MOTIVIC HYPERKÄHLER RESOLUTION CONJECTURE : II. HILBERT SCHEMES OF K3 SURFACES

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Abstract. This is the second piece of work on the investigation of the *Motivic HyperKähler Resolution Conjecture* (MHRC for short) and its applications. The conjecture says that given a smooth projective holomorphic symplectic variety M endowed with an action of a finite group G by symplectic automorphisms, the orbifold Chow ring (resp. orbifold Chow motive) of the quotient stack $[M/G]$ is isomorphic as $\mathbf{C}$-algebras (resp. algebra objects in the category of Chow motives) to the usual Chow ring (resp. Chow motivic algebra) of any symplectic resolution of M/G. In this paper, we prove the MHRC, as well as its K-theoretic version proposed by Jarvis-Kaufmann-Kimura, in the case of Hilbert-Chow resolution $S^{[n]} \to S^{(n)}$ for the n-th symmetric product of a projective K3 surface S. In particular, we obtain an explicit description of the ring structure of $\mathrm{CH}^*(S^{[n]})$ in terms of the Chow rings of self-products of S. As an application, we prove the weak splitting conjecture of Beauville and a stronger version proposed by Voisin in some new cases. Moreover, we also give a refinement of the result of Vial on the multiplicative Chow-Künneth decomposition, which is conjectured by Beauville for any hyperKähler variety, of the Hilbert scheme of K3 surfaces. A feature of the proof is the use of Gromov-Witten theory together with Voisin's theory on universally defined cycles.

Contents

Lie Fu is supported by the Agence Nationale de la Recherche (ANR) through ECOVA and LABEX MILYON (ANR-10-LABX-0070) within the program *Investissements d'Avenir* (ANR-11-IDEX-0007). Zhiyu Tian is supported by the project IDEX "IRS-Initiative de Recherche Stratégique" of Université Grenoble-Alpes. Lie Fu and Zhiyu Tian are supported by the Agence Nationale de la Recherche (ANR) through HodgeFun, and by *Projet Exploratoire Premier Soutien* (PEPS) Jeunes chercheur-e-s 2016 operated by Insmi and *Projet Inter-Laboratoire* 2016 and 2017 by Fédération de Recherche en Mathématiques Rhône-Alpes/Auvergne CNRS 3490.

1. Introduction

Considerations from string theory of orbifolds ([21], [22]) lead to the general philosophy that (stringy) topological invariants of an orbifold should be equal, or at least related to the corresponding invariants of a crepant resolution. Let us mention the work [3], [4], [64] and [43] on Euler numbers and Hodge numbers/structures. In [54], [55], Ruan puts this relation into a bigger picture of *stringy topology / geometry*, relating the quantum cohomology theory of an orbifold to that of its crepant resolutions. More precisely, among other deep speculations, he makes in [55] the following *Cohomological Crepant Resolution Conjecture* (see [15], [19] for more sophisticated versions):

Conjecture 1.1 (Ruan's CCRC)**.** *Let $\mathcal{X}$ be a smooth compact complex orbifold with underlying (possibly singular) variety X. Assume that X verifies the* hard Lefschetz condition[1]*. If there is a crepant resolution $Y \to X$, then we have an isomorphism of graded $\mathbf{C}$-algebras: $H^*_{qc}(Y, \mathbf{C}) \simeq H^*_{orb}(\mathcal{X}, \mathbf{C})$.*

Here on the left hand side, H^*_{qc} is the *quantum corrected* cohomology, whose underlying graded vector space is the same as the singular cohomology while the product is a modification of the cup product by the Gromov-Witten invariants of contracted rational curve classes (see §5) ; on the right hand side, H^*_{orb} is the *orbifold cohomology* defined by Chen and Ruan in [16] (see [1] for an algebro-geometric construction), whose underlying graded vector space is the singular cohomology of the inertia stack with degree shifted by *ages* while the product structure is highly non-trivial. We will give a detailed and down-to-earth definition as well as its generalization to the Chow setting in the global quotient case in §2, following [23], [31], [33].

A special case of Conjecture 1.1 is particularly interesting: when the crepant resolution Y is *hyperKähler* (or more generally *holomorphic symplectic*), as all Gromov-Witten invariants of Y vanish, there are no quantum corrections at all. Moreover the hard Lefschetz condition is always satisfied in the hyperKähler situation (see Remark 3.3). We get in this case the following *Cohomological Hyperkähler Resolution Conjecture* of [54]:

Conjecture 1.2 (Ruan's CHRC)**.** *Let $\mathcal{X}$ be a smooth compact complex orbifold with underlying (possibly singular) variety X. If there is a crepant resolution $Y \to X$ with Y being hyperKähler, then we have an isomorphism of graded $\mathbf{C}$-algebras: $H^*(Y, \mathbf{C}) \simeq H^*_{orb}(\mathcal{X}, \mathbf{C})$.*

The known results on CHRC are quite limited: (*a*) The cases of Hilbert schemes of K3 surfaces and abelian surfaces are proved by Fantechi-Göttsche [23], and independently by Uribe [58], based on Lehn-Sorger [37]. (*b*) The case of generalized Kummer varieties is proved as a byproduct in our previous work [28, Theorem 1.5] based on Nieper-Wisskirchen [48]. (c) If one drops the compactness condition in Conjecture 1.2, the 'local' case of a symplectic resolution of a symplectic vector space quotient by a finite group of symplectic automorphisms is proved in Ginzburg-Kaledin [29, Theorem 1.2].

In the recent joint work with Charles Vial, we propose to study [28, Conjecture 1.2] the *motivic* version of Ruan's HyperKähler Resolution Conjecture (MHRC). Here is the global quotient case [28, Conjecture 3.2] which is already broad enough to contain many situations that we want to investigate:

Conjecture 1.3 (Motivic HyperKähler Resolution Conjecture [28])**.** *Let M be a projective holomorphic symplectic manifold and G be a finite subgroup of the group of symplectic automorphisms of M. If $Y \to M/G$*

[1] This condition is included later in Bryan and Graber's work [15] based on the computations of [18].

is a symplectic resolution of singularities, then we have an isomorphism of algebra-objects in the category of complex Chow motives:

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb}([M/G]) \quad in \quad \mathrm{CHM}_{\mathbf{C}}.$$

In particular, there is an isomorphism of graded **C***-algebras:*

$$\mathrm{CH}^*(Y)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([M/G])_{\mathbf{C}}.$$

In [31, Conjecture 1.2], Jarvis, Kaufmann and Kimura propose a closely related conjecture (KHRC) from K-theoretic point of view:

Conjecture 1.4 (K-theoretic HyperKähler Resolution Conjecture [31])**.** *In the same situation as in Conjecture 1.3, we have an isomorphism of* **C***-algebras:*

$$K(Y)_{\mathbf{C}} \simeq K_{orb}([M/G])_{\mathbf{C}}.$$

The basic notions of symplectic singularities and symplectic resolutions are recalled in Definition 3.1. Let us explain briefly the construction of orbifold Chow rings CH^*_{orb} appeared above and refer to §2 for the details and the extension to orbifold Chow motives $\mathfrak{h}_{orb}$ and orbifold K-theory K_{orb}. Let M and G be as in Conjecture 1.3. The *orbifold Chow ring* of the stack $[M/G]$ is defined to be the subring of G-invariants:

$$\mathrm{CH}^*_{orb}([M/G]) := \left(\bigoplus_{g\in G} \mathrm{CH}^{*-\mathrm{age}(g)}(M^g), \star_{orb}\right)^G,$$

where for any $g \in G$, M^g is the fixed locus of the symplectic automorphism g, $\mathrm{age}(g)$ is some locally constant integer-valued function determined by the local data of the quotient singularities (Definition 2.1) and the G-action is the natural one ; while the orbifold product $\star_{orb}$, which is compatible with the G-grading, age-shifting and G-action, is given by the following rule: let $g, h \in G$ and $\alpha \in \mathrm{CH}^{i-\mathrm{age}(g)}(M^g)$, $\beta \in \mathrm{CH}^{j-\mathrm{age}(h)}(M^h)$, then

$$\alpha \star_{orb} \beta := \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right) \in \mathrm{CH}^{i+j-\mathrm{age}(gh)}(M^{gh}), \tag{1}$$

where $M^{<g,h>} = M^g \cap M^h$ (with the reduced structure), $\iota : M^{<g,h>} \hookrightarrow M$ is the natural embedding and $F_{g,h}$ is the *obstruction bundle* on $M^{<g,h>}$ whose class in $K_0(M^{<g,h>})_{\mathbf{Q}}$ involves natural inclusions and various normal bundles between fixed loci (see §2, Definiton 2.2).

There are many interesting symplectic resolutions which fit into the context of Motivic HyperKähler Resolution Conjecture 1.3. For instance,

Example 1.5. Beauville [6] provides some fundamental examples:

(1) Let S be a projective K3 surface or an abelian surface. Let $G = \mathfrak{S}_n$ act naturally on $M = S^n$ by permutations, then the n-th *Hilbert scheme* $S^{[n]}$ together with the Hilbert-Chow morphism

$$\rho : S^{[n]} \to S^{(n)}$$

is a symplectic resolution of the n-th symmetric product of S.

(2) Let A be an abelian surface. Let $G = \mathfrak{S}_{n+1}$ act naturally on $M = \mathrm{Ker}\left(+ : A^{n+1} \to A\right)$, then the *generalized Kummer variety* $K_n(A)$ together with the restriction the Hilbert-Chow morphism

$$\rho : K_n(A) \to M/G$$

is a symplectic resolution.

The MHRC 1.3 is proved in [28] in the cases of Hilbert schemes of abelian surfaces and generalized Kummer varieties. The main result of the paper is the following, confirming Conjecture 1.3 and Conjecture 1.4 in the case of Hilbert schemes of projective K3 surfaces:

Theorem 1.6 (=Theorem 3.4+ Corollary 3.5. MHRC and KHRC for Hilbert schemes of K3 surfaces)**.** *Let S be a projective K3 surface. Let the symmetric group $\mathfrak{S}_n$ act on S^n by permutations. We have an isomorphism of algebra objects in the category of complex Chow motives:*

$$\mathfrak{h}\left(S^{[n]}\right) \simeq \mathfrak{h}_{orb}\left([S^n/\mathfrak{S}_n]\right) \ \ \textit{in} \ \ \mathrm{CHM}_{\mathbf{C}}.$$

In particular, there are isomorphisms of (graded in the first one) $\mathbf{C}$*-algebras:*

$$\mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{C}};$$

$$K(S^{[n]})_{\mathbf{C}} \simeq K_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}}.$$

Remark 1.7 (Coefficient fields and discrete torsion)**.** In fact, one can replace in the statement the coefficient field $\mathbf{C}$ by $\mathbf{Q}(\sqrt{-1})$, because the only (non-real) complex coefficients appeared in the proof, as well as the result that we use [40], are $\pm\sqrt{-1}$. We want to warn the reader that the statements are no longer true if we use rational coefficients! However, a sign change in the definition of the orbifold product will obtain the so-called orbifold Chow motive (*resp.* Chow ring, K-theory, cohomology) *with discrete torsion*, denoted by $\mathfrak{h}_{orb,dt}$, $\mathrm{CH}^*_{orb,dt}$ *etc.* and Theorem 1.6 can be restated as the isomorphism of algebra objects $\mathfrak{h}\left(S^{[n]}\right) \simeq \mathfrak{h}_{orb,dt}\left([S^n/\mathfrak{S}_n]\right)$ in $\mathrm{CHM}_{\mathbf{Q}}$ and isomorphisms of $\mathbf{Q}$-algebras $\mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{Q}} \simeq \mathrm{CH}^*_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}}$ and $K(S^{[n]})_{\mathbf{Q}} \simeq K_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{Q}}$ for all projective K3 surface S, see Corollary 1.9. For a more careful treatment of discrete torsions, see [31, §7] and our previous work [28] where the main results are stated in this context.

Remark 1.8. The CHRC 1.2 for Hilbert schemes of K3 surfaces is proved by Fantechi-Göttsche [23], and independently by Uribe [58], using the result of Lehn-Sorger [37]. Our proof is not an alternative one for CHRC in this case. Actually, one essential ingredient of our proof is the (far-reaching) strengthening of their results due to Li-Qin [40] proving the CCRC 1.1 for the Hilbert-Chow resolutions for *all* simply-connected surfaces.

Let us talk about our original motivations and some applications of this work.

1.1. **Motivation (I) : Multiplication table of Chow ring.** Let S be a projective K3 surface, as a hyperKähler variety, $S^{[n]}$ has infinite dimensional Chow groups ([45]). As far as the authors' knowledge goes, unlike the situation for cohomology $H^*(S^{[n]}, \mathbf{Q})$ (*cf.* [37]), the ring structure of $\mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{Q}}$ was poorly understood. The case of $n=2$ is classical; the case of $n=3$ can be worked out from the geometric construction of the Hilbert cube (*cf.* [57]). It was open for $n \geq 4$ due to the lack of explicit construction for $S^{[n]}$. However if we take the Chow rings of self-products of S as basic information, Theorem 1.6 provides the following complete and explicit description, namely a multiplication table, of the Chow ring $\mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{Q}}$ for all n. The slogan is that *the intersection product of Chow ring is given by exactly the same formula for the cup product of cohomology ring*. More precisely:

Corollary 1.9 (Ring structure of $\mathrm{CH}^*\left(S^{[n]}\right)$)**.** *Let S be a projective K3 surface and $n \in \mathbf{N}$. Using the following injective map of De Cataldo-Migliorini* [20] *(see also §4)*

$$\phi : \mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{Q}} \hookrightarrow \bigoplus_{g \in \mathfrak{S}_n} \mathrm{CH}^{*-\mathrm{age}(g)}\left((S^n)^g\right)_{\mathbf{Q}},$$

the intersection product on $\mathrm{CH}^*(S^{[n]})_{\mathbf{Q}}$ *is determined as follows: for any* $g, h \in G$ *and* $\alpha \in \mathrm{CH}^{i-\mathrm{age}(g)}((S^n)^g)$, $\beta \in \mathrm{CH}^{j-\mathrm{age}(h)}((S^n)^h)$, *then*

$$\alpha \star \beta = \epsilon(g,h) \cdot \iota_* \left(\alpha|_{(S^n)^{<g,h>}} \cdot \beta|_{(S^n)^{<g,h>}} \cdot c_{g,h} \right) \in \mathrm{CH}^{i+j-\mathrm{age}(gh)}((S^n)^{gh}),$$

where $\epsilon(g,h) := (-1)^{\frac{\mathrm{age}(g)+\mathrm{age}(h)-\mathrm{age}(gh)}{2}}$ *is a sign change (discrete torsion),* $c_{g,h}$ *is the obstruction class computed in (37) and* $\iota : (S^n)^{<g,h>} \hookrightarrow (S^n)^{gh}$ *is the inclusion.*

To illustrate the effectiveness of Corollary 1.9, we work out some interesting intersection products in §9.

1.2. **Motivation (II) : Beauville's splitting property.** The starting point of this whole story is the following fundamental result of Beauville and Voisin [10, Theorem 1 and Proposition 3.2] on the Chow rings of K3 surfaces:

Theorem 1.10 (Beauville-Voisin [10])**.** *Let S be a projective K3 surface. There exists a canonical element* $c_S \in \mathrm{CH}_0(S)$ *of degree 1, such that*

(1) $\mathrm{Im}\left(\mathrm{CH}^1(S) \otimes \mathrm{CH}^1(S) \xrightarrow{\cdot} \mathrm{CH}_0(S)\right) = \mathbf{Z} \cdot c_S$*;*
(2) $c_2(T_S) = 24 c_S$ *in* $\mathrm{CH}_0(S)$*;*
(3) $\Delta_*(D) = D \times c_S + c_S \times D$ *in* $\mathrm{CH}_1(S \times S)_{\mathbf{Q}}$ *for any* $D \in \mathrm{CH}^1(S)$*;*
(4) $\delta_S - (\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}) + \left(\mathrm{pr}_1^* c_S + \mathrm{pr}_2^* c_S + \mathrm{pr}_3^* c_S\right) = 0$ *in* $\mathrm{CH}_2(S \times S \times S)_{\mathbf{Q}}$*, where* δ_S *is the small diagonal of* $S \times S \times S$*.*

Note that the canonical zero-cycle c_S, the so-called *Beauville-Voisin class*, is represented by any point on any rational curve on S. This theorem contrasts to Mumford's result ([45]) that $\mathrm{CH}_0(S)$ is *infinite dimensional* in a precise (and strong!) sense. In Theorem 1.10, the last equality (4) on the decomposition of the small diagonal implies the rest of the theorem, but its proof requires to establish at least (1) first. A key ingredient in the proof of Theorem 1.10 is the Bogomolov-Mumford theorem, documented in Mori-Mukai [44], on the existence of rational curves in linear systems of K3 surfaces.

Inspired by his result on algebraic cycles on abelian varieties [5] and [7], Beauville proposes in [9] to regard Theorem 1.10 as a canonical splitting of the Bloch-Beilinson filtration on Chow rings of K3 surfaces and he conjectures that such splittings should exist for all hyperKähler varieties :

Conjecture 1.11 (Splitting Property [9])**.** *Let X be a smooth projective holomorphic symplectic variety. Then there exists a natural multiplicative bigrading of the rational Chow ring* $\mathrm{CH}^*(X)_{\mathbf{Q}}$*: for any* $0 \le i \le \dim(X)$*, there is a finite direct sum decomposition*

$$\mathrm{CH}^i(X)_{\mathbf{Q}} = \bigoplus_s \mathrm{CH}^i(X)_s,$$

such that

- *(Multiplicativity)* $\mathrm{CH}^i(X)_s \cdot \mathrm{CH}^{i'}(X)_{s'} \subset \mathrm{CH}^{i+i'}(X)_{s+s'}$*;*
- *(Bloch-Beilinson) The associated filtration* $F^j\,\mathrm{CH}^i(X)_{\mathbf{Q}} := \oplus_{s \ge j} \mathrm{CH}^i(X)_s$ *satisfies the Bloch-Beilinson conjecture. In particular,*
  - ($F^0 = \mathrm{CH}$). *For any* $s < 0$*, we have* $\mathrm{CH}^i(X)_s = 0$ *;*
  - ($F^1 = \mathrm{CH}_{hom}$). *The restriction of the cycle class map* $\mathrm{cl} : \bigoplus_{s>0} \mathrm{CH}^i(X)_s \to H^{2i}(X, \mathbf{Q})$ *is zero ;*
  - *(Injectivity) The restriction of the cycle class map* $\mathrm{cl} : \mathrm{CH}^i(X)_0 \to H^{2i}(X, \mathbf{Q})$ *is injective.*

As the existence of the Bloch-Beilinson filtrations is wildly open (see [60, Conjecture 11.21]), Conjecture 1.11 is hard to work with directly[2]. Instead, we will be interested in two closely related conjectures (Conjecture 1.12 and Conjecture 1.15) whose statements are down-to-earth. The first one is the following Beauville's Weak Splitting Property in [9]. As is explained in *loc.cit.*, it follows easily from Conjecture 1.11.

Conjecture 1.12 (Weak Splitting Property [9]). *Let X be a smooth projective irreducible holomorphic symplectic variety. Then the restriction of the cycle class map to the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(X)_{\mathbf{Q}}$ generated by the divisors is injective :*

$$\mathrm{cl} : \langle\ \mathrm{CH}^1(X)\ \rangle \hookrightarrow H^*(X, \mathbf{Q}).$$

The Weak Splitting Property conjecture is investigated and further strengthened by Voisin in [61], where she also takes into account of Chern classes of the tangent bundle:

Conjecture 1.13 (Beauville-Voisin [9], [61]). *Let X be a smooth projective irreducible holomorphic symplectic variety. Then the restriction of the cycle class map to the $\mathbf{Q}$-subalgebra of $\mathrm{CH}^*(X)_{\mathbf{Q}}$ generated by the divisors and Chern classes of T_X is injective :*

$$\mathrm{cl} : \langle\ \mathrm{CH}^1(X), c_i(T_X); i \in \mathbf{N}\ \rangle \hookrightarrow H^*(X, \mathbf{Q}).$$

For known cases of Conjectures 1.12 and 1.13 see §10. We would like to prove in §10 the following partial result towards Weak Splitting Property Conjecture 1.12 and Beauville-Voisin Conjecture 1.13 for Hilbert schemes of projective K3 surfaces, improving the bound provided by Voisin [61].

