompositions. Note that for Fermat hypersurfaces of general type, we have counter-examples even in dimension 1, see Example I.3.9.

The draw back of the previous Fermat examples is that the existence of multiplicative Chow–Künneth decompositions for their deformations is unknown (and we expect a negative answer in general). To this end, in [**FLV19**], we provide two complete families of Fano fourfolds admitting multiplicative Chow–Künneth decompositions.

Theorem I.3.17 ([**FLV19**]). *Smooth cubic fourfolds and Küchle fourfolds of type c7 have a multiplicative Chow–Künneth decomposition.*

Here Küchle fourfolds of type c7 are the zero loci of global sections of the vector bundle $\bigwedge^2 Q(1) \oplus \mathcal{O}(1)$ on the Grassmannian variety $\mathrm{Gr}(3,8)$, where Q is the rank-5 tautological quotient bundle and $\mathcal{O}(1) = \det(Q)$ is the Plücker line bundle. They are one of the families of Fano fourfolds of index 1 that can be obtained as zero loci of sections of homogeneous vector bundles on Grassmannian varieties classified by [**Küc95**].

Idea of the proof. There is a canonical candidate Chow–Künneth decomposition for a cubic fourfold X, namely $\pi^0 = \frac{1}{3}H^4 \times 1_X$, $\pi^2 = \frac{1}{3}H^3 \times H$, $\pi^6 = \frac{1}{3}H \times H^3$, $\pi^8 = \frac{1}{3}1_X \times H^4$,

$\pi^4 = \Delta_X - \pi^0 - \pi^2 - \pi^6 - \pi^8$ and $\pi^1 = \pi^3 = \pi^5 = \pi^7 = 0$, where $H = c_1(\mathcal{O}_X(1))$. The multiplicativity of this Chow–Künneth decomposition is then equivalent to the following decomposition of the small diagonal:

$$\delta_X = \frac{1}{3}\left(p_{12}^*(\Delta_X)p_3^*(H^4) + p_{13}^*(\Delta_X)p_2^*(H^4) + p_{23}^*(\Delta_X)p_1^*(H^4)\right) + P(p_1^*H, p_2^*H, p_3^*H),$$

where P is some universal polynomial. By using Laterveer's motivic relation among $X, X^{[2]}$ and the Fano variety of lines $F(X)$ established in [**Lat17**], we reduce the problem to showing the vanishing of some homologically trivial tautological cycle in $F(X) \times F(X)$. This is the content of the so-called *Franchetta property* (Chapter V), which is proved for $F(X) \times F(X)$ in [**FLVS19**], see Theorem V.2.2.
As to the Küchle fourfolds of type c7, we use Kuznetsov's result [**Kuz15**] that they can be obtained as blow-ups of cubic fourfolds along some Veronese surfaces. With some extra work using Proposition I.3.6, the MCK decomposition of Küchle fourfolds of type c7 follows from that of cubic fourfolds. □

QUESTION I.3.18. Cubic fourfolds and Küchle fourfolds of type c7 are examples of varieties of *cohomological K3 type*, that is, $2m$-dimensional varieties X with Hodge numbers $h^{p,q}(X) = 0$ for all $p \neq q$ except for the middle cohomology $h^{m-1,m+1}(X) = h^{m+1,m-1}(X) = 1$. We ask in [**FLV19**] which varieties of cohomological K3 type have a multiplicative Chow–Künneth decomposition.

A general-type and regular example. In dimension ≥ 2, apart from varieties with Tate-type motive and products of curves of general type, it is quite hard to construct examples of multiplicative Chow–Künneth decomposition on varieties of general type. In the upcoming work [**FLV19**], we provide the first examples of regular surfaces of general type with MCK decompositions. Recall that a *Todorov* surface [**Tod81**] is a smooth projective surface S of general type with $p_g = 1$, $q = 0$ and such that the bicanonical map induces a degree-two quotient to a (singular) K3 surface with at worst rational double points. Such surfaces were originally constructed to provide counter-examples to the global Torelli theorem.

$$\begin{tikzcd} S \arrow[rr, "\iota"] \arrow[dr, "\Phi_{2K_S}"'] & & S \arrow[dl, "\Phi_{2K_S}"] \\ & S/\iota & \end{tikzcd}$$

Two fundamental invariants (α, k) of Todorov surfaces were introduced by Morrison [**Mor88**]: 2^α is the order of the 2-torsion group of $\mathrm{Pic}(S)$ and $k = K_S^2 + 8$. The possible values of (α, k) are: $(0,9), (0,10), (0,11), (1,10), (1,11), (1,12), (2,12), (2,13), (3,14), (4,15), (5,16)$.
We can establish an MCK decomposition for two families of Todorov surfaces.

THEOREM I.3.19 ([**FLV19**]). *Todorov surfaces with fundamental invariants* $(0,9)$ *or* $(1,10)$ *have multiplicative Chow–Künneth decompositions.*

Fano counter-examples. One natural strategy to produce a Fano variety without a multiplicative Chow–Künneth decomposition is to start with an ambiant Fano variety admitting an (essentially unique) MCK decomposition, then blow it up along a subvariety which does not admit any MCK decomposition in such a way that the blow-up stays of Fano type. One such example was already in Beauville [**Bea07**, Example 1.7]: he considers the blow-up of $\mathbf{P}^3$ along some curve B of genus at least 2 embedded via a divisor non-proportional to K_B, and shows that the cycle class map restricted to the subalgebra generated by divisors is not injective.

One hopes naturally to construct Fano counter-examples which are "minimal" in the sense of not being blow ups. The author does not have one in mind, but cubic threefolds seem to be an interesting candidate to start with.

CHAPTER II

Motives of hyper-Kähler varieties: the framework

In the sequel, we concentrate our attention on varieties with trivial canonical bundle. By the Beauville–Bogomolov decomposition theorem [**Bea83c**], any smooth projective variety with vanishing first Chern class admits a finite étale cover which is a product of abelian varieties, (strict) Calabi–Yau varieties and hyper-Kähler varieties. The third ones will be the main object of study in this memoir. Let us first recall its definition.

Definition II.0.1 (Hyper-Kähler varieties). A compact Kähler manifold X is called *hyper-Kähler* or *irreducible holomorphic symplectic* if it is simply connected and $H^0(X, \Omega_X^2)$ is generated by a holomorphic symplectic form σ_X.

Clearly, 2-dimensional compact hyper-Kähler manifolds are nothing else but K3 surfaces.

Remark II.0.2. Thanks to Yau's theorem [**Yau78**], any compact Kähler manifold with vanishing first Chern class admits a Ricci-flat curvature. From this differential geometric point of view, compact hyper-Kähler manifolds can also be defined as those with holonomy group the compact symplectic group.

Compact hyper-Kähler manifolds are natural higher-dimensional generalizations of K3 surfaces and they do share very similar geometry. The most illuminating analogy is arguably their deformation theory, culminating in the global Torelli theorem [**Ver13**], [**Mar11**], [**Huy12**]. The study of compact hyper-Kähler manifolds (and their singular counterparts) has developed rapidly in the last three decades and manifests intriguing relations to geometry, topology, algebra, arithmetic, dynamic systems, and mathematical physics. Here let me simply refer to [**GHJ03**, Part III] and [**Mar11**] for some pointers to works on this subject.

The main goal of the memoir is to study algebraic cycles as well as their intersections on projective hyper-Kähler varieties, from a motivic point of view. Recall that by convention, all Chow groups are with rational coefficients.

II.1. The starting point: K3 surfaces

The whole story began with the case of K3 surfaces worked out in the seminal paper of Beauville and Voisin [**BV04**].

Theorem II.1.1 (Beauville–Voisin). *Let S be a projective K3 surface. There is a canonical 0-cycle class $c_S \in \mathrm{CH}_2(S)$ of degree 1 such that*

(*i*) $\mathrm{Im}\left(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathrm{CH}^2(S)\right) = \mathbf{Q}c_S$.

(*ii*) $c_2(T_S) = 24c_S$ *in* $\mathrm{CH}^2(S)$.

(*iii*) $\delta_S = p_{12}^*(\Delta_S)p_3^*(c_S) + p_{23}^*(\Delta_S)p_1^*(c_S) + p_{13}^*(\Delta_S)p_2^*(c_S) - p_1^*(c_S)p_2^*(c_S) - p_1^*(c_S)p_3^*(c_S) - p_2^*(c_S)p_3^*(c_S)$. *In other words, the modified small diagonal* $\Gamma_3(S, c_S)$ *vanishes in* $\mathrm{CH}^4(S^3)$.

The canonical element c_S is called the *Beauville–Voisin class*. It is represented by any point on any rational curve on S. The well-definedness and (*i*) are deduced from Bogomolov–Mumford's result on the existence of rational curves in primitive ample linear system [**MM83**]. The proof of (*iii*) uses the fact that K3 surfaces are swept out by a family of elliptic curves. (*ii*) is obtained by making both sides of (*iii*) act on Δ_S.

Theorem II.1.1 is a surprising and remarkable result because Mumford's theorem [**Mum68**] (see Example I.1.2) says $\mathrm{CH}^2(S)$ is infinite-dimensional, while Beauville–Voisin's theorem says that there is nevertheless a 1-dimensional subspace (generated by the canonical class c_S) in this huge space $\mathrm{CH}^2(S)$ that receives all the intersection products and Chern classes.

By Proposition I.3.11, Beauville–Voisin's theorem II.1.1 can be reinterpreted as follows, a fact that was observed by Shen and Vial.

Corollary II.1.2 ([**SV16a**]). *Let S be a projective K3 surface and let c_S be the Beauville–Voisin class. Then S has a canonical multiplicative Chow–Künneth decomposition*

$$\mathfrak{h}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S),$$

where $\mathfrak{h}^0(S) = (S, c_S \times 1_S, 0) \simeq \mathbb{1}$ *and* $\mathfrak{h}^4(S) = (S, 1_S \times c_S, 0) \simeq \mathbb{1}(-2)$. *In terms of the projectors, we have*

$$\pi^0 = c_S \times 1_S, \pi^4 = 1_S \times c_S, \pi^2 = \Delta_S - \pi^0 - \pi^4, \pi^1 = \pi^3 = 0.$$

Moreover, Conjecture I.3.4 is satisfied for S.

Remark II.1.3 (Splitting property). By §I.3.2, the MCK decomposition of the projective K3 surface S in Corollary II.1.2 implies a multiplicative splitting of its Chow ring $\mathrm{CH}^*(S)$, where the subalgebra of grade-0 cycles is very explicit:

$$\mathrm{CH}^*(S)_{(0)} = \mathbf{Q} \cdot 1_S \oplus \mathrm{NS}(S)_{\mathbf{Q}} \oplus \mathbf{Q} \cdot c_S.$$

By Theorem II.1.1, $\mathrm{CH}^*(S)_{(0)}$ is also equal to the so-called Beauville–Voisin subring $R^*(S)$, that is, the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(S)$ generated by divisors and Chern classes of S. The natural cycle class map induces an isomorphism $\mathrm{CH}^*(S)_{(0)} \xrightarrow{\sim} \overline{\mathrm{CH}}^*(S) \simeq H^*_{\mathrm{alg}}(S, \mathbf{Q})$, where $\overline{\mathrm{CH}}^*$ denotes the group of cycles modulo numerical equivalence and H^*_{alg} is the algebra of algebraic cohomology classes. In other words, the $\mathbf{Q}$-algebra homomorphism $\mathrm{CH}^*(S) \to \overline{\mathrm{CH}}^*(S)$ has a canonical section (as $\mathbf{Q}$-algebras).

Our main goal is to generalize all these properties (except the equality $R^* = \mathrm{CH}^*_{(0)}$) to higher-dimensional hyper-Kähler varieties. See §II.3 for precise formulations.

II.2. The paradigm: abelian varieties

Hyper-Kähler varieties behave in almost every aspect like abelian varieties;[1] the intersection theory is certainly among such similarities. We collect in this section some known facts about motives and Chow rings of abelian varieties, for two purposes: on the one hand, these facts will be the key ingredients for the proof of some of the main results presented in the subsequent chapters; on the other hand, the study of the intersection theory of algebraic cycles on hyper-Kähler varieties will be (conjecturally) modeled on the corresponding much better developed theory of abelian varieties; see §II.3.

[1] What is the reason behind this phenomenon? The Kuga–Satake construction may partially explain it.

II.2.1. Decomposition of Chow rings. Let A be an abelian variety of dimension g and let $\hat{A} := \mathrm{Pic}^0(A)$ be its dual abelian variety. Mukai [**Muk81**] proved that the Fourier–Mukai transform $\mathrm{D}^b(A) \to \mathrm{D}^b(\hat{A})$ with kernel the Poincaré line bundle $\mathcal{P}$ on $A \times \hat{A}$ is an equivalence of triangulated categories interchanging the tensor product and the convolution product (up to a shift). Beauville [**Bea83a**] [**Bea86**] performed this transformation on the level of Chow groups: the cycle[2] $\exp(c_1(\mathcal{P})) \in \mathrm{CH}^*(A \times \hat{A})$ viewed as a correspondence from A to $\hat{A}$ induces an (ungraded) isomorphism between $\mathrm{CH}^*(A)$ and $\mathrm{CH}^*(\hat{A})$, interchanging the intersection product and the Pontryagin product (up to a sign). This map is called the *Fourier transform* on the Chow groups of abelian varieties. As $\exp(c_1(\mathcal{P}))$ is not of pure dimension, the Fourier transform sends in general a cycle of pure dimension on an abelian variety to a cycle of mixed dimensions on the dual abelian variety. According to the dimension of the Fourier image, the Chow ring $\mathrm{CH}^*(A)$ acquires a new (motivic) grading. We can summarize the main result of Beauville as follows.

Theorem II.2.1 ([**Bea83a**] [**Bea86**]). *Let A be an abelian variety of dimension g and let $\hat{A}$ be its dual abelian variety.*

(*i*) *For any $0 \le i \le g$, there is a canonical decomposition*

$$\mathrm{CH}^i(A) = \bigoplus_{s=i-g}^{i} \mathrm{CH}^i(A)_{(s)}, \tag{II.1}$$

with $\mathrm{CH}^i(A)_{(s)} := \{\alpha \in \mathrm{CH}^i(A) \mid \forall m \in \mathbf{Z}, \mathbf{m}^*(\alpha) = m^{2i-s}\alpha\}$, where $\mathbf{m}$ is the multiplication-by-m map on A. This bigrading is obviously multiplicative: $\mathrm{CH}^i(A)_{(s)} \cdot \mathrm{CH}^{i'}(A)_{(s')} \subset \mathrm{CH}^{i+i'}(A)_{(s+s')}$.

(*ii*) *The Fourier transform from $\mathrm{CH}^*(A)$ to $\mathrm{CH}^*(\hat{A})$ induces an isomorphism between $\mathrm{CH}^i(A)_{(s)}$ and $\mathrm{CH}^{g-i+s}(\hat{A})_{(s)}$.*

(*iii*) *If $f : A \to B$ is a morphism of abelian varieties, then f^* and f_* preserve the s-grading.*

Remark II.2.2 (Splitting). According to the definition of Beauville's decomposition, the subspace $\mathrm{CH}^i(A)_{(s)}$ behaves under the action $\mathbf{m}$ like cohomology classes of degree $2i - s$. In the spirit of Murre's conjecture I.2.3, Beauville's decomposition (II.1) should be a canonical splitting of the Bloch–Beilinson–Murre filtration:

$$\bigoplus_{s \ge j} \mathrm{CH}^i(A)_{(s)} = F^j \, \mathrm{CH}^i(A).$$

Using a polarization, Beauville [**Bea86**, Proposition 4] shows that $\mathrm{CH}^i(A)_{(s)}$ is non-zero for $0 \le s \le i \le g$. This group is expected to be infinite-dimensional in the sense of Mumford [**Mum68**] for $s \ge 2$, according to the Bloch–Beilinson conjecture.

As was pointed out by Beauville in [**Bea86**], Remark II.2.2 allows one to reformulate the Bloch–Beilinson–Murre conjecture for abelian varieties.

Conjecture II.2.3 (Beauville [**Bea86**]). *Let A be an abelian variety of dimension g. Then for any $0 \le i \le g$,*

(*i*) $\mathrm{CH}^i(A)_{(s)} = 0$ *for all $s < 0$.*

(*ii*) *The cycle class map restricted to $\mathrm{CH}^i(A)_{(0)}$ is injective.*

(*iii*) *The Abel–Jacobi map restricted to $\mathrm{CH}^i(A)_{(1)} \to J^{2i-1}(A)_{\mathbf{Q}}$ is injective.*

[2] This is nothing else but the *Mukai vector* of the Fourier–Mukai kernel $\mathcal{P}$.

Here are some general known cases: (i), called the *Beauville vanishing conjecture*, holds for $i = 0, 1, g-2, g-1, g$; (ii), called the *Beauville injectivity conjecture*, holds for $i = 0, 1, g-1, g$; (iii) holds for $i = 1, g$.

II.2.2. Motivic decomposition. The above theory on Chow rings of abelian varieties of Beauville was further developed for motives by Deninger and Murre [**DM91**], Künnemann [**Kün94**], Kimura [**Kim05**].

Theorem II.2.4. *Let A be an abelian variety of dimension g.*

(i) *(Deninger–Murre [**DM91**]). There is a canonical self-dual Chow–Künneth decomposition*

$$\mathfrak{h}(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A),$$

such that $\mathrm{CH}^i(A)_{(s)} = \mathrm{CH}^i(\mathfrak{h}^{2i-s}(A))$, for any i and s. Moreover, $\mathfrak{h}^0(A) \cong \mathbb{1}$ and $\mathfrak{h}^{2g}(A) \cong \mathbb{1}(-g)$.

(ii) *(Künnemann [**Kün94**, Theorem 3.3.1]). For any $0 \leq i \leq 2g$, there is a canonical isomorphism $\mathfrak{h}^i(A) \simeq \mathrm{Sym}^i \mathfrak{h}^1(A)$, and they together provide an isomorphism $\mathfrak{h}(A) \simeq \mathrm{Sym}^\bullet \mathfrak{h}^1(A)$ of algebra objects in the category of rational Chow motives. Here the symmetric power is understood in the super sense.*

(iii) *(Kimura [**Kim05**], cf. [**And05**]). One has $\mathrm{Sym}^i \mathfrak{h}^1(A) = 0$ for all $i > 2g$. In particular, A is Kimura finite-dimensional.*

Let us record the following consequence which fits perfectly in our perspective described in Chapter I, especially §I.3.

Corollary II.2.5. *The Deninger–Murre decomposition [**DM91**] provides for any abelian variety a canonical multiplicative Chow–Künneth decomposition (Definition I.3.1). Its corresponding multiplicative bigrading (§I.3.2) is Beauville's decomposition (Theorem II.2.1) and the Bloch–Beilinson–Murre Conjecture I.3.4 becomes the Beauville Conjecture II.2.3.*

Proof. The multiplicativity of the decomposition given by Deninger–Murre (see Theorem II.2.4) is a consequence of the result of Künnemann recalled in Theorem II.2.4. The fact that item (iii) (equivalently (iii')) in Conjecture I.3.4 holds for abelian varieties is contained in Beauville's theorem II.2.1 (i). □

II.2.3. Symmetrically distinguished cycles. Let us have a closer look at the Beauville injectivity conjecture II.2.3 (ii). Apart from the trivial cases of fundamental class and 0-cycles, it is easy to see that this conjecture holds for divisors: for an abelian variety A,

$$\mathrm{CH}^1(A) = \mathrm{CH}^1(A)_{(1)} \oplus \mathrm{CH}^1(A)_{(0)};$$

this decomposition is nothing else but the basic fact that

$$\mathrm{Pic}(A)_{\mathbf{Q}} = \mathrm{Pic}^0(A)_{\mathbf{Q}} \oplus \mathrm{Pic}^s(A)_{\mathbf{Q}},$$

which identifies the space of symmetric $\mathbf{Q}$-line bundles $\mathrm{Pic}^s(A)$ with the Néron–Severi space $\mathrm{NS}(A)_{\mathbf{Q}}$ and identifies $\mathrm{Pic}^0(A)_{\mathbf{Q}}$ with the space of anti-symmetric $\mathbf{Q}$-line bundles. The next known case is that of 1-cycles proved by Beauville in [**Bea86**]: using Fourier transform (Theorem II.2.1), the injectivity of cycle class map on $\mathrm{CH}^{g-1}(A)_{(0)}$ is equivalent to the injectivity of the cycle class map on $\mathrm{CH}^1(\hat{A})_{(0)}$ for the dual abelian variety, which is known. Other than these general results, this conjecture is wide open.

Let us denote by $\overline{\mathrm{CH}}^*(A)$ the quotient of the Chow ring $\mathrm{CH}^*(A)$ modulo the numerical equivalence relation, which is also the image of the cycle class map $\mathrm{CH}^*(A) \to H^*(A, \mathbf{Q})$ by the validity of the Lefschetz standard conjecture [**Kle94**]. Then Conjecture II.2.3 (*ii*) can be equivalently reformulated as saying that the composition

$$\mathrm{CH}^*(A)_{(0)} \hookrightarrow \mathrm{CH}^*(A) \twoheadrightarrow \overline{\mathrm{CH}}^*(A)$$

is an isomorphism. In other words, the subalgebra $\mathrm{CH}^*(A)_{(0)}$ should provide a canonical section of $\mathbf{Q}$-algebras for the natural epimorphism $\mathrm{CH}^*(A) \twoheadrightarrow \overline{\mathrm{CH}}^*(A)$. Let us call it the *section property*. From this point of view, the most remarkable breakthrough towards the Beauville injectivity conjecture is the work of Peter O'Sullivan [**O'S11b**] where the section property is established using the so-called *symmetrically distinguished cycles*, which is the natural generalization of symmetric $\mathbf{Q}$-divisors. Let us first recall the definition.

Definition II.2.6 (Symmetrically distinguished cycles on abelian varieties [**O'S11b**]). Let A be an abelian variety and let $\alpha \in \mathrm{CH}^*(A)$. For each integer $m \geq 0$, denote by $V_m(\alpha)$ the $\mathbf{Q}$-vector subspace of $\mathrm{CH}^*(A^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \times \alpha^{r_2} \times \cdots \times \alpha^{r_n}),$$

where $n \leq m$, $r_j \geq 0$ are integers, and where $p : A^n \to A^m$ is a closed immersion, each component $A^n \to A$ of which is either a projection or the composite of a projection with the involution $[-1] : A \to A$. Then α is *symmetrically distinguished* if for every m, the restriction of the projection $\mathrm{CH}^*(A^m) \to \overline{\mathrm{CH}}^*(A^m)$ to $V_m(\alpha)$ is injective. The subgroup of symmetrically distinguished cycles is denoted by $\mathrm{DCH}^*(A)$.

Here is the main result of O'Sullivan [**O'S11b**] which confirms the section property for abelian varieties.

Theorem II.2.7 (O'Sullivan [**O'S11b**, Theorem 6.2.5]). *Let A be an abelian variety. Then the symmetrically distinguished cycles in $\mathrm{CH}^*(A)$ form a graded $\mathbf{Q}$-subalgebra $\mathrm{DCH}^*(A)$ that contains symmetric $\mathbf{Q}$-divisors and that is stable under pull-backs and push-forwards along homomorphisms of abelian varieties. Moreover the composition*

$$\mathrm{DCH}^*(A) \hookrightarrow \mathrm{CH}^*(A) \twoheadrightarrow \overline{\mathrm{CH}}^*(A)$$

is an isomorphism of graded $\mathbf{Q}$-algebras.

As a conseqence, $\mathrm{DCH}^*(A)$ is a graded subalgebra of $\mathrm{CH}^*(A)_{(0)}$ and the content of Beauville's injectivity conjecture II.2.3 (*ii*) is that they are equal:

$$\mathrm{DCH}^*(A) = \mathrm{CH}^*(A)_{(0)}.$$

The approach of O'Sullivan is categorical [**O'S11b**] and the main input is the fact that motives of abelian varieties are Kimura finite-dimensional (Theorem II.2.4 (*iii*)) and carry a canonical Hopf algebra structure.

Remark II.2.8. The projectors of the canonical multiplicative Chow–Künneth decomposition (Theorem II.2.4) of Deninger–Murre [**DM91**] are given explicitly in [**Kün94**], and we see immediately from Theorem II.2.7 that these Chow–Künneth projectors are symmetrically distinguished cycles.

REMARK II.2.9. Similarly to the case of K3 surfaces, we can define the *Beauville–Voisin ring* of an abelian variety A as the $\mathbf{Q}$-subalgebra

$$R^*(A) := \langle \mathrm{CH}^1(A)_{(0)} \rangle$$

of $\mathrm{CH}^*(A)$ generated by *symmetric* $\mathbf{Q}$-divisors.[3] Then we have

$$R^*(A) \subset \mathrm{DCH}^*(A) \subset \mathrm{CH}^*(A)_{(0)} \subset \mathrm{CH}^*(A) \twoheadrightarrow \overline{\mathrm{CH}}^*(A),$$

where the first inclusion is in general strict and the second inclusion is conjecturally an equality. Moreover, as the natural map $\mathrm{DCH}^*(A) \to \overline{\mathrm{CH}}^*(A)$ is an isomorphism by Theorem II.2.7, the natural map $R^*(A) \to \overline{\mathrm{CH}}^*(A)$ is injective. This was originally conjectured by Voisin. An alternative proof was provided by Moonen [Moo16].

II.3. The conjectural picture for hyper-Kähler varieties

The main purpose of the memoir is to understand the multiplicative structure of the Chow rings and the Chow motives of projective hyper-Kähler varieties. In this section, we describe the expected general picture, which is largely conjectural. In the subsequent chapters, we will present some of its established parts for certain examples.

II.3.1. Conjectures for Chow rings. Although our point of view will eventually be motivic, let us collect here the relevant conjectures for Chow rings.

First of all, just like K3 surfaces (§II.1) and abelian varieties (§II.2), the conjectural Bloch–Beilinson–Murre filtration is conjectured to have a canonical splitting.

CONJECTURE II.3.1 (Beauville splitting property [Bea07]). *Let X be a holomorphic symplectic*[4] *variety of dimension $2n$. Then there is a canonical multiplicative*[5] *bigrading of $\mathrm{CH}^*(X)$: for any i,*

$$\mathrm{CH}^i(X) = \bigoplus_s \mathrm{CH}^i(X)_{(s)}, \tag{II.2}$$

whose associated (ring) filtration $F^j\,\mathrm{CH}^i(X) := \bigoplus_{s\geq j} \mathrm{CH}^i(X)_{(s)}$ satisfies the Bloch–Beilinson conjecture. In particular, for all i:

(*i*) $\mathrm{CH}^i(X)_{(s)} = 0$ *for all $s < 0$.*
(*ii*) *The cycle class map vanishes on $\mathrm{CH}^i(X)_{(s)}$ for all $s > 0$;*
(*iii*) *The restriction of the cycle class map to $\mathrm{CH}^i(X)_{(0)}$ is injective.*

As a projective hyper-Kähler variety X is regular, we always have $\mathrm{CH}^1(X) = \mathrm{CH}^1(X)_{(0)}$. As a special case of Conjecture II.3.1 (*iii*) above, Beauville [Bea07] considers the subalgebra of $\mathrm{CH}^*(X)$ generated by divisors $\mathrm{r}^*(X) := \langle \mathrm{CH}^1(X) \rangle$ and conjectures, under the name of *weak splitting property*, that the cycle class map restricted to $\mathrm{r}^*(X)$ is injective. The insight of Voisin [Voi08] is that one can also include Chern classes into this "tautological ring."

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[3] All Chern classes of A vanish as the tangent bundle is trivial.

[4] A smooth projective variety is called *holomorphic symplectic* if it admits a holomorphic 2-form that is nowhere degenerate. Examples of such varieties include hyper-Kähler varieties, even-dimensional abelian varieties and products of them. We do not restrict to *irreducible* holomorphic symplectic varieties in the statement of the conjecture, because we think it is plausible in this more general setting.

[5] Here the multipicativity means $\mathrm{CH}^i(X)_{(s)} \cdot \mathrm{CH}^{i'}(X)_{(s')} \subset \mathrm{CH}^{i+i'}(X)_{(s+s')}$ for all i, i', s, s'.

DEFINITION II.3.2 (Beauville–Voisin ring). Let X be a projective hyper-Kähler variety of dimension $2n$. The *Beauville–Voisin* ring is the $\mathbf{Q}$-subalgebra

$$R^*(X) := \langle \mathrm{CH}^1(X), c_{2i}(T_X); 1 \leq i \leq n \rangle$$

of $\mathrm{CH}^*(X)$ generated by the $\mathbf{Q}$-divisors and the Chern classes of the tangent bundle of X. Note that all the Chern classes of odd degree vanish because T_X is isomorphic to its dual bundle Ω^1_X via the symplectic form.

The following Beauville–Voisin conjecture plays a central role in the subject.

CONJECTURE II.3.3 (Beauville–Voisin [**Bea07**] [**Voi08**]). *Let X be a projective hyper-Kähler variety. The restriction of the cycle class map to the Beauville–Voisin ring $R^*(X)$ is injective.*

Here are some known cases:

- (Beauville [**Bea07**]) For S a K3 surface, its Hilbert square $S^{[2]}$ and Hilbert cube $S^{[3]}$ satisfy the weak splitting property;
- (Voisin [**Voi08**]) For S a K3 surface and $n \leq 2b_{2,tr}(S)+4$, the Hilbert scheme $S^{[n]}$ satisfies the Beauville-Voisin Conjecture, where $b_{2,tr} = b_2 - \rho$ is the second transcendental Betti number;
- (Voisin [**Voi08**]) The variety of lines on a smooth cubic fourfold satisfies the Beauville-Voisin conjecture;
- (Ferretti [**Fer12**]) A very general double EPW sextic satisfies the Beauville-Voisin conjecture;
- (Fu [**Fu15**]) For A an abelian surface and n a natural number, the generalized Kummer variety $K_n(A)$ satisfies the Beauville-Voisin conjecture;
- (Rieß [**Rie16**]) For X a projective hyper-Kähler variety with an isotropic non-trivial line bundle, if X satisfies the rational Lagrangian fibration conjecture (for example, when X is deformation equivalent to Hilbert schemes of K3 surfaces or generalized Kummer varieties), X satisfies the weak splitting property;
- (Yin [**Yin15**]) For S a K3 surface which is Kimura finite dimensional and n a natural number, the Hilbert scheme $S^{[n]}$ satisfies the Beauville-Voisin conjecture;
- (Fu–Tian [**FT17**]) For S a K3 surface and n a natural number, the Hilbert scheme $S^{[n]}$ satisfies the weak splitting property for $n \leq 506$ and satisfies the Beauville–Voisin conjecture for $n < (b_2(S)_{tr} + 1)(b_2(S)_{tr} + 2)$;
- (Maulik–Negut [**MN19**]) For S a K3 surface and n a natural number, the Hilbert scheme $S^{[n]}$ satisfies the weak splitting property for all n.

Voisin [**Voi16a**] goes even further than the Beauville–Voisin conjecture II.3.3 and proposes to put the classes of the algebraically coisotropic subvarieties into this "tautological ring."

CONJECTURE II.3.4 (Voisin [**Voi16a**]). *Let X be a projective hyper-Kähler variety. Then the restriction of the cycle class map to the following* extended Beauville–Voisin ring

$$\widetilde{R}^*(X) := \langle R^*(X), [Z]; Z \textit{ algebraically coisotropic} \rangle$$

is injective.

We can view $\mathrm{r}^*(X)$, $R^*(X)$, and $\widetilde{R}^*(X)$ as better and better approximations of the conjectural subalgebra $\mathrm{CH}^*(X)_{(0)}$, which is supposed to provide a section of the natural surjective $\mathbf{Q}$-algebra homomorphism $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$, where $\overline{\mathrm{CH}}^*(X)$ is the image of the cycle class map, or assuming the standard conjecture, the Chow ring modulo numerical equivalence.

The bigrading being conjectural for now, the author and Vial propose in [**FV19a**] the following conjecture.

Conjecture II.3.5 (Section property [**FV19a**]). *Let X be a projective holomorphic symplectic variety. The natural surjection $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$ has a section*[6] *of graded $\mathbf{Q}$-algebras whose image contains all the Chern classes of X. In other words, there is a graded $\mathbf{Q}$-subalgebra $\mathrm{DCH}^*(X)$ of $\mathrm{CH}^*(X)$ containing the Chern classes of X, such that the composition $\mathrm{DCH}^*(X) \hookrightarrow \mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$ is an isomorphism. In this case, the elements in $\mathrm{DCH}^*(X)$ are called* distinguished cycles.

Roughly speaking, a distinguished cycle is the "best" lift in the Chow ring of the corresponding algebraic cycle class modulo numerical equivalence. The Section Property conjecture II.3.5 obviously imply the Beauville–Voisin conjecture II.3.3. The point is that, given a hyper-Kähler variety X, the subalgebra $\mathrm{DCH}^*(X)$ of its distinguished cycles, once constructed, is expected to be unique and contains much more elements than the Beauville–Voisin ring $R^*(X)$. One example of such extra cycles is the aforementioned algebraically coisotropic subvarieties studied by Voisin [**Voi16a**] (Conjecture II.3.4); another important example is the *generically defined cycles*, which are by definition the restrictions of cycles defined on the universal family over some moduli space of polarized hyper-Kähler varieties that it belongs to. The idea that generically defined cycles are distinguished is the so-called generalized Franchetta conjecture for hyper-Kähler varieties (the K3 surface case was conjectured by O'Grady [**O'G13**]).

