Hochschild cohomology of Hilbert schemes of points on surfaces

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Abstract

We compute the Hochschild cohomology of Hilbert schemes of points on surfaces and observe that it is, in general, not determined solely by the Hochschild cohomology of the surface, but by its "Hochschild–Serre cohomology"—the bigraded vector space obtained by taking Hochschild homologies with coefficients in powers of the Serre functor. As applications, we obtain various consequences on the deformation theory of the Hilbert schemes; in particular, we recover and extend results of Fantechi, Boissière, and Hitchin.

Our method is to compute more generally for any smooth proper algebraic variety X the Hochschild–Serre cohomology of the symmetric quotient stack $[X^n/\mathfrak{S}_n]$, in terms of the Hochschild–Serre cohomology of X.

Contents

1 Introduction

1.1 Hochschild cohomology of Hilbert schemes of points on surfaces

For a smooth projective surface S, the Hilbert scheme of points $\mathrm{Hilb}^n S$ is again a smooth projective variety [24]. The geometry and invariants of $\mathrm{Hilb}^n S$ are controlled by those of S. Probably the most famous result in this direction is the identification of the direct sum of the singular cohomologies of all the Hilbert schemes of points on S with the Fock space representation of the Heisenberg Lie algebra associated with the cohomology of the surface S; see [28, 30, 31, 58]. This means, in particular, that we have an isomorphism of graded vector spaces (up to some degree shift)

$$\bigoplus_{n\geqslant 0} \mathrm{H}^*(\mathrm{Hilb}^n S, \mathbb{C})t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}\mathrm{H}^*(S,\mathbb{C})t^i\right), \tag{1}$$

where the symmetric power $\mathrm{Sym}^\bullet$ on the right-hand side is graded for the grading of the cohomology, but ordinary for the grading given by exponents of the formal variable t; see Section 3.1 for details on graded symmetric powers.

Due to the identification of singular cohomology and Hochschild homology via the Hochschild–Kostant–Rosenberg isomorphism, we get an analogous isomorphism of graded vector spaces (without degree shift) for the *Hochschild homology* of all the Hilbert schemes taken together.

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Hilb}^n S)t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}\mathrm{HH}_*(S)t^i\right). \tag{2}$$

In particular, the Hochschild homology of $\mathrm{Hilb}^n S$ is determined by that of S.

Hochschild cohomology is another invariant of varieties, whose definition is similar to that of Hochschild homology, but whose behavior is, in general, quite different. For some explicit examples of calculations and its behavior in families, the reader is referred to [5, 8]. Hochschild cohomology plays an important role in the deformation theory of varieties and their (derived) categories of coherent sheaves, as discussed in Section 3.5.

The motivation for this paper was whether the Hochschild cohomology of the Hilbert schemes of points $\mathrm{HH}^*(\mathrm{Hilb}^n S)$ can be expressed by a formula similar to (1) and (2), or at least determined by the Hochschild cohomology $\mathrm{HH}^*(S)$ of the surface. It turns out that the information of the Hochschild cohomology of S, even jointly with its Hochschild homology, is not enough for this purpose in general (see Section 4.3 for concrete examples using bielliptic surfaces), but one needs the full non-positive part ($k \leqslant 0$) of the *Hochschild–Serre cohomology*

$$\mathrm{HS}_\bullet(S) := \bigoplus_{k\in\mathbb{Z}} \mathrm{HS}_k(S) \quad \text{where} \quad \mathrm{HS}_k(S) := \mathbf{R}\mathrm{Hom}_{S\times S}(\Delta_*\mathcal{O}_S, \Delta_*\omega_S^{\otimes k}[k\dim S]). \tag{3}$$

This invariant involving all powers of the Serre functor was first defined for varieties and shown to be a derived invariant in [63, page 535] (see also [40, page 139]). The definition of Hochschild–Serre cohomology, as well as its derived invariance, can be naturally extended to orbifolds (Definition 2.3, Corollary 3.19), and even to smooth proper dg categories (Appendix A.1). In [40, 63] this is referred to as the *Hochschild algebra*, as it is a bigraded algebra which contains the Hochschild cohomology $\mathrm{HS}_0(S) = \mathrm{HH}^*(S)$ as a graded subalgebra, and Hochschild homology $\mathrm{HS}_1(S) = \mathrm{HH}_*(S)$ as a graded submodule. But to avoid confusion with the usual algebra structure on Hochschild cohomology, we will refer to the entire bigraded structure in (3) as *Hochschild–Serre cohomology.*

Our first main result is a formula expressing the Hochschild–Serre cohomology of all the Hilbert schemes of points using the Hochschild–Serre cohomology of the surface S. We should point out that we only compute the Hochschild–Serre cohomology of Hilbert schemes as a bigraded vector space. So for now, we have to leave the computation of the algebra structure as an open problem.

Theorem A (Corollary 3.23)**.** *For any smooth projective surface S defined over a field of characteristic zero, and any integer k, we have*

$$\bigoplus_{n\geqslant 0} \mathrm{HS}_k(\mathrm{Hilb}^n S)t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}\mathrm{HS}_{1+(k-1)i}(S)t^i\right). \tag{4}$$

In particular, considering $k = 0$, the Hochschild cohomology of the Hilbert schemes is given by

$$(5) \qquad \bigoplus_{n\geqslant 0} \mathrm{HH}^*(\mathrm{Hilb}^n S)t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HS}_{1-i}(S)t^i\right).$$

Notice on the right-hand side of (5), pieces of Hochschild–Serre cohomology of S other than $\mathrm{HH}^*(S)$ do come into play.

We are in particular interested in $\mathrm{HH}^1(\mathrm{Hilb}^n S)$ and $\mathrm{HH}^2(\mathrm{Hilb}^n S)$, given the role of these spaces in understanding symmetries and deformations of $\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S)$, and thus also of $\mathrm{Hilb}^n S$, as recalled in Section 3.5. As an application of Theorem A, the following corollary bootstraps the calculation of Theorem A to describe one of the summands in the Hochschild–Kostant–Rosenberg decomposition of $\mathrm{HH}^2(\mathrm{Hilb}^n S)$ which describes the *classical* deformation theory of $\mathrm{Hilb}^n S$. It reproves (with different methods) and generalizes results of Fantechi [22, Theorems 0.1 and 0.3] and Hitchin [39, §4.1], so that arbitrary surfaces can now be considered.

Corollary B. *Let S be a smooth projective surface defined over a field $\mathbf{k}$ of characteristic zero. For all $n \geqslant 2$, there exists an isomorphism*

$$(6) \qquad \mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) \cong \mathrm{H}^1(S, \mathrm{T}_S) \oplus \left(\mathrm{H}^0(S, \mathrm{T}_S) \otimes_{\mathbf{k}} \mathrm{H}^1(S, \mathcal{O}_S)\right) \oplus \mathrm{H}^0(S, \omega_S^\vee).$$

This corollary suggests that one should study the deformation theory of Hilbert schemes of points on bielliptic surfaces more closely. In Example 5.6 we explain how they admit an additional deformation direction.

Another applications of Theorem A is Corollary 5.1, where we give an alternative proof of Boissière's result [11, Corollaire 1] stating that the infinitesimal automorphisms of S and $\mathrm{Hilb}^n S$ agree, by means of the equality $\dim \mathrm{Aut}^0(S) = \dim \mathrm{Aut}^0(\mathrm{Hilb}^n S)$.

1.2 Hochschild–Serre cohomology for symmetric quotient stacks

Our approach to Theorem A is "non-commutative". We deduce it from the following more general result involving the derived category of the *symmetric quotient stack* $[\mathrm{Sym}^n X]$, namely, the quotient stack $[X^n/\mathfrak{S}_n]$ where the symmetric group $\mathfrak{S}_n$ acts on the cartesian power X^n by permuting the factors. The result works for varieties of arbitrary dimension, not only for surfaces.

Theorem C (Corollary 3.14)**.** *Let X be a smooth proper algebraic variety of dimension d_X over a field $\mathbf{k}$ of characteristic zero. Let k be a fixed positive integer.*

(i) If $(k-1)d_X$ is even, we have

$$(7) \qquad \bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HS}_{1+(k-1)i}(X)t^i\right).$$

(ii) If $(k-1)d_X$ is odd, we have

$$(8) \qquad \bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{\substack{i\geqslant 1\\ i \text{ odd}}} \mathrm{HS}_{1+(k-1)i}(X)t^i\right).$$

Theorem C is a special case of the even more general Theorem 3.9, which deals with general Hochschild homology with coefficients, not only Hochschild–Serre cohomology.

The relation between Theorem A and Theorem C stems from the equivalence of categories

$$(9) \qquad \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S) \cong \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n S])$$

obtained by combining Bridgeland–King–Reid's derived McKay correspondence [15] with Haiman's description of the isospectral Hilbert scheme [35]. Thus if one is interested in derived invariants of Hilbert schemes of points on surfaces, such as Hochschild–Serre cohomology, one can use the symmetric quotient stack for computations. In particluar, Theorem A becomes a special case of Theorem C.

Taking $k = 1$ in Theorem C, we deduce the following higher-dimensional generalization of (2):

Corollary D. *Let X be a smooth proper algebraic variety of dimension d_X over a field $\mathbf{k}$ of characteristic zero. We have*

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HH}_*(X)t^i\right). \tag{10}$$

Consequently, the vector space $\bigoplus_{n\geqslant 0} \mathrm{HH}_*([\mathrm{Sym}^n X])$ can be identified with the Fock space representation of the Heisenberg algebra associated with $\mathrm{HH}_*(X)$.

Despite the similarity between the formulas in Theorem C for Hochschild–Serre cohomology and (10) for Hochschild homology, we do not see a way to equip $\bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n$ with the structure of a Fock space for $k \neq 1$; see Section 3.7 for some further discussion on this.

Remark 1.1. We expect Corollary D to hold more generally (Conjecture 3.24): for any smooth proper dg category $\mathcal{T}$, we should have

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{T})t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HH}_*(\mathcal{T})t^i\right). \tag{11}$$

We can indeed provide ample evidence to (11), for instance, for various Kuznetsov components of (Fano) varieties. See Corollary 3.26.

1.3 Twisted Hodge groups: Boissière's conjecture and a revised version

As mentioned before, our most general statement, Theorem 3.9, computes the *Hochschild homology with coefficients* $\mathrm{HH}_*([\mathrm{Sym}^n X], F^{\{n\}})$ of symmetric quotient stacks $[\mathrm{Sym}^n X]$ with coefficients in external tensor powers of objects $F \in \mathbf{D}^{\mathrm{b}}(X)$; see Definition 2.1 and Notation 3.1 for the definitions. The proof uses the orbifold version of the Hochschild–Kostant–Rosenberg isomorphism due to Arinkin–Căldăraru–Hablicsek [2] (actually, we need a straightforward generalisation of their result; see Proposition 2.8).

Taking the coefficient F to be some powers of the appropriately shifted canonical bundle $\omega_X[d_X]$, we obtain Hochschild–Serre cohomology, and hence Theorem A. Another interesting special case of Theorem 3.9 occurs when we take the coefficient F to be a line bundle L, hence $L^{\{n\}}$ is the naturally induced line bundle on the symmetric quotient stack (corresponding to, in the surface case, the line bundle L_n on the Hilbert scheme). More precisely, we prove in Corollary 3.22 the following more general version of Theorem A:

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Hilb}^n S, L_n)t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HH}_*(S, L^{\otimes i})t^i\right). \tag{12}$$

Via the Hochschild–Konstant–Rosenberg isomorphism, this is closely related to a conjecture of Boissière [11, Conjecture 1] (disproven in [38, Appendix B]) on twisted Hodge numbers of Hilbert schemes $\mathrm{H}^{p,q}(\mathrm{Hilb}^n S, L_n)$.

In Section 5.3, an alternative counterexample to Boissière's conjecture (see Example 5.7) is provided, and we speculate about the bigraded version of the decomposition (12) and propose the following revision of Boissière's conjecture.

Conjecture E (Conjecture 5.8)**.** *Let S be a smooth projective surface, and $L \in \mathrm{Pic}\, S$. Then*

$$\sum_{n\geqslant 0}\sum_{p=0}^{2n}\sum_{q=0}^{2n} \mathrm{h}^{p,q}(\mathrm{Hilb}^n S, L_n)x^p y^q t^n = \prod_{k\geqslant 1}\prod_{p=0}^{2}\prod_{q=0}^{2}\left(1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k\right)^{-(-1)^{p+q}\mathrm{h}^{p,q}(S,L^{\otimes k})}. \tag{13}$$

We verify the conjecture for $x = 0$, $y = 0$, $y = -1$, or $x = y^{-1}$, and finally and most notably, in Section 5.4, we use a result of Nieper–Wißkirchen [61] to show that Conjecture 5.8 holds for line bundles L admitting a unitary flat connection; see Theorem 5.20.

Examples We work out some examples of the formula (5) (and its generalization for arbitrary X) to illustrate the methods.

In Section 4.1 we compute $\mathrm{HH}^*([\mathrm{Sym}^2\mathbb{P}^1])$ using our main result, and verify it by calculating the Hochschild cohomology of a derived equivalent finite-dimensional algebra.

In Section 4.2 we compute $\mathrm{HH}^*([\mathrm{Sym}^2\mathbb{P}^2]) \cong \mathrm{HH}^*(\mathrm{Hilb}^2\mathbb{P}^2)$ using our main result, and improve upon it by calculating the Hochschild–Kostant–Rosenberg decomposition using an explicit geometric model for the Hilbert square of $\mathbb{P}^2$.

In Section 4.3 we show by means of an example involving bielliptic surfaces that $\mathrm{HH}^*(S)$ and $\mathrm{HH}_*(S)$ do not determine $\mathrm{HH}^*(\mathrm{Hilb}^n S)$ in general. In other words, the Hochschild–Serre cohomology of a surface contains strictly more information than just its Hochschild (co)homology. See Corollary 4.10 for the precise statement.

Conventions In this paper, $\mathbf{k}$ is a base field. An *orbifold* means a tame separated Deligne–Mumford stack of finite type over $\mathbf{k}$. In contrast to the usual definition, we do not impose trivial generic stabilizer. Unless otherwise specified, all the fiber products are over $\mathrm{Spec}\,\mathbf{k}$. We denote by d_X the dimension of a variety (or stack) X.

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2 Hochschild (co)homology, Hochschild–Serre cohomology and their decomposition

2.1 Hochschild (co)homology with coefficients and Hochschild–Serre cohomology

Hochschild (co)homology of varieties, and more generally orbifolds, can be defined in various ways, we will use the approach using Fourier–Mukai transforms as used, e.g., in [16]. In [63, page 535] (see also [40, page 139]) a generalisation is introduced, which incorporates all powers of the Serre functor, and which we will call *Hochschild–Serre cohomology*; see Definition 2.3. We will use the slightly more general notion of *Hochschild (co)homology with coefficients*, which includes all parts of the Hochschild–Serre cohomology as special cases.

Throughout, let $\mathcal{X}$ be a smooth *proper* orbifold. We denote by $\omega_{\mathcal{X}}$ its canonical bundle and $d_{\mathcal{X}}$ its dimension.

The Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{X})$ is given by

(14) $\mathrm{S}_{\mathcal{X}} := - \otimes \omega_{\mathcal{X}}[d_{\mathcal{X}}],$

see [18, Section 2.2], [62], and also [52] in the presence of a projective coarse moduli space. If $\mathcal{X} = [M/G]$ is a global quotient stack by a finite group G, then under the usual identification of coherent sheaves on the quotient stack $[M/G]$ with G-equivariant coherent sheaves on M, the canonical bundle $\omega_{\mathcal{X}}$ corresponds to ω_M equipped with the linearisation given by pullback of top forms along the group action.

Definition 2.1. For $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$, we define the *Hochschild homology of* $\mathcal{X}$ *with coefficients in* $\mathcal{E}$ as

(15) $\mathrm{HH}_*(\mathcal{X}, \mathcal{E}) := \mathbf{R}\Gamma(\mathcal{X} \times \mathcal{X}, \Delta_*\mathcal{O}_{\mathcal{X}} \otimes^{\mathbf{L}} \Delta_*\mathcal{E}) = \mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\mathcal{O}_{\mathcal{X}\times\mathcal{X}}, \Delta_*\mathcal{O}_{\mathcal{X}} \otimes^{\mathbf{L}} \Delta_*\mathcal{E}),$

and the *Hochschild cohomology of* $\mathcal{X}$ *with coefficients in* $\mathcal{E}$ as

(16) $\mathrm{HH}^*(\mathcal{X}, \mathcal{E}) := \mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*\mathcal{E})\,,$

where $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal morphism. By Grothendieck duality, these two invariants are related as follows (see, for example, [50, Lemma 2.1]):

(17) $\mathrm{HH}^*(\mathcal{X}, \mathrm{S}_{\mathcal{X}}\,\mathcal{E}) \cong \mathrm{HH}_*(\mathcal{X}, \mathcal{E})\,.$

Concretely,

(18) $\mathrm{HH}_*(\mathcal{X}, \mathcal{E}) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*(\mathrm{S}_{\mathcal{X}}\,\mathcal{E})) = \mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}] \otimes \mathcal{E}))\,,$

There is a slight abuse of notation happening here, where we will not always distinguish between an object in $\mathbf{D}^{\mathrm{b}}(\mathbf{k})$ and its cohomology, which is a graded vector space.

If we spell out what happens when we take cohomology, then for any $j \in \mathbb{Z}$,

$$(19) \quad \mathrm{HH}_j(\mathcal{X}, \mathcal{E}) = \mathrm{Ext}^j_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}] \otimes \mathcal{E})).$$

Remark 2.2. It is possible to define Hochschild (co)homology with values in bimodules, i.e., elements of $\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})$, but we will not consider this. The reason is that the Hochschild–Kostant–Rosenberg decomposition only works for symmetric bimodules (for the affine setting of this statement, see [54, §1.3]), and thus only for objects in the essential image of Δ_*.

For the special case where $\mathcal{E}$ is a tensor power of the shifted canonical bundle $\omega_{\mathcal{X}}[d_{\mathcal{X}}]$, we introduce the following terminology.

Definition 2.3. The *Hochschild–Serre cohomology* of $\mathcal{X}$ is

$$(20) \quad \mathrm{HS}_\bullet(\mathcal{X}) := \bigoplus_{k\in\mathbb{Z}} \mathrm{HS}_k(\mathcal{X})$$

where for any $k \in \mathbb{Z}$,

$$(21) \quad \mathrm{HS}_k(\mathcal{X}) := \mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*\omega_{\mathcal{X}}^{\otimes k}[kd_{\mathcal{X}}])$$

and thus

$$(22) \quad \mathrm{HS}_k(\mathcal{X}) \cong \mathrm{HH}^*(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k}[kd_{\mathcal{X}}]) \cong \mathrm{HH}_*(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k-1}[(k-1)d_{\mathcal{X}}]).$$

Taking cohomology, we get the bigraded algebra

$$(23) \quad \mathrm{HS}^*_\bullet(\mathcal{X}) := \bigoplus_{j,k\in\mathbb{Z}} \mathrm{HS}^j_k(\mathcal{X}),$$

with

$$(24) \quad \mathrm{HS}^j_k(\mathcal{X}) := \mathrm{HH}^j(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k}[kd_{\mathcal{X}}]) = \mathrm{Ext}^{j+kd_{\mathcal{X}}}_{\mathcal{X}\times\mathcal{X}}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*\omega_{\mathcal{X}}^{\otimes k}),$$

where the algebra structure is given by composition in $\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})$.

The objective of this paper is to compute, given a smooth and proper variety X, the Hochschild–Serre cohomology of the symmetric quotient stack $[X^n/\mathfrak{S}_n]$, in terms of the Hochschild–Serre cohomology of X.

Remark 2.4. We recover some classical invariants as certain graded pieces of Hochschild–Serre cohomology:

- Hochschild cohomology, namely $\mathrm{HH}^*(\mathcal{X}) \cong \mathrm{HS}^*_0(\mathcal{X})$,
- Hochschild homology, namely $\mathrm{HH}_*(\mathcal{X}) \cong \mathrm{HS}^*_1(\mathcal{X})$,
- the canonical ring, namely $\mathrm{R}(\mathcal{X}) = \bigoplus_{k\geqslant 0} \mathrm{HS}^{-kd_{\mathcal{X}}}_k(\mathcal{X})$.

A generalisation of Hochschild–Serre cohomology for varieties is used in [55] to find new derived invariants, through the formalism of cohomological support loci.

Remark 2.5. In [63, §2.1] and [40, §6.1] this definition is given for smooth and projective varieties, but a different notation and grading is used. The notation in op. cit. is $\mathrm{HA}_{\bullet,*}(X)$ for what we denote $\mathrm{HS}^*_\bullet(X)$, and the Serre functor is not used with the shift. To make the definition generalise to the more general setting of smooth and proper dg categories (see Appendix A.1 and the next remark) it is necessary to use the *actual* Serre functor, and not just $-\otimes\omega_{\mathcal{X}}$, as there is no notion of an unshifted Serre functor in general.

Remark 2.6 (Generalisation to dg categories)**.** This definition has an obvious generalisation for a smooth and proper dg category, so that a Serre functor exists. This is discussed in Appendix A.1. See also [65, Section 4] where similar notions are discussed in the context of stable ∞-categories, but the coefficients used are related to group actions and not powers of the Serre functor.

Remark 2.7 (Derived invariance)**.** Hochschild–Serre cohomology of a variety is a derived invariant [63, Theorem 2.1.8], thus implying that Hochschild cohomology and Hochschild homology are derived invariants. Using the general definition of Appendix A.1 for dg categories, it follows from Theorems A.4 and A.5 that Hochschild–Serre cohomology, as a bigraded algebra, is also a derived invariant for smooth proper Deligne–Mumford stacks, which is important for geometric applications to Hilbert schemes of points on surfaces. See also Corollary 3.19 for a direct proof in the geometric setting without appealing to dg categories. This is a special case of Proposition 3.17, which shows that, under a certain compatibility condition, Hochschild homology with coefficients is a derived invariant.

It is moreover possible to upgrade the functoriality properties of Hochschild–Serre cohomology of varieties so that it becomes functorial for étale morphisms, but we will not need this in the main body of the text, so this discussion is relegated to Appendix A.2.

2.2 The Hochschild–Kostant–Rosenberg decomposition

In this section we consider the case where $\mathcal{X}$ is isomorphic to a global quotient stack by a finite group. Let M be a smooth proper variety defined over $\mathbf{k}$, and G a finite group acting on M. Let $\mathcal{X} := [M/G]$ be the quotient stack, which is a smooth proper orbifold.

For any $g \in G$, we denote by M^g the fixed locus of $g \in \mathrm{Aut}(M)$, which is a smooth closed subvariety of M. Their disjoint union is denoted by

$$(25)\quad \mathrm{I}_G M := \coprod_{g\in G} M^g,$$

which is sometimes called the *inertia variety* of the action G on M. The inertia variety $\mathrm{I}_G M$ admits a natural G-action as follows: for any $g, h \in G$, there is an isomorphism

$$(26)\quad h\cdot : M^g \xrightarrow{\simeq} M^{hgh^{-1}}.$$

Taking the disjoint union for g running through G, we can define the action of h (and G) on $\mathrm{I}_G M$.