Theorem 1.14. *Let S be a smooth projective K3 surface, n a natural number and $X := S^{[n]}$ be the Hilbert scheme of length-n subschemes on S. Then*

(1) If $n < 506$, then X satisfies the weak splitting property: the restriction of the cycle class map to the $\mathbf{Q}$-sub-algebra of $\mathrm{CH}^*(X)$ generated by divisors is injective.
(2) If $n < (b_{2,tr} + 1)(b_{2,tr} + 2)$, then X satisfies the Beauville-Voisin conjecture: the restriction of the cycle class map to the $\mathbf{Q}$-sub-algebra of $\mathrm{CH}^*(X)$ generated by divisors and Chern classes of T_X is injective.

The second conjecture closely related to Conjecture 1.11 is the following motivic enhancement of Beauville's original splitting property. The idea is that the multiplicative bigrading on Chow ring should come from a decomposition at the level of Chow motives. The key notion is the so-called *multiplicative Chow-Künneth decomposition* ([56]), see Definition 11.1.

Conjecture 1.15 (Motivic Splitting Property [28, Conjecture 7.4]). *Let X be a smooth projective holomorphic symplectic variety of dimension $2n$. Then we have a canonical multiplicative Chow–Künneth decomposition of $\mathfrak{h}(X)$ of Bloch–Beilinson type, that is, a direct sum decomposition in the category of rational Chow motives :*

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X) \tag{2}$$

satisfying the following properties :

(1) (Chow–Künneth) The cohomology realization of the decomposition gives the Künneth decomposition : for each $0 \le i \le 4n$, $H^*(\mathfrak{h}^i(X)) = H^i(X)$.

[2]Beauville's words in [9]: "...Now asking for a conjectural splitting of a conjectural filtration may look like a rather idle occupation..."

(2) *(Multiplicativity) The product* $\mu : \mathfrak{h}(X)\otimes\mathfrak{h}(X) \to \mathfrak{h}(X)$ *given by the small diagonal* $\delta_X \subset X\times X\times X$ *respects the decomposition: the restriction of* μ *on the summand* $\mathfrak{h}^i(X)\otimes\mathfrak{h}^j(X)$ *factorizes through* $\mathfrak{h}^{i+j}(X)$.

(3) *(Bloch–Beilinson–Murre) for any* $i, j \in \mathbf{N}$,
- $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j < i$;
- $\mathrm{CH}^i(\mathfrak{h}^j(X)) = 0$ *if* $j > 2i$;
- *the realization induces an injective map* $\mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}(-i), \mathfrak{h}^{2i}(X)\right) \to \mathrm{Hom}_{\mathbf{Q}-HS}\left(\mathbf{Q}(-i), H^{2i}(X)\right)$.

Our main result Theorem 1.6 then provides immediately a canonical multiplicative Chow-Künneth decomposition for Hilbert schemes of K3 surfaces (*i.e.* satisfying (1) and (2) above), which is the candidate decomposition for Conjecture 1.15 (*i.e.* also satisfying (3)). Precisely,

Theorem 1.16. *Given a projective K3 surface S and a natural number n, let* $S^{[n]}$ *be its n-th Hilbert scheme of points. We have the following canonical decomposition:*

$$\mathfrak{h}\left(S^{[n]}\right) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i\left(S^{[n]}\right),$$

where via the isomorphism of Theorem 1.6,

$$\mathfrak{h}^i\left(S^{[n]}\right) := \left(\bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}^{i-2\,\mathrm{age}(g)}\left((S^n)^g\right)(-\,\mathrm{age}(g))\right)^{\mathfrak{S}_n}.$$

(1) *This is a multiplicative Chow-Künneth decomposition for* $S^{[n]}$ *in the sense of Definition 11.1.*

(2) *For any* $0 \le i, j \le 2n$*, the restriction of the multiplication* $\mu : \mathfrak{h}(S^{[n]})\otimes\mathfrak{h}(S^{[n]}) \to \mathfrak{h}(S^{[n]})$ *induced by the small diagonal* $\delta_{S^{[n]}}$ *to* $\mu : \mathfrak{h}^i(S^{[n]})\otimes\mathfrak{h}^j(S^{[n]}) \to \mathfrak{h}^{i+j}(S^{[n]})$ *is determined in the following way: for any* $g, h \in \mathfrak{S}_n$*, let* $d := \mathrm{age}(g) + \mathrm{age}(h) - \mathrm{age}(gh)$*, the morphism*

$$\mathfrak{h}^i\left((S^n)^g\right)\otimes\mathfrak{h}^j\left((S^n)^h\right) \to \mathfrak{h}^{i+j+2d}\left((S^n)^{gh}\right)(d)$$

is given by $\delta_*\left(c_{g,h}\right)$*, where* $c_{g,h}$ *is the obstruction cycle computed in (37) and* $\delta : (S^n)^{<g,h>} \hookrightarrow (S^n)^g \times (S^n)^h \times (S^n)^{gh}$ *is the diagonal embedding.*

The first part of this result is not new: it is obtained by Vial in [59, Theorem 1]. The second part of Theorem 1.16 is a refinement: writing $X = S^{[n]}$, we not only show that $\mathfrak{h}^i(X)\otimes\mathfrak{h}^j(X) \to \mathfrak{h}^k(X)$ is zero map when $i + j \neq k$, but also describe the multiplication $\mathfrak{h}^i(X)\otimes\mathfrak{h}^j(X) \to \mathfrak{h}^k(X)$ when $i + j = k$. See §11 for the proof of Theorem 1.16 as well as the consequences.

Covention: All Chow rings are with rational coefficients unless otherwise stated, whose intersection product is denoted by $\cdot : \mathrm{CH}^*\otimes\mathrm{CH}^* \to \mathrm{CH}^*$. K-theory $K(-)$ means the Grothendieck group/ring $K_0(-)$. The category of Chow motives with rational coefficients is denoted by CHM and $\mathfrak{h}$ is the contravariant functor that associates a variety its Chow motive.

Acknowledgements: We want to thank Alessandro Chiodo, Tom Graber, Julien Grivaux, Jérémy Guéré, Daniel Halpern-Leistner, Daniel Huybrechts, Jun Li, Bruno Kahn, Étienne Mann, Marco Robalo, Yongbin Ruan, Burt Totaro, Gabriel Vezzosi, Charles Vial, Claire Voisin and Qizheng Yin for stimulating discussions. This project was initiated when we were members (funded by NSF) of the I.A.S. during the special year *Topology and Algebraic Geometry* 2014-2015. We thank the Institute for the excellent working and living conditions.

2. Orbifold Chow rings, orbifold K-theory, and orbifold motives

In this section, we recall the constructions of the orbifold Chow motives and orbifold Chow rings associated to (Gorenstein) global quotient stacks. For a more complete and more general treatment, see our previous work [28, §2]. Our approach follows the down-to-earth treatment of [31] (*cf.* also [33]).

Let M be a smooth projective variety endowed with an action of a finite group G by automorphisms. Throughout this section, we will assume that the canonical bundle of M is locally preserved by G, namely,

Assumption $(*)$ **:** For any $x \in M$, the action of the stabilizer Stab_x on the tangent space T_xM factors through $\mathrm{SL}(T_xM)$.

Note that this assumption amounts to require the quotient M/G to be Gorenstein. In the situation of MHRC (*i.e.* G acts by symplectic automorphisms on a holomorphic symplectic variety M), the assumption $(*)$ is always satisfied.

For any $g \in G$, M^g denotes the fixed locus of g. Recall the following notion of *age* in [52]:

Definition 2.1 (Age)**.** For any $g \in G$, let r be its order. The *age* of g, denoted by $\mathrm{age}(g)$, is the locally constant function on M^g whose value on a connected component Z of M^g is given as follows:

$$\mathrm{age}(g)\big|_Z = \frac{1}{r}\sum_{j=0}^{r-1} jm_j,$$

where m_j is the multiplicity of the eigenvalue $e^{2\pi\sqrt{-1}\frac{j}{r}}$ of the endomorphism $Dg_x \in \mathrm{SL}(T_xM)$ for an/any $x \in Z$.
Observe that $\det(Dg_x) = 1$, which is imposed by Assumption $(*)$, implies that the age function takes values in natural numbers here (see [28, Remark 2.7]). It is easy to check that the age function is invariant under conjugation and $\mathrm{age}(g) + \mathrm{age}(g^{-1}) = \mathrm{codim}(M^g)$.

Definition 2.2 (Orbifold Chow ring, orbifold cohomology and orbifold K-theory)**.** Given a finite group G acting on a smooth projective variety M, under the assumption $(*)$, we define an auxiliary (in general non-commutative) graded $\mathbf{Q}$-algebra $\mathrm{CH}^*(M,G)$ in several steps and the *orbifold Chow ring* of $[M/G]$ will be defined as the subalgebra of invariants:

$$\mathrm{CH}^*_{orb}([M/G]) := \mathrm{CH}^*(M,G)^G.$$

(1°) For any $i \in \mathbf{N}$, let

$$\mathrm{CH}^i(M,G) := \bigoplus_{g\in G} \mathrm{CH}^{i-\mathrm{age}(g)}\left(M^g\right).$$

Note that it is nonzero only if $0 \le i \le \dim M$.

(2°) As a graded vector space (with an extra G-grading),

$$\mathrm{CH}^*(M,G) := \bigoplus_{i=0}^{\dim M} \mathrm{CH}^i(M,G).$$

(3°) There is a natural G-action on $\mathrm{CH}^*(M,G)$: for any $h \in G$, the action of h sends for any $g \in G$, $\mathrm{CH}(M^g)$ isomorphically to $\mathrm{CH}(M^{hgh^{-1}})$ via the isomorphism

$$\begin{aligned} h: M^g &\to M^{hgh^{-1}} \\ x &\mapsto h.x \end{aligned}$$

Note that the G-action is compatible with the G-grading and the age shifting, thanks to the conjugacy invariance of the age funciton.

(4°) For any $g \in G$, let r be its order. The natural automorphism Dg on the vector bundle $TM|_{M^g}$, gives rise to the eigen-subbundle decomposition:

$$TM|_{M^g} = \bigoplus_{j=0}^{r-1} W_j,$$

where W_j is associated to the eigenvalue $e^{2\pi\sqrt{-1}\frac{j}{r}}$. We define the virtual bunlde:

$$V_g := \sum_{j=0}^{r-1} \frac{j}{r}[W_j] \in K_0(M^g)_{\mathbf{Q}}, \tag{3}$$

whose virtual rank is age(g) by Definition 2.1.

(5°) For any $g_1, g_2, g_3 \in G$ with $g_1g_2g_3 = 1$, let $M^{<g_1,g_2>} := M^{g_1} \cap M^{g_2}$ with the reduced structure. Define the virtual bundle:

$$F_{g_1,g_2} := V_{g_1}\big|_{M^{<g_1,g_2>}} + V_{g_2}\big|_{M^{<g_1,g_2>}} + V_{g_3}\big|_{M^{<g_1,g_2>}} + TM^{<g_1,g_2>} - TM|_{M^{<g_1,g_2>}} \in K_0\left(M^{<g_1,g_2>}\right)_{\mathbf{Q}}. \tag{4}$$

(6°) Now one can define the product $\star_{orb}$ on $\mathrm{CH}^*(M, G)$ as follows: for any $g, h \in G$ and $\alpha \in \mathrm{CH}^{i-\mathrm{age}(g)}(M^g)$, $\beta \in \mathrm{CH}^{j-\mathrm{age}(h)}(M^h)$, then

$$\alpha \star_{orb} \beta := \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right) \in \mathrm{CH}^{i+j-\mathrm{age}(gh)}(M^{gh}), \tag{5}$$

where $\iota : M^{<g,h>} \hookrightarrow M$ is the natural inclusion. It is a non-trivial fact that $\star_{orb}$ is associative (see [31, Lemma 5.4]).

(7°) It is easy to check that the product $\star_{orb}$ is additive with respect to the G-grading, the age-shifted cohomological degrees and invariant by the G-action. Hence one can define a graded $\mathbf{Q}$-algebra, called the *orbifold Chow ring* of $[M/G]$:

$$\mathrm{CH}^*_{orb}([M/G]) := (\mathrm{CH}^*(M, G), \star_{orb})^G.$$

It turns out that $\mathrm{CH}^*_{orb}([M/G])$ is a commutative graded $\mathbf{Q}$-algebra. Moreover, the orbifold Chow ring depends only on the stack $[M/G]$ (not the pair (M, G)).

(8°) The *orbifold cohomology* is defined in the same fashion as the orbifold Chow ring (except that the age-shifting should be doubled):

$$H^*_{orb}([M/G], \mathbf{Q}) := \left(\bigoplus_{g\in G} H^{*-2\,\mathrm{age}(g)}(M^g, \mathbf{Q}), \star_{orb}\right)^G,$$

which is a graded-commutative graded $\mathbf{Q}$-algebra.

(9°) The *orbifold K-theory* of $[M/G]$, denoted by $K_{orb}([M/G])$, can be defined in a similar way:

$$K_{orb}([M/G]) := (K(M, G), \star_{orb})^G,$$

where in the definition of orbifold product $\star_{orb}$, on uses λ_{-1} of the dual of the obstruction bundle instead of its top Chern class (see [31, Definition 1.7]). Moreover, we have a $\mathbf{Q}$-algebra isomorphism $K_{orb}([M/G]) \simeq \mathrm{CH}^*_{orb}([M/G])$ ([31, Main result 3]) induced by the orbifold Chern character.

Let CHM be the category of rational Chow motives, which is naturally a symmetric monoïdal category with tensor unit $\mathbb{1}$ given by the motive of a point. See [2] for basics about Chow motives. Recall that an *algebra object* in CHM is just a motive $\mathfrak{h} \in \mathrm{Obj(CHM)}$ together with a product morphism $\mu : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ and a unit morphism $\mathbb{1} \to \mathfrak{h}$ satisfying the usual axioms. An algebra object $(\mathfrak{h}, \mu, \mathbb{1})$ is called *commutative* if $\mu \circ \iota = \mu$ for $\iota : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ being the natural swapping morphism. Note that for any smooth projective variety X, its Chow motive $\mathfrak{h}(X)$ is naturally a commutative algebra object in CHM, where the unit morphism is given by its fundamental class $\mathbb{1}_X \in \mathrm{CH}^0(X) = \mathrm{Hom}(\mathbb{1}, \mathfrak{h}(X))$ and the multiplication morphism is given by the small diagonal $\delta_X := \{(x, x, x) | x \in X\}$,

$$\begin{aligned} \delta_X \in \mathrm{CH}^{2\dim X}(X^3) &= \mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}, \mathfrak{h}(X^2)(2\dim X) \otimes \mathfrak{h}(X)\right) \\ &= \mathrm{Hom}_{\mathrm{CHM}}\left(\mathbb{1}, \mathfrak{h}(X^2)^{\vee} \otimes \mathfrak{h}(X)\right) = \mathrm{Hom}_{\mathrm{CHM}}\left(\mathfrak{h}(X)^{\otimes 2}, \mathfrak{h}(X)\right), \end{aligned}$$

where the second equality is the Poicaré duality in CHM. Unless stated otherwise, the algebra object structure of the motive of a smooth projective variety is always this one in this paper.

Definition 2.3 (Orbifold motive [28])**.** Let M and G be as before. The *orbifold Chow motive* of $[M/G]$ is the commutative algebra object in CHM given by:

$$\mathfrak{h}_{orb}([M/G]) := \mathfrak{h}(M, G)^G,$$

where

$$\mathfrak{h}(M, G) := \bigoplus_{g \in G} \mathfrak{h}(M^g)(-\operatorname{age}(g))$$

endowed with the product $\star_{orb}$, defined in a parallel way of Definition 2.2, where all steps are the same except in (6°) the product morphism $\star_{orb}$ is defined as the analogue of (5) using the language of algebraic correspondences, namely, first of all, it respects the G-grading and for any $g, h \in G$, then the *orbifold product* morphism

$$\star_{orb} : \mathfrak{h}(M^g)(-\operatorname{age}(g)) \otimes \mathfrak{h}(M^h)(-\operatorname{age}(h)) \to \mathfrak{h}(M^{gh})(-\operatorname{age}(gh))$$

is determined by the correspondence

$$\delta_*(c_{top}(F_{g,h})) \in \mathrm{CH}^{\dim M^g + \dim M^h + \operatorname{age}(g) + \operatorname{age}(h) - \operatorname{age}(gh)}(M^g \times M^h \times M^{gh}), \tag{6}$$

where $\delta : M^{<g,h>} \to M^g \times M^h \times M^{gh}$ is the natural morphism sending x to (x, x, x) and c_{top} denotes the top Chern class of $F_{g,h}$. See our previous work [28, Definition 2.5] for other details.

3. Motivic HyperKähler Resolution Conjecture and main results

Definition 3.1 (Symplectic singularities and resolutions)**.** The original sources are [8], [47].

- A normal projective variety X is called *symplectic* if its regular part carries a holomorphic symplectic form (*i.e.* a nowhere degenerate holomorphic 2-form) whose pull-back to a/any resolution of singularities extends to a holomorphic 2-form. One can show that it is equivalent to require the existence of a holomorphic symplectic form on the regular part together with the condition that X is Gorenstein and has rational singularities. Basic examples of (singular) symplectic varieties include the quotient of a smooth holomorphic symplectic variety by a finite group of symplectic automorphisms.
- A resolution of singularities $f : Y \to X$ is called a *symplectic resolution* (or *hyperKähler resolution*) if the pullback of a/any holomorphic symplectic form on X_{reg} extends to a holomorphic *symplectic* form on Y. One can show that it is equivalent to the condition that the resolution is *crepant*: $f^*\omega_X = \omega_Y$.

As explained in §1 Introduction, string theory leads Ruan to formulate in [54] the *Cohomological HyperKähler Resolution Conjecutre* 1.2 which relates the orbifold cohomology ring of a complex (Gorenstein) orbifold to the cohomology ring of its hyperKähler resolutions. In the algebro-geometric setting, our insight is that this relation should hold true already at the motivic level and CHRC becomes the cohomological realization of the following MHRC initiated in [28]:

Conjecture 3.2 (=Conjecture 1.3 MHRC). *Let M be a smooth projective holomorphic symplectic variety and G be a finite group acting faithfully upon M by symplectic automorphisms. For any symplectic resolution of singularities $Y \to M/G$, we have an isomorphism of algebra objects in the category of complex Chow motives:*

$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb}([M/G]) \quad in \quad \mathrm{CHM}_{\mathbf{C}}.$$

In particular, there are isomorphisms of (graded for the first two) $\mathbf{C}$-algebras:

$$\begin{aligned} \mathrm{CH}^*(Y)_{\mathbf{C}} &\simeq \mathrm{CH}^*_{orb}([M/G])_{\mathbf{C}}; \\ H^*(Y, \mathbf{C}) &\simeq H^*_{orb}([M/G], \mathbf{C}); \\ K(Y)_{\mathbf{C}} &\simeq K_{orb}([M/G])_{\mathbf{C}}. \end{aligned}$$

Remark 3.3 (Hard Lefschetz condition). In the setting of Conjecture 3.2, the *Hard Lefschetz condition* ([15]) is automatically satisfied. Indeed, it is equivalent to the condition that $\mathrm{age}(g) = \mathrm{age}(g^{-1})$ for any $g \in G$ at any fixed point, which is easy to check: clearly $M^g = M^{g^{-1}}$ and by Definition 2.1 of the age functions, we only need to apply the following exercise in linear algebra to $V = T_{M,x}$ and $\varphi = D_{g_x}$ for any point $x \in M^g$: Let (V, σ) be a (finite dimensional) vector space endowed with a symplectic form and $\varphi \in \mathrm{Sp}(V, \sigma)$ be a symplectic automorphism of finite order, then φ and φ^{-1} have the same eigenvalues with same multiplicities. We leave the exercise to the reader.

Known cases of MHRC: it is proved in the following situations:

- The surface case is done in [27] as a global version of the *multiplicative motivic McKay correspondence* for surfaces: let S be a projective K3 or abelian surface endowed with an action of a finite group G by symplectic automorphisms. One has the minimal (=symplectic) resolution $\Sigma \to S/G$. Then $\mathfrak{h}(\Sigma) \simeq \mathfrak{h}_{orb}([S/G])$ as algebra objects in $\mathrm{CHM}_{\mathbf{C}}$ and $\mathrm{CH}^*(\Sigma)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([S/G])_{\mathbf{C}}$ as $\mathbf{C}$-algebras.
- The case of Example 1.5 (1) with S being an abelian surface [28, Theorem 1.3].
- The case of Example 1.5 (2) for generalized Kummer varieties [28, Theorem 1.4].

This paper deals with the case of Example 1.5 (1) with S being a projective K3 surface. Our main result is as follows which confirms that the Motivic Hyper-Kähler Resolution Conjecture is true in this case. It is stated in §1 Introduction, Theorem 1.6.

