

On the action of symplectic automorphisms on the CH_0 -groups of some hyper-Kähler fourfolds

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Abstract We prove that for any polarized symplectic automorphism of the Fano variety of lines of a smooth cubic fourfold (equipped with the Plücker polarization), the induced action on the Chow group of 0-cycles is identity, as predicted by Bloch–Beilinson conjecture. We also prove the same result for the Chow group of homologically trivial 2-cycles up to torsion.

1 Introduction

In this paper we are interested in an analogue of Bloch’s conjecture for the action on 0-cycles of a symplectic automorphism of an irreducible holomorphic symplectic variety. First of all, let us recall the Bloch conjecture and the general philosophy of the Bloch–Beilinson conjecture which motivate our result.

The Bloch conjecture for 0-cycles on algebraic surfaces states the following (cf. [6, p. 17]):

Conjecture 1.1 (Bloch) *Let Y be a smooth projective variety, X be a smooth projective surface and $\Gamma \in \mathrm{CH}^2(Y \times X)$ be a correspondence between them. If the cohomological correspondence $[\Gamma]^* : H^{2,0}(X) \rightarrow H^{2,0}(Y)$ vanishes, then the Chow-theoretic correspondence*

$$\Gamma_* : \mathrm{CH}_0(Y)_{\mathrm{alb}} \rightarrow \mathrm{CH}_0(X)_{\mathrm{alb}}$$

vanishes as well, where $\mathrm{CH}_0(\bullet)_{\mathrm{alb}} := \mathrm{Ker}(\mathrm{alb} : \mathrm{CH}_0(\bullet)_{\mathrm{hom}} \rightarrow \mathrm{Alb}(\bullet))$ denotes the group of the 0-cycles of degree 0 whose albanese classes are trivial.

The special case in Bloch’s conjecture where $X = Y$ is a surface S and $\Gamma = \Delta_S \in \mathrm{CH}^2(S \times S)$ states: *if a smooth projective surface S admits no non-zero holomorphic 2-forms, i.e. $H^{2,0}(S) = 0$, then $\mathrm{CH}_0(S)_{\mathrm{alb}} = 0$.* This has been proved for surfaces which are not of

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general type [7], for surfaces rationally dominated by a product of curves (by Kimura's work [18] on the nilpotence conjecture, cf. [26, Theorem 3.30]), and for Catanese surfaces and Barlow surfaces [25], etc.

What is more related to the present paper is another particular case of Bloch's conjecture: let S be a smooth projective surface with irregularity $q = 0$. If an automorphism f of S acts on $H^{2,0}(S)$ as identity, i.e. it preserves any holomorphic 2-forms, then f also acts as identity on $CH_0(S)$. This version is obtained from Conjecture 1.1 by taking $X = Y = S$ a surface and $\Gamma = \Delta_S - \Gamma_f \in CH^2(S \times S)$, where Γ_f is the graph of f . We would like to remark that it is also a consequence of the more general Bloch–Beilinson–Murre conjecture.

Recently Voisin [23] and Huybrechts [15] proved this conjecture for any symplectic automorphism of finite order of a projective K3 surface (see also [16]):

Theorem 1.2 (Voisin, Huybrechts) *Let f be an automorphism of finite order of a projective K3 surface S . If f is symplectic, i.e. $f^*(\omega) = \omega$, where ω is a generator of $H^{2,0}(S)$, then f acts as identity on $CH_0(S)$.*

The purpose of the paper is to investigate the analogous results in higher dimensional situation. The natural generalizations of K3 surfaces in higher dimensions are the so-called *irreducible holomorphic symplectic varieties* or *hyperkähler manifolds* (cf. [3]), which by definition is a simply connected compact Kähler manifold with $H^{2,0}$ generated by a symplectic form (i.e. nowhere degenerate holomorphic 2-form). We can conjecture the following vast generalization of Theorem 1.2:

Conjecture 1.3 *Let f be an automorphism of finite order of an irreducible holomorphic symplectic projective variety X . If f is symplectic: $f^*(\omega) = \omega$, where ω is a generator $H^{2,0}(X)$. Then f acts as identity on $CH_0(X)$.*

Like Theorem 1.2 is predicted by Bloch's Conjectures 1.1, 1.3 is predicted by the more general Bloch–Beilinson conjecture (cf. [2,5, Chapitre 11], [17,21, Chapter 11]). Instead of the most ambitious version involving the conjectural category of mixed motives, let us formulate it only for the 0-cycles and in the down-to-earth fashion ([21, Conjecture 11.22]), parallel to Conjecture 1.1:

Conjecture 1.4 (Bloch–Beilinson) *There exists a decreasing filtration F^\bullet on $CH_0(X)_{\mathbf{Q}} := CH_0(X) \otimes \mathbf{Q}$ for each smooth projective variety X , satisfying:*

- (i) $F^0 CH_0(X)_{\mathbf{Q}} = CH_0(X)_{\mathbf{Q}}$, $F^1 CH_0(X)_{\mathbf{Q}} = CH_0(X)_{\mathbf{Q},hom}$;
- (ii) F^\bullet is stable under algebraic correspondences;
- (iii) Given a correspondence $\Gamma \in CH^{\dim X}(Y \times X)_{\mathbf{Q}}$. If the cohomological correspondence $[\Gamma]^* : H^{i,0}(X) \rightarrow H^{i,0}(Y)$ vanishes, then the Chow-theoretic correspondence $Gr_F^i \Gamma_* : Gr_F^i CH_0(Y)_{\mathbf{Q}} \rightarrow Gr_F^i CH_0(X)_{\mathbf{Q}}$ on the i -th graded piece also vanishes.
- (iv) $F^{\dim X+1} CH_0(X)_{\mathbf{Q}} = 0$.

The implication from the Bloch–Beilinson Conjectures 1.3–1.4 is quite straightforward: as before, we take $Y = X$ to be the symplectic variety. If f is of order n , then define two projectors in $CH^{\dim X}(X \times X)_{\mathbf{Q}}$ by $\pi^{inv} := \frac{1}{n} (\Delta_X + \Gamma_f + \dots + \Gamma_{f^{n-1}})$ and $\pi^\# := \Delta_X - \pi^{inv}$. Since $H^{2j-1,0}(X) = 0$ and $H^{2j,0}(X) = \mathbf{C} \cdot \omega^j$, the assumption that f preserves the symplectic form ω implies that $[\pi^\#]^* : H^{i,0}(X) \rightarrow H^{i,0}(X)$ vanishes for any i . By (iii), $Gr_F^i(\pi^\#)_* : Gr_F^i CH_0(X)_{\mathbf{Q}} \rightarrow Gr_F^i CH_0(X)_{\mathbf{Q}}$ vanishes for any i . In other words, for the Chow motive $(X, \pi^\#)$, $Gr_F^i CH_0(X, \pi^\#) = 0$ for each i . Therefore by five-lemma and the finiteness condition (iv), we have $CH_0(X, \pi^\#) = 0$, that is, $\text{Im}(\pi^\#) = 0$. Equivalently, for

any $z \in \text{CH}_0(X)_{\mathbf{Q}}$, $\pi^{inv}(z) = z$, i.e. f acts as identity on $\text{CH}_0(X)_{\mathbf{Q}}$. Thanks to Roitman’s theorem on the torsion of $\text{CH}_0(X)$, the same still holds true for \mathbf{Z} -coefficients.

In [4], Beauville and Donagi provide an example of a 20-dimensional family of 4-dimensional irreducible holomorphic symplectic projective varieties, namely the Fano varieties of lines contained in smooth cubic fourfolds. In this paper, we propose to study Conjecture 1.3 for finite order symplectic automorphisms of this particular family. Our main result is the following:

Theorem 1.5 *Let f be an automorphism of a smooth cubic fourfold X . If the induced action on its Fano variety of lines $F(X)$, denoted by \hat{f} , preserves the symplectic form, then \hat{f} acts on $\text{CH}_0(F(X))$ as identity. Equivalently, the polarized symplectic automorphisms of $F(X)$ act as identity on $\text{CH}_0(F(X))$.*

We will show in Sect. 3 (cf. Corollary 3.3) how to deduce the above main theorem from the following result:

Theorem 1.6 (cf. Theorem 4.3) *Let f be an automorphism of a smooth cubic fourfold X acting as identity on $H^{3,1}(X)$. Then f acts as the identity on $\text{CH}_1(X)_{\mathbf{Q}}$.*

As a consequence of the main theorem, we will deduce in the last section the following consequence:

Corollary 1.7 *Under the same hypothesis as in Theorem 1.5: if \hat{f} is a polarized symplectic automorphism of the Fano variety of lines $F(X)$ of a smooth cubic fourfold X , then \hat{f} acts on $\text{CH}_2(F(X))_{\mathbf{Q},hom}$ as identity.*

Let us explain the main strategy of the proof of Theorem 1.6: we use the techniques of spread as in Voisin’s paper [24]. More precisely, we can summarize as follows the main steps. Let f and X be as in Theorem 1.6.

- (a) Let $\Gamma_f \subset X \times X$ be the graph of f . Let n be the order of f and $\pi^{inv} := \sum_{i=0}^{n-1} \Gamma_{f^i} \in \text{CH}^4(X \times X)_{\mathbf{Q}}$ be the projector onto the invariant part of X . In order to prove that f acts trivially on $\text{CH}_1(X)_{\mathbf{Q}}$ it suffices to show that there exists a decomposition in $\text{CH}^4(X \times X)_{\mathbf{Q}}$:

$$\Delta_X - \pi^{inv} = \Gamma'_0 + Z' + Z'', \tag{1}$$

where Γ'_0 is supported on $Y \times Y$ for a codimension 2 closed algebraic subset $Y \subset X$, and Z', Z'' are the pull-back of cycles on $X \times \mathbf{P}^5$ and $\mathbf{P}^5 \times X$ respectively, cf. (37).