Using this action, the *inertia stack* $\mathrm{I}\mathcal{X} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ can itself be described as a global quotient stack by the same finite group G:

$$(27)\quad \mathrm{I}\mathcal{X} \cong [\mathrm{I}_G M/G].$$

The usual Hochschild–Kostant–Rosenberg isomorphism for varieties (which we state in Proposition 2.11) admits the following orbifold version[1], essentially due to Arinkin–Căldăraru–Hablicsek [2].

Proposition 2.8 (Hochschild–Kostant–Rosenberg for global quotient stacks)**.** *Let $\mathcal{X} = [M/G]$ be a smooth global quotient orbifold defined over $\mathbf{k}$ as above. Let $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}([M/G])$ be given by a G-linearised object $E \in \mathbf{D}^{\mathrm{b}}(M)$. Then*

$$(28)\quad \mathrm{HH}_*(\mathcal{X}, \mathcal{E}) \cong \left(\bigoplus_{g\in G} \mathrm{H}^*\left(M^g, \mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E\right)\right)^G.$$

Equivalently,

$$(29)\quad \mathrm{HH}_*(\mathcal{X}, \mathcal{E}) \cong \bigoplus_{[g]\in \mathrm{Conj}(G)} \mathrm{H}^*\left(M^g, \mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E\right)^{\mathrm{C}(g)}.$$

Here, M^g is the fixed locus of g and $i_g\colon M^g \hookrightarrow M$ is the closed immersion, $\mathrm{Conj}(G)$ is the set of conjugacy classes of G, and $\mathrm{C}(g)$ is the centraliser of g in G. Here the action on the right-hand sides of (28) *and* (29) *are induced by the action* (26).

[1] The tools to prove the Hochschild–Kostant–Rosenberg decomposition for arbitrary smooth Deligne–Mumford stacks, which are not global quotients by finite groups, seem to be lacking from the literature.

The graded vector spaces occuring on the right-hand sides of (28) and (29) denote (hyper-)cohomology of the complex $\mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E$. In other words,

$$\mathrm{H}^* \left(M^g, \mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E\right) := \mathbf{R}\Gamma \left(M^g, \mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E\right). \tag{30}$$

Proof. Since in [2] only the cases $\mathcal{E} = \mathcal{O}_\mathcal{X}$ and $\mathcal{E} = \omega_\mathcal{X}^{-1}[-d_\mathcal{X}]$ (which give Hochschild homology and Hochschild cohomology without coefficients) are explicitly worked out, we include a full proof here for the sake of completeness.

We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{L}\mathcal{X} \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\
& \mathrm{I}\mathcal{X} \arrow[ul] \arrow[r, "p"] \arrow[d, "q"] & \mathcal{X} \arrow[d, "\Delta"] \\
& \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd} \tag{31}$$

where $\mathrm{L}\mathcal{X} := \mathcal{X} \times^{\mathbf{R}}_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is the derived fiber product, called the *free loop space*.

By [2, Corollary 1.17], p' and q' are homotopic, hence by base change and the projection formula, we have

$$\mathbf{L}\Delta^*\Delta_* \cong \mathbf{R}q'_*\mathbf{L}p'^* \cong \mathbf{R}p'_*\mathbf{L}p'^* \cong - \otimes \mathbf{R}p'_*(\mathcal{O}_{\mathrm{L}\mathcal{X}}) \cong - \otimes \mathbf{R}p_* \mathrm{Sym}^\bullet(\Omega^1_{\mathrm{I}\mathcal{X}}[1]), \tag{32}$$

where the last isomorphism uses [2, Theorem 1.15], which says that $\mathrm{L}\mathcal{X}$ is isomorphic (over $\mathcal{X} \times \mathcal{X}$) to the total space of the shifted tangent bundle of $\mathrm{I}\mathcal{X}$.

Therefore,

$$\begin{aligned}
\mathrm{HH}_*(\mathcal{X}, \mathcal{E}) &\cong \mathbf{R}\Gamma(\mathcal{X} \times \mathcal{X}, \Delta_*\mathcal{O}_\mathcal{X} \otimes \Delta_*\mathcal{E}) \\
&\cong \mathbf{R}\Gamma(\mathcal{X} \times \mathcal{X}, \Delta_*\mathbf{L}\Delta^*\Delta_*\mathcal{E}) \\
&\cong \mathbf{R}\Gamma(\mathcal{X}, \mathbf{L}\Delta^*\Delta_*\mathcal{E}) \\
&\cong \mathbf{R}\Gamma(\mathcal{X}, \mathcal{E} \otimes \mathbf{R}p_* \mathrm{Sym}^\bullet(\Omega^1_{\mathrm{I}\mathcal{X}}[1])) \\
&\cong \mathbf{R}\Gamma(\mathrm{I}\mathcal{X}, p^*\mathcal{E} \otimes \mathrm{Sym}^\bullet(\Omega^1_{\mathrm{I}\mathcal{X}}[1])) \\
&\cong \left(\bigoplus_{g\in G} \mathbf{R}\Gamma \left(M^g, \mathrm{Sym}^\bullet(\Omega^1_{M^g}[1]) \otimes i_g^* E\right)\right)^G
\end{aligned} \tag{33}$$

where the fourth isomorphism uses (32) and the last isomorphism follows the fact that $-^G$ is an exact functor since we are in characteristic zero. This proves (28), and (29) follows immediately. □

Remark 2.9. For later use, let us make the group action on the right-hand sides of (28) and (29) more precise. In the statement of Proposition 2.8, for any $h \in G$, the action of h sends the summand indexed by g isomorphically to the one indexed by hgh^{-1} via (26):

$$h\colon M^g \xrightarrow{\simeq} M^{hgh^{-1}}. \tag{34}$$

In particular, $\mathrm{C}(g)$ acts on M^g. The isomorphisms $\Omega^1_{M^g} \xrightarrow{\simeq} h^*\Omega^1_{M^{hgh^{-1}}}$ are the canonical ones. The isomorphisms $h^* i^*_{hgh^{-1}}(E) \cong i_g^* h^*(E) \xrightarrow{\simeq} i_g^* E$ come from the linearisation of E.

Remark 2.10. In the case that $M = \operatorname{Spec} R$ is affine, coherent sheaves on $[M/G]$ are the same as finitely generated modules over the crossed product algebra $R\#G$. For the Hochschild cohomology of these crossed product algebras, decompositions analogous to (29) are given in [72, Section 3] and [1]. There, it is also described what happens with the ring structure of Hochschild cohomology under this decomposition. For M non-affine this remains an open problem; see [17] and [59] for some partial result and speculation on this.

2.3 Orbifold Hochschild–Kostant–Rosenberg in terms of Hodge groups of inertia

As a motivation for the definitions that will follow, and calculations in the surface case, we first spell out Proposition 2.8 in the special case where G is the trivial group.

Proposition 2.11 (Hochschild–Kostant–Rosenberg with coefficients on a variety)**.** *Let X be a smooth proper variety over $\mathbf{k}$ of dimension d_X. For any $E \in \mathbf{D}^{\mathrm{b}}(X)$, we have*

$$(35)\quad \mathrm{HH}_*(X, E) \cong \mathbf{R}\Gamma(X, \mathrm{Sym}^\bullet(\Omega_X^1[1]) \otimes E),$$

where $\mathrm{Sym}^\bullet$ *is taken in the graded sense, so that* $\mathrm{Sym}^\bullet(\Omega_X^1[1]) = \bigoplus_{p\geqslant 0} \bigwedge^p \Omega_X^1[p]$. *Taking cohomology, for any $j \in \mathbb{Z}$,*

$$(36)\quad \mathrm{HH}_j(X, E) \cong \bigoplus_{q-p=j} \mathrm{H}^q(X, \Omega_X^p \otimes E).$$

In particular,

$$(37)\quad \mathrm{HS}_k^j(X) \cong \bigoplus_{p+q=j+kd_X} \mathrm{H}^q\left(X, \bigwedge^p \mathrm{T}_X \otimes \omega_X^{\otimes k}\right),$$

which is (possibly) non-zero only in degrees $[-kd_X, 2d_X - kd_X]$.

For $k = 0$ resp. $k = 1$ we obtain the usual Hochschild–Kostant–Rosenberg decomposition for Hochschild cohomology resp. Hochschild homology.

Definition 2.12. Let X be a smooth and proper variety. Let $E \in \mathbf{D}^{\mathrm{b}}(X)$. For any integers p, q, define the *twisted Hodge group*

$$(38)\quad \mathrm{H}^{p,q}(X, E) \coloneqq \mathrm{H}^q(X, \Omega_X^p \otimes E).$$

Then (36) says that

$$(39)\quad \mathrm{HH}_j(X, E) \cong \bigoplus_{q-p=j} \mathrm{H}^{p,q}(X, E).$$

In order to give an analogue of (39) for quotient stacks, we introduce the following analogue of twisted Hodge groups for orbifolds.

Definition 2.13. Let $\mathcal{X}$ be a smooth and proper orbifold. Let $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$. For any integer p, q, define the *twisted Hodge group*

$$(40)\quad \mathrm{H}^{p,q}(\mathcal{X}, \mathcal{E}) \coloneqq \mathrm{H}^q(\mathcal{X}, \Omega_{\mathcal{X}}^p \otimes \mathcal{E}).$$

Of particular interest to us are the twisted Hodge groups of the inertia stack. In the sequel, we often abuse notation to denote $\mathrm{H}^{p,q}(\mathrm{I}\mathcal{X}, p^*\mathcal{E})$ simply by $\mathrm{H}^{p,q}(\mathrm{I}\mathcal{X}, \mathcal{E})$, where $p\colon \mathrm{I}\mathcal{X} \to \mathcal{X}$ is the canonical map.

When $\mathcal{X} = [M/G]$, $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$, and $E \in \mathbf{D}^{\mathrm{b}}(M)$ be as in Proposition 2.8, we have

$$(41)\quad \mathrm{H}^{p,q}(\mathrm{I}\mathcal{X}, \mathcal{E}) = \left(\bigoplus_{g\in G} \mathrm{H}^{p,q}(M^g, E|_{M^g})\right)^G \cong \bigoplus_{[g]\in \mathrm{Conj}(G)} \mathrm{H}^{p,q}\left(M^g, E|_{M^g}\right)^{\mathrm{C}(g)}.$$

Corollary 2.14. *With notation as in Proposition 2.8 and for any $j \in \mathbb{Z}$, we have*

$$(42)\quad \mathrm{HH}_j(\mathcal{X}, \mathcal{E}) \cong \bigoplus_{q-p=j} \mathrm{H}^{p,q}(\mathrm{I}\mathcal{X}, \mathcal{E}).$$

Proof. We take cohomology in (28) and plug in (41). □

We can collect the Hodge groups with coefficients in the bigraded vector space

$$\mathrm{H}^{\#,\star}(\mathrm{I}\mathcal{X},\mathcal{E}) := \left(\bigoplus_{g\in G}\mathrm{H}^{\#,\star}(M^g, E|_{M^g})\right)^G \cong \bigoplus_{[g]\in\mathrm{Conj}(G)} \mathrm{H}^{\#,\star}\left(M^g, E|_{M^g}\right)^{\mathrm{C}(g)}, \tag{43}$$

where $\mathrm{H}^{\#,\star}(M^g, E|_{M^g})) = \mathrm{H}^{\star}(M^g, \Omega^{\#}_{M^g}\otimes E_{|M^g})$. Then we can rephrase Corollary 2.14 as

Corollary 2.15. *Turning the bigraded vector space* $\mathrm{H}^{\#,\star}(\mathcal{X},\mathcal{E})$ *into a (single-)graded vector space with grading* $* = \star - \#$*, this graded vector space is isomorphic to the Hochschild homology with coefficients in* $\mathcal{E}$*:*

$$\mathrm{H}^{\#,\star}(\mathrm{I}\mathcal{X},\mathcal{E}) \cong \mathrm{HH}_*(\mathcal{X},\mathcal{E}) \quad \textit{for } * = \star - \#\,. \tag{44}$$

3 Symmetric quotient stacks

This section contains the proof of our result Theorem 3.9 describing the Hochschild homology of symmetric quotient stacks with a certain type of coefficients. Afterwards, we work out most of the results stated in the introduction as corollaries.

Let X be a smooth proper variety defined over a field $\mathbf{k}$ of characteristic zero. For any integer $n \geqslant 1$, let the symmetric group $\mathfrak{S}_n$ act on X^n by permuting the factors. Denote by $[\mathrm{Sym}^n X] := [X^n/\mathfrak{S}_n]$ the quotient stack, called the *nth symmetric quotient stack of* X.

Notation 3.1. For any $F \in \mathbf{D}^{\mathrm{b}}(X)$, endow $F^{\boxtimes n} \in \mathbf{D}^{\mathrm{b}}(X^n)$ with the natural $\mathfrak{S}_n$-linearisation, giving rise to an object $F^{\{n\}} \in \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n X])$. The pull-back of $F^{\{n\}}$ to the inertia stack $\mathrm{I}[\mathrm{Sym}^n X]$ via the natural morphism $\mathrm{I}[\mathrm{Sym}^n X] \to [\mathrm{Sym}^n X]$ is also denoted by the same notation.

For any element $g \in \mathfrak{S}_n$, we view it as a permutation of the set $\{1,\dots,n\}$, and denote by $\mathrm{O}(g)$ the set of the orbits of the permutation g. For each orbit $o \in \mathrm{O}(g)$, let $\#o$ be the cardinality (or length) of the orbit o. In what follows, for any finite set S, we use the notation X^S to denote the variety $\mathrm{Map}(\mathrm{Spec}(\prod_S \mathbf{k}), X)$ parametrizing $\mathbf{k}$-morphisms from $S := \coprod_S \mathrm{Spec}\,\mathbf{k}$ to X.

3.1 Yoga on symmetric powers of (multi)graded vector spaces

We recall some basic properties of symmetric powers of (multi)graded vector spaces.

Graded case Given a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$, there are two ways to form the symmetric product

$$\mathrm{Sym}^\bullet(V) = \bigoplus_{n\geqslant 0}\mathrm{Sym}^n V. \tag{45}$$

- For the *ordinary* symmetric product, we consider the symmetric group $\mathfrak{S}_n$ acting on $V^{\otimes n}$ by permutation of the tensor factors: $\sigma\cdot(v_1\otimes\cdots\otimes v_n) := v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}$. Then $\mathrm{Sym}^n V \subset V^{\otimes n}$ is defined to be the subspace of the invariants under this action. As we are working in characteristic zero, this is isomorphic to the coinvariants.
- For the *graded* symmetric product, we consider the $\mathfrak{S}_n$-action on $V^{\otimes n}$ by permutation of the tensor factors together with a minus sign whenever two factors of odd degree switch positions. Concretely, for a transposition of neighbours $\tau = (j,\ j{+}1)$, we set $\tau\cdot(v_1\otimes\cdots\otimes v_n) := (-1)^{\deg(v_j)\cdot\deg(v_{j+1})} v_{\tau^{-1}(1)}\otimes\cdots\otimes v_{\tau^{-1}(n)}$. Then $\mathrm{Sym}^n V \subset V^{\otimes n}$ denotes the space of the invariants under this action.

If not stated otherwise $\mathrm{Sym}^\bullet$ always denotes the graded symmetric product in what follows.

Multigraded case In several places in this paper, we will have multigraded vector spaces, that is, a $\mathbb{Z}^m$-graded (also referred to as m-multigraded) vector space, for some integer $m \geqslant 1$.

Given an m-multigraded vector space V, and some multi-index $\mathbf{d} \in \mathbb{Z}^m$, we define the *shift* of V by $\mathbf{d}$ as the m-multigraded vector space $V[\mathbf{d}]$ whose graded pieces are given by $(V[\mathbf{d}])_{\mathbf{i}} := V_{\mathbf{i}+\mathbf{d}}$, for any $\mathbf{i} \in \mathbb{Z}^m$.

In the sequel, when we take the symmetric product of some multigraded vector space, it is taken in the graded sense with respect to some degrees, but in the ordinary sense with respect to the other degrees. Let us make this precise. Let $m \in \mathbb{N}$, and let

$$(46) \quad V = \bigoplus_{(i_1,\dots,i_m)\in\mathbb{Z}^m} V_{(i_1,\dots,i_m)}$$

be an m-multigraded vector space. For an homogeneous element $v \in V_{(i_1,\dots,i_m)}$, and $k \in \{1,\dots,m\}$, we set $\deg_k(v) := i_k$. Let $K \subset \{1,\dots,m\}$. We define the symmetric product $\mathrm{Sym}^n V$ which is *graded with respect to the K-degrees, and ordinary with respect to the other degrees* as the subspace of $\mathfrak{S}_n$-invariants of $V^{\otimes n}$ under the action given for $\tau = (j,\ j+1)$ by

$$(47) \quad \tau \cdot (v_1 \otimes \cdots \otimes v_n) := (-1)^{\sum_{k\in K} \deg_k(v_j)\cdot\deg_k(v_{j+1})} v_1 \otimes \cdots v_{j+1} \otimes v_j \otimes \cdots \otimes v_n.$$

We stress the dependence of $\mathrm{Sym}^n V$ on the choice of the "super-degrees" $K \subset \{1,\dots,m\}$, although not reflected in the notation.

For a fixed subset $K \subset \{1,\dots,m\}$, it is straightforward to check that for two m-multigraded vector spaces U and V, we have the following canonical isomorphism of m-multigraded vector spaces:

$$(48) \quad \mathrm{Sym}^\bullet(U \oplus V) \cong \mathrm{Sym}^\bullet(U) \otimes \mathrm{Sym}^\bullet(V).$$

Generating series The *generating series* of a m-multigraded vector space V which is finite-dimensional in each degree is the Laurent series in m variables $s_1,\dots,s_m$ given by

$$(49) \quad \mathbb{E}_V := \sum_{\mathbf{i}\in\mathbb{Z}^m} \dim_{\mathbf{k}}(V_{\mathbf{i}}) s^{\mathbf{i}} \quad \text{where} \quad s^{\mathbf{i}} = s_1^{i_1}\cdots s_m^{i_m} \quad \text{for} \quad \mathbf{i} = (i_1,\dots,i_m)\,.$$

For the following two lemmas, we fix some subset $K \subset \{1,\dots,m\}$. The symmetric product of any m-multigraded vector space is then meant to be graded with respect to the K-degrees and ordinary with respect to the other degrees.

Lemma 3.2. *Let V be a finite-dimensional vector space, which we consider as an m-multigraded vector space concentrated in degree $\mathbf{0} = (0,\dots,0)$. For $\mathbf{d} = (d_1,\dots,d_m) \in \mathbb{Z}^m$, we set $|\mathbf{d}|_K := \sum_{k\in K} d_k$ to be total K-degree. We have*

$$(50) \quad \mathbb{E}_{\mathrm{Sym}^\bullet(V[-\mathbf{d}])} = (1 - (-1)^{|\mathbf{d}|_K} s^{\mathbf{d}})^{-(-1)^{|\mathbf{d}|_K}\dim_{\mathbf{k}} V}\,.$$

Proof. The multigraded vector space $\mathrm{Sym}^\bullet(V[-\mathbf{d}])$ is concentrated in degrees which are non-negative integral multiples of $\mathbf{d}$. Furthermore, for any $a \in \mathbb{N}$, we have

$$(51) \quad \mathrm{Sym}^\bullet(V[-\mathbf{d}])_{a\mathbf{d}} = \begin{cases} \mathrm{Sym}^a V & \text{if } |\mathbf{d}|_K \text{ is even,} \\ \bigwedge^a V & \text{if } |\mathbf{d}|_K \text{ is odd.} \end{cases}$$

Indeed, by (47), the action of $\tau = (j,\ j+1)$ on $v_1 \otimes \cdots \otimes v_n$ with $v_i \in V[-\mathbf{d}]$ is given by

$$(52) \quad \tau \cdot (v_1 \otimes \cdots \otimes v_n) := (-1)^{\sum_{k\in K} d_k^2} v_1 \otimes \cdots v_{j+1} \otimes v_j \otimes \cdots \otimes v_n;$$

and we have

$$(53) \quad (-1)^{\sum_{k\in K} d_k^2} = (-1)^{\sum_{k\in K} d_k} = (-1)^{|\mathbf{d}|_K}\,.$$

Therefore, we get the asserted

$$(54) \quad \sum_{\mathbf{i}\in\mathbb{Z}^m} \dim_{\mathbf{k}}\big(\mathrm{Sym}^\bullet(V[-\mathbf{d}])_{\mathbf{i}}\big) s^{\mathbf{i}} = \begin{cases} \sum_{a\in\mathbb{N}} \dim_{\mathbf{k}}\big(\mathrm{Sym}^a V\big) s^{a\mathbf{d}} = (1 - s^{\mathbf{d}})^{-\dim_{\mathbf{k}} V} & \text{for } |\mathbf{d}|_K \text{ even,} \\ \sum_{a\in\mathbb{N}} \dim_{\mathbf{k}}\big(\bigwedge^a V\big) s^{a\mathbf{d}} = (1 + s^{\mathbf{d}})^{\dim_{\mathbf{k}} V} & \text{for } |\mathbf{d}|_K \text{ odd,} \end{cases}$$

where the second equality (in each case) is well-known; see, e.g., [46, Lemma A.3]. □

Lemma 3.3. *Let W be a finite dimensional m-multigraded vector space. Then the generating series of its total symmetric power $\operatorname{Sym}^\bullet W$ is given as follows:*

$$\mathbb{E}_{\operatorname{Sym}^\bullet(W)} = \prod_{\mathbf{d}\in\mathbb{Z}^m} (1-(-1)^{|\mathbf{d}|_K} s^{\mathbf{d}})^{-(-1)^{|\mathbf{d}|_K} \dim_{\mathrm{k}}(W_{\mathbf{d}})}, \tag{55}$$

where again $|\mathbf{d}|_K = \sum_{k\in K} d_k$ for $\mathbf{d} = (d_1,\dots,d_m)$.

Proof. The compatibility of the symmetric product with direct sums (48) gives an isomorphism of graded vector spaces

$$\operatorname{Sym}^\bullet W = \operatorname{Sym}^\bullet\left(\bigoplus_{\mathbf{d}\in\mathbb{Z}^m} W_{\mathbf{d}}[-\mathbf{d}]\right) \cong \bigotimes_{\mathbf{d}\in\mathbb{Z}^m} \operatorname{Sym}^\bullet(W_{\mathbf{d}}[-\mathbf{d}]). \tag{56}$$

Since the generating series of a tensor product is the product of generator series of its factors, we obtain

$$\mathbb{E}_{\operatorname{Sym}^\bullet(W)} = \prod_{\mathbf{d}\in\mathbb{Z}^m} \mathbb{E}_{\operatorname{Sym}^\bullet(W_{\mathbf{d}}[-\mathbf{d}])}. \tag{57}$$

Combining this with Lemma 3.2 gives the desired formula. □

3.2 Twisted Hodge groups of the inertia

We first express the twisted Hodge groups of the inertia stack of the symmetric quotient stack $[\operatorname{Sym}^n X]$ in terms of the twisted Hodge groups of X and the action of the symmetric group.

