ERRATUM FOR "ON THE MOTIVE OF O'GRADY'S TEN-DIMENSIONAL HYPER-KÄHLER VARIETIES"

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Two proofs in our paper [FFZ20] have gaps due to an incorrect application of Lemma A.5 (whose proof is instead correct). The two proofs affected are those of Theorem A.4 and Lemma 6.10. We thank Ben Moonen for pointing this out to us.

We present below a full proof of Theorem A.4 in §1. The conclusion of Lemma 6.10 remains instead only conjectural in general. This does not affect any of the main results in the paper, but requires small adjustments of some statements and proofs, see §2.

1. Proof of Theorem A.4

We refer to [FFZ20, Appendix A] for the notation. To fix the proof of Theorem A.4 we use the Hodge-maximality of Hodge structures of K3-type proven by Moonen–Cadoret [CM18].

Proof of Theorem A.4. Assume given Hodge structures $W_1, W_2 \in \mathrm{HS}^{\mathrm{pol}}_{\mathbf{Q}}$, both of odd weight, and such that $\langle W_i\rangle^{\mathrm{ev}} = \langle V\rangle$, for $i = 1, 2$. We consider $W_1 \oplus W_2$. We have surjective homomorphisms $q_i\colon \mathrm{MT}(W_1 \oplus W_2) \to \mathrm{MT}(W_i)$, and a commutative diagram with exact rows

$$\begin{array}{ccccc} \ker(q_1) & \longrightarrow & \mathrm{MT}(W_1 \oplus W_2) & \xrightarrow{q_1} & \mathrm{MT}(W_1) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle q_2} & & \downarrow \\ \langle \iota_2 \rangle & \longrightarrow & \mathrm{MT}(W_2) & \longrightarrow & \mathrm{MT}(V) \end{array}$$

We claim that q_1 and q_2 are both isomorphisms. Equivalently, j is the trivial map. Indeed, if j is trivial then $\ker(q_1) = \ker(q_2)$, which implies $\langle W_1\rangle = \langle W_2\rangle \subset \mathrm{HS}^{\mathrm{pol}}_{\mathbf{Q}}$.

Assume by contradiction that there exists $\tau \in \ker(q_1)$ with $j(\tau) \neq 1$. Then, by construction, $\tau = (\mathrm{id}_{W_1}, -\mathrm{id}_{W_2}) \in \mathrm{GL}(W_1 \oplus W_2)$. Let $\mathsf{C} \subset \langle W_1 \oplus W_2\rangle$ be the subcategory on which τ acts trivially. Then $\mathsf{C} \subset \mathrm{HS}^{\mathrm{pol}}_{\mathbf{Q}}$ is the Tannakian subcategory generated by W_1 and $\langle W_2\rangle^{\mathrm{ev}}$; it follows that $\mathsf{C} = \langle W_1\rangle$. Thus, the induced homomorphism $q_1\colon \mathrm{MT}(W_1 \oplus W_2) \to \mathrm{MT}(W_1)$ is the quotient by $\langle\tau\rangle$. Now, by Lemma A.5 in [FFZ20],

$$\mathrm{MT}(\langle W_1 \oplus W_2\rangle^{\mathrm{ev}}) \to \mathrm{MT}(V)$$

is an isogeny of degree 2. Since Mumford–Tate groups are connected, this contradicts the Hodge maximality of V, see [CM18, Proposition 6.2]. □

2. Avoiding Lemma 6.10

Contrarily to what claimed in Lemma 6.10 of [FFZ20], we are not able to prove that for any hyper-Kähler variety X with non-trivial odd cohomology we have $\mathcal{H}^1(A) \in \langle \mathcal{H}(X)\rangle_{\mathrm{AM}}$ for any Kuga-Satake variety for $H^2(X)$. Note however that for the only known type of hyper-Kähler varieties with non-trivial odd cohomology, that is, varieties X of generalized Kummer type, we in fact prove that $\langle \mathcal{H}^1(A)\rangle_{\mathrm{AM}} = \langle \mathcal{H}(X)\rangle_{\mathrm{AM}}$ for any Kuga-Satake variety A for $H^2(X)$.