Conjecture II.3.6 (Generalized Franchetta conjecture, *cf.* [**FLVS19**] [**BL19a**]). *Let $\mathcal{F}$ be the moduli stack of a locally complete family of polarized hyper-Kähler varieties and let $\mathcal{X} \to \mathcal{F}$ be the universal family. Let $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$. If it is fiber-wise homologically trivial, its restriction to any fiber is (rationally equivalent to) zero.*

The generalized Franchetta conjecture II.3.6 is closely related to the Beauville–Voisin conjecture II.3.3 as well as to Voisin's refinement Conjecture II.3.4. On the one hand, the generalized Franchetta conjecture implies the part of the Beauville–Voisin conjecture involving only Chern classes and the polarization, and also some interesting parts of Conjecture II.3.4 involving algebraically coisotropic subvarieties (see V.5 for precise examples); on the other hand, in many cases where we can establish the generalized Franchetta conjecture, the proof often makes use of some known cases of the Beauville–Voisin conjecture (especially [**Voi08**]).

Let us summarize: we have a chain

$$\mathrm{r}^*(X) \subset R^*(X) \subset \mathrm{DCH}^*(X) \subset \mathrm{CH}^*(X)_{(0)} \subset \mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X) \tag{II.3}$$

of subalgebras. We have in (II.3)

- the middle two subalgebras are conjectured and when constructed, they are conjecturally equal;
- the first two inclusions are strict in general (but are equalities for K3 surfaces);
- $\mathrm{DCH}^*(X)$ is expected to contain $\widetilde{R}^*(X)$ and the inclusions $R^*(X) \subset \widetilde{R}^*(X) \subset \mathrm{DCH}^*(X)$ are all strict in general;
- the injectivity of $\mathrm{r}^*(X) \to \overline{\mathrm{CH}}^*(X)$ is Beauville's weak splitting property conjecture;
- the injectivity of $R^*(X) \to \overline{\mathrm{CH}}^*(X)$ is the Beauville–Voisin conjecture II.3.3;

[6] It is conjectured to be canonical if X is regular, that is, $H^1(X, \mathcal{O}_X) = 0$.

- the section property conjecture II.3.5 says that $\mathrm{DCH}^*(X) \to \overline{\mathrm{CH}}^*(X)$ is an isomorphism;
- $\mathrm{DCH}^*(X)$ is also conjectured to contain the generically defined cycles. Together with the section property, this gives the generalized Franchetta conjecture II.3.6;
- Beauville's splitting property conjecture II.3.1 implies that $\mathrm{CH}^*(X)_{(0)} \to \overline{\mathrm{CH}}^*(X)$ is an isomorphism.

II.3.2. Conjectures on motives. We now lift the conjectures on Chow rings discussed in §II.3.1 to the category of Chow motives and we argue that their motivic analogues are more fundamental and at the same time behave better under various operations.

II.3.2.1. *Motivic splitting.* The motivic lift of Beauville's splitting property conjecture II.3.1 is the combination of the following two conjectures.

Conjecture II.3.7 (Multiplicative Chow–Künneth decomposition [SV16a]). *Let X be a smooth projective holomorphic symplectic variety of dimension $2n$. Then there exists a (canonical) multiplicative Chow–Künneth decomposition*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X).$$

Under this conjecture, one defines as in §I.3.2 the induced multiplicative bigrading

$$\mathrm{CH}^i(X)_{(s)} := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X))$$

of the Chow ring and reformulates the Bloch–Beilinson–Murre conjecture I.3.4 (in the presence of MCK decomposition).

Conjecture II.3.8 (Bloch–Beilinson–Murre with MCK). *Keep the same notation. One has*

(*i*) $\mathrm{CH}^i(X)_{(s)} = 0$ *for all $s < 0$ and all i.*
(*ii*) *The cycle class map restricted to the* $\mathbf{Q}$*-subalgebra* $\mathrm{CH}^*(X)_{(0)}$ *is injective.*
(*iii*) $\mathrm{CH}^i(X)_{(s)} = 0$ *for all $s > i$.*

Obviously, Conjecture II.3.7 and Conjecture II.3.8 together imply Beauville's splitting property conjecture II.3.1.

We point out that even assuming the validity of Conjectures II.3.7 and II.3.8, the following important questions remain.

Question II.3.9 (Multiplication table). Let X be a projective hyper-Kähler variety with a multiplicative Chow–Künneth decomposition. For all positive integers i, i' and s, s', how to describe the morphisms

$$\mathfrak{h}^i(X) \otimes \mathfrak{h}^{i'}(X) \to \mathfrak{h}^{i+i'}(X)$$

and

$$\mathrm{CH}^i(X)_{(s)} \otimes \mathrm{CH}^{i'}(X)_{(s')} \to \mathrm{CH}^{i+i'}(X)_{(s+s')}$$

given by the intersection product?

This question is one of the motivations to study the *motivic hyper-Kähler resolution conjecture* II.3.17, which will be introduced in §II.3.4 and studied in detail in Chapter III.

II.3.2.2. *Distinguished cycles.* The motivic version of the section property conjecture II.3.5 is the so-called *distinguished marking conjecture* proposed in the author's joint work with Vial [**FV19a**, Conjecture 2] and will be dealt with in greater detail in Chapter IV. This conjecture treats specifically (not necessarily symplectic) varieties with motive of abelian type,[7], by introducing the notion of *distinguished cycles* on them, which depends *a priori* on the choice of a *marking*: a marking for a variety X is an isomorphism $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} M$ of Chow motives, where M is a direct summand of a Chow motive of the form $\oplus_i \mathfrak{h}(A_i)(n_i)$ cut out by an idempotent matrix P of symmetrically distinguished cycles (Definition II.2.6), where A_i is an abelian variety and $n_i \in \mathbf{Z}$. Given such a marking, the group of *distinguished cycles* $\mathrm{DCH}^*_\phi(X)$ consists of the image under P_* of the symmetrically distinguished cycles on each A_i, transported *via* the induced isomorphism $\phi_* : \mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(M)$.

It is easy to see that O'Sullivan's theorem II.2.7 implies that $\mathrm{DCH}_\phi(X)$ provides a section as graded vector spaces of the epimorphism $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$. The question is the following: what are sufficient conditions on a marking ϕ for $\mathrm{DCH}^*_\phi(X)$ to be a subalgebra of $\mathrm{CH}^*(X)$ that contains the Chern classes?

To this end, we formulate in [**FV19a**] the following two conditions:[8]

$$(\star_{\mathrm{Mult}}) \quad \text{The small diagonal } \delta_X \text{ belongs to } \mathrm{DCH}^*_{\phi^{\otimes 3}}(X^3).$$

$$(\star_{\mathrm{Chern}}) \quad \text{All Chern classes of } X \text{ belong to } \mathrm{DCH}^*_\phi(X),$$

where $\phi^{\otimes 3} : \mathfrak{h}(X^3) \xrightarrow{\simeq} M^{\otimes 3}$ is the natural marking induced from ϕ. If a marking satisfying these two conditions for X exists, we will say that X *satisfies* $(\star)$ or X *has a distinguished marking*. Our conjecture is the following.

Conjecture II.3.10 (Distinguished marking [**FV19a**]). *A smooth projective holomorphic symplectic variety admits a marking that satisfies* $(\star)$.

As we will see in Chapter IV, a marking satisfying $(\star)$ gives rise to a section of the $\mathbf{Q}$-algebra epimorphism $\mathrm{CH}^* \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$ whose image contains the Chern classes of X. In short, the distinguished marking conjecture II.3.10 implies the section property conjecture II.3.5. The difference is that the section property conjecture does not behave well enough under basic operations, for instance, products, blow ups, quotients *etc.*; however, the closely related condition $(\star)$ is essentially motivic and behaves much better; we refer to Chapter IV for more details.

Finally, let us record a relation between two motivic conjectures: Conjecture II.3.10 for a variety X implies Conjecture II.3.7 for the same variety X (see Proposition IV.3.5).

II.3.3. Derived categories: multiplicative Orlov conjecture. Orlov conjectures that the bounded derived category of coherent sheaves of a smooth projective variety determines its rational Chow motive.

[7] By the generalized Kuga–Satake construction [**KSV19**] we expect that all hyper-Kähler varieties have motives of abelian type.

[8] The condition $(\star_{\mathrm{Chern}})$ is not only esthetically pleasing, it is also essential to establish that the condition $(\star)$ is stable under natural constructions such as blow ups.

Conjecture II.3.11 (Orlov [**Orl05**]). *Let X and Y be derived equivalent smooth projective varieties. Then their Chow motives are isomorphic.*

This conjecture was proved for K3 surfaces by Huybrechts [**Huy18**] and for all smooth projective surfaces in the recent work [**FV19b**, Theorem 1.1]. See also [**FV19b**, Proposition 1.6] for some new cases in dimension 3 and 4.

Recall that the motive of a smooth projective variety is naturally a Frobenius algebra object in the category of rational Chow motives (§I.2.4). In view of Theorem I.2.15, which says that derived equivalent K3 surfaces have isomorphic Chow motives as Frobenius algebra objects, we naturally ask that under what circumstances one could expect a "multiplicative Orlov conjecture."

Question II.3.12. When can we expect that a derived equivalence between two smooth projective varieties implies an isomorphism between their rational Chow motives as *Frobenius algebra objects*?

Note that in concrete terms, the fact that the motives of two d-dimensional smooth projective varieties X and Y are isomorphic as Frobenius algebra objects means the existence of a correspondence $\Gamma \in \mathrm{CH}^d(X \times Y)$ satisfying $\Gamma^{-1} = {}^t\Gamma$ and $(\Gamma \otimes \Gamma \otimes \Gamma)_*(\delta_X) = \delta_Y$.

According to the celebrated theorem of Bondal–Orlov [**BO01**], Question II.3.12 has a positive answer for varieties with ample or anti-ample canonical bundle, since any two such derived equivalent varieties must be isomorphic. More generally, Question II.3.12 also has a positive answer for varieties that do not have non-isomorphic Fourier–Mukai partners, such as curves [**Huy06**, Corollary 5.46].

The situation gets more intriguing for varieties with trivial canonical bundle, where one cannot expect in general a positive answer to Question II.3.12. In fact, if $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$ are isomorphic as Frobenius algebra objects then, by applying the Betti realization functor, their cohomology are isomorphic as Frobenius algebras, that is, there is a (graded) isomorphism $H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ of $\mathbf{Q}$-algebras sending the class of a point on X to the class of a point on Y. However, as we see in the next example, this is not the case in general for derived equivalent varieties.

Example II.3.13. Borisov and Căldăraru [**BC09**] constructed derived equivalent (but non-birational) Calabi–Yau threefolds X and Y with the following properties: $\mathrm{Pic}(X) = \mathbf{Z}H_X$ with $\deg(H_X^3) = 14$ and $\mathrm{Pic}(Y) = \mathbf{Z}H_Y$ with $\deg(H_Y^3) = 42$; hence there is no graded $\mathbf{Q}$-algebra isomorphism between $H^*(X, \mathbf{Q})$ and $H^*(Y, \mathbf{Q})$ that respects the point class. Therefore, $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$ are not isomorphic as Frobenius algebra objects. Nevertheless, it is easy to see that $H^*(X, \mathbf{Q})$ and $H^*(Y, \mathbf{Q})$ are Hodge isomorphic as graded $\mathbf{Q}$-algebras, see [**FV19b**, Proposition 4.4] for a generalization.

For abelian varieties, we have the following result.

Proposition II.3.14 (Isogenous abelian varieties [**FV19b**, Proposition 4.5]). *Let A and B be isogenous abelian varieties of dimension g. Then*

(*i*) *$\mathfrak{h}(A)$ and $\mathfrak{h}(B)$ are isomorphic as algebra objects.*
(*ii*) *The following conditions are equivalent:*
  (*a*) *There is an isomorphism of Frobenius algebra objects between $\mathfrak{h}(A)$ and $\mathfrak{h}(B)$.*
  (*b*) *There is a graded Hodge isomorphism of Frobenius algebras between $H^*(A, \mathbf{Q})$ and $H^*(B, \mathbf{Q})$.*
  (*c*) *There exists an isogeny of degree m^{2g} between A and B for some $m \in \mathbf{Z}_{>0}$.*

As a consequence, given two derived equivalent abelian varieties, in general there is no isomorphism of Frobenius algebra objects between their Chow motives (or their cohomology). Indeed, by Proposition II.3.14(*ii*), the motives of two derived equivalent abelian varieties that cannot be related by an isogeny of degree equal to the $2g$-th power of some positive integer are not isomorphic as Frobenius algebra objects. For instance, if one considers an abelian variety A with Néron–Severi group generated by one ample line bundle L, any isogeny between A and $\hat{A}$ is of degree $\chi(L)^2 m^{4g}$ for some $m \in \mathbf{Z}_{>0}$. But in general, $\chi(L)$ is not a g-th power in $\mathbf{Z}$. On the other hand, A and $\hat{A}$ are always derived equivalent by Mukai's classical result [**Muk81**].

Although we do not provide much evidence beyond the case of K3 surfaces (Theorem I.2.15), we are tempted to propose the following.

Conjecture II.3.15 (Multiplicative Orlov conjecture [**FV19b**, Conjecture 4.6]). *Let X and Y be projective hyper-Kähler varieties. If there is an exact equivalence between the triangulated categories $\mathrm{D}^b(X)$ and $\mathrm{D}^b(Y)$, there exists an isomorphism between the Chow motives $\mathfrak{h}(X)$ and $\mathfrak{h}(Y)$, as (Frobenius) algebra objects in the category of Chow motives. In particular, their Chow rings, as well as their cohomology rings, are isomorphic.*

As a tiny piece of evidence, derived equivalent projective hyper-Kähler varieties have isomorphic complex cohomology algebras (*cf.* [**FV19b**, Proposition 4.7]).

Let us test Conjecture II.3.15 in the (few) available examples of derived equivalent hyper-Kähler varieties:

- If S and S' are derived equivalent projective K3 surfaces, then, by combining the results of Bridgeland–King–Reid [**BKR01**] and Haiman [**Hai01**], we know that their Hilbert schemes $S^{[n]}$ and $S'^{[n]}$ are derived equivalent. In [**FV19b**, Corollary 1], by combining Theorem I.2.15 ([**FV19b**]) and Theorem III.7.1 ([**FT17**]), we show that the Chow motives of $S^{[n]}$ and $S'^{[n]}$ are indeed isomorphic as Frobenius algebra objects.
- Conjecturally, birationally equivalent hyper-Kähler varieties are derived equivalent [**Kaw02**] [**Kaw18**] (*cf.* [**Huy06**, Conjecture 6.24]). Thanks to the main result of Rieß [**Rie14**], or rather its proof, we know that birational hyper-Kähler varieties have isomorphic Chow motives as Frobenius algebra objects, hence consistent with Conjecture II.3.15. There are by now some cases where the derived equivalence of birational hyper-Kähler varieties is known. The easiest example might be the so-called Mukai flop. Another instance of interest is the moduli spaces of stable sheaves with a fixed Mukai vector on a projective K3 surfaces with respect to various stability conditions. They are birationally equivalent via *wall-crossings* (*cf.* [**BM14a**] [**BM14b**]) and their derived equivalence has been announced by Halpern-Leistner in [**HL19**]. See also §III.8.7.

II.3.4. Birational hyper-Kähler orbifolds. If one is willing to enlarge the category of hyper-Kähler varieties to that of hyper-Kähler *orbifolds*, the orbifold analogue of the multiplicative Orlov conjecture II.3.15 is closely related to the so-called *motivic hyper-Kähler resolution conjecture* introduced and studied in [**FTV19**], [**FT17**], [**FT19**], [**FN19**]. Let us introduce it here in an even broader conjectural context of birational hyper-Kähler orbifolds and come back to its detailed study in Chapter III.

In this memoir, an (algebraic) *orbifold* is by definition a smooth proper Deligne–Mumford stack with trivial generic stabilizer and projective coarse moduli space. An orbifold is called hyper-Kähler if it admits a holomorphic symplectic 2-form which is unique up to scalar and the underlying variety is simply connected. The notion of birational equivalence naturally extends to orbifolds by taking Zariski open subsets avoiding the stacky locus. For an orbifold $\mathcal{X}$, the right notion of its motive is the notion of the *orbifold motive* (see §III.2), denoted by $\mathfrak{h}_{\mathrm{orb}}(\mathcal{X})$, which is an algebra object[9] in the category of rational Chow motives and coincides with the canonical algebra structure on the usual Chow motive defined in §I.2.3 if $\mathcal{X}$ is a smooth projective variety. Similarly, we have theories of *orbifold Chow ring*, *orbifold K-theory*, and *orbifold cohomology ring* ([**CR04**]), generalizing the usual Chow ring, K-theory and cohomology ring for smooth projective varieties; see Chapter III for details.

We want to propose the following conjecture, which can be considered as a motivic version of the *K-equivalence conjecture*, or the *quantum minimal model conjecture* of Ruan [**Rua06**] for hyper-Kähler orbifolds.

Conjecture II.3.16 (Motivic hyper-Kähler K-equivalence conjecture). *Let $\mathcal{X}$ and $\mathcal{X}'$ be hyper-Kähler orbifolds. If $\mathcal{X}$ and $\mathcal{X}'$ are birationally equivalent, their complex orbifold motives $\mathfrak{h}_{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}}$ and $\mathfrak{h}_{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}}$ are isomorphic as algebra objects*[10] *in the category of complex Chow motives* $\mathrm{CHM}_{\mathbf{C}}$*. In particular, there is an isomorphism*

$$\mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X},\bullet)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X}',\bullet)_{\mathbf{C}}$$

of bigraded $\mathbf{C}$*-algebras between their orbifold higher Chow rings (defined in §III.2).*

When $\mathcal{X}$ and $\mathcal{X}'$ are hyper-Kähler varieties, the main result of Rieß [**Rie14**], based on the work of Huybrechts [**Huy99**], says that birationally equivalent projective hyper-Kähler varieties have isomorphic (rational) Chow rings. In fact, her proof shows the stronger property that they have isomorphic Chow motives as algebra objects. So the non-orbifold case of Conjecture II.3.16 is known.

If $\mathcal{X}'$ is a (smooth projective) hyper-Kähler variety, it is called a hyper-Kähler resolution of the underlying coarse moduli space of $\mathcal{X}$. In this case, Conjecture II.3.16 reduces to the following.

Conjecture II.3.17 (Motivic hyper-Kähler resolution conjecture [**FTV19**]). *Let $\mathcal{X}$ be a hyper-Kähler orbifold with underlying coarse moduli space a (singular) symplectic variety X. If there is a symplectic resolution $Y \to X$, we have an isomorphism $\mathfrak{h}(Y) \simeq \mathfrak{h}_{\mathrm{orb}}(\mathcal{X})$ of commutative algebra objects in* $\mathrm{CHM}_{\mathbf{C}}$*, hence in particular an isomorphism of bigraded* $\mathbf{C}$*-algebras* $\mathrm{CH}^*(Y,\bullet)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X},\bullet)_{\mathbf{C}}$.

Conjecture II.3.17 is the main subject of study in Chapter III, where several interesting evidences and ample applications are provided.

Remark II.3.18 (Relations between conjectures). For hyper-Kähler orbifolds $\mathcal{X}$ and $\mathcal{X}'$, consider the following three conditions:

(*i*) (K-equivalence) $\mathcal{X}$ and $\mathcal{X}'$ are birationally equivalent;
(*ii*) (D-equivalence) $\mathrm{D}^b(\mathcal{X}) \simeq \mathrm{D}^b(\mathcal{X}')$;
(*iii*) (Motivic equivalence) $\mathfrak{h}_{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}} \simeq \mathfrak{h}_{\mathrm{orb}}(\mathcal{X}')_{\mathbf{C}}$ as algebra objects.

Then (*i*) $\Longrightarrow$ (*ii*) is Kawamata's DK-hypothesis conjecture [**Kaw02**] (generalized to quotient stacks in [**Kaw18**]).
(*ii*) $\Longrightarrow$ (*iii*) is the multiplicative Orlov conjecture II.3.15 generalized to the orbifold case.
(*i*) $\Longrightarrow$ (*iii*) is the motivic hyper-Kähler K-equivalence conjecture II.3.16. If one of the orbifold

[9] The author is working on the possibility to put a Frobenius algebra object structure on it.

[10] Once there is a natural Frobenius algebra object structure on the orbifold motives, the author believes the conjecture can be stated more strongly for an isomorphism as Frobenius algebra objects.

is a variety, then it is the motivic hyper-Kähler resolution conjecture II.3.17.
Therefore, in some sense, Conjecture II.3.16 is implied by the combination of Kawamata's conjecture and Conjecture II.3.15.

To be more concrete, let us give a particularly interesting situation. Let M be a projective holomorphic symplectic variety endowed with a faithful action of a finite group G by symplectic automorphisms. The quotient stack $[M/G]$ is a hyper-Kähler (or rather symplectic) orbifold. If Y is a symplectic (or equivalently crepant) resolution of the singular symplectic variety M/G, then one expects an equivalence $\mathrm{D}^b(Y) \simeq \mathrm{D}^b([M/G])$. It is indeed the case when the main component of the G-invariant Hilbert scheme $Y = G{-}\mathrm{Hilb}(M)$ is a symplectic resolution: such a derived equivalence was established by Bridgeland–King–Reid [**BKR01**, Corollary 1.3]. On the other hand, the motivic hyper-Kähler resolution conjecture II.3.17 (or the more general Conjecture II.3.16) predicts that the orbifold motive of $[M/G]$ endowed with the *orbifold product* is isomorphic to the motive of Y as algebra objects.

CHAPTER III

The motivic hyper-Kähler resolution conjecture

This chapter presents the general theory of orbifold motives and studies the relation between the motive of a hyper-Kähler orbifold and the motive of its crepant resolutions, as algebra objects. The motivic hyper-Kähler resolution conjecture II.3.17 discussed in §II.3.4 will be the main subject of study. The application in mind is getting information on the Chow motive (or Chow ring) of the resolution from that of the orbifold. This chapter is based on the papers [FTV19], [FT17], [FT19], [FN19].

III.1. Origin: Ruan's crepant resolution conjectures

From the topological string theory of orbifolds in [DHVW85], [DHVW86], one observes that the stringy topological invariants of an orbifold, *e.g.* the orbifold Euler number and the orbifold Hodge numbers, should be related to the corresponding invariants of a crepant resolution ([Bat98], [BD96], [Yas04], [LP04]). Ruan brings forward a much deeper relation and puts this relation into a bigger picture of *stringy topology / geometry*, relating the quantum cohomology theory of an orbifold to that of its crepant resolutions. More precisely, among other speculations, he makes in [Rua06] the following *cohomological crepant resolution conjecture* (see [BG09], [CR13] for more sophisticated versions).

Conjecture III.1.1. *Let $\mathcal{X}$ be a smooth compact complex orbifold with underlying (possibly singular) variety X. Assume that X verifies the* hard Lefschetz condition.[1] *If there is a crepant resolution $Y \to X$, there is an isomorphism $H^*_{qc}(Y, \mathbf{C}) \simeq H^*_{\mathrm{orb}}(\mathcal{X}, \mathbf{C})$ of graded $\mathbf{C}$-algebras.*

On the left-hand side, H^*_{qc} is the *quantum corrected* cohomology, whose underlying graded vector space is the same as the singular cohomology while the product is a modification of the cup product by the Gromov–Witten invariants of contracted rational curve classes; on the right-hand side, H^*_{orb} is the *orbifold cohomology* defined by Chen and Ruan in [CR04] for any complex orbifold $\mathcal{X}$. As a $\mathbf{Q}$-vector space, it is the cohomology of its inertia variety $H^*(I\mathcal{X})$ (with degrees shifted by some rational numbers called *age*), but it is endowed with a highly non-trivial ring structure coming from moduli spaces of curves mapping to X. An algebro-geometric treatment is contained in Abramovich–Graber–Vistoli's work [AGV08], based on the construction of the moduli stack of twisted stable maps in [AV02]. In the global quotient case,[2] some equivalent definitions are available: see for example [FG03], [JKK07], [Kim08]. We will give a detailed and down-to-earth definition as well as a generalization to the Chow setting in §III.2.

A special case of Conjecture III.1.1 is particularly interesting: when the crepant resolution Y is *hyper-Kähler* (or more generally *holomorphic symplectic*), since all the Gromov–Witten

[1] This condition was added later in Bryan and Graber's work [BG09], following the computations of [CIT09].

[2] An orbifold is called a global quotient if it is the quotient stack of a smooth projective variety by a *finite* group.

invariants of Y vanish, there are no quantum corrections at all. Moreover the hard Lefschetz condition is always satisfied in the hyper-Kähler situation. We get in this case the following *cohomological hyper-Kähler resolution conjecture* of Ruan [**Rua02**].

Conjecture III.1.2. *Let $\mathcal{X}$ be a smooth compact complex orbifold with Gorenstein underlying (possibly singular) variety X. If there is a crepant resolution $Y \to X$ with Y hyper-Kähler, we have an isomorphism $H^*(Y, \mathbf{C}) \simeq H^*_{\mathrm{orb}}(\mathcal{X}, \mathbf{C})$ of graded $\mathbf{C}$-algebras.*

One observes that the construction of the orbifold product can be expressed using algebraic correspondences (*cf.* [**AGV08**] and §III.2), so we can construct in an analogous way the orbifold Chow ring $\mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X})$, or even the orbifold higher Chow ring $\mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X}, \bullet)$ and even better, the orbifold Chow motive as an algebra object (see Definition III.2.4 for the global quotient case) for an algebraic orbifold $\mathcal{X}$. As the Chow-theoretic and motivic analogue of Conjecture III.1.2, we propose to investigate Conjecture II.3.17 discussed in §II.3.4. Let us restate it.

Conjecture III.1.3 (Motivic hyper-Kähler resolution conjecture). *Let $\mathcal{X}$ be an (algebraic) orbifold*[3] *with coarse moduli space a (singular) symplectic variety X. If there is a symplectic resolution $Y \to X$, there is an isomorphism $\mathfrak{h}(Y) \simeq \mathfrak{h}_{\mathrm{orb}}(\mathcal{X})$ of commutative algebra objects in the category of Chow motives with complex coefficients* $\mathrm{CHM}_{\mathbf{C}}$, *hence in particular an isomorphism* $\mathrm{CH}^*(Y, \bullet)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X}, \bullet)_{\mathbf{C}}$ *of bigraded $\mathbf{C}$-algebras.*

Recall that a *symplectic variety* is a normal complex algebraic variety whose smooth part admits a symplectic form whose pull-back to a/any resolution extends to a holomorphic 2-form. By Namikawa [**Nam01**], a normal variety is symplectic if and only if it has rational Gorenstein singularities and its smooth part admits a symplectic form. The main examples that we are dealing with are quotients of smooth symplectic variety by a finite group of symplectic automorphisms. A *symplectic resolution* or *hyper-Kähler resolution* of a singular symplectic variety is a resolution $f : Y \to X$ such that the pull-back of a symplectic form on the smooth part of X extends to a symplectic form on Y. Note that a resolution is symplectic if and only if it is *crepant*: $f^*\omega_X = \omega_Y$. The definition is independent of the choice of the symplectic form.

The construction of orbifold Chow motives and orbifold Chow rings will be the content of §III.2, first in the global quotient case following the author's joint work with Tian and Vial [**FTV19**], then in the general case following the author's joint work with Nguyen [**FN19**].

As most of the cases where we can solve Conjecture III.1.3 are when the orbifold is a global quotient, let us record the following special case of the motivic hyper-Kähler resolution conjecture.

Conjecture III.1.4 (MHRC: global quotient case). *Let M be a smooth projective holomorphic symplectic variety equipped with a faithful action of a finite group G by symplectic automorphisms of M. If Y is a symplectic resolution of the quotient variety M/G, we have an isomorphism*

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{\mathrm{orb}}([M/G]) \text{ in } \mathrm{CHM}_{\mathbf{C}}$$

of (commutative) algebra objects in the category of Chow motives with complex *coefficients. In particular, we have an isomorphism of bigraded $\mathbf{C}$-algebras*

$$\mathrm{CH}^*(Y, \bullet)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}([M/G], \bullet)_{\mathbf{C}}.$$

[3] Recall our convention that an orbifold is a smooth proper Deligne–Mumford stack with trivial generic stabilizer and with projective underlying coarse moduli space.

Remark III.1.5 (Relation to birational equivalence). The argument of Rieß in [Rie14] implies that birationally equivalent projective hyper-Kähler varieties have isomorphic Chow motives as algebra objects. We remark that the motivic hyper-Kähler resolution conjecture can be viewed as claiming that the same result holds true more generally for hyper-Kähler *orbifolds*; see §II.3.4. Indeed, the symplectic resolution Y and the orbifold $\mathcal{X}$ are birationally equivalent in the category of Deligne–Mumford stacks and the natural notion of motive (as an algebra object) for an orbifold is the orbifold motive. The author wonders whether the approach of Rieß [Rie14] (or rather the essential tool [Huy99]) somehow works at least partially in the orbifold setting in order to give new evidence to the motivic hyper-Kähler resolution conjecture III.1.3 or the more general motivic hyper-Kähler K-equivalence conjecture II.3.16.

III.2. The orbifold product

We define in this section the orbifold motive and the orbifold product on it. We start with the usual Chow ring in the global quotient case. In this setting, the definition is particularly down-to-earth.

III.2.1. Global quotients. Let M be an m-dimensional smooth projective complex variety equipped with a faithful action of a finite group G. We adapt the constructions in [FG03] and [JKK07] to define the orbifold motive of the smooth proper Deligne–Mumford stack $[M/G]$. For any $g \in G$, $M^g := \{x \in M \mid gx = x\}$ is the fixed locus of the automorphism g, which is a smooth subvariety of M. For simplicity, we always assume that $g_* \in \mathrm{SL}(T_xM)$ for all $x \in M^g$, so that the *age* function defined below takes integer values.

Definition III.2.1 (Age [Rei02]). Given an element $g \in G$, let $r \in \mathbf{N}$ be its order. The *age* of g, denoted by $\mathrm{age}(g)$, is the locally constant function on M^g defined as follows. Let Z be a connected component of M^g. Choosing any point $x \in Z$, we have the induced automorphism $g_* \in \mathrm{SL}(T_xM)$, whose eigenvalues, repeated according to multiplicities, are

$$\left\{e^{2\pi\sqrt{-1}\frac{\alpha_1}{r}}, \dots, e^{2\pi\sqrt{-1}\frac{\alpha_m}{r}}\right\},$$

with $0 \le \alpha_i \le r-1$. One defines

$$\mathrm{age}(g)|_Z := \frac{1}{r}\sum_{i=1}^{m} \alpha_i.$$

It is obvious that the value of $\mathrm{age}(g)$ on Z is independent of the choice of $x \in Z$ and it takes values in $\mathbf{N}$, since $g_* \in \mathrm{SL}(T_xM)$. Also immediate from the definition, we have $\mathrm{age}(g) + \mathrm{age}(g^{-1}) = \mathrm{codim}(M^g \subset M)$ as locally constant functions. Thanks to the natural isomorphism $h : M^g \to M^{hgh^{-1}}$ sending x to $h.x$, for any $g, h \in G$, the age function is invariant under conjugation.

Example III.2.2. Let S be a smooth projective variety of dimension d and let n be a positive integer. The symmetric group $\mathfrak{S}_n$ acts by permutation on $M = S^n$. For each $g \in \mathfrak{S}_n$, a straightforward computation shows that $\mathrm{age}(g)$ is the constant function $\frac{d}{2}(n - |O(g)|)$, where $O(g)$ is the set of orbits of the permutation g of $\{1, \dots, n\}$. For example, when S is a surface (*i.e.*, $d = 2$), the age is always a non-negative integer and we have $\mathrm{age}(\mathrm{id}) = 0$, $\mathrm{age}(12\dots r) = r-1$, $\mathrm{age}(12)(345) = 3$ *etc.*

Definition III.2.3 (Orbifold Chow motive [FTV19]). We define first of all an auxiliary (in general non-commutative) algebra object $\mathfrak{h}(M, G)$ of CHM in several steps:

(i) As a Chow motive, $\mathfrak{h}(M,G)$ is the direct sum over G of the motives of fixed loci twisted à la Tate by minus the age:

$$\mathfrak{h}(M,G) := \bigoplus_{g\in G} \mathfrak{h}(M^g)\,(-\operatorname{age}(g))\,.$$

(ii) $\mathfrak{h}(M,G)$ is equipped with a natural G-action: each element $h\in G$ induces for each $g\in G$ an isomorphism $h: M^g \to M^{hgh^{-1}}$ by sending x to $h.x$, hence an isomorphism between the direct summands $\mathfrak{h}(M^g)(-\operatorname{age}(g))$ and $\mathfrak{h}(M^{hgh^{-1}})(-\operatorname{age}(hgh^{-1}))$ by the conjugation invariance of the age function.

(iii) For any $g\in G$, let r be its order. We have a natural automorphism g_* of the vector bundle $TM|_{M^g}$. Consider its eigen-subbundle decomposition:

$$TM|_{M^g} = \bigoplus_{j=0}^{r-1} W_{g,j},$$

where $W_{g,j}$ is the subbundle associated to the eigenvalue $e^{2\pi\sqrt{-1}\frac{j}{r}}$. Define

$$S_g := \sum_{j=0}^{r-1} \frac{j}{r}[W_{g,j}] \in K_0(M^g)_{\mathbf{Q}}.$$

Note that the virtual rank of S_g is nothing but $\operatorname{age}(g)$ by Definition III.2.1.

(iv) For any $g_1, g_2 \in G$, let $M^{<g_1,g_2>} = M^{g_1}\cap M^{g_2}$ and $g_3 = g_2^{-1}g_1^{-1}$. Define the following element in $K_0(M^{<g_1,g_2>})_{\mathbf{Q}}$:

$$F_{g_1,g_2} := S_{g_1}\big|_{M^{<g_1,g_2>}} + S_{g_2}\big|_{M^{<g_1,g_2>}} + S_{g_3}\big|_{M^{<g_1,g_2>}} + TM^{<g_1,g_2>} - TM|_{M^{<g_1,g_2>}}\,.$$

Note that its virtual rank is

$$\operatorname{rank} F_{g_1,g_2} = \operatorname{age}(g_1)+\operatorname{age}(g_2)-\operatorname{age}(g_1g_2)-\operatorname{codim}(M^{<g_1,g_2>}\subset M^{g_1g_2}). \tag{III.1}$$

In fact, this class in the Grothendieck group is represented by an obstruction vector bundle constructed in [**FG03**] (*cf.* [**JKK07**]). In particular, $\operatorname{age}(g_1)+\operatorname{age}(g_2)-\operatorname{age}(g_1g_2)$ is always an integer.