Proposition 3.4. *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $F^{\{n\}}$ be as in Notation 3.1. We have the following isomorphism of bigraded vector spaces:*

$$\mathrm{H}^{\#,\star}\left(\mathrm{I}[\operatorname{Sym}^n X], F^{\{n\}}\right) \cong \left(\bigoplus_{g\in\mathfrak{S}_n}\bigotimes_{o\in\mathrm{O}(g)} \mathrm{H}^{\#,\star}(X, F^o|_X)\right)^{\mathfrak{S}_n}. \tag{58}$$

Here $F^o = \bigotimes_{t\in o} \operatorname{pr}_t^*(F) \in \mathbf{D}^{\mathrm{b}}(X^o)$ and $F^o|_X$ is its restriction to the small diagonal (identified with X) of X^o.

Proof. We first observe the following canonical identifications: for any $g \in \mathfrak{S}_n$,

- $(X^n)^g = X^{\mathrm{O}(g)}$, which is a partial diagonal of X^n; we denote the closed immersion $i_g\colon X^{\mathrm{O}(g)} \hookrightarrow X^n$;
- $\Omega_{X^{\mathrm{O}(g)}} = \bigoplus_{o\in\mathrm{O}(g)} p_o^*\Omega_X$, where the projection $p_o\colon X^{\mathrm{O}(g)} \to X$ is induced by $\{o\} \subset \mathrm{O}(g)$;
- $i_g^*(F^{\boxtimes n}) = \boxtimes_{o\in\mathrm{O}(g)} F^o|_X$ where for each o, X is identified with the small diagonal of X^o. Note that $F^o|_X$ is isomorphic to $F^{\otimes\# o}$, but we keep the canonical identification here to remember the action.

Plugging these into (43), we find that

$$\mathrm{H}^{\#,\star}(\mathrm{I}[\operatorname{Sym}^n X], F^{\{n\}}) \cong \left(\bigoplus_{g\in\mathfrak{S}_n} \mathrm{H}^\star\left(X^{\mathrm{O}(g)}, \bigwedge^{\#}\left(\bigoplus_{o\in\mathrm{O}(g)} p_o^*\Omega_X\right) \otimes \left(\boxtimes_{o\in\mathrm{O}(g)} F^o|_X\right)\right)\right)^{\mathfrak{S}_n}. \tag{59}$$

Using that the total wedge power of a direct sum is the tensor product of the total wedge powers of the summands, and the Künneth formula, we have the following isomorphisms of bigraded vector spaces:

$$\begin{aligned}
\mathrm{H}^{\#,\star}(\mathrm{I}[\operatorname{Sym}^n X], F^{\{n\}}) &\cong \left(\bigoplus_{g\in\mathfrak{S}_n} \mathrm{H}^\star\left(X^{\mathrm{O}(g)}, \boxtimes_{o\in\mathrm{O}(g)} \left(\bigwedge^{\#}\Omega_X \otimes F^o|_X\right)\right)\right)^{\mathfrak{S}_n}\\
&\cong \left(\bigoplus_{g\in\mathfrak{S}_n}\bigotimes_{o\in\mathrm{O}(g)} \mathrm{H}^\star\left(X, \bigwedge^{\#}\Omega_X \otimes F^o|_X\right)\right)^{\mathfrak{S}_n}\\
&\cong \left(\bigoplus_{g\in\mathfrak{S}_n}\bigotimes_{o\in\mathrm{O}(g)} \mathrm{H}^{\#,\star}(X, F^o|_X)\right)^{\mathfrak{S}_n}.
\end{aligned} \tag{60}$$

□

Before further simplifying (58), we first recall the following well-known result on centralisers of permutations.

Lemma 3.5. *Let $g \in \mathfrak{S}_n$ viewed as a permutation on $\{1, \dots, n\}$, and let $\mathrm{O}(g)$ be the set of its orbits. Let λ_j be the number of length-j orbits. Then the centraliser of g is*

$$\text{(61)} \quad \mathrm{C}(g) = \left(\prod_{o \in \mathrm{O}(g)} \mu_o \right) \rtimes \prod_{j=1}^{n} \mathfrak{S}_{\lambda_j},$$

where μ_o is the cyclic group of order $\#o$ generated by $g|_o$ (the permutation that is g on o but id *elsewhere), and $\mathfrak{S}_{\lambda_j}$ permutes the length-j orbits (with fixed bijections between elements among those orbits).*

Proof. See, e.g., [67, Proposition 1.1.1]. □

Notation 3.6. For $n \in \mathbb{N}\backslash\{0\}$, we consider the cyclic permutation $\sigma_n := (1\,2\,\cdots\,n) \in \mathfrak{S}_n$ of order n. For $F \in \mathbf{D}^{\mathrm{b}}(X)$, we consider $F^{\otimes n} \in \mathbf{D}^{\mathrm{b}}(X)$ equipped with the $\mathfrak{S}_n$-action given by permuting the tensor factors, and denote its subobject of σ_n-invariants by

$$\text{(62)} \quad F^{\langle n \rangle} := \left(F^{\otimes n}\right)^{\langle \sigma_n \rangle} \in \mathbf{D}^{\mathrm{b}}(X)\,.$$

To avoid confusion, we stress that $F^{\langle n \rangle}$ is an object on the variety X, but not on the orbifold $[\mathrm{Sym}^n X]$.

Theorem 3.7. *Let X be a smooth proper algebraic variety over a field $\mathbf{k}$ of characteristic zero, and let $F \in \mathbf{D}^{\mathrm{b}}(X)$. For every $n \in \mathbb{N}^*$, we have*

$$\text{(63)} \quad \mathrm{H}^{\#,\star}(\mathrm{I}[\mathrm{Sym}^n X], F^{\{n\}}) \cong \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{H}^{\#,\star}(X, F^{\langle i \rangle}) \right).$$

where ν runs through all partitions of n, and λ_i is the number of i's in the partition ν.

Proof. We use Proposition 3.4. In (58), by passing to the conjugacy classes, we obtain that

$$\text{(64)} \quad \begin{aligned} \mathrm{H}^{\#,\star}(\mathrm{I}[\mathrm{Sym}^n X], F^{\{n\}}) &\cong \bigoplus_{[g] \in \mathrm{Conj}(\mathfrak{S}_n)} \left(\bigotimes_{o \in \mathrm{O}(g)} \mathrm{H}^{\#,\star}\left(X, F^o|_X\right) \right)^{\mathrm{C}(g)} \\ &\cong \bigoplus_{[g] \in \mathrm{Conj}(\mathfrak{S}_n)} \left(\bigotimes_{o \in \mathrm{O}(g)} \mathrm{H}^{\#,\star}\left(X, F^o|_X\right)^{\mu_o} \right)^{\mathfrak{S}_\lambda}, \end{aligned}$$

where the groups μ_o and $\mathfrak{S}_\lambda := \prod_{j=1}^n \mathfrak{S}_{\lambda_j}$ are the semi-direct factors of $\mathrm{C}(g)$ as described in Lemma 3.5.

The cyclic group $\mu_o \cong \mu_{\#o}$ acts on $F^o|_X \cong F^{\#o}$ by cyclic permutation of the tensor factors, and acts trivially on X, hence also on Ω_X. Hence,

$$\text{(65)} \quad \mathrm{H}^{\#,\star}\left(X, F^o|_X\right)^{\mu_o} \cong \mathrm{H}^{\#,\star}\left(X, F^{\langle \#o \rangle}\right),$$

where $F^{\langle \#o \rangle}$ is as in Notation 3.6. The group $\mathfrak{S}_\lambda$ acts by permuting the orbits of the same lengths. Hence,

$$\text{(66)} \quad \left(\bigotimes_{o \in \mathrm{O}(g)} \mathrm{H}^{\#,\star}\left(X, F^o|_X\right)^{\mu_o} \right)^{\mathfrak{S}_\lambda} \cong \left(\bigotimes_{o \in \mathrm{O}(g)} \mathrm{H}^{\#,\star}\left(X, F^{\langle \#o \rangle}\right) \right)^{\mathfrak{S}_\lambda} \cong \bigotimes_{i \geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{H}^{\#,\star}(X, F^{\langle i \rangle})\,.$$

The assertion follows by the well-known bijection between conjugacy classes of $\mathfrak{S}_n$ and partitions of n. □

We can summarise the formulae (63) for fixed X and F but varying n in the following way.

Corollary 3.8. *We have an isomorphism of trigraded vector spaces*

$$\bigoplus_{n\geqslant 0} \mathrm{H}^{\#,\star}(\mathrm{I}[\mathrm{Sym}^n X], F^{\{n\}})t^n \cong \mathrm{Sym}^{\bullet}\left(\bigoplus_{i\geqslant 1} \mathrm{H}^{\#,\star}(X, F^{\langle i\rangle})t^i\right), \tag{67}$$

where the symmetric power on the right-hand side is graded with respect to the double grading of $\mathrm{H}^{\#,\star}$*, but ordinary with respect to the grading given by exponents of the formal variable* t*.*

Proof. By (63) in Theorem 3.7, we have

$$\bigoplus_{n\geqslant 0} \mathrm{H}^{\#,\star}(\mathrm{I}[\mathrm{Sym}^n X], F^{\{n\}})t^n \cong \bigoplus_{\lambda_1,\lambda_2,\cdots}\left(\bigotimes_{i\geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{H}^{\#,\star}(X, F^{\langle i\rangle})\right)t^{\sum_i i\lambda_i} \cong \bigoplus_{\lambda_1,\lambda_2,\cdots}\bigotimes_{i\geqslant 1} \mathrm{Sym}^{\lambda_i}\left(\mathrm{H}^{\#,\star}(X, F^{\langle i\rangle})t^i\right) \tag{68}$$

which is the right-hand side of (67). □

3.3 Hochschild homology with coefficients

We can deduce our main result from Theorem 3.7 by collapsing the bigrading to a single grading.

Theorem 3.9. *Let* X *be a smooth proper algebraic variety over a field* $\mathbf{k}$ *of characteristic zero, and let* $F \in \mathbf{D}^{\mathrm{b}}(X)$*. For every* $n \in \mathbb{N}$*, let* $F^{\{n\}} \in \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n X])$ *and* $F^{\langle n\rangle} \in \mathbf{D}^{\mathrm{b}}(X)$ *as in Notation 3.1 and Notation 3.6 respectively. we have*

$$\mathrm{HH}_*([\mathrm{Sym}^n X], F^{\{n\}}) \cong \bigoplus_{\nu\dashv n}\left(\bigotimes_{i\geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{HH}_*(X, F^{\langle i\rangle})\right), \tag{69}$$

where λ_i *is the number of* i*'s in the partition* ν*. Furthermore, collecting these isomorphisms for varying* n*, we get*

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*([\mathrm{Sym}^n X], F^{\{n\}})t^n \cong \mathrm{Sym}^{\bullet}\left(\bigoplus_{i\geqslant 1} \mathrm{HH}_*(X, F^{\langle i\rangle})t^i\right), \tag{70}$$

where the symmetric power $\mathrm{Sym}^{\bullet}$ *on the right-hand side is graded with respect to the grading of* HH_**, but ordinary with respect to the grading given by exponents of the formal variable* t*.*

Proof. Turn the isomorphisms of multigraded vector spaces in (63) and (67) into isomorphisms of (single-)graded (or bigraded if we take into account the powers of t in (67)) vector spaces by defining the new grading by $* = \star - \#$. Note that this process of collapsing gradings is compatible with the direct sums, tensor products and symmetric products. By the Hochschild–Kostant–Rosenberg isomorphism for orbifolds Corollary 2.14 (or more directly, by Corollary 2.15), we get the desired result on the Hochschild homology with coefficients. □

We now can reformulate Theorem 3.9 in terms of the generating function for the dimensions of the graded pieces.

Corollary 3.10. *For* X *a smooth proper variety defined over a field* $\mathbf{k}$ *of characteristic zero, and* $F \in \mathbf{D}^{\mathrm{b}}(X)$*, we have the following formula for the generating series of the Hochschild homology of* $[\mathrm{Sym}^n X]$*:*

$$\sum_{n\geqslant 0}\sum_{i\in\mathbb{Z}} \dim_{\mathbf{k}} \mathrm{HH}_i([\mathrm{Sym}^n X], F^{\{n\}})s^i t^n = \prod_{k\geqslant 1}\prod_{j\in\mathbb{Z}} (1-(-s)^j t^k)^{-(-1)^j \mathrm{hh}_j(X, F^{\langle k\rangle})}, \tag{71}$$

where $\mathrm{hh}_j(X, F^{\langle k\rangle}) := \dim_{\mathbf{k}} \mathrm{HH}_j(X, F^{\langle k\rangle})$.

Proof. This follows by applying Lemma 3.3 to (70). □

Line bundle coefficients In the case that $F = L$ is a line bundle on the variety X, the $\langle\sigma_i\rangle$-action on $L^{\otimes i}$ permuting the tensor factors is the trivial action, hence

$$(72)\quad L^{\langle i\rangle} = L^{\otimes i}.$$

Therefore, Theorem 3.7, Corollary 3.8 and Theorem 3.9 specialise to the following:

Corollary 3.11. *For $L \in \operatorname{Pic}(X)$ and $n \in \mathbb{N}\backslash\{0\}$, we have isomorphisms of bigraded vector spaces*

$$(73)\quad \mathrm{H}^{\#,\star}(\mathrm{I}[\operatorname{Sym}^n X], L^{\{n\}}) \cong \bigoplus_{\nu\dashv n}\left(\bigotimes_{i\geqslant 1}\operatorname{Sym}^{\lambda_i}\mathrm{H}^{\#,\star}(X, L^{\otimes i})\right),$$

where λ_i is the number of i's in the partition ν. Collecting these isomorphisms for varying n gives

$$(74)\quad \bigoplus_{n\geqslant 0}\mathrm{H}^{\#,\star}(\mathrm{I}[\operatorname{Sym}^n X], L^{\{n\}})t^n \cong \operatorname{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}\mathrm{H}^{\#,\star}(X, L^{\otimes i})t^i\right),$$

$$(75)\quad \bigoplus_{n\geqslant 0}\mathrm{HH}_*([\operatorname{Sym}^n X], L^{\{n\}})t^n \cong \operatorname{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}\mathrm{HH}_*(X, L^{\otimes i})t^i\right).$$

The symmetric product on the right-hand sides are graded with respect to the grading of the Hodge groups and the Hochschild homology, but ordinary with respect to the grading coming from the exponents of the formal variable t.

3.4 Hochschild–Serre cohomology

We will now specialise the previous results to the setting of Hochschild–Serre cohomology. Let X be a smooth proper variety of dimension d_X.

Lemma 3.12. *In Notation 3.1, we have:*

$$(76)\quad \omega_{[\operatorname{Sym}^n X]}[nd_X] \cong (\omega_X[d_X])^{\{n\}}.$$

Proof. Note that the canonical bundle of the symmetric quotient stack $[\operatorname{Sym}^n X]$ is given by

$$(77)\quad \omega_{[\operatorname{Sym}^n X]} \cong \omega_X^{\{n\}} \otimes \operatorname{sgn}^{\otimes d_X} = \begin{cases}\omega_X^{\{n\}} & \text{if } d_X \text{ is even,}\\ \omega_X^{\{n\}}\otimes\operatorname{sgn} & \text{if } d_X \text{ is odd,}\end{cases}$$

where sgn denotes the alternating representation of $\mathfrak{S}_n$ (i.e., the one-dimensional representation on which $g \in \mathfrak{S}_n$ acts by multiplication by $\operatorname{sgn}(g)$); see [48, Lem. 5.10]. The lemma follows from the simple fact that for any $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $d \in \mathbb{Z}$, we have

$$(78)\quad (F[d_X])^{\{n\}} \cong F^{\{n\}}\otimes\operatorname{sgn}^{\otimes d_X}[nd_X] = \begin{cases}F^{\{n\}}[nd_X] & \text{for } d_X \text{ even,}\\ F^{\{n\}}\otimes\operatorname{sgn}[nd_X] & \text{for } d_X \text{ odd.}\end{cases}$$

Note that the dimension shift conveniently takes care of the sign, and (77) and (78) together proof the assertion. □

Lemma 3.13. *Let i be a positive integer.*
If $(k-1)d_X$ is even, then we have an isomorphism of objects in $\mathbf{D}^{\mathrm{b}}(X)$:

$$(79)\quad \left(\omega_X^{\otimes k-1}[(k-1)d_X]\right)^{\langle i\rangle} \cong \omega_X^{\otimes i(k-1)}[i(k-1)d_X].$$

If $(k-1)d_X$ is odd, then

$$(80)\quad \left(\omega_X^{\otimes k-1}[(k-1)d_X]\right)^{\langle i\rangle} \cong \begin{cases}\omega_X^{\otimes i(k-1)}[i(k-1)d_X] & \text{for } i \text{ odd,}\\ 0 & \text{for } i \text{ even.}\end{cases}$$

Proof. In analogy with (78), we also have

$$(F[d_X])^{\otimes i} \cong F^{\otimes i} \otimes \mathrm{sgn}^{\otimes d_X}[id_X] = \begin{cases} F^{\otimes i}[id_X] & \text{for } d_X \text{ even,} \\ F^{\otimes i} \otimes \mathrm{sgn}[id_X] & \text{for } d_X \text{ odd.} \end{cases} \tag{81}$$

The case distinction for $(k-1)d_X$ in the statement comes from the fact that $\mathrm{sgn}(\sigma_i) = (-1)^{i-1}$, where $\sigma_i = (1\,2\,\ldots\,i)$ is the cyclic permutation; see Notation 3.6. Hence, the $\langle \sigma_i \rangle$-action on $\omega_X^{\otimes i(k-1)} \otimes \mathrm{sgn}$ is trivial for i odd, but non-trivial for i even . This leads to the invariants being the entire $\omega_X^{\otimes i(k-1)}$ or 0, respectively. □

Recall Definition 2.3:

$$\mathrm{HS}_k^*(\mathcal{X}) = \mathrm{HH}_*(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k-1}[(k-1)d_{\mathcal{X}}])\,, \tag{82}$$

which (for fixed k) is a graded vector space, whereas we write $\mathrm{HS}_k(\mathcal{X})$ for the object in $\mathbf{D}^{\mathrm{b}}(\mathbf{k})$. We can now show Theorem C. We restate it with some more details:

Corollary 3.14. *Let X be a smooth proper algebraic variety of dimension d_X over a field $\mathbf{k}$ of characteristic zero. Let k be a fixed positive integer.*

(i) If $(k-1)d_X$ is even, we have an isomorphism of graded vector spaces:

$$\mathrm{HS}_k([\mathrm{Sym}^n X]) \cong \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{HS}_{1+(k-1)i}(X) \right), \tag{83}$$

where λ_i is the number of i's in the partition ν. Collecting these isomorphisms together by varying n, we have an isomorphism of bigraded vector spaces:

$$\bigoplus_{n \geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet \left(\bigoplus_{i \geqslant 1} \mathrm{HS}_{1+(k-1)i}(X)t^i \right). \tag{84}$$

Here the $\mathrm{Sym}^\bullet$ is graded with respect to the grading of the Hochschild–Serre cohomology and ordinary with respect to the grading given by the exponents of the formal variable t.

(ii) If $(k-1)d_X$ is odd, we have an isomorphism of graded vector spaces:

$$\mathrm{HS}_k([\mathrm{Sym}^n X]) \cong \bigoplus_{\substack{\nu \dashv n \\ \text{all } \nu_j \text{ odd}}} \left(\bigotimes_{\substack{i \geqslant 1 \\ (i \text{ odd})}} \mathrm{Sym}^{\lambda_i} \mathrm{HS}_{1+(k-1)i}(X) \right) \tag{85}$$

and

$$\bigoplus_{n \geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet \left(\bigoplus_{\substack{i \geqslant 1 \\ i \text{ odd}}} \mathrm{HS}_{1+(k-1)i}(X)t^i \right). \tag{86}$$

Proof. We plug $F = \omega_X^{\otimes k-1}[(k-1)d_X]$ into Theorem 3.9 and apply Lemmas 3.12 and 3.13. □

3.5 Hochschild cohomology in low degrees and deformation

Let us briefly recall why we are interested in Hochschild cohomology. Originating from the deformation theory of algebras, introduced by Gerstenhaber, it is now known that Hochschild cohomology of abelian (or derived) categories governs their deformation as a category [56, 57]. The second Hochschild cohomology is the deformation space, and third Hochschild cohomology is the obstruction space. First Hochschild cohomology has an interpretation as the Lie algebra of autoequivalences [42]. There exist geometric definitions of Hochschild cohomology for varieties, and their agreement with the categorical definition can be found in [57].

For a smooth variety X defined over a field of characteristic zero, by the Hochschild–Kostant–Rosenberg decomposition (see Proposition 2.11), we have a decomposition

$$\mathrm{HH}^2(X) \cong \underbrace{\mathrm{H}^0\left(X, \bigwedge^2 \mathrm{T}_X\right)}_{\text{noncommutative}} \oplus \underbrace{\mathrm{H}^1(X, \mathrm{T}_X)}_{\text{geometric}} \oplus \underbrace{\mathrm{H}^2(X, \mathcal{O}_X)}_{\text{gerby}}, \tag{87}$$

with the deformation-theoretic justification for the interpretation of these terms given in [71].

The noncommutative deformations are also known as Poisson deformations. A *Poisson structure* on X is a global section of $\bigwedge^2 \mathrm{T}_X$ for which the Schouten–Nijenhuis self-bracket $[\sigma, \sigma]$ vanishes. This obstruction class lives in the global sections of $\bigwedge^3 \mathrm{T}_X$. By Kontsevich's celebrated formality theorem [44] this means that it admits a formal (and not just first-order) deformation.

In this paper, for applications in deformation theory we are particularly interested in the way $\mathrm{HH}^1([\mathrm{Sym}^n X])$ and $\mathrm{HH}^2([\mathrm{Sym}^n X])$ are determined by the Hochschild–Serre cohomology of X.

We specialise Corollary 3.14 to $k = 0$:

Corollary 3.15. *Let X be a smooth proper algebraic variety of dimension d_X over a field $\mathbf{k}$ of characteristic zero. Let k be a fixed positive integer.*

(i) If d_X is even,

$$\bigoplus_{n\geqslant 0} \mathrm{HH}^*([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HS}_{1-i}(X)t^i\right). \tag{88}$$

(ii) If d_X is odd,

$$\bigoplus_{n\geqslant 0} \mathrm{HH}^*([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{\substack{i\geqslant 1\\ i \text{ odd}}} \mathrm{HS}_{1-i}(X)t^i\right). \tag{89}$$

Let us give in the following corollary the explicit results for HH^1 and HH^2. The interested reader can write down the (more lengthy) description for HH^3, which we will not need it in what follows. One should really understand the Gerstenhaber bracket too, which involves HH^3, so that the obstruction can be computed, but this is an open question.