Lemma 6.10 is used in [FFZ20] to define the defect group in presence of non-trivial odd cohomology. To avoid it, we proceed as follows.

Definition of the defect group in presence of non-trivial odd cohomology, cf. [FFZ20, §1.3]*:* if X is a hyper-Kähler variety with non-trivial odd-cohomology, we first define the *extended* defect group $\widetilde{P}(X)$ as the kernel

$$\widetilde{P}(X) := \operatorname{Ker}\big(\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) \twoheadrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))\big)$$

of the natural morphism of motivic Galois groups coming from the inclusion $\mathcal{H}^2(X) \subset \mathcal{H}(X)$. Since the odd cohomology of X is not trivial, with the notation of [FFZ20, §6.2], the representation

$$\sigma\colon \mathrm{CSpin}(H) \to \prod_i \mathrm{GL}(H^i(X))$$

is faithful; the element $\iota = \sigma(-1)$ acts on $H^i(X)$ as multiplication by $(-1)^i$. Therefore, ι generates a central subgroup of order 2 of $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X))$, which is contained in $\widetilde{P}(X)$. We then define the defect group of X as the quotient

$$P(X) := \widetilde{P}(X)/\langle\iota\rangle.$$

The following statements from [FFZ20] need modification.

- In presence of non-trivial odd cohomology, the first statement of Theorem 6.9 needs to be changed to:
  Let X be a projective hyper-Kähler variety with $b_2(X) \neq 3$. Then, inside $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X))$, the subgroups $\widetilde{P}(X)$ and $\mathrm{MT}(H^*(X))$ commute, intersect in the central subgroup of order 2 generated by ι, and we have
  $$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) = \mathrm{MT}(H^*(X)) \cdot \widetilde{P}(X).$$
  This follows from the last part of the proof of Theorem 6.9: the group $Q(X)$ introduced in that proof is precisely the extended defect group $\widetilde{P}(X)$.
- In the statement of Corollary 7.2.(iii), we need to assume that the conclusion of Lemma 6.10 holds for X.

With the definition above, in presence of non-trivial cohomology in odd degrees the defect group does not a priori act on $H^*(X)$, but only after a degree 2 cover. It should always be possible to identify $P(X)$ with a subgroup of $\widetilde{P}(X)$, as we are going to explain.

If X is a hyper-Kähler variety with non-trivial odd cohomology and $b_2 > 3$, the following statements are equivalent:

(1) the short exact sequence $1 \to \langle\iota\rangle \to \widetilde{P}(X) \to P(X) \to 1$ splits;
(2) the motive $\mathcal{H}^1(A)$ lies in $\langle\mathcal{H}(X)\rangle_{\mathrm{AM}}$, for any Kuga-Satake variety A for $H^2(X)$.

Proof. Assume that (1) holds. We identify $P(X)$ with a subgroup of $\widetilde{P}(X)$ such that $\widetilde{P}(X) = \langle\iota\rangle \times P(X)$. Since $\iota \in \mathrm{MT}(H^*(X))$, Theorem 6.9 (amended as above) implies that

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) = \mathrm{MT}(H^*(X)) \times P(X).$$

The projection $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) \twoheadrightarrow \mathrm{MT}(H^*(X))$ corresponds to a Tannakian subcategory C of $\langle\mathcal{H}(X)\rangle_{\mathrm{AM}}$; by construction, we have $\mathrm{G}_{\mathrm{mot}}(\mathsf{C}) \cong \mathrm{MT}(H^*(X))$. Denoting by $r\colon \mathrm{AM} \to \mathrm{HS}^{\mathrm{pol}}_{\mathbf{Q}}$ the realization functor, we have $r(\mathsf{C}) = \langle H^*(X)\rangle \subset \mathrm{HS}^{\mathrm{pol}}_{\mathbf{Q}}$.