(v) The product structure $\star_{\mathrm{orb}}$ on $\mathfrak{h}(M,G)$ is defined to be multiplicative with respect to the G-grading and for each $g_1, g_2\in G$, the *orbifold product*

$$\star_{\mathrm{orb}} : \mathfrak{h}(M^{g_1})(-\operatorname{age}(g_1))\otimes \mathfrak{h}(M^{g_2})(-\operatorname{age}(g_2)) \to \mathfrak{h}(M^{g_1g_2})(-\operatorname{age}(g_1g_2))$$

is the correspondence determined by the algebraic cycle

$$\delta_*(c_{top}(F_{g_1,g_2})) \in \mathrm{CH}^{\dim M^{g_1}+\dim M^{g_2}+\operatorname{age}(g_1)+\operatorname{age}(g_2)-\operatorname{age}(g_1g_2)}(M^{g_1}\times M^{g_2}\times M^{g_1g_2}),$$

where $\delta: M^{<g_1,g_2>} \to M^{g_1}\times M^{g_2}\times M^{g_1g_2}$ is the natural morphism sending x to (x,x,x) and c_{top} means the top Chern class of F_{g_1,g_2}. One can easily check that the product structure $\star_{\mathrm{orb}}$ is invariant under the action of G.

(vi) The associativity of $\star_{\mathrm{orb}}$ is non-trivial. The proof in [**JKK07**, Lemma 5.4] is completely algebraic hence also works in our motivic case.

(*vii*) Finally, the *orbifold Chow motive* of $[M/G]$, denoted by $\mathfrak{h}_{\mathrm{orb}}([M/G])$, is the G-invariant subalgebra object

$$\mathfrak{h}_{\mathrm{orb}}([M/G]) := \mathfrak{h}(M,G)^G = \left(\bigoplus_{g\in G} \mathfrak{h}(M^g)\left(-\operatorname{age}(g)\right), \star_{\mathrm{orb}}\right)^G \tag{III.2}$$

of $\mathfrak{h}(M,G)$, which turns out to be commutative.

We still use $\star_{\mathrm{orb}}$ to denote the orbifold product on this subalgebra object $\mathfrak{h}_{\mathrm{orb}}([M/G])$.

The definition of the orbifold Chow ring then follows in the standard way and agrees with the one in [**FG03**], [**JKK07**] and [**AGV08**].

Definition III.2.4 (Orbifold Chow ring). The setting is as before. The *orbifold Chow ring* of $[M/G]$ is the commutative graded $\mathbf{Q}$-algebra $\mathrm{CH}^*_{\mathrm{orb}}([M/G]) := \bigoplus_i \mathrm{CH}^i_{\mathrm{orb}}([M/G])$ with

$$\mathrm{CH}^i_{\mathrm{orb}}([M/G]) := \mathrm{Hom}_{\mathrm{CHM}}(\mathbb{1}(-i), \mathfrak{h}_{\mathrm{orb}}([M/G])). \tag{III.3}$$

The ring structure on $\mathrm{CH}^*_{\mathrm{orb}}([M/G])$, called the *orbifold product*, denoted again by $\star_{\mathrm{orb}}$, is determined by the product structure $\star_{\mathrm{orb}} : \mathfrak{h}_{\mathrm{orb}}([M/G]) \otimes \mathfrak{h}_{\mathrm{orb}}([M/G]) \to \mathfrak{h}_{\mathrm{orb}}([M/G])$ in Definition III.2.3. More concretely, $\mathrm{CH}^*_{\mathrm{orb}}([M/G])$ is the G-invariant subalgebra of an auxiliary (non-commutative) finitely graded $\mathbf{Q}$-algebra $\mathrm{CH}^*(M,G)$, which is defined by

$$\mathrm{CH}^*(M,G) := \left(\bigoplus_{g\in G} \mathrm{CH}^{*-\operatorname{age}(g)}(M^g), \star_{orb}\right),$$

where for two elements $g,h \in G$ and $\alpha \in \mathrm{CH}^{i-\operatorname{age}(g)}(M^g)$, $\beta \in \mathrm{CH}^{j-\operatorname{age}(h)}(M^h)$, their orbifold product is the following element

$$\alpha \star_{\mathrm{orb}} \beta := \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right), \tag{III.4}$$

in $\mathrm{CH}^{i+j-\operatorname{age}(gh)}(M^{gh})$, where $\iota : M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion.

Similarly, the orbifold Grothendieck ring is defined as follows.

Definition III.2.5 (Orbifold Grothendieck ring). The *orbifold Grothendieck ring* of $[M/G]$, denoted by $K_0^{\mathrm{orb}}([M/G])$, is the sub-algebra of G-invariant elements of the $\mathbf{Q}$-algebra $K_0(M,G)$, which is defined by

$$K_0(M,G) := \left(\bigoplus_{g\in G} K_0(M^g), \star_{\mathrm{orb}}\right),$$

where for elements $g,h \in G$ and $\alpha \in K_0(M^g)$, $\beta \in K_0(M^h)$, their orbifold product is the following element in $K_0(M^{gh})$:

$$\alpha \star_{\mathrm{orb}} \beta := \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot \lambda_{-1}(F^\vee_{g,h})\right),$$

where $\iota : M^{<g,h>} \hookrightarrow M^{gh}$ is the natural inclusion and $\lambda_{-1}(F^\vee_{g,h})$ is the *K-theoretic Euler class* of $F_{g,h}$.

III.2.2. General case. In the author's joint work [**FN19**] with Manh Toan Nguyen, the above orbifold Chow ring and orbifold Grothendieck ring were generalized respectively to the higher orbifold Chow ring and higher orbifold K-theory; more importantly, they were extended to all (algebraic) orbifolds. By [**EHKV01**, Theorem 2.18], all algebraic orbifolds are quotients of an algebraic space by a linear algebraic group. The idea is then to present the orbifold as such a quotient and use equivariant geometry to define the orbifold theories.

For simplicity, let us assume that the orbifold $\mathcal{X}$ is the quotient of a smooth projective variety X by the action of a linear algebraic group G.

Let $\mathrm{CH}^*_G(X,\bullet) = \bigoplus_n \mathrm{CH}^*_G(X,n)$ be the equivariant (higher) Chow ring of Totaro [**Tot99**] and Edidin–Graham [**EG98**]. Similarly, $K^G_\bullet(X) := K_\bullet([X/G])$ is the equivariant (higher) K-theory, namely, the K-theory of the exact category of G-equivariant vector bundles on X. Let us state the main result of [**FN19**] as follows.

Theorem III.2.6 ([**FN19**]). *Let X be a smooth projective complex variety with an action of a linear algebraic group G. Denote by $I_G(X) := \{(g,x) \in G \times X \mid gx = x\}$ the inertia variety, endowed with the natural G-action given by $h.(g,x) = (hgh^{-1}, hx)$ for all $h \in G$ and $(g,x) \in I_G(X)$. Assume that the action has finite stabilizer,* i.e. *$I_G(X) \to X$ is a finite morphism. Let $\mathcal{X} := [X/G]$ denote the quotient Deligne–Mumford stack and let $I\mathcal{X} := [I_G(X)/G]$ be its inertia stack. Then*

(*i*) *On the equivariant higher Chow group $\mathrm{CH}^*_G(I_G(X),\bullet)$, there is an orbifold product $\star_{c_{\mathbb{T}}}$ which makes it into a (graded) commutative and associative bigraded ring. The orbifold product is independent of the choice of the presentation of the stack $\mathcal{X}$.*

(*ii*) *On the equivariant algebraic K-theory $K^G_\bullet(I_G(X))$, there is an orbifold product $\star_{\mathcal{E}_{\mathbb{T}}}$ which makes it into a (graded) commutative and associative graded ring. The orbifold product is independent of the choice of the presentation of the stack $\mathcal{X}$.*

(*iii*) *There is a natural graded ring homomorphism*

$$\mathfrak{ch} : K^G_\bullet(I_G(X)) \longrightarrow \mathrm{CH}^*_G(I_G(X),\bullet)$$

with respect to the orbifold products, called the orbifold (higher) Chern character map. *It induces an isomorphism*

$$\mathfrak{ch} : K^G_\bullet(I_G(X))^\wedge \xrightarrow{\simeq} \mathrm{CH}^*_G(I_G(X),\bullet),$$

where the left-hand side is the completion with respect to the augmentation ideal of the representation ring of G.

Definition III.2.7. The *orbifold higher Chow ring* of the stack $\mathcal{X}$, denoted by $\mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X},\bullet)$, is the ring $\mathrm{CH}^*_G(I_G(X),\bullet)$. Similarly, the *orbifold K-theory* of the stack $\mathcal{X}$, denoted by $K^{\mathrm{orb}}_\bullet(\mathcal{X})$, is $K^G_\bullet(I_G(X))$.

We refer to [**FN19**, §5] for details on the construction of these orbifold products. The techniques we use, called logarithmic trace and twisted pull-backs, are due to Edidin–Jarvis–Kimura [**EJK10**] (extending the work [**JKK07**]).

III.3. Main results

In the subsequent sections, we will give the precise statements and some ideas of the proof of several cases of the motivic hyper-Kähler resolution conjecture III.1.3, established in the author's series of joint work with Nguyen, Tian, and Vial. The main results are that Conjecture III.1.3 holds true in the following situations. There are isomorphisms of algebra objects in the category of *complex* Chow motives

- between the orbifold motive of a 2-dimensional Gorenstein orbifold and the motive of its minimal resolution [FT19] (note that here we do not even need to impose the hyper-Kähler condition, but only the crepant one; the reason for that is Lemma III.4.1);
- between the motive of the n-th Hilbert scheme of an abelian surface A and the orbifold motive of the quotient stack $[A^n/\mathfrak{S}_n]$ [FTV19];
- between the motive of the n-th generalized Kummer variety associated to an abelian surface A and the orbifold motive of the quotient stack $[\ker(A^{n+1} \to A)/\mathfrak{S}_{n+1}]$ [FTV19];
- between the motive of the n-th Hilbert scheme of a K3 surface S and the orbifold motive of the quotient stack $[S^n/\mathfrak{S}_n]$ [FT17].

In particular, in each case, there is an isomorphism of bigraded $\mathbf{C}$-algebras between the higher Chow ring of the resolution and higher orbifold Chow ring of the stack, with $\mathbf{C}$-coefficients [FN19].

Remark III.3.1 (Coefficient fields and discrete torsion). One can replace the coefficient field $\mathbf{C}$ by $\mathbf{Q}(\sqrt{-1})$ in the last three cases, because the only (non-real) complex coefficients appearing in the proof are $\pm\sqrt{-1}$. We want to warn the reader that the statements are no longer true if we use rational coefficients! However, a sign change in the definition of the orbifold product will give the so-called orbifold Chow motive (*resp.* Chow ring, K-theory, cohomology) *with discrete torsion*, denoted by $\mathfrak{h}_{\mathrm{orb,dt}}$, $\mathrm{CH}^*_{\mathrm{orb,dt}}$ *etc.* and the results can be restated with rational coefficients. For example, there is an isomorphism of algebra objects $\mathfrak{h}\left(S^{[n]}\right) \simeq \mathfrak{h}_{\mathrm{orb,dt}}([S^n/\mathfrak{S}_n])$ in CHM and isomorphisms of $\mathbf{Q}$-algebras $\mathrm{CH}^*\left(S^{[n]}\right) \simeq \mathrm{CH}^*_{\mathrm{orb,dt}}([S^n/\mathfrak{S}_n])$ for all projective K3 surface S. For a more careful treatment of discrete torsions, see [JKK07, §7] and the work [FTV19] where the main results are stated in this context.

III.4. The surface case

We revisit the dimension-2 case of Ruan's cohomological crepant resolution conjecture III.1.1. Let $\mathcal{X}$ be a proper 2-dimensional orbifold, that is, a smooth proper Deligne–Mumford stack of dimension 2 with isolated stacky points. Assume that the underlying singular surface X has only Gorenstein (or equivalently ADE) singularities. Denote by $Y \to X$ the minimal resolution; it is crepant.

The key observation is that in the two-dimensional situation, the Gromov—Witten invariants always vanish by Lemma III.4.1 below, hence there are no quantum corrections involved. As a result, the left-hand side of Conjecture III.1.1 is the usual singular cohomology and Conjecture III.1.1 predicts an isomorphism $H^*(Y, \mathbf{C}) \simeq H^*_{\mathrm{orb}}(\mathcal{X}, \mathbf{C})$ of graded $\mathbf{C}$-algebras.

Lemma III.4.1. *Let X be a surface with ADE singularities and $\pi : Y \to X$ be the minimal resolution. Then the virtual fundamental class of $\overline{M_{0,3}}(Y, \beta)$ is rationally equivalent to zero for any curve class β which is contracted by π.*

Proof. Consider the forgetful-stabilization morphism

$$f : \overline{M_{0,3}}(Y, \beta) \to \overline{M_{0,0}}(Y, \beta).$$

By the general theory, the virtual fundamental class of $\overline{M_{0,3}}(Y, \beta)$ is the pull-back of the virtual fundamental class of $\overline{M_{0,0}}(Y, \beta)$. However, the virtual dimension of $\overline{M_{0,0}}(Y, \beta)$ is

$(\beta \cdot K_Y) + (\dim Y - 3) = -1$ since π is crepant. Therefore, both moduli spaces have zero virtual fundamental class in Chow group, cohomology, or K-theory. $\square$

Remark III.4.2 (Relation to the McKay correspondence). Let $G \subset \mathrm{SL}_2(\mathbf{C})$ be a (non-trivial) finite subgroup acting naturally on the vector space $V := \mathbf{C}^2$. The quotient $X := V/G$ has a unique A-D-E singularity. Let $Y \to X$ be the minimal resolution. The exceptional divisor E consists of a union of (-2)-curves meeting transversally. The classical McKay correspondence (*cf.* [Rei02]) establishes a bijection

$$\begin{array}{ccc} \mathrm{Irr}'(G) & \xrightarrow{\sim} & \mathrm{Irr}(E) \\ \rho & \mapsto & E_\rho. \end{array}$$

between the set $\mathrm{Irr}'(G)$ of non-trivial irreducible representations of G on the one hand and the set $\mathrm{Irr}(E)$ of irreducible components of E on the other hand. Thus $E = \bigcup_{\rho \in \mathrm{Irr}'(G)} E_\rho$. Following Reid [Rei97], one can recast the above McKay correspondence (the bijection) as an isomorphism

$$H^*(Y) \simeq H^*(|I\mathcal{X}|)$$

of vector spaces. Now observe that the orbifold cohomology ring $H^*_{\mathrm{orb}}([V/G])$ has $H^*(|I\mathcal{X}|)$ as underlying vector space. Therefore we have an isomorphism

$$H^*(Y) \simeq H^*_{\mathrm{orb}}([V/G]),$$

where both sides have a natural product structure, namely the cup product on the left-hand side and the orbifold product on the right-hand side.

The "local case" of the cohomological crepant resolution conjecture III.1.1 asks whether there is an isomorphism which respects these multiplicative structures. The existence of such an isomorphism of algebras is known: it is a baby case of the result of Ginzburg–Kaledin [GK04] on symplectic resolutions of symplectic quotient singularities.

The main result of this section, established in a joint work with Tian [FT19], is that Conjecture III.1.1 is also true on the motivic level for surfaces. This confirms in particular the two-dimensional motivic hyper-Kähler resolution conjecture. According to Remark III.4.2, it can also be seen as a motivic multiplicative McKay correspondence.

Theorem III.4.3 (Motivic multiplicative McKay correspondence [FT19]). *Let $\mathcal{X}$ be a smooth proper two-dimensional Deligne–Mumford stack with isolated stacky points. Assume that $\mathcal{X}$ has projective coarse moduli space X with Gorenstein singularities. Let $Y \to X$ be the minimal resolution. Then we have an isomorphism*

$$\mathfrak{h}(Y)_{\mathbf{C}} \simeq \mathfrak{h}_{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}}. \tag{III.5}$$

of algebra objects in the category $\mathrm{CHM}_{\mathbf{C}}$ of Chow motives with complex coefficients. In particular, one has an isomorphism

$$\mathrm{CH}^*(Y, \bullet)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}(\mathcal{X}, \bullet)_{\mathbf{C}}$$

of bigraded $\mathbf{C}$-algebras.

III.4.1. Idea of the proof. We only present the proof in the global quotient case and refer to [FT19] for the general case.

Let S be a smooth projective surface and let G be a finite group acting faithfully on S such that the canonical bundle is locally preserved. Let $X := S/G$ be the quotient surface (with ADE singularities) and let $Y \to X$ be the minimal (crepant) resolution.

For any $x \in S$, let

$$G_x := \{g \in G \mid gx = x\}$$

be the stabilizer. Let $\mathrm{Irr}(G_x)$ be the set of isomorphism classes of irreducible representations of G_x and let $\mathrm{Irr}'(G_x)$ be the subset of non-trivial ones.

On the resolution side, using the motivic decomposition of de Cataldo–Migliorini [dCM04, Theorem 1.0.1], we see that

$$\mathfrak{h}(Y) \simeq \mathfrak{h}(S)^G \oplus \bigoplus_{\overline{x}\in S/G} \bigoplus_{\rho\in\mathrm{Irr}'(G_x)} \mathbb{L}_{\overline{x},\rho} \simeq \left(\mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\rho\in\mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho}\right)^G, \tag{III.6}$$

where $\mathbb{L}_{\overline{x},\rho}$ is a copy of Lefschetz motive indexed by $\overline{x}$ and ρ.

The product structure of $\mathfrak{h}(Y)$ is determined as follows via the above decomposition (III.6), which also uses the classical McKay correspondence. Let $i_x : \{x\} \hookrightarrow S$ be the natural inclusion.

- $\mathfrak{h}(S) \otimes \mathfrak{h}(S) \xrightarrow{\delta_S} \mathfrak{h}(S)$ is the usual product induced by the small diagonal of S^3.
- For any x with non-trivial stabilizer G_x and any $\rho \in \mathrm{Irr}'(G_x)$,

  $$\mathfrak{h}(S) \otimes \mathbb{L}_{x,\rho} \xrightarrow{i_x^*} \mathbb{L}_{x,\rho}$$

  is determined by the class $x \in \mathrm{CH}^2(S) = \mathrm{Hom}(\mathfrak{h}(S) \otimes \mathbb{L}, \mathbb{L})$.
- For any $\rho \in \mathrm{Irr}'(G_x)$ as above,

  $$\mathbb{L}_{x,\rho} \otimes \mathbb{L}_{x,\rho} \xrightarrow{-2i_{x,*}} \mathfrak{h}(S),$$

  is determined by $-2x \in \mathrm{CH}^2(S)$. The reason is that each component of the exceptional divisor is a smooth rational curve of self-intersection number equal to -2.
- For any $\rho_1 \neq \rho_2 \in \mathrm{Irr}'(G_x)$,
  - If they are *adjacent*, that is, ρ_1 appears (with multiplicity 1) in the G_x-module $\rho_2 \otimes T_xS$, then by the classical McKay correspondence, the components in the exceptional divisor over $\overline{x}$ indexed by ρ_1 and ρ_2 intersect transversally at one point. Therefore

    $$\mathbb{L}_{x,\rho_1} \otimes \mathbb{L}_{x,\rho_2} \xrightarrow{i_{x,*}} \mathfrak{h}(S),$$

    is determined by $x \in \mathrm{CH}^2(S)$.
  - If they are not adjacent, the classical McKay correspondence again tells us that the two components indexed by ρ_1 and ρ_2 of the exceptional divisor do not intersect; hence $\mathbb{L}_{x,\rho_1} \otimes \mathbb{L}_{x,\rho_2} \xrightarrow{0} \mathfrak{h}(S)$ is the zero map.
- The other multiplication maps are zero.

The G-action on (III.6) is as follows:

- The G-action of $\mathfrak{h}(S)$ is induced by the original action on S.
- For any $h \in G$, it maps for any $x \in S$ and $\rho \in \mathrm{Irr}'(G_x)$, the Lefschetz motive $\mathbb{L}_{x,\rho}$ isomorphically to $\mathbb{L}_{hx,h\rho}$, where $h\rho \in \mathrm{Irr}'(G_{hx})$ is the representation which makes the

diagram

$$\begin{tikzcd} G_x \arrow[rr, "g\mapsto hgh^{-1}", "\simeq"'] \arrow[dr, "\rho"'] & & G_{hx} \arrow[dl, "h\rho"] \\ & V_\rho & \end{tikzcd} \tag{III.7}$$

commute.

On the orbifold side, $\mathfrak{h}_{\mathrm{orb}}([S/G])$ is the G-invariant part of the algebra object $\mathfrak{h}(S,G)$ defined as

$$\mathfrak{h}(S,G) = \mathfrak{h}(S) \oplus \bigoplus_{\substack{g\in G\\ g\neq \mathrm{id}}} \bigoplus_{x\in S^g} \mathbb{L}_{x,g} = \mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g}, \tag{III.8}$$

where $\mathbb{L}_{x,g}$ is the Lefschetz motive $\mathbb{1}(-1)$ indexed by the fixed point x of g.

Let us determine the product structure: first, the obstruction class is easily computed [**FT19**, Lemma 3.1]; namely, for any $g,h \in G$ different from id, the obstruction class is

$$c_{g,h} = \begin{cases} 1 & \text{if} \quad g = h^{-1} \\ 0 & \text{if} \quad g \neq h^{-1} \end{cases}.$$

Therefore, the orbifold product on $\mathfrak{h}(S,G)$ is given as follows via the decomposition (III.8):

$$\begin{aligned} \mathfrak{h}(S)\otimes\mathfrak{h}(S) &\xrightarrow{\delta_S} \mathfrak{h}(S); \\ \forall x\in S^g, \mathfrak{h}(S)\otimes\mathbb{L}_{x,g} &\xrightarrow{i_x^*} \mathbb{L}_{x,g}; \\ \mathbb{L}_{x,g}\otimes\mathbb{L}_{x,g^{-1}} &\xrightarrow{i_{x,*}} \mathfrak{h}(S), \end{aligned}$$

where the first morphism is the usual product given by the small diagonal; the second and the third morphisms are given by the class $x \in \mathrm{CH}^2(S)$ and $i_x : \{x\} \hookrightarrow S$ is the natural inclusion; all the other possible maps are zero.

With both sides of the correspondence computed, we define the *multiplicative McKay correspondence* morphism

$$\Phi : \mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\rho\in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g} \tag{III.9}$$

in the category $\mathrm{CHM}_{\mathbb{C}}$ of complex Chow motives. It is given by the following 'block diagonal matrix':

- id : $\mathfrak{h}(S) \to \mathfrak{h}(S)$;
- For each $x \in S$ (with non-trivial stabilizer G_x), the morphism
$$\bigoplus_{\rho\in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g}$$
is the 'matrix' with entry $\frac{1}{\sqrt{|G_x|}}\sqrt{\chi_{\rho_0}(g) - 2}\cdot \chi_\rho(g)$ at place $(\rho, g) \in \mathrm{Irr}'(G_x) \times (G_x\backslash\{\mathrm{id}\})$, where χ denotes the character and ρ_0 is the natural 2-dimensional representation T_xS of G_x. Note that $\rho_0(g)$ has determinant 1, hence its trace $\chi_{\rho_0}(g)$ is a real number.
- The other morphisms are zero.

To conclude the proof, one has to show three things: (*i*) Φ is compatible with the G-action; (*ii*) Φ is multiplicative and (*iii*) Φ induces an isomorphism Φ^G of complex Chow motives on G-invariants. These can be checked in a straightforward way. We refer to [**FT19**] for details.

III.5. Hilbert schemes of abelian surfaces

The main result of this section is the following theorem. It is a collaboration with Tian and Vial [**FTV19**], extended in a joint work with Nguyen [**FN19**].

THEOREM III.5.1 (MHRC for $A^{[n]}$). *Let A be an abelian surface and let $A^{[n]}$ be its Hilbert scheme. Let the symmetric group $\mathfrak{S}_n$ act on A^n by permutations. Then we have an isomorphism*

$$\mathfrak{h}\left(A^{[n]}\right)_{\mathbf{C}} \simeq \mathfrak{h}_{\mathrm{orb}}\left([A^n/\mathfrak{S}_n]\right)_{\mathbf{C}}$$

of commutative algebra objects in the category $\mathrm{CHM}_{\mathbf{C}}$ *of Chow motives with complex coefficients. In particular, we have an isomorphism*

$$\mathrm{CH}^*\left(A^{[n]},\bullet\right)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}([A^n/\mathfrak{S}_n],\bullet)_{\mathbf{C}} \tag{III.10}$$

of commutative bigraded $\mathbf{C}$*-algebras.*

Let us give the main steps of the proof. Set $M = A^n$, $G = \mathfrak{S}_n$, and $Y = A^{[n]}$.

Step 1. Additive isomorphism. Recall the notation $\mathfrak{h}(M,G) := \oplus_{g\in G}\,\mathfrak{h}(M^g)(-\operatorname{age}(g))$ from Definition III.2.3. Denote by

$$\iota : \mathfrak{h}(M,G)^G \hookrightarrow \mathfrak{h}(M,G) \quad \text{and} \quad p : \mathfrak{h}(M,G) \twoheadrightarrow \mathfrak{h}(M,G)^G$$

the inclusion of and the projection onto the G-invariant part $\mathfrak{h}(M,G)^G$, which is a direct summand of $\mathfrak{h}(M,G)$ inside CHM.

Let

$$U^g := (A^{[n]} \times_{A^{(n)}} (A^n)^g)_{red} = \left\{(z,x_1,\dots,x_n) \in A^{[n]} \times (A^n)^g \;\middle|\; \rho(z) = [x_1]+\cdots+[x_n]\right\} \tag{III.11}$$

be the incidence variety, where $\rho : A^{[n]} \to A^{(n)}$ is the Hilbert–Chow morphism. As the notation suggests, U^g is the fixed locus of the induced automorphism g on the *isospectral Hilbert scheme*

$$U := U^{\mathrm{id}} = A^{[n]} \times_{A^{(n)}} A^n = \left\{(z,x_1,\dots,x_n) \in A^{[n]} \times A^n \;\middle|\; \rho(z) = [x_1]+\cdots+[x_n]\right\}.$$

Note that $\dim U^g = n + |O(g)| = 2n - \operatorname{age}(g)$ ([**Bri77**]) and $\dim\left(A^{[n]} \times (A^n)^g\right) = 2\dim U^g$.

Let us first construct an *a priori* just additive G-equivariant morphism

$$\Gamma = \sum_g \Gamma_g : \mathfrak{h}(Y) \to \mathfrak{h}(M,G) = \oplus_{g\in G}\,\mathfrak{h}(M^g)(-\operatorname{age}(g))$$

of complex Chow motives given by the correspondences $\left\{\Gamma_g := \sqrt{-1}^{\operatorname{age}(g)} U^g \in \mathrm{CH}(Y\times M^g)_{\mathbf{C}}\right\}_{g\in G}$ inducing an (additive) isomorphism

$$\phi = p\circ\Gamma : \mathfrak{h}(Y) \xrightarrow{\sim} \mathfrak{h}_{\mathrm{orb}}([M/G]) = \mathfrak{h}(M,G)^G.$$

The G-equivariance of Γ can be checked directly [**FTV19**, Lemma 5.1]; hence $\Gamma = \iota\circ\phi$.

Now the result of de Cataldo–Migliorini [**dCM02**] can be reformulated as follows.

PROPOSITION III.5.2. *The morphism ϕ is an isomorphism whose inverse is given by* $\psi := \frac{1}{|G|}{}^t\Gamma\circ\iota$.

See [FTV19, Proposition 5.2] for the proof.

Our goal is then to prove that these morphisms are moreover multiplicative, *i.e.* the diagram

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] \arrow[d, "\phi^{\otimes 2}"'] & \mathfrak{h}(Y) \arrow[d, "\phi"] \\ \mathfrak{h}_{\mathrm{orb}}([M/G])^{\otimes 2} \arrow[r, "\star_{\mathrm{orb}}"'] & \mathfrak{h}_{\mathrm{orb}}([M/G]) \end{tikzcd} \tag{III.12}$$

is commutative, where the algebra structure $\star_{\mathrm{orb}}$ on the Chow motive $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is the symmetrization of the algebra structure $\star_{\mathrm{orb}}$ on $\mathfrak{h}(M, G)$.

The main theorem will then be deduced from the following.

Proposition III.5.3. *Notation being as before, the following two algebraic cycles have the same symmetrization in* $\mathrm{CH}\left(\left(\coprod_{g\in G} M^g\right)^3\right)_{\mathbb{C}}$

- $W := \left(\frac{1}{|G|}\sum_g \Gamma_g \times \frac{1}{|G|}\sum_g \Gamma_g \times \sum_g \Gamma_g\right)_*(\delta_Y)$;
- *The algebraic cycle Z determining the orbifold product (Definition III.2.3(v)):*

$$Z|_{M^{g_1}\times M^{g_2}\times M^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1g_2, \\ \delta_* c_{top}(F_{g_1,g_2}) & \text{if } g_3 = g_1g_2. \end{cases}$$

Here the symmetrization *of a cycle in* $\left(\coprod_{g\in G} M^g\right)^3$ *is the operation*

$$\gamma \mapsto (p\otimes p\otimes p)_*\gamma = \frac{1}{|G|^3}\sum_{g_1,g_2,g_3\in G}(g_1,g_2,g_3)\gamma.$$

Proposition III.5.3 implies Theorems III.5.1. The only thing to show is the commutativity of (III.12), which is of course equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] & \mathfrak{h}(Y) \arrow[d, "\phi"] \\ \mathfrak{h}_{\mathrm{orb}}([M/G])^{\otimes 2} \arrow[r, "\star_{\mathrm{orb}}"'] \arrow[u, "\psi^{\otimes 2}"] & \mathfrak{h}_{\mathrm{orb}}([M/G]). \end{tikzcd}$$

By the definition of ϕ and ψ, we need to show that the diagram

$$\begin{tikzcd} \mathfrak{h}(Y)^{\otimes 2} \arrow[r, "\delta_Y"] & \mathfrak{h}(Y) \arrow[d, "\sum_g \Gamma_g"] \\ \mathfrak{h}(M,G)^{\otimes 2} \arrow[u, "(\frac{1}{|G|}\sum_g {}^t\Gamma_g)^{\otimes 2}"] & \mathfrak{h}(M,G) \arrow[d, "p"] \\ \mathfrak{h}_{\mathrm{orb}}([M/G])^{\otimes 2} \arrow[r, "\star_{\mathrm{orb}}"'] \arrow[u, "\iota^{\otimes 2}"] & \mathfrak{h}_{\mathrm{orb}}([M/G]) \end{tikzcd} \tag{III.13}$$

is commutative. By Lieberman's formula (see *e.g.* [And04, 3.1.4]), the composition $\sum_g \Gamma_g \circ \delta_Y \circ (\frac{1}{|G|}\sum_g {}^t\Gamma_g)^{\otimes 2}$ is the morphism (or correspondence) induced by the cycle W in Proposition

III.5.3. On the other hand, $\star_{\mathrm{orb}}$ for $\mathfrak{h}_{\mathrm{orb}}([M/G])$ is by definition $p\circ Z\circ\iota^{\otimes 2}$. Therefore, the desired commutativity, hence also the main result, amount to the equality $p\circ W\circ\iota^{\otimes 2}=p\circ Z\circ\iota^{\otimes 2}$, which says exactly that the symmetrizations of W and of Z are equal in $\mathrm{CH}\left(\left(\coprod_{g\in G}M^g\right)^3\right)$. ◻

One is therefore reduced to show Proposition III.5.3.

Step 2. Distinguishedness. Observe first that M^g are abelian varieties for all g. We prove that W and Z, as well as their symmetrizations, are symmetrically distinguished in the sense of O'Sullivan [**O'S11b**] (see Definition II.2.6).

Proposition III.5.4. *The algebraic cycles Z and W in Proposition III.5.3, as well as their symmetrizations, are symmetrically distinguished in* $\mathrm{CH}\left(\left(\coprod_{g\in G}M^g\right)^3\right)_{\mathbf{C}}$.

Proof. For W, it amounts to show that for any $g_1,g_2,g_3\in G$, the cycle $(U^{g_1}\times U^{g_2}\times U^{g_3})_*(\delta_{A^{[n]}})$ is symmetrically distinguished in $\mathrm{CH}((A^n)^{g_1}\times(A^n)^{g_2}\times(A^n)^{g_3})$. Indeed, by [**Voi15a**, Proposition 5.6], the cycle $(U^{g_1}\times U^{g_2}\times U^{g_3})_*(\delta_{A^{[n]}})$ is a polynomial in the big diagonals of $(A^n)^{g_1}\times(A^n)^{g_2}\times(A^n)^{g_3}=:A^N$. However, all big diagonals of A^N are clearly symmetrically distinguished since $\Delta_A\in\mathrm{CH}(A\times A)$ is. By Theorem II.2.7, W is symmetrically distinguished. As for Z, for any fixed $g_1,g_2\in G$, the vector bundle F_{g_1,g_2} is easily seen to be trivial, at least virtually, hence its top Chern class is either 0 or 1 (the fundamental class), which is of course symmetrically distinguished. Also recall that (Definition III.2.3)

$$\delta:(A^n)^{<g_1,g_2>}\hookrightarrow(A^n)^{g_1}\times(A^n)^{g_2}\times(A^n)^{g_1g_2},$$

which is a (partial) diagonal inclusion, is in particular a morphism of abelian varieties. Therefore, $\delta_*(c_{top}(F_{g_1,g_2}))$ is symmetrically distinguished by Theorem II.2.7, hence so is Z. Finally, since any automorphism in $G\times G\times G$ preserves symmetrically distinguished cycles, symmetrizations of Z and W remain symmetrically distinguished. ◻

By Theorem II.2.7, in order to show Proposition III.5.3, it suffices to show on the one hand that the symmetrizations of Z and W are both symmetrically distinguished, and on the other hand that they are numerically equivalent. The first part is exactly the previous Proposition III.5.4. So it remains to show that the symmetrizations of Z and W are numerically equivalent.