- (b) To prove (1), we show firstly that there exists an algebraic cycle Γ'_0 supported on $Y \times Y$ for a codimension 2 closed algebraic subset $Y \subset X$, such that $\Delta_X - \pi^{inv} = \Gamma'_0$ has zero cohomology class (with rational coefficients), see Proposition 6.2. Now consider the family $\mathcal{X} \rightarrow B$ of all smooth cubic fourfolds which are mapped to themselves by the automorphism f (here f is a fixed projective automorphism of \mathbf{P}^5). One shows that the cycle Γ'_0 for the varying $X \times X$ fit together to give a cycle Γ' on $\mathcal{X} \times_B \mathcal{X}$, see Proposition 6.3. Of course, the cycles $(\Delta_{X_b} - \pi^{inv}_{X_b})$ fit together to give a cycle Γ on $\mathcal{X} \times_b \mathcal{X}$. Then the cohomology class of $\Gamma - \Gamma'$ restricts to zero on each fiber $X_b \times X_b$. By a Leray spectral sequence argument as in [24], there exist algebraic cycles $\mathcal{Z}', \mathcal{Z}''$ on $\mathcal{X} \times_b \mathcal{X}$, which are the pull-back of cycles on $\mathcal{X} \times \mathbf{P}^5$ and $\mathbf{P}^5 \times \mathcal{X}$ respectively, such that $\Gamma - \Gamma' - \mathcal{Z}' - \mathcal{Z}''$ has zero cohomology class.
- (c) Now comes the core of the proof: one show that given $z \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$ which is homologically trivial there exists a dense open subset $B' \subset B$ such that the restriction of z to the base-changed family $\mathcal{X}' \times_{B'} \mathcal{X}'$ vanishes. There are two main ingredients

in the proof:

- (i) One completes the family $\mathcal{X} \times_B \mathcal{X}$ to a smooth projective variety for which the rational equivalence and cohomological equivalence coincide, once we tensor with \mathbf{Q} .
- (ii) To extend a homologically trivial cycle to a homologically trivial cycle in the compactification (or rather its resolution of singularities), we exploit the fact that the Chow motive of a cubic fourfold decomposes into pieces which do not exceed the size of the Chow motives of surfaces.
- (d) Applying the result in (c) to the cycle $(\Gamma - \Gamma' - \mathcal{Z}' - \mathcal{Z}'')$ gives (1) and hence the result.

We organize the paper as follows. In Sect. 2, we start describing the parameter space of cubic fourfolds with an action satisfying that the induced actions on the Fano varieties of lines are symplectic. In Sect. 3, the main theorem is reduced to a statement about the 1-cycles of the cubic fourfold. By varying the cubic fourfold, in Sect. 4 we reduce the main theorem 1.5 to the form that we will prove, which concerns only the 1-cycles of a general member in the family. The purpose of Sect. 5 is to establish the triviality of Chow groups of some total spaces. The first half Sect. 5.1 shows the triviality of Chow groups of its compactification; then the second half Sect. 5.2 passes to the open part by comparing to surfaces. Section 6 proves the main Theorem 1.5 by combining the strategy of Voisin's paper [24] and the result of Sect. 5. In Sect. 7 we reformulate the hypothesis in the main theorem to the assumption of being 'polarized'. Finally in Sect. 8, we verify another prediction of Bloch–Beilinson conjecture on the Chow group of 2-cycles (Corollary 1.7) from our main result.

We will work over the complex numbers throughout this paper.

2 Basic settings

In this first section, we establish the basic settings for automorphisms of the Fano variety of a cubic fourfold, and work out the condition corresponding to the symplectic assumption.

Let V be a fixed 6-dimensional \mathbf{C} -vector space, and $\mathbf{P}^5 := \mathbf{P}(V)$ be the corresponding projective space of 1-dimensional subspaces of V . Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold, which is defined by a polynomial $T \in H^0(\mathbf{P}^5, \mathcal{O}(3)) = \text{Sym}^3 V^\vee$. Let f be an automorphism of X . Since $\text{Pic}(X) = \mathbf{Z} \cdot \mathcal{O}_X(1)$, any automorphism of X is *induced*: it is the restriction of a linear automorphism of \mathbf{P}^5 preserving X , still denoted by f .

It is classical and well-known that $\text{Aut}(X)$ is a finite group. Let f be of order $n \in \mathbf{N}_+$. Since the minimal polynomial of f is semi-simple, we can assume without loss of generality that $f : \mathbf{P}^5 \rightarrow \mathbf{P}^5$ is given by:

$$f : [x_0 : x_1 : \cdots : x_5] \mapsto [\zeta^{e_0} x_0 : \zeta^{e_1} x_1 : \cdots : \zeta^{e_5} x_5], \quad (2)$$

where $\zeta = e^{\frac{2\pi\sqrt{-1}}{n}}$ is a primitive n -th root of unity and $e_i \in \mathbf{Z}/n\mathbf{Z}$ for $i = 0, \dots, 5$. Now it is clear that X is preserved by f if and only if its defining equation T is contained in an eigenspace of $\text{Sym}^3 V^\vee$, where $\text{Sym}^3 V^\vee$ is endowed with the induced action coming from V .

Let us make it more precise: as usual, we use the coordinates x_0, x_1, \dots, x_5 of \mathbf{P}^5 as a basis of V^\vee , then $\{\underline{x}^\alpha\}_{\alpha \in \Lambda}$ is a basis of $\text{Sym}^3 V^\vee = H^0(\mathbf{P}^5, \mathcal{O}(3))$, where \underline{x}^α denotes $x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_5^{\alpha_5}$, and

$$\Lambda := \left\{ \alpha = (\alpha_0, \dots, \alpha_5) \in \mathbf{N}^5 \mid \alpha_0 + \cdots + \alpha_5 = 3 \right\}. \quad (3)$$

Therefore the eigenspace decomposition of $\text{Sym}^3 V^\vee$ is the following:

$$\text{Sym}^3 V^\vee = \bigoplus_{j \in \mathbf{Z}/n\mathbf{Z}} \left(\bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot \underline{x}^\alpha \right),$$

where for each $j \in \mathbf{Z}/n\mathbf{Z}$, we define the subset of Λ

$$\Lambda_j := \left\{ \alpha = (\alpha_0, \dots, \alpha_5) \in \mathbf{N}^5 \mid \begin{matrix} \alpha_0 + \dots + \alpha_5 = 3 \\ e_0\alpha_0 + \dots + e_5\alpha_5 = j \pmod n \end{matrix} \right\}. \tag{4}$$

and the eigenvalue of $\bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot \underline{x}^\alpha$ is ζ^j . Therefore, explicitly speaking, we have:

Lemma 2.1 *Keeping the notation (2), (3), (4), the cubic fourfold X is preserved by f if and only if there exists a $j \in \mathbf{Z}/n\mathbf{Z}$ such that the defining polynomial $T \in \bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot \underline{x}^\alpha$.*

Let us deal now with the symplectic condition for the induced action on $F(X)$. First of all, let us recall some basic constructions and facts. The following subvariety of the Grassmannian $\text{Gr}(\mathbf{P}^1, \mathbf{P}^5)$

$$F(X) := \left\{ [L] \in \text{Gr}(\mathbf{P}^1, \mathbf{P}^5) \mid L \subset X \right\} \tag{5}$$

is called the *Fano variety of lines*¹ of X . It is well-known that $F(X)$ is a 4-dimensional smooth projective variety equipped with the restriction of the Plücker polarization of the ambient Grassmannian. Consider the incidence variety (i.e. the universal projective line over $F(X)$):

$$P(X) := \{(x, [L]) \in X \times F(X) \mid x \in L\}.$$

We have the following natural correspondence:

$$\begin{array}{ccc} P(X) & \xrightarrow{q} & X \\ \downarrow p & & \\ F(X) & & \end{array}$$

Theorem 2.2 (Beauville–Donagi [4]) *Using the above notation,*

- (i) *$F(X)$ is a 4-dimensional irreducible holomorphic symplectic projective variety, i.e. $F(X)$ is simply-connected and $H^{2,0}(F(X)) = \mathbf{C} \cdot \omega$ with ω a nowhere degenerate holomorphic 2-form.*
- (ii) *The correspondence*

$$p_*q^* : H^4(X, \mathbf{Z}) \rightarrow H^2(F(X), \mathbf{Z})$$

is an isomorphism of Hodge structures.

In particular, $p_*q^* : H^{3,1}(X) \xrightarrow{\cong} H^{2,0}(X)$ is an isomorphism. If X is equipped with an action f as before, we denote by \hat{f} the induced automorphism of $F(X)$. Since the construction of the Fano variety of lines $F(X)$ and the correspondence p_*q^* are both functorial with respect to X , the condition that \hat{f} is *symplectic*, namely $\hat{f}^*(\omega) = \omega$ for ω a generator of $H^{2,0}(F(X))$, is equivalent to the condition that f^* acts as identity on $H^{3,1}(X)$. Working this out explicitly, we arrive at the following

¹ In the scheme-theoretic language, $F(X)$ is defined to be the zero locus of $s_T \in H^0(\text{Gr}(\mathbf{P}^1, \mathbf{P}^5), \text{Sym}^3 S^\vee)$, where S is the universal tautological subbundle on the Grassmannian, and s_T is the section induced by T using the morphism of vector bundles $\text{Sym}^3 V^\vee \otimes \mathcal{O} \rightarrow \text{Sym}^3 S^\vee$ on $\text{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.

Lemma 2.3 *Let f be the linear automorphism in (2), and X be a cubic fourfold defined by equation T . Then the followings are equivalent:*

- f preserves X and the induced action \hat{f} on $F(X)$ is symplectic;
- There exists a $j \in \mathbf{Z}/n\mathbf{Z}$ satisfying the equation

$$e_0 + e_1 + \dots + e_5 = 2j \pmod n, \tag{6}$$

such that the defining polynomial $T \in \bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot \underline{x}^\alpha$, where as in (4)

$$\Lambda_j := \left\{ \underline{\alpha} = (\alpha_0, \dots, \alpha_5) \in \mathbf{N}^5 \mid \begin{matrix} \alpha_0 + \dots + \alpha_5 = 3 \\ e_0\alpha_0 + \dots + e_5\alpha_5 = j \pmod n \end{matrix} \right\}.$$

Proof Firstly, the condition that f preserves X is given in Lemma 2.1. As is remarked before the lemma, \hat{f} is symplectic if and only if f^* acts as identity on $H^{3,1}(X)$. On the other hand, by Griffiths’ theory of the Hodge structures of hypersurfaces (cf. [21, Chapter 18]), $H^{3,1}(X)$ is generated by the residue $\text{Res} \frac{\Omega}{T}$, where $\Omega := \sum_{i=0}^5 (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx}_i \wedge \dots \wedge dx_5$ is a generator of $H^0(\mathbf{P}^5, K_{\mathbf{P}^5}(6))$. f given in (2), we get $f^*\Omega = \zeta^{e_0+\dots+e_5}\Omega$ and $f^*(T) = \zeta^j T$. Hence the action of f^* on $H^{3,1}(X)$ is multiplication by $\zeta^{e_0+\dots+e_5-2j}$, from which we obtain Eq. (6). □

3 Reduction to 1-cycles on cubic fourfolds

The objective of this section is to prove Corollary 3.3. It allows us in particular to reduce the main Theorem 1.5, which is about the action on 0-cycles on the Fano variety of lines, to the study of the action on 1-cycles of the cubic fourfold itself (see Theorem 4.3).