Theorem 3.4 (MHRC for Hilbert schemes of K3 surfaces). *Let S be a projective K3 surface. Let the symmetric group $\mathfrak{S}_n$ act on S^n by permutations. We have an isomorphism of algebra objects in the category of complex Chow motives:*

$$\mathfrak{h}(S^{[n]}) \simeq \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \quad in \quad \mathrm{CHM}_{\mathbf{C}}.$$

In particular, there are isomorphisms of graded $\mathbf{C}$-algebras:

$$\mathrm{CH}^*(S^{[n]})_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}};$$

$$H^*(S^{[n]})_{\mathbf{C}} \simeq H^*_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}}.$$

As a consequence, we get the K-theoretic HyperKähler Resolution Conjecture 1.4 of Jarvis-Kaufmann-Kimura [31, Conjecture 1.2] in this case:

Corollary 3.5 (KHRC for Hilbert schemes of K3 surfaces). *Situation as in Theorem 3.4, we have isomorphisms of $\mathbf{C}$-algebras:*

$$K(S^{[n]})_{\mathbf{C}} \simeq K_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}};$$

$$K^{top}(S^{[n]})_{\mathbf{C}} \simeq K^{top}_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}}.$$

Proof. In [31], an *orbifold Chern character* is constructed, which induces an isomorphism of $\mathbf{C}$-algebras ([31, Main result 3]):

$$\mathrm{ch}_{orb} : K_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}}.$$

The desired isomorphism of algebras is simply the combination[3] of this ch_{orb}, the usual Chern character isomorphism $\mathrm{ch} : K\left(S^{[n]}\right)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{Q}}$ and the isomorphism $\mathrm{CH}^*(S^{[n]})_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}}$ in Theorem 3.4.
The statement for topological K-theory comes similarly from the orbifold topological Chern character, which is also constructed in [31],

$$\mathrm{ch}_{orb} : K^{top}_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}} \xrightarrow{\simeq} H^*_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{Q}},$$

together with $\mathrm{ch} : K^{top}\left(S^{[n]}\right)_{\mathbf{Q}} \xrightarrow{\simeq} H^*\left(S^{[n]}\right)_{\mathbf{Q}}$ and the last isomorphism in Theorem 3.4. □

Remark 3.6. We claim no originality for the statement for topological invariants, namely the last isomorphism in Theorem 3.4 and the last isomorphism in Corollary 3.5, which are included here just for completeness. Indeed, the cohomological version is proved independently in [23] and [58] based on [37], and the topological K-theoretic version is then a direct consequence thanks to the construction in [31], as is explained in the above proof. See also Remark 1.8.

4. The additive isomorphism

Given any smooth projective surface S and a natural number $n \in \mathbf{N}^*$, let the n-th symmetric group $\mathfrak{S}_n$ acts on S^n by permutations. The quotient is the n-th symmetric product $S^{(n)} := S^n/\mathfrak{S}_n$, which admits a crepant resolution given by the Hilbert-Chow morphism $\rho : S^{[n]} \to S^{(n)}$, where $S^{[n]} := \mathrm{Hilb}^n(S)$ denotes the Hilbert scheme of length-n subschemes of S, which is a smooth ([25]) projective variety of dimension $2n$. As a first step towards Theorem 1.6, we establish in this section an (*a priori* just additive) isomorphism in the category of complex Chow motives $\mathrm{CHM}_{\mathbf{C}}$:

$$\phi : \mathfrak{h}(S^{[n]}) \xrightarrow{\simeq} \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]).$$

This is essentially a reformulation of the result of De Cataldo-Migliorini [20].

We start with some constructions in [20]. For any permutation $g \in \mathfrak{S}_n$, note that the fixed locus $(S^n)^g$ is a partial diagonal of S^n. We identify sometimes $(S^n)^g$ with $S^{O(g)}$, where $O(g)$ is the set of orbits of the permutation g. Consider the incidence subvariety (with the reduced structure), also known as the *isospectral Hilbert scheme*:

$$U := S^{[n]} \times_{S^{(n)}} S^n = \left\{(z, x_1, \cdots, x_n) \in S^{[n]} \times S^n \;\middle|\; \rho(z) = x_1 + \cdots + x_n\right\},$$

endowed with a natural $\mathfrak{S}_n$-action via the second factor. Its fixed loci are also an incidence subvarieties:

$$U^g := (S^{[n]} \times_{S^{(n)}} (S^n)^g)_{red} = \left\{(z, x_1, \cdots, x_n) \in S^{[n]} \times (S^n)^g \;\middle|\; \rho(z) = x_1 + \cdots + x_n\right\}.$$

[3] A direct and geometrically meaningful isomorphism between $K(S^{[n]})_{\mathbf{C}}$ and $K_{orb}([S^n/\mathfrak{S}_n])_{\mathbf{C}}$ would be very interesting. Unfortunately, the authors are not able to find one so far.

Note that the generic fibre of the projection $U^g \to (S^n)^g$ is isomorphic to the product of Briançon varieties [14] $\prod_{o\in O(g)} \mathfrak{B}_{|o|}$. Since $\mathfrak{B}_l$ is irreducible of dimension $l-1$ (see *loc.cit.*), we have

$$\dim U^g = \dim (S^n)^g + \sum_{o\in O(g)} (|o|-1) = n + |O(g)| = 2n - \mathrm{age}(g),$$

where the last equality uses the following lemma.

Lemma 4.1. *The natural action of $\mathfrak{S}_n$ on S^n satisfies Assumption* ($*$) *(i.e. the canonique bundle is locally preserved) and for any $g \in \mathfrak{S}_n$,*

$$\mathrm{age}(g) = n - |O(g)|.$$

Proof. For the first statement, given a point $x \in S^n$, observe that the stablizer of a point in S^n is of the form $\mathrm{Stab}_x \simeq \prod_{i=1}^{l} \mathfrak{S}_{\lambda_i}$ for some partition $\lambda = (\lambda_1, \cdots, \lambda_l)$ of n and the tangent space T_xS^n as a Stab_x-representation is $\oplus_{i=1}^{l} V_i \otimes C_i$ where for all i, V_i is a 2-dimensional vector space with trivial action while C_i is the natural λ_i-dimensional representation of $\mathfrak{S}_{\lambda_i}$ by permutations. It is straightforward to check that the determinant is always 1. For the computation of the age function, it suffices to observe that the age(g) is the sum of ages of its disjoint cycles and the age of a cycle of length k is $k-1$ since the $k\times k$ permutation matrix has eigenvalues $1, \zeta, \cdots, \zeta^{k-1}$, where $\zeta = e^{\frac{2\pi\sqrt{-1}}{k}}$. □

We have therefore for any $g \in \mathfrak{S}_n$, an algebraic correspondence (with $\mathbf{C}$-coefficients[4])

$$\Gamma_g := \frac{1}{\sqrt{-1}^{\mathrm{age}(g)}} U^g \in \mathrm{CH}^{2n-\mathrm{age}(g)}\left(S^{[n]} \times (S^n)^g\right)_{\mathbf{C}}, \tag{7}$$

from which we can define two morphisms in the category $\mathrm{CHM}_{\mathbf{C}}$ of complex Chow motives

$$\Gamma := \sum_{g\in G} \Gamma_g : \mathfrak{h}\left(S^{[n]}\right) \to \mathfrak{h}(S^n, \mathfrak{S}_n);$$

$${}^t\Gamma := \sum_{g\in G} {}^t\Gamma_g : \mathfrak{h}(S^n, \mathfrak{S}_n) \to \mathfrak{h}\left(S^{[n]}\right),$$

where $\mathfrak{h}(S^n, \mathfrak{S}_n) := \bigoplus_{g\in G} \mathfrak{h}\left((S^n)^g\right)(-\mathrm{age}(g))$ as in §2. Denote by $\iota : \mathfrak{h}\left(S^n, \mathfrak{S}_n\right)^{\mathfrak{S}_n} \hookrightarrow \mathfrak{h}\left(S^n, \mathfrak{S}_n\right)$ and $p : \mathfrak{h}\left(S^n, \mathfrak{S}_n\right) \twoheadrightarrow \mathfrak{h}\left(S^n, \mathfrak{S}_n\right)^{\mathfrak{S}_n}$ the inclusion of and the projection onto the invariant part. It is easy to check that Γ and ${}^t\Gamma$ are $\mathfrak{S}_n$-invariant (*cf.* [28, Lemma 5.1]) and therefore induce morphisms

$$\phi := p \circ \Gamma : \mathfrak{h}\left(S^{[n]}\right) \to \mathfrak{h}_{orb}\left([S^n/\mathfrak{S}_n]\right), \tag{8}$$

$$\psi := \frac{1}{n!} {}^t\Gamma \circ \iota : \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \to \mathfrak{h}\left(S^{[n]}\right), \tag{9}$$

where $\mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) := \mathfrak{h}\left(S^n, \mathfrak{S}_n\right)^{\mathfrak{S}_n}$ (Definition 2.3). Note that $\Gamma = \iota \circ \phi$ and $\frac{1}{n!} {}^t\Gamma = \psi \circ p$.

The main result of this section is the following:

Proposition 4.2. *ϕ and ψ are a pair of inverse isomorphisms between $\mathfrak{h}(S^{[n]})$ and $\mathfrak{h}_{orb}\left([S^n/\mathfrak{S}_n]\right)$ in $\mathrm{CHM}_{\mathbf{C}}$. In particular, they induce inverse isomorphisms $\mathrm{CH}^*\left(S^{[n]}\right)_{\mathbf{C}} \simeq \mathrm{CH}^*_{orb}\left([S^n/\mathfrak{S}_n]\right)_{\mathbf{C}}$.*

[4]Obviously, one can use $\sqrt{-1}$ instead of $\frac{1}{\sqrt{-1}}$ in the formula (7). Our choice here is not at all essential but only to make accordance with Li-Qin [40]'s isomorphism. For now, one simply remarks that if one makes the other choice, ϕ in (8) would differ by a sign at some components hence Proposition 4.2 remains true.

Proof. We refer to [28, Proposition 5.2] for the detailed proof (which works for any smooth projective surface). We only mention that the idea is to use the following isomorphism of [28, Lemma 5.3]

$$\mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \xrightarrow{\simeq} \bigoplus_{\lambda\in\mathscr{P}(n)} \mathfrak{h}\left(S^{(\lambda)}\right)(|\lambda|-n)$$

to reduce the problem to the following isomorphism established in De Cataldo-Migliorini [20]: let $\mathscr{P}(n)$ be the set of all partitions of n, then

$$\sum_{\lambda\in\mathscr{P}(n)} U^{(\lambda)} : \mathfrak{h}(S^{[n]}) \xrightarrow{\simeq} \bigoplus_{\lambda\in\mathscr{P}(n)} \mathfrak{h}(S^{(\lambda)})(|\lambda|-n),$$

whose inverse is given by

$$\sum_{\lambda\in\mathscr{P}(n)} \frac{1}{m_\lambda}\cdot {}^tU^{(\lambda)} : \bigoplus_{\lambda\in\mathscr{P}(n)} \mathfrak{h}(S^{(\lambda)})(|\lambda|-n) \xrightarrow{\simeq} \mathfrak{h}(S^{[n]}),$$

where $m_\lambda = (-1)^{n-|\lambda|}\prod_{j=i}^{|\lambda|}\lambda_j$. Here $S^{(\lambda)}$ (*resp.* $U^{(\lambda)}$) is the analogue of $(S^n)^g$ (*resp.* U^g). □

An additive isomorphism being established for Theorem 1.6, the objective now is to show that ϕ and/or ψ respect the multiplicative structures, that is, the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{h}(S^{[n]})^{\otimes 2} \arrow[r, "\delta_{S^{[n]}}"] \arrow[d, "\phi^{\otimes 2}"'] & \mathfrak{h}(S^{[n]}) \arrow[d, "\phi"] \\ \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n])^{\otimes 2} \arrow[r, "\star_{orb}"'] & \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \end{tikzcd} \tag{10}$$

where ϕ in the vertical arrows is the one constructed above and the upper horizontal arrow is the (usual) multiplication morphism for the motive of $S^{[n]}$, which is the correspondence given by the small diagonal $\delta_{S^{[n]}} \in \mathrm{CH}_{2n}\left(\left(S^{[n]}\right)^3\right)$; while the lower horizontal arrow $\star_{orb}$ is the orbifold product (see Definition 2.3 (6) or [28, Definition 2.5]).

We reduce the main Theorem 1.6 to the following proposition. Note that

$$\mathrm{CH}^*\left(\mathfrak{h}(S^n,\mathfrak{S}_n)^{\otimes 3}\right) = \bigoplus_{g_1,g_2,g_3\in\mathfrak{S}_n} \mathrm{CH}^{*-\sum_{i=1}^3 \mathrm{age}(g_i)}\left((S^n)^{g_1}\times(S^n)^{g_2}\times(S^n)^{g_3}\right).$$

Proposition 4.3 (⇒ main Theorem 1.6)**.** *In* $\oplus_{g_1,g_2,g_3\in\mathfrak{S}_n}\mathrm{CH}\left((S^n)^{g_1}\times(S^n)^{g_2}\times(S^n)^{g_3}\right)$*, the algebraic cycle*

$$W := \frac{1}{(n!)^2}\left(\sum_g \Gamma_g \times \sum_g \Gamma_g \times \sum_g \Gamma_g\right)_* (\delta_{S^{[n]}}) \tag{11}$$

is equal to the symmetrization *of the following algebraic cycle Z determining the orbifold product:*

$$Z|_{(S^n)^{g_1}\times(S^n)^{g_2}\times(S^n)^{g_3}} = \begin{cases} 0 & \text{if } g_3 \neq g_1g_2 \\ \delta_* c_{top}(F_{g_1,g_2}) & \text{if } g_3 = g_1g_2. \end{cases} \tag{12}$$

where the morphism δ is as in Definition 2.3 (6) and the vector bundle F_{g_1,g_2} is as in Definition 2.2 (5°). Here the symmetrization *of a cycle γ is by definition the cycle $\frac{1}{(n!)^3}\sum_{g_1,g_2,g_3\in\mathfrak{S}_n}(g_1,g_2,g_3).\gamma$. Note that W is already symmetrized.*

The proof of this proposition occupies §5, §6, §7, §8. Admitting Proposition 4.3, let us show our main result Theorem 1.6:

Proposition 4.3 ⇒ Theorem 1.6. Thanks to Proposition 4.2, the commutativity of the diagram (10) is equivalent to the commutativity of

$$\begin{tikzcd} \mathfrak{h}(S^{[n]})^{\otimes 2} \arrow[r, "\delta_{S^{[n]}}"] & \mathfrak{h}(S^{[n]}) \arrow[d, "\phi"] \\ \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n])^{\otimes 2} \arrow[u, "\psi^{\otimes 2}"] \arrow[r, "\star_{orb}"'] & \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \end{tikzcd}$$

which, by the definition of ϕ and ψ in (8) and (9), is equivalent to the commutativity of the following one:

(13)
$$\begin{tikzcd} \mathfrak{h}(S^{[n]})^{\otimes 2} \arrow[r, "\delta_{S^{[n]}}"] & \mathfrak{h}(S^{[n]}) \arrow[d, "\Gamma"] \\ \mathfrak{h}(S^n, \mathfrak{S}_n)^{\otimes 2} \arrow[u, "(\frac{1}{n!}{}^t\Gamma)^{\otimes 2}"] & \mathfrak{h}(S^n, \mathfrak{S}_n) \arrow[d, two heads, "p"] \\ \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n])^{\otimes 2} \arrow[u, hook, "\iota^{\otimes 2}"] \arrow[r, "\star_{orb}"'] & \mathfrak{h}_{orb}([S^n/\mathfrak{S}_n]) \end{tikzcd}$$

On the one hand, it is easy to see (*cf.* for example [2, 3.1.4]) that the composition of the upper three correspondences in (13)

$$\Gamma \circ \delta_{S^{[n]}} \circ \left(\frac{1}{n!}{}^t\Gamma\right)^{\otimes 2} := \left(\sum_g \Gamma_g\right) \circ \delta_{S^{[n]}} \circ \left(\frac{1}{n!}\sum_g {}^t\Gamma_g\right)^{\otimes 2}$$

is induced by the cycle W.
On the other hand, since the product morphism $\star_{orb}$ for $\mathfrak{h}_{orb}([S^n/\mathfrak{S}_n])$ is the restriction of the orbifold product for $\mathfrak{h}(S^n, \mathfrak{S}_n)$, which is given by the cycle Z in (12) by Definition 2.3 (6), we have that $\star_{orb} = p \circ Z \circ \iota^{\otimes 2}$. Therefore, the commutativity of (13) is equivalent to the equality

$$p \circ W \circ \iota^{\otimes 2} = p \circ Z \circ \iota^{\otimes 2},$$

which is equivalent to (using $p \circ \iota = \mathrm{id}$): $\iota \circ p \circ W \circ \iota^{\otimes 2} \circ p^{\otimes 2} = \iota \circ p \circ Z \circ \iota^{\otimes 2} \circ p^{\otimes 2}$, which is equivalent to the condition that the symmetrizations of Z and W are the same. □

As is mentioned in Remark 1.8, CHRC is known in the case of Hilbert schemes of K3 surfaces ([37], [23], [58]), which means we know the cohomological version of Proposition 4.3, namely the symmetrizations of the cycles W and Z have the same cohomology class. However, it is somehow surprising to us that in order to prove Proposition 4.3, we actually need much more cohomological information, namely, the Cohomological Crepant Resolution Conjecture for Hilbert-Chow resolutions for *all* simply-connected surfaces, solved by Li-Qin [40], to which we now turn.

5. Quantum corrections and CCRC for Hilbert-Chow resolutions

Let S be a smooth projective surface and $n \in \mathbf{N}^*$. The Hilbert-Chow resolution $\rho : S^{[n]} \to S^{(n)}$ has its fibres isomorphic to different products of Briançon varieties ([14]). In particular, the fibre of ρ over a generic point on the (big) diagonal of $S^{(n)}$ is isomorphic to $\mathbf{P}^1$: it is the projectivization of the tangent space of the multiplicity-two point, parametrizing length-two non-reduced scheme structures supported on that point. Let us denote the class of this rational curve by $\beta \in H_2(S^{[n]}, \mathbf{Z})$.

Note that the group of the curve classes contracted by ρ is infinite cyclic and generated by β. In particular, ρ is *non-degenerated* in the sense of [55, §2].

For any $d \in \mathbf{N}$, let $\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)$ be the (proper) moduli space of stable maps ([34]) from genus zero curves with three marked points to $S^{[n]}$ with curve class $d\beta \in H_2\left(S^{[n]}, \mathbf{Z}\right)$. Since β is contracted by ρ, which is crepant, the moduli space has virtual dimension $2n$ and moreover it is endowed with a virtual fundamental cycle class (*cf.* [11], [12]) in the rational Chow group:

$$\left[\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right]^{vir} \in \mathrm{CH}_{2n}\left(\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right).$$

There is a natural evaluation morphism $ev : \overline{M_{0,3}}\left(S^{[n]}, d\beta\right) \to \left(S^{[n]}\right)^3$. We define

$$\gamma_d := ev_*\left(\left[\overline{M_{0,3}}\left(S^{[n]}, d\beta\right)\right]^{vir}\right) \in \mathrm{CH}_{2n}\left(\left(S^{[n]}\right)^3\right). \tag{14}$$

We write $[\gamma_d] \in H_{4n}\left(\left(S^{[n]}\right)^3, \mathbf{Q}\right)$ for its (co)homology class.

Let us remark that for $d = 0$, the moduli space $\overline{M_{0,3}}\left(S^{[n]}, 0\right)$ is naturally identified with $S^{[n]}$, its virtual fundamental class is simply its fundamental class and therefore $\gamma_0 = \delta_{S^{[n]}}$ is the small diagonal of $\left(S^{[n]}\right)^3$. Now we can define as follows the *quantum corrections* to the cup product of $H^*(S^{[n]}, \mathbf{Q})$. By Poincaré duality, one sees easily that our definition is equivalent to the one in [55].