Corollary 3.16. *Let X be a geometrically connected smooth proper variety of dimension at least 1 defined over a field $\mathbf{k}$ of characteristic zero. For all $n \geqslant 2$ we have that*

$$\mathrm{HH}^1([\mathrm{Sym}^n X]) \cong \mathrm{HH}^1(X) \tag{90}$$

and

$$\mathrm{HH}^2([\mathrm{Sym}^n X]) \cong \begin{cases} \mathrm{HH}^2(X) \oplus \bigwedge^2 \mathrm{HH}^1(X) & d_X \geqslant 3 \\ \mathrm{HH}^2(X) \oplus \bigwedge^2 \mathrm{HH}^1(X) \oplus \mathrm{HS}^2_{-1}(X) & d_X = 2 \\ \mathrm{HH}^2(X) \oplus \bigwedge^2 \mathrm{HH}^1(X) \oplus \mathrm{HS}^2_{-2}(X) & d_X = 1 \end{cases} \tag{91}$$

except when $d_X = 1$ and $n = 2$, in which case

$$\mathrm{HH}^2([\mathrm{Sym}^2 X]) \cong \mathrm{HH}^2(X) \oplus \bigwedge^2 \mathrm{HH}^1(X). \tag{92}$$

Proof. We apply Corollary 3.15. Note that under our assumption on X, we have $\mathrm{HH}^0(X) = \mathbf{k}$.

Observe that $\mathrm{HS}_{-k}(X)$ is concentrated in degrees $\geqslant kd_X$, which explains the absence of those contributions in (90) resp. (91) if $d_X \geqslant 2$ resp. $d_X \geqslant 3$, whereas $\mathrm{HH}^1(X)$ is always present. To conclude the computation for (90) if $d_X = 1$ it suffices to observe that we exclude even i, therefore $\mathrm{HS}_{-1}(X)$ does not contribute.

For similar degree reasons we observe that $\mathrm{HH}^2(X)$ is always present in (91). Now we wish to understand the other contributions to $\mathrm{HH}^2([\mathrm{Sym}^n X])t^n$. The first one arises as a summand of Sym^n in (84) and (86), namely

$$\mathrm{Sym}^{n-2}(\mathrm{HH}^0(X)t) \otimes \mathrm{Sym}^2(\mathrm{HH}^1(X)t) \cong \bigwedge^2 \mathrm{HH}^1(X)t^{n+2}, \tag{93}$$

so it is present for all $d_X \geqslant 1$ and $n \geqslant 2$. The summand

$$\mathrm{Sym}^{n-2}(\mathrm{HH}^0(X)t) \otimes \mathrm{HS}^2_{-1}(X)t^2 \cong \mathrm{HS}^2_{-1}(X)t^n \tag{94}$$

of Sym^{n-1} in (84) contributes for $d_X = 2$ only, as $\mathrm{HS}^2_{-1}(X) = 0$ for $d_X \geqslant 3$ by Proposition 2.11 or it is excluded in (86).

Finally, if $d_X = 1$ and $n \geqslant 3$ then for $i = 2$ we have a contribution by $\mathrm{HS}^2_{-2}(X)$ in degree 2, so Sym^{n-2} in (86) will have the summand

$$\mathrm{Sym}^{n-3}(\mathrm{HH}^0(X)t) \otimes \mathrm{HS}^2_{-2}(X)t^3 \cong \mathrm{HS}^2_{-2}(X)t^n. \tag{95}$$

□

3.6 Consequences for Hilbert schemes of points on surfaces

In the case that X is a smooth projective surface we will write $X = S$, and we have the *derived McKay correspondence*

$$\Psi\colon \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n S]) \xrightarrow{\cong} \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S) \tag{96}$$

of [15, 35] for every $n \in \mathbb{N}$, identifying the derived categories of the symmetric quotient stacks and that of the Hilbert schemes of points. Concretely, Ψ is the Fourier–Mukai transform along

$$\mathcal{O}_{\mathcal{Z}} \in \mathbf{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n \times \mathrm{Hilb}^n S) \cong \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n X] \times \mathrm{Hilb}^n S) \tag{97}$$

where $\mathcal{Z} \subset S^n \times \mathrm{Hilb}^n S$ is the universal family of $\mathfrak{S}_n$-clusters. Using this, we can deduce formulas for Hochschild homology with values in natural line bundles and the Hochschild–Serre cohomology of Hilbert schemes of points.

Given a line bundle $L \in \mathrm{Pic}\, S$, the equivariant line bundle $L^{\{n\}}$ on S^n descends to a line bundle $L^{(n)}$ on the symmetric quotient *variety* $S^{(n)} = S^n/\mathfrak{S}_n$. Concretely, if $\pi\colon S^n \to S^{(n)}$ denotes the quotient morphism, we have $L^{(n)} = \pi_*(L^{\{n\}})^{\mathfrak{S}_n}$. Pulling back by the Hilbert–Chow morphism $\mu\colon \mathrm{Hilb}^n S \to S^{(n)}$ gives the *natural line bundle on* $\mathrm{Hilb}^n S$ *induced by* L :

$$L_n \coloneqq \mu^* L^{(n)} \in \mathrm{Pic}(\mathrm{Hilb}^n S). \tag{98}$$

We first prove a technical result, of a similar nature to Corollary A.6, but we explicitly use Fourier–Mukai transforms, unlike the quoted result. We will prove it using the formalism described in [63, §2.1]. Recall that a Fourier–Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{X}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{Y})$ given by kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{Y})$ induces an equivalence $\mathrm{Ad}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\simeq} \mathbf{D}^{\mathrm{b}}(\mathcal{Y} \times \mathcal{Y})$ given by the kernel $\mathcal{P} \boxtimes \mathcal{P} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y}) \cong \mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Y})$ We denote by $\circ$ the convolution product of Fourier–Mukai kernels. The kernel for the inverse of $\Phi_{\mathcal{P}}$ will be denoted $\mathcal{P}^{\mathrm{T}}$, so that $\mathcal{P}^{\mathrm{T}} \circ \mathcal{P} \cong \Delta_*(\mathcal{O}_{\mathcal{Y}})$.

Proposition 3.17. *Let* $\mathcal{X}, \mathcal{Y}$ *be smooth and proper orbifolds, such that* $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{Y})$ *induces an equivalence* $\Phi_{\mathcal{P}}$. *Let* $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$ *and consider a morphism*

$$\Delta_*\mathcal{E} \circ \mathcal{P} \to \mathcal{P} \circ \Delta_*(\Phi_{\mathcal{P}}(\mathcal{E})) \tag{99}$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{X} \times \mathcal{Y})$. *Then* $\mathrm{Ad}_{\mathcal{P}}$ *induces a morphism*

$$\mathrm{HH}_*(\mathcal{X}, \mathcal{E}) \to \mathrm{HH}_*(\mathcal{Y}, \Phi_{\mathcal{P}}(\mathcal{E})), \tag{100}$$

which is an isomorphism if (99) *is an isomorphism.*

Proof. We have the sequence of morphisms

$$\begin{aligned}
\mathrm{HH}_*(\mathcal{X},\mathcal{E}) &= \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}]\otimes\mathcal{E}))\\
&= \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})}(\Delta_*\mathcal{O}_{\mathcal{X}}, \Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}])\circ\Delta_*(\mathcal{E})))\\
&\cong \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{Y}\times\mathcal{Y})}(\Delta_*\mathcal{O}_{\mathcal{Y}}, \mathcal{P}^{\mathrm{T}}\circ\Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}])\circ\Delta_*(\mathcal{E})\circ\mathcal{P})\\
&\to \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{Y}\times\mathcal{Y})}(\Delta_*\mathcal{O}_{\mathcal{Y}}, \mathcal{P}^{\mathrm{T}}\circ\Delta_*(\omega_{\mathcal{X}}[d_{\mathcal{X}}])\circ\mathcal{P}\circ\Delta_*(\Phi_{\mathcal{P}}(\mathcal{E})))\\
&\cong \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{Y}\times\mathcal{Y})}(\Delta_*\mathcal{O}_{\mathcal{Y}}, \mathcal{P}^{\mathrm{T}}\circ\mathcal{P}\circ\Delta_*(\omega_{\mathcal{Y}}[d_{\mathcal{Y}}])\circ\Delta_*(\Phi_{\mathcal{P}}(\mathcal{E})))\\
&= \mathrm{Hom}^*_{\mathbf{D}^{\mathrm{b}}(\mathcal{Y}\times\mathcal{Y})}(\Delta_*\mathcal{O}_{\mathcal{Y}}, \Delta_*(\omega_{\mathcal{Y}}[d_{\mathcal{Y}}])\circ\Delta_*(\Phi_{\mathcal{P}}(\mathcal{E})))\\
&= \mathrm{HH}_*(\mathcal{Y},\Phi_{\mathcal{P}}(\mathcal{E})),
\end{aligned}\tag{101}$$

where on the third line we used (in the first argument) that $\mathrm{Ad}_{\mathcal{P}}$ preserves the identity functor and on the fifth line we used (in the second argument) that the Serre functor commutes with equivalences. The morphism on the fourth line is an isomorphism if (99) is one. □

Remark 3.18. The condition (99) implies the following isomorphism of functors:

$$\Phi_{\mathcal{P}}(-\otimes\mathcal{E}) \cong \Phi_{\mathcal{P}}(-)\otimes\Phi_{\mathcal{P}}(\mathcal{E}).\tag{102}$$

Corollary 3.19. *Let $\mathcal{X},\mathcal{Y}$ be smooth and proper orbifolds, such that $\mathcal{P}\in\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{Y})$ induces an equivalence $\Phi_{\mathcal{P}}$. Then $\Phi_{\mathcal{P}}$ induces also an isomorphism of bigraded vector spaces*[2] *:*

$$\mathrm{HS}^*_\bullet(\mathcal{X}) \cong \mathrm{HS}^*_\bullet(\mathcal{Y}).\tag{103}$$

Proof. By the uniqueness of the Serre functor (hence it commutes with equivalences), the condition (102) is satisfied with $\mathcal{E}=\omega^k_{\mathcal{X}}[kd_{\mathcal{X}}]$ for any $k\in\mathbb{Z}$. □

Without the intertwining compatibility (99) between the derived equivalence and the objects of coefficients, the following example that we do not necessarily get an isomorphism of the Hochschild homologies with coefficients.

Example 3.20. Let E be an elliptic curve, and consider the Fourier–Mukai equivalence given by the Poincaré bundle $\mathcal{P}$ on $E\times E$. If $e\in E$ is the origin, then $\Phi_{\mathcal{P}}(\mathcal{O}_e)=\mathcal{O}_E$, i.e., the skyscraper at the origin is sent to the structure sheaf. Then one can compute that

$$\mathrm{HH}_*(E,\mathcal{O}_e)\cong\mathbf{k}[0]\oplus\mathbf{k}[1]\tag{104}$$

whereas

$$\mathrm{HH}_*(E,\mathcal{O}_E)\cong\mathrm{HH}_*(E)\cong\mathbf{k}[-1]\oplus\mathbf{k}^{\oplus2}[0]\oplus\mathbf{k}[1].\tag{105}$$

Proposition 3.21. *Let S be a smooth projective surface. Let L be a line bundle on S. The derived invariance and agreement from Proposition 3.17 induce isomorphisms*

$$\mathrm{HH}_*(\mathrm{Hilb}^n S, L_n)\cong\mathrm{HH}_*([\mathrm{Sym}^n S], L^{\{n\}}).\tag{106}$$

This result does not hold more generally with L replaced by a higher-rank vector bundle F. The main reason is that the $\mathfrak{S}_n$-equivariant vector bundle $F^{\{n\}}$ on S^n is in general not the pull-back of a vector bundle on $S^{(n)}$ and, if we set $F_n:=\Psi(F^{\{n\}})$, the condition (102) does not hold in this situation.

[2]In fact $\Phi_{\mathcal{P}}$ induces an isomorphism of bigraded *algebras*, see Theorem A.4 and Theorem A.5.

Proof. By Proposition 3.17, it suffices to check that the Bridgeland–King–Reid–Haiman equivalence [15, 35] satisfies the intertwining property that there is an isomorphism (99). Recall that Ψ is the Fourier–Mukai transform along $\mathcal{O}_{\mathcal{Z}}$, and that we have a commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"] & S^n \arrow[d, "\pi"] \\ \operatorname{Hilb}^n S \arrow[r, "\mu"] & S^{(n)} = S^n/\mathfrak{S}_n \end{tikzcd} \tag{107}$$

where p and q are the restrictions of the projections of the product $S^n \times \operatorname{Hilb}^n S$ to its factors. Denoting the embedding of the universal family by $i\colon \mathcal{Z} \hookrightarrow S^n \times \operatorname{Hilb}^n S$, a standard computation[3] for Fourier–Mukai kernels shows that

$$\Delta_* L^{\{n\}} \circ \mathcal{O}_{\mathcal{Z}} \cong i_* p^* L^{\{n\}} \quad , \quad \mathcal{O}_{\mathcal{Z}} \circ \Delta_* L_n \cong i_* q^* L_n\,. \tag{108}$$

As $L^{\{n\}} \cong \pi^* L^{(n)}$ and $L_n = \mu^* L^{(n)}$, commutativity of (107) gives an isomorphsim $p^* L^{\{n\}} \cong q^* L_n$. Hence, we have an isomorphism between the two objects in (108), which is exactly what we need to conclude by Proposition 3.17. □

We can now state the two main corollaries of this subsection.

Corollary 3.22. *Let S be a smooth projective surface defined over a field* $\mathbf{k}$ *of characteristic zero. For $L \in \operatorname{Pic}(S)$ and $n \in \mathbb{N}\backslash\{0\}$, we have*

$$\operatorname{HH}_*(\operatorname{Hilb}^n S, L_n) \cong \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \operatorname{Sym}^{\lambda_i} \operatorname{HH}_*(S, L^{\otimes i}) \right), \tag{109}$$

where λ_i is the number of i's in the partition ν. Collecting these isomorphisms for varying n, we get

$$\bigoplus_{n \geqslant 0} \operatorname{HH}_*(\operatorname{Hilb}^n S, L_n) t^n \cong \operatorname{Sym}^\bullet \left(\bigoplus_{i \geqslant 1} \operatorname{HH}_*(S, L^{\otimes i}) t^i \right). \tag{110}$$

The symmetric product on the right-hand side of (110) *is graded with respect to the grading of the Hochschild homology, but ordinary with respect to the grading coming from the exponents of t. In terms of generating functions for the dimensions of the graded pieces, this means*

$$\sum_{n \geqslant 0} \sum_{i=-2n}^{2n} \dim_{\mathbf{k}} \operatorname{HH}_i(\operatorname{Hilb}^n S, L_n) s^i t^n = \prod_{k \geqslant 1} \prod_{j=-2}^{2} (1 - (-s)^j t^k)^{-(-1)^j \operatorname{hh}_j(S, L^{\otimes k})} \tag{111}$$

Proof. This follows from Corollary 3.11, (96), and Proposition 3.21. For (111), we apply Lemma 3.3 to (110). □

We also get a formula for the Hochschild–Serre cohomology of the Hilbert schemes of points, which was our original goal.

Corollary 3.23. *For every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have*

$$\operatorname{HS}_k(\operatorname{Hilb}^n S) \cong \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \operatorname{Sym}^{\lambda_i} \operatorname{HS}_{1+(k-1)i}(S) \right). \tag{112}$$

Collecting the formula for varying n, but fixed k, we get

$$\bigoplus_{n \geqslant 0} \operatorname{HS}_k(\operatorname{Hilb}^n S) t^n \cong \operatorname{Sym}^\bullet \left(\bigoplus_{i \geqslant 1} \operatorname{HS}_{1+(k-1)i}(S) t^i \right). \tag{113}$$

Proof. This follows from Corollary 3.14, (96), and Corollary 3.19. □

[3]For example, one can note that [63, Proposition 2.1.6] still works for orbifolds in place of varieties and for convolution products of kernels in place of compositions of functors. Then (108) are just two instances of this general statement.

3.7 On the Fock space structure

In this section we work over $\mathbf{k} = \mathbb{C}$. Let V be a vector space together with a bilinear form $\langle -, - \rangle$. One associates the *Heisenberg algebra* H_V to the pair $(V, \langle -, - \rangle)$. It is generated by elements $p_\alpha^{(n)}$ and $q_\alpha^{(n)}$ for $\alpha \in V$ and $n \in \mathbb{N}$. The p-generators commute among each other, as do the q-generators. In addition, there are more complicated relations between mixed two-letter words in the generators; see, e.g., [34, §2.2] for details.

Note that we do not require $\langle -, - \rangle$ to be symmetric, and the Mukai pairing on Hochschild homology is indeed not symmetric in general. This means that we have a well-defined Heisenberg algebra H_V, but not necessarily the Heisenberg Lie algebra $\mathfrak{h}_V$; see again [34, §2.2] for details.

The algebra H_V has an irreducible representation called the *Fock space*. It is given by the quotient H_V/I where I is the left ideal generated by the q-generators. The Fock space can be identified with the symmetric power of an infinite direct sum of copies of V via the isomorphism of vector spaces

$$(114)\quad F_V = H_V/I \xrightarrow{\simeq} \mathrm{Sym}^\bullet\left(\bigoplus_{i \geqslant 1} V t^i\right) : \overline{p_{\alpha_1}^{(\nu_1)} p_{\alpha_2}^{(\nu_2)} \cdots p_{\alpha_\ell}^{(\nu_\ell)}} \mapsto (\alpha_1 t^{\nu_1}) \cdot (\alpha_2 t^{\nu_2}) \cdots (\alpha_\ell t^{\nu_\ell})$$

The main examples for us are:

- $V = \mathrm{H}^*(S, \mathbb{C})$ the singular cohomology of a smooth projective surface together with the cup product pairing. By the work of Göttsche [30], Nakajima [58], and Grojnowski [33], the cohomology $\bigoplus_{n \geqslant 0} \mathrm{H}^*(\mathrm{Hilb}^n S, \mathbb{C}) t^n$ can be identified with the Fock space $F_{\mathrm{H}^*(S,\mathbb{C})}$.
- $V = \mathrm{HH}_*(X)$ the Hochschild homology of a smooth projective variety X equipped with the Mukai pairing. By Corollary D, $\bigoplus_{n \geqslant 0} \mathrm{HH}_*([\mathrm{Sym}^n X]) t^n$ can be identified with the Fock space $F_{\mathrm{HH}_*(X)}$.

These two examples are related: by the Hochschild–Kostant–Rosenberg isomorphism, we have the isomorphism of ungraded vector spaces $\mathrm{HH}_*(\mathrm{Hilb}^n S) \cong \mathrm{H}^*(\mathrm{Hilb}^n S)$. Hence, after summing over n, we obtain the isomorphism

$$(115)\quad F_{\mathrm{H}^*(S,\mathbb{C})} \cong F_{\mathrm{HH}_*(S)}.$$

Finally, the derived McKay correspondence of Bridgeland–King–Reid [15] and Haiman [35] gives equivalences $\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S) \cong \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n S])$ which induce isomorphisms $\mathrm{HH}_*(\mathrm{Hilb}^n S) \cong \mathrm{HH}_*([\mathrm{Sym}^n S])$ for every n, as discussed in Section 3.6.

We formulate the following generalization of Corollary D as a conjecture. In Appendix A.1 we recall the definition of Hochschild homology in this generality (and generalize it to Hochschild–Serre cohomology for smooth and proper dg categories, which we will use shortly). When $\mathcal{T}$ is a dg category, then $\mathrm{Sym}^n \mathcal{T}$ denotes its nth symmetric power in the sense of Ganter–Kapranov [27]. If $\mathcal{T}$ is smooth and proper, then so is $\mathrm{Sym}^n \mathcal{T}$.

Conjecture 3.24. *Let $\mathcal{T}$ be a smooth proper dg category. Then we have*

$$(116)\quad \bigoplus_{n \geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{T}) t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i \geqslant 1} \mathrm{HH}_*(\mathcal{T}) t^i\right).$$

Proposition 3.25. *Let $\mathcal{T}$ be a smooth proper dg category. Let $\mathcal{T} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_m \rangle$ be a semiorthogonal decomposition. If Conjecture 3.24 holds for all categories except one among $\mathcal{T}, \mathcal{A}_1, \cdots, \mathcal{A}_m$, then it also holds for the remaining one.*

Proof. By induction, we only need to deal with the case $m = 2$. Assume $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. By [45, Theorem 1.1], we have a semiorthogonal decomposition for any $n \geqslant 0$,

$$(117)\quad \mathrm{Sym}^n \mathcal{T} = \langle \mathrm{Sym}^n \mathcal{A}, \mathrm{Sym}^{n-1} \mathcal{A} \otimes \mathcal{B}, \cdots, \mathrm{Sym}^n \mathcal{B} \rangle.$$

By additivity [43, §1.5, §1.12] and the Künneth formula [68, Theorem 2.8] for Hochschild homology, we get

$$(118)\quad \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{T}) \cong \bigoplus_{i=0}^{n} \mathrm{HH}_*(\mathrm{Sym}^i \mathcal{A}) \otimes \mathrm{HH}_*(\mathrm{Sym}^{n-i} \mathcal{B}).$$

Taking the sum over n, we get

$$\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{T})t^n \cong \left(\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{A})t^n\right) \otimes \left(\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{B})t^n\right). \tag{119}$$

On the other hand, again by the additivity of HH_*, we have

$$F_{\mathrm{HH}_*(\mathcal{T})} = \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HH}_*(\mathcal{T})t^i\right) \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1}(\mathrm{HH}_*(\mathcal{A})t^i \oplus \mathrm{HH}_*(\mathcal{B})t^i)\right) = F_{\mathrm{HH}_*(\mathcal{A})} \otimes F_{\mathrm{HH}_*(\mathcal{B})}. \tag{120}$$

By comparing (119) and (120), it is clear that if $\mathcal{A}$ and $\mathcal{B}$ satisfy Conjecture 3.24 then so does $\mathcal{T}$. Supposing $\mathcal{A}$ and $\mathcal{T}$ satisfy Conjecture 3.24, then since the generating series $\sum_{n\geqslant 0} \dim \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{A})t^n$ is an invertible power series (because the constant coefficient is 1), $\bigoplus_{n\geqslant 0} \mathrm{HH}_*(\mathrm{Sym}^n \mathcal{B})t^n$ and $F_{\mathrm{HH}_*(\mathcal{B})}$ have the same generating series, hence we have an isomorphism of graded vector spaces as in Conjecture 3.24 for $\mathcal{B}$. □

The following consequence asserts that Conjecture 3.24 holds for many dg categories arising as the so-called Kuznetsov component of a variety.

Corollary 3.26. *Let X be a smooth proper variety defined over a field $\mathbf{k}$ of characteristic zero. Suppose there is an exceptional collection $E_1, \ldots, E_m$ in $\mathbf{D}^{\mathrm{b}}(X)$. Denote by $\mathcal{A}_X$ the (left or right) orthogonal of $\langle E_1, \ldots, E_m\rangle$. Then Conjecture 3.24 holds for $\mathcal{A}_X$.*

Proof. We can apply Proposition 3.25 since Conjecture 3.24 holds for $\mathbf{D}^{\mathrm{b}}(X)$ by Corollary D and for each $\langle E_i\rangle \cong \mathbf{D}^{\mathrm{b}}(\mathrm{Spec}\,\mathbf{k})$. □

Corollary 3.27. *Let $\mathcal{T}$ be a smooth proper dg category. Let $\mathcal{T} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_m, \mathcal{B}\rangle$ be a semiorthogonal decomposition such that $\mathcal{B}$ is a quasi-phantom category, i.e., $\mathrm{HH}_*(\mathcal{B}) = 0$. If Conjecture 3.24 holds for all the categories $\mathcal{T}, \mathcal{A}_1, \ldots, \mathcal{A}_m$, then* $\mathrm{Sym}^n \mathcal{B}$ *is again a quasi-phantom category for all n.*

Proof. By Proposition 3.25, we know that Conjecture 3.24 holds for $\mathcal{B}$. The assertion now just follows from the simple fact the the symmetric power of the zero vector space is the zero vector space. □

If $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(S)$ for a smooth projective surface S, and $\mathcal{A}_i = \langle E_i\rangle$ for some exceptional object E_i for every $i = 1, \ldots, m$, Corollary 3.27 specialises to [45, Lemma 4.4]. We were inspired to include Corollary 3.27 in the second version of this article by Koseki (the author of op. cit.) asking in a talk about possible generalisations of his result.