To this end, we want to make use of Voisin’s equality (see Proposition 3.1(ii)) in the Chow group of 0-cycles of the Fano variety of a cubic fourfold. Let X be a (smooth) cubic fourfold, $F := F(X)$ be its Fano variety of lines and $P := P(X)$ be the universal projective line over F fitting into the diagram below:

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ p \downarrow & & \\ & & F \end{array}$$

For any line L contained in X , we denote the corresponding point in F by l . Define $S_l := \{l' \in F \mid L' \cap L \neq \emptyset\}$ to be the surface contained in F parameterizing lines in X meeting a give line L . As algebraic cycles,

$$L = q_* p^*(l) \in \text{CH}_1(X); \tag{7}$$

$$S_l = p_* q^*(L) \in \text{CH}_2(F). \tag{8}$$

The following relations are discovered by Voisin in [22]:

Proposition 3.1 *Let $I := \{(l, l') \in F \times F \mid L \cap L' \neq \emptyset\}$ be the incidence subvariety. We denote by $g \in \text{CH}^1(F)$ the Plücker polarization, and by $c \in \text{CH}^2(F)$ the second Chern class of the restriction to F of the tautological rank 2 subbundle on $\text{Gr}(\mathbf{P}^1, \mathbf{P}^5)$.*

(i) *There is a quadratic relation in $\text{CH}^4(F \times F)$:*

$$I^2 = \alpha \Delta_F + I \cdot \Gamma + \Gamma',$$

where $\alpha \neq 0$ is an integer, Γ is a degree 2 polynomial in $\text{pr}_1^* g, \text{pr}_2^* g$, and Γ' is a weighted degree 4 polynomial in $\text{pr}_1^* g, \text{pr}_2^* g, \text{pr}_1^* c, \text{pr}_2^* c$.

(ii) For any $l \in F$, we have an equality in $\text{CH}_0(F)$:

$$S_l^2 = \alpha \cdot l + \beta S_l \cdot g^2 + \Gamma'', \tag{9}$$

where $\alpha \neq 0$ and β are constant integers, Γ'' is a polynomial in g^2 and c of degree 2 with integral coefficients independent of l .

Proof For the first equality (i), cf. [22, Proposition 3.3]. For (ii), we restrict the relation in (i) to a fiber $\{l\} \times F$, then $I|_{\{l\} \times F} = S_l$ and $\Delta_F|_{\{l\} \times F} = l$. Hence the Eq. (9). \square

Corollary 3.2 *Given an automorphism f of a cubic fourfold X , let L be a line contained in X and $l \in \text{CH}_0(F)$, $S_l \in \text{CH}_2(F)$ be the cycles as above. Then the followings are equivalent:*

- (i) $\hat{f}(l) = l$ in $\text{CH}_0(F)$;
- (ii) $f(L) = L$ in $\text{CH}_1(X)$;
- (iii) $\hat{f}(S_l) = S_l$ in $\text{CH}_2(F)$.

The same equivalences hold also for Chow groups with rational coefficients.

Proof (i) \Rightarrow (ii): by (7) and the functorialities of p and q .

(ii) \Rightarrow (iii): by (8) and the functorialities of q and p .

(iii) \Rightarrow (i): by (9) and the fact that g, c are all invariant by \hat{f} , we obtain $\alpha(l - \hat{f}(l)) = 0$ in $\text{CH}_0(F)$ with $\alpha \neq 0$. However by Roitman theorem $\text{CH}_0(F)$ is torsion-free, thus $l = \hat{f}(l)$ in $\text{CH}_0(F)$.

Of course, the same² proof gives the same equivalences for Chow groups with rational coefficients. \square

In particular, we have:

Corollary 3.3 *Let f be an automorphism of a cubic fourfold X and F be the Fano variety of lines of X , equipped with the induced action \hat{f} . Then the followings are equivalent:*

- (i) \hat{f} acts on $\text{CH}_0(F)$ as identity;
- (ii) \hat{f} acts on $\text{CH}_0(F)_{\mathbf{Q}}$ as identity;
- (iii) f acts on $\text{CH}_1(X)$ as identity;
- (iv) f acts on $\text{CH}_1(X)_{\mathbf{Q}}$ as identity.

Proof By the result in [20]³ that $\text{CH}_1(X)$ is generated by the lines contained in X , the previous corollary gives the equivalences (i) \Leftrightarrow (iii) and (ii) \Leftrightarrow (iv). On the other hand, F is simply-connected and thus its Albanese variety is trivial. Therefore $\text{CH}_0(F)$ is torsion-free by Roitman’s theorem, hence (i) \Leftrightarrow (ii). \square

We remark that this corollary allows us to reduced Theorems 1.5–1.6 which is stated purely in terms of the action on the Chow group of the 1-cycles of the cubic fourfold itself.

² In fact easier, because we do not need to invoke Roitman theorem.

³ For rational coefficients it can be easily deduced by the argument in [13].

4 Reduction to the general member of the family

Our basic approach to the main theorem 1.5 is to vary the cubic fourfold in family and make use of certain good properties of the total space (cf. Sect. 5) to get some useful information for a member of the family. To this end, we give in this section a family version of previous constructions, and then by combining with Corollary 3.3, we reduce the main Theorems 4.3–1.5 which is a statement for 1-cycles of a general member in the family.

Fix $n \in \mathbf{N}_+$, fix f as in (2) and fix a solution $j \in \mathbf{Z}/n\mathbf{Z}$ of (6). Consider the projective space parameterizing certain possibly singular cubic hypersurfaces in \mathbf{P}^5 .

$$\overline{B} = \mathbf{P} \left(\bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot x^\alpha \right),$$

where Λ_j is defined in (4). Let us denote the universal family by

$$\begin{array}{c} \overline{\mathcal{X}} \\ \downarrow \pi \\ \overline{B} \end{array}$$

whose fibre over the a point $b \in \overline{B}$ is a cubic hypersurface in \mathbf{P}^5 denoted by X_b . Let $B \subset \overline{B}$ be the Zariski open subset parameterizing the smooth ones. By base change, we have over B the universal family of smooth cubic fourfolds with a (constant) fiberwise action f , and similarly the universal Fano variety of lines \mathcal{F} equipped with the corresponding fiberwise action \hat{f} :

$$\begin{array}{ccc} \hat{f} \circ \mathcal{F} & \xrightarrow{\quad} & \mathcal{X} \hookrightarrow B \times \mathbf{P}^5 \circ f \\ & \searrow & \downarrow \pi \quad \swarrow \text{pr}_1 \\ & & B \end{array} \tag{10}$$

The fibre over $b \in B$ of \mathcal{F} is denoted by $F_b = F(X_b)$, on which \hat{f} acts symplectically by construction.

By the following general fact, we claim that to prove the main theorem 1.5, it suffices to prove it for a very general member in the family:

Lemma 4.1 *Let $\mathcal{F} \rightarrow B$ be a smooth projective fibration with a fiberwise action \hat{f} (for example in the situation (10) before). If for a general point $b \in B$, \hat{f} acts as identity on $\text{CH}_0(F_b)$, then the same thing holds true for any $b \in B$.*

Proof For any $b_0 \in B$, we want to show that \hat{f} acts as identity on $\text{CH}_0(F_{b_0})$. Given any 0-cycle $Z \in \text{CH}_0(F_{b_0})$, we can find a generically finite dominant base-change

$$\begin{array}{ccc} \mathcal{F}' & \longrightarrow & \mathcal{F} \\ \downarrow & \square & \downarrow \\ B' & \longrightarrow & B \end{array}$$

and a cycle $\mathcal{Z} \in \text{CH}_{\dim B'}(\mathcal{F}')$, such that $\mathcal{Z}|_{F'_{b'_0}} = Z$, where $b'_0 \in B'$ is a preimage of $b_0 \in B$. (For example, when Z is just one point, we can take B' to be the transversal intersection of

$(\dim \mathcal{F} - \dim B)$ general hyperplane sections of \mathcal{F} passing through this point. By iterating this construction, one can treat the general case when Z is a 0-cycle.) Hence $F'_{b'_0} = F_{b_0}$. Now consider $\Gamma := \hat{f}^* \mathcal{L} - \mathcal{L} \in \text{CH}_{\dim B'}(\mathcal{F}')$, by assumption it satisfies $\Gamma|_{F'_{b'}} = 0$ in $\text{CH}_0(F'_{b'})$ for a general point $b' \in B'$. However, by an argument of Hilbert scheme (cf. [21, Chapter 22]), the set of points $b' \in B'$ satisfying $\Gamma|_{F'_{b'}} = 0$ is a countable union of closed algebraic subsets. Thus together with the preimage of the locus where $b \in B$ does not satisfy the condition in the lemma (which is again a countable union of proper closed algebraic subsets by assumption), they cover B' . By Baire theorem, in this countable collection there exists one which is in fact the entire B' , i.e. $\Gamma|_{F'_{b'}} = 0$ holds for every $b' \in B'$. In particular, for b'_0 , we have $\hat{f}^* Z - Z = \Gamma|_{F_{b_0}} = \Gamma|_{F'_{b'_0}} = 0$. \square

Remark 4.2 Thanks to this lemma, instead of defining B as the parameter space of smooth cubic fourfolds, we can and we will feel free to shrink B to any of its Zariski open subsets whenever we want to in the rest of the paper.

To summarize this section, we reduce the main Theorem 1.5 into the following statement:

Theorem 4.3 *Let $n = p^m$ be a power of a prime number, f be an automorphism of \mathbf{P}^5 given by (2):*

$$f : [x_0 : x_1 : \dots : x_5] \mapsto [\zeta^{e_0} x_0 : \zeta^{e_1} x_1 : \dots : \zeta^{e_5} x_5],$$

and $j \in \mathbf{Z}/n\mathbf{Z}$ be a solution to (6): $e_0 + e_1 + \dots + e_5 = 2j \pmod n$.