Definition 5.1 (Quantum corrections)**.** Let $\alpha_1, \alpha_2 \in H^*(S^{[n]}, \mathbf{Q})$, their *quantum corrected product* is by definition the 'infinite alternating sum'

$$\alpha_1 \smile_{qc} \alpha_2 := \lim_{q \to -1^+} \sum_{d=0}^{\infty} (\mathrm{pr}_3)_*\left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\gamma_d]\right) q^d, \tag{15}$$

where pr_i is the projection from $\left(S^{[n]}\right)^3$ to its i-th factor ($i = 1, 2, 3$). Here for any $d > 0$, the d-th term $\mathrm{pr}_{3,*}\left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\gamma_d]\right)$ is called the d-th *quantum correction* to the (usual) cup product, which corresponds to the term $d = 0$, since $\alpha_1 \smile \alpha_2 = \mathrm{pr}_{3,*}\left(\mathrm{pr}_1^*(\alpha_1) \smile \mathrm{pr}_2^*(\alpha_2) \smile [\delta_{S^{[n]}}]\right)$. The convergence property is a consequence of [40] (see Remark 5.3 below).

Without 'evaluating' q to -1, one would give the so-called *quatum product* on the vector space $H^*\left(S^{[n]}, \mathbf{Q}\right)[\![q]\!]$. Therefore in some sense, the quantum corrected cup product on $H^*\left(S^{[n]}, \mathbf{Q}\right)$ is the classical limit of the quantum product (*cf.* [55]). It is a non-trivial fact that the quantum product, hence also the quantum corrected product, is associative. The graded vector space $H^*\left(S^{[n]}, \mathbf{Q}\right)$ equipped with the product $\smile_{qc}$ is called the *quantum corrected cohomology algebra* of $S^{[n]}$, denoted by $H^*_{qc}\left(S^{[n]}, \mathbf{Q}\right)$.

The main result of Wei-Ping Li and Zhenbo Qin [40] (based on a number of works [36], [41], [42], [51], [38], [39], [17], see also the upcoming book [50]) can be stated as follows:

Theorem 5.2 (CCRC for Hilbert-Chow [40])**.** *Let S be a smooth projective simply-connected surface and n a natural number. There is an isomorphism of* $\mathbf{C}$*-algebras*

$$H^*_{qc}\left(S^{[n]}, \mathbf{C}\right) \simeq H^*_{orb}\left([S^n/\mathfrak{S}_n], \mathbf{C}\right).$$

Remark 5.3. Their result is stronger: [40, Theorems 1.2 and 1.3] basically says that there is a universal way (*i.e.* depending only on d, n and the canonical bundle of the surface) to compute the quantum corrections (essentially the three-point Gromov-Witten invariants $<\alpha_1, \alpha_2, \alpha_3>_{0,d\beta}$). This universality result allows them to reduce to the case of toric surfaces, which is already proved by Cheong [17] based on the work of Bryan-Graber [15] and Okounkov-Pandharipande [49]. In particular, as the q-power series in (15) in Definition 5.1, whose terms can be computed in a universal way ([40]), has convergence radius at least 1 and has a continuation across $q=-1$ when S is a toric surface, it must be the case for all smooth projective surfaces. The authors thank Wei-Ping Li and Zhenbo Qin for explaining to us with great patience on this point.

To draw the cohomological information needed from Theorem 5.2, let us compare the cohomological realization of our isomorphism in §4 Proposition 4.2 to Li-Qin's isomorphism [40]. Note that the isomorphism in the original paper [40] is somewhat implicit, see however [50, Theorem 15.21]) for the explicit formula that we use below (17) in the proof.

Proposition 5.4 (Comparison)**.** *The isomorphism of* $\mathbf{C}$*-algebras in Theorem 5.2 coincides with the cohomological realization of* ϕ *constructed in (8).*

Proof. To simplify the notation, we will denote by $S^{(m)}$ the stack $[S^m/\mathfrak{S}_m]$ in this proof. Let us first recall the two isomorphisms in play here. On the one hand, the cohomological realization of ϕ is given by the sum, over all $g\in\mathfrak{S}_n$, of the following cohomological correspondences (see (7)):

$$\frac{1}{\sqrt{-1}^{\operatorname{age}(g)}}[U^g]_*: H^*(S^{[n]}) \to H^*\left((S^n)^g\right),\ \forall g\in\mathfrak{S}_n,$$

where U^g is the incidence subvariety in $S^{[n]}\times(S^n)^g$.
On the other hand, the isomorphism that Li-Qin [40] used in Theorem 5.2 is the following (we drop the subscript 'qc' on the left hand side, as the product is irrelevant here):

$$\Phi: H^*\left(S^{[n]}\right) \quad\to\quad H^*_{orb}(S^{(n)}) \tag{16}$$

$$\mathfrak{a}_{-\lambda_1}(\alpha_1)\cdots\mathfrak{a}_{-\lambda_l}(\alpha_l)\,\mathbb{1} \quad\mapsto\quad \sqrt{-1}^{\,n-l}\,\mathfrak{p}_{-\lambda_1}(\alpha_1)\cdots\mathfrak{p}_{-\lambda_l}(\alpha_l)\,\mathbb{1} \tag{17}$$

where $\alpha_i\in H^*(S)$, the *vacuum* $\mathbb{1}\in H^0(S^{[0]})=H^0(S^{(0)})$ is the fundamental class of a point, $\lambda=(\lambda_1,\cdots,\lambda_l)$ is a partition of n and on the left hand side, $\mathfrak{a}$ is the Nakajima (creation) operator ([46]) while on the right hand side, the Heisenberg (creation) operator $\mathfrak{p}_{-m}(\alpha)$ is defined as the composition (see [51, §3.2] or [50, §10.2]):

$$\mathfrak{p}_{-m}(\alpha): H^*_{orb}(S^{(k)}) \xrightarrow{\omega_m(\alpha)\times\bullet} H^*_{orb}(S^{(m)})\otimes H^*_{orb}(S^{(k)}) \simeq H^*_{orb}([S^m\times S^k/\mathfrak{S}_m\times\mathfrak{S}_m]) \xrightarrow{Ind^{\mathfrak{S}_{m+k}}_{\mathfrak{S}_m\times\mathfrak{S}_k}} H^*_{orb}(S^{(m+k)}),$$

where $\omega_m(\alpha)\in H^*_{orb}(S^{(m)})$ is given by $m\alpha$ on each direct summand $H^*((S^m)^g)\simeq H^*(S)$ indexed by an m-cycle g and zero on other summands ; the induction map for a subgroup $K\subset G$ is defined by $Ind^G_K := \frac{1}{|K|}\sum_{g\in G} g$. (see [51, §3.1]).

Recall that the proof of the similar result in our previous work [28, Proposition 5.8] says that

$$[U^g]_*\left(\mathfrak{a}_{-\lambda_1}(\alpha_1)\cdots\mathfrak{a}_{-\lambda_l}(\alpha_l)\,\mathbb{1}\right)=0 \text{ if } g\notin\lambda;\text{ and} \tag{18}$$

$$\sum_{g\in\lambda}(-1)^{\operatorname{age}(g)}[U^g]_*\left(\mathfrak{a}_{-\lambda_1}(\alpha_1)\cdots\mathfrak{a}_{-\lambda_l}(\alpha_l)\,\mathbb{1}\right)=n!\cdot\operatorname{Sym}\left(\alpha_1\times\cdots\times\alpha_l\right), \tag{19}$$

where $g\in\lambda$ means g is of type λ, namely the lengths of orbits of g are $\lambda_1,\cdots,\lambda_l$; $\operatorname{Sym} := \frac{1}{n!}\sum_{h\in\mathfrak{S}_n} h$. is the symmetrization operator (on $H(S^n,\mathfrak{S}_n)$) ; and $\alpha_1\times\cdots\times\alpha_l$ is explained as follows: one chooses any permutation $g \in \mathfrak{S}_n$ of type λ and a numbering of its orbits $\{1, 2, \cdots, l\} \simeq O(g)$ such that the length of the j-th orbit is λ_j, then $\alpha_1 \times \cdots \times \alpha_l$ is considere as an element of the direct summand $H^*((S^n)^g)$ of $H(S^n, \mathfrak{S}_n)$ indexed by g, via the natural isomorphisms $S^l \simeq S^{O(g)} \simeq (S^n)^g$. It is clear that $\mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l)$ does not depend on the choice of g or the numbering.

Thanks to (18) and (19),

$$\sum_{g \in \mathfrak{S}_n} \frac{1}{\sqrt{-1}^{\mathrm{age}(g)}} [U^g]_* \left(\mathfrak{a}_{-\lambda_1}(\alpha_1) \cdots \mathfrak{a}_{-\lambda_l}(\alpha_l) \, \mathbb{1} \right) = \sqrt{-1}^{n-l} n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l).$$

Therefore it remains to show that in $H^*_{orb}(S^{(n)})$, we have

$$\mathfrak{p}_{-\lambda_1}(\alpha_1) \cdots \mathfrak{p}_{-\lambda_l}(\alpha_l) \, \mathbb{1} = n! \cdot \mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l).$$

As clearly $\mathrm{Sym}(\alpha_1 \times \cdots \times \alpha_l) = \mathrm{Sym}(\alpha_1 \times \mathrm{Sym}(\alpha_2 \times \cdots \times \alpha_l))$ (partial symmetrizations would not affect the result of the global symmetrization), by induction on l (=the length of λ), we are reduced to prove the following equality for any $m, k \in \mathbf{N}$, $\alpha \in H^*(S)$ and $\beta \in H^*_{orb}(S^{(k)})$:

$$\mathfrak{p}_{-m}(\alpha)(\beta) = \frac{(m+k)!}{k!} \mathrm{Sym}(\alpha \times \beta).$$

But this is just a reformulation of the definition of $\mathfrak{p}$: firstly, note that there are $(m-1)!$ different m-cycles in $\mathfrak{S}_m$, thus $\omega_m(\alpha)$ is nothing else but the symmetrization $\mathrm{Sym}\,(m! \cdot \alpha)$ where $\alpha \in H^*((S^m)^c) \simeq H^*(S)$ for a fixed m-cycle c. Secondly, $Ind_{\mathfrak{S}_m \times \mathfrak{S}_k}^{\mathfrak{S}_{m+k}}(\bullet)$ by definition is simply $\frac{(m+k)!}{m!k!} \mathrm{Sym}(\bullet)$. Putting these together, we obtain

$$\mathfrak{p}_{-m}(\alpha)(\beta) = Ind_{\mathfrak{S}_m \times \mathfrak{S}_k}^{\mathfrak{S}_{m+k}} (\mathrm{Sym}\,(m! \cdot \alpha) \times \beta) = \frac{(m+k)!}{m! \cdot k!} \mathrm{Sym}\,(\mathrm{Sym}(m! \cdot \alpha) \times \beta) = \frac{(m+k)!}{k!} \mathrm{Sym}(\alpha \times \beta),$$

as desired. □

Thanks to Proposition 5.4, we can relate at least the cohomological classes of the algebraic cycles W and Z defined in Proposition 4.3. To this end, we introduce a series of algebraic cycles W_d accompanying W: use the same notation, define for any $d \in \mathbf{N}$

$$W_d := \frac{1}{(n!)^2} \left(\sum_{g_1} \Gamma_{g_1} \times \sum_{g_2} \Gamma_{g_2} \times \sum_{g_3} \Gamma_{g_3} \right)_* (\gamma_d) \in \bigoplus_{g_1, g_2, g_3 \in \mathfrak{S}_n} \mathrm{CH}\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3} \right), \tag{20}$$

where $\gamma_d \in \mathrm{CH}_{2n}\left(\left(S^{[n]} \right)^3 \right)$ is defined in (14). As is remarked before, $\gamma_0 = \delta_{S^{[n]}}$ and thus $W_0 = W$. Now one can reformulate Theorem 5.2 in the following way that we will be using later. Let $[-]$ denote the cohomology class of an algebraic cycle.

Corollary 5.5. *Let S be a smooth projective simply-connected surface. The (vector-valued) q-power series $\sum_{d=0}^{\infty} [W_d] \cdot q^d$ has convergence radius at least 1 and has a continuation across $q = -1$ with value equal to the cohomology class of the symmetrization of Z:*

$$\lim_{q \to -1^+} \sum_{d=0}^{\infty} [W_d] \cdot q^d = [\mathrm{Sym(Z)}] \tag{21}$$

in $\oplus_{g_1, g_2, g_3 \in \mathfrak{S}_n} H^*\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3} \right)$. Here the cycle Z and the symmetrization operator Sym *are as in Proposition 4.3.*

Proof. As all cohomology groups involved here are finite dimensional vector spaces, the affirmation concerning the convergence radius and the continuation across -1 follows directly from the fact that the quantum corrected product is well-defined in the Hilbert-Chow case, which is a consequence

of the main theorem of Li-Qin [40]: see Remark 5.3.
Now Theorem 5.2 together with Proposition 5.4 says that the cohomological realization of our isomorphism ϕ induces an isomorphism of $\mathbf{C}$-algebras

$$[\phi]_* : H^*_{qc}\left(S^{[n]}\right) \xrightarrow{\simeq} H^*_{orb}\left([S^n/\mathfrak{S}_n]\right).$$

The rest of the proof is similar to the proof of *Proposition* 4.3 ⇒ *Theorem* 1.6 in the end of §4. Let us give a sketch: by Poincaré duality and Künneth formula for cohomology, the fact that $[\phi]_*$ is an isomorphism of algebra amounts to an equality of cohomological correspondences:

$$[\phi]_* \circ \left(\lim_{q\to -1^+} \sum_{d=0}^{\infty} [\gamma_d]\cdot q^d\right)_* = [\mathrm{Sym}(Z)]_* \circ [\phi\times\phi]_*, \tag{22}$$

where γ_d defined in (14) control the quantum corrections. By the formulae (8) and (9) for ϕ and its inverse ψ, (22) yields that

$$\frac{1}{(n!)^2}\left(\sum_{g_1}[\Gamma_{g_1}]\times\sum_{g_2}[\Gamma_{g_2}]\times\sum_{g_3}[\Gamma_{g_3}]\right)_* \left(\lim_{q\to -1^+}\sum_{d=0}^{\infty}[\gamma_d]\cdot q^d\right) = [\mathrm{Sym}(Z)],$$

which is exactly the desired equality (21) according to the definition of W_d in (20). □

The idea now is to upgrade the equality (21) between cohomology classes to an equality of algebraic cycles in rational Chow groups. We will not do this for any simply-connected surface but only for (projective) K3 surfaces (but the argument uses (21) for non-K3 simply-connected surfaces). The key point to make such an upgrade from homological equivalence to rational equivalence is to show that the cycles W_d and Z are of some special type, so-called *tautological* (see Definition 6.2). This is the theory of Voisin's *universally defined cycles*, which is the content of the next section §6.

5.1. **A digression on relative virtual fundamental classes.** For later use (Proposition 6.5), we would like to make a brief digression on a generalization of the construction of the virtual fundamental class. Namely, we work in a relative situation over a smooth base B and only assume the surfaces to be quasi-projective.

For each family of smooth quasi-projective surface over a smooth quasi-projective base $\mathcal{S}/B$, one can form the relative n-th Hilbert scheme $\mathcal{S}^{[n]/B}$ which is smooth and quasi-projective of relative dimension $2n$ over B and provides a fiberwise resolution of singularities, by the relative Hilbert-Chow morphism, of the relative n-th symmetric product $\mathcal{S}^{(n)/B}$. As in the absolute situation, there is a unique effective curve class generator $\beta \in H_2\left(\mathcal{S}^{[n]/B}, \mathbf{Z}\right)$ contracted by this resolution.

Let $\mathcal{M} := \overline{\mathcal{M}_{0,3}}\left(\mathcal{S}^{[n]/B}, d\beta\right)$ be the relative moduli space of genus zero stable maps with three marked points whose curve classes are $d\beta$, and let

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{S}^{[n]/B} \\ \overline{\mathcal{M}_{0,3}}\left(\mathcal{S}^{[n]/B}, d\beta\right) & \end{tikzcd}$$

be the universal family of stable maps to $\mathcal{S}^{[n]/B}$.

The general construction of a virtual fundamental class in the relative situation can be found in [12, Section 7]. We give a quick introduction below.

The construction of the relative intrinsic normal cone is as follows. Choose an étale cover $\{B_i, , i \in I\}$ of B and an étale cover $\{U_{i,j}, i \in I, j \in J\}$ of $\mathcal{M}|_{B_i}$, such that each morphism $U_{i,j} \to B_i$ is affine and factors through a local immersion $U_{i,j} \to V_{i,j} \to B_i$, where $V_{i,j} \to B_i$ is affine and smooth. The relative normal cone $C_{i,j}$ is defined as $\mathrm{Spec}\left(\oplus_k I^k_{i,j}/I^{k+1}_{i,j}\right)$, where $I_{i,j}$ is the ideal defining $U_{i,j}$ in $V_{i,j}$. This cone admits an action of the restriction of the relative tangent bundle $T_{V_{i,j}/B_i}|_{U_{i,j}}$ (See [12, Lemma 3.2] for the case B is a point). Denote by $\mathfrak{C}_{i,j}$ the stack quotient $[C_{i,j}/T_{V_{i,j}/B_i}]$. These algebraic stacks satisfy the obvious compatibility and glue together to give the *relative intrinsic normal cone* $\mathfrak{C}_{\mathcal{M}/B}$.

To define the virtual fundamental class, we proceed as follows. Denote by $\mathcal{M}_{0,3}$ be the smooth algebraic stack of *pre-stable* genus 0 curves with three marked points over S. There is a forgetful morphism $\mathcal{M} \to \mathcal{M}_{0,3}$, simply forgetting the morphism to $\mathcal{S}^{[n]/B}$. We have a relative perfect obstruction theory given by

$$\left(R\pi_* f^* T_{\mathcal{S}^{[n]}/B}\right)^\vee \to \mathbb{L}^\bullet_{\mathcal{M}/\mathcal{M}_{0,3}},$$

where $\mathbb{L}^\bullet_{\mathcal{M}/\mathcal{M}_{0,3}}$ is the relative cotangent complex of the forgetful morphism $\mathcal{M} \to \mathcal{M}_{0,3}$. This induces a closed immersion of the relative intrinsic normal cone $\mathfrak{C}_{\mathcal{M}/B} \to h^1/h^0(R\pi_* f^* T_{\mathcal{S}^{[n]}/B})$ (See [12, Section 2] for definitions of $h^1/h^0(R\pi_* f^* T_{\mathcal{S}^{[n]}/B})$ and see Section 5 *ibid.* for details of this construction when B is a point). The moduli space $\mathcal{M}$ embeds into the cone stack $h^1/h^0(R\pi_* T_{\mathcal{S}^{[n]}/B})$ as the zero section $\mathfrak{o}$. The relative virtual fundamental class is the intersection of $\mathfrak{o}$ and $\mathfrak{C}_{\mathcal{M}/B}$. The necessary properties of Chow groups of an Artin stack required to make this construction work are developed by Kresch [35].

We will need the following proposition later in the proof of Proposition 6.5.

Proposition 5.6. *The above construction commutes with base change $B' \to B$ (with B' smooth) and restriction to a Zariski open $\mathcal{S}' \subset \mathcal{S}$.*

Proof. See[12, Proposition 7.5] for the base change case. The restriction to the Zariski open subset case is immediate from the construction, since the relative moduli space of stable maps for the family $\mathcal{S}'^{[n]/B}$ is an open substack of $\mathcal{S}^{[n]/B}$. □

6. Universally defined cycles are tautological

In this section, we introduce the recent powerful theory of Voisin [63] on universally defined cycles and apply her theorem to the cycles W_d (and Z) studied in the previous sections.

We start by the definition of universally defined cycles (see also [62, §5.2]), which is a system of algebraic cycles with compatibility conditions. Recall that all Chow groups are with rational coefficients.

Definition 6.1 (Universally defined cycles [63])**.** For any fixed natural number m. A *universally defined cycle* (for m copies of surfaces) consists of the following data: for each family of smooth quasi-projective surface[5] over a smooth quasi-projective base $\mathcal{S}/B$, an element $z_{\mathcal{S}/B} \in \mathrm{CH}^*\left(\mathcal{S}^{m/B}\right)$ is given, satisfying two axioms:

[5]That is, the structural morphism $\mathcal{S} \to B$ is smooth and quasi-projective of pure relative dimension 2.

- (*Base change*) For any morphism between smooth quasi-projective varieties $B' \to B$ and any family of smooth quasi-projective surfaces $\mathcal{S}/B$, let $\mathcal{S}'/B'$ be the base-changed family and $f : \mathcal{S}'^{m/B'} \to \mathcal{S}^{m/B}$ be the natural morphism, then $f^* z_{\mathcal{S}/B} = z_{\mathcal{S}'/B'}$ in $\mathrm{CH}^*\left(\mathcal{S}'^{m/B'}\right)$.
- (*Zariski open subset*) For any family of smooth quasi-projective surfaces $\mathcal{S}/B$ over a smooth quasi-projective base B and any Zariski open subset $\mathcal{U}$ of $\mathcal{S}$, let $j : \mathcal{U}^{m/B} \hookrightarrow \mathcal{S}^{m/B}$ be the natural open immersion, then the restriction $j^* z_{\mathcal{S}/B} = z_{\mathcal{U}/B}$ in $\mathrm{CH}^*\left(\mathcal{U}^{m/B}\right)$.