If Conjecture 3.24 holds in full generality, this would imply that every symmetric power of every quasi-phantom category is again a quasi-phantom category.

Is Hochschild–Serre cohomology of Hilbert schemes a Fock space? For $k \neq 1$, Corollary 3.14 still gives an identification of $\bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n$ with a certain total symmetric power, namely, when d_X is even,

$$\bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n \cong \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HS}_{1+(k-1)i}(X)t^i\right). \tag{121}$$

However, the symmetric power is not taken of an infinite direct sum of copies of one vector space, but of an infinite direct sum of *different* vector spaces. In other words, we still have a basis of $\bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n$ consisting of elements of the form $(\alpha_1 t^{\nu_1}) \cdot (\alpha_2 t^{\nu_2}) \cdots (\alpha_\ell t^{\nu_\ell})$ as in (114), but now the α_i are elements of *different* vector spaces, depending on the exponent ν_i of t. Hence, it seems to us like there is no identification of $\bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n$ with a Fock space for $k \neq 1$.

One could hope that the full Hochschild–Serre cohomology

$$\bigoplus_{k\in\mathbb{Z}} \bigoplus_{n\geqslant 0} \mathrm{HS}_k([\mathrm{Sym}^n X])t^n = \bigoplus_{n\geqslant 0} \mathrm{HS}_\bullet([\mathrm{Sym}^n X])t^n \tag{122}$$

can be identified with the Fock space associated to $V = \bigoplus_{k\in\mathbb{Z}} \mathrm{HS}_k(X) = \mathrm{HS}(X)$, but this does not work either. Indeed, (122) is generated by words of the form

$$(\alpha_1 t^{\nu_1}) \cdot (\alpha_2 t^{\nu_2}) \cdots (\alpha_\ell t^{\nu_\ell}), \tag{123}$$

but we cannot take arbitrary $\alpha_i \in V$ and integers ν_i. Instead, there must be some k such that $\alpha_i \in \mathrm{HS}_{1+(k-1)\nu_i}(X)$ for all $i = 1, \ldots, \ell$ if (123) is an element of (122). Hence,

$$\bigoplus_{n\geqslant 0} \mathrm{HS}_\bullet([\mathrm{Sym}^n X])t^n \subsetneq \mathrm{Sym}^\bullet\left(\bigoplus_{i\geqslant 1} \mathrm{HS}_\bullet(X)t^i\right) \tag{124}$$

is a proper subspace of the symmetric power.

Geometric and categorical Heisenberg action For X a smooth projective variety of arbitrary dimension, by [47], there is a categorical Heisenberg action of $\mathbf{D}^{\mathrm{b}}(X)$ on $\bigoplus_{n\geqslant 0} \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n X])$. It seems worthwile to study how it relates to the Heisenberg action of $\mathrm{HH}_*(X)$ on $\bigoplus_{n\geqslant 0} \mathrm{HH}_*([\mathrm{Sym}^n X])$ given by Corollary D.

Similarly, in the case $X = S$ is a surface, one could study how Nakajima's action of $\mathrm{H}^*(S)$ on $\bigoplus_{n\geqslant 0} \mathrm{H}^*(\mathrm{Hilb}^n S)$ is related to these actions, under the identification given by Hochschild–Kostant–Rosenberg isomorphism and the McKay correspondence.

The categorical action of [47] was generalised in [34] from derived categories of smooth projective varieties to dg-enhanced triangulated categories. So the same question can be asked whenever we have an instance of Conjecture 3.24.

4 Examples

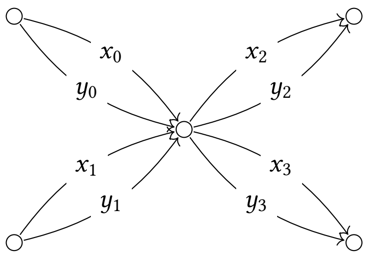
We will compute some examples, to illustrate some interesting phenomena.

4.1 Symmetric square stack of $\mathbb{P}^1$

The first case we consider is $[\mathrm{Sym}^2 \mathbb{P}^1]$, where we will assume $\mathbf{k}$ algebraically closed. We will compute its Hochschild cohomology both geometrically using the main theorem of this paper, and algebraically, using finite-dimensional algebras. To get started with the latter, using [49, Proposition 4.5] we obtain the following result.

Lemma 4.1. *There exists a derived equivalence* $\mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^2 \mathbb{P}^1]) \cong \mathbf{D}^{\mathrm{b}}(\mathbf{k}Q/I)$ *where* Q *is the quiver*

(125)



and $I = (x_0y_2 - y_0x_2, x_1y_3 - y_1x_3, x_0x_3, y_0y_3, x_0y_3 + y_0x_3, x_1x_2, y_1y_2, x_1y_2 + y_1x_2)$.

Proof. Taking the full exceptional collection $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$ for $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ yields the full 3-block exceptional collection

$$\mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^2 \mathbb{P}^1]) = \left\langle \begin{matrix} \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1} \\ \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1} \otimes \mathrm{sgn} \end{matrix}, \mathrm{Ind}(\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)), \begin{matrix} \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \\ \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{sgn} \end{matrix} \right\rangle, \tag{126}$$

where objects in the same block are completely orthogonal. Denoting $V = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ we have that morphisms between objects in consecutive blocks are given by V. The other Hom-spaces are accordingly given by $\mathrm{Sym}^2 V$ resp. $\bigwedge^2 V$. If we choose a basis x, y of V and label the bases of the Hom-spaces using x_i, y_i, we get the quiver in (125) with the claimed relations. $\square$

This allows us to compute the Hochschild cohomology of $[\operatorname{Sym}^2 \mathbb{P}^1]$ purely algebraically.

Lemma 4.2. *For the finite-dimensional algebra $\mathbf{k}Q/I$ from Lemma 4.1 we have that it is of global dimension 2, and*

$$\sum_{i=0}^{2} \dim_{\mathbf{k}} \mathrm{HH}^i(\mathbf{k}Q/I)t^i = 1 + 3t + 3t^2. \tag{127}$$

Proof. The proof is similar to the method used in [4, §3]. The quiver is acyclic, thus $\mathrm{HH}(\mathbf{k}Q/I) \cong \mathbf{k}$. The algebra is not hereditary by the presence of relations, and its global dimension is bounded above by 2 because of the length of a maximal path, thus precisely 2. Because $\operatorname{gl\,dim} A = \operatorname{proj\,dim}_{A^e} A$ for A a finite-dimensional algebra over an algebraically closed field [36, Lemma 1.5], we only need to determine $\mathrm{HH}^1(\mathbf{k}Q/I)$ and $\mathrm{HH}^2(\mathbf{k}Q/I)$.

By [37] we have

$$\chi(\mathrm{HH}^\bullet(A)) = -\operatorname{tr} \mathrm{C}_{\mathbf{k}Q/I} = -1 \tag{128}$$

where $\mathrm{C}_{\mathbf{k}Q/I}$ is the Coxeter matrix, thus $\mathrm{HH}^1(\mathbf{k}Q/I) \cong \mathrm{HH}^2(\mathbf{k}Q/I)$. The Coxeter matrix is readily computed from the Cartan matrix $\mathrm{A}_{\mathbf{k}Q/I}$, as $\mathrm{C}_{\mathbf{k}Q/I} = -\mathrm{A}_{\mathbf{k}Q/I}^{-1}\mathrm{A}_{\mathbf{k}Q/I}^{\mathrm{t}}$. For the choice of basis given by the exceptional objects in Lemma 4.1 the Cartan matrix is given by

$$\mathrm{A}_{\mathbf{k}Q/I} = \begin{pmatrix} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{129}$$

To determine $\mathrm{HH}^1(\mathbf{k}Q/I)$, recall that the first Hochschild cohomology (as a Lie algebra) is isomorphic to the Lie algebra of the outer automorphism group of $\mathbf{k}Q/I$, see, e.g., [70, Proposition 1.1]. In our case, observe that the natural action of $\mathrm{GL}(V)$ on each individual Hom-space gives us an identification $\mathrm{Out}^0(\mathbf{k}Q/I) \cong \mathrm{PGL}(V)$: the ideal I forces the automorphisms to lie in the diagonal subgroup in the product of the $\mathrm{GL}(V)$, and the passage to outer automorphisms yields $\mathrm{PGL}(V)$. Hence $\mathrm{HH}^1(\mathbf{k}Q/I) = \operatorname{Lie}\mathrm{Out}^0(\mathbf{k}Q/I) \cong \mathfrak{sl}_2$, □

On the other hand, in Corollary 3.14 the only contribution for $n = 2$ is given by $i = 0$ on the right-hand side by degree reasons, for which $\mathrm{HS}_0^*(\mathbb{P}^1) \cong \mathrm{HH}^*(\mathbb{P}^1) \cong \mathbf{k}[0] \oplus \mathfrak{sl}_2[-1]$. We obtain the dimensions in (127), see also Corollary 3.16. Thus it agrees with Lemma 4.2.

4.2 Hilbert square of $\mathbb{P}^2$

Another interesting example is that of $[\operatorname{Sym}^2 \mathbb{P}^2]$, which by (9) is derived equivalent to $\operatorname{Hilb}^2 \mathbb{P}^2$. Using Section 3.4 we can compute the Hochschild cohomology of $\operatorname{Hilb}^2 \mathbb{P}^2$ and obtain

Lemma 4.3. *We have that*

$$\sum_{i=0}^{8} \dim_{\mathbf{k}} \mathrm{HH}^i(\operatorname{Hilb}^2 \mathbb{P}^2)t^i = 1 + 8t + 48t^2 + 115t^3 + 83t^4. \tag{130}$$

But as mentioned before, we cannot compute the Hochschild–Kostant–Rosenberg decomposition of $\operatorname{Hilb}^2 \mathbb{P}^2$, which is a bigraded decomposition, using symmetric quotient stacks.

To do this, we can use the isomorphism

$$\operatorname{Hilb}^2 \mathbb{P}^2 \cong \mathbb{P}_{\mathbb{P}^2}(\operatorname{Sym}^2 \Omega_{\mathbb{P}^2}(1)), \tag{131}$$

from, e.g., [64, §3.2]. This description is explicit enough to compute the Hochschild–Kostant–Rosenberg decomposition as in the following proposition.

Proposition 4.4. *We have that*

$$
\begin{aligned}
\mathrm{HH}^1(\mathrm{Hilb}^2\mathbb{P}^2) &\cong \mathrm{H}^0(\mathrm{Hilb}^2\mathbb{P}^2, \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \\
&\cong \mathbf{k}^8 \\
\mathrm{HH}^2(\mathrm{Hilb}^2\mathbb{P}^2) &\cong \mathrm{H}^1(\mathrm{Hilb}^2\mathbb{P}^2, \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \oplus \mathrm{H}^0(\mathrm{Hilb}^2\mathbb{P}^2, \textstyle\bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \\
&\cong \mathbf{k}^{10} \oplus \mathbf{k}^{38} \\
\mathrm{HH}^3(\mathrm{Hilb}^2\mathbb{P}^2) &\cong \mathrm{H}^1(\mathrm{Hilb}^2\mathbb{P}^2, \textstyle\bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \oplus \mathrm{H}^0(\mathrm{Hilb}^2\mathbb{P}^2, \textstyle\bigwedge^3 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \\
&\cong \mathbf{k}^{35} \oplus \mathbf{k}^{80} \\
\mathrm{HH}^4(\mathrm{Hilb}^2\mathbb{P}^2) &\cong \mathrm{H}^1(\mathrm{Hilb}^2\mathbb{P}^2, \textstyle\bigwedge^3 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \oplus \mathrm{H}^0(\mathrm{Hilb}^2\mathbb{P}^2, \textstyle\bigwedge^4 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}) \\
&\cong \mathbf{k}^{28} \oplus \mathbf{k}^{55} \\
\mathrm{HH}^{\geqslant 5}(\mathrm{Hilb}^2\mathbb{P}^2) &= 0
\end{aligned}
\tag{132}
$$

Proof. The cohomology of $\mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}$ is computed in Corollary B.4, the cohomology of $\bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}$ is computed in Corollary B.7, and the cohomology of $\bigwedge^3 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2}$ is computed in Corollary B.9. For $\bigwedge^4 \mathrm{T}_{\mathrm{Hilb}^2\mathbb{P}^2} \cong \omega_{\mathrm{Hilb}^2\mathbb{P}^2}$ the cohomology is readily computed using that $\mathrm{Hilb}^2\mathbb{P}^2 \to \mathrm{Sym}^2\mathbb{P}^2$ is a crepant resolution. □

Because the proof requires some Borel–Weil–Bott calculations we relegate the details of the computation to Appendix B.

Remark 4.5. A similar computation for $\mathrm{Hilb}^2\mathbb{P}^n \cong \mathbb{P}_{\mathrm{Gr}(2,n+1)}(\mathrm{Sym}^2\mathcal{S})$, see Corollary B.4 for more details, allows to recover the rigidity result from [6, Corollary 37] in this special case.

4.3 Hilbert squares of bielliptic surfaces

For $\mathrm{HH}_*(\mathrm{Hilb}^n S)$ we have that $\mathrm{HH}_*(S)$ determines it completely as a graded vector space. Thus a natural question to ask is:

Is $\mathrm{HH}^*(\mathrm{Hilb}^n S)$ *determined by* $\mathrm{HH}^*(S)$ *(and* $\mathrm{HH}_*(S)$*)?*

Equation (5) in Theorem A suggests that the answer to the question is negative. However, in order to really prove this, we need an example of a pair of surfaces, which have the same HH^* and HH_*, but different negative parts $\mathrm{HS}_{\leqslant -1}$ of Hochschild–Serre cohomology. In the following, we will find such a pair among the class of bielliptic surfaces, which is a class of surfaces of Kodaira dimension 0.

Bielliptic surfaces A *bielliptic surface* S is a surface of the form $(E \times F)/G$, where

- E and F are elliptic curves,
- G is a finite abelian group acting diagonally on $E \times F$,
- G acts by automorphisms on E (so that $E/G \cong \mathbb{P}^1$) and by translation on F.

There are 7 deformation families, given by the Bagnera–de Franchis list. Each family depends on the isomorphism type of E and the shape of the action of G, and the full list is given in Table 1.

Their Hodge diamond is of the form

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
0 & & 2 & & 0 \\
 & 1 & & 1 & \\
 & & 1 & &
\end{array}
\tag{133}
$$

We recall the properties of bielliptic surfaces as explained in [51, §4.D]. The Albanese morphism

$$\mathrm{alb}\colon S \to \mathrm{Pic}^0(S) = C \tag{134}$$

	$j(E)$	G	action of G	order of ω_S
1	arbitrary	$\mathbb{Z}/2\mathbb{Z}$	$e \mapsto -e$	2
2	arbitrary	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$e \mapsto -e$, $e \mapsto e + c$ where $c = -c$	2
3	0	$\mathbb{Z}/3\mathbb{Z}$	$e \mapsto \zeta e$	3
4	0	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$e \mapsto \zeta e$, $e \mapsto e + c$ where $c = \zeta c$	3
5	1728	$\mathbb{Z}/2\mathbb{Z}$	$e \mapsto \mathrm{i}e$	4
6	1728	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$e \mapsto \mathrm{i}e$, $e \mapsto e + c$ where $c = \mathrm{i}c$	4
7	0	$\mathbb{Z}/6\mathbb{Z}$	$e \mapsto -\zeta e$	6

Table 1: Classification of bielliptic surfaces

is a smooth surjection onto an elliptic curve, so we have a short exact sequence

(135) $$0 \to \mathcal{O}_S \to \Omega^1_S \to \Omega^1_{S/C} \to 0$$

which moreover can be shown to split. Defining

(136) $$L \coloneqq \mathrm{R}^1\mathrm{alb}_*\mathcal{O}_S,$$

we have $\Omega^1_{S/C} \cong \mathrm{alb}^* L^\vee$. Thus we obtain

(137) $$\begin{aligned} \mathrm{T}_S &\cong \mathcal{O}_S \oplus \mathrm{alb}^* L \\ \omega_S^\vee &\cong \mathrm{alb}^* L. \end{aligned}$$

The following straightforward cohomology computation is the key lemma.

Lemma 4.6. *We have that*

(138) $$\mathrm{H}^\bullet(S, \mathrm{alb}^* L) \cong \begin{cases} 0 & \mathrm{ord}\,\omega_S = 3,4,6 \\ \mathbf{k}[-1] \oplus \mathbf{k}[-2] & \mathrm{ord}\,\omega_S = 2 \end{cases}$$

and

(139) $$\mathrm{H}^\bullet(S, \mathrm{alb}^* L^{\otimes 2}) \cong \begin{cases} 0 & \mathrm{ord}\,\omega_S = 4,6 \\ \mathbf{k}[-1] \oplus \mathbf{k}[-2] & \mathrm{ord}\,\omega_S = 3 \\ \mathbf{k}[0] \oplus \mathbf{k}[-1] & \mathrm{ord}\,\omega_S = 2. \end{cases}$$

Proof. This follows from the isomorphisms

(140) $$\begin{aligned} \mathbf{R}\mathrm{alb}_* \circ \mathrm{alb}^* L &\cong L \oplus L^{\otimes 2}[-1], \\ \mathbf{R}\mathrm{alb}_* \circ \mathrm{alb}^* L^{\otimes 2} &\cong L^{\otimes 2} \oplus L^{\otimes 3}[-1] \end{aligned}$$

obtained via the projection formula, using that $\mathbf{R}\mathrm{alb}_*\mathcal{O}_S \cong \mathcal{O}_C[0] \oplus L[-1]$ (as $\dim C = 1$ and alb has connected fibers), and the fact that non-trivial torsion line bundles are cohomology-free on C so the only cohomology appears when $L^{\otimes i} \cong \mathcal{O}_C$. □

We thus obtain the following.

Lemma 4.7. *Let S be a bielliptic surface. Then*

(141) $$\sum_{i=0}^{4} \dim_{\mathbf{k}} \mathrm{HH}^i(S) t^i = \begin{cases} 1 + 2t + t^2 & \mathrm{ord}\,\omega_S = 3,4,6 \\ 1 + 2t + 2t^2 + 2t^3 + t^4 & \mathrm{ord}\,\omega_S = 2. \end{cases}$$

Proof. This is a straightforward computation using the Hochschild–Kostant–Rosenberg decomposition (35), with the cohomology of T_S and $\omega_S^\vee$ being determined by Lemma 4.6, which gives

$$\sum_{i=0}^{2} \mathrm{h}^i(S,\mathrm{T}_S)t^i = \begin{cases} 1+t & \operatorname{ord}\omega_S = 3,4,6 \\ 1+2t+t^2 & \operatorname{ord}\omega_S = 2 \end{cases} \tag{142}$$

and

$$\sum_{i=0}^{2} \mathrm{h}^i(S,\omega_S^\vee)t^i = \begin{cases} 0 & \operatorname{ord}\omega_S = 3,4,6 \\ t+t^2 & \operatorname{ord}\omega_S = 2. \end{cases} \tag{143}$$

□

Next we compute a slice of the Hochschild–Serre cohomology.

Lemma 4.8. *Let S be a bielliptic surface. Then*

$$\sum_{i=0}^{4} \dim_{\mathbf{k}} \mathrm{HS}^i_{-1}(S)t^i = \begin{cases} 0 & \operatorname{ord}\omega_S = 4,6 \\ t^4+2t^5+t^6 & \operatorname{ord}\omega_S = 3 \\ 2t^3+4t^4+2t^5 & \operatorname{ord}\omega_S = 2. \end{cases} \tag{144}$$

Proof. This proof is analogous to that of Lemma 4.7, now using (37) and Lemma 4.6 to compute the Hochschild–Kostant–Rosenberg decomposition (35). □

As an application of the description in Section 3.4 we obtain the following.

Proposition 4.9. *Let S be a bielliptic surface. Then*

$$\begin{aligned} &\sum_{i=0}^{8} \dim_{\mathbf{k}} \mathrm{HH}^i(\mathrm{Hilb}^2 S)t^i \\ &= \begin{cases} 1+2t+2t^2+2t^3+t^4 & \operatorname{ord}\omega_S = 4,6 \\ 1+2t+2t^2+2t^3+2t^4+2t^5+t^6 & \operatorname{ord}\omega_S = 3 \\ 1+2t+3t^2+8t^3+12t^4+8t^5+3t^6+2t^7+t^8 & \operatorname{ord}\omega_S = 2. \end{cases} \end{aligned} \tag{145}$$

Proof. We consider the partitions $\nu = (2)$ resp. $\nu = (1,1)$, with $\lambda = (0,1)$ resp. $\lambda = (2)$. The first partition gives a copy of $\mathrm{HS}^*_{-1}(S)$, described in Lemma 4.8. The second partition gives a copy of $\mathrm{Sym}^2(\mathrm{HH}^*(S))$, which is computed using Lemma 4.7. □

All bielliptic surfaces have the same Hodge diamond (133). Hence, by the Hochschild–Kostant–Rosenberg theorem, they also have the same Hochschild homology. Combining this with Lemma 4.7 and Proposition 4.9 we obtain the following.

Corollary 4.10. *Let S be a bielliptic surface with* $\operatorname{ord}\omega_S = 3$ *and let T be a bielliptic surface with* $\operatorname{ord}\omega_T \in \{4,6\}$. *Then* $\mathrm{HH}^*(S) \cong \mathrm{HH}^*(T)$ *and* $\mathrm{HH}_*(S) \cong \mathrm{HH}_*(T)$ *as graded vector spaces, but* $\mathrm{HH}^*(\mathrm{Hilb}^2 S) \ncong \mathrm{HH}^*(\mathrm{Hilb}^2 T)$.

Remark 4.11. We see that $\mathrm{HH}^2(\mathrm{Hilb}^2 S)$ is strictly bigger than $\mathrm{HH}^2(S)$, regardless of $\operatorname{ord}\omega_S$, and we have that $\mathrm{H}^2(\mathrm{Hilb}^2 S, \mathcal{O}_{\mathrm{Hilb}^2 S}) = 0$. These considerations in fact hold for arbitrary $n \geqslant 2$, using the degree considerations in Section 3.4. Thus (infinitesimally) there might be new commutative or noncommutative deformations of Hilbert schemes of points of bielliptic surfaces. We come back to this in Example 5.6.