If for a general point $b \in \overline{B} := \mathbf{P} \left(\bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot \underline{x}^\alpha \right)$, X_b is smooth, then f acts as identity on $\text{CH}_1(X_b)_{\mathbf{Q}}$, where

$$\Lambda_j := \left\{ \underline{\alpha} = (\alpha_0, \dots, \alpha_5) \in \mathbf{N}^5 \mid \begin{matrix} \alpha_0 + \dots + \alpha_5 = 3 \\ e_0 \alpha_0 + \dots + e_5 \alpha_5 = j \pmod n \end{matrix} \right\}$$

as in (4).

Theorem 4.3 \Rightarrow Theorem 1.5. First of all, in order to prove the main Theorem 1.5, we can assume that the order of f is a power of a prime number: suppose the prime factorization of the order of f is

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}.$$

Let $g_i = f^{n p_i^{-a_i}}$ for any $1 \leq i \leq r$, then g_i is of order $p_i^{a_i}$. Since \hat{f} acts symplectically on $F(X)$, so do the g_i 's. Then by assumption, \hat{g}_i acts as identity on $\text{CH}_0(F(X))$ for any i . Now by Chinese remainder theorem, there exist $b_1, \dots, b_r \in \mathbf{N}$ such that $f = \prod_{i=1}^r g_i^{b_i}$. Therefore, $\hat{f} = \prod_{i=1}^r \hat{g}_i^{b_i}$ acts as identity on $\text{CH}_0(F(X))$ as well. Secondly, the parameter space \overline{B} comes from the constraints we deduced in Lemma 2.3. Thirdly, Lemma 4.1 allows us to reduce the statement to the case of a (very) general member in the family. Fourthly, we can switch from $\text{CH}_0(F_b)$ to $\text{CH}_1(X_b)_{\mathbf{Q}}$ by Corollary 3.3. Finally, the reformulation in terms of the polarization is explained in Proposition 7.1. \square

5 The Chow group of the total space

As a key step toward the proof of Theorem 4.3, we establish in this section the following result.

Proposition 5.1 Consider the direct system consisting of the open subsets B of \overline{B} , then we have

$$\varinjlim_B \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}, \text{hom}} = 0.$$

More precisely, for an open subset B of \overline{B} , and for any homologically trivial codimension 4 \mathbf{Q} -cycle z of $\mathcal{X} \times_B \mathcal{X}$, there exists a dense open subset $B' \subset B$, such that the restriction of z to the base changed family $\mathcal{X}' \times_{B'} \mathcal{X}'$ is rationally equivalent to 0.

We achieve this in two steps: the first one is to show that homological equivalence and rational equivalence coincide on a resolution of singularities of the compactification $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$ (see Proposition 5.2 below); in the second step, to pass to the open variety $\mathcal{X} \times_B \mathcal{X}$, we need to ‘extend’ a homologically trivial cycle of the open variety to a cycle homologically trivial of the compactification or rather its resolution (see Proposition 5.3 below). More precisely, let B be an open subset of \overline{B} :

Proposition 5.2 (Step 1) There exists a resolution of singularities $\tau : W \rightarrow \overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$, such that the rational equivalence and homological equivalence coincide on W when tensored with \mathbf{Q} (see Definition 5.4 below). In particular, $\text{CH}^4(W)_{\mathbf{Q}, \text{hom}} = 0$.

Proposition 5.3 (Step 2) Let $\tau : W \rightarrow \overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$ be a resolution of singularities. For any homologically trivial cycle $z \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}, \text{hom}}$, there exist a dense open subset $B' \subset B$ and a homologically trivial cycle $\bar{z} \in \text{CH}^4(W)_{\mathbf{Q}, \text{hom}}$, such that

$$z|_{\mathcal{X}' \times_{B'} \mathcal{X}'} = \tau'_*(\bar{z}|_{W'}) \in \text{CH}^4(\mathcal{X}' \times_{B'} \mathcal{X}')_{\mathbf{Q}}, \tag{11}$$

where $\mathcal{X}' = \mathcal{X} \times_B B'$, $W' = W \times_{\overline{B}} B'$ are obtained by base change. We denote by $\tau' : W' \rightarrow \mathcal{X}' \times_{B'} \mathcal{X}'$ the restriction of τ to W' .

Propositions 5.2 and 5.3 will be proved in Sects. 5.1 and 5.2 respectively. Admitting them, Proposition 5.1 becomes obvious:

Prop. 5.2 + Prop. 5.3 \Rightarrow *Prop. 5.1*. Let W, τ be as in Proposition 5.2. For a given $z \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}, \text{hom}}$, let $B', \mathcal{X}', \bar{z}, \tau', W'$ be as in Proposition 5.3. Since \bar{z} is homologically trivial, \bar{z} is (rationally equivalent to) zero by Proposition 5.2, hence so is its restriction $\bar{z}|_{W'}$ to the open subset W' . Therefore by (11),

$$z|_{\mathcal{X}' \times_{B'} \mathcal{X}'} = \tau'_*(\bar{z}|_{W'}) = 0.$$

□

5.1 The Chow group of the compactification

In this subsection, we prove Proposition 5.2.

We first recall the following notion due to Voisin [24, §2.1]:

Definition 5.4 We say a smooth projective variety X satisfies *property* \mathcal{P} , if the cycle class map is an isomorphism

$$[-] : \text{CH}^*(X)_{\mathbf{Q}} \xrightarrow{\cong} H^*(X, \mathbf{Q}).$$

Here we provide some examples and summarize some operations that preserve this property \mathcal{P} . For details, cf. [24].

Lemma 5.5 (i) *Homogeneous variety of the form G/P satisfies property \mathcal{P} , where G is a linear algebraic group and P is a parabolic subgroup. For example, projective spaces, Grassmannians, flag varieties, etc.*

- (ii) *If X and Y satisfy property \mathcal{P} , then so does $X \times Y$.*
- (iii) *If X satisfies property \mathcal{P} , and E is a vector bundle on it, then the projective bundle $\mathbf{P}(E)$ satisfies property \mathcal{P} .*
- (iv) *If X satisfies property \mathcal{P} , and $Z \subset X$ is a smooth subvariety satisfying property \mathcal{P} , then so is the blow up variety $\text{Bl}_Z X$.*
- (v) *Let $f : X \rightarrow X'$ be a surjective generic finite morphism. If X satisfies property \mathcal{P} , then so does X' .*

Since some toric geometry will be needed in the sequel, let us also recall some standard definitions and properties, see [11, 14] for details. Given a lattice N and a fan Δ in $N_{\mathbf{R}}$, one can construct a toric variety of dimension $\text{rank}(N)$, which will be denoted by $X(\Delta)$. By definition, $X(\Delta)$ is the union of affine toric varieties $\text{Spec}(\mathbf{C}[N^{\vee} \cap \sigma^{\vee}])$, where N^{\vee} is the dual lattice, σ^{\vee} is the dual cone in $N^{\vee}_{\mathbf{R}}$ and σ runs over the cones in Δ . A fan Δ is said *regular* if each cone in Δ is generated by a part of a \mathbf{Z} -basis of N . Let N' be another lattice and Δ' be a fan in $N'_{\mathbf{R}}$. Then a homomorphism (as abelian groups) $f : N \rightarrow N'$ induces a rational map of the toric varieties $\phi : X(\Delta) \dashrightarrow X(\Delta')$. Such maps are called *equivariant* or *monomial*.

Proposition 5.6 *Using the above notation for toric geometry, then we have:*

- (i) *$X(\Delta)$ is smooth if and only if Δ is regular.*
- (ii) *$\phi : X(\Delta) \dashrightarrow X(\Delta')$ is a morphism if and only if for any cone $\sigma \in \Delta$, there exists a cone $\sigma' \in \Delta'$ such that f sends σ into σ' .*
- (iii) *Any fan admits a refinement consisting of regular cones.*
- (iv) *Any smooth projective toric variety satisfies property \mathcal{P} .*
- (v) *ϕ admits an elimination of indeterminacies:*

$$\begin{array}{ccc}
 X(\tilde{\Delta}) & & \\
 \downarrow \tau & \searrow \tilde{\phi} & \\
 X(\Delta) & \xrightarrow{\phi} & X(\Delta')
 \end{array}$$

such that $X(\tilde{\Delta})$ is smooth projective satisfying property \mathcal{P} .

Proof For (i), [11, Theorem 3.1.19]; for (ii), [11, Theorem 3.3.4]; for (iii), [11, Theorem 11.1.9]; for (iv), [11, Theorem 12.5.3]. Finally, (v) is a consequence of the first four: by (iii), we can find a regular refinement of $\Delta \cup f^{-1}(\Delta')$, denoted by $\tilde{\Delta}$, then $X(\tilde{\Delta})$ is smooth by (i) and satisfies property \mathcal{P} by (iv). Moreover, (ii) implies that $\phi \circ \tau : X(\tilde{\Delta}) \rightarrow X(\Delta')$ is a morphism. □

Turning back to our question, we adopt the previous notation as in Theorem 4.3.

We can view $\overline{B} = \mathbf{P}(\bigoplus_{\alpha \in \Lambda_j} \mathbf{C} \cdot x^{\alpha})$ as an *incomplete* linear system on \mathbf{P}^5 associated to the line bundle $\mathcal{O}_{\mathbf{P}^5}(3)$. We remark that by construction in Sect. 2, each member of \overline{B} (which is a possibly singular cubic fourfold) is preserved under the action of f . Consider the rational map associated to this linear system:

$$\phi := \phi_{|\overline{B}|} : \mathbf{P}^5 \dashrightarrow \overline{B}^{\vee},$$

where \overline{B}^{\vee} is the dual projective space consisting of the hyperplanes of \overline{B} . We remark that since \overline{B}^{\vee} has a basis given by monomials, the above rational map ϕ is a *monomial* map between two toric varieties (cf. the definition before Proposition 5.6).

Lemma 5.7 (i) *There exists an elimination of indeterminacies of ϕ :*

$$\begin{array}{ccc} \tilde{\mathbf{P}}^5 & & \\ \tau \downarrow & \searrow \tilde{\phi} & \\ \mathbf{P}^5 & \xrightarrow{\phi} & \overline{B}^\vee \end{array}$$

such that $\tilde{\mathbf{P}}^5$ is smooth projective satisfying property \mathcal{P} .

(ii) *The strict transform of $\overline{\mathcal{X}} \subset \mathbf{P}^5 \times \overline{B}$, denoted by $\widetilde{\overline{\mathcal{X}}}$, is the incidence subvariety in $\tilde{\mathbf{P}}^5 \times \overline{B}$:*

$$\widetilde{\overline{\mathcal{X}}} = \left\{ (x, b) \in \tilde{\mathbf{P}}^5 \times \overline{B} \mid b \in \tilde{\phi}(x) \right\}.$$

Proof (i) By Proposition 5.6(v).