Here $\mathcal{S}^{m/B} := \underbrace{\mathcal{S} \times_B \cdots \times_B \mathcal{S}}_{m}$ denotes the m-th self fiber product of $\mathcal{S}$ over B.

Roughly speaking, a universally defined cycle is the cycle that can be defined for (the m-th power of) all quasi-projective surfaces in a universal way, whence the name. Note that it is essential to include in the definition *all* (families of) open surfaces, which allows one to do degeneration argument in the proof of Theorem 6.4 below. As one would expect, such universally defined cycles should be rather canonical (*i.e.* does not involve any special properties of a given surface) in the following sense[6]:

Definition 6.2 (Tautological cycles)**.** Given a natural number m, a *tautological cycle* (for m-copies of surfaces) is a system of algebraic cycles $z_S \in \mathrm{CH}^*(S^m)$ indexed by all smooth projective surfaces S, such that there exists a universal[7] polynomial

$$P \in \mathbf{Q}[X_{i,j}, Y_k, E_l\ ;\ 1 \le i,\ j,\ k,\ l \le m;\ i < j],$$

such that for any smooth projective surface S, we have in $\mathrm{CH}^*(S^m)$:

$$z_S = P(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)),$$

where $\Delta_{i,j} = \left\{(x_1, \cdots, x_m) \in S^m \;\middle|\; x_i = x_j\right\}$ is a big diagonal, $\mathrm{pr}_k : S^m \to S$ is the k-th natural projection and $c_i(S)$ is the i-th Chern class of the tangent bundle of S.

One observes immediately the following easy fact :

Lemma 6.3. *Let $n, m \in \mathbf{N}$. Given indices $j_1, \cdots, j_n \in \{1, \cdots, m\}$. For any smooth projective surface S, consider the morphism*

$$\begin{array}{rcl} f_S : S^m & \to & S^n \\ (x_1, \cdots x_m) & \mapsto & (x_{j_1}, \cdots, x_{j_n}). \end{array}$$

Then the pull-back and push-forward by f of tautological cycles remain tautological. Precisely:

(i) For any system of tautological cycles $\{z_S \in \mathrm{CH}^*(S^n)\}_S$ indexed by smooth projective surfaces, the pull-back system $\left\{f_S^*(z_S) \in \mathrm{CH}^*(S^m)\right\}_S$ is a tautological cycle.

(ii) Suppose moreover[8] that f_S is injective[9]. Then for any system of tautological cycles $\{z_S \in \mathrm{CH}^*(S^m)\}_S$ indexed by smooth projective surfaces, the push-forward system $\{f_{S,*}(z_S) \in \mathrm{CH}^*(S^n)\}_S$ is a tautological cycle.

[6]We learned this terminology from a talk of Qizheng Yin. Yet our definition is a bit different: on the one hand it is stronger than the usual one since we require that the polynomial P is universal for all surfaces; on the other hand, we do not include divisor classes other than the canonical divisor.

[7]That is, independent of S.

[8]This assumption can be removed, but this is all we need.

[9]It is obviously equivalent to the condition that the index map $j : \{1, \cdots, n\} \to \{1, \cdots, m\}$ is surjective.

Proof. For the pull-backs, it is clear: the pull-back of a big diagonal of S^n is either a big diagonal of S^m or the fundamental class of S^m; while the pull-back of a Chern class of a factor of S^n is also a Chern class of a factor of S^m. The polynomial obtained is obviously independent of S.
For the push-forwards, by induction, the problem is reduced to the special case where $n = m + 1$ and $f_S : S^m \hookrightarrow S^{m+1}$ is given by $(x_1, \cdots, x_{m-1}, x_m) \mapsto (x_1, \cdots, x_{m-1}, x_m, x_m)$. Then the push-forward of a tautological cycle is given by multiplying $\Delta_{m-1,m}$. □

Now we can state Voisin's result as follows :

Theorem 6.4 (Voisin [63]). *For any fixed natural number m. Let z be a universally defined cycle (for m copies of surfaces). Then the system of cycles* $\left\{z_{S/\mathbf{C}} \in \mathrm{CH}^*(S^m)\right\}_S$ *indexed by all smooth projective surfaces is* tautological, *i.e. there exists a polynomial* $P \in \mathbf{Q}[X_{i,j}, Y_k, E_l\ ;\ 1 \le i,\ j,\ k,\ l \le m;\ i < j]$, *such that for any smooth projective surface S,*

$$z_{S/\mathbf{C}} = P(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)) \in \mathrm{CH}^*(S^m).$$

Theorem 6.4 allows us to show that the algebraic cycles in (21) are tautological:

Proposition 6.5. *For any* $g_1, g_2, g_3 \in \mathfrak{S}_n$*, the restrictions on* $(S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}$ *of the algebraic cycles* W_d *(in particular W) defined in* (20) *and Z defined in (*12*), with S varying through all smooth projective surfaces, are tautological cycles.*

Proof. We first deal with Z. By its definition (12), we can assume $g_3 = g_1 g_2$ and then on this component we need to show that $Z = \delta_* c_{top}\left(F_{g_1,g_2}\right)$ is tautological, where $\delta : (S^n)^{<g_1,g_2>} \hookrightarrow (S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_1 g_2}$ is the natural inclusion and F_{g_1,g_2} is the obstruction bundle defined in Definition 2.2 (5°). By Lemma 6.3 (*ii*), it suffices to show that the Chern classes of $F_{g_1,g_2} \in \mathrm{CH}^*\left((S^n)^{<g_1,g_2>}\right)$ are tautological. Regarding the definition of F_{g_1,g_2} in (4), Lemma 6.3 (*i*) reduces us to show that for any $g \in \mathfrak{S}_n$, the Chern classes of $V_g \in \mathrm{CH}^*\left((S^n)^g\right)$ defined in (3) is tautological. To this end, suppose that the permutation g has l orbits, whose lengths are $\lambda_1, \cdots, \lambda_l$, let us identify $(S^n)^g = S^{O(g)}$ with S^l, then

$$T_{S^n}|_{(S^n)^g} = \bigoplus_{j=1}^{l} \mathrm{pr}_j^* T_S^{\oplus \lambda_j},$$

with each $T_S^{\oplus \lambda_j}$ equipped with the cyclic $\mathbf{Z}/\lambda_j\mathbf{Z}$-action. Therefore, $V_g = \sum_{j=1}^{l} \frac{\lambda_j - 1}{2}[\mathrm{pr}_j^* T_S]$ and hence

$$c_t(V_g) = \prod_{j=1}^{l} \mathrm{pr}_j^* \left(c_t(T_S)\right)^{\frac{\lambda_j - 1}{2}},$$

which is in the form of a tautological cycle. Note that with rational coefficients, the above calculus of Chern classes make sense as c_t has leading term 1 thus has well-defined square root. In conclusion, Z is a tautological cycle. Note that we did not invoke Voisin's Theorem 6.4 for Z.

Now we show that for each $d \in \mathbf{N}$, the algebraic cycle W_d is tautological. To apply Theorem 6.4, we check that W_d belongs to a system of universally defined cycle. Fix $g_1, g_2, g_3 \in \mathfrak{S}_n$. For each family of smooth quasi-projective surface over a smooth quasi-projective base $\mathcal{S}/B$, one can form the relative n-th Hilbert scheme $\mathcal{S}^{[n]/B}$ which is smooth and quasi-projective of relative dimension $2n$ over B and provides a fiberwise resolution of singularities, by the relative Hilbert-Chow morphism, of the relative n-th symmetric product $\mathcal{S}^{(n)/B}$. As in the absolute situation, there is a unique effective curve class generator $\beta \in H_2\left(\mathcal{S}^{[n]/B}, \mathbf{Z}\right)$ contracted by this resolution. Let

$\mathcal{M} := \overline{\mathcal{M}_{0,3}}\left(\mathcal{S}^{[n]/B}, d\beta\right)$ be the relative moduli space of genus zero stable maps with three marked points whose curve classes are $d\beta$, and let

$$ev : \mathcal{M} := \overline{\mathcal{M}_{0,3}}\left(\mathcal{S}^{[n]/B}, d\beta\right) \to \left(\mathcal{S}^{[n]/B}\right)^{3/B}$$

be the evaluation morphism to $\mathcal{S}^{[n]/B}$. Note that any curves of class $d\beta$ is contained in a fiber of the relative Hilbert-Chow morphism $\mathcal{S}^{[n]/B} \to \mathcal{S}^{(n)/B}$, which is a proper subvariety of $\mathcal{S}^{[n]/B}$. Thus, even though the variety $\mathcal{S}^{[n]/B}$ is only quasi-projective and the moduli space of stable maps considered above is not proper, the evaluation morphism is still proper. The theory of virtual fundamental class ([12, 11]) gives rise to a class $[\mathcal{M}]^{vir} \in \mathrm{CH}(\mathcal{M})$. See the discussion in the digression §5.1 for details. By construction, this virtual fundamental class has the following property. Let $\mathcal{S}_b$ be the fiber of the family $\mathcal{S} \to B$ over a point $b \in B$ and $\overline{\mathcal{S}_b}$ be any smooth projective compactification of $\mathcal{S}_b$. Then the fiber over b of the moduli space $\mathcal{M}_b = \overline{\mathcal{M}_{0,3}}\left(\mathcal{S}^{[n]/B}, d\beta\right)_b$ is an open substack of $\overline{M_{0,3}}\left(\overline{\mathcal{S}_b}^{[n]}, d\beta\right)$, and the restriction of this virtual fundamental class $[\mathcal{M}]^{vir}$ to the fiber $\mathcal{M}_b$ is nothing but the restriction of the virtual fundamental class of $\overline{M_{0,3}}\left(\overline{\mathcal{S}_b}^{[n]}, d\beta\right)$. Define

$$\gamma_{d,\mathcal{S}/B} := ev_*\left([\mathcal{M}]^{vir}\right) \in \mathrm{CH}^*\left(\left(\mathcal{S}^{[n]/B}\right)^{3/B}\right).$$

Finally, for each g, let Γ_g be the correspondence (relative to B) between $\mathcal{S}^{[n]/B}$ and $(\mathcal{S}^{n/B})^g$ given by incidence subvarieties, which is proper with respect to two projections, we define on the component $(\mathcal{S}^{n/B})^{g_1} \times (\mathcal{S}^{n/B})^{g_2} \times (\mathcal{S}^{n/B})^{g_3}$,

$$\mathcal{W}_{d,\mathcal{S}/B} := \frac{1}{(n!)^2}\left(\Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}\right)_*\left(\gamma_{d,\mathcal{S}/B}\right) \in \mathrm{CH}^*\left((\mathcal{S}^{n/B})^{g_1} \times (\mathcal{S}^{n/B})^{g_2} \times (\mathcal{S}^{n/B})^{g_3}\right).$$

We claim that the system $\mathcal{W}_{d,\mathcal{S}/B}$ for all families $\mathcal{S}/B$ is a universally defined cycle (for $(|O(g_1)| + |O(g_2)| + |O(g_3)|)$-copies of surfaces). Indeed, it suffices to show that the construction of the cycle $\gamma_{d,\mathcal{S}/B}$ commutes with base change and restriction to a Zariski open subset, which in turn follows from the corresponding properties of the virtual fundamental class (Proposition 5.6). By Voisin's Theorem 6.4, we have in particular, the cycle W_d, for all d and all smooth projective surfaces S, defined in (20), is tautological. □

7. Cohomological relations between tautological cycles

Before the proof of the main theorem, we need to classify the algebraic relations between cohomological classes of tautological cycles. Recall that $[-] : \mathrm{CH}^* \to H^{2*}$ stands for the class of an algebraic cycle in cohomology.

Theorem 7.1 (Yin [65], see also Voisin [62, Theorem 5.9])**.** *Let S be a simply-connected smooth projective surface and m be a natural number. Let $a_S := (K_S^2)$ be the self-intersection number of its canonical bundle and $b_S := b_2(S) = \deg(c_2(S)) - 2 > 0$ be its second Betti number. Consider the following homomorphism* *of graded* $\mathbf{Q}$*-algebras:*

$$
\begin{array}{rcl}
\mathrm{cl}^H_S : \mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t\ ;\ 1 \leq i,\ j,\ k,\ l,\ t \leq m\,; i \neq j] & \to & H^{2*}(S^m, \mathbf{Q}) \\
A & \mapsto & a_S \cdot [\mathbb{1}] \\
B & \mapsto & b_S \cdot [\mathbb{1}] \\
X_{i,j} & \mapsto & [\Delta_{i,j}] \\
Y_k & \mapsto & \mathrm{pr}_k^*[c_1(S)] \\
E_l & \mapsto & \mathrm{pr}_l^*[c_2(S)] \\
Z_t & \mapsto & \mathrm{pr}_t^*(o_S),
\end{array}
$$

where cl^H *stands for the 'cohomology class map'; on the left hand side, the degrees of variables are given by* $\deg(A) = \deg(B) = 0$*,* $\deg(X_{i,j}) = \deg(E_l) = \deg(Z_t) = 2$ *and* $\deg(Y_k) = 1$*; on the right hand side,* $[\mathbb{1}] \in H^0(S^m, \mathbf{Q})$ *is the cohomological fundamental class of* S^m*, and* $o_S \in H^4(S, \mathbf{Q})$ *is the class of a point of* S*. Then the ideal* $\mathrm{Ker}\left(\mathrm{cl}^H_S\right)$ *is generated by the following four sets of relations:*

(i) (Trivial relations) For all $1 \leq i, j, k \leq m$ *distinct,*
- $X_{i,j} - X_{j,i}$
- Z_i^2
- $Y_i Z_i$
- $X_{i,j}X_{j,k} - X_{i,k}X_{j,k}$
- $X_{i,j}Z_i - Z_iZ_j$

(ii) (Beauville-Voisin relations) For all $1 \leq i, j, k \leq m$ *distinct*
- $Y_i^2 - AZ_i$
- $E_i - BZ_i$
- $X_{i,j}Y_i - Y_iZ_j - Y_jZ_i$
- $X_{i,j}^2 - BZ_iZ_j$
- $X_{i,j}X_{j,k} - \left(X_{i,j}Z_k + X_{i,k}Z_j + X_{j,k}Z_i\right) + \left(Z_iZ_j + Z_iZ_k + Z_jZ_k\right)$

(iii) (Numerical relations)
- $A - a_S$
- $B - b_S$

(iv) (Kimura relations)
- *For all* $1 \leq j_1, \cdots, j_{2b_S+2} \leq m$ *distinct,*
  $\sum_{\sigma \in \mathfrak{S}_{b_S+1}} \mathrm{sgn}(\sigma) \prod_{i=1}^{b_S+1} (X_{j_i, j_{b_S+1+\sigma(i)}} - Z_{j_i} - Z_{j_{b_S+1+\sigma(i)}})$
- *For all* $1 \leq j_1, \cdots, j_{2b_S} \leq m$ *distinct,*
  $\sum_{\sigma \in \mathfrak{S}_{b_S}} \mathrm{sgn}(\sigma) \prod_{i=1}^{b_S} \left(A(X_{j_i, j_{b_S+1+\sigma(i)}} - Z_{j_i} - Z_{j_{b_S+1+\sigma(i)}}) - Y_{j_i} Y_{j_{b_S+1+\sigma(i)}}\right)$

This result is essentially due to Yin [65], based on the work of Hanlon and Wales [30]. Although the original statement in [65] is only for K3 surfaces, it is pointed out by Voisin in [62] and also in the end of the Yin's paper [65, §3.8 Final remark (iii)] that the argument in fact gives the corresponding result for *all* regular[10] smooth projective surfaces. Note that in [65], Yin takes into account of all divisors, and the Kimura relation in *loc.cit.* is of degree $2(b_{2,tr}(S) + 1)$, however his proof works equally when one only considers one divisor class. The reason why we have two types of relations in *(iv)* is to take care of the surfaces with $(K_S^2) = 0$ in which case the Kimura relations of the first type, together with *(i)* – *(iii)* suffice to generate $\mathrm{Ker}\left(\mathrm{cl}^H_S\right)$. Our statement is

[10]That is, $q(S) := \dim H^1(X, \mathcal{O}_X) = 0$. For our purpose, it is enough to look at only the simply-connected surfaces.

slightly different (but obviously equivalent) in the sense that we add two formal variables A and B (and then kill them by the numerical relations (iii)), which will be convenient for later use.

Let $R := \mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t]$ be the ambiant polynomial ring. Let $J := \langle (i), (ii) \rangle$ be the ideal generated by the first two sets of relations in Theorem 7.1:

(i) (Trivial relations) For all $1 \le i, j, k \le m$ distinct,
- $X_{i,j} - X_{j,i}$
- Z_i^2
- $Y_i Z_i$
- $X_{i,j}X_{j,k} - X_{i,k}X_{j,k}$
- $X_{i,j}Z_i - Z_i Z_j$

(ii) (Beauville-Voisin relations) For all $1 \le i, j, k \le m$ distinct,
- $Y_i^2 - AZ_i$
- $E_i - BZ_i$
- $X_{i,j}Y_i - Y_i Z_j - Y_j Z_i$
- $X_{i,j}^2 - BZ_i Z_j$
- $X_{i,j}X_{j,k} - \left(X_{i,j}Z_k + X_{i,k}Z_j + X_{j,k}Z_i\right) + \left(Z_i Z_j + Z_i Z_k + Z_j Z_k\right)$

Let us show a completely algebraic result:

Proposition 7.2. *As a $\mathbf{Q}[A, B]$-module, the quotient R/J is free of finite rank*[11].

Proof. Let Mon be the set of all monomials on $X_{i,j}$, Y_k and Z_t ($i, j, k, t \in \{1, \cdots, m\}$) without repeated indices and for any $d \in \mathbf{N}$, let Mon^d be the subset of monomials with weighted degree d. It is easy to count that

$$r := |\mathrm{Mon}| = \sum_i \frac{m!}{2^i i!(m-2i)!} 3^{m-2i}$$

and

$$|\mathrm{Mon}^d| = \sum_{2i+j+2k=d} \frac{m!}{2^i i! j! k! (m-2i-j-k)!}$$

Write the elements of Mon as :

$$\mathrm{Mon} = \{M_1, \cdots, M_r\}.$$

Let us show the following morphism of $\mathbf{Q}[A, B]$-modules is an isomorphism:

$$\begin{aligned} \varphi : \mathbf{Q}[A, B]^{\oplus r} &\to R/J \\ (f_1, \cdots, f_r) &\mapsto \sum_{i=1}^r f_i \cdot M_i \end{aligned}$$

The surjectivity of φ follows from the argument of 'elimination of repeated indices' due to Voisin [61, Proof of Lemma 2.3, P. 6-10] and Yin [65]: observe that the relations in (i) and (ii) tell us precisely that, modulo J, for any polynomial in R one can first get rid of E_i's by using the second relation in (ii) and then each time one sees a repeated index in a monomial in $X_{i,j}$'s, Y_k's and Z_t's, one can use a suitable relation in (i) or (ii) to decrease the total number of repetitions of indices by at least 1. This algorithm must stop after finite number of replacements and gives a new representative, modulo J, of the original polynomial such that each term of the new representative has no repeated index. In other words, in the image of ϕ.

[11]The rank is given by the number r in the proof.