Step 3. Cohomological realization.

Proposition III.5.5. *The cohomology realization of the (additive) isomorphism*

$$\phi:\mathfrak{h}(A^{[n]})\xrightarrow{\sim}\left(\oplus_{g\in G}\mathfrak{h}((A^n)^g)(-\operatorname{age}(g))\right)^{\mathfrak{S}_n}$$

is an isomorphism

$$\overline{\phi}:H^*(A^{[n]},\mathbf{C})\xrightarrow{\simeq}H^*_{\mathrm{orb}}([A^n/\mathfrak{S}_n],\mathbf{C})$$

of $\mathbf{C}$*-algebras. In other words,* $\mathrm{Sym}(W)$ *and* $\mathrm{Sym}(Z)$ *are homologically equivalent.*

Proof. The existence of an isomorphism of $\mathbf{Q}$-algebras between the cohomology rings $H^*(A^{[n]})$ and $H^*_{\mathrm{orb}}([A^n/\mathfrak{S}_n])$ with some sign change in the orbifold product (discrete torsion) was established by Fantechi and Göttsche [**FG03**, Theorem 3.10] following the work of Lehn and Sorger [LS03]. Therefore by the definition of ϕ in Step 1, it suffices to show that the cohomological correspondence

$$\Gamma_* := \sum_{g\in\mathfrak{S}_n} (-1)^{\mathrm{age}(g)} {U^g}_* : H^*(A^{[n]}) \to \bigoplus_{g\in\mathfrak{S}_n} H^{*-2\,\mathrm{age}(g)}((A^n)^g)$$

coincides with the inverse

$$\begin{array}{rcl} \Phi : H^*(A^{[n]}) & \to & \bigoplus_{g\in\mathfrak{S}_n} H^{*-2\,\mathrm{age}(g)}((A^n)^g) \\ \mathfrak{p}_{\lambda_1}(\alpha_1)\dots\mathfrak{p}_{\lambda_l}(\alpha_l)\,\mathbb{1} & \mapsto & n!\cdot \mathrm{Sym}(\alpha_1\times\dots\times\alpha_l) \end{array}$$

of the isomorphism Ψ used in Fantechi–Göttsche [FG03, Theorem 3.10], where $\mathfrak{p}$ is the Nakajima operator [Nak99]. This is a rather straightforward computation. See [FTV19, Proposition 5.8] for details. □

Finally, once the theory of higher orbifold Chow ring is established ([FN19]), it suffices to apply the (higher) Chow-theoretic realization functor to go from an isomorphism of Chow motives to an isomorphism of higher Chow rings. The proof of Theorem III.5.1 is complete.

III.6. Generalized Kummer varieties

The main result of this section is the following. It is a collaboration with Tian and Vial [FTV19], extended in joint work with Nguyen [FN19].

THEOREM III.6.1 (MHRC for $K_n(A)$). *Let $K_n(A)$ be the $2n$-dimensional generalized Kummer variety associated with an abelian surface A. Endow $A_0^{n+1} := \mathrm{Ker}\left(+ : A^{n+1} \to A\right)$ with the natural $\mathfrak{S}_{n+1}$-action. Then we have an isomorphism*

$$\mathfrak{h}\left(K_n(A)\right)_{\mathbf{C}} \simeq \mathfrak{h}_{\mathrm{orb}}\left(\left[A_0^{n+1}/\,\mathfrak{S}_{n+1}\right]\right)_{\mathbf{C}}$$

of commutative algebra objects in the category of complex Chow motives $\mathrm{CHM}_{\mathbf{C}}$. In particular, we have an isomorphism

$$\mathrm{CH}^*\left(K_n(A),\bullet\right)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}\left(\left[A_0^{n+1}/\,\mathfrak{S}_{n+1}\right],\bullet\right)_{\mathbf{C}} \tag{III.14}$$

of commutative graded $\mathbf{C}$-algebras.

The proof proceeds in three steps as the proof of Theorem III.5.1 explained in §III.5, with each step requiring some extra arguments.

Set

$$M := A_0^{n+1} := \mathrm{Ker}\left(A^{n+1} \xrightarrow{+} A\right),$$

endowed with the canonical action of $G = \mathfrak{S}_{n+1}$ and set $X := A_0^{(n+1)} := M/G$. Then the restriction of the Hilbert–Chow morphism to the *generalized Kummer variety*

$$K_n(A) =: Y \xrightarrow{f} A_0^{(n+1)}$$

is a symplectic resolution.

Step 1. Additive isomorphism. In place of U^g, we consider for any $g\in G$ the analogous incidence correspondence :

$$V^g := (K_n(A)\times_{A_0^{(n+1)}} M^g)_{red} \subset K_n(A)\times M^g. \tag{III.15}$$

Note that $Y \to X$ is semismall. Instead of using [**dCM02**], one applies [**dCM04**] on the motivic decomposition of semismall morphisms to obtain that the analogue of Proposition III.5.2 holds. More precisely, $\phi = p \circ \Gamma : \mathfrak{h}(Y) \to \mathfrak{h}(M, G)^G$ is an isomorphism with inverse $\psi = \frac{1}{|G|}{}^t\Gamma \circ \iota$, where

$$\Gamma = \sum_g \Gamma_g : \mathfrak{h}(Y) \to \mathfrak{h}(M, G) = \oplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g))$$

is given by the correspondences $\left\{\Gamma_g := \sqrt{-1}^{\operatorname{age}(g)} V^g \in \mathrm{CH}(Y \times M^g)_{\mathbf{C}}\right\}_{g \in G}$.

Therefore, by exactly the same reasoning in §III.5, we are reduced to show Proposition III.5.3 that the cycles Z and W have the same symmetrization in $\mathrm{CH}\left(\left(\coprod_{g \in G} M^g\right)^3\right)_{\mathbf{C}}$. Note that our notation is chosen so that the cycles Z and W are defined by the same formula.

Step 2. Distinguishedness. The idea is again to show that the cycles Z and W are symmetrically distinguished. Observe that M^g is in general not connected, but each connected component is isomorphic to an abelian variety. However, the problem is that there is no canonical choice of the origin and the notion of symmetric distinguishedness depends crucially on this choice (see Definition II.2.6, or simply think about symmetric divisors!). The key observation in [**FTV19**] is that these connected components have nevertheless a canonical choice of the subset of torsion points (without knowing which element is the origin) and we can generalize O'Sullivan's theory of symmetrically distinguished cycles on abelian varieties to the broader setting of so-called *abelian torsors with torsion structures*, which is, roughly speaking, a variety that is isomorphic to an abelian variety, together with a choice of the subset of torsion points.

We skip the formal definition of the category of abelian torsors with torsion structures and refer to [**FTV19**, §6.2]. Once the basic theory of such geometric objects is settled, by using the fact that torsion translations act trivially on rational Chow groups of abelian varieties [**FTV19**, Lemma 6.7], one checks in a straightforward way that the notion of symmetrically distinguished cycles naturally extends to this setting and the main result of O'Sullivan Theorem II.2.7 holds more generally for abelian torsors with torsion structures.

Now observe that each connected component of M^g is canonically an abelian torsor with torsion structure and the analogue of Proposition III.5.4, which says that Z and W are symmetrically distinguished, can be proved similarly.

Step 3. Cohomological realization. As in §III.5, by Step 2, it suffices to check that the symmetrizations of W and Z are numerically equivalent. Instead of using [**FG03**] and [**LS03**], we use Nieper–Wißkirchen's following description [**NW09**] of the cohomology ring $H^*(K_n(A), \mathbf{C})$ to show the analogue of Proposition III.5.5.

Let $s : A^{[n+1]} \to A$ be the composition of the Hilbert–Chow morphism followed by the summation map. Recall that s is an isotrivial fibration. We have a commutative diagram (all cohomology groups are with complex coefficients)

$$\begin{tikzcd} H^*(A) \arrow[r, "s^*"] \arrow[d, "\varepsilon"'] & H^*(A^{[n]}) \arrow[d, "restr."] \\ \mathbf{C} \arrow[r] & H^*(K_n(A)), \end{tikzcd}$$

where the upper arrow s^* is the pull-back by s, the lower arrow is the unit map sending 1 to the fundamental class $1_{K_n(A)}$, the map ε is the quotient by the ideal consisting of elements of positive degree, and the right arrow is the restriction map. The commutativity comes from the fact that $K_n(A) = s^{-1}(O_A)$ is a fiber. Thus one has a ring homomorphism

$$R : H^*(A^{[n]}) \otimes_{H^*(A)} \mathbf{C} \to H^*(K_n(A)).$$

Then [**NW09**, Theorem 1.7] asserts that this is an isomorphism of $\mathbf{C}$-algebras. Using this description of the algebra $H^*(K_n(A))$, one can show that ϕ induces an algebra isomorphism on cohomology. We refer to [**FTV19**, §6.3] for the rest of the proof.

III.7. Hilbert schemes of K3 surfaces

The main result of this section is the following. This is joint work with Tian [**FT17**], extended in joint work with Nguyen [**FN19**].

THEOREM III.7.1 (MHRC for Hilbert schemes of K3 surfaces). *Let S be a projective K3 surface. Let the symmetric group $\mathfrak{S}_n$ act on S^n by permutations. We have an isomorphism*

$$\mathfrak{h}\left(S^{[n]}\right)_{\mathbf{C}} \simeq \mathfrak{h}_{\mathrm{orb}}\left([S^n/\,\mathfrak{S}_n]\right)_{\mathbf{C}} \quad \textit{in} \quad \mathrm{CHM}_{\mathbf{C}}$$

of algebra objects in the category of complex Chow motives. In particular, there are isomorphisms

$$\mathrm{CH}^*\left(S^{[n]}, \bullet\right)_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}\left([S^n/\,\mathfrak{S}_n], \bullet\right)_{\mathbf{C}}$$

of bigraded $\mathbf{C}$-algebras.

The proof follows a similar pattern as the proof of Theorem III.5.1 explained in §III.5. For some reason which will become clear later in Step 4, we want to consider not only K3 surface but *all* smooth projective simply-connected surfaces. Let S be such a surface and set $M = S^n$, $G = \mathfrak{S}_n$ and $Y = S^{[n]}$.

Step 1. Additive isomorphism. By exactly the same proof as in the case of abelian surfaces in §III.5, Proposition III.5.2 holds true for any surfaces S. So we are reduced to show Proposition III.5.3 for K3 surfaces (it would not be true for all simply-connected surfaces in general).

Step 2. Quantum corrections and cohomological realization. Note that $S^{[n]} \to S^{(n)}$ is a always a crepant resolution and recall that Ruan's cohomological crepant resolution conjecture III.1.1 says that there is an isomorphism

$$H^*_{qc}(Y, \mathbf{C}) \simeq H^*_{\mathrm{orb}}([M/G], \mathbf{C})$$

of $\mathbf{C}$-algebras, where the left-hand side is the *quantum corrected* cohomology: it has the same graded vector space $H^*(Y, \mathbf{C})$ but the product is the cup product corrected by quantum corrections. Let us recall briefly the basic notions.

For any $d \in \mathbf{N}$, let $\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)$ be the (proper) moduli space of stable maps ([**KM94**]) from genus zero curves with three marked points to $S^{[n]}$ with curve class $d\beta \in H_2\left(S^{[n]}, \mathbf{Z}\right)$. Since β is contracted by ρ, which is crepant, the moduli space has virtual dimension $2n$ and moreover it is endowed with a virtual fundamental cycle class (*cf.* [**Beh97**], [**BF97**])

$$\left[\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right]^{\mathrm{vir}} \in \mathrm{CH}_{2n}\left(\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right)$$

in the rational Chow group. There is a natural evaluation morphism $ev : \overline{M_{0,3}}\left(S^{[n]}, d\beta\right) \to \left(S^{[n]}\right)^3$. We define

$$\gamma_d := ev_*\left(\left[\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right]^{\mathrm{vir}}\right) \in \mathrm{CH}_{2n}\left(\left(S^{[n]}\right)^3\right). \tag{III.16}$$

We write $[\gamma_d] \in H_{4n}\left(\left(S^{[n]}\right)^3, \mathbf{Q}\right)$ for its (co)homology class.

For $d = 0$, the moduli space $\overline{M_{0,3}}\left(S^{[n]}, 0\right)$ is naturally identified with $S^{[n]}$, its virtual fundamental class is simply its fundamental class, and therefore $\gamma_0 = \delta_{S^{[n]}}$ is the small diagonal of $\left(S^{[n]}\right)^3$. Now we can define the *quantum corrections* to the cup product of $H^*(S^{[n]}, \mathbf{Q})$ as follows. By Poincaré duality, one sees easily that our definition is equivalent to the one in [**Rua06**].

Definition III.7.2 (Quantum corrections). Let $\alpha_1, \alpha_2 \in H^*(S^{[n]}, \mathbf{Q})$. Their *quantum corrected product* is by definition the "infinite sum"

$$\alpha_1 \smile_{qc} \alpha_2 := \lim_{q \to -1^+} \sum_{d=0}^{\infty} (\mathrm{pr}_3)_* \left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\gamma_d]\right) q^d, \tag{III.17}$$

where pr_i is the projection from $\left(S^{[n]}\right)^3$ to its i-th factor ($i \in \{1,2,3\}$). For any $d > 0$, the d-th term $\mathrm{pr}_{3,*}\left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\gamma_d]\right)$ is called the d-th *quantum correction* to the (usual) cup product. The latter corresponds to the term $d = 0$, since $\alpha_1 \smile \alpha_2 = \mathrm{pr}_{3,*}\left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\delta_{S^{[n]}}]\right)$. The convergence property is a consequence of [**LQ16**]. The graded vector space $H^*\left(S^{[n]}, \mathbf{Q}\right)$ equipped with the product $\smile_{qc}$ is called the *quantum corrected cohomology algebra* of $S^{[n]}$, denoted by $H^*_{qc}\left(S^{[n]}, \mathbf{Q}\right)$.

The main result of Wei-Ping Li and Zhenbo Qin [**LQ16**] (based on a number of works [**Leh99**], [**LQW02**], [**LQW03**], [**QW02**], [**Li06**], [**LL11**], [**Che13**], see also the upcoming book [**Qin17**]) says that the cohomological crepant resolution conjecture III.1.1 is true for the Hilbert-Chow resolution for simply-connected surfaces.

Theorem III.7.3 ([**LQ16**]). *Let S be a smooth projective simply-connected surface and n a natural number. There is an isomorphism*

$$H^*_{qc}\left(S^{[n]}, \mathbf{C}\right) \simeq H^*_{\mathrm{orb}}\left([S^n/\mathfrak{S}_n], \mathbf{C}\right)$$

of $\mathbf{C}$*-algebras.*

The isomorphism in the previous theorem is somewhat implicitly hidden in their series of works, but one can nevertheless check that it coincides with our isomorphism in Step 1.

Proposition III.7.4 ([FT17, Proposition 5.4]). *The isomorphism of* $\mathbf{C}$*-algebras in Theorem III.7.3 coincides with the cohomological realization of* ϕ *constructed in Step 1.*

Thanks to Proposition III.7.4, we can relate the cohomological classes of the algebraic cycles W and Z defined in Proposition III.5.3. To this end, we introduce a series of algebraic cycles W_d accompanying W: using the same notation, we define for any $d \in \mathbf{N}$

$$W_d := \frac{1}{(n!)^2}\Big(\sum_{g_1}\Gamma_{g_1} \times \sum_{g_2}\Gamma_{g_2} \times \sum_{g_3}\Gamma_{g_3}\Big)_*(\gamma_d) \in \bigoplus_{g_1,g_2,g_3\in\mathfrak{S}_n} \mathrm{CH}\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}\right), \tag{III.18}$$

where $\gamma_d \in \mathrm{CH}_{2n}\big(\big(S^{[n]}\big)^3\big)$ is defined in (III.16). As remarked before, $\gamma_0 = \delta_{S^{[n]}}$ and thus $W_0 = W$. Now one can reformulate Theorem III.7.3 in the following way.

Corollary III.7.5 ([FT17, Corollary 5.5]). *Let S be a smooth projective simply-connected surface. The (vector-valued) q-power series $\sum_{d=0}^{\infty}[W_d]\cdot q^d$ has convergence radius at least 1 and has a continuation across $q=-1$ with value equal to the cohomology class of the symmetrization of Z:*

$$\lim_{q\to -1^+}\sum_{d=0}^{\infty}[W_d]\cdot q^d = [\mathrm{Sym}(Z)] \tag{III.19}$$

in $\bigoplus_{g_1,g_2,g_3\in\mathfrak{S}_n} H^*\left((S^n)^{g_1}\times(S^n)^{g_2}\times(S^n)^{g_3}\right)$.

Step 3. Distinguishedness. Using the construction of relative moduli spaces of stable maps, we see that the cycles W_d and Z also exist in the relative setting for quasi-projective smooth surfaces over a quasi-projective smooth base and behave well with respect to base-change and restriction to Zariski open subsets. Then Voisin's theory of *universally defined cycles* [Voi19] (see also [Voi15a, §5.2] and [FT17, §6]) allows us to conclude that W_d and Z are *tautological* cycles.

Proposition III.7.6. *Let S be a smooth projective surface. For any $g_1, g_2, g_3 \in \mathfrak{S}_n$, and $d \in \mathbf{N}$, the restrictions to $(S^n)^{g_1}\times(S^n)^{g_2}\times(S^n)^{g_3}$ of the algebraic cycles W_d (in particular W) defined in (III.18) and Z defined in Proposition III.5.3 are tautological cycles, that is, cycles of the form*

$$P(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)) \in \mathrm{CH}^*(S^m),$$

where P is a polynomial.

Step 4. Cohomological relations. The idea is due to Voisin [Voi15a]. Thanks to a result of Yin [Yin15], all cohomological relations among tautological cycles on a power S^m of a simply-connected surface S can be generated by three types of "basic" relations:

(*i*) Trivial relations.
(*ii*) Beauville–Voisin relations, which essentially say that there is a decomposition of the small diagonal in terms of the big diagonals, or equivalently, the modified small diagonal vanishes.
(*iii*) The Kimura relation, which essentially says that $\bigwedge^N H^2(S)$ vanishes when N is larger than the second Betti number.

Corollary III.7.7. *Given a natural number m, let P be a polynomial such that for any simply-connected smooth projective surface S,*

$$P\left([\Delta_{i,j}], \mathrm{pr}_k^*[c_1(S)], \mathrm{pr}_l^*[c_2(S)]\right) = 0 \text{ in } H^{2*}(S^m, \mathbf{Q}).$$

Then, for any projective K3 surface S, we have

$$P\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) = 0 \text{ in } \mathrm{CH}^*(S^m).$$

Proof. The following is a heuristic explanation. The actually proof is quite involved, see [**FT17**, §7]. The assumption on the vanishing in cohomology for all simply-connected surface S tells us that P belongs to the ideal generated by the "basic" relations (i), (ii), (iii) by Yin's theorem. However, as the Kimura relation involves more and more variables when the second Betti number of S gets larger and larger (and this cannot be bounded), we see that P is actually in the ideal generated by first two sets of relations (i) and (ii) above. Therefore, it suffices to check all these relations for S a projective K3 surface. The *trivial relations* in (i) hold in fact for all surfaces and are very easy to check, while the relations in (ii) hold for K3 surfaces thanks to Theorem II.1.1, due to Beauville and Voisin [**BV04**]. □

Step 5. Conclude. Ignoring the convergence problem (see [**FT17**, §8] for a rigorous proof), a combination of Corollary III.7.5, Proposition III.7.6, and Corollary III.7.7 implies that for a projective K3 surface S, we have

$$\lim_{q\to -1^+} \sum_{d=0}^{\infty} W_d \cdot q^d = \mathrm{Sym}(Z) \tag{III.20}$$

in $\bigoplus_{g_1,g_2,g_3\in\mathfrak{S}_n} \mathrm{CH}^*\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}\right)_{\mathbf{C}}$. However, since S is a K3 surface, $S^{[n]}$ is a holomorphic symplectic variety, hence all its Gromov-Witten invariants vanish in a strong sense.

Lemma III.7.8. *Let S be a projective K3 surface. Let $\gamma_d \in \mathrm{CH}_{2n}\left((S^{[n]})^3\right)$ be the algebraic cycle defined in (III.16). Then, $\gamma_d = 0$ in $\mathrm{CH}_{2n}\left((S^{[n]})^3\right)$ for all $d > 0$.*

Proof. Recall that in (III.16), γ_d was defined as the push-forward by the evaluation morphism of the virtual fundamental cycle $[M]^{\mathrm{vir}} \in \mathrm{CH}_{2n}(M)$, where $M := \overline{M_{0,3}}\left(S^{[n]}, d\beta\right)$ is the moduli space of stable maps from genus zero curves with three marked points to $S^{[n]}$ with class $d\beta \in H_2(S^{[n]}, \mathbf{Z})$. We will actually prove that in $\mathrm{CH}_{2n}(M)$, we have

$$[M]^{\mathrm{vir}} = 0.$$

This result is well known. Consider a family of smooth projective holomorphic symplectic varieties $\pi : \mathcal{X} \to B$ whose central fiber over $b_0 \in B$ is isomorphic to $S^{[n]}$, such that the class β is not a Hodge class over a general point of B. Let $\mathcal{M}/B$ be the relative moduli space of stable maps from genus zero curves with three marked points to fibers of π with class $d\beta$. Then we know that the fiber of $\mathcal{M}/B$ over a general point $b \in B$ is the moduli space $\mathcal{M}_b \simeq \overline{M_{0,3}}(\mathcal{X}_b, d\beta)$, which is empty since $d\beta$ is not a curve class, hence $[\mathcal{M}_b]^{\mathrm{vir}} = 0$ for a general point $b \in B$. By specialization, we find that $[M]^{\mathrm{vir}} = \left[\mathcal{M}_{b_0}\right]^{\mathrm{vir}} = 0$ in $\mathrm{CH}(M)$. □

This implies that $W_d = 0$ in $\mathrm{CH}^N(S^m)$ for all $d > 0$ by their definition (III.18). Therefore, in (III.20), all terms with $d \geq 1$ vanish and it actually says

$$W := W_0 = \mathrm{Sym}(Z)$$

in $\mathrm{CH}^N(S^m)$. Proposition III.5.3 is proved, so is Theorem III.7.1.

III.8. Applications

We present here plenty of consequences of the established cases of the motivic hyper-Kähler resolution conjecture. We want to deliver the message that an isomorphism as predicted by the conjecture usually leads to highly non-trivial implications on the Chow ring of the hyper-Kähler resolution.

III.8.1. Multiplicative Chow-Künneth decomposition. Combining our main results Theorem III.5.1, Theorem III.6.1, and Theorem III.7.1 with the fact that abelian varieties and K3 surfaces admit a canonical multiplicative Chow–Künneth decomposition (§I.3), we obtain some evidence for Conjecture II.3.7 on multiplicative Chow–Künneth decompositions.

Corollary III.8.1. *The Hilbert schemes of abelian surfaces and of K3 surfaces, and the generalized Kummer varieties have canonical multiplicative Chow–Künneth decompositions.*

We point out that in the first two cases, the existence of multiplicative Chow–Künneth decomposition was previously proved by Vial in [Via17]. The main advantage of establishing the relation with orbifolds is that we get precise formulas for the product on Chow rings or Chow motives for the non-zero parts (for example for the morphism $\mathfrak{h}^i \otimes \mathfrak{h}^j \to \mathfrak{h}^{i+j}$, where the multiplicative Chow–Künneth decomposition says nothing). This is the next application.

III.8.2. Multiplicative structure on Chow rings. Let S be a projective K3 surface. As a hyper-Kähler variety, $S^{[n]}$ has infinite dimensional Chow groups ([Mum68]). As far as the author's knowledge goes, unlike the situation for cohomology $H^*(S^{[n]}, \mathbf{Q})$ (*cf.* [LS03]), the ring structure of $\mathrm{CH}^*\left(S^{[n]}\right)$ is poorly understood. The case $n = 2$ is classical; the case $n = 3$ can be probably worked out from the geometric construction of the Hilbert cube. It was open for $n \geq 4$ because of the lack of explicit construction for $S^{[n]}$. However if we take the Chow rings of self-products of S as basic information, Theorem III.7.1 provides the following complete and explicit description, namely a multiplication table of the Chow rings $\mathrm{CH}^*\left(S^{[n]}\right)$ for all n. Roughly speaking, *the intersection product for the Chow rings is given by exactly the same formula as for the cup product for the cohomology rings*.

Corollary III.8.2 (Ring structure of $\mathrm{CH}^*\left(S^{[n]}\right)$, [FT17, Corollary 1.9]). *Let S be a projective K3 surface and let $n \in \mathbf{N}$. Using the injective map*

$$\phi : \mathrm{CH}^*\left(S^{[n]}\right) \hookrightarrow \bigoplus_{g \in \mathfrak{S}_n} \mathrm{CH}^{*-\mathrm{age}(g)}\left((S^n)^g\right)$$

of de Cataldo-Migliorini [dCM02], the intersection product on $\mathrm{CH}^*(S^{[n]})_{\mathbf{Q}}$ is determined as follows: for any $g, h \in G$ and $\alpha \in \mathrm{CH}^{i-\mathrm{age}(g)}((S^n)^g)$, $\beta \in \mathrm{CH}^{j-\mathrm{age}(h)}((S^n)^h)$, we have

$$\alpha \star \beta = \varepsilon(g,h) \cdot \iota_*\left(\alpha|_{(S^n)^{<g,h>}} \cdot \beta|_{(S^n)^{<g,h>}} \cdot c_{g,h}\right) \in \mathrm{CH}^{i+j-\mathrm{age}(gh)}((S^n)^{gh}).$$

Here $\iota : (S^n)^{<g,h>} \hookrightarrow (S^n)^{gh}$ is the inclusion, $\varepsilon(g,h) := (-1)^{\frac{\mathrm{age}(g)+\mathrm{age}(h)-\mathrm{age}(gh)}{2}}$ is a sign change (discrete torsion) and $c_{g,h}$ is the obstruction class

$$c_{g,h} = \begin{cases} 0, & \text{if } \exists\, o \in O(g,h) \quad d_{g,h}(o) \geq 2; \\ \prod_{o \in I}\left(24\,\mathrm{pr}_o^*(c_S)\right), & \text{if } \forall o \in O(g,h) \quad d_{g,h}(o) = 0 \text{ or } 1, \end{cases} \tag{III.21}$$

where $d_{g,h}$ is a combinatorial datum called graph defect, $I := \{o \in O(g,h) \mid d_{g,h}(o) = 1\}$.

In [FT17, Example 9.4], we worked out some interesting intersection products.

In a similar vein, for an abelian surface A, Theorem III.5.1 and Theorem III.6.1 allow us to determine the multiplication table of the Chow rings of the Hilbert scheme $A^{[n]}$ and the generalized Kummer variety $K_n(A)$ explicitly in terms of the cycles on the abelian varieties A^m.

III.8.3. The Beauville–Voisin conjecture. We can deduce some new evidence for the Beauville–Voisin conjecture II.3.3.

- Theorem III.6.1 implies the Beauville–Voisin conjecture II.3.3 for generalized Kummer varieties, which was originally proved in [Fu15].
- Theorem III.7.1 allows us to improve the known bound of the Beauville–Voisin conjecture for Hilbert schemes $S^{[n]}$ of K3 surfaces to $n < (b_{2,\mathrm{tr}} + 1)(b_{2,\mathrm{tr}} + 2)$ (it was proved for $n \leq 2b_2(S)_{\mathrm{tr}} + 4$ by Voisin [Voi08]), where $b_{2,\mathrm{tr}}$ is the transcendental second Betti number of the K3 surface.

III.8.4. Cohomological crepant resolution conjecture. Passing from Chow motives to their realizations, we automatically get the original cohomological versions of the crepant / hyper-Kähler resolution conjectures of Ruan.

Taking the Betti cohomological realization in Theorem III.4.3, we obtain the 2-dimensional case of Ruan's cohomological crepant resolution conjecture III.1.1:

Corollary III.8.3 (Cohomological multiplicative McKay correspondence for surfaces). *Let $\mathcal{X}$ be a smooth proper two-dimensional Deligne–Mumford stack with isolated stacky points. Assume that $\mathcal{X}$ has a projective coarse moduli space X with Gorenstein singularities. Let $Y \to X$ be the minimal resolution. Then we have an isomorphism*

$$H^*(Y, \mathbf{C}) \simeq H^*_{\mathrm{orb}}(\mathcal{X}, \mathbf{C})$$

of graded $\mathbf{C}$*-algebras.*

As far as the author knows, this version of multiplicative McKay correspondence has never been checked in the literature. Some relevant works are [BGP08], from where our key formula is borrowed, and [GK04], where the local case is proved.

Taking the Betti cohomological realization in Theorem III.6.1, we confirm Ruan's original cohomological hyper-Kähler resolution conjecture III.1.2 in the case of generalized Kummer varieties.

Theorem III.8.4 ([FTV19]). *Let $K_n(A)$ be the $2n$-dimensional generalized Kummer variety associated with an abelian surface A. Endow $A_0^{n+1} := \mathrm{Ker}\left(+ : A^{n+1} \to A\right)$ with the natural $\mathfrak{S}_{n+1}$-action. Then we have an isomorphism*

$$H^*\left(K_n(A), \mathbf{C}\right) \simeq H^*_{\mathrm{orb}}\left(\left[A_0^{n+1} / \mathfrak{S}_{n+1}\right], \mathbf{C}\right)$$

of graded $\mathbf{C}$*-algebras.*

The cohomological hyper-Kähler resolution conjecture III.1.2 for Hilbert schemes of abelian surfaces and K3 surfaces are of course also consequences of our Theorems III.5.1 and III.7.1, but they were proved in [FG03], based on [LS03], and their results are used in our proof.

III.8.5. The K-theoretic hyper-Kähler resolution conjecture. From the K-theoretic point of view, we also have the following closely related conjecture proposed in [**JKK07**, Conjecture 1.2] for Grothendieck rings and extended in the author's joint work with Nguyen [**FN19**] to higher algebraic K-theory. Recall that the *orbifold K-theory* is defined in a similar way with the top Chern class replaced by the K-theoretic Euler class; see Definition III.2.5 for details.

Conjecture III.8.5 (K-theoretic HyperKähler resolution conjecture [**JKK07**], [**FN19**]). *Let $\mathcal{X}$ be a smooth proper complex Deligne–Mumford stack with underlying coarse moduli space a (singular) symplectic variety X. If there is a symplectic resolution $Y \to X$, we have isomorphisms*

$$K_\bullet(Y)_{\mathbf{C}} \simeq K_\bullet^{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}};$$
$$K_{top}(Y)_{\mathbf{C}} \simeq K_{top}^{\mathrm{orb}}(\mathcal{X})_{\mathbf{C}}$$

of graded $\mathbf{C}$*-algebras.*

Using Theorems III.5.1, III.6.1, and III.7.1, we provide evidence for Conjecture III.8.5.

Corollary III.8.6. *Let A be an abelian surface, let S be a projective K3 surface, and let n be a natural number. There are isomorphisms*

$$\begin{aligned}
K_\bullet\left(A^{[n]}\right)_{\mathbf{C}} &\simeq K_\bullet^{\mathrm{orb}}\left([A^n/\mathfrak{S}_n]\right)_{\mathbf{C}};\\
K_{\mathrm{top}}\left(A^{[n]}\right)_{\mathbf{C}} &\simeq K_{\mathrm{top}}^{\mathrm{orb}}\left([A^n/\mathfrak{S}_n]\right)_{\mathbf{C}};\\
K_\bullet\left(S^{[n]}\right)_{\mathbf{C}} &\simeq K_\bullet^{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{C}};\\
K_{\mathrm{top}}\left(S^{[n]}\right)_{\mathbf{C}} &\simeq K_{\mathrm{top}}^{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{C}};\\
K_\bullet\left(K_n(A)\right)_{\mathbf{C}} &\simeq K_\bullet^{\mathrm{orb}}\left(\left[A_0^{n+1}/\mathfrak{S}_{n+1}\right]\right)_{\mathbf{C}};\\
K_{\mathrm{top}}\left(K_n(A)\right)_{\mathbf{C}} &\simeq K_{\mathrm{top}}^{\mathrm{orb}}\left(\left[A_0^{n+1}/\mathfrak{S}_{n+1}\right]\right)_{\mathbf{C}}.
\end{aligned}$$

of graded commutative $\mathbf{C}$*-algebras.*

Proof. We only give the proof for the case of Hilbert schemes of K3 surfaces, the other cases being similar. In [**JKK07**], an *orbifold Chern character* was constructed for the Grothendieck ring, which induces an isomorphism ([**JKK07**, Main result 3])

$$\mathrm{ch}_{\mathrm{orb}} : K_0^{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*_{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)$$

of $\mathbf{Q}$-algebras. This construction was extended to higher K-theory in [**FN19**] to get an isomorphism (see Theorem III.2.6)

$$\mathfrak{ch} : K_\bullet^{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*_{\mathrm{orb}}\left([S^n/\mathfrak{S}_n], \bullet\right).$$

The desired isomorphism of algebras is simply the combination of this extended Chern character $\mathfrak{ch}$, the usual Chern character isomorphism $\mathrm{ch} : K_\bullet\left(S^{[n]}\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*\left(S^{[n]}, \bullet\right)_{\mathbf{Q}}$ and the isomorphism $\mathrm{CH}^*(S^{[n]})_{\mathbf{C}} \simeq \mathrm{CH}^*_{\mathrm{orb}}([S^n/\mathfrak{S}_n])_{\mathbf{C}}$ in Theorem III.7.1.
The statement for topological K-theory comes similarly from the orbifold topological Chern character,

$$\mathrm{ch}_{\mathrm{orb}} : K_{top}^{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}} \xrightarrow{\simeq} H^*_{\mathrm{orb}}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}},$$

which was also constructed in [**JKK07**], together with ch : $K_{top}\left(S^{[n]}\right)_{\mathbf{Q}} \xrightarrow{\simeq} H^*\left(S^{[n]}\right)_{\mathbf{Q}}$ and an isomorphism in cohomology, which can be obtained by applying the realization functor to Theorem III.7.1. □

When the Conjecture III.8.5 was proposed in [**JKK07**], there was hardly any evidence and actually [**JKK07**, Main result 3] reduces their conjecture to the Chow-theoretic version. Note that the proof presented in this chapter uses in an essential way Chow motives. A direct and geometrically meaningful isomorphism (in terms of Fourier–Mukai transforms for example) between $K(S^{[n]})_{\mathbb{C}}$ and $K^{\mathrm{orb}}([S^n/\mathfrak{S}_n])_{\mathbb{C}}$ would be very interesting. Unfortunately, the author was not able to find one so far.