5 Applications to Hilbert schemes

In this section we discuss various applications of our computations for the Hochschild(–Serre) cohomology of symmetric quotient stacks to the study of some *classical problems* about Hilbert schemes of points on surfaces. In Section 5.1, we discuss $\mathrm{HH}^1(\mathrm{Hilb}^n S)$ and applications to automorphisms. In Section 5.2 we discuss $\mathrm{HH}^2(\mathrm{Hilb}^n S)$ and applications to their deformation theory. In Section 5.3 we suggest a revised version of a conjecture by Boissière (see Conjecture 5.8), and in Section 5.4 we explain how results of Nieper-Wißkirchen can be used to provide evidence for the revised conjecture. Throughout this section, S is a smooth projective surface defined over a field of characteristic zero.

5.1 Infinitesimal automorphisms of Hilbert schemes

The following corollary to Theorem A is already proven in [11, Corollaire 1] in the context of complex analytic surfaces, using a generalisation of a computation of Göttsche. For us, using non-commutative methods, it is an easy consequence of the degree shifts as they appear in Section 3.4, *without* appealing to [11, Proposition 1] (see also Proposition 5.14).

Corollary 5.1 (Boissière). *Let S be a smooth projective surface. Then* $\dim \mathrm{Aut}^0(S) = \dim \mathrm{Aut}^0(\mathrm{Hilb}^n S)$ *for all* $n \geqslant 1$.

Proof of Corollary 5.1 using Theorem A. By (90) in Corollary 3.16 and the Bridgeland–King–Reid–Haiman equivalence (9) we obtain the identification

(146) $\mathrm{HH}^1(S) \cong \mathrm{HH}^1([\mathrm{Sym}^n S]) \cong \mathrm{HH}^1(\mathrm{Hilb}^n S)$.

By the Hochschild–Kostant–Rosenberg decomposition we have

(147) $\mathrm{HH}^1(\mathrm{Hilb}^n S) \cong \mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{O}_{\mathrm{Hilb}^n S}) \oplus \mathrm{H}^0(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S})$.

Taking $\mathfrak{S}_n$-invariant sections of $\mathrm{H}^1(S^n, \mathcal{O}_{S^n})$, we get

(148) $\mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{O}_{\mathrm{Hilb}^n S}) \cong \mathrm{H}^1(S, \mathcal{O}_S)$.

Hence, we conclude that

(149) $\dim \mathrm{Aut}^0(S) = \dim_{\mathbf{k}} \mathrm{H}^0(S, \mathrm{T}_S) = \dim_{\mathbf{k}} \mathrm{H}^0(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) = \dim \mathrm{Aut}^0(\mathrm{Hilb}^n S)$. ◻

5.2 Deformations of Hilbert schemes

It is an interesting question to understand how the deformation theory of a smooth projective surface S determines the deformation theory of the Hilbert scheme $\mathrm{Hilb}^n S$. Using the relative Hilbert scheme one observes that a deformation of S induces a deformation of $\mathrm{Hilb}^n S$, thus we obtain an *inclusion of deformation functors* $\mathrm{Def}_S \to \mathrm{Def}_{\mathrm{Hilb}^n S}$. It is therefore important to understand whether it is an isomorphism, or what measures the difference between the two. In particular, we are interested in comparing $\mathrm{H}^1(S, \mathrm{T}_S)$ to $\mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S})$.

It was shown by Fantechi that for *surfaces of general type* the two deformation theories are the same [22, Theorems 0.1 and 0.3]. More generally, she proves the following.

Theorem 5.2 (Fantechi). *Let S be a surface for which*

- $\mathrm{H}^0(S, \mathrm{T}_S) = 0$ *or* $\mathrm{H}^1(S, \mathcal{O}_S) = 0$*;*
- $\mathrm{H}^0(S, \omega_S^\vee) = 0$.

Then the natural morphism $\mathrm{Def}_S \to \mathrm{Def}_{\mathrm{Hilb}^n S}$ *is an isomorphism. In particular*

(150) $\mathrm{H}^1(S, \mathrm{T}_S) \cong \mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S})$.

On the other hand, Hitchin proves in [39, §4.1] the following.

Theorem 5.3 (Hitchin). *Let S be a surface for which* $\mathrm{H}^1(S, \mathcal{O}_S) = 0$. *Then there is a natural split exact sequence*

(151) $0 \to \mathrm{H}^1(S, \mathrm{T}_S) \to \mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) \to \mathrm{H}^0(S, \omega_S^\vee) \to 0$.

Thus the existence of Poisson structures on S gives geometric deformations of $\mathrm{Hilb}^n S$ which are induced by noncommutative deformations of S. In [6] this result is considered from the point-of-view of derived categories. The geometric deformations of $\mathrm{Hilb}^n S$ are studied in e.g. [20, 60] for noncommutative deformations of $\mathbb{P}^2$, and in [53] for S a (noncommutative) deformation of a del Pezzo surface.

Finally, the following was shown in [12, 13] by Bottacin.

Theorem 5.4 (Bottacin)**.** *Let S be a surface, and let σ be a non-trivial Poisson structure on S. Then there is a natural non-trivial Poisson structure $\sigma^{[n]}$ on* $\mathrm{Hilb}^n S$.

Using Theorem A we can prove a unified version of Theorems 5.2 and 5.3 (and heuristically understand Theorem 5.4) using stacky methods and the Bridgeland–King–Reid–Haiman equivalence (9).

Proof of Corollary B. By (91) in Corollary 3.16, the derived McKay correspondence (9), the derived invariance of Hochschild cohomology in Corollary A.6, and the Hochschild–Kostant–Rosenberg decomposition for S we have that

$$\begin{aligned}\mathrm{HH}^2(\mathrm{Hilb}^n S) &\cong \mathrm{HH}^2([\mathrm{Sym}^n S]) \\ &\cong \mathrm{H}^2(S,\mathcal{O}_S)\oplus \mathrm{H}^1(S,\mathrm{T}_S)\oplus \mathrm{H}^0(S,\omega_S^\vee) \\ &\quad \oplus \bigwedge^2 \mathrm{H}^1(S,\mathcal{O}_S)\oplus \bigwedge^2 \mathrm{H}^0(S,\mathrm{T}_S)\oplus \left(\mathrm{H}^1(S,\mathcal{O}_S)\otimes_{\mathbf{k}} \mathrm{H}^0(S,\mathrm{T}_S)\right) \\ &\quad \oplus \mathrm{H}^0(S,\omega_S^\vee).\end{aligned} \tag{152}$$

We wish to match this to the Hochschild–Kostant–Rosenberg decomposition for $\mathrm{Hilb}^n S$, which reads

$$\mathrm{HH}^2(\mathrm{Hilb}^n S)\cong \mathrm{H}^2(\mathrm{Hilb}^n S,\mathcal{O}_{\mathrm{Hilb}^n S})\oplus \mathrm{H}^1(\mathrm{Hilb}^n S,\mathrm{T}_{\mathrm{Hilb}^n S})\oplus \mathrm{H}^0(\mathrm{Hilb}^n S,\bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^n S}). \tag{153}$$

For the first summand in (153) we can take $\mathfrak{S}_n$-invariant sections of $\mathrm{H}^2(S^n,\mathcal{O}_{S^n})$ to obtain

$$\mathrm{H}^2(\mathrm{Hilb}^n S,\mathcal{O}_{\mathrm{Hilb}^n S})\cong \mathrm{H}^2(S,\mathcal{O}_S)\oplus \bigwedge^2 \mathrm{H}^1(S,\mathcal{O}_S). \tag{154}$$

The third summand in (153) is given by

$$\mathrm{H}^0(\mathrm{Hilb}^n S,\bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^n S})\cong \bigwedge^2 \mathrm{H}^0(S,\mathrm{T}_S)\oplus \mathrm{H}^0(S,\omega_S^\vee). \tag{155}$$

as can be extracted from [11, Proposition 1], or, more directly as the # = 2 case of Corollary 5.15 below. Thus, by cancelling (155) and (154) in (152) we obtain the identification (6). ◻

Remark 5.5. An argument similar to the computation for (155) tells us that

$$\mathrm{H}^0(\mathrm{Hilb}^n S,\bigwedge^3 \mathrm{T}_{\mathrm{Hilb}^n S})\cong \begin{cases}\mathrm{H}^0(S,\mathrm{T}_S)\otimes_{\mathbf{k}} \mathrm{H}^0(S,\omega_S^\vee) & n=2\\ \mathrm{H}^0(S,\mathrm{T}_S)\otimes_{\mathbf{k}} \mathrm{H}^0(S,\omega_S^\vee)\oplus \bigwedge^3 \mathrm{H}^0(S,\mathrm{T}_S) & n\geqslant 3.\end{cases} \tag{156}$$

Thus, if $\mathrm{H}^0(S,\mathrm{T}_S)=0$ every pre-Poisson structure on $\mathrm{Hilb}^n S$ is automatically Poisson. It would be interesting to compare this to Theorem 5.4, which is a more intrinsic recipe to induce Poisson structures on $\mathrm{Hilb}^n S$ from S.

Let us point out the following observation.

Example 5.6. The case of bielliptic surfaces (see Section 4.3) was not yet covered by Theorems 5.2 and 5.3. We already computed in Proposition 4.9 that $\mathrm{HH}^2(\mathrm{Hilb}^2 S)$ is strictly bigger than $\mathrm{HH}^2(S)\cong \mathrm{H}^1(S,\mathrm{T}_S)$. Now, by Corollary B we in fact have for all $n\geqslant 2$ that

$$\begin{aligned}\mathrm{HH}^2(\mathrm{Hilb}^n S) &\cong \mathrm{H}^1(\mathrm{Hilb}^n S,\mathrm{T}_{\mathrm{Hilb}^n S}) \\ &\cong \mathrm{H}^1(S,\mathrm{T}_S)\oplus \mathrm{H}^0(S,\mathrm{T}_S)\otimes_{\mathbf{k}} \mathrm{H}^1(S,\mathcal{O}_S),\end{aligned} \tag{157}$$

thus all first-order deformations are commutative, and

$$\mathrm{h}^1(\mathrm{Hilb}^n S,\mathrm{T}_{\mathrm{Hilb}^n S})=\begin{cases}2 & \operatorname{ord}\omega_S=3,4,6\\ 3 & \operatorname{ord}\omega_S=2,\end{cases} \tag{158}$$

by (142). Thus we get precisely one new deformation direction.

By [66, Corollary 2] we have that deformations of smooth projective varieties with torsion canonical bundle are unobstructed. Thus, Hilbert schemes of points on bielliptic surfaces have *genuinely new deformations*. It would be interesting to understand these deformations.

5.3 On Boissière's conjecture

Recall from Section 3.6 that for every line bundle L on a smooth projective surface S, there is an associated line bundle L_n on the Hilbert scheme $\mathrm{Hilb}^n S$.

Boissière conjectured in [11, Conjecture 1] a generating formula for the twisted Hodge numbers

$$\text{(159)}\quad \mathrm{h}^{p,q}(\mathrm{Hilb}^n S, L_n) := \mathrm{h}^q(\mathrm{Hilb}^n S, \Omega^p_{\mathrm{Hilb}^n S} \otimes L_n),$$

which predicts the equality

$$\text{(160)}\quad \sum_{n=0}^{+\infty}\sum_{p=0}^{2n}\sum_{q=0}^{2n} \mathrm{h}^{p,q}(\mathrm{Hilb}^n S, L_n)x^p y^q t^n = \prod_{k\geqslant 1}\prod_{p=0}^{2}\prod_{q=0}^{2}\left(1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k\right)^{-(-1)^{p+q}\mathrm{h}^{p,q}(S,L)}.$$

The case of $L = \mathcal{O}_S$ thus describes the Hodge numbers, and this is precisely the result of [31].

The case $L = \omega_S^{\otimes i}$ relates to the Hochschild–Serre cohomology of $\mathrm{Hilb}^n S$: for $L = \omega_S^{\otimes i}$ where $i \in \mathbb{Z}$ we get

$$\text{(161)}\quad (\omega_S^{\otimes i})_n \cong \omega^{\otimes i}_{\mathrm{Hilb}^n S},$$

and thus the twisted Hodge numbers in (159) are the dimensions of different pieces of the Hochschild–Serre cohomology; see (37) and (39).

In [38, Appendix B] a counterexample was found to the the conjecture, using S an Enriques surface and $n \geqslant 2$. Alternatively, the computation in Section 4.2 gives another explicit counterexample.

Example 5.7. The right-hand side of (160) for $\mathbb{P}^2$ and $L = \omega^\vee_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3)$ where we take only $k = 1, 2$ into account becomes

$$\text{(162)}\quad \begin{aligned}&\left(\frac{1}{1-t}\right)^{10}\left(\frac{1}{1+xt}\right)^{-8}\left(\frac{1}{1-x^2t}\right)\left(\frac{1}{1-xyt^2}\right)^{10}\left(\frac{1}{1+x^2yt^2}\right)^{-8}\left(\frac{1}{1-x^3yt^2}\right)\\ &\qquad\equiv 1+(10+8x+x^2)t+(55+80x+38x^2+10xy+8x^3+8x^2y+x^4+x^3y)t^2 \bmod t^3.\end{aligned}$$

Using Proposition 4.4 we see that the coefficient of x^3yt^2 should be 10 (not 1), the coefficient of x^2yt^2 should be 35 (not 8) and the coefficient of xyt^2 should be 28 (not 10).

A new conjecture Taking inspiration from our main result we propose a new conjecture, which is compatible with collapsing the bigrading on twisted Hodge numbers into a single grading on Hochschild homology with coefficients.

Conjecture 5.8. *Let S be a smooth projective surface, and $L \in \mathrm{Pic}\, S$. Then*

$$\text{(163)}\quad \sum_{n\geqslant 0}\sum_{p=0}^{2n}\sum_{q=0}^{2n} \mathrm{h}^{p,q}(\mathrm{Hilb}^n S, L_n)x^p y^q t^n = \prod_{k\geqslant 1}\prod_{p=0}^{2}\prod_{q=0}^{2}\left(1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k\right)^{-(-1)^{p+q}\mathrm{h}^{p,q}(S,L^{\otimes k})}.$$

Note that the only difference to Boissière's conjecture (159) is the occurence of the exponent k of L on the right.

By Lemma 3.3, we see that Conjecture 5.8 is equivalent to the claim that we have an isomorphism

$$\text{(164)}\quad \bigoplus_{n\geqslant 0} \mathrm{H}^{\#,\star}(\mathrm{Hilb}^n S, L_n)t^n \overset{?}{\cong} \mathrm{Sym}^\bullet\left(\bigoplus_{k\geqslant 1}\mathrm{H}^{\#,\star}(S, L^{\otimes k})[1-k,1-k]t^k\right),$$

where $[1-k, 1-k]$ denotes the shift of both gradings of $\mathrm{H}^{\#,\star}(S, L^{\otimes k})$ by the same value $1-k$, and the symmetric power is taken in the graded sense with respect to the bigrading of $\mathrm{H}^{\#,\star}(S, L^{\otimes k})$, but in the ordinary sense with respect to the grading by powers of t. Looking at the individual Hilbert schemes, this can also be formulated as

$$\text{(165)}\quad \mathrm{H}^{\#,\star}(\mathrm{Hilb}^n S, L^{\{n\}}) \overset{?}{\cong} \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{H}^{\#,\star}(S, L^{\otimes i}) \right) \left[\sum_i (1-i)\lambda_i, \sum_i (1-i)\lambda_i \right]$$

for every $n \in \mathbb{N}$, where $\nu = 1^{\lambda_1} 2^{\lambda_2} \cdots$. Note that the right-hand side of (165) almost exactly matches the right-hand side of (73), the only difference being the degree shift. This difference can be fixed by introducing, for $\mathcal{X} = [M/G]$ a quotient stack of a smooth proper variety M by a finite group G, and $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(\mathcal{X})$, the *twisted orbifold Hodge groups*

$$\text{(166)}\quad \mathrm{H}^{p,q}_{\mathrm{orb}}(\mathcal{X}, \mathcal{E}) := \left(\bigoplus_{g \in G} \mathrm{H}^{p-\mathrm{age}(g), q-\mathrm{age}(g)}(M^g, E|_{M^g}) \right)^G \cong \bigoplus_{[g] \in \mathrm{Conj}(G)} \mathrm{H}^{p-\mathrm{age}(g), q-\mathrm{age}(g)}(M^g, E|_{M^g})^{\mathrm{C}(g)}$$

The *age function* $\mathrm{age}(-)$, which a priori takes value in $\mathbb{Q}$, is determined by the eigenvalues of the action of g on the normal bundle $\mathrm{N}_{M^g/M}$; see [23] for details. Note that, $\mathrm{H}^{\#,\star}_{\mathrm{orb}}(\mathcal{X}, \mathcal{E})$ and $\mathrm{H}^{\#,\star}(\mathrm{I}\mathcal{X}, \mathcal{E})$ are equal as ungraded vector spaces. But in $\mathrm{H}^{\#,\star}_{\mathrm{orb}}(\mathcal{X}, \mathcal{E})$ shifts for the varying direct summands of $\mathrm{H}^{\#,\star}(\mathrm{I}\mathcal{X}, \mathcal{E})$ corresponding to the components of $\mathrm{I}\mathcal{X}$ are introduced. This is not just some random definition that we make in order to make certain degrees match, but the shift by $\mathrm{age}(g)$ also occurs in the standard convention for the grading of the orbifold cohomology; see [23].

For $M = S^n$, and $g \in \mathfrak{S}_n$ with cycle class λ, the age is given by

$$\text{(167)}\quad \mathrm{age}(g) = \sum_{i \geqslant 1} (i-1)\lambda_i\,;$$

see [23, 25]. Hence, (73) can be rephrased as

$$\text{(168)}\quad \mathrm{H}^{\#,\star}_{\mathrm{orb}}([\mathrm{Sym}^n X], L^{\{n\}}) \cong \bigoplus_{\nu \dashv n} \left(\bigotimes_{i \geqslant 1} \mathrm{Sym}^{\lambda_i} \mathrm{H}^{\#,\star}(S, L^{\otimes i}) \right) \left[\sum_i (1-i)\lambda_i, \sum_i (1-i)\lambda_i \right].$$

Thus, Conjecture 5.8 is equivalent to the following conjecture that the derived McKay correspondence for Hilbert schemes of points on surfaces preserves the twisted Hodge groups.

Conjecture 5.9. *Let S be a smooth projective surface and $L \in \mathrm{Pic}\, S$. Then, for every $n \geqslant 1$, we have an isomorphism of bigraded vector spaces*

$$\text{(169)}\quad \mathrm{H}^{\#,\star}(\mathrm{Hilb}^n S, L_n) \cong \mathrm{H}^{\#,\star}_{\mathrm{orb}}([\mathrm{Sym}^n S], L^{\{n\}}).$$

Some evidence We describe some evidence in favour of Conjecture 5.8, given by various specializations of the identity.

The first specialization reduces to Hochschild homology.

Proposition 5.10. *Conjecture 5.8 holds when specialising to $x = y^{-1}$.*

Proof. Indeed, due to the Hochschild–Kostant–Rosenberg isomorphism formulated in terms of twisted Hodge groups (39), setting $x = s^{-1}$ and $y = s$ in (163) recovers (111). □

The second specialization reduces to the χ_y-genus. Here, for a line bundle L on a variety X, let $\chi_y(X, L)$ denote Hirzebruch's χ_y-genus $\sum_{p \geqslant 0} \chi(X, \Omega^p_X \otimes L) y^p$.

Proposition 5.11. *Conjecture 5.8 holds when specialising to $y = -1$.*
In other words, (renaming the variable x by $-y$), we have

$$\text{(170)}\quad \sum_{n \geqslant 0} \chi_{-y}(\mathrm{Hilb}^n S, L_n) t^n = \prod_{k \geqslant 1} \prod_{p=0}^{2} \left(1 - y^{p+k-1} t^k\right)^{-(-1)^p \chi(S, \Omega^p_S \otimes L^{\otimes k})}.$$

Proof. This is essentially due to Göttsche [29]. Note that in his notation, $\mu(L) = \det(L^{[n]})\otimes\det(\mathcal{O}_S^{[n]})^\vee$ is precisely our L_n, thanks to the formula before [21, Lemma 5.1]. Indeed, by performing the change of variables $p = ty$ in Göttsche's [29, Corollary 1.2] we have that $\sum_{n\geqslant 0}\chi_{-y}(\operatorname{Hilb}^n S, L_n)t^n$ is equal to

(171)
$$\prod_{k\geqslant 1}\left(\frac{(1-t^ky^k)^2}{(1-t^ky^{k+1})(1-t^ky^{k-1})}\right)^{\frac{k^2}{2}(L^2)}\left(\frac{1-t^ky^{k-1}}{1-t^ky^{k+1}}\right)^{\frac{k}{2}(LK_S)}\left(1-t^ky^k\right)^{(K_S^2)}\left((1-t^ky^k)^{10}(1-t^ky^{k+1})(1-t^ky^{k-1})\right)^{-\chi(S,\mathcal{O}_S)}.$$

It suffices to show the right-hand side of (170) is equal to (171). To this end, we apply the Hirzebruch–Riemann–Roch formula and Noether formula to obtain the following:

$$\begin{aligned}\chi(S,L^{\otimes k}) &= \chi(S,\mathcal{O}_S)+\frac{k^2}{2}(L^2)-\frac{k}{2}(LK_S);\\ \chi(S,\Omega_S^1\otimes L^{\otimes k}) &= -10\chi(S,\mathcal{O}_S)+(K_S^2)+k^2(L^2);\\ \chi(S,\Omega_S^2\otimes L^{\otimes k}) &= \chi(S,\mathcal{O}_S)+\frac{k^2}{2}(L^2)+\frac{k}{2}(LK_S).\end{aligned}\tag{172}$$

Plugging (172) into the right-hand side of (170) readily gives (171). □

Remark 5.12. From (170), we can deduce that the χ_y-genus of the line bundle L_n on $\operatorname{Hilb}^n S$ is determined by the χ_y-genera of all the line bundles $L^{\otimes k}$ on S for $k\in\mathbb{N}$. More precisely, we can re-express this identity as

$$\sum_{n\geqslant 0}\chi_{-y}(\operatorname{Hilb}^n S, L_n)t^n = \exp\left(\sum_{m=1}^{\infty}\frac{t^m}{m}\sum_{k=1}^{\infty}(ty)^{(k-1)m}\chi_{-y^m}(S,L^{\otimes k})\right).\tag{173}$$

Compare this to [28, Theorem 2.3.14 (4)], which is the case $L=\mathcal{O}_S$.

The third specialization is concerned with the cohomology of L_n on $\operatorname{Hilb}^n S$.