As for the injectivity of φ, since $\mathbf{Q}[A,B]^{\oplus r}$ is torsion free, we are reduced to show the *generic* injectivity of φ, that is, $\varphi_\eta : \mathbf{Q}(A,B)^{\oplus r} \to (R/J) \otimes_{\mathbf{Q}[A,B]} \mathbf{Q}(A,B)$ is injective. For any $N \in \mathbf{N}$, consider the following subset of $\mathbf{Q}\times\mathbf{Q}$:

$$\Lambda := \left\{(a_S, b_S) \in \mathbf{Q}\times\mathbf{Q} \;\middle|\; S \text{ simply-connected surface with } \chi_{top}(S) > N\right\}, \tag{23}$$

where $a_S = (K_S^2)$ and $b_S = \chi_{top}(S)$. Let us observe firstly the following easy fact :

Lemma 7.3. *For any given $N \in \mathbf{N}$, the above subset Λ is Zariski dense (over $\overline{\mathbf{Q}}$)[12] in $\mathbf{Q}\times\mathbf{Q}$.*

Proof. Suppose that a polynomial $P(X,Y) \in \overline{\mathbf{Q}}[X,Y]$ satisfies $P(a_S, b_S) = 0$ for all simply-connected smooth projective surface S with $b_S > N$. By considering the blow-up of S at $n \in \mathbf{N}$ points, which would decrease a_S by n and increase b_S by n, we find that $P(a_S - n, b_S + n) = 0$ for any $n \in \mathbf{N}$. This implies that there exists a polynomial Q, such that $P(X,Y) = Q(X+Y)$. Now by hypothesis, Q satisfies $Q(a_S + b_S) = 0$ for any simply-connected surface S with $\chi_{top}(S) > N$. By Noether formula,

$$a_S + b_S := (K_S^2) + \chi_{top}(S) = 12\chi(S, \mathcal{O}_S) \xlongequal{\pi_1(S)=\{1\}} 12 + 12h^{2,0}(S).$$

Hence $Q(12 + 12h^{2,0}(S)) = 0$ for all such surfaces. By looking at surfaces in $\mathbf{P}^3$ of higher and higher degrees, we see that $Q = 0$, thus $P = 0$, which proves the Zariski density. □

Return to the proof of Proposition 7.2. Fix any $N \gg m$, by the Zariski density of Λ defined in (23), to show the generic injectivity of φ, it suffices to show that for any $(a_S, b_S) \in \Lambda$, φ is injective on the fiber over (a_S, b_S):

$$\varphi_{a_S,b_S} : \mathbf{Q}^{\oplus r} \hookrightarrow R/\langle J, A - a_S, B - b_S\rangle.$$

Now for any simply-connected surface S with $b_S > N(\gg m)$, recall that we have a cohomology class map $\mathrm{cl}_S^H : R/\langle J, A - a_S, B - b_S\rangle \to H^{2*}(S^m, \mathbf{Q})$ defined in Theorem 7.1. As the injectivity of φ_{a_S,b_S} will follow from the injectivity of the following composition

$$\begin{array}{rcl} \mathrm{cl}_S^H \circ \varphi_{a_S,b_S} : \mathbf{Q}^{\oplus r} & \to & H^{2*}(S^m, \mathbf{Q}) \\ (\mu_1, \cdots, \mu_r) & \mapsto & \sum_{i=1}^r \mu_i [M_i], \end{array}$$

it is enough to show that the cohomology classes $[M_1], \cdots, [M_r]$ are $\mathbf{Q}$-linearly independent in $H^{2*}(S^m, \mathbf{Q})$. However, in [65, P. 510, last paragraph before §3.8 Final remarks], Yin shows that for each d, there is a pairing, compatible with the cup product pairing in cohomology,

$$\mathrm{Mon}^d \times \mathrm{Mon}^{2m-d} \to \mathbf{Q}, \tag{24}$$

whose kernel is entirely generated by the Kimura relations (*iv*), which will not appear in our case thanks to our choice $b_S \gg m$ (as there is not enough variables X_{ij} for any Kimura relation). Therefore, the pairing (24) is perfect for any d. In particular, the cohomology classes of monomials $[M_i]$'s are linearly independent. (Monomials of different weighted degrees are mapped to cohomology groups of different degrees hence automatically linearly independent.) □

The following observation is due to Voisin [62, Corollary 5.10]: by using Theorem 7.1 for *all* (simply-connected) surfaces, one can in some sense get rid of the Kimura relations. The authors thank Claire Voisin for allowing them to write down a proof here with more details and thank Charles Vial and Qizheng Yin for helpful discussions on the proof below.

[12] or any field extension of $\mathbf{Q}$.

Proposition 7.4. *Notation is as before. Consider the intersection ideal*

$$I := \bigcap_{\pi_1(S)=\{1\}} \operatorname{Ker}\left(\mathrm{cl}_S^H\right),$$

where the intersection is indexed by all simply-connected smooth projective surfaces. Let $J := \langle\ (i), (ii)\ \rangle$ be the ideal of $R = \mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t]$ generated by the first two sets of relations in Theorem 7.1. Then these two ideals have the same intersection with the subring of polynomials on variables $X_{i,j}$'s, Y_k's and E_l's:

$$I \cap \mathbf{Q}[X_{i,j}, Y_k, E_l] = J \cap \mathbf{Q}[X_{i,j}, Y_k, E_l].$$

Proof. First of all, since the relations in (i) and (ii) do not depend on surfaces (*i.e.* the numerical invariants a_S, b_S do not appear), they are contained in $\operatorname{Ker}\left(\mathrm{cl}_S^H\right)$ for all S, hence in the intersection ideal I. Therefore $J \subset I$ and so are their intersections with $\mathbf{Q}[X_{i,j}, Y_k, E_l]$.
For the inverse inclusion, let $P \in I \cap \mathbf{Q}[X_{i,j}, Y_k, E_l]$ and let us show that $P \in J$. For any simply-connected surface S with $b_S := b_2(S) > m/2$, since any Kimura relation (iv) requires at least $2b_S + 2$ different indices in its variables X_{ij}, thus cannot appear, P must lie in the ideal $\langle\ J, A - a_S, B - b_S\ \rangle$ by Theorem 7.1, where $a_S := (K_S^2)$. Therefore

$$P(X_{i,j}, Y_k, E_l) \in \bigcap_{(a,b)\in\Lambda} \langle\ J, A - a, B - b\ \rangle, \tag{25}$$

where the index set is as before

$$\Lambda := \left\{(a_S, b_S) \in \mathbf{Q} \times \mathbf{Q} \;\middle|\; S \text{ simply-connected surface with } \chi_{top}(S) > m/2\right\}. \tag{26}$$

which is Zariski dense in $\mathbf{Q} \times \mathbf{Q}$ by Lemma 7.3.
The rest of the proof is mostly commutative algebra/affine algebraic geometry. Let

$$M := \langle X_{i,j}, Y_k, E_l, Z_t; i, j, k, l, t \in \{1, \cdots m\}; i \neq j\ \rangle \tag{27}$$

be the ideal of $R = \mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t]$ generated by the variables $X_{i,j}, Y_k, E_l$ and Z_t. It is easy to verify that there exists $n \in \mathbf{N}$ such that

$$M \supset J \supset M^n. \tag{28}$$

Consider the chain of closed subschemes $\operatorname{Spec}(R/M) \hookrightarrow \operatorname{Spec}(R/J) \hookrightarrow \operatorname{Spec}(R/M^n)$ of $\operatorname{Spec}(R)$ which is an affine space over the affine plane $\operatorname{Spec}(\mathbf{Q}[A, B])$.

Lemma 7.5. *There exists a non-zero polynomial $f \in \mathbf{Q}[A, B]$ such that $f(A, B)P(X_{i,j}, Y_k, E_l) \in J$.*

Proof. On the one hand, by the theorem of generic flatness, there exists a principal open subset $D_f := (f \neq 0)$ of $\operatorname{Spec}(\mathbf{Q}[A, B])$ associated to a (non-zero) polynomial $f \in \mathbf{Q}[A, B]$, over which the closed subschemes $\operatorname{Spec}(R/\langle J, P\rangle) \hookrightarrow \operatorname{Spec}(R/J) \hookrightarrow \operatorname{Spec}(R/M^n)$ are all flat. Hence the lengths of their fibers over any point $(a, b) \in D_f$ are constant and finite. On the other hand, (25) says that the closed immersion $\operatorname{Spec}(R/\langle J, P\rangle) \hookrightarrow \operatorname{Spec}(R/J)$ is an equality on the fibers over any point (a, b) in the subset $\Lambda \subset \mathbf{Q} \times \mathbf{Q}$ defined in (26). By the Zariski density of Λ (proved in Lemma 7.3), we have $\Lambda \cap D_f \neq \emptyset$, which implies that the lengths of fibers of $\operatorname{Spec}(R/\langle J, P\rangle)$ and $\operatorname{Spec}(R/J)$ over D_f are equal, hence the scheme $\operatorname{Spec}(R/\langle J, P\rangle)$ is equal to $\operatorname{Spec}(R/J)$ over D_f. In other words, P is in the ideal of the localization ring R_f generated by the image of J, which is exactly the statement of the lemma (up to replace f by its power). □

Now the proof of Proposition 7.4 is finished: denote by $\overline{P}$ the class of $P(X_{i,j}, Y_k, E_l)$ in R/J. by Lemma 7.5, there is a non-zero element $f \in \mathbf{Q}[A, B]$ such that $f \cdot \overline{P} = 0$ in R/J. However by Proposition 7.2, R/J is torsion-free as a $\mathbf{Q}[A, B]$-module hence $\overline{P} = 0 \in R/J$, *i.e.* $P \in J$ as desired. □

Let us state the following consequence of Proposition 7.4 which will be used in §8 during the proof of the main theorem.

Corollary 7.6. *Notation is as in Theorem 7.1 and Proposition 7.4. Given a natural number m, let $P \in \mathbf{Q}[X_{i,j}, Y_k, E_l\,;\ 1 \le i,\ j,\ k,\ l \le m]$ be a polynomial. If $P \in I$, namely for any simply-connected smooth projective surface S,*

$$\mathrm{cl}^H_S(P) := P\left([\Delta_{i,j}], \mathrm{pr}_k^*[c_1(S)], \mathrm{pr}_l^*[c_2(S)]\right) = 0 \text{ in } H^{2*}(S^m, \mathbf{Q}),$$

then for any projective K3 surface S,

$$P\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) = 0 \text{ in } \mathrm{CH}^*(S^m).$$

Proof. The assumption on the vanishing of $\mathrm{cl}^H_S(P)$ in cohomology for all simply-connected surface S allows us to apply Proposition 7.4 to conclude that P belongs to the ideal J generated by the first two sets of relations (i) and (ii) in Theorem 7.1. Therefore it suffices to check all these relations for S a projective K3 surface and $A = 0, B = 24 \cdot \mathbb{1}, X_{i,j} = \Delta_{i,j}, Y_k = \mathrm{pr}_k^* c_1(S) = 0, E_l = \mathrm{pr}_l^* c_2(S)$ and $Z_t = \mathrm{pr}_t^* c_S$, where $\mathbb{1}$ is the (Chow) fundamental class of S^m and $c_S \in \mathrm{CH}_0(S)$ is the canonical *Beauville-Voisin class* in Theorem 1.10. The *trivial relations* in (i) hold in fact for all surfaces and are very easy to check; while the relations in (ii) hold for K3 surfaces thanks to Theorem 1.10, recalled in §1, due to Beauville and Voisin [10]. □

8. Proof of the main result

With all the ingredients being prepared in the previous sections, namely Corollary 5.5, Proposition 6.5 and Corollary 7.6, we are ready to prove Proposition 4.3, which implies the main Theorem 1.6 as shown in the end of §4.

Proof of Proposition 4.3: Recall that our goal is to show that given a projective K3 surface S,

$$W = \mathrm{Sym}(Z)$$

in $\oplus_{g_1,g_2,g_3\in\mathfrak{S}_n} \mathrm{CH}\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}\right)$. Now for any given triple $g_1, g_2, g_3 \in \mathfrak{S}_n$, let us show that the restrictions on $(S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}$ of W and $\mathrm{Sym}(Z)$, still denoted by W and $\mathrm{Sym}(Z)$ for simplicity, are equal in $\mathrm{CH}^N\left((S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}\right)$, where $N := 4n - \mathrm{age}(g_1) - \mathrm{age}(g_2) - \mathrm{age}(g_3)$ is their codimension[13]. Denote by $m := |O(g_1)| + |O(g_2)| + |O(g_3)|$, and we identify S^m and $(S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}$ in the sequel. For any $d \in \mathbf{N}$, we denote also by $W_d \in \mathrm{CH}^N(S^m)$ the restriction of W_d defined in (20) to the component $(S^n)^{g_1} \times (S^n)^{g_2} \times (S^n)^{g_3}$.

As in §7, we consider the graded polynomial ring $\mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t\,; 1 \le i,\ j,\ k,\ l,\ t \le m]$ with $\deg(A) = \deg(B) = 0, \deg(X_{i,j}) = \deg(E_l) = \deg(Z_t) = 2$ and $\deg(Y_k) = 1$, and its homogenous ideal $I := \cap_S \mathrm{Ker}\left(\mathrm{cl}^H_S\right)$ in Proposition 7.4, where S runs over all simply-connected smooth projective surfaces and cl^H_S is the graded homomorphism of 'cohomology class map' defined in Theorem 7.1. We consider also the (homogenous) subring $\mathbf{Q}[X_{i,j}, Y_k, E_l]$ and let $I' := I \cap \mathbf{Q}[X_{i,j}, Y_k, E_l]$ be the restricted (homogenous) ideal. Choose a graded $\mathbf{Q}$-vector space complement C of I' in $\mathbf{Q}[X_{i,j}, Y_k, E_l]$:

$$C \oplus I' = \mathbf{Q}[X_{i,j}, Y_k, E_l]. \tag{29}$$

[13]This precise formula of N will not be used later.

Then by construction, the product of cohomology cycle class maps restricted to C

$$\prod_S \mathrm{cl}_S^H : C \hookrightarrow \prod_{\pi_1(S)=\{1\}} H^{2*}(S^m, \mathbf{Q})$$

is injective. Recall that N is the codimension of W_d and Z. Since C_N, the degree N-part of C, is finite-dimensional[14], there exists a finite number of simply-connected surfaces $S_1, \cdots, S_r$ such that

$$\prod_{i=1}^{r} \mathrm{cl}_{S_i}^H : C_N \hookrightarrow \prod_{i=1}^{r} H^{2N}(S_i^m, \mathbf{Q}) \tag{30}$$

is already injective.

By Proposition 6.5, there exist universal polynomials of (weighted) degree N

$$P_d, Q \in \mathbf{Q}[X_{i,j}, Y_k, E_l]$$

for all $d \in \mathbf{N}$, such that for any smooth projective surface S and for any $d \in \mathbf{N}$, we have equalities in $\mathrm{CH}^N(S^m)$:

$$W_d = P_d\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right); \tag{31}$$

$$\mathrm{Sym}(Z) = Q\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right). \tag{32}$$

Let for any $d \in \mathbf{N}$, $P_d = P'_d + P''_d$ and $Q = Q' + Q''$ be the decompositions of P_d and Q with respect to (29). In particular $Q', P'_d \in C_N$ and $Q'', P''_d \in I'$.

By taking cohomology classes in (31) and (32), one can interprate Corollary 5.5 as saying that for any simply-connected surface S, the left hand side below has covergence radius at least 1 and

$$\lim_{q \to -1^+} \left(\sum_{d=0}^{\infty} \mathrm{cl}_S^H(P_d) \cdot q^d \right) = \mathrm{cl}_S^H(Q) \ \text{ in } \ H^{2N}(S^m, \mathbf{Q}).$$

As $P''_d, Q'' \in I' \subset I \subset \mathrm{Ker}(\mathrm{cl}_S^H)$, we obtain that for any simply-connected surface S, the convergence radius of the power series below is at least 1 and

$$\lim_{q \to -1^+} \left(\sum_{d=0}^{\infty} \mathrm{cl}_S^H(P'_d) \cdot q^d \right) = \mathrm{cl}_S^H(Q') \ \text{ in } \ H^{2N}(S^m, \mathbf{Q}),$$

which implies that, via the injective linear map (30) between two finite-dimensional vector spaces, the image of the q-power series (with C_N-coefficients) $\sum_{d=0}^{\infty} P'_d \cdot q^d$ has convergence radius at least 1 and has limit equal to the image of Q' when $q \to -1^+$. As the topology on finite dimensional vector spaces is unique and injective linear maps preserve and detect limits, the same holds true in C_N: the q-power series below has convergence radius at least 1 and

$$\lim_{q \to -1^+} \left(\sum_{d=0}^{\infty} P'_d \cdot q^d \right) = Q' \ \text{ in } \ C_N. \tag{33}$$

Let us emphasize that this equality (33), by definition, is really about the coefficient-wise convergences and limits, namely, for any monomial M on $X_{i,j}$'s, Y_k's and E_l's, the numerical q-power series formed by the corresponding coefficients of M in P'_d, has convergence radius at least 1 and has limit equal to the corresponding coefficient of M in Q', when $q \to -1^+$.

[14] This is the reason why we have to restrict to the subring $\mathbf{Q}[X_{i,j}, Y_k, E_l]$: the complement of the ideal I in $\mathbf{Q}[A, B, X_{i,j}, Y_k, E_l, Z_t]$ is not finite-dimensional in each degree because $\deg(A) = \deg(B) = 0$.

Now let us specialize back to the case that S is a smooth projective K3 surface. We evaluate in (33) with $X_{i,j} = \Delta_{i,j}$, $Y_k = \mathrm{pr}_k^* c_1(S) = 0$ and $E_l = \mathrm{pr}_l^* c_2(S) \in \mathrm{CH}(S^m)$ to obtain that

$$\lim_{q\to -1^+}\left(\sum_{d=0}^{\infty} P'_d\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right)\cdot q^d\right) = Q'\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) \text{ in } \mathrm{CH}^N(S^m). \tag{34}$$

On the other hand, thanks to Corollary 7.6, the assumption that $Q'', P''_d \in I'$ guarantees that

$$P''_d\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) = Q''\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) = 0 \text{ in } \mathrm{CH}^N(S^m).$$

Therefore (34) is equivalent to

$$\lim_{q\to -1^+}\left(\sum_{d=0}^{\infty} P_d\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right)\cdot q^d\right) = Q\left(\Delta_{i,j}, \mathrm{pr}_k^* c_1(S), \mathrm{pr}_l^* c_2(S)\right) \text{ in } \mathrm{CH}^N(S^m).$$

which is the following by (31) and (32):

$$\lim_{q\to -1^+}\left(\sum_{d=0}^{\infty} W_d\cdot q^d\right) = \mathrm{Sym}(Z) \text{ in } \mathrm{CH}^N(S^m). \tag{35}$$

However, since S is a K3 surface, $S^{[n]}$ is a holomorphic symplectic variety, hence all its Gromov-Witten invariants vanish in the strong sense, proved in the following Lemma 8.1, that for all $d > 0$,

$$\gamma_d = 0 \in \mathrm{CH}_{2n}(S^{[n]}\times S^{[n]}\times S^{[n]}),$$

where γ_d is the cycle controlling the quantum corrections defined in (14). This implies that $W_d = 0$ in $\mathrm{CH}^N(S^m)$ for all $d > 0$ by their definition (20). Therefore in (35) all terms with $d \geq 1$ vanish and it actually says

$$W := W_0 = \mathrm{Sym}(Z)$$

in $\mathrm{CH}^N(S^m)$. Proposition 4.3 is proved, so is the main Theorem 1.6. □

Lemma 8.1. *Let S be a projective K3 surface. Let $\gamma_d \in \mathrm{CH}_{2n}\left((S^{[n]})^3\right)$ be the algebraic cycle defined in (14). Then $\gamma_d = 0$ in $\mathrm{CH}_{2n}\left((S^{[n]})^3\right)$ for all $d > 0$.*

Proof. Recall that in (14), γ_d is defined as the push-forward by the evaluation morphism of the virtual fundamental cycle $[M]^{vir} \in \mathrm{CH}_{2n}(M)$, where $M := \overline{M_{0,3}}\left(S^{[n]}, d\beta\right)$ is the moduli space of stable maps from genus zero curves with three marked points to $S^{[n]}$ with class $d\beta \in H_2(S^{[n]}, \mathbf{Z})$. We will actually prove that in $\mathrm{CH}_{2n}(M)$,

$$[M]^{vir} = 0.$$

This result is well-known. Consider a family of smooth projective holomorphic symplectic varieties $\pi : \mathcal{X} \to B$ whose central fiber over $b_0 \in B$ is isomorphic to $S^{[n]}$, such that the class β is not a Hodge class over a general point of B. Let $\mathcal{M}/B$ be the relative moduli space of stable maps from genus zero curves with three marked points to fibers of π with class $d\beta$. Then we know that the fiber of $\mathcal{M}/B$ over a general point $b \in B$ is the moduli space $\mathcal{M}_b \simeq \overline{M_{0,3}}(\mathcal{X}_b, d\beta)$, which is empty since $d\beta$ is not a curve class, hence $[\mathcal{M}_b]^{vir} = 0$ for a general point $b \in B$. By specialization, we find that $[M]^{vir} = \left[\mathcal{M}_{b_0}\right]^{vir} = 0$ in $\mathrm{CH}(M)$. □

9. Application (I): The Chow ring structure

In this section, we will give the multiplication table of the Chow rings of Hilbert schemes of K3 surfaces. First of all, let us compute the obstruction bundles $F_{g,h}$, as well as its top Chern classes, in our situation. Denote by $\mathbf{n} := \{1, 2, \cdots, n\}$ and identify S^n with $S^{\mathbf{n}}$. For any $g \in \mathfrak{S}_n$, viewed always as a self-bijection of $\mathbf{n}$, let $O(g) := \mathbf{n}/g$ be the set of orbits of g and we identify naturally $(S^n)^g$ with $S^{O(g)}$. Similarly, for $g, h \in \mathfrak{S}_n$, the set of orbits $O(g,h) := \mathbf{n}/< g, h >$ and $(S^n)^{<g,h>}$ is identified with $S^{O(g,h)}$.