III.8.6. Multiplicative decomposition theorem of rational cohomology. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. Deligne's decomposition theorem [**Del68**] states that

$$R\pi_*\mathbf{Q} \cong \bigoplus_i R^i\pi_*\mathbf{Q}[-i] \tag{III.22}$$

in the derived category of sheaves of $\mathbf{Q}$-vector spaces on B.

Voisin [**Voi12**, Question 0.2] asked if there exists a decomposition as in (III.22) which is multiplicative, *i.e.*, which is compatible with cup products on both sides, maybe over a nonempty Zariski open subset of B. By Deninger–Murre [**DM91**], there does exist such a decomposition for an abelian scheme $\pi : \mathcal{A} \to B$. The main result of [**Voi12**] is that for any smooth projective family of K3 surfaces, there exist a decomposition isomorphism as in (III.22) and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family $\pi|_U : \mathcal{X}|_U \to U$.

Vial [**Via17**, Section 4] generalized this for Hilbert schemes of K3 surfaces. In [**FTV19**, Theorem 8.3], we proved that Voisin's Theorem holds more generally for any smooth projective family whose general fiber admits a multiplicative Chow–Künneth decomposition. Therefore, we can deduce from Theorems III.5.1 and III.6.1 the following.

Corollary III.8.7 ([**FTV19**, Corollary 8.4]). *Let $\pi : \mathcal{A} \to B$ be an abelian surface over B. Consider Case (A): $\mathcal{A}^{[n]} \to B$ the relative Hilbert scheme of length-n subschemes on $\mathcal{A} \to B$; or Case (B): $K_n(\mathcal{A}) \to B$ the relative generalized Kummer variety. In both cases, there exist a decomposition isomorphism as in* (III.22) *and a nonempty Zariski open subset U of B, such that this decomposition becomes multiplicative for the restricted family over U.*

Remark III.8.8. The conclusion of Voisin's theorem as well as our extensions to hyper-Kähler varieties is purely topological and thus makes sense for non-projective ones. But the proof uses algebraic methods (Chow groups *etc.*). It is natural to ask whether similar results hold more generally in the Kähler setting.

III.8.7. Orbifold version of the multiplicative Orlov conjecture. We saw in §II.3.3 that we expect the Chow motives of derived equivalent projective hyper-Kähler varieties to be isomorphic as (Frobenius) algebra objects (§I.2.4). If we consider more generally hyper-Kähler *orbifolds*, the analogue of the multiplicative Orlov Conjecture II.3.15 is closely related to the motivic hyper-Kähler resolution conjecture III.1.3, as explained in §II.3.4.

In this sense, forgetting the Frobenius structure, we can obtain some evidence for the orbifold analogue of Conjecture II.3.15:

- between a K3 orbifold and its minimal resolution by Theorem III.4.3,

- between $[A^n/\mathfrak{S}_n]$ and the n-th Hilbert scheme of an abelian surface A by Theorem III.5.1,
- between $[\ker(A^{n+1} \xrightarrow{+} A)/\mathfrak{S}_n]$ and the n-th generalized Kummer variety associated to an abelian surface A by Theorem III.6.1,
- between $[S^n/\mathfrak{S}_n]$ and the n-th Hilbert scheme of a K3 surface S by Theorem III.7.1.

In each of the above situations, the derived equivalence was established in [**BKR01**].

III.9. A strategy for the general case (project)

As explained in the previous sections, some interesting cases of the motivic hyper-Kähler resolution conjecture were established. The time seems ripe for a serious attack on this conjecture in general. As all our afore-mentioned results rely on the corresponding (highly non-trivial) cohomological results, one strategy would be to first establish the cohomological hyper-Kähler resolution conjecture, then try to lift the homological relations to the motivic level. However, by looking at the sophisticated techniques (e.g. representation of vertex algebras) used in establishing the cohomological version in those cases, it is hard to come up with a reasonable generalization.

Now the idea is to use new techniques from *derived algebraic geometry* as developed in the last decade by Lurie, Toën–Vezzosi et al. More precisely, we can equip the (singular) incidence correspondences between the hyper-Kähler resolution and the stacky loci of the orbifold with their natural derived structures. The hope is that all the subtleties of the orbifold product should appear as the derived version of the excess intersection formula. One technical problem is that the foundations of algebraic cycles of derived schemes are not well established and we need to transfer at this point to the Grothendieck group, or rather the derived categories of the various derived schemes involved. The point is that the base-change theorem for derived categories in the setting of DAG incorporates the excess intersection formula.

The interest of this project is clear: among other potential consequences, it trivially implies the original cohomological hyper-Kähler resolution conjecture, hence in particular gives a new proof of the hard results of [**LS03**] and [**FG03**].

CHAPTER IV

Towards a theory of distinguished cycles

This chapter is based on the joint work with Vial [**FV19a**], which aims at constructing a subalgebra of the rational Chow ring of certain varieties (in particular hyper-Kähler varieties) consisting of canonical lifts of algebraic cohomology classes.

As briefly sketched in §II.3.2.2, given a projective holomorphic symplectic variety X, there is a conjectural $\mathbf{Q}$-subalgebra $\mathrm{DCH}^*(X)$ of the rational Chow ring $\mathrm{CH}^*(X)$, whose elements are called *distinguished cycles*, such that the restriction of the cycle class map to $\mathrm{DCH}^*(X)$ is injective and it is maximal among all such subalgebras. The existence of this subalgebra is a consequence of the combination of the more ambitious conjecture on the existence of a multiplicative Chow–Künneth decomposition II.3.7 and the Bloch–Beilinson–Murre conjecture I.3.4 in this setting: it is simply the subalgebra $\mathrm{CH}^*(X)_{(0)} = \mathrm{CH}^*(\mathfrak{h}^{2*}(X)) := \mathrm{Hom}(\mathbb{1}(-*), \mathfrak{h}^{2*}(X))$.

For abelian varieties, even though the multiplicative Chow–Künneth decomposition was constructed in [**DM91**] (see Theorem II.2.4), the Beauville conjecture II.2.3 is largely open. However, a subalgebra of distinguished cycles, called *symmetrically distinguished* cycles, was constructed unconditionally by O'Sullivan [**O'S11b**], (see §II.2.3). This seems to deliver the message that a theory of distinguished cycles is probably much more realistic than the whole package of splitting and Bloch–Beilinson conjectures. In this chapter, we survey the work [**FV19a**] which is an attempt at investigating the following two questions, even beyond the symplectic setting:

- What kind of varieties can admit a subalgebra of distinguished cycles?
- How to construct such a theory when it is possible?

The key idea is to make systematic use of O'Sullivan's Theorem II.2.7 by comparing the motive of the variety to that of abelian varieties, and to define the distinguished cycles to be the transportation of symmetrically distinguished cycles on abelian varieties.

In this chapter, $\overline{\mathrm{CH}}^*$ denotes the graded ring of cycles with rational coefficients modulo numerical equivalence. There is a natural projection $\mathrm{CH}^* \twoheadrightarrow \overline{\mathrm{CH}}^*$.

IV.1. Abelian motives, markings

Definition IV.1.1 (Motives of abelian type). Let $\mathscr{M}^{ab}$ be the strictly[1] full, thick, and rigid tensor subcategory of CHM generated by the motives of abelian varieties. A motive is said to be *of abelian type* if it belongs to $\mathscr{M}^{ab}$; equivalently, if one of its Tate twists is isomorphic to a direct summand of the motive of an abelian variety.

Example IV.1.2. The Chow motives of the following algebraic varieties belong to the category $\mathscr{M}^{ab}$:

[1]A full subcategory is called *strictly full* if it is closed under isomorphisms.

(*i*) projective spaces, Grassmannian varieties, and more generally projective homogeneous varieties under a linear algebraic group and toric varieties;
(*ii*) smooth projective curves;
(*iii*) Kummer K3 surfaces; K3 surfaces with Picard numbers at least 19 as well as their (nested) Hilbert schemes;
(*iv*) abelian varieties;
(*v*) Hilbert schemes of abelian surfaces;
(*vi*) generalized Kummer varieties;
(*vii*) Fermat hypersurfaces;
(*viii*) projective bundles over the examples above;
(*ix*) products and surjective images of the examples above.

To make a systematic use of O'Sullivan's Theorem II.2.7, we generalize it to the level of motives. Let us introduce the category $\mathscr{M}_{sd}^{ab}$ of symmetrically distinguished abelian motives, constructed in [**FV19a**].

Definition IV.1.3 ([**FV19a**, Definition 2.1]). The category of *symmetrically distinguished abelian motives*, denoted $\mathscr{M}_{sd}^{ab}$, is defined as follows.

(*i*) An object consists of the data of
- a positive integer $r \in \mathbf{N}^*$;
- abelian varieties (thus with fixed origins) $A_1, \ldots, A_r$;
- integers $n_1, \ldots, n_r \in \mathbf{Z}$;
- an $(r \times r)$-matrix $P := \left(p_{i,j}\right)_{1\leq i,j\leq r}$ with $p_{i,j} \in \mathrm{DCH}^{\dim A_i + n_j - n_i}(A_i \times A_j)$ a *symmetrically distinguished* cycle (Definition II.2.6), such that $P \circ P = P$, that is, for all $1 \leq i, j \leq r$, we have

$$\sum_{k=1}^{r} p_{k,j} \circ p_{i,k} = p_{i,j} \text{ in } \mathrm{CH}^{\dim A_i + n_j - n_i}(A_i \times A_j)$$

Such an object is denoted in the sequel by a triple

$$\left(A_1 \sqcup \cdots \sqcup A_r, P = \left(p_{i,j}\right), (n_1, \ldots, n_r)\right).$$

(*ii*) The group of morphisms from $\left(A_1 \sqcup \cdots \sqcup A_r, P = \left(p_{i,j}\right), (n_1, \ldots, n_r)\right)$ to another object $\left(B_1 \sqcup \cdots \sqcup B_s, Q = \left(q_{i,j}\right), (m_1, \ldots, m_s)\right)$ is defined to be the subgroup of

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \mathrm{CH}^{\dim A_i + m_j - n_i}(A_i \times B_j)$$

(whose elements are viewed as $(s \times r)$-matrices) given by

$$Q \circ \left(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \mathrm{CH}^{\dim A_i + m_j - n_i}(A_i \times B_j) \right) \circ P,$$

where the multiplication law is the one between matrices.
(*iii*) The composition is defined as usual by composition of correspondences.

(*iv*) The category $\mathscr{M}_{sd}^{ab}$ is an additive category where the direct sum is given by

$$\begin{aligned}&\left(\bigsqcup_{i=1}^{r} A_i, P, (n_1, \dots, n_r)\right) \oplus \left(\bigsqcup_{j=1}^{s} B_j, Q, (m_1, \dots, m_s)\right)\\ = &\left(\bigsqcup_{i=1}^{r} A_i \sqcup \bigsqcup_{j=1}^{s} B_j, P \oplus Q := \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, (n_1, \dots, n_r, m_1, \dots, m_s)\right)\end{aligned}$$

(*v*) The category $\mathscr{M}_{sd}^{ab}$ is a symmetric monoïdal category where the tensor product is defined by

$$\begin{aligned}&\left(\bigsqcup_{i=1}^{r} A_i, P, (n_1, \dots, n_r)\right) \otimes \left(\bigsqcup_{j=1}^{s} B_j, Q, (m_1, \dots, m_s)\right)\\ = &\left(\bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{s} A_i \times B_j, P \otimes Q, (n_i m_j; 1 \le i \le r, 1 \le j \le s)\right)\end{aligned}$$

where $P \otimes Q$ is the Kronecker product of two matrices.
In particular, for any $m \in \mathbf{Z}$, the m-th *Tate twist*, *i.e.*, the tensor product with the Tate object $\mathbb{1}(m) := (\operatorname{Spec} k, \operatorname{Spec} k, m)$ sends $(A_1 \sqcup \cdots \sqcup A_r, P, (n_1, \dots, n_r))$ to the object $(A_1 \sqcup \cdots \sqcup A_r, P, (n_1 + m, \dots, n_r + m))$. All Tate objects are $\otimes$-invertible.

(*vi*) The category $\mathscr{M}_{sd}^{ab}$ is rigid; the dual of $\left(A_1 \sqcup \cdots \sqcup A_r, P = \left(p_{i,j}\right), (n_1, \dots, n_r)\right)$ is given by $\left(A_1 \sqcup \cdots \sqcup A_r, {}^tP := ({}^tp_{j,i}), (d_1 - n_1, \dots, d_r - n_r)\right)$, where $d_k = \dim A_k$ and the (i, j)-th entry of ${}^tP$ is ${}^tp_{j,i} \in \mathrm{CH}^{d_i + (d_j - n_j) - (d_i - n_i)}(A_i \times A_j)$, the transpose of $p_{j,i} \in \mathrm{CH}^{d_j + n_i - n_j}(A_j \times A_i)$.

There is a natural fully faithful additive tensor functor

$$F : \mathscr{M}_{sd}^{ab} \to \mathscr{M}^{ab},$$

which send an object $\left(A_1 \sqcup \cdots \sqcup A_r, P = \left(p_{i,j}\right), (n_1, \dots, n_r)\right)$ to the Chow motive

$$\operatorname{Im}\left(P : \bigoplus_{i=1}^{r} \mathfrak{h}(A_i)(n_i) \to \bigoplus_{i=1}^{r} \mathfrak{h}(A_i)(n_i)\right).$$

Here we use the facts that CHM is pseudo-abelian and that P induces an idempotent endomorphism of $\bigoplus_{i=1}^{r} \mathfrak{h}(A_i)(n_i)$ by construction.

For any object M in $\mathscr{M}_{sd}^{ab}$ and any $i \in \mathbf{Z}$, the i-th *Chow group* $\mathrm{CH}^i(M)$ is defined to be $\mathrm{CH}^i(F(M))$, which is nothing but $\operatorname{Hom}_{\mathscr{M}_{sd}^{ab}}((\operatorname{Spec} k, \operatorname{Spec} k, -i), M)$.

Despite the technical construction of the category $\mathscr{M}_{sd}^{ab}$, it is, after all, not so different from the category $\mathscr{M}^{ab}$ of abelian motives (Definition IV.1.1). In fact, we can show that $F : \mathscr{M}_{sd}^{ab} \to \mathscr{M}^{ab}$ is an equivalence of categories (see [**FV19a**, Lemma 2.2])

It is natural to define the notion of symmetric distinguishedness for a morphism in $\mathscr{M}_{sd}^{ab}$, see [**FV19a**, Definition 2.3], and we obtain the subcategory of $\mathscr{M}_{sd}^{ab}$ with the same class of objects but only symmetrically distinguished morphisms. It is worth noting that this subcategory is again pseudo-abelian ([**FV19a**, Lemma 2.6]). In fact it is the pseudo-abelian additive envelop of the category of symmetrically distinguished correspondences between abelian varieties.

IV.2. Distinguished cycles

To define distinguished cycles, we need the following notion.

Definition IV.2.1 (Marking). Let X be a smooth projective variety whose Chow motive $\mathfrak{h}(X)$ belongs to $\mathscr{M}^{ab}$. A *marking* for X consists of an object $M \in \mathscr{M}^{ab}_{sd}$ together with an isomorphism

$$\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M) \quad \text{in CHM}.$$

Definition IV.2.2 (Distinguished cycles [**FV19a**, Definition 3.2]). Let X be a smooth projective variety whose Chow motive $\mathfrak{h}(X)$ belongs to $\mathscr{M}^{ab}$. Given a marking $\phi : \mathfrak{h}(X) \xrightarrow{\simeq} F(M)$ with $M \in \mathscr{M}^{ab}_{sd}$, we define the subgroup of *distinguished cycles* of codimension i of X, denoted by $\mathrm{DCH}^i_\phi(X)$, or sometimes $\mathrm{DCH}^i(X)$ if ϕ is clear from the context, to be the pre-image of $\mathrm{DCH}^i(M)$ *via* the induced isomorphism $\phi_* : \mathrm{CH}^i(X) \xrightarrow{\simeq} \mathrm{CH}^i(M)$. Here $\mathrm{DCH}^*(M)$ is defined by writing M as a direct summand of a Tate twist of the motive of an abelian variety A and using O'Sullivan's theory of symmetrically distinguished cycles (Definition II.2.6) on A.

By construction and Theorem II.2.7, the composition

$$\mathrm{DCH}^i_\phi(X) \hookrightarrow \mathrm{CH}^i(X) \twoheadrightarrow \overline{\mathrm{CH}}^i(X)$$

is an isomorphism. In other words, ϕ provides a section (as graded vector spaces) of the natural projection $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$. We remark that the fundamental class 1_X is always distinguished for any choice of marking.

IV.3. The (⋆) condition (distinguished marking) and the section property

Question IV.3.1. Here are the most important properties of distinguished cycles that we are going to investigate:

- When does $\bigoplus_i \mathrm{DCH}^i_\phi(X)$ form a (graded) $\mathbf{Q}$-subalgebra of $\mathrm{CH}(X)$?
- When do the Chern classes of X belong to $\bigoplus_i \mathrm{DCH}^i_\phi(X)$?

To this end, let us introduce the following condition for smooth projective varieties whose Chow motive is of abelian type.

Definition IV.3.2. We say that a smooth projective variety X with $\mathfrak{h}(X) \in \mathscr{M}^{ab}$ satisfies the condition $(\star)$ if there exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\sim} M$ (with $M \in \mathscr{M}^{ab}$) such that

$(\star_{\mathrm{Mult}})$ **(Multiplicativity)** the small diagonal δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 3}}(X^3)$, that is, under the induced isomorphism $\phi_*^{\otimes 3} : \mathrm{CH}(X^3) \xrightarrow{\sim} \mathrm{CH}(M^{\otimes 3})$, the image of δ_X is symmetrically distinguished, *i.e.*, in $\mathrm{DCH}(M^{\otimes 3})$;

$(\star_{\mathrm{Chern}})$ **(Chern classes)** all Chern classes of T_X belong to $\mathrm{DCH}_\phi(X)$.

More generally, if X is a smooth projective variety equipped with an action of a finite group G, we say that (X, G) satisfies $(\star)$ if there exists a marking $\phi : \mathfrak{h}(X) \xrightarrow{\sim} F(M)$ that satisfies, in addition to $(\star_{\mathrm{Mult}})$ and $(\star_{\mathrm{Chern}})$ above,

$(\star_G)$ **(G-invariance)** the graph g_X of $g : X \to X$ belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$ for any $g \in G$.

A marking satisfying (⋆) is called a *distinguished marking*.

Remark IV.3.3. Condition ($\star_{\mathrm{Mult}}$) implies that the diagonal Δ_X belongs to $\mathrm{DCH}_{\phi^{\otimes 2}}(X^2)$. In fact, the classes of all partial diagonals[2] in a self-product of X are distinguished. See [**FV19a**, Lemma 3.8 and Corollary 3.13].

The original motivation to study condition (⋆) is the following.

Proposition IV.3.4 (Subalgebra). *Let X be a smooth projective variety with motive of abelian type. If X satisfies condition ($\star_{\mathrm{Mult}}$), then there is a section, as* graded algebras, *of the natural surjective morphism* $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$. *If moreover ($\star_{\mathrm{Chern}}$) is satisfied, then all Chern classes of X are in the image of this section.*

In other words, under (⋆), we have a graded **Q**-*subalgebra* $\mathrm{DCH}(X)$ *of the Chow ring* $\mathrm{CH}(X)$ *which contains all the Chern classes of X and is mapped isomorphically to* $\overline{\mathrm{CH}}(X)$. *We call elements of* $\mathrm{DCH}(X)$ distinguished cycles *of X.*

Proof. Let $\phi : \mathfrak{h}(X) \xrightarrow{\sim} F(M)$ be a marking, where $M \in \mathscr{M}^{ab}_{sd}$. If ϕ satisfies (⋆), we define $\mathrm{DCH}(X) := \mathrm{DCH}_\phi(X)$ as in Definition IV.2.2, and this provides a section of the epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ as graded vector spaces. To show that it provides a section as *algebras*, one has to show that $\mathrm{DCH}_\phi(X)$ is closed under the intersection product of X (the unit 1_X is automatically distinguished as remarked before). Let $\alpha \in \mathrm{DCH}^i_\phi(X)$ and $\beta \in \mathrm{DCH}^j_\phi(X)$. By definition, the morphisms $\phi \circ \alpha : \mathbb{1}(-i) \to M$ and $\phi \circ \beta : \mathbb{1}(-j) \to M$ determine symmetrically distinguished morphisms. Therefore $(\phi^{\otimes 2}) \circ (\alpha \otimes \beta) = (\phi \circ \alpha) \otimes (\phi \circ \beta) : \mathbb{1}(-i-j) \to M^{\otimes 2}$ also determines symmetrically distinguished morphism, as can be seen in the following diagram.

$$\begin{tikzcd}
\mathbb{1}(-i-j) \arrow[r, "\alpha\otimes\beta"] \arrow[dr] & \mathfrak{h}(X)^{\otimes 2} \arrow[r, "\delta_X"] \arrow[d, "\phi^{\otimes 2}", "\simeq"'] & \mathfrak{h}(X) \arrow[d, "\phi", "\simeq"'] \\
 & F(M)^{\otimes 2} \arrow[r, "F(\mu)"] & F(M).
\end{tikzcd}$$

Condition (⋆) implies that $\mu : M^{\otimes 2} \to M$, which is determined by the above commutative diagram, is a symmetrically distinguished morphism. Therefore, the composition $\phi \circ \delta_X \circ (\alpha \otimes \beta)$ in the above diagram determines a symmetrically distinguished morphism, which means that $\alpha \cdot \beta = \delta_{X,*}(\alpha \otimes \beta)$ is in $\mathrm{DCH}_\phi(X)$. The assertion concerning Chern classes is tautological. □

As a consequence, for holomorphic symplectic varieties, the distinguished marking conjecture II.3.10 implies the section property conjecture II.3.5. Moreover, distinguished markings have a strong relation with multiplicative Chow–Künneth decompositions discussed in §I.3.

Proposition IV.3.5 (Distinguished marking and MCK decomposition [**FV19a**, Prop. 6.1]). *Let X be a smooth projective variety with a marking ϕ that satisfies ($\star_{\mathrm{Mult}}$). Then X has a multiplicative Chow–Künneth decomposition with the property that* $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$. *Moreover, equality holds if Murre's conjecture I.2.3 (D) is true.*

[2]A *partial diagonal* of a self-product X^n is a subvariety of the form $\{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for all } i \sim j\}$ for an equivalence relation $\sim$ on $\{1, \ldots, n\}$.

Proof. It is easy to check that if X and Y are smooth projective varieties endowed with markings satisfying ($\star_{\mathrm{Mult}}$), the product marking on $X \times Y$ also satisfies ($\star_{\mathrm{Mult}}$). Moreover, the graphs of the projection morphisms are distinguished for the product markings. Therefore, compositions of distinguished correspondences are distinguished.

Let A be an abelian variety and let $p \in \mathrm{DCH}(A \times A)$ be a symmetrically distinguished projector. The Deninger–Murre Chow–Künneth projectors π^i_A in [DM91] of A are symmetrically distinguished. Since the Chow–Künneth projectors are central modulo homological equivalence, we see that $p \circ \pi^i_A = \pi^i_A \circ p \in \mathrm{CH}^*(A \times A)$ and in particular, these provide distinguished Chow–Künneth projectors for (A, p).

It follows that, assuming X has a marking ϕ that satisfies ($\star_{\mathrm{Mult}}$), X admits a distinguished Chow–Künneth decomposition. We conclude that X has a multiplicative Chow–Künneth decomposition by noting that since a Künneth decomposition is always multiplicative, any distinguished Chow–Künneth decomposition is also multiplicative.

Finally, the inclusion $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) \subseteq \mathrm{CH}^*(X^n)_{(0)}$ is a consequence of the following three facts: the product Chow–Künneth decomposition $\{\pi^i_{X^n}\}$ is distinguished, the cycle $(\pi^i_{X^n})_*\alpha$ is homologically trivial (and hence numerically trivial) for all $\alpha \in \mathrm{CH}^j(X^n)$ and all $i \neq 2j$, and $(\pi^i_{X^n})_*\alpha$ is distinguished if α is. Murre's conjecture (D) for X^n stipulates that $\mathrm{CH}^i(X^n)_{(0)}$ should inject in cohomology *via* the cycle class map, and in particular that the surjective quotient morphism $\mathrm{CH}^i(X^n) \to \overline{\mathrm{CH}}^*(X^n)$ is an isomorphism when restricted to $\mathrm{CH}^i(X^N)_{(0)}$. Since the quotient morphism is surjective when restricted to $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n)$, Murre's conjecture implies $\mathrm{DCH}^*_{\phi^{\otimes n}}(X^n) = \mathrm{CH}^*(X^n)_{(0)}$. ∎

Remark IV.3.6. Recently, a generalized Kuga–Satake construction [KSV19] was worked out for all compact hyper-Kähler varieties: it embeds the whole cohomology of the hyper-Kähler variety as a sub-Hodge structure into the cohomology of some complex torus, in such a way that the embedding is compatible with the Poincaré pairing and the Lie algebra action of Looijenga–Lunts–Verbitsky [LL97] [Ver96]. In the projective setting and assuming the Hodge conjecture, the embedding, as well as its left inverse, is algebraic and the motive of the hyper-Kähler variety is of abelian type; in other words, it acquires a marking in the sense of Definition IV.2.1. We expect that the theory developed in this chapter should produce the space of distinguished cycles in full generality.

IV.3.1. Negative results. As a consequence of Proposition IV.3.5, we cannot expect all smooth projective varieties to have distinguished markings: there are already counter-examples for the existence of multiplicative Chow–Künneth decomposition provided in §I.3.3. For instance, Example I.3.9 implies that a very general curve of genus > 2 or a Fermat curve of degree $4 \leq d \leq 100$ admits no distinguished markings.

Another obstruction to the existence of a distinguished marking is the Beauville–Voisin property: if the variety X is regular (*i.e.* $q(X) = 0$) and satisfies the section property (for example when it admits a distinguished marking), there is a distinguished 0-cycle c_X such that the top self-intersection of any divisor, as well as the top Chern class of T_X is proportional to c_X. One can easily produce a counter-example: the blow-up of a K3 surface at a point that does not represent the Beauville–Voisin class. A less artificial example is O'Grady's result discussed in Proposition I.3.14: he showed in [O'G16] the existence of surfaces in $\mathbf{P}^3$ violating the Beauville–Voisin property, for any degree $d \geq 7$. For a general surface in $\mathbf{P}^3$ of degree $d \geq 7$, even if the Beauville–Voisin property as well as the section property is (trivially) satisfied, we expect that they have no distinguished markings: assuming that its motive is Kimura finite dimensional, then by Remark I.2.4, the degree-0 and top degree part of a multiplicative Chow–Künneth decomposition is of the form $\pi^0 = c_X \times 1_X$ and $\pi^{2d} = 1_X \times c_X$. However, Proposition I.3.14 shows that there is no such multiplicative Chow–Künneth decompositions, hence X has no distinguished markings by Proposition IV.3.5.

IV.4. Positive results

To give examples of varieties admitting a distinguished marking, we established in [**FV19a**] a list of standard operations that produce new distinguished markings out of old ones, then apply them to "basic building blocks."

IV.4.1. Operations. Let us summarize the operations here and refer to [**FV19a**, §4] for the proofs.

Products [**FV19a**, Proposition 4.1]. Assume X and Y are smooth projective varieties satisfying the condition ($\star$). Then the natural marking on the product $X \times Y$ satisfies ($\star$) and has the additional property that the graphs of the two natural projections are distinguished.

If in addition X and Y are equipped with the action of a finite group G and the respective markings satisfy ($\star_G$), then the product marking on $X \times Y$ satisfies ($\star_G$).

Projective bundles [**FV19a**, Proposition 4.5]. Let X be a smooth projective variety and let E be a vector bundle over X of rank $(r+1)$. Let $\pi : \mathbf{P}(E) \to X$ be the associated projective bundle. If we have a marking for X satisfying ($\star$) such that all Chern classes of E are distinguished, then $\mathbf{P}(E)$ has a natural marking such that $\mathbf{P}(E)$ satisfies ($\star$) and such that the projection $\pi : \mathbf{P}(E) \to X$ is distinguished.

If in addition X is equipped with the action of a finite group G such that E is G-equivariant and such that the marking of X satisfies ($\star_G$), then the natural marking of $\mathbf{P}(E)$ satisfies ($\star_G$).

Blow-ups [**FV19a**, Proposition 4.8]. Let X be a smooth projective variety and let $i : Y \hookrightarrow X$ be a smooth closed subvariety. If we have markings satisfying the condition ($\star$) for X and Y such that the graph of the inclusion morphism $i : Y \hookrightarrow X$ is distinguished, then $\widetilde{X}$, the blow-up of X along Y, has a natural marking that satisfies ($\star$) and is such that the natural morphisms are all distinguished [3].

If in addition X is equipped with the action of a finite group G such that $G \cdot Y = Y$ and such that the markings of X and Y satisfy ($\star_G$), then the natural marking of $\widetilde{X}$ also satisfies ($\star_G$).

Generically finite surjections [**FV19a**, Propositions 4.9 and 4.11]. Let $\pi : X \to Y$ be a generically finite and surjective morphism between smooth projective varieties. If X has a marking satisfying ($\star_{\mathrm{Mult}}$) and such that the cycle ${}^t\Gamma_\pi \circ \Gamma_\pi$ is distinguished in $\mathrm{CH}(X \times X)$, then Y has a natural marking that satisfies ($\star_{\mathrm{Mult}}$) and is such that the graph of π is distinguished.

If moreover, π is étale and the marking for X satisfies ($\star_{\mathrm{Chern}}$), then the natural marking for Y also satisfies ($\star_{\mathrm{Chern}}$).

[3]The exceptional divisor E is endowed with the natural marking by its projective bundle structure over Y.

Quotients [**FV19a**, Proposition 4.12]. Let X be a smooth projective variety endowed with an action of a finite group G such that the quotient $Y := X/G$ is smooth. If there is a marking for (X, G) satisfying $(\star_{\mathrm{Mult}})$ and $(\star_G)$, then Y has a natural marking that satisfies $(\star_{\mathrm{Mult}})$ and is such that the quotient morphism $\pi : X \to Y$ is distinguished.

Moreover, if $\pi : X \to Y$ is étale or a cyclic covering along a divisor D such that $D \in \mathrm{DCH}(X)$ and if the marking for X satisfies $(\star_{\mathrm{Chern}})$, then the natural marking for Y also satisfies $(\star_{\mathrm{Chern}})$.

(Nested) Hilbert schemes [**FV19a**, Propositions 4.13 and 4.14]. Assume X is a smooth projective variety with a marking that satisfies $(\star)$. Then $X^{[2]}, X^{[1,2]}$, and $X^{[2,3]}$ have natural markings that satisfy $(\star)$ and are such that the universal subschemes are distinguished.

Birational hyper-Kähler varieties [**Rie14**], *see* [**FV19a**, Corollary 4.17]. Let X and Y be birationally isomorphic hyper-Kähler varieties. If X has a marking that satisfies $(\star)$, then so does Y.

IV.4.2. Main result and applications. We find some basic varieties admitting distinguished markings (*i.e.*satisfying the condition $(\star)$), for example, varieties with Tate motives (*e.g.* homogeneous varieties, toric varieties) [**FV19a**, §5.1], hyperelliptic curves [**FV19a**, §5.2], abelian varieties, cubic Fermat hypersurfaces [**FV19a**, §5.3], K3 surfaces with Picard rank ≥ 19 [**FV19a**, §5.4], generalized Kummer varieties [**FV19a**, §5.5] etc. By applying the operations in §IV.4.1, we obtain the following result.