(ii) follows from the fact that for $x \in \mathbf{P}^5$ not in the base locus of \overline{B} , $b \in \phi(x)$ if and only if $(x, b) \in \overline{\mathcal{X}}$. □

Corollary 5.8 $\widetilde{\overline{\mathcal{X}}}$ is smooth and satisfies property \mathcal{P} .

Proof Thanks to Lemma 5.7(iii), $\widetilde{\overline{\mathcal{X}}} \subset \tilde{\mathbf{P}}^5 \times \overline{B}$ is the incidence subvariety with respect to $\tilde{\phi} : \tilde{\mathbf{P}}^5 \rightarrow \overline{B}^\vee$. Therefore the first projection $\widetilde{\overline{\mathcal{X}}} \rightarrow \tilde{\mathbf{P}}^5$ is a projective bundle (whose fiber over $x \in \tilde{\mathbf{P}}^5$ is the hyperplane of \overline{B} determined by $\tilde{\phi}(x) \in \overline{B}^\vee$), hence smooth. By Lemma 5.5(iii), $\widetilde{\overline{\mathcal{X}}}$ satisfies property \mathcal{P} . □

We remark that the action of f on \mathbf{P}^5 lifts to $\tilde{\mathbf{P}}^5$ because the base locus of \overline{B} is clearly preserved by f . Correspondingly, the linear system \overline{B} pulls back to $\tilde{\mathbf{P}}^5$ to a base-point-free linear system, still denoted by \overline{B} , with each member preserved by f , and the morphism $\tilde{\phi}$ constructed above is exactly the morphism associated to this linear system.

To deal with the (possibly singular) variety $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$, we follow the same recipe as before (see Lemmas 5.9, 5.10 and Proposition 5.2). The morphism $\tilde{\phi} \times \tilde{\phi} : \tilde{\mathbf{P}}^5 \times \tilde{\mathbf{P}}^5 \rightarrow \overline{B}^\vee \times \overline{B}^\vee$ induces a rational map

$$\varphi : \tilde{\mathbf{P}}^5 \times \tilde{\mathbf{P}}^5 \dashrightarrow \text{Bl}_{\Delta_{\overline{B}^\vee}}(\overline{B}^\vee \times \overline{B}^\vee).$$

We remark that this rational map φ is again monomial, simply because $\phi : \mathbf{P}^5 \dashrightarrow \overline{B}^\vee$ is so.

Lemma 5.9 *There exists an elimination of indeterminacies of ϕ :*

$$\begin{array}{ccc} \widetilde{\tilde{\mathbf{P}}^5 \times \tilde{\mathbf{P}}^5} & \xrightarrow{\tilde{\varphi}} & \text{Bl}_{\Delta}(\overline{B}^\vee \times \overline{B}^\vee) \\ \tau \downarrow & \nearrow \varphi & \downarrow \\ \tilde{\mathbf{P}}^5 \times \tilde{\mathbf{P}}^5 & \xrightarrow{\tilde{\phi} \times \tilde{\phi}} & \overline{B}^\vee \times \overline{B}^\vee \end{array}$$

such that $\widetilde{\tilde{\mathbf{P}}^5 \times \tilde{\mathbf{P}}^5}$ is smooth projective satisfying property \mathcal{P} .

Proof It is a direct application of Proposition 5.6(v). □

Consider the rational map $\overline{B}^\vee \times \overline{B}^\vee \dashrightarrow \text{Gr}(\overline{B}, 2)$ defined by ‘intersecting two hyperplanes’, where $\text{Gr}(\overline{B}, 2)$ is the Grassmannian of codimension 2 sub-projective spaces of \overline{B} . Blowing up the diagonal will resolve the indeterminacies:

$$\begin{array}{ccc} \text{Bl}_{\Delta_{\overline{B}^\vee}}(\overline{B}^\vee \times \overline{B}^\vee) & & \\ \downarrow & \searrow & \\ \overline{B}^\vee \times \overline{B}^\vee & \dashrightarrow & \text{Gr}(\overline{B}, 2) \end{array}$$

Composing it with $\tilde{\varphi}$ constructed in the previous lemma, we obtain a morphism

$$\psi : \widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5} \rightarrow \text{Gr}(\overline{B}, 2).$$

As in Lemma 5.7, we have

Lemma 5.10 Consider the following incidence subvariety of $\widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5} \times \overline{B}$ with respect to ψ :

$$W := \left\{ (z, b) \in \widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5} \times \overline{B} \mid b \in \psi(z) \right\}.$$

- (i) The first projection $W \rightarrow \widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5}$ is a projective bundle, whose fiber over $z \in \widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5}$ is the codimension 2 sub-projective space determined by $\psi(z) \in \text{Gr}(\overline{B}, 2)$.
- (ii) W has a birational morphism onto $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$.

Proof (i) is obvious.

(ii) We have a natural morphism $W \rightarrow \mathbf{P}^5 \times \mathbf{P}^5 \times \overline{B}$. We claim that this morphism is birational onto its image $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$: since for two general points x_1, x_2 in \mathbf{P}^5 , $\psi(x_1, x_2) = \phi(x_1) \cap \phi(x_2)$, thus $(x_1, x_2, b) \in W$ is by definition equivalent to $b \in \phi(x_1) \cap \phi(x_2)$, which is equivalent to $(x_1, x_2, b) \in \overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$. \square

Now we can accomplish our first step of this section:

Proof of Proposition 5.2 Since W is a projective bundle (Lemma 5.10(i)) over the variety $\widetilde{\mathbf{P}^5} \times \widetilde{\mathbf{P}^5}$ satisfying property \mathcal{P} (Lemma 5.9(ii)), W satisfies also property \mathcal{P} (Lemma 5.5(iii)). Then we conclude by Lemma 5.10(ii).

5.2 Extension of homologically trivial algebraic cycles

In this subsection we prove Proposition 5.3.

To pass from the compactification $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$ to the space $\overline{\mathcal{X}} \times_B \overline{\mathcal{X}}$ which concerns us, we would like to mention Voisin’s ‘conjecture N’ ([24, Conjecture 0.6]):

Conjecture 5.11 (Conjecture N) Let X be a smooth projective variety, and let $U \subset X$ be an open subset. Assume an algebraic cycle $Z \in \text{CH}^i(X)_{\mathbf{Q}}$ has cohomology class $[Z] \in H^{2i}(X, \mathbf{Q})$ which vanishes when restricted to $H^{2i}(U, \mathbf{Q})$. Then there exists another cycle $Z' \in \text{CH}^i(X)_{\mathbf{Q}}$, which is supported on $X \setminus U$ and such that $[Z'] = [Z] \in H^{2i}(X, \mathbf{Q})$.

This Conjecture N is equivalent to the surjectivity of $\text{CH}^i(X)_{\mathbf{Q}, \text{hom}} \rightarrow \text{CH}^i(U)_{\mathbf{Q}, \text{hom}}$, hence implies the following conjecture (cf. [26, Lemma 4.20]):

Conjecture 5.12 *Let X be a smooth projective variety, and U be an open subset of X . If $\text{CH}^i(X)_{\mathbf{Q},\text{hom}} = 0$, then $\text{CH}^i(U)_{\mathbf{Q},\text{hom}} = 0$.*

According to this conjecture, Proposition 5.2 would have implied the desired result Proposition 5.1. To get around this conjecture N, our starting point is the following observation in [24, Lemma 1.1].

Lemma 5.13 *Conjecture N is true for $i \leq 2$. In particular, for $i \leq 2$ and for any $Z^o \in \text{CH}^i(U)_{\mathbf{Q},\text{hom}}$, there exists $W \in \text{CH}^i(X)_{\mathbf{Q},\text{hom}}$ such that $W|_U = Z^o$.*

Now the crucial observation is that *the Chow motive of a cubic fourfold does not exceed the size of Chow motives of surfaces*, so that we can reduce the problem to a known case of Conjecture N, namely Lemma 5.13. To illustrate, we first investigate the situation of one cubic fourfold (absolute case), then make the construction into the family version.

Absolute case

Let X be a (smooth) cubic fourfold. Recall the following diagram as in the proof of the unirationality of cubic fourfold:

$$\begin{array}{c} \mathbf{P}(T_X|_L) \xrightarrow{q} X \\ \downarrow \\ L \end{array}$$

Here we fix a line L contained in X , and the vertical arrow is the natural \mathbf{P}^3 -fibration, and the rational map q is defined in the following classical way: for any $(x, v) \in \mathbf{P}(T_X|_L)$ where v is a non-zero tangent vector of X at $x \in L$, then as long as the line $\mathbf{P}(\mathbf{C} \cdot v)$ generated by the tangent vector v is not contained in X , the intersection of this line $\mathbf{P}(\mathbf{C} \cdot v)$ with X should be three (not necessarily distinct) points with two of them x . Let y be the remaining intersection point. We define $q : (x, v) \mapsto y$. By construction, the indeterminacy locus of q is $\{(x, v) \in \mathbf{P}(T_X|_L) \mid \mathbf{P}(\mathbf{C} \cdot v) \subset X\}$. Note that q is dominant of degree 2.

By Hironaka’s theorem, we have an elimination of indeterminacies:

$$\begin{array}{ccc} \widetilde{\mathbf{P}(T_X|_L)} & & \\ \tau \downarrow & \searrow \tilde{q} & \\ \mathbf{P}(T_X|_L) & \xrightarrow{q} & X \\ \downarrow & & \\ L & & \end{array}$$

where τ is the composition of a series of successive blow ups along smooth centers of dimension ≤ 2 , and \tilde{q} is surjective thus generically finite (of degree 2).