It follows that $\mathrm{G}_{\mathrm{mot}}(\mathsf{C}) = \mathrm{MT}(r(\mathsf{C}))$: any Hodge class in C is motivated. By [FFZ20, Proposition 6.4.(ii)], the category $r(\mathsf{C}) \subset \mathrm{HS}^{\mathrm{pol}}$ is the Kuga–Satake category attached to $H^2(X)$ in the sense of Theorem A.4. It follows that C consists of abelian motives. Then, by [FFZ20, Corollary A.5], we have $\mathsf{C} = \langle\mathcal{H}^1(A)\rangle$ for any Kuga–Satake variety A for $H^2(X)$. This shows that (1) implies (2).

Assume conversely that (2) holds. By Theorem A.4, the group

$$P'(X) := \operatorname{Ker}\big(\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) \xrightarrow{\pi_{A,\mathrm{mot}}} \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^1(A))\big).$$

is independent of the Kuga-Satake variety A for $H^2(X)$, and by the argument in the proof of [FFZ20, Theorem 6.9] we obtain

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X)) = P'(X) \times \mathrm{MT}(H^*(X)).$$

Clearly, $P'(X)$ is contained in the extended defect group $\widetilde{P}(X)$. Moreover $\iota \notin P'(X)$, since ι is multiplication by -1 on $H^1(A)$. Hence, the quotient morphism $\widetilde{P}(X) \to P(X)$ restricts to an isomorphism $P'(X) \xrightarrow{\sim} P(X)$, and we have $\widetilde{P}(X) \cong P(X) \times \langle \iota \rangle$. □

We do not need to modify any other of the statements in our article besides the already mentioned Theorem 6.9 and Corollary 7.2.(iii). In presence of non-trivial odd cohomology, the arguments from [FFZ20] can be easily adapted using the extended defect group, as we briefly indicate.

- In the proof of Corollary 6.11, we use the observation above to show that (i) implies (ii).
- In the proof of Theorem 6.12, the argument given yields a local system $\widetilde{P}(\mathfrak{X}/S)$ of algebraic groups over S, with fibre at $s \in S$ the extended defect groups of X_s; hence, the extended defect group is constant in families. Moreover, $\widetilde{P}(\mathfrak{X}/S)$ contains a central sub-local system $\langle \iota \rangle / S$ with fibre the subgroup $\langle \iota \rangle \subset \widetilde{P}(X_s)$ at each $s \in S$, and the quotient local system $P(X/S)$ has fibre at $s \in S$ the defect group $P(X_s)$. Hence, the defect groups of all fibres are isomorphic.
- Finally, in the last part of the proof of Proposition 7.6, an additional short argument is needed to show that if $P(X)$ is finite then $\mathrm{MT}(H^*_{\mathrm{B}}(X)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X_{\bar{k}}))^0$: since $P(X)$ is finite, also $\widetilde{P}(X)$ is finite, and hence $\mathrm{MT}(H^*_{\mathrm{B}}(X))$ is a closed subgroup of the connected algebraic group $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X_{\bar{k}}))^0$ of the same dimension; hence, we have

$$\mathrm{MT}(H^*_{\mathrm{B}}(X)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}(X_{\bar{k}}))^0.$$

References

[CM18] Anna Cadoret and Ben Moonen, *Integral and adelic aspects of the Mumford-Tate conjecture*, Journal of the Institute of Mathematics of Jussieu **19** (2018), no. 3, 869–890, https://doi.org/10.1017/S1474748018000233.

[FFZ20] Salvatore Floccari, Lie Fu, and Ziyu Zhang, *On the motive of O'Grady's ten-dimensional hyper-Kähler varieties*, Communications in Contemporary Mathematics (2020), https://doi.org/10.1142/S0219199720500340.