Theorem IV.4.1 ([**FV19a**, Theorem 2]). *Let E be the smallest collection of isomorphism classes of smooth projective complex varieties that contains varieties with Chow groups of finite rank (as $\mathbf{Q}$-vector spaces), abelian varieties, hyperelliptic curves, cubic Fermat hypersurfaces, K3 surfaces with Picard rank ≥ 19, and generalized Kummer varieties, and that is stable under the following operations:*

- *(i) if X and Y belong to E, then $X \times Y$ belongs to E;*
- *(ii) if X belongs to E, then $\mathbf{P}(\oplus_i \mathbb{S}_{\lambda_i} T_X)$ belongs to E, where T_X is the tangent bundle of X, the λ_i is a non-increasing sequence of integers and $\mathbb{S}_{\lambda_i}$ is the Schur functor associated to λ_i;*
- *(iii) if X belongs to E, then the Hilbert scheme of length-2 subschemes $X^{[2]}$, as well as the nested Hilbert schemes $X^{[1,2]}$ and $X^{[2,3]}$ belong to E;*
- *(iv) if X is a curve or a surface that belongs to E, then for any $n \in \mathbf{N}$, the Hilbert scheme of length-n subschemes $X^{[n]}$, as well as the nested Hilbert schemes $X^{[n,n+1]}$ belong to E.*
- *(v) if one of two birationally equivalent hyper-Kähler varieties belongs to E, so does the other.*

If X is a smooth projective variety that belongs to E, then X admits a marking that satisfies $(\star)$, so that the section property is satisfied: the $\mathbf{Q}$-algebra epimorphism $\mathrm{CH}(X) \twoheadrightarrow \overline{\mathrm{CH}}(X)$ admits a section (as $\mathbf{Q}$-algebras) whose image contains the Chern classes of X. In particular, if X is moreover regular, the restriction of the cycle class map to the Beauville–Voisin subalgebra $R^*(X)$ of $\mathrm{CH}^*(X)$ generated by divisors and Chern classes of the T_X is injective.

Combining this with Proposition IV.3.5, all varieties in E have multiplicative Chow–Künneth decompositions.

Theorem IV.4.1 implies some evidences for the distinguished marking conjecture II.3.10 and hence the section property conjecture II.3.5.

Corollary IV.4.2. *Let X be a product of holomorphic symplectic varieties that are birational to either the Hilbert scheme of length-n subschemes of an abelian surface, or a Kummer surface, or a K3* *surface with Picard number ≥ 19, or a generalized Kummer variety. Then Conjectures II.3.10 and II.3.5 hold for X.*

We therefore recover immediately the author's old result [**Fu15**] confirming the Beauville–Voisin conjecture II.3.3 for all generalized Kummer varieties.

COROLLARY IV.4.3 ([**Fu15**]). *Let A be an abelian surface and let n be a positive integer. Denote by $K_n(A)$ the n-th generalized Kummer variety associated with A. Then any cycle that is a polynomial in $\mathbf{Q}$-divisors and Chern classes of $K_n(A)$ (i.e. in the Beauville–Voisin ring) is rationally equivalent to zero if and only if it is numerically equivalent to zero.*

CHAPTER V

Cycles on the universal family: the Franchetta property

In this chapter, we are interested in hyper-Kähler varieties defined over function fields [**FLVS19**]. It is especially useful to study the generic fiber of the universal family of polarized hyper-Kähler variety over the moduli space, for example to find new interesting evidences to the Beauville–Voisin conjecture II.3.3 and its refinement II.3.4 due to Voisin [**Voi16a**]. The expectation, called the Franchetta property, is that the Chow groups of this generic fiber are simple, in the sense that the cycle class map is injective. Let us first recall the background.

V.1. Origin: cycles on universal curves and universal K3 surfaces

The original Franchetta conjecture [**Fra54**] (proved in [**Har83**], see also [**Mes87**] and [**AC87**]) states the following.

Theorem V.1.1. *For an integer $g \geq 2$, let $\mathcal{M}_g$ be the moduli stack of smooth projective curves of genus g, and let $\mathcal{C} \to \mathcal{M}_g$ be the universal curve. Then for any line bundle L on $\mathcal{C}$ and any closed point $b \in \mathcal{M}_g$, the restriction of L to the fiber C_b is a power of the canonical bundle of C_b.*

In other words, the Picard group of the generic fiber of $\mathcal{C} \to \mathcal{M}_g$ is of rank 1 and generated by the (relative) canonical bundle.

In the case of the universal family of K3 surfaces, O'Grady proposed in [**O'G13**] the following analogue of the Franchetta conjecture.

Conjecture V.1.2 (O'Grady [**O'G13**]). *Let $\mathcal{F}_g$ be the moduli stack of polarized K3 surfaces of genus g and $\mathcal{S} \to \mathcal{F}_g$ be the universal family of K3 surfaces. Then for any algebraic cycle $z \in \mathrm{CH}^2(\mathcal{S})$ and any point $b \in \mathcal{F}_g$, the restriction of z to the fiber K3 surface S_b is a multiple of the Beauville–Voisin class of S_b.*

Equivalently, the conjecture of O'Grady says that the generic fiber of $\mathcal{S} \to \mathcal{F}_g$, which is a K3 surface defined over the function field of $\mathcal{F}_g$, has 1-dimensional CH^2, generated by the second Chern class of the relative tangent bundle (which is 24 times the relative Beauville–Voisin class). Using Mukai models, Conjecture V.1.2 was verified in [**PSY17**] for K3 surfaces of genus $g \leq 10$ and $g \in \{12, 13, 16, 18, 20\}$. Otherwise, Conjecture V.1.2 is still wide open.

The main goal of this chapter is to investigate Conjecture II.3.6, which is the higher-dimensional analogue of O'Grady's Conjecture V.1.2. Let us restate it here.

Conjecture V.1.3 (Generalized Franchetta conjecture, *cf.* [**FLVS19**] [**BL19a**]). *Let $\mathcal{F}$ be the moduli stack of a locally complete family of polarized hyper-Kähler varieties, and let $\mathcal{X} \to \mathcal{F}$ be the universal family. For any $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$, if its restriction to a very general fiber is homologically trivial then its restriction to any fiber is (rationally equivalent to) zero.*

We note that a cycle is homologically trivial when restricted to a very general fiber if and only if it is homologically trivial when restricted to any fiber. More generally, given any smooth family of projective varieties $\mathcal{X} \to \mathcal{F}$ with $\mathcal{F}$ smooth, we will say that $\mathcal{X} \to \mathcal{F}$ satisfies the *Franchetta property* if for any $z \in \mathrm{CH}^*(\mathcal{X})_{\mathbf{Q}}$ which is fiberwise homologically trivial, its restriction to any fiber is (rationally equivalent to) zero.

It would be too optimistic[1] to believe that the Franchetta property is satisfied by self-products of hyper-Kähler varieties $\mathcal{X} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} \mathcal{X} \to \mathcal{F}$. We may nevertheless ask, given a locally complete family $\mathcal{X} \to \mathcal{F}$ of polarized hyper-Kähler varieties, for which integers n, the n-th relative fiber product $\mathcal{X} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} \mathcal{X} \to \mathcal{F}$ satisfies the Franchetta property. We provide some results in that direction in Theorems V.3.1, V.2.2, and V.4.1 below.

V.2. Fano varieties of lines of universal cubic fourfolds

For the universal family of Fano varieties of lines of cubic fourfolds, which form a locally complete family of projective hyper-Kähler fourfolds of $\mathrm{K3}^{[2]}$-type ([**BD85**]), we have the following slightly stronger result than predicted by Conjecture V.1.3.

Theorem V.2.1 ([**FLVS19**, Theorem 1.9]). *Let C be the moduli stack of smooth cubic fourfolds, let $\mathcal{X} \to C$ be the universal family and let $\mathcal{F} \to C$ be the universal family of Fano varieties of lines in the fibers of $\mathcal{X}/C$. Then for any $z \in \mathrm{CH}^*(\mathcal{F})_{\mathbf{Q}}$ and any $b \in C$, the restriction of z to the fiber F_b is numerically trivial if and only if it is (rationally equivalent to) zero.*

The following analogous result on the relative square of the universal family of Fano varieties of lines will be used in order to study the Lehn–Lehn–Sorger–van Straten hyper-Kähler eightfold (Theorem V.4.1).

Theorem V.2.2 ([**FLVS19**, Theorem 1.10]). *Notation is as in Theorem V.2.1. Then $\mathcal{F} \times_C \mathcal{F} \to C$ satisfies the Franchetta property: for $z \in \mathrm{CH}^*(\mathcal{F} \times_C \mathcal{F})_{\mathbf{Q}}$ and any $b \in C$, the restriction of z to the fiber $F_b \times F_b$ is numerically trivial if and only if it is (rationally equivalent to) zero.*

The proof of Theorem V.2.1 (resp. Theorem V.2.2) consists of two steps:

- First, we show that cycles that belong to the image of the restriction map $\mathrm{CH}^*(\mathcal{F})_{\mathbf{Q}} \to \mathrm{CH}^*(F_b)_{\mathbf{Q}}$ (*resp.* $\mathrm{CH}^*(\mathcal{F} \times_C \mathcal{F})_{\mathbf{Q}} \to \mathrm{CH}^*(F_b \times F_b)_{\mathbf{Q}}$) are *tautological*, that is, they are in the subring generated by the Plücker polarization and the second Chern class of F_b (*resp.* the subring generated by the diagonal, the Plücker polarization, the second Chern class on both factors, and finally the incidence subvariety I parametrizing pairs of intersecting lines).
- Second, we completely determine in terms of generators and relations the rings of tautological cycles for F_b and $F_b \times F_b$. In the case of F_b, we conclude by the result of Voisin [**Voi08**] confirming the Beauville–Voisin conjecture II.3.3 for Fano varieties of lines on cubic fourfolds. In the case of $F_b \times F_b$, all relations but one de Cataldohad been established in [**Voi08**] and [**SV16a**]. The remaining relation is established in the appendix of [**FLVS19**].

V.3. Hilbert schemes of universal K3 surfaces

Concerning the Hilbert schemes of polarized K3 surfaces, we have the following main result.

[1]When $g \geq 4$, the relative square of the universal curve of genus g does not satisfy the Franchetta property because the degree-0 0-cycle $p_1^*K_C \cdot p_2^*K_C - \deg(K_C)p_1^*K_C \cdot \Delta_C$ is not rationally trivial for C very general of genus $g \geq 4$; see [**GG03**].

THEOREM V.3.1 ([**FLVS19**, Theorems 1.4 and 1.6]). *Let $\mathcal{M}$ be the moduli stack of smooth K3 surfaces of genus g and let $\mathcal{S} \to \mathcal{M}$ be the universal family. Then the following families satisfy the Franchetta property:*

(i) $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \mathcal{S}$, $\mathcal{S} \times_{\mathcal{M}} \mathrm{Hilb}^2_{\mathcal{M}} \mathcal{S}$, *and* $\mathrm{Hilb}^3_{\mathcal{M}} \mathcal{S}$, *where* $\mathcal{S} \to \mathcal{M}$ *is the universal family of smooth K3 surfaces of genus 2, 4, or 5.*
(ii) $\mathrm{Hilb}^{r_1}_{\mathcal{M}} \mathcal{S} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathrm{Hilb}^{r_m}_{\mathcal{M}} \mathcal{S}$, *where* $\mathcal{S} \to \mathcal{M}$ *is the universal family of smooth quartic (i.e. genus 3) K3 surfaces and* $r_1 + \cdots + r_m \leq 5$.
(iii) *The relative square and relative Hilbert square of the universal family of K3 surfaces of genus 6, 7, 8, 9, 10, or 12.*

The case of relative Hilbert schemes is immediately reduced to the case of relative powers, thanks to the result of de Cataldo–Migliorini [**dCM02**]. For the case of relative powers, the strategy is as in §V.2: show first that the restriction to fibers of cycles coming from the universal family are tautological, then use known cases of the Beauville–Voisin conjecture II.3.3 to conclude. Let us first make precise the notion of tautological cycles.

DEFINITION V.3.2 (Tautological ring). Let (S, H) be a polarized K3 surface and let $r \in \mathbf{N}$. Set $h := c_1(H) \in \mathrm{CH}^1(S)$. The *tautological ring* $R^*(S^r)$ is the subring of the (rational) Chow ring $\mathrm{CH}^*(S^r)$ generated by the big diagonals $\Delta_{i,j}$ $(1 \leq i < j \leq r)$, the polarization classes $h_i := \mathrm{pr}_i^*(h)$ and the Beauville–Voisin classes $\mathfrak{o}_i := \mathrm{pr}_i^*(c_S)$ $(1 \leq i \leq r)$.

REMARK V.3.3. Using [**BV04**, Proposition 2.6], we see that the tautological rings of different powers of a K3 surface are stable under push-forwards and pull-backs along all kinds of (partial) diagonal inclusions.

The techniques used in the proof depend on the available Mukai models of K3 surfaces. Recall that for a natural number g, we say that a *Mukai model* for K3 surfaces of genus g exists if there exist an ambient homogeneous space $G = G_g$ (often a Grassmannian) and a globally generated homogeneous vector bundle $E = E_g$ on G such that the zero locus of a general section of E gives a general K3 surface of genus g. For the available constructions of Mukai models and the corresponding G and E, we refer to [**PSY17**] as well as to the original sources [**Muk88**], [**Muk92**], [**Muk06**], [**Muk16**]. Accordingly, we have a universal family

$$\begin{tikzcd} \mathcal{S} \arrow[r, "p"] \arrow[d, "\pi"'] & G \\ B = H^0(G, E) & \end{tikzcd}$$

and we denote $B^\circ \subset B$ the locus parameterizing smooth K3 surfaces of genus g.

The crucial condition for our techniques to work is the following.

DEFINITION V.3.4. For $r \in \mathbf{N}^*$, we say that the Mukai model (G, E) satisfies the condition $(\star_r)$ if

$(\star_r)$: for any $x_1, \ldots, x_r$ distinct points of G, the evaluation map

$$H^0(G, E) \to \bigoplus_{i=1}^{r} E_{x_i}$$

is surjective. Or equivalently, $H^0(G, E \otimes I_{x_1} \otimes \cdots \otimes I_{x_r})$ is of codimension $r \cdot \mathrm{rank}(E)$ in $H^0(G, E)$. Clearly, $(\star_r)$ implies $(\star_k)$ for all $k \leq r$.

In the sequel, for a S-scheme X, we will denote by $X^{n/S}$ the fiber product $X \times_S \cdots \times_S X$ with n-factors.

Proposition V.3.5. *The notation is as above. Fix a genus g for which a Mukai model exists for K3 surfaces of genus g and fix such a Mukai model which satisfies condition $(\star_r)$. Assume that*

$$\mathrm{Im}\,(\mathrm{CH}^*(\mathcal{S}) \to \mathrm{CH}^*(S_b)) = R^*(S_b),$$

for any $b \in B^\circ$. Then

$$\mathrm{Im}\left(\mathrm{CH}^*(\mathcal{S}^{r/B}) \to \mathrm{CH}^*(S_b^r)\right) = R^*(S_b^r),$$

for any $b \in B^\circ$.

Proof. We proceed by induction on r. Consider the evaluation map $q : \mathcal{S}^{r/B} \to G^r$, which is a stratified projective bundle (see [FLVS19, Definition 5.1]) where the stratification on G^r is given by the different types of incidence relations for r points of G:

(V.1)
$$\begin{tikzcd} \mathcal{X}_n = \mathcal{S} \arrow[r, hook] \arrow[d, "q_n=p"] \arrow[dr, phantom, "\square"] & \ldots \arrow[r, hook] \arrow[dr, phantom, "\square"] & \mathcal{X}_1 \arrow[r, hook] \arrow[d, "q_1"] \arrow[dr, phantom, "\square"] & \mathcal{X}_0 = \mathcal{S}^{r/B} \arrow[r] \arrow[d, "q_0=q"] & B \\ Y_n = G \arrow[r, hook] & \ldots \arrow[r, hook] & Y_1 \arrow[r, hook] & Y_0 = G^r. \end{tikzcd}$$

By Proposition [FLVS19, Proposition 5.2], for any $b \in B^\circ$, one has

$$\mathrm{Im}\left(\mathrm{CH}^*(\mathcal{S}^{r/B}) \to \mathrm{CH}^*(S_b^r)\right) = \sum_{i=0}^{n} \iota_{i,*}\,\mathrm{Im}\left(\mathrm{CH}^*(Y_i) \to \mathrm{CH}^*(\mathcal{X}'_{ib})\right), \tag{V.2}$$

where $\mathcal{X}'_i$ is the Zariski closure of $\mathcal{X}_i \backslash \mathcal{X}_{i+1}$. Let us show that each term of the sum in (V.2) is in the tautological ring $R^*(S_b)$ by ascending order for $0 \le i \le n$:

- If $i = 0$, since the Chow ring of G satisfies the Künneth formula, we only need to show that
$$\mathrm{Im}\,(\mathrm{CH}^*(G) \to \mathrm{CH}^*(S_b)) \subset R^*(S_b),$$
which is true by assumption.
- If a general point of Y_i parameterizes r points of G where at least two of them coincide, the contribution of the i-th term of the sum in (V.2) factors through $R^*(S_b^{r-1})$ (via the diagonal push-forward) by the induction hypothesis, hence is contained in $R^*(S_b^r)$ (Remark V.3.3).
- If a general point of Y_i parameterizes r distinct points of G, the hypothesis $(\star_r)$ means precisely that any r distinct points of G impose independent conditions on B, each of codimension rank(E). Therefore, $\mathcal{X}'_i$, the Zariski closure of $\mathcal{X}_i \backslash \mathcal{X}_{i+1}$, has codimension in $\mathcal{X}_{i-1}$ equal to $\mathrm{codim}_{Y_{i-1}}(Y_i)$. The excess intersection formula ([Ful98, §6.3]) applied to the cartesian diagram
$$\begin{tikzcd} \mathcal{X}_i = \mathcal{X}_{i+1} \cup \mathcal{X}'_i \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{i-1} \arrow[d] \\ Y_i \arrow[r, hook] & Y_{i-1} \end{tikzcd}$$
tells us that modulo the $(i+1)$-th term of the sum in (V.2), the contribution of the i-th term is contained in the $(i-1)$-th term.

$\square$

To summarize, we have the following.

THEOREM V.3.6. *Fix a genus g for which a Mukai model exists for K3 surfaces of genus g, and fix such a Mukai model. Assume that*

(i) the Mukai model satisfies the condition $(\star_r)$;
(ii) the O'Grady conjecture V.1.2 is true for the universal family $\mathcal{S} \to B$ of K3 surfaces of genus g;
(iii) the cycle class map restricted to the tautological ring $R^*(S^r)$ is injective for the very general K3 surface S of genus g.

Then the Franchetta property holds for $\mathcal{S}^{[r_1]/B} \times_B \cdots \times_B \mathcal{S}^{[r_m]/B}$, for any $r_1, \ldots, r_m$ whose sum is $\leq r$.

PROOF. The case of relative powers $\mathcal{S}^{k/B}$, for any $k \leq r$, is a direct consequence of Proposition V.3.5 and the hypothesis on the injectivity of the cycle class map on the tautological ring. The other cases reduce to the cases of $\mathcal{S}^{k/B}$ for all $1 \leq k \leq r$ by making use of de Cataldo–Migliorini's result [dCM02] for Chow motives of Hilbert schemes of surfaces. □

We apply Theorem V.3.6 to some Mukai models to get concrete unconditional results.

PROOF OF THEOREM V.3.1. Assumption (*ii*) was proven for $g \in \{2, \ldots, 10\} \cup \{12\}$ in [PSY17]. Assumption (*iii*) is taken care of for $r \leq 43$ by [Voi08, Proposition 2.2]. It remains to check assumption (*i*) of Theorem V.3.6; we proceed by a case-by-case analysis of the positivity of the homogeneous bundle in the Mukai model, (see Mukai's series of papers [Muk88], [Muk92], [Muk06], [Muk16] for more information on the geometry of these models).

- K3 surfaces of genus $g = 2$ are smooth degree 6 hypersurfaces in the weighted projective space $\mathbf{P} := \mathbf{P}(1,1,1,3)$.[2] The Mukai model for this family is thus $(G, E) = (\mathbf{P}, \mathcal{O}(6))$. Note that the K3 surfaces in this family all avoid the singular point $O := [0,0,0,1]$. Let us check the condition $(\star_3)$, *i.e.*, that the evaluation map
$$H^0(\mathbf{P}, \mathcal{O}(6)) \to \bigoplus_{i=1}^{3} \mathbf{C}_{x_i}$$
is surjective for distinct $x_1, x_2, x_3 \neq O$, where $\mathbf{C}_x$ denotes the fiber of $\mathcal{O}(6)$ at x. It is easy to see that $\mathbf{P}(1,1,1,3)$ is isomorphic to the projective cone over the third Veronese embedding of $\mathbf{P}^2$ and O is the vertex. By upper-semicontinuity, it is enough to treat the most degenerate case for three distinct points of $\mathbf{P} \setminus \{O\}$, which is when they lie in the same ruling of the projective cone. In this case, since the restriction of $\mathcal{O}(6)$ to the ruling is $\mathcal{O}(2)$, condition $(\star_3)$ follows from the surjections
$$H^0(\mathbf{P}, \mathcal{O}(6)) \twoheadrightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2)) \twoheadrightarrow \bigoplus_{i=1}^{3} \mathbf{C}_{x_i},$$
where $\mathbf{P}^1$ is the ruling which contains x_1, x_2, x_3.
- For quartic surfaces ($g = 3$), let us first show that $(\mathbf{P}^3, \mathcal{O}(4))$ satisfies $(\star_5)$, *i.e.*, that the evaluation map
$$H^0(\mathbf{P}^3, \mathcal{O}(4)) \to \bigoplus_{i=1}^{5} \mathbf{C}_{x_i}$$
is surjective for distinct x_i. Again, it is enough to treat the most degenerate cases, namely:

[2] Equivalently, these K3 surfaces are also double covers of $\mathbf{P}^2$ ramified along smooth sextic curves.

– when $x_1, \dots, x_5$ are collinear, then this follows from the surjectivity of the restriction and the evaluation

$$H^0(\mathbf{P}^3, \mathcal{O}(4)) \twoheadrightarrow H^0(\mathbf{P}^1, \mathcal{O}(4)) \twoheadrightarrow \bigoplus_{i=1}^{5} \mathbf{C}_{x_i},$$

where $\mathbf{P}^1$ is the line containing these points.

– when $x_1, \dots, x_5$ are on a conic C. Then the Koszul resolution provides an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-3) \to \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_C \to 0,$$

which allows us to see that the restriction map $H^0(\mathbf{P}^3, \mathcal{O}(4)) \to H^0(C, \mathcal{O}_C(8))$ is surjective. Since $H^0(C, \mathcal{O}_C(8)) \to \bigoplus_{i=1}^{5} \mathbf{C}_{x_i}$ is clearly surjective, we are done.

Condition $(\star_5)$ is proven.

- For $g = 6$, the Mukai model is $(G, E) = (\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2))$, where $\mathcal{O}(1)$ is the Plücker line bundle. It is clear that the condition $(\star_2)$ is equivalent to the surjectivity of
$$H^0(G, \mathcal{O}(1)) \to \mathbf{C}_{x_1} \oplus \mathbf{C}_{x_2}$$
for any two distinct points $x_1, x_2 \in G$. This last condition follows from the very ampleness of the Plücker line bundle $\mathcal{O}(1)$.
- For $g = 7$, the Mukai model is $(G, E) = \left(\mathrm{OGr}(5,10), U^{\oplus 8}\right)$, where $\mathrm{OGr}(5,10)$ is the orthogonal Grassmannian parameterizing isotropic subspaces of dimension 5 in a vector space of dimension 10 equipped with a non-degenerate quadratic form and U is a line bundle corresponding to the half spinor representation. The proof is similar to the previous case: one uses the very ampleness of U.
- For $g = 8$, the Mukai model is $(G, E) = \left(\mathrm{Gr}(2,6), \mathcal{O}(1)^{\oplus 6}\right)$, where $\mathcal{O}(1)$ is the Plücker line bundle. The proof goes as for $g = 6$ by the very ampleness of the Plücker line bundle.
- For $g = 9$, the Mukai model is $(G, E) = \left(\mathrm{LGr}(3,6), \mathcal{O}(1)^{\oplus 4}\right)$, where $\mathrm{LGr}(3,6)$ is the symplectic Grassmannian parameterizing Lagrangian subspaces in a 6-dimensional vector space equipped with a symplectic form and $\mathcal{O}(1)$ is the restriction of the Plücker line bundle of $\mathrm{Gr}(3,6)$. The proof goes as before: one uses the very ampleness of $\mathcal{O}(1)$.
- For $g = 10$, the Mukai model is $(G, E) = (G_2/P, \mathcal{O}(1)^{\oplus 3})$, where G is the 5-dimensional quotient of the simply-connected semi-simple algebraic group of type G_2 by a maximal parabolic subgroup P and $\mathcal{O}(1)$ is the line bundle associated with the adjoint representation of G_2; in other words, $G = G_2/P \hookrightarrow \mathbf{P}(\mathfrak{g}_2^\vee)$. Again, we can conclude by the very ampleness of $\mathcal{O}(1)$.
- For $g = 12$, we use a slight variant of the above argument. Indeed, the general K3 surface of genus 12 can be constructed as an anti-canonical section in a smooth prime Fano threefold X of genus 12 (*cf.* [**IM07**, Section 3.1]). The Fano threefold X has very ample anti-canonical bundle and $H^3(X, \mathbf{Q}) = 0$ ([**Sha99**, Corollary 4.3.5]), so that X has trivial Chow groups[3] (this Fano threefold X is the variety denoted by $X_{22} \subset \mathbf{P}^{13}$ in [**Sha99**, Propositions 4.1.11 and 4.1.12]; actually X is an intersection of

[3]Following Voisin [**Voi13**], we say a smooth projective variety has *trivial Chow groups* if the cycle map $\mathrm{cl}^i : \mathrm{CH}^i(X)_{\mathbf{Q}} \to H^{2i}(X, \mathbf{Q})$ is injective for any i.

quadrics). We now consider a variant of Theorem V.3.6, replacing G by X and E by $-K_X$. The very ampleness of $-K_X$ ensures that condition $(\star_2)$ holds. As X has trivial Chow groups, there is a Chow–Künneth formula for products of X, and so one is reduced to the statement for the K3 surface S_b, which is [PSY17].

□

V.4. Lehn–Lehn–Sorger–van Straten eightfolds

Lehn–Lehn–Sorger–van Straten (LLSvS) considered in [LLSvS17] twisted cubic curves on a cubic fourfold not containing a plane and show that the base of the maximal rationally connected (MRC) quotient of the moduli space of such curves is a hyper-Kähler eightfold. Later, Addington and M. Lehn showed in [AL17] that this hyper-Kähler eightfold is of $\mathrm{K3}^{[4]}$-deformation type. For the universal family of LLSvS hyper-Kähler eightfolds, we have the following result, which confirms the 0-cycle and codimension-2 cases of the generalized Franchetta conjecture.

Theorem V.4.1 ([FLVS19, Theorem 1.11]). *Let C° be the moduli stack of smooth cubic fourfolds not containing a plane and let $\mathcal{Z} \to C^\circ$ be the universal family of LLSvS hyper-Kähler eightfolds ([LLSvS17]). Then*

(i) for any $b \in C^\circ$ and for any $\gamma \in \mathrm{CH}^8(\mathcal{Z})$ which is fiber-wise of degree 0, the restriction of γ to the fiber Z_b is (rationally equivalent to) zero;

(ii) for any $b \in C^\circ$ and for any $\gamma \in \mathrm{CH}^2(\mathcal{Z})_{\mathbf{Q}}$, its restriction to the fiber Z_b is zero if and only if its cohomology class vanishes.

The proof uses in a crucial way the Franchetta property for $\mathcal{F} \times_C \mathcal{F} \to C$ proved in Theorem V.2.2, together with the degree-6 rational dominant map $F \times F \dashrightarrow Z$ constructed by Voisin in [Voi16a, Proposition 4.8].

V.5. Applications

V.5.1. Voisin's refinement of the Beauville–Voisin conjecture. As consequences of the results mentioned before, we obtain some partial confirmation Voisin's conjecture II.3.4 involving algebraically coisotropic subvarieties.

Corollary V.5.1 ([FLVS19, Corollary 1.6]). *Let S be a general K3 surface of genus $g \leq 10$ or 12, and let X be the Hilbert square $X = \mathrm{Hilb}^2(S)$. Let $R^*(X) \subset \mathrm{CH}^*(X)_{\mathbf{Q}}$ denote the $\mathbf{Q}$-subalgebra generated by the polarization class h, the Chern classes c_i, and the Lagrangian surface $T \subset X$ constructed in* [IM07, Proposition 4]. *Then $R^*(X)$ injects into the cohomology of X by the cycle class map.*

Corollary V.5.2 ([FLVS19, Corollary 1.7]). *Let $S \subset \mathbf{P}^3$ be a quartic K3 surface, and let $X = \mathrm{Hilb}^5 S$, $\mathrm{Hilb}^2 S \times \mathrm{Hilb}^2 S \times S$, $\mathrm{Hilb}^2 S \times S^3$, or $\mathrm{Hilb}^2 S \times \mathrm{Hilb}^3 S$. Let $R^*(X) \subset \mathrm{CH}^*(X)_{\mathbf{Q}}$ denote the $\mathbf{Q}$-subalgebra generated by the polarization class h, the Chern classes c_i, the coisotropic subvarieties E_μ of* [Voi16a, 4.1 item 1)], *the Lagrangian surface $T \subset \mathrm{Hilb}^2 S$ constructed in* [IM07, Proposition 4], *and the surface of bitangents $U \subset \mathrm{Hilb}^2 S$. Then $R^*(X)$ injects into the cohomology of X by the cycle class map.*

Corollary V.5.3 ([FLVS19, Corollary 1.12]). *Given any smooth cubic fourfold X containing no planes, let Z be the LLSvS hyper-Kähler eightfold associated to X. Denote by h the polarization* *class. Then the classes*

$$h^8, c_2h^6, c_2^2h^4, c_2^3h^2, c_2^4, c_4h^4, c_2c_4h^2, c_2^2c_4, c_6h^2, c_2c_6, c_4^2, c_8 \in \mathrm{CH}_0(Z)_{\mathbf{Q}}$$

are all proportional, where $c_i := c_i(T_Z)$. We call the generator of degree 1 in this one-dimensional subspace the canonical 0-cycle class *or the* Beauville–Voisin class *of Z, denoted by c_Z.*
Moreover, let $R^*(Z)$ be the $\mathbf{Q}$-subalgebra generated by the polarization class h and the Chern classes c_i together with the following classes of coisotropic subvarieties of Z:

- *the embedded cubic fourfold $X \subset Z$ ([**LLSvS17**]);*
- *the space of twisted cubics contained in a general hyperplane section of X ([**SS17**]);*
- *the coisotropic subvarieties of codimension* 1, 2, 3, 4 *constructed by Voisin* [**Voi16a**, Corollary 4.9]*;*
- *the fixed locus of the anti-symplectic involution ι of Z ([**LLMS18**]);*
- *the images by ι of all the above subvarieties.*

Then $R^8(Z) = \mathbf{Q} \cdot c_Z$.

V.5.2. The Bloch conjecture for the Beauville involution. Another consequence concerns the Bloch conjecture for Beauville's anti-symplectic involution on Hilbert squares of quartic surfaces.

Corollary V.5.4 ([**FLVS19**, Corollary 1.8]). *Let $X = \mathrm{Hilb}^2 S$ be the Hilbert square of a quartic K3 surface S, and let $\iota\colon X \to X$ be Beauville's anti-symplectic involution* [**Bea83b**]*. Then*

$$\iota^* = -\,\mathrm{id}\colon \quad \mathrm{CH}^i(X)_{(2)} \to \mathrm{CH}^i(X)_{(2)} \quad (i \in \{2,4\}),$$
$$\iota^* = \mathrm{id}\colon \quad \mathrm{CH}^4(X)_{(j)} \to \mathrm{CH}^4(X)_{(j)} \quad (j \in \{0,4\}).$$

(Here, the notation $\mathrm{CH}^*(X)_{(*)}$ is the multiplicative bigrading constructed in [**SV16a**]*.)*

V.5.3. Constructing multiplicative Chow–Künneth decompositions. The Franchetta property of the relative square of the Fano varieties of lines of cubic fourfolds, proved in Theorem V.2.2, was used in [**FLV19**] to establish the existence of a multiplicative Chow–Künneth decomposition for cubic fourfolds (Theorem I.3.17).

V.6. Some projects

One project I have in mind is to show in full the Franchetta property for the LLSvS hyper-Kähler eightfold, extending our published result in [**FLVS19**] (see §V.4). The idea is to view the LLSvS hyper-Kähler eightfold as a moduli space of stable objects of the Kuznetsov component of the derived category of the cubic fourfold [**LPZ18**]. By running the arguments of Bülles [**Bü18**], one should be able to show that the Chow motive of the hyper-Kähler eightfold is controlled by the fourth power of the cubic fourfold and hence reduce the problem to the Franchetta property of fourth powers of cubic fourfolds. I believe that the techniques of [**FLVS19**] on stratified projective bundles, explained in the context of universal K3 surfaces §V.3, would then allow us to conclude.

Another family for which I want very much to investigate the Franchetta property is the hyper-Kähler compactification of the (twisted) jacobian fibration associated with the universal hyperplane section of a cubic fourfold, worked out by Laza–Saccà–Voisin [**LSV17**] and Voisin [**Voi18**]. These are hyper-Kähler varieties of O'Grady-10 deformation type. The general idea would also be to control the motive of the hyper-Kähler tenfold by the motive of powers of the cubic fourfold, which is related to another work in progress of Li–Pertusi–Zhao.

CHAPTER VI

Supersingular irreducible symplectic varieties

In this chapter, which is based on the joint work [FL18] with Zhiyuan Li, we treat some arithmetic aspects of hyper-Kähler geometry by studying *supersingular* objects, which live only over fields of positive characteristic. As the base field is no longer the complex numbers and we will lose completely the metric aspect in particular the hyper-Kähler rotations, we decide to rather call them *irreducible symplectic varieties*. In some sense, among irreducible symplectic varieties in positive characteristics, the supersingular ones are the "most different" from their counterparts in characteristic zero. On the one hand, some powerful tools, especially Hodge theory, are not available in positive characteristics; on the other hand, the supersingular ones are actually, at least conjecturally, simpler than the complex ones: they are expected to be unirational, to have motives of Tate type if their odd Betti numbers vanish etc. We will give the general conjectural picture and then study more specifically the moduli spaces of sheaves on K3 and abelian surfaces.