Proposition 5.13. *Conjecture 5.8 holds when specialising to $x = 0$, which says that*

$$\begin{aligned}\sum_{n\geqslant 0}\sum_{q=0}^{2n}\mathrm{h}^q(\operatorname{Hilb}^n S, L_n)y^qt^n &= \prod_{q=0}^{2}(1-(-1)^qy^qt)^{-(-1)^q\mathrm{h}^q(S,L)}\\ &= \frac{(1+yt)^{\mathrm{h}^1(S,L)}}{(1-t)^{\mathrm{h}^0(S,L)}(1-y^2t)^{\mathrm{h}^2(S,L)}}.\end{aligned}\tag{174}$$

Proof. Indeed, let $\rho\colon \operatorname{Hilb}^n S\to S^{(n)}$ be the Hilbert–Chow morphism. The above equality follows from the isomorphism

$$\mathrm{H}^*(\operatorname{Hilb}^n S, L_n)\cong \mathrm{H}^*(\operatorname{Hilb}^n S,\rho^*L^{(n)})\cong \mathrm{H}^*(S^{(n)},\mathbf{R}\rho_*\rho^*L^{(n)})\cong \mathrm{H}^*(S^{(n)},L^{(n)})\cong \operatorname{Sym}^n\mathrm{H}^*(S,L),\tag{175}$$

where we used the fact that $S^{(n)}$ has rational singularities hence $\mathbf{R}\rho_*\mathcal{O}_{\operatorname{Hilb}^n S}=\mathcal{O}_{S^{(n)}}$, and Sym^n is taken in the graded sense. □

The final specialization is concerned with the global sections of $\Omega^p_{\operatorname{Hilb}^n S}\otimes L_n$.

Proposition 5.14. *Conjecture 5.8 holds when specialising to $y = 0$, which says that*

$$\begin{aligned}\sum_{n\geqslant 0}\sum_{p=0}^{2n}\mathrm{h}^0(\operatorname{Hilb}^n S,\Omega^p_{\operatorname{Hilb}^n S}\otimes L_n)x^pt^n &= \prod_{p=0}^{2}\left(1-(-1)^px^pt\right)^{-(-1)^p\mathrm{h}^{p,0}(S,L)}\\ &= \frac{(1+xt)^{\mathrm{h}^0(S,\Omega_S^1\otimes L)}}{(1-t)^{\mathrm{h}^0(S,L)}(1-x^2t)^{\mathrm{h}^0(S,\omega_S^\vee\otimes L)}}\end{aligned}\tag{176}$$

Proof. This is proved by Boissière in [11, Proposition 1]. The key point is that by [69, Lemma 1.11] we have an isomorphism $\rho_* \Omega^p_{\mathrm{Hilb}^n S} \cong \widetilde{\Omega}^p_{S^{(n)}}$, where the right-hand side denotes the sheaf of reflexive p-differentials on $S^{(n)}$. Hence,

$$\text{(177)}\quad \mathrm{H}^0(\mathrm{Hilb}^n S, \Omega^p_{\mathrm{Hilb}^n S} \otimes L_n) \cong \mathrm{H}^0(S^{(n)}, \widetilde{\Omega}^p_{S^{(n)}} \otimes L^{(n)}) \cong \mathrm{H}^0(S^n, \Omega^p_{S^n} \otimes L^{\boxtimes n})^{\mathfrak{S}_n},$$

and the Künneth formula allows us to conclude. □

As a corollary, there is the following formula for polyvector fields on the Hilbert scheme.

Corollary 5.15. *For every $n \in \mathbb{N}$, we have an isomorphism of graded vector spaces*

$$\text{(178)}\quad \mathrm{H}^0\left(\mathrm{Hilb}^n S, \bigwedge^{2n-\#} \mathrm{T}_{\mathrm{Hilb}^n S}\right) \cong \mathrm{Sym}^n\left(\underbrace{\mathrm{H}^0(S, \omega_S^\vee)}_{\deg 0} \oplus \underbrace{\mathrm{H}^0(S, \mathrm{T}_S)}_{\deg 1} \oplus \underbrace{\mathrm{H}^0(S, \mathcal{O}_S)}_{\deg 2}\right)$$

Proof. Translating from generating functions to vector spaces, Proposition 5.14 says that the $\# = 0$ case of (165) is true. Due to the shifts on the right-hand side, the only contribution to $\# = 0$ comes when $\lambda_i = 0$ for all $i > 1$, in which case $\lambda_1 = n$. Hence, we have

$$\text{(179)}\quad \mathrm{H}^0\left(\mathrm{Hilb}^n S, \Omega^\#_{\mathrm{Hilb}^n S} \otimes L_n\right) \cong \mathrm{Sym}^n\left(\underbrace{\mathrm{H}^0(S, L)}_{\deg 0} \oplus \underbrace{\mathrm{H}^0(S, \Omega^1_S \otimes L)}_{\deg 1} \oplus \underbrace{\mathrm{H}^0(S, \omega_S \otimes L)}_{\deg 2}\right)$$

The assertion is just the case $L = \omega_S^\vee$ of (179). □

The results of Section 4.2 give an explicit instance in which the conjecture is checked.

Example 5.16. The case $S = \mathbb{P}^2$, $n = 2$, and $L = \omega_S^\vee$ can be compared with the results in Section 4.2: we have that

$$\text{(180)}\quad (\mathrm{h}^{p,q}(\mathbb{P}^2, \omega_S^\vee))_{p,q} = \begin{pmatrix} 10 & 0 & 0 \\ 8 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\text{(181)}\quad (\mathrm{h}^{p,q}(\mathbb{P}^2, \omega_S^\vee \otimes \omega_S^\vee))_{p,q} = \begin{pmatrix} 28 & 0 & 0 \\ 35 & 0 & 0 \\ 10 & 0 & 0 \end{pmatrix},$$

and a computation shows the output of Conjecture 5.8 agrees with Proposition 4.4.

Remark 5.17. By taking $L = \mathcal{O}_S$ (hence $L_n = \mathcal{O}_{\mathrm{Hilb}^n S}$), Conjecture 5.8 reduces to the classical formula of Hodge numbers of Hilbert schemes due to Göttsche and Soergel [31]. More generally, by using [61, Remark 2.7], we will see in Theorem 5.20 that Conjecture 5.8 holds in the case that L admits a unitary flat connection.

5.4 Relation to Nieper–Wißkirchen's work

Recall that for a smooth projective variety X equipped with a rank-1 local system $\mathbb{L}$ of complex vector spaces, the cohomology group $\mathrm{H}^k(X, \mathbb{L})$ carries a weight-k $\mathbb{C}$-Hodge structure[4]. The Hodge components are described as follows. Under the non-abelian Hodge correspondence, $\mathbb{L}$ corresponds to a Higgs line bundle (L, θ), where L is a holomorphic line bundle of degree 0 and $\theta \in \mathrm{H}^0(X, \Omega^1_X)$ is a holomorphic 1-form. Then the Hodge decomposition takes the following form:

$$\text{(182)}\quad \mathrm{H}^k(X, \mathbb{L}) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X, (L, \theta)),$$

[4]A $\mathbb{C}$-Hodge structure of weight k is nothing but a $\mathbb{C}$-vector space together with a direct sum decomposition into subspaces $V = \bigoplus_{p+q=k} V^{p,q}$.

where $\mathrm{H}^{p,q}(X,(L,\theta))$ is by definition the cohomology of the complex

(183) $\mathrm{H}^q(X, L\otimes\Omega_X^{p-1}) \xrightarrow{\theta} \mathrm{H}^q(X, L\otimes\Omega_X^{p}) \xrightarrow{\theta} \mathrm{H}^q(X, L\otimes\Omega_X^{p+1})$.

Therefore, we denote

(184) $\mathrm{H}^{p,q}(X,\mathbb{L}) := \mathrm{H}^{p,q}(X,(L,\theta))$.

Again by the non-abelian Hodge correspondence, $\mathbb{L}$ is unitary if and only if $L\cong\mathbb{L}\otimes\mathcal{O}_X$ and $\theta=0$. In this case, (182) becomes

(185) $$\mathrm{H}^k(X,\mathbb{L}) \cong \bigoplus_{p+q=k}\mathrm{H}^{p,q}(X,L),$$

where $\mathrm{H}^{p,q}(X,L) = \mathrm{H}^q(X, L\otimes\Omega_X^p)$, consistent with our notation in previous sections.

In the rest of this section, we specialise to the case of dimension 2. Let S be a smooth projective surface and $\mathbb{L}$ a rank-1 local system of $\mathbb{C}$-vector spaces on S.

As $\pi_1(\mathrm{Hilb}^n S)\cong\pi_1(S^{(n)})\cong\pi_1(S)^{\mathrm{ab}}\cong\mathrm{H}_1(S,\mathbb{Z})$ (see [3, §6, Lemma 1]), the local system $\mathbb{L}$ induces a rank-1 local system on $\mathrm{Hilb}^n S$, denoted by $\mathbb{L}_n$. Similarly, any line bundle L induces a line bundle L_n on $\mathrm{Hilb}^n S$, and we have a canonical identification $\mathrm{H}^{1,0}(S)\cong\mathrm{H}^{1,0}(\mathrm{Hilb}^n S)$. One can easily check the following compatibility:

Lemma 5.18. *Let $\mathbb{L}$ be a rank-1 local system on S corresponding to the Higgs line bundle (L,θ). Then the local system $\mathbb{L}_n$ on $\mathrm{Hilb}^n S$ corresponds to the Higgs line bundle (L_n,θ). In particular, if $\mathbb{L}$ is unitary, then so is $\mathbb{L}_n$ and $L_n\cong\mathbb{L}_n\otimes\mathcal{O}_{\mathrm{Hilb}^n S}$, $\theta=0$.*

In [61, Theorem 1.2, Remark 2.7], Nieper–Wißkirchen proved the following result.

Theorem 5.19 (Nieper–Wißkirchen)**.** *In the above notation,*

(186) $$\sum_{n\geqslant 0}\sum_{p=0}^{2n}\sum_{q=0}^{2n}\mathrm{h}^{p,q}(\mathrm{Hilb}^n S,\mathbb{L}_n)x^py^qt^n = \prod_{k\geqslant 1}\prod_{p=0}^{2}\prod_{q=0}^{2}\left(1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k\right)^{-(-1)^{p+q}\mathrm{h}^{p,q}(S,\mathbb{L}^{\otimes k})}.$$

In particular, by specialising to $x=y$, we have

(187) $$\sum_{n\geqslant 0}\sum_{i=0}^{4n}\mathrm{h}^{i}(\mathrm{Hilb}^n S,\mathbb{L}_n)x^it^n = \prod_{k\geqslant 1}\prod_{i=0}^{4}\left(1-(-1)^{i}x^{i+2k-2}t^k\right)^{-(-1)^{i}\mathrm{h}^{i}(S,\mathbb{L}^{\otimes k})}.$$

Combining Lemma 5.18 and Theorem 5.19, we obtain that Conjecture 5.8 holds for line bundles arising from rank-1 unitary local systems:

Theorem 5.20. *Let $\mathbb{L}$ be a rank-1 unitary local system on a smooth projective surface S. Define $L:=\mathbb{L}\otimes\mathcal{O}_S$. Let L_n be the associated line bundle on $\mathrm{Hilb}^n S$. Then we have the following equality.*

(188) $$\sum_{n\geqslant 0}\sum_{p=0}^{2n}\sum_{q=0}^{2n}\mathrm{h}^{p,q}(\mathrm{Hilb}^n S,L_n)x^py^qt^n = \prod_{k\geqslant 1}\prod_{p=0}^{2}\prod_{q=0}^{2}\left(1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k\right)^{-(-1)^{p+q}\mathrm{h}^{p,q}(S,L^{\otimes k})}.$$

A Hochschild–Serre cohomology for dg categories and functorialities

The definition of Hochschild–Serre cohomology in Definition 2.3 using Fourier–Mukai functors has a counterpart using dg bimodules as alluded to in Remark 2.6. In Appendix A.1 we will describe this, and use it to show that it is a derived invariant. We will also consider more generally Hochschild homology with coefficients, and discuss how it is a derived invariant under a certain natural compatibility between the equivalence and the coefficients.

Using the Fourier–Mukai definition, we will point out in Appendix A.2 how Hochschild–Serre cohomology moreover satisfy functorialities for étale morphisms.

A.1 Hochschild–Serre cohomology for dg categories and Morita invariance

Throughout we fix a base field $\mathbf{k}$. Let $\mathcal{A}$ be a dg category, and $\operatorname{Perf}\mathcal{A}$ be the subcategory of $\mathbf{D}(\mathcal{A})$ consisting of perfect right $\mathcal{A}$-modules. We let $\mathcal{A}^{\mathrm{e}}$ denote its enveloping category, defined as $\mathcal{A}\otimes_k \mathcal{A}^{\mathrm{op}}$. For an introduction to the Morita theory of dg categories in the setup that we will use, we refer to [14, §2].

As in [14, §2.3], if $\mathcal{A}$ is smooth and proper, we will consider

- the diagonal bimodule $\mathcal{A}$ as an object of $\operatorname{Perf}\mathcal{A}^{\mathrm{e}}$, representing the identity functor;
- the right dual $\mathcal{A}^*$ as an object of $\operatorname{Perf}\mathcal{A}^{\mathrm{e}}$, representing the Serre functor;
- the left dual $\mathcal{A}^!$ as an object of $\operatorname{Perf}\mathcal{A}^{\mathrm{e}}$, representing the inverse Serre functor.

This allows us to generalise Definition 2.3 as follows, where we use the tensor product of dg bimodules to encode iterated (inverse) powers of the Serre functor.

Definition A.1. Let $\mathcal{A}$ be a smooth and proper dg category, and let $\mathrm{S}_{\mathcal{A}}$ denote its Serre functor. The *Hochschild–Serre cohomology of* $\mathcal{A}$ is the bigraded algebra

$$\mathrm{HS}^*_\bullet(\mathcal{A}) \coloneqq \bigoplus_{k\in\mathbb{Z}} \mathrm{HS}^*_k(\mathcal{A}) = \bigoplus_{k\in\mathbb{Z}}\bigoplus_{j\in\mathbb{Z}} \mathrm{HS}^j_k(\mathcal{A}) \tag{189}$$

where

$$\mathrm{HS}^j_k(\mathcal{A}) \coloneqq \mathrm{H}^j\,\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{e}}}(\mathrm{id}_{\mathcal{A}}, \mathrm{S}^{\circ k}_{\mathcal{A}}) = \begin{cases} \mathrm{H}^j\,\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{e}}}(\mathcal{A}, \mathcal{A}^{*,\otimes^{\mathrm{L}}k}) & k \geqslant 0 \\ \mathrm{H}^j\,\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{e}}}(\mathcal{A}, \mathcal{A}^{!,\otimes^{\mathrm{L}}-k}) & k < 0. \end{cases} \tag{190}$$

and the multiplication is induced by the composition in $\operatorname{Perf}\mathcal{A}^{\mathrm{e}}$.

As in Remark 2.4, this definition incorporates the Hochschild cohomology (resp. homology) of the dg category $\mathcal{A}$, as

$$\mathrm{HH}^*(\mathcal{A}) \cong \mathrm{HS}^*_0(\mathcal{A}) \tag{191}$$

resp.

$$\mathrm{HH}_*(\mathcal{A}) \cong \mathrm{HS}^*_1(\mathcal{A}). \tag{192}$$

Remark A.2. The definition in Definition A.1 suffices for our purposes, we do not need to give a chain-level definition via a generalization of the Hochschild (co)chain complex. To study certain algebraic and higher structures which exist on Hochschild (co)homology, and might possess a generalization to Hochschild–Serre cohomology, this could however be useful.

Moreover, both Hochschild homology and cohomology can be defined for arbitrary dg categories, not just smooth and proper ones. It would be interesting to find a definition of Hochschild–Serre cohomology which works in this generality.

Künneth formula Let $\mathcal{A}$ and $\mathcal{B}$ be smooth proper dg categories. We have a canonical equivalence

$$(\mathcal{A}\otimes\mathcal{B})^{\mathrm{e}} \cong \mathcal{A}^{\mathrm{e}}\otimes\mathcal{B}^{\mathrm{e}}, \tag{193}$$

under which the bimodule $\mathcal{A}\otimes\mathcal{B}$ is identified with $\mathcal{A}\boxtimes\mathcal{B}$, and similarly $(\mathcal{A}\otimes\mathcal{B})^*$ with $\mathcal{A}^*\boxtimes\mathcal{B}^*$, and $(\mathcal{A}\otimes\mathcal{B})^!$ with $\mathcal{A}^!\boxtimes\mathcal{B}^!$. In other words, as dg endofunctors of $\mathcal{A}\otimes\mathcal{B}$,

$$\begin{aligned} \mathrm{id}_{\mathcal{A}\otimes\mathcal{B}} &\cong \mathrm{id}_{\mathcal{A}}\otimes\mathrm{id}_{\mathcal{B}}, \\ \mathrm{S}_{\mathcal{A}\otimes\mathcal{B}} &\cong \mathrm{S}_{\mathcal{A}}\otimes\mathrm{S}_{\mathcal{B}}\,. \end{aligned} \tag{194}$$

As a consequence, we have the Künneth formula for Hochschild–Serre cohomology:

Proposition A.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be smooth proper dg categories. We have an isomorphism of bigraded algebras:*

$$\mathrm{HS}^*_\bullet(\mathcal{A}\otimes\mathcal{B}) \cong \mathrm{HS}^*_\bullet(\mathcal{A})\otimes\mathrm{HS}^*_\bullet(\mathcal{B}). \tag{195}$$

Morita invariance The following theorem explains that, as generalisation of Hochschild (co)homology, Hochschild–Serre cohomology is still a Morita invariant for dg categories. This is a generalisation of [63, Theorem 2.1.8] which proves it for derived categories of smooth projective varieties.

Theorem A.4 (Morita invariance). *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a Morita equivalence between smooth and proper dg categories. Then there is a naturally induced isomorphism of bigraded algebras*

$$\mathrm{HS}^*_\bullet(\mathcal{A}) \cong \mathrm{HS}^*_\bullet(\mathcal{B}). \tag{196}$$

Proof. The functor F induces a functor $-\otimes^{\mathbf{L}}_{\mathcal{A}} M\colon \operatorname{Perf}\mathcal{A} \to \operatorname{Perf}\mathcal{B}$, with quasi-inverse G given as $-\otimes^{\mathbf{L}}_{\mathcal{B}} N$, where the bimodules M and N are perfect. Then the Morita equivalence F induces a Morita equivalence

$$F^{\mathrm{e}}\colon \operatorname{Perf}\mathcal{A}^{\mathrm{e}} \to \operatorname{Perf}\mathcal{B}^{\mathrm{e}} \tag{197}$$

which we can write using bimodules as $N \otimes^{\mathbf{L}}_{\mathcal{A}} - \otimes^{\mathbf{L}}_{\mathcal{A}} M$.

By the Morita theory for dg categories this functor always preserves the identity functor. The categories $\mathcal{A}^{\mathrm{e}}$ and $\mathcal{B}^{\mathrm{e}}$ are again smooth and proper, thus the equivalence F^{e} is compatible with duality, and therefore sends the bimodules $\mathcal{A}^*$ (resp. $\mathcal{A}^!$) representing the Serre functor (resp. inverse Serre functor) to $\mathcal{B}^*$ (resp. $\mathcal{B}^!$). Therefore, we obtain an isomorphism of vector spaces

$$\mathrm{HS}^j_k(\mathcal{A}) \xrightarrow{\sim} \mathrm{HS}^j_k(\mathcal{B}), \tag{198}$$

and taking the direct sum we obtain the isomorphism of vector spaces in (196).

The isomorphism of bigraded vector spaces is compatible with the algebra structure, as this is induced from the compositions in the equivalent categories $\mathbf{D}^{\mathrm{b}}(\mathcal{A}^{\mathrm{e}})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{B}^{\mathrm{e}})$. ◻

Agreement between the two approaches We wish to show how the dg version from Definition A.1 agrees with the Fourier–Mukai approach Definition 2.3 for smooth and proper orbifolds. The case of Hochschild cohomology was already discussed in [59, Appendix A], and the method is very similar.

Theorem A.5 (Agreement). *Let $\mathcal{X}$ be a smooth and proper orbifold, and let $\mathbf{D}^{\mathrm{b}}(\mathcal{X})$ be a dg enhancement of its bounded derived category of coherent sheaves. Then Definition A.1 for $\mathcal{A} = \mathbf{D}^{\mathrm{b}}(\mathcal{X})$ and Definition 2.3 are isomorphic as bigraded algebras.*

Before giving the proof we first explain why we can apply Definition A.1.

- To consider $\mathbf{D}^{\mathrm{b}}(\mathcal{X})$ as a dg category, we can use the enhancement from [10, Example 5.5]. By [19, Proposition 6.10] the derived category has a unique dg enhancement, i.e., all dg enhancements are quasi-equivalent.
- In [10, Theorem 6.6] it is shown that the derived category of a smooth and proper orbifold is an admissible subcategory in the derived category of a smooth and proper variety. As smoothness and properness for dg categories is inherited by admissible subcategories, see, e.g., [10, Proposition 5.20] we obtain that any dg enhancement of $\mathbf{D}^{\mathrm{b}}(\mathcal{X})$ is indeed a smooth and proper dg category.

Proof of Theorem A.5. By the Morita invariance from Theorem A.4 and uniqueness of the dg enhancement we can ignore the choice of enhancement. By [9, Theorem 1.2] we have that $\mathbf{D}^{\mathrm{b}}(\mathcal{X})^{\mathrm{e}} \cong \mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})$. Next, observe that

- the diagonal bimodule corresponds to the identity functor $\Delta_*\mathcal{O}_{\mathcal{X}}$;
- the Serre functor corresponds to $\Delta_*\omega_{\mathcal{X}}[d_X]$;
- the inverse Serre functor corresponds to $\Delta_*\omega_{\mathcal{X}}^\vee[-d_X]$.

The correspondence for the Serre functor follows from, e.g., [62, Proposition 2.31] when $\mathbf{k}$ is algebraically closed, or [52, Theorem 1] when $\mathbf{k}$ is arbitrary and $\mathcal{X}$ has projective coarse moduli space. Thus the definitions in Definition A.1 and Definition 2.3 agree, because the bigraded algebra structures correspond to composition in the derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{X}\times\mathcal{X})$. ◻

We record the following corollary of the Bridgeland–King–Reid–Haiman equivalence (9), which motivates the approach taken in the main body of the paper.

Corollary A.6. *Let S be a smooth, projective surface. Then for all $n \geqslant 0$ there exists an isomorphism*

$$\mathrm{HS}^*_\bullet(\mathrm{Hilb}^n S) \cong \mathrm{HS}^*_\bullet([\mathrm{Sym}^n S]) \tag{199}$$

of bigraded algebras.

Proof. The (necessarily unique) dg enhancements of $\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S)$ and $\mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n S])$ are unique, so by the agreement of the definitions in Theorem A.5 we are done. □

A.2 Étale functoriality of Hochschild–Serre cohomology

The original definition of Hochschild–Serre cohomology of smooth projective varieties was only shown to be functorial for equivalences. Hochschild homology on the other hand is functorial for arbitrary morphisms, and even functors [43]. As suggested in [44, Claim in §8.4], Hochschild cohomology should be functorial for étale morphisms.

In this section we show that Hochschild–Serre cohomology (and thus Hochschild cohomology) is indeed functorial for étale morphisms, at least as vector spaces. This takes on two forms:

- a covariant functoriality, for which we provide a more general criterion in Proposition A.7 leading to an *étale pushforward* in Corollary A.8;
- a contravariant functoriality in Proposition A.10, i.e., an *étale pullback*.

Already for Hochschild cohomology we are not aware of a written reference where this is proven, even on the level of vector spaces. We are content with showing that there is a naturally induced morphism for Hochschild–Serre cohomology, we do not work out the compatibility with composition. We also do not address whether Hochschild–Serre cohomology satisfies some functoriality with respect to fully faithful functors, which is discussed for Hochschild cohomology in [41].