We follow the notation of [2] to denote the category of Chow motives with rational coefficients by $\text{CHM}_{\mathbf{Q}}$, and to write \mathfrak{h} for the Chow motive of a smooth projective variety, which is a *contravariant* functor

$$\mathfrak{h} : \text{SmProj}^{op} \rightarrow \text{CHM}_{\mathbf{Q}}.$$

Denote $M := \mathbf{P}(T_X|_L)$ and $\tilde{M} := \widetilde{\mathbf{P}(T_X|_L)}$. Let S_i be the blow up centers of τ and $c_i = \text{codim } S_i \in \{2, 3, 4\}$. By the blow up formula and the projective bundle formula for

Chow motives (cf. [2, 4.3.2]),

$$\mathfrak{h}(\tilde{M}) = \mathfrak{h}(\mathbf{P}(T_X|_L)) \oplus \bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l) = \left(\bigoplus_{l=0}^3 \mathfrak{h}(L)(-l) \right) \oplus \left(\bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l) \right),$$

and since $L \simeq \mathbf{P}^1$,

$$\mathfrak{h}(\tilde{M}) = (\mathbb{1} \oplus \mathbb{1}(-1)^{\oplus 2} \oplus \mathbb{1}(-2)^{\oplus 2} \oplus \mathbb{1}(-3)^{\oplus 2} \oplus \mathbb{1}(-4)) \oplus \left(\bigoplus_i \bigoplus_{l=1}^{c_i-1} \mathfrak{h}(S_i)(-l) \right), \tag{12}$$

where $\mathbb{1} := \mathfrak{h}(\text{pt})$ is the trivial motive. On the other hand, since $\tilde{q}_* \tilde{q}^* = \text{deg}(\tilde{q}) = 2 \cdot \text{id}$, $\mathfrak{h}(X)$ is a direct factor of $\mathfrak{h}(\mathbf{P}(\tilde{T}_X|_L))$, which has been decomposed in (12). This gives a precise explanation of what we mean by saying that $\mathfrak{h}(X)$ does not exceed the size of motives of surfaces above.

By the monoidal structure of $\text{CHM}_{\mathbf{Q}}$ (cf. [2, 4.1.4]), the motive of $\tilde{M} \times \tilde{M}$ has the following form:

$$\mathfrak{h}(\tilde{M} \times \tilde{M}) = \bigoplus_{k \in J} \mathfrak{h}(V_k \times V'_k)(-l_k), \tag{13}$$

where J is the index set parameterizing all possible products, and $V_k \times V'_k$ is of one of the following forms:

- $\text{pt} \times \text{pt}$ and $l_k = 0$ or 1 ;
- $\text{pt} \times S_i$ or $S_i \times \text{pt}$ and $l_k = 1$;
- $l_k \geq 2$.

For each summand $\mathfrak{h}(V_k \times V'_k)(-l_k)$ in (13), the inclusion of this direct factor

$$t_k \in \text{Hom}_{\text{CHM}_{\mathbf{Q}}}(\mathfrak{h}(V_k \times V'_k)(-l_k), \mathfrak{h}(\tilde{M} \times \tilde{M})) \tag{14}$$

determines a natural correspondence from $V_k \times V'_k$ to $\tilde{M} \times \tilde{M}$.

Similarly, for each $k \in J$, the projection to the k -th direct factor

$$p_k \in \text{Hom}_{\text{CHM}_{\mathbf{Q}}}(\mathfrak{h}(\tilde{M} \times \tilde{M}), \mathfrak{h}(V_k \times V'_k)(-l_k)) \tag{15}$$

determines also a natural correspondence from $\tilde{M} \times \tilde{M}$ back to $V_k \times V'_k$. By construction

$$p_k \circ t_k = \text{id} \in \text{End}_{\text{CHM}_{\mathbf{Q}}}(\mathfrak{h}(V_k \times V'_k)(-l_k)), \quad \text{for any } k \in J;$$

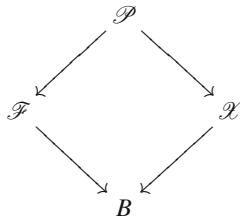
$$\sum_{k \in J} t_k \circ p_k = \text{id} \in \text{End}_{\text{CHM}_{\mathbf{Q}}}(\mathfrak{h}(\tilde{M} \times \tilde{M})).$$

Equivalently, the last equation says

$$\sum_{k \in J} t_k \circ p_k = \Delta_{\tilde{M} \times \tilde{M}} \text{ in } \text{CH}^*(\tilde{M} \times \tilde{M} \times \tilde{M} \times \tilde{M})_{\mathbf{Q}}. \tag{16}$$

Construction in family We now turn to the family version of the above constructions. To this end, we need to choose a specific line for each cubic fourfold in the family, and also a specific point on the chosen line. Therefore a base change (i.e. $T \rightarrow B$ constructed below) will be necessary to construct the family version of the previous p_k and t_k 's (see Lemma 5.14).

Precisely, consider the universal family \mathcal{X} of cubic fourfolds over B , and the universal family of Fano varieties of lines \mathcal{F} as well as the universal incidence varieties \mathcal{P} :



By taking general hyperplane sections of \mathcal{P} , we get T a subvariety of it, such that the induced morphism $T \rightarrow B$ is generically finite. In fact, by shrinking⁴ B (and also T correspondingly), we can assume $T \rightarrow B$ is finite and étale, and hence T is smooth.

By base change construction, we have over T a universal family of cubic fourfolds \mathcal{Y} , a universal family of lines \mathcal{L} contained in \mathcal{Y} and a section $\sigma : T \rightarrow \mathcal{L}$ corresponding to the universal family of the chosen points in \mathcal{L} . We summarize the situation in the following diagram:

$$\begin{array}{ccccc}
 \mathcal{L} & \longrightarrow & \mathcal{Y} & \xrightarrow{r} & \mathcal{X} \\
 \swarrow \sigma & & \downarrow \pi' & \square & \downarrow \pi \\
 & & T & \longrightarrow & B
 \end{array} \tag{17}$$

where for any $t \in T$ with image b in B , the fiber $Y_t = X_b, L_t$ is a line contained in it and $\sigma(t) \in L_t$. As $T \rightarrow B$ is finite and étale, so is $r : \mathcal{Y} \rightarrow \mathcal{X}$.

Now we define

$$\mathcal{M} := \mathbf{P}(T_{\mathcal{Y}/T}|\mathcal{L}),$$

and a dominant rational map of degree 2

$$q : \mathcal{M} \dashrightarrow \mathcal{Y}.$$

Over $t \in T$, the fiber of \mathcal{M} is $M_t = \mathbf{P}(T_{Y_t}|L_t)$, and the restriction of q to this fiber is exactly the rational map $\mathbf{P}(T_{Y_t}|L_t) \dashrightarrow Y_t$ constructed before in the absolute case.

By Hironaka’s theorem, we have an elimination of indeterminacies of q :

$$\begin{array}{ccc}
 \widetilde{\mathcal{M}} & & \\
 \tau \downarrow & \searrow \widetilde{q} & \\
 \mathcal{M} & \xrightarrow{q} & \mathcal{Y} \\
 \downarrow & & \uparrow \\
 \mathcal{L} & & \\
 \downarrow & & \\
 T & &
 \end{array} \tag{18}$$

⁴ Recall that we are allowed to shrink B whenever we want, see Remark 4.2.

such that, up to shrinking B (and also T correspondingly), τ consists of blow ups along smooth centers which are *smooth* over T (by generic smoothness theorem) of relative dimension (over T) at most 2. Suppose the blow up centers are \mathcal{S}_i , whose codimension is denoted by $c_i \in \{2, 3, 4\}$. A fiber of $\widetilde{\mathcal{M}}, \mathcal{M}, \mathcal{S}_i$ is exactly \widetilde{M}, M, S_i respectively constructed in the absolute case. In the same fashion, we denote by \mathcal{V}_k and \mathcal{V}'_k the family version of the varieties V_k and V'_k in the absolute case, which is nothing else but of the form $\mathcal{S}_i \times_T \mathcal{S}_j$ or $T \times_T \mathcal{S}_i$ etc. Let d_k (resp. d'_k) be the dimension of V_k (resp. V'_k), i.e. when $V_k = \text{pt}$, $\mathcal{V}_k = \sigma(T)$ and $d_k = 0$; when $V_k = S_i$, $\mathcal{V}_k = \mathcal{S}_i$ and $d_k = 4 - c_i$, similarly for \mathcal{V}'_k .

We can now globalize the correspondences (14) and (15) into their following family versions. Here we use the same notation:

Lemma 5.14 (i) *For any $k \in J$, there exist natural correspondences (over T)*

$$\begin{aligned} \iota_k &\in \text{CH}^{d_k+d'_k+l_k}(\mathcal{V}_k \times_T \mathcal{V}'_k \times_T \widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}})_{\mathbf{Q}}, \\ p_k &\in \text{CH}^{8-l_k}(\widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}} \times_T \mathcal{V}_k \times_T \mathcal{V}'_k)_{\mathbf{Q}}, \end{aligned}$$

such that the following two identities hold on each fiber: for any $t \in T$, we have

$$\begin{aligned} (p_k \circ \iota_k)_t &= \Delta_{V_{k,t} \times V'_{k,t}} \quad \text{for any } k \in J; \\ \sum_{k \in J} (\iota_k \circ p_k)_t &= \Delta_{\widetilde{M}_t \times \widetilde{M}_t}. \end{aligned} \tag{19}$$

(ii) *Up to shrinking B and T correspondingly, we have in $\text{CH}^*(\widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}})_{\mathbf{Q}}$,*

$$\sum_{k \in J} \iota_k \circ p_k = \Delta_{\widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}}}. \tag{20}$$

Proof (i). For the existence, it suffices to remark that the correspondences ι_k and p_k can in fact be universally defined over T , because when we make the canonical isomorphisms (12) or (13) precise by using the projective bundle formula and blow up formula, they are just compositions of inclusions, pull-backs, intersections with the relative $\mathcal{O}(1)$ of projective bundles, each of which can be defined in family over T . Note that in this step of the construction, we used the section σ to make the isomorphism $\mathfrak{h}(L_t) \simeq \mathbb{1} \oplus \mathbb{1}(-1)$ well-defined in family over T , because this isomorphism amounts to choose a point on the line.

Finally, equality (19) is exactly equality (16) in the absolute case.