Lemma 9.1. *For any $g \in \mathfrak{S}_n$, the class in $K_0\left(S^{O(g)}\right)_{\mathbf{Q}}$ of the (virtual) bundle V_g defined in Definition 2.2 (3) is*

$$V_g = \sum_{o \in O(g)} \frac{|o| - 1}{2} \operatorname{pr}_o^* TS,$$

where $\operatorname{pr}_o : S^{O(g)} \twoheadrightarrow S$ *is induced by* $\{o\} \hookrightarrow O(g)$*. In particular, the (virtual) rank of* V_g *is* $\operatorname{age}(g) = n - |O(g)|$ *and* $V_g = V_{g^{-1}}$*.*

Proof. It is a direct computation from the definition. □

For any $g, h \in \mathfrak{S}_n$, we have natural surjective maps between orbit sets:

$$\begin{tikzcd}
O(g) \arrow[dr, two heads, "\pi_g"] & \\
O(h) \arrow[r, two heads, "\pi_h"] & O(g,h) \\
O(gh) \arrow[ur, two heads, "\pi_{gh}"'] &
\end{tikzcd}$$

corresponding to natural embeddings of $S^{O(g,h)}$ into $S^{O(g)}$, $S^{O(h)}$ and $S^{O(gh)}$. For the combinatorics of orbits, we have the following important number:

Lemma 9.2 (Graph defect [37, Lemma 2.7])**.** *For any $g, h \in \mathfrak{S}_n$, we have that for each $o \in O(g,h)$, the* graph defect *defined by*

$$d_{g,h}(o) := \frac{2 + |o| - |\pi_g^{-1}(o)| - |\pi_h^{-1}(o)| - |\pi_{gh}^{-1}(o)|}{2} = \frac{2 + |o| - |o/g| - |o/h| - |o/gh|}{2}, \tag{36}$$

is always a non-negative integer.

Lemma 9.3. *For any $g, h \in \mathfrak{S}_n$, let $d_{g,h}(o)$ be the graph defect defined in (36) for any $o \in O(g,h)$.*

(i) The class in $K_0\left(S^{O(g,h)}\right)_{\mathbf{Q}}$ of the obstruction bundle $F_{g,h}$ defined in (4) is

$$F_{g,h} = \sum_{o \in O(g,h)} d_{g,h}(o) \operatorname{pr}_o^* TS.$$

(ii) The virtual rank of $F_{g,h}$ is $r_{g,h} := n + 2|O(g,h)| - |O(g)| - |O(h)| - |O(gh)| = 2|O(g,h)| + \operatorname{age}(g) + \operatorname{age}(h) + \operatorname{age}(gh) - 2n$*, which is an even non-negative integer.*

(iii) The top Chern class of $F_{g,h}$, called the obstruction class $c_{g,h} \in \mathrm{CH}\left(S^{O(g,h)}\right)$ *is*

$$c_{g,h} := c_{top}\left(F_{g,h}\right) = \begin{cases} 0, & \text{if } \exists\, o \in O(g,h) \text{ with } d_{g,h}(o) \geq 2; \\ \prod_{o \in I}\left(24\,\mathrm{pr}_o^*(c_S)\right), & \text{if } \forall o \in O(g,h) \text{ has } d_{g,h}(o) = 0 \text{ or } 1. \end{cases} \tag{37}$$

where $I := \{o \in O(g,h) \mid d_{g,h}(o) = 1\}$.

Proof. The definition of $F_{g,h}$ becomes the following in our situation:

$$F_{g,h} := V_g\big|_{S^{O(g,h)}} + V_h|_{S^{O(g,h)}} + V_{gh}\big|_{S^{O(g,h)}} + TS^{O(g,h)} - TS^n|_{S^{O(g,h)}} \in K_0\left(S^{O(g,h)}\right)_{\mathbf{Q}}.$$

By Lemma 9.1, $V_g\big|_{S^{O(g,h)}} = \sum_{o \in O(g,h)} \frac{|o| - |\pi_g^{-1}(o)|}{2}\,\mathrm{pr}_o^*\, TS$, similarly for h and gh. The formula (i) for $F_{g,h}$ then follows.
(ii) and (iii) follow immediately from (i), Lemma 9.2 and Theorem 1.10 (2). □

The main result of this section is Corollary 1.9, which is a multiplication table for $\mathrm{CH}^*\left(S^{[n]}\right)$. See the precise statement in Introduction §1.1.

Proof of Corollary 1.9. As is pointed out in Remark 1.7, the isomorphisms in Theorem 1.6 hold true for $\mathbf{Q}$-coefficient if one modifies the sign of the orbifold product in the following way: For any $g, h \in \mathfrak{S}_n$, let $\epsilon(g,h) := (-1)^{\frac{\mathrm{age}(g)+\mathrm{age}(h)-\mathrm{age}(gh)}{2}}$, define the orbifold product *with discrete torsion* of $\alpha \in \mathrm{CH}\left((S^n)^g\right)$ and $\beta \in \mathrm{CH}\left((S^n)^h\right)$ to be

$$\alpha \star_{orb,dt} \beta := \epsilon(g,h) \cdot \alpha \star_{orb} \beta.$$

The resulting $\mathbf{Q}$-algebra is then denoted by $\mathrm{CH}^*_{orb,dt}\left([S^n/\,\mathfrak{S}_n]\right)$. For details on discrete torsion and how it allows us to work with rational coefficients, see our previous work [28].

In short, we have an injective homomorphism of $\mathbf{Q}$-algebras:

$$\mathrm{CH}^*\left(S^{[n]}\right) \hookrightarrow \mathrm{CH}^*_{orb,dt}\left([S^n/\,\mathfrak{S}_n]\right),$$

hence the product structure on the right hand side determines that of the left hand side. Now the statement of Corollary 1.9 becomes a reformulation of the definition of the orbifold product (Definition 2.2) together with the computation of the obstruction class $c_{g,h}$ in (37). □

The statement of Corollary 1.9 may seem abstract, but it is actually an effective algorithm. Let us show some non-trivial examples here.

Example 9.4. Let S be a projective K3 surface. We are interested in the powers of divisors of $S^{[n]}$. We know that $H^2(S^{[n]}) \simeq H^2(S) \oplus \mathbf{Q}[E]$, where E is the exceptional divisor and for each divisor L on S, $\sum_{i=1}^n \mathrm{pr}_i^* L$ on S^n descends to a divisor on $S^{(n)}$, which is pulled back to a divisor $\widetilde{L}$ on $S^{[n]}$. Therefore a $\mathbf{Q}$-divisor on $S^{[n]}$ is of the form $\widetilde{L} + \mu E$ with $L \in \mathrm{CH}^1(S)$ and $\mu \in \mathbf{Q}$. Let us compute some powers of $\widetilde{L} + \mu E$. Via the injective homomorphism of Corollary 1.9, $\widetilde{L}$ is identified with $L_1 + \cdots + L_n$ on the component S^n and E is identified with the element which is the fundamental class in the component $(S^n)^{(ij)} \simeq S^{n-1}$ for each transposition $(ij) \in \mathfrak{S}_n$.

In the notation of the following computations, '=' means the image of the left hand side, via the injective homomorphism, is equal to the sum of all terms on the right hand side ; $c := c_S \in \mathrm{CH}_0(S)$ is the Beauville-Voisin class ; $L_i := \mathrm{pr}_i^*(L)$ is the pull-back by the i-th projection of L, similarly for c_i ; $\mathbb{1} \in \mathrm{CH}^0$ means the fundamental class ; $a := (L^2)$ is the self-intersection number and $e := 24$ is the Euler characteristic of the K3 surface S.

(1) ($n = 2$). This is very classically and well-known due to the geometric construction of $S^{[2]}$ given by the quotient of $\mathrm{Bl}_\Delta(S \times S)$ by involution. Our computation recovers known results without using the geometric construction.

g	id	(12)
$(S \times S)^g$	$S \times S$	S
$\widetilde{L} + \mu E =$	$L_1 + L_2$	$\mu \mathbb{1}$
$(\widetilde{L} + \mu E)^2 =$	$a(c_1 + c_2) + 2L_1L_2 - \mu^2\Delta_S$	$4\mu L$
$(\widetilde{L} + \mu E)^3 =$	$3a(c_1L_2 + c_2L_1) - 6\mu^2\Delta_*(L)$	$(12\mu a - e\mu^3)c$

(2) ($n = 3$).

g	id	(12), (13) and (23)	(123) and (132)
$(S^3)^g$	$S \times S \times S$	$S \times S$	S
$\widetilde{L} + \mu E =$	$L_1 + L_2 + L_3$	$\mu \mathbb{1}$	0
$(\widetilde{L} + \mu E)^2 =$	$a(c_1+c_2+c_3)$ $+2(L_1L_2+L_2L_3+L_1L_3)$ $-\mu^2(\Delta_{12}+\Delta_{23}+\Delta_{13})$	$2\mu(2L_1 + L_2)$	$3\mu^2 \mathbb{1}$
$(\widetilde{L} + \mu E)^3 =$	$3a(c_1+c_2+c_3)(L_1+L_2+L_3)$ $+6L_1L_2L_3$ $-3\mu^2(L_1+L_2+L_3)(\Delta_{12}+\Delta_{23}+\Delta_{13})$	$3\mu(4ac_1+ac_2+4L_1L_2)$ $-\mu^3(ec_1+8\Delta)$	$27\mu^2L$
$(\widetilde{L} + \mu E)^4 =$	$(L_1+L_2+L_3)^4$ $-6\mu^2(L_1+L_2+L_3)^2(\Delta_{12}+\Delta_{23}+\Delta_{13})$ $+\mu^4(e(c_1c_2+c_2c_3+c_3c_1)+24\delta)$	$24a\mu(2c_1L_2+c_2L_1)$ $-4\mu^3ec_1L_2$ $-96\mu^3(c_1L_2+c_2L_1)$	$162\mu^2ac$ $-27\mu^4ec$

10. Application (II): Weak splitting property and the Beauville-Voisin Conjecture

As is presented in the Introduction §1.2, Beauville-Voisin's Theorem 1.10 on the Chow rings of K3 surfaces, together with his earlier results on the multiplicative decomposition of Chow rings of abelian varieties [7], leads Beauville to speculate in [9] the Splitting Property Conjecture 1.11 for all holomorphic symplectic varieties. To avoid speaking of the conjectural Bloch-Beilinson filtration in the statement, we are actually interested in its down-to-earth consequences, namely the Weak Splitting Property Conjecture 1.12 and its strengthening, Beauville-Voisin Conjecture 1.13. These conjectures recently attract a lot of research interest:

Known cases:

- (Beauville [9]) For S a K3 surface, its Hilbert square $S^{[2]}$ and Hilbert cube $S^{[3]}$ satisfy the Weak Splitting Property Conjecture;
- (Voisin [61]) For S a K3 surface, and $n \leq 2b_2(S)_{tr} + 4$, then the Hilbert scheme $S^{[n]}$ satisfies the Beauville-Voisin Conjecture, where $b_{2,tr} = b_2 - \rho$ is the second transcendental Betti number;
- (Voisin [61]) For X a smooth cubic fourfold, its Fano variety of lines F satisfies the Beauville-Voisin Conjecture;
- (Ferretti [24]) A very general double EPW sextic satisfies the Beauville-Voisin conjecture;
- (Fu [26]) For A an abelian surface and n a natural number. The generalized Kummer variety $K_n(A)$ satisfies the Beauville-Voisin Conjecture;
- (Rieß [53]) For X an irreducible holomorphic symplectic variety with an isotropic non-trivial line bundle, if X satisfies the rational Lagrangian fibration conjecture (for example, when

X is deformation equivalent to Hilbert schemes of K3 surfaces or generalized Kummer varieties), then X satisfies the Weak Splitting Property conjecture;
- (Yin [65]) For S a K3 surface which is Kimura finite dimensional[15], and n a natural number, then the Hilbert scheme $S^{[n]}$ satisfies the Beauville-Voisin Conjecture.

The goal of this section is to improve the results of Voisin [61] mentioned above for Hilbert schemes of K3 surfaces. See Theorem 1.14 in Introduction §1.2 for the precise statement.

Let S be a projective K3 surface. Given $m \in \mathbf{N}$ and divisors $L_1, \ldots, L_\rho$ on S with $(L_s \cdot L_t) = a_{s,t}$. Assume these divisors are linearly independent (hence ρ is at most the Picard number of S). Denote by $b := 22 - \rho$. Consider the following homomorphism of graded $\mathbf{Q}$-algebras:

$$\begin{array}{rcl} \mathrm{cl}^{CH} : \mathbf{Q}[X_{i,j}, Y_{k,t}, Z_l \,;\, 1 \le i,\ j,\ k,\ l \le m, 1 \le t \le \rho; i \neq j] & \to & \mathrm{CH}^*(S^m)_{\mathbf{Q}} \\ X_{i,j} & \mapsto & \Delta_{i,j} \\ Y_{k,t} & \mapsto & L_{k,t} := \mathrm{pr}_k^*(L_t) \\ Z_l & \mapsto & c_l := \mathrm{pr}_l^*(c_S), \end{array}$$

where $\mathrm{cl}_L^{\mathrm{CH}}$ stands for the 'Chow class map'; on the left hand side, the degrees of variables are given by $\deg(X_{i,j}) = \deg(Z_l) = 2$ and $\deg(Y_{k,t}) = 1$; on the right hand side, $c_S \in \mathrm{CH}^2(S, \mathbf{Q})$ is the Beauville-Voisin class of S.
Denote by $[-] : \mathrm{CH}^* \to H^{2*}$ the cycle class map. Define

$$\mathrm{cl}^H : \mathbf{Q}[X_{i,j}, Y_{k,t}, Z_l \,;\, 1 \le i,\ j,\ k,\ l \le m; 1 \le t \le \rho; i \neq j] \to H^{2*}(S^m, \mathbf{Q})$$

to be the composition $[-] \circ \mathrm{cl}^{\mathrm{CH}}$. One key ingredient for the proof of Theorem 1.14 is again Yin's result [65] in the following form (compare it to Theorem 7.1).

Theorem 10.1 (Yin [65])**.** *Notation is as before. Then the ideal* $\mathrm{Ker}\left(\mathrm{cl}^H\right)$ *is generated by the following three sets of relations:*

(i) (Trivial relations) For all $1 \le i, j, k \le m$ *with* i, j, k *distinct,*
- $X_{ij} - X_{ji}$
- Z_i^2
- $Y_{i,r} Z_i$
- $X_{i,j} X_{j,k} - X_{i,k} X_{j,k}$
- $X_{i,j} Z_i - Z_i Z_j$

(ii) (Beauville-Voisin relations) For all $1 \le i, j, k \le m, 1 \le t, s \le \rho$ *with* i, j, k *distinct,*
- $Y_{i,t} Y_{i,s} - a_{s,t} Z_i$
- $X_{i,j} Y_{i,t} - Y_{i,t} Z_j - Y_{j,t} Z_i$
- $X_{i,j}^2 - 24 Z_i Z_j$
- $X_{i,j} X_{j,k} - \left(X_{i,j} Z_k + X_{i,k} Z_j + X_{j,k} Z_i\right) + \left(Z_i Z_j + Z_i Z_k + Z_j Z_k\right)$

(iii) (Kimura relation) For any two subsets $I = \{i_1, \cdots, i_{b+1}\}; I' = \{i'_1, \cdots, i'_{b+1}\}$ *of* $\{1, \cdots, m\}$ *satisfying* $|I| = |I'| = b + 1 = 22 - \rho + 1$ *and* $I \cap I' = \emptyset$,
- $\sum_{\sigma \in \mathfrak{S}_{b+1}} \mathrm{sgn}(\sigma) \prod_{k=1}^{b+1} T_{i_k, i'_{\sigma(k)}}$ *where* $T_{i,j} := X_{i,j} - Z_i - Z_j - \sum_{s,t=1}^{\rho} a'_{s,t} Y_{i,s} Y_{j,t}$, $b = 22 - \rho$ *and* $(a'_{s,t})$ *is the inverse of the matrix* $(a_{s,t})$.

We will need the following fact on combinatorics of permutations:

[15]Conjecturally ([32]), all K3 surfaces, even all varieties, should be Kimura finite dimensional. However the Kimura finite dimensionality of K3 surfaces is wide open.

Lemma 10.2. *If $n < (b+1)(b+2)$, then for any $g \in \mathfrak{S}_n$, either $|O(g)| < 2(b+1)$ or for any $2(b+1)$ distinct orbits of g: $o_1, \cdots, o_{b+1}, o'_1, \cdots, o'_{b+1} \in O(g)$, there exist $i \neq j$, such that o_i and o_j have the same length or o'_i and o'_j have the same length.*

Proof. It suffices to compute the smallest n which admits a permutation $g \in \mathfrak{S}_n$ with $2(b+1)$ distinct orbits $o_1, \cdots, o_{b+1}, o'_1, \cdots, o'_{b+1}$ such that $o_1, \cdots, o_{b+1}$ have distinct lengths and $o'_1, \cdots, o'_{b+1}$ have distinct lengths. The minimal n is $2 \times (1 + 2 + \cdots + b + (b+1))) = (b+1)(b+2)$. □

Proof of Theorem 1.14. (1) For Conjecture 1.12, as is pointed out by Beauville [9, Corollary 2.3], the result of Verbitsky and Bogomolov [13] reduces the weak splitting property to show that for any $\mathbf{C}$-divisor $D \in \mathrm{CH}^1(X)_{\mathbf{C}}$ isotropic with respect to the Beauville-Bogomolov-Fujiki quadratic form q satisfies $D^{n+1} = 0$ in $\mathrm{CH}^{n+1}(X)_{\mathbf{C}}$. Let $D = \widetilde{L} + \mu E$ be such a $\mathbf{C}$-divisor, where $\mu \in \mathbf{C}$, E is the exceptional divisor and $L \in \mathrm{CH}^1(S)_{\mathbf{C}}$ which gives rise to an $\mathfrak{S}_n$-invariant divisor $\sum_{i=1}^n \mathrm{pr}_i^*(L)$ on S^n, which descends to a divisor on $S^{(n)}$, and $\widetilde{L}$ denotes its pull-back to $S^{[n]}$ via the Hilbert-Chow morphism. The condition $q(D) = 0$ is equivalent to the condition on the self-intersection number

$$(L^2) = 2\mu^2(n-1).$$

If $\mu = 0$, *i.e.* $(L^2) = 0$, then by Theorem 1.10 (*i*), we have $L^2 = 0$ in $\mathrm{CH}_0(S)$ and then it is easy to see that $D^{n+1} = \widetilde{L}^{n+1} = 0$ in $\mathrm{CH}^{n+1}(X)_{\mathbf{C}}$. We assume from now on that $\mu \neq 0$, by rescaling L, one can further assume that $\mu = 1$, *i.e.*

$$D = \widetilde{L} + E \ \text{ with } \ (L^2) = 2(n-1).$$

The image of D via the injective ring homomorphism of Corollary 1.9 is the following element in $\bigoplus_{g \in \mathfrak{S}_n} \mathrm{CH}^{1-\mathrm{age}(g)}((S^n)^g)_{\mathbf{C}}$ (*cf.* Example 9.4):

$$\alpha = (L_1 + \cdots + L_n)_{\mathrm{id}} + \sum_{i<j} (\mathbb{1})_{(ij)}, \tag{38}$$

here and in the sequel the subscripts $g \in \mathfrak{S}_n$ are to keep track of the component $(S^n)^g$ where the cycle lives on. The value of a cycle $\gamma \in \bigoplus_{g \in \mathfrak{S}_n} \mathrm{CH}^*((S^n)^g)_{\mathbf{C}}$ on the summand indexed by $g \in \mathfrak{S}_n$ is denoted by $\gamma_g \in \mathrm{CH}^*((S^n)^g)_{\mathbf{C}}$.