Let k be an algebraically closed field of characteristic $p > 0$. Denote by $W = W(k)$ the ring of Witt vectors of k and by K the field of fractions of W.

VI.1. Origin: supersingular K3 surfaces

Our theory takes into account the recent progress on supersingular K3 surfaces.

Recall that over an arbitrary field, a *K3 surface* is a smooth projective surface with trivial canonical bundle and vanishing first cohomology of the structure sheaf. Let X be a K3 surface. On the one hand, X is called *Artin supersingular* if its formal Brauer group $\widehat{\mathrm{Br}}(X)$ is the formal additive group $\widehat{\mathbb{G}}_a$ (see [Art74], [AM77]), or equivalently, if the Newton polygon associated to the second crystalline cohomology $H^2_{\mathrm{cris}}(X/W(k))$ is a straight line (of slope 1). On the other hand, Shioda [Shi74] introduced another notion of supersingularity for K3 surfaces by considering the algebraicity of the ℓ-adic cohomology classes of degree 2, for any $\ell \neq p$: we say that X is *Shioda supersingular* if the first Chern class map

$$c_1 : \mathrm{Pic}(X) \otimes \mathbf{Q}_\ell \to H^2_{\acute{e}t}(X, \mathbf{Q}_\ell(1))$$

is surjective. This condition is independent of ℓ, as it is equivalent to the maximality of the Picard rank, *i.e.* $\rho_X = b_2(X) = 22$. It is easy to see that Shioda supersingularity implies Artin supersingularity. Conversely, the Tate conjecture [Tat65] for K3 surfaces over finite fields, solved in [Nyg83], [NO85], [Mau14], [Cha13], [MP15], [Cha16] and [KMP16], implies that these two notions actually coincide for any algebraically closed field of positive characteristic, *cf.* [Lie16, Theorem 4.8]:

$$\text{Shioda supersingularity} \Leftrightarrow \text{Artin supersingularity}. \tag{VI.1}$$

Supersingularity being essentially a *cohomological* notion, it is natural to look for its relation to *geometric* properties. Unlike complex K3 surfaces, there exist *unirational* K3 surfaces over fields of positive characteristic: the first examples were constructed by Shioda in [Shi74] and Rudakov–Šafarevič in [RŠ78]. Then Artin [Art74] and Shioda [Shi74] observed that unirational K3 surfaces must have maximal Picard rank 22, hence are supersingular. Conversely, one expects that unirationality is a geometric characterization of the supersingularity for K3 surfaces.

Conjecture VI.1.1 (Artin [Art74], Shioda [Shi74], Rudakov–Šafarevič [RŠ78]). *A K3 surface is supersingular if and only if it is unirational.*

This conjecture has been confirmed in characteristic 2 by Rudakov–Šafarevič [RŠ78] via the existence of quasi-elliptic fibration. In Liedtke [Lie15] and Bragg–Lieblich [BL18], two proofs of Conjecture VI.1.1 are given, but recently a common gap in the proofs was discovered [BL19b]. Note that the conjectural unirationality of supersingular K3 surfaces implies that the Chow motive is of Tate type and in particular that the Chow group of 0-cycles is isomorphic to $\mathbf{Z}$, thus contrasting drastically with the situation over the complex numbers, where CH_0 is infinite dimensional by Mumford's celebrated observation in [Mum68].

The objective of this chapter is to generalize the theory of supersingular K3 surfaces to higher dimensional irreducible symplectic varieties.

VI.2. Basic definitions

VI.2.1. Irreducible symplectic varieties. In positive characteristic, there seems to be no commonly accepted definition of *irreducible symplectic varieties* (see however [Cha16]). In this paper, we define them as follows.

Definition VI.2.1. Let X be a geometrically connected smooth projective variety defined over k of characteristic $p > 0$ and let $\Omega^2_{X/k}$ be the locally free sheaf of algebraic 2-forms over k. The variety X is called *irreducible symplectic* if

(1) $\pi_1^{\text{ét}}(X) = 0$;
(2) the Frölicher spectral sequence $E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k)$ degenerates at E_1;
(3) $H^0(X, \Omega^2_{X/k})$ is spanned by a nowhere degenerate *closed* algebraic 2-form.

In particular, X is even-dimensional with trivial canonical bundle.

Remark VI.2.2. Conditions (1) and (2) in Definition VI.2.1 together imply

$$H^1(X, \mathcal{O}_X) = H^0(X, \Omega^1_X) = H^0(X, T_X) = 0.$$

Due to the lack of Hodge symmetry in positive characteristic, we do not know whether $H^2(X, \mathcal{O}_X) \simeq k$, although we expect it is the case.

VI.2.2. Artin supersingularity. In positive characteristic, the cohomology theory we will use is crystalline cohomology. For any $i \in \mathbf{N}$, we denote by $H^i_{\mathrm{cris}}(X/W)$ the i-th integral crystalline cohomology of X, which is a W-module whose rank is equal to the i-th Betti number of X. We set

$$H^i(X) := H^i_{\mathrm{cris}}(X/W)/\text{torsion}, \ H^i(X)_K = H^i_{\mathrm{cris}}(X/W) \otimes_W K.$$

Then $H^i(X)$ is a free W-module and it is endowed with a natural σ-linear map as following, where σ is the induced Frobenius map on W:

$$\varphi : H^i(X) \to H^i(X)$$

induced from the absolute Frobenius morphism $F : X \to X$ by functoriality. Moreover, by Poincaré duality, φ is injective.

The pair $(H^i(X), \varphi)$ (*resp.* $(H^i(X)_K, \varphi_K)$) forms therefore an F-crystal (*resp.* F-isocrystal) with *Newton polygon* $\mathrm{Nt}^i(X)$ and *Hodge polygon* $\mathrm{Hdg}^i(X)$.

In general, we say that a smooth projective variety X over k is i-th *Artin supersingular* if the F-crystal $H^i(X)$ is supersingular, that is, its Newton polygon is a straight line. For abelian varieties, the first Artin supersingularity coincides with the classical notion, and implies also that it is Artin supersingular in all degrees.

As over the complex numbers, we expect that the second cohomology of an irreducible symplectic variety should control most of its geometry, up to birational equivalence. This motivates us to pay special attention to the 2^{nd}-Artin supersingularity. We recall the the definition.

Definition VI.2.3. Let X be an irreducible symplectic variety of dimension $2n$ over k. X is called 2^{nd}-*Artin supersingular* if the F-crystal $(H^2(X), \varphi)$ is supersingular, *i.e.* the Newton polygon $\mathrm{Nt}^2(X)$ is a straight line (of slope 1).

For a K3 surface X, Artin defined its supersingularity originally in [**Art74**] by looking at its *formal Brauer group* $\widehat{\mathrm{Br}}(X)$, which turns out to be equivalent to the supersingularity of the F-crystal $(H^2(X), \varphi)$ discussed before. More generally, Artin and Mazur made the observation in [**AM77**] that the formal Brauer group $\widehat{\mathrm{Br}}(X)$ actually fits into a whole series of formal groups. For any $i \in \mathbf{N}$, consider the functor

$$\begin{aligned} \Phi^i_X : (\text{Artin local } k\text{-algebras}) &\to (\text{Abelian groups}) \\ R &\mapsto \ker\left(H^i_{\text{ét}}(X \times_k R, \mathbb{G}_m) \to H^i_{\text{ét}}(X, \mathbb{G}_m)\right). \end{aligned}$$

If $H^{i-1}(X, \mathcal{O}_X) = 0$, this functor is pro-representable by a formal group $\widehat{\Phi}^i(X)$, called the i-th *Artin–Mazur formal group* of X. In particular, $\widehat{\Phi}^1(X) = \widehat{\mathrm{Pic}}(X)$ is the formal Picard group and $\widehat{\Phi}^2(X) = \widehat{\mathrm{Br}}(X)$ is the formal Brauer group of X. Moreover, if $H^{i+1}(X, \mathcal{O}_X)$ vanishes, the functor Φ^i_X, as well as $\widehat{\Phi}^i(X)$, is formally smooth with abelian Lie algebra $H^i(X, \mathcal{O}_X)$.

In particular, for an irreducible symplectic variety X, we always have the vanishing of $H^1(X, \mathcal{O}_X)$, which implies that the formal Brauer group $\widehat{\mathrm{Br}}(X)$ is a well-defined formal group, *i.e.* the functor $\widehat{\Phi}^2$ is pro-representable. As an analogue of Artin's notion for supersingularity of K3 surfaces in [**Art74**], we make the following definition.

Definition VI.2.4. An irreducible symplectic variety X is called *Artin* $\widehat{\mathrm{Br}}$-*supersingular* if $\widehat{\mathrm{Br}}(X)$ is isomorphic to the formal additive group $\widehat{\mathbb{G}}_a$.

Remark VI.2.5. Let X be an irreducible symplectic variety defined over k.

- $\widehat{\mathrm{Br}}(X)$ is expected to be formally smooth. It is the case when $H^3(X, \mathcal{O}_X)$ vanishes (for instance, for $K3^{[n]}$-type varieties).
- Provided that $\dim H^2(X, \mathcal{O}_X) = 1$ and $\widehat{\mathrm{Br}}(X)$ is formally smooth (both hold for most known examples of irreducible symplectic varieties of $K3^{[n]}$-type), $\widehat{\mathrm{Br}}(X)$ is a smooth one-dimensional formal group and it is uniquely determined by its *height* ([**Man63**], [**AM77**]). In this case, $\widehat{\mathrm{Br}}(X)$ is isomorphic to $\widehat{\mathbb{G}}_a$ if and only if the height is ∞, if and

only if the F-crystal $(H^2(X), \varphi)$ is supersingular. In other words, in this case, Artin $\widehat{\mathrm{Br}}$-supersingularity is equivalent to 2^{nd}-Artin supersingularity.

VI.2.3. Shioda supersingularity. Let X be a smooth projective variety defined over k. For any $r \in \mathbf{N}$, there is a crystalline cycle class map

$$\mathrm{cl}^r\colon \mathrm{CH}^r(X) \otimes_{\mathbf{Z}} K \longrightarrow H^{2r}(X)_K := H^{2r}_{\mathrm{cris}}(X/W) \otimes_W K \tag{VI.2}$$

whose image lands in the eigenspace of eigenvalue p^r with respect to the action of φ on $H^{2r}(X)$.

Definition VI.2.6 (Shioda supersingularity). A smooth projective variety X defined over k is called

- $(2r)^{th}$-*Shioda supersingular* if (VI.2) is surjective;
- *even Shioda supersingular*[1] if (VI.2) is surjective for all r;
- $(2r+1)^{th}$-*Shioda supersingular*[2] if there exist a supersingular abelian variety A and an algebraic correspondence $\Gamma \in \mathrm{CH}^{\dim X - r}(X \times A)$ such that the cohomological correspondence $\Gamma_* : H^{2r+1}_{\mathrm{cris}}(X/W)_K \to H^1_{\mathrm{cris}}(A/W)_K$ is an isomorphism;
- *odd Shioda supersingular* if it is $(2r+1)^{th}$-Shioda supersingular for all r;
- *fully Shioda supersingular* if it is both even and odd Shioda supersingular.

Remark VI.2.7 ("Shioda implies Artin"). By looking at the action of the Frobenius on algebraic cycles, one sees easily that each notion of Shioda supersingularity in Definition VI.2.6 implies the corresponding notion of Artin supersingularity. The converse follows from the crystalline Tate conjecture (*cf.* [**And04**, 7.3.3.2]). Note that supersingular abelian varieties are fully Shioda supersingular.

VI.3. The conjectural picture and general results

We propose in this section many conjectures on the geometric and motivic aspects of supersingular irreducible symplectic varieties inspired by the theory of supersingular K3 surfaces. Some established implications between these conjectures are also included here.

VI.3.1. Artin supersingularity *vs.* Shioda supersingularity. As mentioned in Remark VI.2.7, 2^{nd}-Artin supersingularity is *a priori* weaker than 2^{nd}-Shioda supersingularity. In the other direction, for irreducible symplectic varieties, we have the Tate conjecture for certain irreducible symplectic varieties.

Theorem VI.3.1 (Charles [**Cha13**]). *Let Y be an irreducible symplectic variety variety of dimension $2n$ over $\mathbf{C}$, let L be an ample line bundle on Y, and set $d = c_1(L)^{2n}$. Assume that p is prime to d and that $p > 2n$. Suppose that Y can be defined over a finite unramified extension of $\mathbf{Q}_p$ and that Y has good reduction at p. If the Beauville-Bogomolov form of Y induces a non-degenerate quadratic form on the reduction modulo p of the primitive lattice in the second cohomology group of Y, then the reduction of Y at p, denoted by X, satisfies the Tate conjecture for divisors.*

[1] This is called *fully rigged* in [**vdGK03**, Definition 5.4].

[2] One could come up with slightly different definitions, for example that the $(2r+1)$-th cohomology of the variety is isomorphic as an isocrystal to the first cohomology of a supersingular abelian variety. The condition that this isomorphism is induced by an algebraic correspondence is predicted by the Tate conjecture. Here we used the strongest definition, because later we can prove it in some cases.

In particular, when Y is an irreducible symplectic variety of K3 type and $p > 2n, p \nmid d$, then X satisfies the Tate conjecture for divisors.

This yields the following consequence.

Corollary VI.3.2. *Suppose X is an irreducible symplectic variety defined over k satisfying all the conditions in Theorem VI.3.1. Then X is 2^{nd}-Artin supersingular if and only if it is 2^{nd}-Shioda supersingular.*

Example VI.3.3. If X is the Fano variety of lines on a smooth cubic fourfold and $p \geq 5$, then X is 2^{nd}-Artin supersingular if and only if it is 2^{nd}-Shioda supersingular.

A more difficult question is to go beyond the second cohomology and ask whether X is *fully* Shioda supersingular and hence *fully* Artin supersingular if X is 2^{nd}-Artin supersingular.

Conjecture VI.3.4 (**Equivalence conjecture**). *Let X be an irreducible symplectic variety defined over an algebraically closed field of positive characteristic. The following conditions are equivalent:*

- *X is 2^{nd}-Artin supersingular;*
- *X is 2^{nd}-Shioda supersingular;*
- *X is fully Artin supersingular;*
- *X is fully Shioda supersingular.*

There are some easy implications in this conjecture, see Remark VI.2.7:

$$\begin{tikzcd} \text{fully Shioda supersingular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{fully Artin supersingular} \arrow[d, Rightarrow] \\ 2^{nd}\text{-Shioda supersingular} \arrow[r, Rightarrow] & 2^{nd}\text{-Artin supersingular.} \end{tikzcd}$$

VI.3.2. Unirationality. Motivated by the unirationality Conjecture VI.1.1 for K3 surfaces, as a geometric characterization for the cohomological notion of supersingularity, we propose the following.

Conjecture VI.3.5 (**Unirationality conjecture**). *An irreducible symplectic variety is 2^{nd}-Artin supersingular if and only if it is unirational.*

The previous conjecture has the following weaker version.

Conjecture VI.3.6 (**RCC conjecture**). *An irreducible symplectic variety is 2^{nd}-Artin supersingular if and only if it is rationally chain connected.*

The result below, which says that *rational chain connectedness implies algebraicity of H^2*, is well known in characteristic 0, and it holds in positive characteristics as well.

Theorem VI.3.7 (*cf.* [GJ17, Theorem 1.2]). *Let X be smooth projective variety over k. If X is rationally chain connected, then the first Chern class map induces an isomorphism $\mathrm{Pic}(X) \otimes \mathbf{Q}_\ell \cong H^2_{\text{ét}}(X, \mathbf{Q}_\ell(1))$ for all $\ell \neq p$. In particular, $\rho(X) = b_2(X)$ and X is 2^{nd}-Shioda supersingular.*

As a consequence we have some implications:

$$\text{Unirational} \Longrightarrow \text{RCC} \Longrightarrow 2^{nd}\text{-Shioda supersingular} \Longrightarrow 2^{nd}\text{-Artin supersingular.}$$

VI.3.3. Motives and cycles. We also expect that algebraic cycles on a supersingular irreducible symplectic variety are "as easy as possible." The most fundamental way to formulate this is in the category of motives:

Definition VI.3.8 (Supersingular abelian motives). Let CHM be the category of rational Chow motives over k. Let $\mathcal{M}^{\mathrm{ssab}}$ be the idempotent-complete symmetric monoidal subcategory of CHM generated by motives of supersingular abelian varieties. A smooth projective variety X is said to have *supersingular abelian motive* if its rational Chow motive $\mathfrak{h}(X)$ belongs to $\mathcal{M}^{\mathrm{ssab}}$.

Remark VI.3.9. The category $\mathcal{M}^{\mathrm{ssab}}$ contains the Tate motives by definition. Thanks to [**FL18**, Theorem 2.13], $\mathcal{M}^{\mathrm{ssab}}$ is actually generated, as an idempotent-complete tensor category, by the Tate motives together with $\mathfrak{h}^1(E)$ for a/any supersingular elliptic curve E. It can be shown that any object in $\mathcal{M}^{\mathrm{ssab}}$ is a direct summand of the motive of some supersingular abelian variety. Therefore, for a smooth projective variety X, the condition of having supersingular abelian motive is exactly Fakhruddin's notion of "strong supersingularity" in [**Fak02**].

Conjecture VI.3.10 (**Supersingular abelian motive conjecture**). *The rational Chow motive of a* 2^{nd}*-Artin supersingular irreducible symplectic variety is a supersingular abelian motive, that is, a direct summand of the motive of a supersingular abelian variety.*

We have a pretty good understanding of motives of supersingular abelian varieties, and more generally supersingular abelian motives.

Corollary VI.3.11 ([**Fak02**] and [**FL18**, Corollary 2.16]). *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$, such that X has supersingular abelian motive (Definition VI.3.8). Let b_i be the i-th Betti number of X. Then we have the following:*

(*i*) *In the category* CHM, *we have*

$$\mathfrak{h}(X) \simeq \bigoplus_{i=0}^{n} \mathbb{1}(-i)^{\oplus b_{2i}} \oplus \bigoplus_{i=0}^{n-1} \mathfrak{h}^1(E)(-i)^{\oplus \frac{1}{2}b_{2i+1}}, \tag{VI.3}$$

where E is a supersingular elliptic curve.

(*ii*) *X is fully Shioda supersingular (Definition VI.2.6):*
   - *(a) For any i and any prime number $\ell \neq p$, the cycle class map $\mathrm{CH}^i(X)_{\mathbf{Q}_\ell} \to H^{2i}_{\acute{e}t}(X, \mathbf{Q}_\ell)$ is surjective.*
   - *(b) For any i, the cycle class map $\mathrm{CH}^i(X)_K \to H^{2i}_{\mathrm{cris}}(X/W)_K$ is surjective.*
   - *(c) For any i, there exist a supersingular abelian variety B together with an algebraic correspondence $\Gamma \in \mathrm{CH}^{n-i}(X \times B)$ such that $\Gamma_* \colon H^{2i+1}_{\mathrm{cris}}(X/W)_K \to H^1_{\mathrm{cris}}(B/W)_K$ is an isomorphism.*

   In particular, X is fully Artin supersingular.

(*iii*) *Numerical equivalence and algebraic equivalence coincide. In particular, for any i, the Griffiths group is torsion:* $\mathrm{Griff}^i(X)_{\mathbf{Q}} = 0$.

(*iv*) $\mathrm{CH}^i(X) = \mathrm{CH}^i(X)_{(0)} \oplus \mathrm{CH}^i(X)_{(1)}$ *with* $\mathrm{CH}^i(X)_{(0)} \simeq \mathbf{Q}^{\oplus b_{2i}}$, *providing a* $\mathbf{Q}$*-structure for cohomology and* $\mathrm{CH}^i(X)_{(1)} \simeq E^{\frac{1}{2}b_{2i-1}} \otimes_{\mathbf{Z}} \mathbf{Q}$ *is the algebraically trivial part.*

(*v*) $\mathrm{CH}^i(X)_{\mathrm{alg}}$ *has an algebraic representative (ν_i, Ab^i) with* $\ker(\nu_i)$ *finite and Ab^i a supersingular abelian variety of dimension $\frac{1}{2}b_{2i-1}$.*

(*vi*) *The intersection product restricted to* $\mathrm{CH}^*(X)_{\mathrm{alg}}$ *is zero.*

As a consequence, having supersingular abelian motive implies that the variety is fully Shioda supersingular (*cf.* Corollary VI.3.11). Therefore to summarize, we expect that the notions in the following diagram of implications are all equivalent for irreducible symplectic varieties:

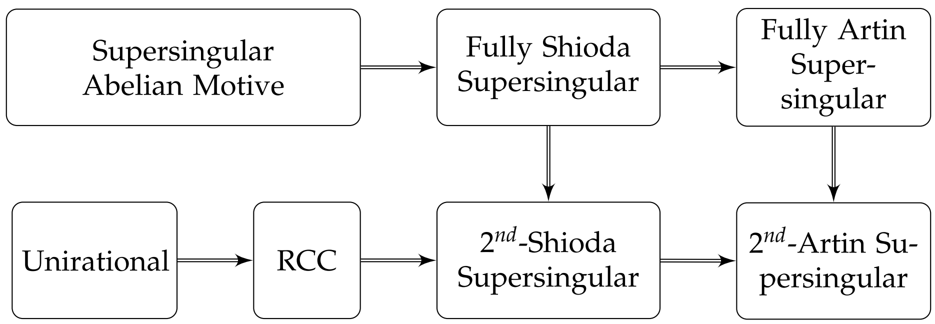

FIGURE 1. Characterizations of supersingularity for ISV

We can reformulate the previous theorem into the following supersingular version of the Bloch–Beilinson conjecture, which gives a quite complete description of the additive structure of the rational Chow groups of supersingular irreducible symplectic varieties. Recall that $\mathrm{CH}^i(X)_{\mathrm{alg}}$ is the subgroup of the Chow group $\mathrm{CH}^i(X)$ consisting of algebraically trivial cycles. Denote also by $\overline{\mathrm{CH}}^i(X)$ the i^{th} rational Chow group modulo numerical equivalence.

CONJECTURE VI.3.12 (**Supersingular Bloch–Beilinson conjecture**). *Let X be a supersingular irreducible symplectic variety. Then for any $0 \le i \le \dim(X)$,*

- *Numerical equivalence and algebraic equivalence coincide on* $\mathrm{CH}^i(X)$. *In particular, the Griffiths group* $\mathrm{Griff}^i(X)$ *is trivial.*
- *There exists a regular surjective homomorphism*

$$v_i : \mathrm{CH}^i(X)_{\mathrm{alg}} \to \mathrm{Ab}^i(X)$$

  to an abelian variety $\mathrm{Ab}^i(X)$, *called the* algebraic representative, *which is universal for regular homomorphisms from* $\mathrm{CH}^i(X)_{\mathrm{alg}}$ *to abelian varieties.*
- *The kernel of v_i is finite and* $\mathrm{Ab}^i(X)$ *is a supersingular abelian variety of dimension* $\frac{1}{2}b_{2i-1}(X)$.
- *The intersection product restricted to the subring* $\mathrm{CH}^*(X)_{\mathrm{alg}}$ *is zero.*

In particular, the kernel of the algebra epimorphism $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$ *is a square zero graded ideal given by the supersingular abelian varieties* $\mathrm{Ab}^*(X)_{\mathbf{Q}} := \bigoplus_i \mathrm{Ab}^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

For irreducible symplectic varieties, the rational Chow ring has a supplementary feature, namely there is Beauville's insight that the Bloch–Beilinson filtration should possess a canonical multiplicative splitting [**Bea07**], [**Voi08**] (see Conjecture II.3.1). Inspired by this conjecture, the section property conjecture II.3.5 was proposed and studied in [**FV19a**] mainly over the complex numbers. We expect it to hold in any characteristic (see Conjecture II.3.5), but let us restate it again only in the supersingular situation.

CONJECTURE VI.3.13 (**Supersingular section property conjecture** *cf.* II.3.5). *Let X be a supersingular irreducible symplectic variety. Then for any $0 \leq i \leq \dim(X)$, there is a subspace $\mathrm{DCH}^i(X) \subset \mathrm{CH}^i(X)$, whose elements are called* distinguished cycles, *such that*

- *the composition $\mathrm{DCH}^i(X) \hookrightarrow \mathrm{CH}^i(X) \twoheadrightarrow \overline{\mathrm{CH}}^i(X)$ is an isomorphism;*
- *$\mathrm{DCH}^*(X) := \bigoplus_i \mathrm{DCH}^i(X)$ forms a $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(X)$;*
- *the Chern class $c_i(T_X) \in \mathrm{DCH}^i(X)$ for any i.*

In other words, there exists a section of the natural algebra *epimorphism $\mathrm{CH}^*(X) \twoheadrightarrow \overline{\mathrm{CH}}^*(X)$ whose image contains all Chern classes of X.*

Combining the equivalence conjecture VI.3.4, the supersingular Bloch–Beilinson conjecture VI.3.12, and the section property conjecture VI.3.13, we have the following rather complete conjectural description for the structure of the rational Chow ring of a supersingular irreducible symplectic variety, which qualifies as the supersingular version of Beauville's splitting conjecture II.3.1 ([**Bea07**]).

CONJECTURE VI.3.14 (**Supersingular splitting property**). *Let X be a supersingular irreducible symplectic variety. Then the rational Chow ring of X has a multiplicative decomposition*

$$\mathrm{CH}^*(X) = \mathrm{DCH}^*(X) \oplus \mathrm{CH}^*(X)_{\mathrm{alg}}, \tag{VI.4}$$

such that

- *$\mathrm{DCH}^*(X)$ is a graded $\mathbf{Q}$-subalgebra which contains all the Chern classes of X and is mapped isomorphically to $\overline{\mathrm{CH}}^*(X)$ via the natural projection. It also provides a $\mathbf{Q}$-structure for the ℓ-adic even cohomology ring $H^{2*}_{\text{ét}}(X, \mathbf{Q}_\ell)$ for all $\ell \neq p$, i.e. the restriction of the cycle class map $\mathrm{DCH}^*(X)_{\mathbf{Q}_\ell} \xrightarrow{\simeq} H^{2*}(X, \mathbf{Q}_\ell)$ is an isomorphism;*
- *The algebraically trivial cycles $\mathrm{CH}^*(X)_{\mathrm{alg}}$ form a square zero graded ideal, which is mapped isomorphically to supersingular abelian varieties given by the algebraic representatives $\mathrm{Ab}^*(X)_{\mathbf{Q}}$.*

In other words, $\mathrm{CH}^*(X)$ is the square zero extension of a graded subalgebra isomorphic to $\overline{\mathrm{CH}}^*(X)$ by a graded module $\mathrm{Ab}^*(X)_{\mathbf{Q}}$.

REMARK VI.3.15. The decomposition (VI.4) is expected to be canonical. Moreover, the $\overline{\mathrm{CH}}^*(X)$-module structure on $Ab^*(X)_{\mathbf{Q}}$ should be determined by, or at least closely related to, the $H^{2*}(X)$-module structure on $H^{2*-1}(X)$, where H is some Weil cohomology theory.

To summarize, we have a diagram

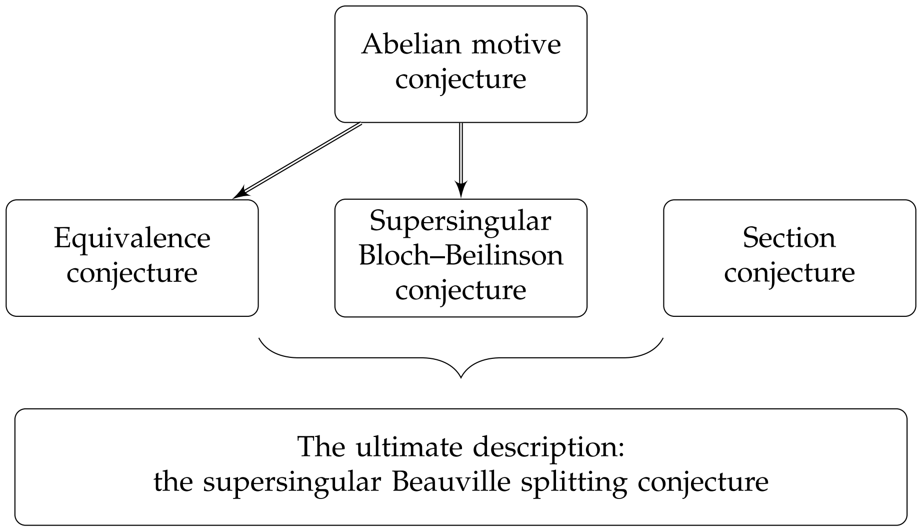

FIGURE 2. Conjectures on algebraic cycles of supersingular ISV

VI.4. Main results: moduli spaces of sheaves

The main evidence we provide for the conjectures proposed before concerns the moduli spaces of sheaves on K3 surfaces and abelian surfaces.

For moduli spaces of sheaves on K3 surfaces, the key result is the following Theorem VI.4.1, which relates these moduli spaces birationally to punctual Hilbert schemes of K3 surfaces, in the supersingular situation.

THEOREM VI.4.1 ([FL18]). *Let k be an algebraically closed field of characteristic $p > 0$ and let S be a K3 surface defined over k. Let H be an ample line bundle on S and let X be the moduli space of H-semistable sheaves on S with Mukai vector $v = (r, c_1, s)$ satisfying $\langle v, v\rangle \geq 0$ and $r > 0$.*

(1) If H is general with respect to v, then X is an irreducible symplectic variety of dimension $2n = \langle v, v\rangle + 2$ and deformation equivalent to the n^{th} Hilbert scheme of points of S. Moreover, X is 2^{nd}-Artin supersingular if and only if S is supersingular.

(2) If S is supersingular and v is coprime to p, then X is a 2^{nd}-Artin supersingular irreducible symplectic variety and it is birational to the Hilbert scheme $S^{[n]}$, where $n = \frac{\langle v,v\rangle+2}{2}$.

The condition that v is coprime to p is natural for X to be smooth.

The birational equivalence in the theorem above can be chosen to be the composition of several birational equivalences that are liftable to characteristic zero bases between liftable irreducible symplectic varieties (see [FL18, Theorem 4.16]). Combining this with an analysis of the motives of Hilbert schemes of supersingular K3 surfaces ([FL18, Proposition 4.13]), we can prove most of the conjectures in §II.3 for the moduli spaces considered in Theorem VI.4.1.

COROLLARY VI.4.2 ([FL18]). *The notation and assumptions are as in Theorem VI.4.1. If v is coprime to p and H is general with respect to v, the following conditions are equivalent:*

(i) S is supersingular.
(ii) X is 2^{nd}-Artin supersingular.
(ii′) X is fully Artin supersingular.

(iii) *X is 2^{nd}-Shioda supersingular.*
(iii′) *X is fully Shioda supersingular.*
(iv) *The Chow motive of X is of Tate type (i.e. a direct sum of Tate motives).*

When the above conditions hold, the cycle class maps induce isomorphisms $\mathrm{CH}^*(X)_K \simeq H^*_{\mathrm{cris}}(X/W)_K$ and $\mathrm{CH}^*(X)_{\mathbf{Q}_\ell} \simeq H^*_{\acute{e}t}(X, \mathbf{Q}_\ell)$ for all $\ell \neq p$.
If moreover S is unirational, then X is unirational too.

Similarly, for irreducible symplectic varieties of generalized Kummer type, we have the following results.

Theorem VI.4.3 ([**FL18**]). *Let k be an algebraically closed field of characteristic $p > 0$. Let A be an abelian surface defined over k. Let H be an ample line bundle on A and let X be the Albanese fiber of the projective moduli space of H-stable sheaves on A with Mukai vector $v = (r, c_1, s)$ satisfying $\langle v, v\rangle \geq 2$ and $r > 0$.*

(1) If H is general with respect to v and X is smooth over k, then X is an irreducible symplectic variety of dimension $2n := \langle v, v\rangle - 2$ and deformation equivalent to the n^{th} generalized Kummer variety. Moreover, X is 2^{nd}-Artin supersingular if and only if A is supersingular.
(2) Suppose A is supersingular and $p \nmid \frac{1}{2}\langle v, v\rangle$. Then X is 2^{nd}-Artin supersingular and it is birational to the n^{th} generalized Kummer variety associated to some supersingular abelian surface, with $n = \frac{\langle v,v\rangle - 2}{2}$.

The numerical condition on v is natural to ensure the smoothness of X. The birational equivalence here is again liftable to characteristic zero. We can then deduce most of the conjectures in §II.3 for most Kummer type moduli spaces of sheaves on abelian varieties.

Corollary VI.4.4. *The notation and assumptions are as in Theorem VI.4.3. When $p \nmid \frac{1}{2}\langle v, v\rangle$ and H is general with respect to v, the following conditions are equivalent:*

(i) *A is supersingular.*
(ii) *X is 2^{nd}-Artin supersingular.*
(ii′) *X is fully Artin supersingular.*
(iii) *X is 2^{nd}-Shioda supersingular.*
(iii′) *X is fully Shioda supersingular.*
(iv) *X is rationally chain connected.*
(v) *The Chow motive of X is a supersingular abelian motive.*

If one of these conditions are satisfied, all the conclusions in the supersingular Bloch-Beilinson conjecture *VI.3.12, the* section property conjecture *VI.3.13 and the* supersingular splitting property conjecture *VI.3.14 hold for X.*

Unfortunately, we do not have an idea to approach the unirationality conjecture VI.3.4 for generalized Kummer varieties.

VI.5. Some ingredients of the proof

We refrain from giving the details of the proof of the main results in the previous section. Let us only provide some key ingredients used in the case of moduli spaces of sheaves on K3 surfaces and refer the [**FL18**] for the full proof.

VI.5.1. Elliptic fibrations. The existence of elliptic fibrations plays an important role in the study of supersingular K3 surfaces [**Lie15**]. For our purpose, we need a more refined analysis of elliptic fibrations on supersingular K3 surfaces.