(ii). Equation (20) is a direct consequence of (19), thanks to the Bloch–Srinivas type argument of spreading rational equivalences (cf. [8,21, Corollary 10.20]). \square

Keeping the notation in Diagram (17) and Diagram (18), we consider the generic finite morphism

$$\tilde{q} \times \tilde{q} : \widetilde{\mathcal{M}} \times_T \widetilde{\mathcal{M}} \rightarrow \mathcal{Y} \times_T \mathcal{Y},$$

and the finite étale morphism

$$r \times r : \mathcal{Y} \times_T \mathcal{Y} \rightarrow \mathcal{X} \times_B \mathcal{X}.$$

For each $k \in J$, composing the graphs of these two morphisms with ι_k , we get a correspondence over B from $\mathcal{V}_k \times_T \mathcal{V}'_k$ to $\mathcal{X} \times_B \mathcal{X}$:

$$\Gamma_k := \Gamma_{r \times r} \circ \Gamma_{\tilde{q} \times \tilde{q}} \circ \iota_k \in \text{CH}^{d_k+d'_k+l_k}(\mathcal{V}_k \times_T \mathcal{V}'_k \times_B \mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}; \tag{21}$$

similarly, composing p_k with the transposes of their graphs, we obtain a correspondence over B in the other direction:

$$\Gamma'_k := p_k \circ ({}^t\Gamma_{\bar{q} \times \bar{q}}) \circ ({}^t\Gamma_{r \times r}) \in \text{CH}^{8-l_k}(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{Y}'_k \times_T \mathcal{Y}'_k)_{\mathbf{Q}}. \tag{22}$$

Lemma 5.15 *The sum of compositions of the above two correspondences satisfies:*

$$\sum_{k \in J} \Gamma_k \circ \Gamma'_k = 4 \deg(T/B) \cdot \text{id},$$

as correspondences from $\mathcal{X} \times_B \mathcal{X}$ to itself.

Proof It is an immediate consequence of Eq. (20) and the projection formula (note that $\deg(r \times r) = \deg(T/B)$ and $\deg(\bar{q} \times \bar{q}) = 4$). □

For any $k \in J$, fix a smooth projective compactification $\overline{\mathcal{Y}'_k \times_T \mathcal{Y}'_k}$ of $\mathcal{Y}'_k \times_T \mathcal{Y}'_k$ such that the composition $\overline{\mathcal{Y}'_k \times_T \mathcal{Y}'_k} \dashrightarrow T \rightarrow B \rightarrow \overline{B}$ is a morphism. Recall that $\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$ is a (in general singular) compactification of $\mathcal{X} \times_B \mathcal{X}$ and $\tau: W \rightarrow \overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}}$ is a resolution of singularities. Put $W^o := W \times_{\overline{B}} B$, and $\tau^o: W^o \rightarrow \mathcal{X} \times_B \mathcal{X}$ the restriction of τ . (See the end of this section for a diagram.) Consider the composition of the correspondence $\Gamma_k \in \text{CH}^{d_k+d'_k+l_k}(\mathcal{Y}'_k \times_T \mathcal{Y}'_k \times_B \mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$ constructed above, with the pull-back by τ^o viewed as a correspondence ${}^t\Gamma_{\tau^o} \in \text{CH}^8(\mathcal{X} \times_B \mathcal{X} \times_B W^o)$, we have a correspondence

$${}^t\Gamma_{\tau^o} \circ \Gamma_k \in \text{CH}^{d_k+d'_k+l_k}(\mathcal{Y}'_k \times_T \mathcal{Y}'_k \times_B W^o)_{\mathbf{Q}}.$$

Taking its closure, we obtain a correspondence between their smooth compactifications:

$$\overline{{}^t\Gamma_{\tau^o} \circ \Gamma_k} \in \text{CH}^{d_k+d'_k+l_k}(\overline{\mathcal{Y}'_k \times_T \mathcal{Y}'_k} \times_{\overline{B}} W)_{\mathbf{Q}}.$$

For some technical reasons in the proof below, we have to separate the index set J into two parts and deal with them differently. Recall that below Eq. (13), we observed that there are three types of elements $(V_k \times V'_k, l_k)$ in J . Define the subset consisting of elements of the third type:

$$J' := \{k \in J \mid (V_k \times V'_k, l_k) \text{ satisfies } l_k \geq 2\}.$$

And define J'' to be the complement of J' : elements of the first two types. Note that for any $k \in J''$, the corresponding $(V_k \times V'_k, l_k)$ satisfies always

$$\dim(V_k \times V'_k) < 4 - l_k \quad \text{for any } k \in J''. \tag{23}$$

We can now accomplish the main goal of this subsection:

Proof of Proposition 5.3 Let $z \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}, \text{hom}}$. To simplify the notation, we will omit the lower star for the correspondences Γ_k, Γ'_k and ${}^t\Gamma_{\tau^o} \circ \Gamma_k$ since we never use their transposes.

An obvious remark: when $k \in J''$, for any $b \in B$, the fiber $(\Gamma'_k)_t(z_b) \in \text{CH}^{4-l_k}(V_{k,t} \times V'_{k,t})_{\mathbf{Q}} = 0$ by dimension reason (cf. (23)), thus

$$\left(\sum_{k \in J''} \Gamma_k \circ \Gamma'_k(z) \right)_b = 0. \tag{24}$$

As a result, for any $b \in B$, in $\text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$,

$$\left(\sum_{k \in J'} \Gamma_k (\Gamma'_k(z)) \right)_b = \left(\sum_{k \in J} \Gamma_k (\Gamma'_k(z)) \right)_b = 4 \deg(T/B) \cdot z_b, \tag{25}$$

where the first equality comes from (24) and the second equality is by Lemma 5.15.

On the other hand, $\Gamma'_k(z) \in \text{CH}^{4-l_k}(\mathcal{Y}_k \times_T \mathcal{Y}'_k)_{\mathbf{Q}, \text{hom}}$ is homologically trivial. We claim that for any $k \in J'$, the cycle $\Gamma'_k(z)$ ‘extends’ to a homologically trivial algebraic cycle in the compactification, i.e. there exists $\xi_k \in \text{CH}^{4-l_k}(\overline{\mathcal{Y}_k \times_T \mathcal{Y}'_k})_{\mathbf{Q}, \text{hom}}$ such that $\xi_k|_{\mathcal{Y}_k \times_B \mathcal{Y}'_k} = \Gamma'_k(z)$. Indeed, since $4 - l_k \leq 2$ for $k \in J'$ and $\mathcal{Y}_k \times_T \mathcal{Y}'_k$ is smooth by construction, we can apply Lemma 5.13 to find ξ_k .

Now for any $k \in J'$, let us consider the cycle $\overline{{}^t\Gamma_{\tau^o} \circ \Gamma_k}(\xi_k) \in \text{CH}^4(W)_{\mathbf{Q}, \text{hom}}$. Its fiber over a point $b \in B$ is:

$$\overline{{}^t\Gamma_{\tau^o} \circ \Gamma_k}(\xi_k)_b = \left({}^t\Gamma_{\tau^o} \circ \Gamma_k(\xi_k|_{\mathcal{Y}_k \times_B \mathcal{Y}'_k}) \right)_b = ({}^t\Gamma_{\tau^o} \circ \Gamma_k \circ \Gamma'_k(z))_b.$$

Therefore by (25), the restrictions of the following two cycles⁵ to a fiber W_b

$$\beta := \sum_{k \in J'} \overline{{}^t\Gamma_{\tau^o} \circ \Gamma_k}(\xi_k) \text{ and } 4 \deg(T/B) \cdot \tau^{o*}(z)$$

are the same in $\text{CH}^4(W_b)_{\mathbf{Q}}$ for any $b \in B$, where $W_b := W \times_B \{b\}$. Again by the argument of Bloch and Srinivas (cf. [8, 21, §10.2], [26, §2]), there exists a dense open subset $B' \subset B$, such that

$$\beta|_{W'} = (4 \deg(T/B) \cdot \tau^{o*}(z))|_{W'} = \tau'^* (4 \deg(T/B) \cdot z|_{\mathcal{X}' \times_{B'} \mathcal{X}'}) \in \text{CH}^4(W')_{\mathbf{Q}}, \tag{26}$$

where $\mathcal{X}' = \mathcal{X} \times_B B'$, $W' = W \times_{\overline{B}} B'$ are obtained by base change:

$$\begin{array}{ccccc} W'^{\subset} & \longrightarrow & W^{o\subset} & \longrightarrow & W \\ \tau' \downarrow & & \downarrow \tau^o & & \downarrow \tau \\ \mathcal{X}' \times_{B'} & \longrightarrow & \mathcal{X} \times_B & \longrightarrow & \overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}} \\ \downarrow & & \downarrow & & \downarrow \\ B'^{\subset} & \longrightarrow & B^{\subset} & \longrightarrow & \overline{B} \end{array}$$

Define $\overline{z} := \frac{1}{4 \deg(T/B)} \beta \in \text{CH}^4(W)_{\mathbf{Q}}$. By (26) and the projection formula for τ' , we have the required property (11) in Proposition 5.3:

$$\tau'_*(\overline{z}|_{W'}) = \tau'_* \tau'^* (z|_{\mathcal{X}' \times_{B'} \mathcal{X}'}) = z|_{\mathcal{X}' \times_{B'} \mathcal{X}'} \in \text{CH}^4(\mathcal{X}' \times_{B'} \mathcal{X}')_{\mathbf{Q}},$$

To conclude Proposition 5.3, it suffices to remark that \overline{z} is by construction homologically trivial: since the ξ_k ’s are homologically trivial, so is $\beta = \sum_{k \in J'} \overline{{}^t\Gamma_{\tau^o} \circ \Gamma_k}(\xi_k)$ and hence \overline{z} . □

⁵ Here τ^{o*} is well-defined cause W^o and $\mathcal{X} \times_B \mathcal{X}$ are both smooth.

6 Proof of Theorem 4.3

The content of this section is the proof of Theorem 4.3.

We keep the notation $n, f, j, \Lambda_j, \bar{B}$ as in the statement of Theorem 4.3. Let B be an open subset of \bar{B} parameterizing smooth cubic fourfolds. In the following diagram,

$$\begin{array}{ccc}
 f_b \curvearrowright X_b & \xrightarrow{\quad} & \mathcal{X} \curvearrowleft f \\
 \downarrow & \square & \downarrow \pi \\
 b & \xrightarrow{\quad} & B
 \end{array} \tag{27}$$

π is the universal family equipped with a fiberwise action f ; for each $b \in B$, we write f_b the restriction of f on the cubic fourfold X_b if we need to distinguish it from f . By construction, for any $b \in B$, f_b is an automorphism of X_b of order n which acts as identity on $H^{3,1}(X_b)$.

To begin with, we study the Hodge structures of the fibers:

Lemma 6.1 *For any $b \in B$,*

- (i) f^* is an order n automorphism of the Hodge structure $H^4(X_b, \mathbf{Q})$ and $f_* = (f^*)^{-1} = (f^*)^{n-1}$.
- (ii) There is a direct sum decomposition into sub-Hodge structures

$$H^4(X_b, \mathbf{Q}) = H^4(X_b, \mathbf{Q})^{inv} \oplus^\perp H^4(X_b, \mathbf{Q})^\#, \tag{28}$$

where the first summand is the f^* -invariant part, and the second summand is its orthogonal complement with respect to the intersection product $\langle -, - \rangle$.