To show $D^{n+1} = 0$, it suffices to show that $\gamma := \alpha^{\star(n+1)} = 0$, where $\star$ is the orbifold product with a sign change (the *discrete torsion*, *cf.* Remark 1.7 and Corollary 1.9). Deforming D to a purely $(2,0)$-type cohomology class shows that $[D^{n+1}] = 0$ in $H^{2(n+1)}(X, \mathbf{C})$ (*cf.* [13]). By the cohomological version of Theorem 1.6 ([37], [23], [58]), this implies that the cohomology class of $\alpha^{\star(n+1)}$ is trivial on each component, *i.e.*

$$[\gamma_g] = 0 \in H^{2(n+1)-2\,\mathrm{age}(g)}((S^n)^g, \mathbf{C}), \ \forall g \in \mathfrak{S}_n.$$

Now observe that the cycle α in (38) satisfies two properties:

(a) it is symmetric for all indices, *i.e.* under the action of $\mathfrak{S}_n$.
(b) it is a polynomial of big diagonals Δ_{ij} and the line bundle L on factors L_k.

As a result, for any $g \in \mathfrak{S}_n$, let $m := |O(g)|$, the formula for orbifold product implies that γ_g satisfies correspondingly the following two properties:

(a) γ_g is symmetric with respect to the natural action of $\mathfrak{S}_g$ on $(S^n)^g = S^{O(g)}$, where $\mathfrak{S}_g := C_g/\mathrm{Ker}(C_g \to \mathfrak{S}_{O(g)})$ with C_g being the centralizer of g. Concretely, if the partition of n (by lengths of orbits) associated to g is $(1^{a_1}2^{a_2}\cdots n^{a_n})$ (with $\sum_{i=1}^n ia_i = n$), then $\mathfrak{S}_g \simeq \mathfrak{S}_{a_1}\times\cdots\times\mathfrak{S}_{a_n}$, the product of symmetric groups permuting orbits with same lengths.

(b) γ_g is a polynomial of big diagonals Δ_{ij}, the line bundle L on factors S_k and the Beauville-Voisin class c_S on factors c_l. In other words, there exists a polynomial $P \in \mathbf{C}[X_{ij}, Y_k, Z_l\ ;\ i,\ j,\ k,\ l \in O(g); i\neq j]$ such that $\gamma_g = \mathrm{cl}^{\mathrm{CH}}(P)$.

Together, these two properties show that there exists an $\mathfrak{S}_g$-invariant polynomial P such that $\gamma_g = \mathrm{cl}^{\mathrm{CH}}(P)$. Since $[\gamma_g] = 0$, *i.e.* $P \in \mathrm{Ker}\left(\mathrm{cl}^H\right)$, hence P lies in the ideal generated by the trivial relations (i), Beauville-Voisin relations (ii) and Kimura relations (iii) in Theorem 10.1.

We use the following.

Lemma 10.3. *Keep the same notations. Assume furthermore that $n < (b+1)(b+2)$. Consider the following quotient of the sub-algebra of $\mathfrak{S}_g$-invariant polynomials by the $\mathfrak{S}_g$-invariant ideal generated by the relations in (i) and (ii) of Theorem 10.1:*

$$M := \frac{\mathbf{C}[X_{ij}, Y_{k,t}, Z_l\ ;\ i,\ j,\ k,\ l \in O(g); i\neq j; 1\leq t\leq \rho]^{\mathfrak{S}_g}}{\langle\ (i), (ii)\ \rangle}, \tag{39}$$

then any class of $\mathfrak{S}_g$-invariant polynomial P in M that is also contained in the kernal of cl^H is zero.

Proof of Lemma 10.3. We use proof by contradiction. Assume that the image of the polynomial P in M is non-zero and write it as

$$P = \sum_{(I,I')} Q_{I,I'}R_{I,I'} \text{ in } M, \tag{40}$$

where the summation is indexed by all (unordered) couples (I, I') with $I, I' \subset O(g)$ satisfying $|I| = |I'| = b+1$ and $I \cap I' = \varnothing$; $R_{I,I'}$ is the Kimura relation (iii) for the two subsets I, I', and $Q_{I,I'}$ is just a polynomial. By the argument of elimination of repeated indices (due to Voisin [61, Proof of Lemma 2.3, P. 6-10] and Yin [65], see also the proof of Proposition 7.2), modulo $\langle\ (i), (ii)\ \rangle$, one can always assume that no elements of $I \sqcup I'$ appear as indices of any variable involved in the polynomial $Q_{I,I'}$.

Consider the following representations of $\mathfrak{S}_g$:

$$V := \mathrm{Span}_{\mathbf{C}}\left\{X_{i,j}, Y_{k,t}, Z_l\ ;\ i,\ j,\ k,\ l \in O(g); i\neq j; 1\leq t\leq\rho\right\};$$

$$W := \mathrm{Span}_{\mathbf{C}}\left\{R_{I,I'}\ ;\ I, I' \subset O(g), |I| = |I'| = b+1, I\cap I' = \varnothing\right\}.$$

As W is a sub-representation of $\mathrm{Sym}^{2b+2} V$ (here and in the sequel, the symmetric product of a representation is always the weighted symmetric product), we have a natural morphism of representations

$$\mathrm{Sym}^{d-2b-2} V \otimes W \to \mathrm{Sym}^d V,$$

where $d = n + 1 - \mathrm{age}(g)$ is the degree of P. The expression (40) implies that P is in the image of this morphism. Take a preimage of P, say,

$$\widetilde{P} = \sum_{(I,I')} Q_{I,I'} \otimes R_{I,I'} \in \mathrm{Sym}^{d-2b-2} V \otimes W$$

satisfying the condition that elements of $I \sqcup I'$ do not appear as indices of variables in $Q_{I,I'}$. Replacing $\widetilde{P}$ by $\sum_{\sigma\in\mathfrak{S}_g} {}^{\sigma}\widetilde{P}$ if necessary, one can further assume that $\widetilde{P}$ is $\mathfrak{S}_g$-invariant. Let $\widetilde{P} = \sum_{(I,I')} Q_{I,I'} \otimes R_{I,I'}$ be such a preimage of P with least non-zero terms.

Now take any non-zero term of $\widetilde{P}$, say, the term $Q_{I,I'} \otimes R_{I,I'}$ indexed by $I = \{o_1, \cdots, o_{b+1}\}$ and $I' = \{o'_1, \cdots, o'_{b+1}\}$, where $o_1, \cdots, o_{b+1}$ and $o'_1, \cdots, o'_{b+1}$ are $2b+2$ distinct elements of $O(g)$. As $n < (b+1)(b+2)$ by hypothesis, Lemma 10.2 implies that there exist $1 \leq i \neq j \leq b+1$ such that o_i and o_j have the same length or o'_i and o'_j have the same length. The latter case being similar, let us assume the former: $|o_i| = |o_j|$. Note that the transposition exchanging o_i and o_j is an element of $\mathfrak{S}_g$ and $\widetilde{P}$ is $\mathfrak{S}_g$-invariant, therefore $\widetilde{P}$ stays invariant if one exchanges the two indices o_i and o_j. Let us decompose $W = W_1 \oplus W_2$ as

$$W_1 := \mathrm{Span}_{\mathbf{C}}\left\{R_{I,I'}\ ;\ I, I' \subset O(g), |I| = |I'| = b+1, I \cap I' = \varnothing \text{ and } \{o_i, o_j\} \subset I \text{ or } \{o_i, o_j\} \subset I'\right\};$$

$$W_2 := \mathrm{Span}_{\mathbf{C}}\left\{R_{I,I'}\ ;\ I, I' \subset O(g), |I| = |I'| = b+1, I \cap I' = \varnothing \text{ and } o_i, o_j \text{ do not belong simultaneously to } I \text{ or to } I'\right\}.$$

Then the two components of $\widetilde{P}$ in the induced decomposition $(\mathrm{Sym}^{d-2b-2} \otimes W_1) \oplus (\mathrm{Sym}^{d-2b-2} \otimes W_2)$ are also invariant under the transposition $(o_i o_j)$. In particular,

$$\sum_{\substack{(I,I') \\ \{o_i,o_j\}\subset I \text{ or } \{o_i,o_j\}\subset I'}} Q_{I,I'} \otimes R_{I,I'} \tag{41}$$

is invariant under the transposition $(o_i o_j)$. However, in this sum, on the one hand the $Q_{I,I'}$'s do not involve o_i or o_j and are thus invariant ; on the other hand $(o_i o_j)$ maps $R_{I,I'}$ to $-R_{I,I'}$. Therefore (41) is equal to zero and $\widetilde{P}$ has an expression with strictly less terms, which contradicts our minimality assumption. Therefore P is already zero in M. □

Since in the first case only one divisor class L is involved, we set $\rho = 1$ (thus $b = 21$ in Theorem 10.1 and $n < 506$ in Lemma 10.2). By the above lemma and Beauville-Voisin's Theorem 1.10, we have $\gamma_g = 0$ in $\mathrm{CH}^{n+1-\mathrm{age}(g)}\left(S^{O(g)}\right)_{\mathbf{C}}$. As $g \in \mathfrak{S}_n$ is fixed arbitrarily, $\gamma = 0$ hence $D^{n+1} = 0$ for any Beauville-Bogomolov-isotropic $D \in \mathrm{CH}^1(X)_{\mathbf{C}}$.

(2) For the statement involving Chern classes of the Hilbert scheme, the proof is similar. Indeed, the cycles involved are polynomials of Chern classes, diagonals, and line bundles, which are also invariant under the group $\mathfrak{S}_g$. By the same argument, the Kimura relation does not appear when $n < (b_{2,\mathrm{tr}} + 1)(b_{2,\mathrm{tr}} + 2)$ and any polynomial of line bundles and Chern classes of the Hilbert scheme vanishing in cohomology vanishes in Chow group. □

11. Application (III): Multiplicative Chow-Künneth decomposition

Let us start by introducing the following notion, which is studied in [59], [57] and [28] *etc.*

Definition 11.1 (Multiplicative Chow-Künneth decomposition)**.** Let X be a smooth projective variety. A *multiplicative Chow-Künneth decomposition* of X is a direct sum decomposition of its rational Chow motive:

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2\dim X} \mathfrak{h}^i(X),$$

such that

- (Chow-Künneth) The Betti realization gives the Künneth components: $H^*\left(\mathfrak{h}^i(X)\right) = H^i(X)$;
- (Multiplicativity) The multiplication $\mu : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ (given by the small diagonal) respects the decomposition: the restriction of μ to the summand $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$ factorizes through $\mathfrak{h}^{i+j}(X)$ for all i, j.

Equivalently, in the language of correspondences, a multiplicative Chow-Künneth decomposition of X is the data of a family of self-correspondences $\pi^0, \cdots, \pi^{2\dim X} \in \mathrm{CH}^{\dim X}(X \times X)$ satisfying:

- (Projector) $\pi^i \circ \pi^i = \pi^i$ for all i;
- (Orthogonality) $\pi^i \circ \pi^j = 0$ for all $i \neq j$;
- (Completeness) $\pi^0 + \cdots + \pi^{2\dim X} = \Delta_X$;
- (Künneth) $\mathrm{Im}(\pi^i_* : H^*(X) \to H^*(X)) = H^i(X)$;
- (Multipliativity) $\pi^k \circ \delta_X \circ (\pi^i \otimes \pi^j) = 0$ for all $k \neq i + j$, where δ_X is the small diagonal.

With this key notion defined, let us come back to Beauville's Splitting Property Conjecture 1.11. As explained in the Introduction §1.2, we believe that a more fundamental way to understand the Splitting Property is via decompositions of the Chow motives. Our ultimate goal becomes the Motivic Splitting Property Conjecture 1.15. It is indeed an enhancement of the original Conjecture 1.11: given the multiplicative Chow-Künneth decomposition of Bloch-Beilinson type

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2\dim X} \mathfrak{h}^i(X),$$

it suffices to define:

$$\mathrm{CH}^i(X)_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(X)) = \mathrm{Im}\left(\pi^{2i-s}_* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X)\right),$$

and the multiplicativity of the decomposition in Definition 11.1 implies the multiplicativity of the bigrading while the Bloch-Beilinson conditions in Conjecture 1.15 correspond to the Bloch-Beilinson conditions in Conjecture 1.11.

Now comes the main result of this section: Theorem 1.16, see §1 Introduction for the statement. In the case of Hilbert scheme $S^{[n]}$ of a K3 surface S, we would like to deduce from Theorem 1.6 a natural multiplicative Chow-Künneth decomposition for it, which is part of the content of Theorem 1.16. Before proving it, let us do some preparations: denote always by $c_S \in \mathrm{CH}_0(S)$ the Beauville-Voisin cycle, then Beauville-Voisin Theorem 1.10 yields a natural multiplicative Chow-Künneth decomposition for S:

$$\mathfrak{h}(S) = \mathfrak{h}^0(S) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S); \tag{42}$$

where $\mathfrak{h}^0(S) \simeq \mathbb{1}$ and $\mathfrak{h}^4(S) \simeq \mathbb{1}(-2)$ are given by the projectors $S \times c_S$ and $c_S \times S$ respectively. For any $m \in \mathbf{N}$, taking m-th tensor power of (42) obtains a multiplicative Chow-Künneth decomposition for S^m:

$$\mathfrak{h}\left(S^m\right) = (\mathfrak{h}(S))^{\otimes m} = \left(\mathfrak{h}^0(S) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S)\right)^{\otimes m} = \bigoplus_{i=0}^{2m} \mathfrak{h}^{2i}\left(S^m\right), \tag{43}$$

where

$$\mathfrak{h}^{2i}\left(S^m\right) = \bigoplus_{j_1+\cdots+j_m=i} \bigotimes_{k=1}^{m} \mathfrak{h}^{2j_k}(S).$$

We will need the following easy facts:

Lemma 11.2. *Let S be a K3 surface. Let $\varphi : I \twoheadrightarrow J$ be a surjective map between two finite sets. Denote by $f : S^J \hookrightarrow S^I$ the induced (partial diagonal) embedding between powers of S. Then with respect to the multiplicative Chow-Künneth decompositions for S^I and S^J above, we have*

(1) $f^* := \Gamma_f : \mathfrak{h}(S^I) \to \mathfrak{h}(S^J)$ maps $\mathfrak{h}^k(S^I)$ to $\mathfrak{h}^k(S^J)$ for any k.

(2) $f_* := {}^t\Gamma_f : \mathfrak{h}(S^J) \to \mathfrak{h}(S^I)(2|I| - 2|J|)$ *maps* $\mathfrak{h}^k(S^J)$ *to* $\mathfrak{h}^{k+4(|I|-|J|)}(S^I)(2|I| - 2|J|)$ *for any* k.

Proof. We start by observing that the part (2) on the push-forward follows from the part (1) on pull-back by the Poincaré duality for pure motives. For (1), by induction, one is reduced to the case of $|J| = |I| - 1$, which is immediately further reduced to the case when $|J| = 1$ and $|I| = 2$ by the projection formula. Thus we only need to consider the diagonal embedding $f = \Delta : S \hookrightarrow S \times S$. Now the statement for the pull-back Δ^* follows from the fact that (42) is multiplicative. □

Lemma 11.3. *Let S be a K3 surface and $m \in \mathbf{N}$. Let $c \in \mathrm{CH}(S^m)$ be a multiple of a cycle of the form $\prod_{i\in I} \mathrm{pr}_i^* c_S$ for some subset $I \subset \{1, \cdots, m\}$. Then with respect to the multiplicative Chow-Künneth decomposition for S^m defined above, the morphism $\cdot c := \Delta_*(c) : \mathfrak{h}(S^m) \to \mathfrak{h}(S^m)(2|I|)$ maps $\mathfrak{h}^i(S^m)$ to $\mathfrak{h}^{i+4|I|}(S^m)(2|I|)$.*

Proof. By the projection formula, we are reduced to the case that $I = \{1, \cdots, m\}$ and $c = \prod_{i=1}^m \mathrm{pr}_i^* c_S$. Hence the morphism $\mathfrak{h}(S^m) \to \mathfrak{h}(S^m)(2m)$ is the correspondence given by the point $(\underbrace{c_S, \cdots, c_S}_{2m}) \in S^m \times S^m$. It is non-zero only on the summand $\mathfrak{h}^0(S^m)$ which is clearly sent to $\mathfrak{h}^{4m}(S^m)(2m)$. □

Proof of Theorem 1.16. Let us fist deal with part (1), *i.e.* construct a multiplicative Chow-Künneth decomposition. Via the isomorphism in Theorem 1.6:

$$\mathfrak{h}\left(S^{[n]}\right) \simeq \left(\bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}\left((S^n)^g\right)(-\operatorname{age}(g))\right)^{\mathfrak{S}_n}$$

as algebra objects in CHM, it suffices to construct a multiplicative Chow-Künneth decomposition for the (non-commutative[16]) algebra object $\mathfrak{h}(S^n, \mathfrak{S}_n) := \bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}\left((S^n)^g\right)(-\operatorname{age}(g))$. For each $g \in \mathfrak{S}_n$ and $0 \le i \le 2|O(g)|$, the summand $\mathfrak{h}^i((S^n)^g)$ is defined in (43), the notation is coherent with Tate twists. Define

$$\mathfrak{h}^i(S^n, \mathfrak{S}_n) := \bigoplus_{g\in\mathfrak{S}_n} \mathfrak{h}^{i-2\operatorname{age}(g)}\left((S^n)^g\right)(-\operatorname{age}(g)). \tag{44}$$

It is clearly a Chow-Künneth decomposition by construction. To show its multiplicativity, one only needs to check that for any $g, h \in \mathfrak{S}_n$, the orbifold product maps $\mathfrak{h}^i\left((S^n)^g\right) \otimes \mathfrak{h}^j\left((S^n)^h\right)$ to $\mathfrak{h}^{i+j+2d}\left((S^n)^{gh}\right)(d)$ with $d = \operatorname{age}(g) + \operatorname{age}(h) - \operatorname{age}(gh)$. Recall that the orbifold product is defined as the following composition

$$\mathfrak{h}\left((S^n)^g\right) \otimes \mathfrak{h}\left((S^n)^h\right) \xrightarrow{\iota_1^*\otimes\iota_2^*} \mathfrak{h}\left((S^n)^{<g,h>}\right) \otimes \mathfrak{h}\left((S^n)^{<g,h>}\right) \xrightarrow{\mu} \mathfrak{h}\left((S^n)^{<g,h>}\right) \xrightarrow{\cdot c_{g,h}} \mathfrak{h}\left((S^n)^{<g,h>}\right) \xrightarrow{\iota_{3,*}} \mathfrak{h}\left((S^n)^{gh}\right)$$

where ι_1, ι_2, ι_3 are the embeddings of $(S^n)^{<g,h>}$ into $(S^n)^g$, $(S^n)^h$ and $(S^n)^{gh}$ respectively and the obstruction class $c_{g,h}$ is defined in (37). Now in the above composition, the summand $\mathfrak{h}^i\left((S^n)^g\right) \otimes \mathfrak{h}^j\left((S^n)^h\right)$ is sent by the first morphism to $\mathfrak{h}^i\left((S^n)^{<g,h>}\right) \otimes \mathfrak{h}^j\left((S^n)^{<g,h>}\right)$ thanks to Lemma 11.2 (1) ; which is then sent by the second morphism to the summand $\mathfrak{h}^{i+j}\left((S^n)^{<g,h>}\right)$ by the multiplicativity of the decomposition for $(S^n)^{<g,h>}$; which is then sent by the third morphism to $\mathfrak{h}^{i+j+4e}\left((S^n)^{<g,h>}\right)(2e)$ with $e := \frac{n-|O(g)|-|O(h)|-|O(gh)|}{2} + |O(g,h)|$, by Lemma 11.3 and the formula for $c_{g,h}$ in (37); which is

[16]The notion of *multiplicative Chow-Künneth decomposition* still makes sense for non-commutative algebra objects : one uses the same definition as in 11.1. See [28, Defintion 7.3] for details.

finally sent by the last morphism to $\mathfrak{h}^{i+j+4e+4|O(gh)|-4|O(g,h)|}\left((S^n)^{gh}\right)(2e+2|O(gh)|-2|O(g,h)|)$ by Lemma 11.2 (2). We only need to check the equality

$$e+|O(gh)|-|O(g,h)|=\frac{n-|O(g)|-|O(h)|+|O(gh)|}{2}=\frac{\mathrm{age}(g)+\mathrm{age}(h)-\mathrm{age}(gh)}{2}=\frac{d}{2}.$$

The multiplicativity of (44) is proved. Taking the $\mathfrak{S}_n$-invariant part yields, via the isomorphism of Theorem 1.6, a multiplicative Chow-Künneth decomposition for $S^{[n]}$.
As for part (2) on the non-trivial multiplications, it is simply a reformulation of the main theorem together with the definition of the orbifold product. □

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