THEOREM VI.5.1 ([**FL18**]). *Let S be a supersingular K3 surface defined over an algebraic closed field k of characteristic $p > 0$. Let $v = (r, c_1, s) \in \widetilde{H}(S)$ be a Mukai vector coprime to p. Then up to changing the Mukai vector v via the following auto-equivalences*

- *tensoring with a line bundle;*
- *spherical twist associated to a line bundle or $\mathcal{O}_C(-1)$ for some smooth rational curve C on S,*

there exists an elliptic fibration $\pi : S \to \mathbf{P}^1$ such that $\gcd(r, c_1 \cdot E) = 1$, where $E \in \mathrm{NS}(S)$ is the fiber class of π.

VI.5.2. Bridgeland's theorem. The following result is essentially due to Bridgeland [**Bri98**]. It allows us to relate a moduli space of sheaves on an elliptic surface to a Hilbert scheme.

THEOREM VI.5.2 (Bridgeland [**Bri98**], see also [**FL18**]). *Let $\pi : X \to C$ be a smooth relatively minimal elliptic surface over an algebraically closed field $\overline{K}$. We denote by $f \in \mathrm{NS}(X)$ the fiber class of π. Given $v = (r, c_1, s) \in \widetilde{H}(X)$ satisfying $r > 0$ and assume $\gcd(r, c_1 \cdot f) = 1$. If K is of characteristic zero, there exists an ample line bundle H and a birational morphism*

$$M_H(X, v) \dashrightarrow \mathrm{Pic}^0(Y) \times Y^{[n]} \tag{VI.5}$$

where Y is an smooth elliptic surface. In particular, if X is a K3 surface or an abelian surface, so is Y and the assertion also holds for some liftably birational map (VI.5) *when K is of positive characteristic.*

VI.5.3. Motives of Hilbert schemes of supersingular K3 surfaces. For a supersingular K3 surface S, using the result of de Cataldo–Migliorini [**dCM02**], we can reduce the study of the motive of its n-th Hilbert scheme $S^{[n]}$ to the motives of powers S^m. Hence it is enough to see that the motive of S is of Tate type. Assuming that S is unirational ([**Lie15**] [**BL18**]), S is related by blow-ups and blow-downs to $\mathbf{P}^2$ and $\mathfrak{h}(S)$ is clearly of Tate type. Without using the conjectural unirationality of S, one knows nevertheless that $\mathfrak{h}(S)$ is of Tate type by using the elliptic fibration structure, thanks to Fakhruddin [**Fak02**].

VI.6. Further questions and projects

The study on supersingular irreducible symplectic varieties is only initiated here and there are a lot of uncharted areas to be explored. I list here some of the questions that I plan to investigate in the future.

VI.6.1. Unirationality of supersingular generalized Kummer type varieties. The unirationality of the supersingular Kummer K3 surfaces, established by Shioda [**Shi74**], is at the very origin of all the other known unirationality results on supersingular K3 surfaces or supersingular irreducible symplectic varieties. Thus I believe that a thorough understanding of the unirationality of generalized Kummer varieties of dimension at least 4 will constitute a genuine breakthrough in the unirationality conjecture. By using the isogenies between supersingular elliptic curves, one can reduce the problem to consider the unirationality of $E^{2n}/\mathfrak{S}_{n+1}$ for one (preferred) supersingular elliptic curve E.

VI.6.2. Supersingular cubic fourfolds. Fano varieties of lines of cubic fourfolds provide a locally complete family of irreducible symplectic fourfolds [**BD85**]. It is a natural continuation of our work [**FL18**] to try to show that for a supersingular cubic fourfold, its Fano variety of lines is uirational and its Chow motive is of Tate type. Together with Zhiyuan Li, we have started to work on this problem and we have two strategies: the first one is to reduce to our previous work [**FL18**] on the moduli space of twisted sheaves on K3 surfaces, by using the work of Addington–Thomas [**AT14**] together with the techniques of lifting-reduction. The second approach, only aiming for the Chow motives, is to implement and compare the work of Shen–Vial [**SV16a**] on the motivic decomposition of the Fano variety of lines. The point is to show the vanishing of certain components in their motivic decomposition by looking at the Chow-theoretic Fourier transform constructed in [**SV16a**].

VI.6.3. Singular moduli spaces of sheaves on supersingular K3 surfaces. The aim is to study the hyper-Kähler varieties of dimension 6 and 10 constructed by O'Grady [**O'G99**], [**O'G03**]. For the conjectures on algebraic cycles, the idea is to adapt the method in the recent preprint [**Bü18**]. For the unirationality of supersingular hyper-Kähler varieties of O'Grady-10-type and O'Grady-6-type, possible approaches are to use the Laza–Saccà–Voisin compacitification [**LSV17**] and the Mongardi–Rapagnetta–Saccà involution [**MRS18**] respectively.

VI.6.4. Ordinary irreducible symplectic varieties. Going transversally to the study mentioned above on supersingular hyper-Kähler varieties, I also intend to study hyper-Kähler varieties in positive characteristics which are closest to complex hyper-Kähler varieties, namely the *ordinary* ones. One important theme would be their deformation theory and to obtain a global Torelli-like result.

CHAPTER VII

Finiteness results on the automorphism group

In this chapter, which is based on the joint work [**CF19**] with Andrea Cattaneo, we study the automorphism groups of compact hyper-Kähler varieties, not only projective ones as in the previous chapters, but also non-projective ones.

VII.1. Main results

Our results are some general finiteness properties of automorphism groups.

VII.1.1. Finite generation. The classical work of Sterk [**Ste85**] implies that the automorphism group of a projective K3 surface is always finitely generated, *cf.* [**Huy16**, Corollary **15**.2.4]. It is natural to ask whether this finiteness property also holds for automorphism groups, or bimeromorphic automorphism groups, of all compact hyper-Kähler manifolds.

On the one hand, in the non-projective case, the following result of Oguiso provides a quite satisfying and precise answer.

THEOREM VII.1.1 ([**Ogu08**]). *Let X be a* non-projective *compact hyper-Kähler manifold. Its group of bimeromorphic automorphisms* $\mathrm{Bir}(X)$ *is an almost abelian group of rank at most* $\max\{1, \rho(X)-1\}$*, where $\rho(X)$ is the Picard rank of X. Hence the same conclusion holds for the automorphism group* $\mathrm{Aut}(X)$ *as well. In particular,* $\mathrm{Bir}(X)$ *and* $\mathrm{Aut}(X)$ *are both finitely presented.*

Here an *almost abelian group of rank r* is a group isomorphic to $\mathbf{Z}^r$ up to finite kernel and cokernel, see [**Ogu08**, §8] for the precise definition.

On the other hand, for a *projective* hyper-Kähler variety X, $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are of more complicated nature. For example, in [**Ogu06**] and [**Ogu07**, Theorem 1.6], Oguiso showed that these two groups are not necessarily *almost abelian* (see [**Ogu08**, §8]). Nevertheless, using the Global Torelli Theorem ([**Ver13**], [**Mar11**], [**Huy12**]), Boissière and Sarti [**BS12**, Theorem 2] prove that $\mathrm{Bir}(X)$ is finitely generated. The finite-generation problem for $\mathrm{Aut}(X)$ remained open ever since ([**Ogu06**, Question 1.5], [**BS12**, Question 1]). Our first main result is to give this question an affirmative, and stronger, answer.

THEOREM VII.1.2 ([**CF19**, Theorem 1.6]). *For any projective hyper-Kähler manifold X, the automorphism group* $\mathrm{Aut}(X)$ *and the birational automorphism group* $\mathrm{Bir}(X)$ *are finitely presented and satisfy the* (FP_∞) *property.*

See §VII.2 for the definition of the (FP_∞) property and the proof of Theorem VII.1.2. This result contrasts with the examples of Lesieutre [**Les17**] and Dinh–Oguiso [**DO18**], where smooth projective varieties with non-finitely generated automorphism groups are constructed.

VII.1.2. Finite subgroups. Our second main result concerns the finite subgroups of automorphism groups.

THEOREM VII.1.3 ([**CF19**, Theorem 1.4]). *For a compact hyper-Kähler manifold, the automorphism group, as well as the birational automorphism group, contain only finitely many conjugacy classes of finite subgroups. In particular, any compact hyper-Kähler manifold has only finitely many faithful finite group actions up to equivalence.*

The case of K3 surfaces is [**DIK00**, Theorem D.1.1].

To equivalently reformulate Theorem VII.1.3 using group cohomology, we can show ([**CF19**, Lemma 4.7]) without much difficulty that for a group A, there are only finitely many conjugacy classes of finite subgroups of A if and only if the following two conditions are satisfied:

(1) the cardinalities of finite subgroups of A are bounded;
(2) for any finite group G, the set $H^1(G, A)$ is finite, where A is endowed with the trivial G-action.

Moreover, if A satisfies this property then so does any subgroup of A of finite index.

See §VII.3 for the proof of Theorem VII.1.3.

VII.1.3. Real structures. This was actually our original motivation. Recall that a *real structure* on a complex manifold is an anti-holomorphic involution; two real structures are equivalent if they are conjugate by a holomorphic automorphism. It is a central question in real algebraic geometry to classify the real structures on a given complex manifold. It is therefore natural to first ask the following quesition.

QUESTION VII.1.4. Are there only finitely many real structures on a complex manifold up to equivalence?

Assuming the existence of real structures on a complex manifold X, we have the following cohomological "classification" of real structures due to Borel–Serre [**BS64**]: the set of equivalence classes of real structures on X, hence the set of $\mathbf{R}$-isomorphism classes of real forms of X in the projective setting, is in bijection with the (non-abelian) group cohomology $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{Aut}(X))$, where $\mathbf{Z}/2\mathbf{Z}$ is naturally identified with the Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$, $\mathrm{Aut}(X)$ is the group of holomorphic automorphisms of X, and the action of the non-trivial element of $\mathbf{Z}/2\mathbf{Z}$ on $\mathrm{Aut}(X)$ is given by the conjugation by σ.

This cohomological interpretation, together with the finiteness result [**BS64**, Théorème 6.1], allows us to answer the above question on the finiteness of the set of real structures in the affirmative when $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$, the group of components of $\mathrm{Aut}(X)$, is a finite group or an arithmetic group: for instance, Fano varieties [**DIK00**, D.1.10], abelian varieties (or more generally complex tori) [**DIK00**, D.1.11], and varieties of general type, *etc.*, in particular, when $\dim X = 1$. For the next case where X is a complex projective surface, there is an extensive study carried out mainly by the Russian school (Degtyarev, Itenberg, Kharlamov, Kulikov, Nikulin *et al*). We know that there are only finitely many real structures for del Pezzo surfaces, minimal algebraic surfaces [**DIK00**], algebraic surfaces with Kodaira dimension ≥ 1, *etc*. The remaining biggest challenge for surfaces seems to be the case of rational surfaces and in fact recently, Benzerga [**Ben16**] showed that a rational surface with infinitely many non-equivalent real structures, if it exists, must be a blow up of the projective plane at at least 10 points and possesses an automorphism of positive entropy.

It turns out that the answer to the finiteness question is negative in general. The fiblow uprst counter-example is due to Lesieutre in [Les17], where he constructed a 6-dimensional projective manifold with infinitely many non-equivalent real structures and discrete non-finitely generated automorphism group. Inspired by Lesieutre's work, Dinh and Oguiso [DO18] showed that suitable blow ups of some K3 surfaces have the same non-finiteness properties, and hence produce such examples in each dimension ≥ 2.

The finiteness question for higher-dimensional (≥ 3) varieties in general can be very delicate, and apart from the general positive results and the counter-examples mentioned above, it is far from being well understood. The third main result of this chapter is to give a whole class of varieties for which we can show finiteness of the set of real structures.

Theorem VII.1.5 ([CF19, Theorem 1.1]). *Any compact hyper-Kähler manifold has only finitely many real structures up to equivalence.*

The case of K3 surfaces is due to [DIK00, Theorem D.1.1]. See [CF19, §5.3] for several interesting constructions of real structures on various known examples of hyper-Kähler manifolds.

Let us explain in some detail the proofs of the above results in the subsequent sections.

VII.2. Finite presentation

Let us briefly recall various finiteness properties involved. Some standard references are [Bro82] and [Bie81].

Definition VII.2.1 (Finiteness properties of groups [Bro82]). Let Γ be a group.

(1) Γ is called *of type* (FL) (*resp. of length* $\leq n$) if the trivial $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}$ has a finite resolution (*resp.* of length n)
$$0 \longrightarrow \mathbf{Z}[\Gamma]^{m_n} \longrightarrow \dots \longrightarrow \mathbf{Z}[\Gamma]^{m_1} \longrightarrow \mathbf{Z}[\Gamma]^{m_0} \longrightarrow \mathbf{Z} \longrightarrow 0$$
by free $\mathbf{Z}[\Gamma]$-modules of finite rank.

(2) Γ is said to be *of type* (FP) (*resp. of length* $\leq n$) if the trivial $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}$ admits a finite resolution (*resp.* of length n)
$$0 \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0$$
by finitely generated projective $\mathbf{Z}[\Gamma]$-modules.

(3) Let $n \in \mathbf{N}$, we say that Γ is *of type* (FP_n) if the trivial $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}$ has a length-n partial resolution
$$P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0$$
by finitely generated projective $\mathbf{Z}[\Gamma]$-modules. We say Γ is *of type* (FP_∞) if it is *of type* (FP_n) for all $n \geq 0$.

(4) We say Γ *virtually* satisfies a property if it admits a finite-index subgroup satisfying this property. We can therefore define properties like *virtual* (FL) and *virtual* (FP), denoted by (VFL) and (VFP) respectively.

It follows from the definitions that Γ is of type (FP) if and only if Γ is of type (FP_∞) and the ring $\mathbf{Z}[\Gamma]$ is of finite cohomological dimension ([Bro82, Chapter VIII, Proposition 6.1]). For any $0 \leq n \leq \infty$, the property (FP_n) is equivalent to the same condition for any finite-index subgroup ([Bro82, Chapter VIII, Proposition 5.1]). Hence the "virtual (FP_n) property" coincides with (FP_n) itself and (VFP) implies (FP_∞).

The following diagram summarizes some known implications (*cf.* [**Bro82**, Chapter VIII]):

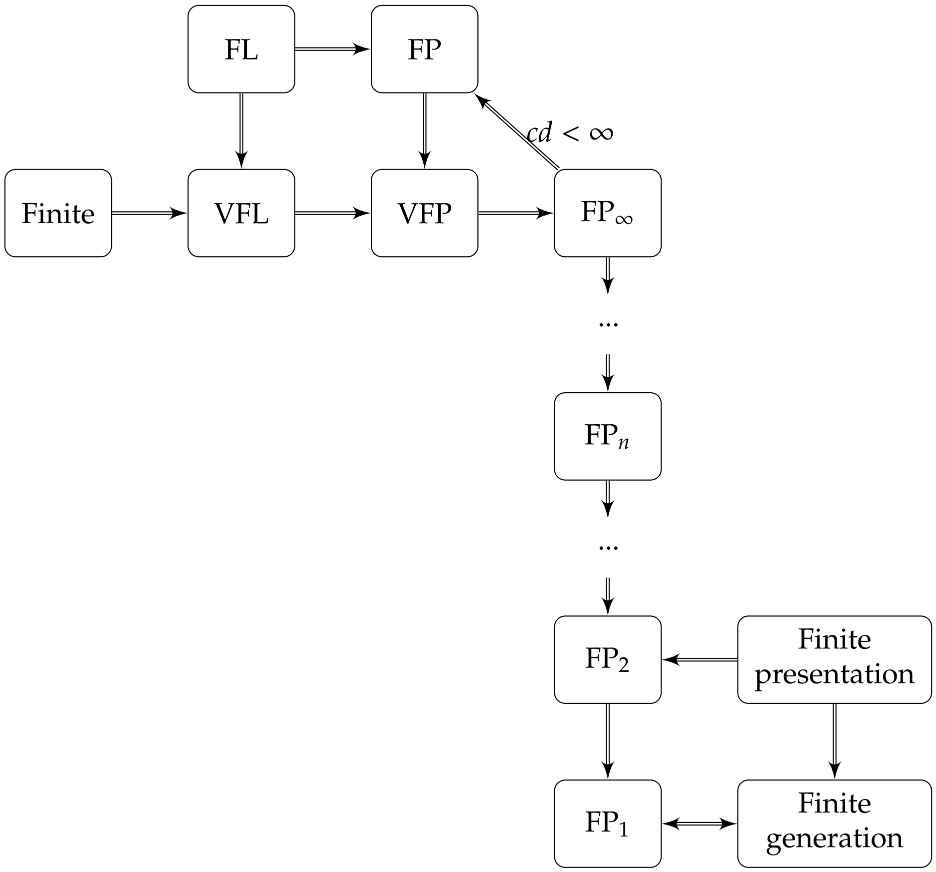

FIGURE 1. Finiteness properties of groups

REMARK VII.2.2. All the finiteness properties in Figure 1 are all preserved under extensions ([**Bro82**, Chapter VIII, §6, Exercise 8], [**Bie81**, P.23, Exercise]), except for (VFL) and (VFP), where one has to require moreover the condition of virtual torsion-freeness ([**Bro82**, Chapter VIII, §11, Exercise 2]).

The key ingredient in our proof is the following result on convex geometry.

PROPOSITION VII.2.3 ([**Loo14**, Corollaries 4.15 and 4.16]). *Let C be a non-degenerate open convex cone in a finite dimensional real vector space V equipped with a $\mathbf{Q}$-structure. Let Γ be a subgroup of* $\mathrm{GL}(V)$ *which preserves C and some lattice in $V(\mathbf{Q})$. If there exists a polyhedral cone Π in C^+, the rational closure of C, such that $\Gamma \cdot \Pi \supseteq C$, then Γ is finitely presented and of type* (VFL) *of length* $\leq \dim(V) - 1$.

Now we have all the ingredients to show our finiteness result.

PROOF OF THEOREM VII.1.2. Let X be a projective hyper-Kähler manifold. We first consider its automorphism group. It fits into an exact sequence

$$1 \longrightarrow \mathrm{Aut}^{\#}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}^{*}(X) \longrightarrow 1\ ,$$

where $\mathrm{Aut}^{\#}(X)$ and $\mathrm{Aut}^{*}(X)$ are respectively the kernel and the image of the natural representation $\mathrm{Aut}(X) \to O(\mathrm{NS}(X))$. On the one hand, the existence of a polyhedral fundamental domain for the action of $\mathrm{Aut}^{*}(X)$ on the rational closure of the ample cone ([**AV17**]) allows us to apply Proposition VII.2.3 and conclude that $\mathrm{Aut}^{*}(X)$ is finitely presented and of type (VFL). On the other hand, it is easy to see that $\mathrm{Aut}^{\#}(X)$ is a finite group, which is of course finitely presented and of type (VFL). As a result, $\mathrm{Aut}(X)$ is an extension of two finitely presented groups of type (VFL), hence is of type (FP_∞), see Figure 1. By Remark VII.2.2, $\mathrm{Aut}(X)$ is also finitely presented and of type (FP_∞).

The above argument applies equally to the birational automorphism group $\mathrm{Bir}(X)$. Indeed, Markman [**Mar11**] showed that the action of $\mathrm{Bir}(X)$, or rather its image $\mathrm{Bir}^{*}(X)$ under the restriction map to the Néron–Severi space, on the rational closure of the movable cone has a rational polyhedral fundamental domain. Looijenga's result Proposition VII.2.3 implies that $\mathrm{Bir}^{*}(X)$ is finitely presented and of type (VFL). We still have the finiteness of $\mathrm{Bir}^{\#}(X) = \ker(\mathrm{Bir}(X) \to \mathrm{Bir}^{*}(X))$ and so one can conclude as in the case of $\mathrm{Aut}(X)$ using Remark VII.2.2. □

Remark VII.2.4 (Bir *vs.* Aut). It was asked in [**Ogu06**, Question 1.6] whether, for a projective hyper-Kähler variety X, the index of $\mathrm{Aut}(X)$ inside $\mathrm{Bir}(X)$ is always finite or not. The answer to this question is negative in general. The first counter-example was constructed by Hassett–Tschinkel [**HT10**, Theorem 7.4, Remark 7.5] (where $\mathrm{Aut}(X)$ is trivial while $\mathrm{Bir}(X)$ is infinite) using Fano varieties of lines of special cubic fourfolds; then Oguiso gave a systematic study in the Picard rank-two case [**Ogu14**, Theorem 1.3].

VII.3. Finiteness of finite subgroups

As we see before, the goal is to bound the order of the subgroups of $\mathrm{Aut}(X)$ as well as to show the finiteness of the group cohomology. The following fact is the key algebraic gadget in the proof. We refer to [**BS64**] and [**CF19**, §4] for the generality of (non-abelian) group cohomology.

Lemma VII.3.1 ([**CF19**, Lemma 4.9]). *Let A be a group. Assume that there is a finite filtration*

$$\{1\} = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A$$

by normal subgroups of A, such that for any $0 \leq i \leq n-1$, A_i/A_{i+1} is either a finite group or an abelian group of finite type. Then there are only finitely many conjugacy classes of finite subgroups in A. Moreover, for any finite group G and any G-action on A preserving the filtration, $H^1(G, A)$ is finite.

VII.3.1. Non-projective case. Although the Beauville–Bogomolov lattice $H^2(X, \mathbf{Z})$ is non-degenerate of signature $(3, b_2(X) - 3)$, there are in general three possibilities for its restriction to the Néron–Severi lattice $\mathrm{NS}(X)$ (*cf.* [**Ogu08**]):

(1) a *hyperbolic* lattice of signature $(1, 0, \rho - 1)$,
(2) an *elliptic* lattice of signature $(0, 0, \rho)$,
(3) a *parabolic* lattice of signature $(0, 1, \rho - 1)$,

where $\rho = \rho(X)$ is the Picard rank of X. It is a theorem of Huybrechts [**Huy99**, Thm. 3.11] that projectivity of X is equivalent to hyperbolicity of $\mathrm{NS}(X)$ (case (1)).

In the sequel of this section, let X be a non-projective compact hyper-Kähler manifold. Hence $\mathrm{NS}(X)$ with the restriction of the Beauville–Bogomolov form q, is either elliptic or

parabolic. Let
$$R := \ker(q|_{\mathrm{NS}(X)})$$
be the *radical* of $\mathrm{NS}(X)$, which is either trivial or isomorphic to $\mathbf{Z}$.

Proof of Theorem VII.1.3 in the non-projective case. Notation is as before. The strategy is to apply Lemma VII.3.1 to natural filtrations of $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$. In the following proof, let A denote either $\mathrm{Aut}(X)$ or $\mathrm{Bir}(X)$. Consider the following normal subgroups

- $A_1 := \{f \in A \mid f^*|_R = \mathrm{id}\}$;
- $A_2 := \left\{f \in A \mid f^*|_R = \mathrm{id}; f^*|_{\mathrm{NS}(X)/R} = \mathrm{id}\right\}$;
- $A_3 := \left\{f \in A \mid f^*|_{\mathrm{NS}(X)} = \mathrm{id}\right\}$;
- $A_4 := \left\{f \in A \mid f^*|_{\mathrm{NS}(X)} = \mathrm{id}; f^*|_{H^{2,0}(X)} = \mathrm{id}\right\}$;
- $A_5 := \left\{f \in A \mid f^*|_{H^2(X)} = \mathrm{id}\right\}$,

of A which form a filtration
$$1 \subseteq A_5 \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \subseteq A.$$
Let us verify that the successive graded subquotients of this filtration satisfy the hypotheses of Lemma VII.3.1, *i.e.* are finite or abelian of finite type:

- A/A_1 is a subgroup of $\mathrm{Aut}(R)$, which is either $\{\pm 1\}$ when R is of rank 1, or zero when R is trivial. In any case, it is finite.
- A_1/A_2 is by construction isomorphic to a subgroup of the automorphism group of the elliptic (*i.e.* negative definite) lattice $\mathrm{NS}(X)/R$, which is obviously a finite group.
- A_2/A_3 is by construction isomorphic to a subgroup of $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{NS}(X)/R, R)$ which is a free abelian group of finite rank (possibly zero).
- A_3/A_4 is by construction isomorphic to a subgroup of the image of
$$\mathrm{Bir}(X) \to \mathrm{GL}(H^{2,0}(X)) \simeq \mathbf{C}^*,$$
which is either $\mathbf{Z}$ or trivial by Oguiso [**Ogu08**, Theorem 2.4, Propositions 4.3, 4.4].
- Finally, we have $A_4 = A_5$, and A_5 is finite by [**Huy99**, Proposition 9.1].

Therefore, we see that all graded pieces of the filtration are either finite or abelian of finite type. One can conclude for $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ by Lemma VII.3.1. □

VII.3.2. Projective case. The key input of our proof in the projective case is the recent result of Amerik–Verbitsky [**AV17**] which solves the Morrison–Kawamata cone conjecture for hyper-Kähler manifolds, as well as Markman's previous analogous work on the action of birational automorphism group on the movable cone.

Theorem VII.3.2 (Cone conjectures). *Let X be a projective hyper-Kähler manifold. Then*

(1) *([**Mar11**, Theorem 6.25]) There exists a rationally polyhedral cone Δ which is a fundamental domain for the action of* $\mathrm{Bir}^*(X)$ *on* $MV^+(X)$.
(2) *([**AV17**, Theorem 5.6]) There exists a rationally polyhedral cone Π which is a fundamental domain for the action of* $\mathrm{Aut}^*(X)$ *on* $\mathcal{A}^+(X)$.

Here $MV(X)$ is the movable cone, $\mathcal{A}(X)$ is the ample cone, and $^+$ means the rational closure ([**CF19**, Definition 6.2]).

The proof of Theorem VII.1.3 in the projective case then has a similar structure as in the non-projective case. We consider the action of $\mathrm{Aut}(X)$ on the ample cone. Denoting by

$\mathrm{Aut}^{\#}(X)$ and $\mathrm{Aut}^*(X)$ its kernel and image respectively, we obtain a short exact sequence of groups

$$1 \longrightarrow \mathrm{Aut}^{\#}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}^*(X) \longrightarrow 1. \tag{VII.1}$$

Similarly, considering the action of $\mathrm{Bir}(X)$ on the movable cone, there is an exact sequence analogous to (VII.1)

$$1 \longrightarrow \mathrm{Bir}^{\#}(X) \longrightarrow \mathrm{Bir}(X) \longrightarrow \mathrm{Bir}^*(X) \longrightarrow 1, \tag{VII.2}$$

where $\mathrm{Bir}^*(X)$ and $\mathrm{Bir}^{\#}(X)$ are the image and the kernel of the action of $\mathrm{Bir}(X)$ on the movable cone.

It is then easy to show that $\mathrm{Aut}^{\#}(X)$ and $\mathrm{Bir}^{\#}(X)$ are finite groups. By the argument of group cohomology (Lemma VII.3.1), we reduce the proof to the following crucial property.

Proposition VII.3.3. *Let X be a projective hyper-Kähler manifold. There are only finitely many conjugacy classes of finite subgroups of* $\mathrm{Aut}^*(X)$ *and* $\mathrm{Bir}^*(X)$.

Proof. Let G be a finite subgroup of $\mathrm{Aut}^*(X)$. Fix a rationally polyhedral fundamental domain Π for the action of $\mathrm{KAut}^*(X)$ on $\mathcal{A}^+(X)$, whose existence was proved in Theorem VII.3.2. First of all, we observe that there exists a point $x \in \mathcal{A}(X)$ such that $g.x = x$ for every $g \in G$. Indeed, $x = \sum_{g\in G} g.y$ for any point $y \in \mathcal{A}(X)$ will work. Hence there exists $h \in \mathrm{Aut}(X)$ such that $x_0 = h^*(x) \in \Sigma$. It follows that for every $g \in G$ we have

$$h^* \circ g \circ (h^{-1})^*(x_0) = h^* \circ g \circ (h^{-1})^* \circ h^*(x) = h^* x = x_0,$$

i.e. the element $h^* \circ g \circ (h^{-1})^*$ fixes x_0 for every $g \in G$. This means that the subgroup h^*Gh^{*-1} acts on $\mathcal{A}(X)$ and fixes a point of Π. Therefore

$$h^*Gh^{*-1} \subseteq \{\varphi \in \mathrm{Aut}^*(X) \,|\, \varphi(\Pi) \cap \Pi \neq \{0\}\} =: \mathcal{S}.$$

We claim that $\mathcal{S}$ is a finite set. By definition, for any $\varphi \in \mathcal{S}$, $\varphi(\Pi)$ and Π share at least a ray. On the one hand, Π has only finitely many rays; and on the other hand, for each ray of Π, there are only finitely many translates of Π by $\mathrm{Aut}^*(X)$ sharing it, thanks to the Siegel property ([**Loo14**, Theorem 3.8], see also [**CF19**, Proposition 6.3]). Therefore $\{\varphi(\Pi) \,|\, \varphi \in \mathcal{S}\}$ is a finite set, which implies the finiteness of $\mathcal{S}$ since Π is a fundamental domain. In conclusion, any finite subgroup of $\mathrm{Aut}^*(X)$ is conjugate to a subset of a given finite set $\mathcal{S}$, which admits of course only finitely many subsets.

The proof for $\mathrm{Bir}^*(X)$ is exactly the same, provided we replace $\mathrm{Aut}^*(X)$, $\mathcal{A}(X)$, $\mathcal{A}^+(X)$, and Π by $\mathrm{Bir}^*(X)$, $MV(X)$, $MV^+(X)$, and Δ respectively. □

VII.4. Finiteness of real structures

To treat real structures, we have to consider the so-called *Klein automorphism group* $\mathrm{KAut}(X)$, which is by definition the group of holomorphic or anti-holomorphic automorphisms. The group $\mathrm{Aut}(X)$ is a normal subgroup of $\mathrm{KAut}(X)$ of index at most 2.

Proof of Theorem VII.1.5. We claim that Theorem VII.1.3 also holds for $\mathrm{KAut}(X)$ ([**CF19**, Theorems 1.3, 1.4]). To show this, we apply the same argument as in §VII.3: the non-projective case goes through without change; while in the projective case, the only difference is that we need to show the analogue of the Morrison–Kawamata cone conjecture (Theorem VII.3.2) for $\mathrm{KAut}(X)$. Indeed, the natural action of $\mathrm{Aut}(X)$ on the ample cone naturally extends to $\mathrm{KAut}(X)$ (see [**CF19**, §3]) and by using Looijenga's result [**Loo14**, Proposition

4.1 and Application 4.14], we can show ([CF19, Theorem 6.6]) that there exists a rationally polyhedral cone Σ which is a fundamental domain for the action of $\mathrm{KAut}^*(X)$ on $\mathcal{A}^+(X)$. See [CF19, §7] for details.

Now assume that there exists at least one real structure σ. In this case, we have a splitting short exact sequence

$$1 \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{KAut}(X) \overset{\varepsilon}{\longrightarrow} \{\pm 1\} \longrightarrow 1 \tag{VII.3}$$

and we need to show finiteness of the cohomology set $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{Aut}(X))$, where $\mathrm{Aut}(X)$ is endowed with the action of conjugation by σ. The short exact sequence (VII.3) induces an exact sequence of pointed sets:

$$\cdots \to \{\pm 1\} \to H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{Aut}(X)) \to H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{KAut}(X)) \to \dots .$$

Since $\{\pm 1\}$ is finite, it suffices, by [BS64, Corollaire 1.13], to show that the cohomology set $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{KAut}(X))$ is finite. However, the action of $\mathbf{Z}/2\mathbf{Z}$ on $\mathrm{KAut}(X)$ is given by conjugation by σ, *i.e.* an inner automorphism, so by [BS64, Proposition 1.5], $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{KAut}(X))$ is in bijection with $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{KAut}(X)_{\mathrm{triv}})$ where $\mathrm{KAut}(X)_{\mathrm{triv}}$ is endowed with the trivial $\mathbf{Z}/2\mathbf{Z}$-action. Finally, the complement of the base point (the trivial cocycle) in $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{KAut}(X)_{\mathrm{triv}})$ is naturally identified with the set of conjugacy classes of subgroups of order 2 in $\mathrm{KAut}(X)$, thus its finiteness is a special case of our claim above. □

VII.5. Some follow-up projects

VII.5.1. Enriques manifolds. In [DIK00], it was shown that the main results presented in this chapter (the finiteness of finite subgroups, the finiteness of real structures and the finite generation of automorphism groups) hold for Enriques surfaces. The higher dimensional analogue of Enriques surfaces are the so-called *Enriques manifolds* [OS11a] (see also [BNWkS11]). Those are (necessarily projective) varieties whose universal cover is hyper-Kähler. To show our results for Enriques manifolds, there are two possible approaches.

The first one is to use directly our established finiteness properties of the covering hyper-Kähler varieties. However, the relation between the automorphism group of the Enriques manifold and that of its universal cover seems unclear. In my opinion, the proof for Enriques surfaces in [DIK00, Corollary D.1.2] is actually flawed at this point; but the result can be saved (see the second approach below).

The second approach is to run the whole argument again for Enriques manifolds. The main missing ingredient is then the Morrison–Kawamata cone conjecture for Enriques manifolds, which is known for Enriques surfaces [Nam85] [Kaw97]. I plan to study this special but interesting case of the cone conjecture in the future.

VII.5.2. Potential density. It was shown by Bogomolov–Tschinkel [BT00] that a K3 surface defined over a number field and with infinite automorphism group has potentially dense rational points. I want to generalize this result for hyper-Kähler varieties. Just as in [BT00], by applying Theorem VII.1.2 on finite generation of the automorphism group, we see that up to taking a finite field extension, all automorphisms are defined over the base field. The remaining (main) issue is to find an appropriate replacement for the geometric arguments on the curves on K3 surfaces. The study of uniruled divisors, which is an active research topic now, will be very much related.

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