Proposition A.7 (Covariant functoriality)**.** *Let $f\colon X \to Y$ be a morphism between smooth projective varieties. Let $\omega_f \coloneqq \omega_X \otimes f^*\omega_Y^\vee$ be the relative canonical bundle, and let $d_f \coloneqq \dim(X) - \dim(Y)$ be the relative dimension.*

For integers $k, m \in \mathbb{Z}$ and an element $\sigma \in \mathrm{H}^{m+(1-k)d_f}(X, \omega_f^{\otimes(1-k)})$, there is a natural morphism for any $i \in \mathbb{Z}$:

$$f_*^\sigma\colon \mathrm{HS}^i_k(X) \to \mathrm{HS}^{i+m}_k(Y). \tag{200}$$

Proof. First note that we have canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}_Y(\mathbf{R}f_*(\omega_X[d_X])^{\otimes k}, (\omega_Y[d_Y])^{\otimes k}[m]) &\cong \mathrm{Hom}_X(\omega_X^{\otimes k}, f^!\omega_Y^{\otimes k}[m - kd_f]) \\ &\cong \mathrm{Hom}_X(\omega_X^{\otimes k}, f^*\omega_Y^{\otimes k} \otimes \omega_f[d_f + m - kd_f]) \\ &\cong \mathrm{H}^m(X, (\omega_f[d_f])^{\otimes(1-k)}). \end{aligned} \tag{201}$$

Hence we can view the element σ as a morphism in $\mathbf{D}^{\mathrm{b}}(Y)$

$$\sigma\colon \mathbf{R}f_*(\omega_X[d_X])^{\otimes k} \to (\omega_Y[d_Y])^{\otimes k}[m]. \tag{202}$$

Now given an element $\alpha \in \mathrm{HS}^i_k(X)$, viewed as a morphism in $\mathbf{D}^{\mathrm{b}}(X \times X)$,

$$\alpha\colon \Delta_{X,*}\mathcal{O}_X \to \Delta_{X,*}(\omega_X[d_X])^{\otimes k}[i], \tag{203}$$

we apply the derived pushforward along $f \times f\colon X \times X \to Y \times Y$ to get

$$\mathbf{R}(f \times f)_*\Delta_{X,*}\mathcal{O}_X \xrightarrow{\mathbf{R}(f\times f)_*(\alpha)} \mathbf{R}(f \times f)_*\Delta_{X,*}(\omega_X[d_X])^{\otimes k}[i]. \tag{204}$$

Since $(f \times f) \circ \Delta_X = \Delta_Y \circ f$, we have by functoriality

$$\Delta_{Y,*}\mathbf{R}f_*\mathcal{O}_X \xrightarrow{\mathbf{R}(f\times f)_*(\alpha)} \Delta_{Y,*}\mathbf{R}f_*(\omega_X[d_X])^{\otimes k}[i]. \tag{205}$$

Using the natural map $\mathcal{O}_Y \to \mathbf{R}f_*\mathcal{O}_X$ and the morphism (202) we get a morphism

$$\Delta_{Y,*}\mathcal{O}_Y \to \Delta_{Y,*}\mathbf{R}f_*\mathcal{O}_X \xrightarrow{\mathbf{R}(f\times f)_*(\alpha)} \Delta_{Y,*}\mathbf{R}f_*(\omega_X[d_X])^{\otimes k}[i] \xrightarrow{\Delta_{Y,*}(\sigma)} \Delta_{Y,*}(\omega_Y[d_Y])^{\otimes k}[i+m], \tag{206}$$

which can be viewed as an element in $\mathrm{HS}_k^{i+m}(Y)$, and defined to be the image of α. It is clear that the obtained map $\mathrm{HS}_k^i(X) \to \mathrm{HS}_k^{i+m}(Y)$ is linear. □

In the above proposition, the case $k = 1$ and $m = 0$ (with σ the canonical element of $\mathrm{H}^0(X, \mathcal{O}_X)$ corresponding to the constant function 1) should recover the covariant functoriality of Hochschild homology for a morphism. Another interesting instance is the case where f is étale.

Corollary A.8 (Étale pushforward)**.** *An étale morphism $f\colon X \to Y$ between smooth projective varieties induces a natural morphism*

$$f_*\colon\ \mathrm{HS}_k^i(X) \to \mathrm{HS}_k^i(Y). \tag{207}$$

Proof. Take $m = 0$. It suffices note that $d_f = 0$ and $\omega_f \cong \mathcal{O}_X$ in this case. Here σ is taken to be the canonical element corresponding to the constant function 1. □

Remark A.9. Using the Hochschild–Kostant–Rosenberg decomposition for Hochschild–Serre cohomology (37), one can get an a priori different covariant functoriality for Hochschild–Serre cohomology. Let the assumption be as in Proposition A.7. Then for each $p, q \in \mathbb{Z}$, we have natural morphisms

$$\begin{aligned}
\mathrm{H}^p(X, \textstyle\bigwedge^q \mathrm{T}_X \otimes \omega_X^{\otimes k}) &\cong \mathrm{H}^p(Y, \mathbf{R}f_*(\textstyle\bigwedge^q \mathrm{T}_X \otimes \omega_X^{\otimes k}))\\
&\to \mathrm{H}^p(Y, \mathbf{R}f_*(f^* \textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_X^{\otimes k}))\\
&\cong \mathrm{H}^p(Y, \textstyle\bigwedge^q \mathrm{T}_Y \otimes \mathbf{R}f_*(\omega_X^{\otimes k}))\\
&\to \mathrm{H}^p(Y, \textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k}[m - kd_f])\\
&= \mathrm{H}^{p+m-kd_f}(Y, \textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k}),
\end{aligned} \tag{208}$$

where the last morphism uses the input datum $\sigma \in \mathrm{H}^m(X, (\omega_f[d_f])^{\otimes(1-k)})$ as in (202). Taking the direct sum over all p, q with $p + q = i + kd_X$, or equivalently $p + q + m - kd_f = i + m + kd_Y$, by (37), we get a morphism

$${}'f_*^\sigma\colon\ \mathrm{HS}_k^i(X) \to \mathrm{HS}_k^{i+m}(Y). \tag{209}$$

It is an interesting question to compare ${}'f_*^\sigma$ with f_*^σ, or rather, in case they are different, to find a suitable modified isomorphism as in (37) to make them equal.

As for contravariant functoriality, we have the following. Observe that strictly speaking we prove étale functoriality only *after* the Hochschild–Kostant–Rosenberg decomposition for Hochschild–Serre cohomology, but this suffices to get contravariant étale functoriality on the level of vector spaces.

Proposition A.10 (Étale pullback)**.** *Let $f\colon X \to Y$ be an étale morphism between smooth projective varieties. Then f induces a natural morphism*

$$f^*\colon\ \mathrm{HS}_k^i(Y) \to \mathrm{HS}_k^i(X). \tag{210}$$

Proof. For any $p, q \in \mathbb{Z}$, we have the following natural morphisms:

$$\begin{aligned}
\mathrm{H}^p(Y, \textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k}) &\to \mathrm{H}^p(Y, \mathbf{R}f_*\mathcal{O}_X \otimes \textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k})\\
&\cong \mathrm{H}^p(Y, \mathbf{R}f_*f^*(\textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k}))\\
&\cong \mathrm{H}^p(X, f^*(\textstyle\bigwedge^q \mathrm{T}_Y \otimes \omega_Y^{\otimes k}))\\
&\cong \mathrm{H}^p(X, \textstyle\bigwedge^q \mathrm{T}_X \otimes \omega_X^{\otimes k}),
\end{aligned} \tag{211}$$

where the first morphism is induced by the natural map $\mathcal{O}_Y \to \mathbf{R}f_*\mathcal{O}_X$, and the last isomorphism uses that f is étale hence induces isomorphisms $f^*\mathrm{T}_Y \cong \mathrm{T}_X$ and $f^*\omega_Y \cong \omega_X$.

Taking the direct sum over p, q with $p + q = i + kd_X = i + kd_Y$, and use (37), we get the desired morphism f^*. □

B Computations for Section 4.2

In this appendix we collect the computations for Proposition 4.4 and Remark 4.5. We will denote $\mathbb{P} = \mathbb{P}^n = \mathbb{P}(V)$ where V is an $(n+1)$ -dimensional vector space V. We write

$$
\begin{aligned}
G &:= \mathrm{Gr}(2, V) \\
H &:= \mathrm{Hilb}^2 \mathbb{P}^n,
\end{aligned}
\tag{212}
$$

where the former comes equipped with the tautological sub- and quotient bundles $\mathcal{S}$ and $\mathcal{Q}$.

We are mostly interested in the case $n = 2$ (for the proof of Proposition 4.4), where we have $G = \mathbb{P}^{2,\vee}$, and $\mathcal{S} \cong \Omega^1_G(1)$ resp. $\mathcal{Q} \cong \mathcal{O}_G(1)$. The methods in this section can be used more generally to compute (pieces of) the Hochschild–Kostant–Rosenberg decomposition of $\mathrm{HH}^*(\mathrm{Hilb}^2 \mathbb{P}^n)$, but we will not work out all the details, and we are content with establishing the rigidty result from Remark 4.5 for $n \geqslant 3$.

Consider the morphism

$$\pi \colon H \to G \tag{213}$$

obtained by sending 2 points, possibly infinitesimally near, to the line they span. The following lemma (for n arbitrary) is probably well-known, and a weaker version was already used in [7, §5]. For $n = 2$ it also appears in [64, §3.2].

Lemma B.1. *The morphism π in* (213) *is identified with the $\mathbb{P}^2$-bundle $\mathbb{P}_G(\mathrm{Sym}^2 \mathcal{S}) \to G$.*

Proof. We can write H as $(\mathbb{P}_G(\mathcal{S}) \times_G \mathbb{P}_G(\mathcal{S}))/(\mathbb{Z}/2\mathbb{Z})$. We can rewrite this as

$$
\begin{aligned}
(\mathbb{P}_G(\mathcal{S}) \times_G \mathbb{P}_G(\mathcal{S}))/(\mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Proj}_G(\mathrm{Sym}^\bullet \mathcal{S}^\vee) \times_G \mathrm{Proj}_G(\mathrm{Sym}^\bullet \mathcal{S}^\vee)/(\mathbb{Z}/2\mathbb{Z}) \\
&\cong \mathrm{Proj}_G(\mathrm{Sym}^\bullet \mathcal{S}^\vee \otimes \mathrm{Sym}^\bullet \mathcal{S}^\vee)/(\mathbb{Z}/2\mathbb{Z}) \\
&\cong \mathrm{Proj}_G(\mathrm{Sym}^2 \mathrm{Sym}^\bullet \mathcal{S}^\vee) \\
&\cong \mathrm{Proj}_G(\mathrm{Sym}^\bullet \mathrm{Sym}^2 \mathcal{S}^\vee) \\
&\cong \mathbb{P}_G(\mathrm{Sym}^2 \mathcal{S}),
\end{aligned}
\tag{214}
$$

where we used Hermite reciprocity (e.g., as in [26, Exercise 6.18]) which states that taking Sym^q and Sym^p of a rank-2 bundle commutes for all p and q. □

From Lemma B.1 we obtain the relative Euler sequence

$$0 \to \mathcal{O}_H \to \pi^*(\mathrm{Sym}^2 \mathcal{S}) \otimes \mathcal{O}_\pi(1) \to \mathrm{T}_\pi \to 0 \tag{215}$$

and the relative tangent sequence

$$0 \to \mathrm{T}_\pi \to \mathrm{T}_H \to \pi^*\mathrm{T}_G \to 0. \tag{216}$$

Let us first compute the cohomology of the tangent bundle for H. The following lemma is standard, where we denote by $\mathbb{S}_\lambda$ the Schur functor associated to a partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_\ell)$ of length ℓ. Recall that $\mathbb{S}_\lambda \mathcal{E}^\vee \cong \mathbb{S}_\mu \mathcal{E}$ where μ is $-\lambda$ reordered so that the entries are decreasing.

Lemma B.2. *We have that*

$$\mathrm{H}^\bullet(G, \mathrm{T}_G) \cong \mathrm{H}^0(G, \mathrm{T}_G) \cong \mathbb{S}_{(1,0,\ldots,0,-1)} V \tag{217}$$

and

$$\mathrm{H}^\bullet(G, \bigwedge^2 \mathrm{T}_G) \cong \mathrm{H}^0(G, \bigwedge^2 \mathrm{T}_G) \cong \begin{cases} \mathbb{S}_{(2,-1,-1)} V & n = 2 \\ \mathbb{S}_{(2,0,\ldots,-1,-1)} V \oplus \mathbb{S}_{(1,1,0,\ldots,0,-2)} V & n \geqslant 3. \end{cases} \tag{218}$$

Proof. This conveniently follows from the description in [8, Theorem B] as the Grassmannian G is cominuscule. □

Now we compute the cohomology of the first term in the relative tangent sequence (216).

Lemma B.3. *We have that*

$$\begin{aligned}\mathbf{R}\pi_*\mathrm{T}_\pi &\cong \mathrm{R}^0\pi_*\mathrm{T}_\pi \\ &\cong \mathbb{S}_{(2,-2)}\mathcal{S}\oplus\mathbb{S}_{(1,-1)}\mathcal{S},\end{aligned} \tag{219}$$

and

$$\mathrm{H}^\bullet(H,\mathrm{T}_\pi)\cong\begin{cases}\mathrm{H}^1(H,\mathrm{T}_\pi)\cong\mathbb{S}_{(1,1,-2)}V & n=2\\ 0 & n\geqslant 3.\end{cases} \tag{220}$$

Proof. By applying $\mathbf{R}\pi_*$ to the relative Euler sequence (215) we obtain, using the identifications $\mathbf{R}\pi_*\mathcal{O}_H\cong\mathcal{O}_G$ and $\mathbf{R}\pi_*\mathcal{O}_\pi(1)\cong\mathrm{Sym}^2\mathcal{S}^\vee$, the short exact sequence

$$0\to\mathcal{O}_G\to\mathrm{Sym}^2\mathcal{S}\otimes\mathrm{Sym}^2\mathcal{S}^\vee\to\mathrm{R}^0\pi_*\mathrm{T}_\pi\to 0. \tag{221}$$

The middle term can be rewritten as

$$\begin{aligned}\mathrm{Sym}^2\mathcal{S}\otimes\mathrm{Sym}^2\mathcal{S}^\vee &\cong \mathrm{Sym}^2\mathcal{S}\otimes\mathrm{Sym}^2\mathcal{S}\otimes(\det\mathcal{S})^{\otimes -2}\\ &\cong(\mathrm{Sym}^4\mathcal{S}\otimes(\det\mathcal{S})^{\otimes -2})\oplus(\mathrm{Sym}^2\mathcal{S}\otimes(\det\mathcal{S})^{\otimes -1})\oplus\mathcal{O}_G\end{aligned} \tag{222}$$

where we used [26, Exercise 11.11]. The inclusion in (221) allows us to cancel $\mathcal{O}_G$ in (222). The rest follows by applying Borel–Weil–Bott. □

Combining these two lemmas we obtain the following from the long exact sequence associated to the relative tangent sequence (216).

Corollary B.4. *If $n=2$ then*

$$\mathrm{H}^i(H,\mathrm{T}_H)\cong\begin{cases}\mathbb{S}_{(1,0,-1)}V & i=0\\ \mathbb{S}_{(1,1,-2)}V & i=1\\ 0 & i\geqslant 2\end{cases} \tag{223}$$

whilst for $n\geqslant 3$

$$\mathrm{H}^i(H,\mathrm{T}_H)\cong\begin{cases}\mathbb{S}_{(1,0,\ldots,0,-1)}V & i=0\\ 0 & i\geqslant 1.\end{cases} \tag{224}$$

Next we compute the cohomology of $\bigwedge^2\mathrm{T}_H$. The second exterior square of (216) produces a filtration with associated graded pieces

- $\bigwedge^2\mathrm{T}_\pi$,
- $\mathrm{T}_\pi\otimes\pi^*\mathrm{T}_G$, and
- $\pi^*\bigwedge^2\mathrm{T}_G$.

The cohomology of the first is computed as follows.

Lemma B.5. *We have that*

$$\begin{aligned}\mathbf{R}\pi_*\bigwedge^2\mathrm{T}_\pi &\cong \mathrm{R}^0\pi_*\bigwedge^2\mathrm{T}_\pi\\ &\cong\mathbb{S}_{(3,-3)}\mathcal{S}\oplus\mathbb{S}_{(1,-1)}\mathcal{S}.\end{aligned} \tag{225}$$

In particular we have for $n=2$ that

$$\mathrm{H}^\bullet(H,\bigwedge^2\mathrm{T}_\pi)\cong\mathrm{H}^1(H,\bigwedge^2\mathrm{T}_\pi)\cong\mathbb{S}_{(2,1,-3)}V. \tag{226}$$

Proof. The second exterior square of the relative Euler sequence (215) induces the short exact sequence

$$0 \to \mathrm{T}_\pi \to \pi^*(\bigwedge^2 \mathrm{Sym}^2 \mathcal{S}) \otimes \mathcal{O}_\pi(2) \to \bigwedge^2 \mathrm{T}_\pi \to 0. \tag{227}$$

By applying $\mathbf{R}\pi_*$ to it we obtain, using the description of $\mathbf{R}\pi_*\mathrm{T}_\pi$ from Lemma B.3, the short exact sequence

$$0 \to \mathbb{S}_{(2,-2)}\mathcal{S} \oplus \mathbb{S}_{(1,-1)}\mathcal{S} \to \bigwedge^2 \mathrm{Sym}^2 \mathcal{S} \otimes \mathrm{Sym}^2 \mathrm{Sym}^2 \mathcal{S}^\vee \to \pi_*(\bigwedge^2 \mathrm{T}_\pi) \to 0. \tag{228}$$

The vanishing of $\mathrm{R}^{\geqslant 1}\pi_*(\bigwedge^2 \mathrm{T}_\pi)$ can be shown using cohomology and base change. Using Lemma B.3 and the isomorphisms

$$\begin{aligned} \bigwedge^2 \mathrm{Sym}^2 \mathcal{S} &\cong \mathbb{S}_{(3,1)}\mathcal{S} \\ \mathrm{Sym}^2 \mathrm{Sym}^2 \mathcal{S}^\vee &\cong \mathbb{S}_{(4,0)}\mathcal{S}^\vee \oplus \mathbb{S}_{(2,2)}\mathcal{S}^\vee \end{aligned} \tag{229}$$

which are standard plethysms, we can rewrite (228) as

$$0 \to \mathbb{S}_{(2,-2)}\mathcal{S} \oplus \mathbb{S}_{(1,-1)}\mathcal{S} \to \mathbb{S}_{(3,-3)}\mathcal{S} \oplus \mathbb{S}_{(2,-2)}\mathcal{S} \oplus (\mathbb{S}_{(1,-1)}\mathcal{S})^{\oplus 2} \to \pi_*(\bigwedge^2 \mathrm{T}_\pi) \to 0. \tag{230}$$

As all the morphisms in this sequence are equivariant we can cancel the corresponding summands and thus we obtain the first part of the lemma. The rest is an application of the Borel–Weil–Bott theorem. ◻

For the second graded piece we have the following.

Lemma B.6. *We have that*

$$\begin{aligned} \mathbf{R}\pi_*(\mathrm{T}_\pi \otimes \pi^*\mathrm{T}_G) &\cong \mathrm{R}^0\pi_*(\mathrm{T}_\pi \otimes \pi^*\mathrm{T}_G) \\ &\cong \mathcal{Q} \otimes \left(\mathbb{S}_{(2,-3)}\mathcal{S} \oplus (\mathbb{S}_{(1,-2)}\mathcal{S})^{\oplus 2} \oplus \mathcal{S}^\vee\right). \end{aligned} \tag{231}$$

In particular we have for $n = 2$ that

$$\mathrm{H}^\bullet(H, \mathrm{T}_\pi \otimes \pi^*\mathrm{T}_G) \cong \mathrm{H}^0(H, \mathrm{T}_\pi \otimes \pi^*\mathrm{T}_G) \cong (\mathbb{S}_{(1,1,-2)}V)^{\oplus 2} \oplus \mathbb{S}_{(1,0,-1)}V. \tag{232}$$

We leave the case $n \geqslant 3$ to the interested reader, here, and in what follows.

Proof. Using the description from Lemma B.3 we obtain that the derived direct image is concentrated in degree zero, and that it is isomorphic to

$$(\mathbb{S}_{(2,-2)}\mathcal{S} \oplus \mathbb{S}_{(1,-1)}\mathcal{S}) \otimes \mathcal{S}^\vee \otimes \mathcal{Q}, \tag{233}$$

using $\mathrm{T}_G \cong \mathcal{S}^\vee \otimes \mathcal{Q}$. This proves the first part of the lemma. The rest is an application of the Borel–Weil–Bott theorem. ◻

Because the cohomology of the third term can be computed using Lemma B.2 we can use the filtration and the previous two lemmas to obtain the following.

Corollary B.7. *If $n = 2$ then*

$$\mathrm{H}^i(H, \bigwedge^2 \mathrm{T}_H) \cong \begin{cases} (\mathbb{S}_{(1,1,-2)}V)^{\oplus 2} \oplus \mathbb{S}_{(1,0,-1)}V \oplus \mathbb{S}_{(2,-1,-1)}V & i = 0 \\ \mathbb{S}_{(2,1,-3)}V & i = 1 \\ 0 & i \geqslant 2 \end{cases} \tag{234}$$

Finally, to compute the cohomology of $\bigwedge^3 \mathrm{T}_H$ one could use the methods used before in the proof of Corollary B.4, starting from the isomorphism $\bigwedge^3 \mathrm{T}_H \cong \Omega^1_H \otimes \omega_H^\vee$, and twisting the duals of (215) and (216) by $\omega_H^\vee \cong \mathcal{O}_\pi(3)$. But by bootstrapping from Lemma 4.3 we can now also compute the cohomology of $\bigwedge^3 \mathrm{T}_H$ using a shortcut. First we need the following Euler characteristic calculation.

Lemma B.8. *We have that*

$$\chi(H, \bigwedge^3 \mathrm{T}_H) = 52. \tag{235}$$

Proof. This can be computed using the Schubert2 package of Macaulay2 [32]:

```
loadPackage "Schubert2";
G := flagBundle({2, 1});
H := projectiveBundle symmetricPower(2, G.SubBundles#1);
chi(exteriorPower(3, tangentBundle H))
```

□

Corollary B.9. *We have that*

$$\mathrm{H}^i(H, \bigwedge^3 \mathrm{T}_H) \cong \begin{cases} \mathbf{k}^{80} & i = 0 \\ \mathbf{k}^{28} & i = 1 \\ 0 & i \geqslant 2. \end{cases} \tag{236}$$

Proof. By Lemma 4.3 we get the vanishing for $i \geqslant 3$. Combining Corollaries B.4 and B.7 with Lemma 4.3 we obtain $\mathrm{H}^0(H, \bigwedge^3 \mathrm{T}_H) \cong \mathbf{k}^{80}$, thus Lemma B.8 allows us to compute H^1. □

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