- (iii) $H^{3,1}(X_b) \subset H^4(X_b, \mathbf{C})^{inv}$.
- (iv) $H^4(X_b, \mathbf{Q})^\#$ is generated by the classes of some codimension 2 algebraic cycles.

Proof (i) is obvious since f^* must preserve the Hodge structure. The last equality comes from $f_* f^* = f^* f_* = \text{id}$ and $(f^*)^n = \text{id}$.

- (ii) Since f^* is of finite order, it is semi-simple: $H^4(X_b, \mathbf{Q})$ decomposes as direct sum of eigenspaces, where $H^4(X_b, \mathbf{Q})^{inv}$ corresponds to eigenvalue 1 and $H^4(X_b, \mathbf{Q})^\#$ is the sum of other eigenspaces. Moreover, f^* preserves the intersection pairing $\langle -, - \rangle$, thus the invariant eigenspace is orthogonal to any other eigenspace.
- (iii) This is our assumption that f^* acts as identity on $H^{3,1}(X)$.
- (iv) By (iii), $H^4(X_b, \mathbf{Q})^\# \subset H_{\mathbf{R}}^{3,1}(X_b)^\perp = H_{\mathbf{R}}^{2,2}(X_b)$, i.e. $H^4(X_b, \mathbf{Q})^\#$ is generated by rational Hodge classes of degree 4. However, the Hodge conjecture is known to be true for cubic fourfolds [27]. We deduce that $H^4(X_b, \mathbf{Q})^\#$ is generated by the classes of some codimension 2 subvarieties in X_b .

□

Define the algebraic cycle

$$\pi^{inv} := \frac{1}{n} (\Delta_{\mathcal{X}} + \Gamma_f + \dots + \Gamma_{f^{n-1}}) \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}. \tag{29}$$

Here π^{inv} can be viewed as a family of self-correspondences of X_b parameterized by B , more precisely:

$$\pi^{inv}|_{X_b \times X_b} =: \pi_b^{inv} = \frac{1}{n} (\Delta_{X_b} + \Gamma_{f_b} + \dots + \Gamma_{f_b^{n-1}}) \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}. \tag{30}$$

It is now clear that π_b^{inv} acts on $H^4(X_b, \mathbf{Q})$ by:

$$[\pi_b^{\text{inv}}]^* = \frac{1}{n} \left(\text{Id} + f_b^* + \dots + (f_b^*)^{n-1} \right),$$

which is exactly the orthogonal projector onto the invariant part in the direct sum decomposition (28). On the other hand, π_b^{inv} acts as identity on $H^0(X_b, \mathbf{Q})$, $H^2(X_b, \mathbf{Q})$, $H^6(X_b, \mathbf{Q})$, $H^8(X_b, \mathbf{Q})$.

Define another cycle

$$\Gamma := \Delta_{\mathcal{X}} - \pi^{\text{inv}} \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}. \tag{31}$$

Then $\Gamma_b := \Gamma|_{X_b \times X_b} = \Delta_{X_b} - \frac{1}{n} \left(\Delta_{X_b} + \Gamma_{f_b} + \dots + \Gamma_{f_b^{n-1}} \right) \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$ acts on $H^4(X_b, \mathbf{Q})$ as the orthogonal projector onto $H^4(X_b, \mathbf{Q})^{\#}$ and acts as zero on $H^0(X_b, \mathbf{Q})$, $H^2(X_b, \mathbf{Q})$, $H^6(X_b, \mathbf{Q})$, $H^8(X_b, \mathbf{Q})$. Now we have some control over the cohomology class of ‘fibers’ of Γ :

Proposition 6.2 *For any $b \in B$,*

- (i) *Let $\Gamma_b := \Gamma|_{X_b \times X_b} \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$. Then its cohomology class $[\Gamma_b] \in H^8(X_b \times X_b, \mathbf{Q})$ is contained in $H^4(X_b, \mathbf{Q})^{\#} \otimes H^4(X_b, \mathbf{Q})^{\#}$.*
- (ii) *The cohomology class $[\Gamma_b]$ is supported on $Y_b \times Y_b$, where Y_b is a closed algebraic subset of X_b of codimension at least 2.*
- (iii) *Moreover, there exists an algebraic cycle $\Gamma'_b \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$, which is supported on $Y_b \times Y_b$ and such that $[\Gamma_b] = [\Gamma'_b]$ in $H^8(X_b \times X_b, \mathbf{Q})$*

Proof (i). By Künneth formula, we make the identification

$$\begin{aligned} H^8(X_b \times X_b) &= (H^0 \otimes H^8) \oplus (H^2 \otimes H^6) \oplus (H^6 \otimes H^2) \oplus (H^8 \otimes H^0) \\ &\oplus (H^{4,\text{inv}} \otimes H^{4,\text{inv}}) \oplus (H^{4,\text{inv}} \otimes H^{4,\#}) \oplus (H^{4,\#} \otimes H^{4,\text{inv}}) \\ &\oplus (H^{4,\#} \otimes H^{4,\#}). \end{aligned}$$

By construction and Poincaré duality, the cohomology class $[\Gamma_b]$ can only have the last component non-zero.

(ii) is a consequence of (iii).

(iii). By Lemma 6.1(iv), $H^4(X_b, \mathbf{Q})^{\#}$ is generated by the classes of some codimension 2 subvarieties in X_b . We thus assume $H^4(X_b, \mathbf{Q})^{\#} = \mathbf{Q}[W_1] \oplus \dots \oplus \mathbf{Q}[W_r]$, where W_i ’s are subvarieties of codimension 2 in X_b . We can now take $Y_b := \bigcup_{i=1}^r W_i$ to be the codimension 2 closed algebraic subset, and take Γ'_b to be of the form $\sum_{i,j=1}^r b_{ij} W_i \times W_j \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$. □

Roughly speaking, the previous proposition says that when restricted to each fiber, the cycle Γ becomes homologically equivalent to a cycle supported on a codimension 2 algebraic subset of the fiber. Now here comes the crucial proposition, which allows us to get some global information about Γ from its fiberwise property. The proposition appeared in Voisin’s paper [24]:

Proposition 6.3 *In the above situation as in Proposition 6.2, there exist a closed algebraic subset \mathcal{Y} in \mathcal{X} of codimension 2, and an algebraic cycle $\Gamma' \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$ which is supported on $\mathcal{Y} \times_B \mathcal{Y}$, such that for any $b \in B$, $[\Gamma'|_{X_b \times X_b}] = [\Gamma|_{X_b \times X_b}]$ in $H^8(X_b \times X_b, \mathbf{Q})$.*

Proof See [24] Proposition 2.7. □

Let $\mathcal{Y} \subset \mathcal{X}$ be the codimension 2 closed algebraic subset introduced above, and $\Gamma' \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$ be the cycle supported on $\mathcal{Y} \times_B \mathcal{Y}$, as constructed in Proposition 6.3. Define

$$\mathcal{Z} := \Gamma - \Gamma' \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}. \tag{32}$$

Then by construction, for any $b \in B$, the ‘fiber’ $Z_b := \mathcal{Z}|_{X_b \times X_b} \in \text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$ has trivial cohomology class:

$$[Z_b] = 0 \in H^8(X_b \times X_b, \mathbf{Q}), \quad \text{for any } b \in B. \tag{33}$$

The next step is to prove the following decomposition of the projector (Proposition 6.4) from the fiberwise cohomological triviality of \mathcal{Z} in (33).

Proposition 6.4 *There exist a closed algebraic subset \mathcal{Y} in \mathcal{X} of codimension 2, an algebraic cycle $\Gamma' \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$ supported on $\mathcal{Y} \times_B \mathcal{Y}$, $\mathcal{Z}' \in \text{CH}^4(\mathcal{X} \times \mathbf{P}^5)_{\mathbf{Q}}$ and $\mathcal{Z}'' \in \text{CH}^4(\mathbf{P}^5 \times \mathcal{X})_{\mathbf{Q}}$ such that we have an equality in $\text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}$:*

$$\Delta_{\mathcal{X}} - \frac{1}{n} (\Delta_{\mathcal{X}} + \Gamma_f + \dots + \Gamma_{f^{n-1}}) = \Gamma' + \mathcal{Z}'|_{\mathcal{X} \times_B \mathcal{X}} + \mathcal{Z}''|_{\mathcal{X} \times_B \mathcal{X}}. \tag{34}$$

To pass from the fiberwise equality (33) to the global equality (34) above, we have to firstly deduce from (33) some global equality up to homological equivalence, then use the result of Sect. 5 to get an equality up to rational equivalence. The argument of Leray spectral sequence due to Voisin [24, Lemma 2.11] (in our equivariant case) can accomplish the first step.

By Deligne’s theorem [12], the Leray spectral sequence associated to the smooth projective morphism $\pi \times \pi : \mathcal{X} \times_B \mathcal{X} \rightarrow B$ degenerates at E_2 :

$$E_{\infty}^{p,q} = E_2^{p,q} = H^p(B, R^q(\pi \times \pi)_* \mathbf{Q}) \Rightarrow H^{p+q}(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}).$$

In other words,

$$\text{Gr}_L^p H^{p+q}(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}) = H^p(B, R^q(\pi \times \pi)_* \mathbf{Q}),$$

where L^{\bullet} is the resulting *Leray filtration* on $H^*(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q})$. The property (33) is thus equivalent to

Lemma 6.5 *The cohomology class $[\mathcal{Z}] \in L^1 H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q})$.*

Proof The image of $[\mathcal{Z}] \in H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q})$ in the first graded piece $\text{Gr}_L^0 H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}) = H^0(B, R^8(\pi \times \pi)_* \mathbf{Q})$ is a section of the local system $R^8(\pi \times \pi)_* \mathbf{Q}$, whose fiber over $b \in B$ is $H^8(X_b \times X_b, \mathbf{Q})$. The value of this section on b is given exactly by $[Z_b] \in H^8(X_b \times X_b, \mathbf{Q})$, which vanishes by (33). Therefore $[\mathcal{Z}] \in H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q})$ becomes zero in the quotient Gr_L^0 , hence is contained in L^1 . □

Consider the Leray spectral sequences associated to the following three smooth projective morphisms to the base B :

$$\begin{array}{ccc}
 \mathcal{X} \times \mathbf{P}^5 & \xleftarrow{\text{id} \times i} & \mathcal{X} \times_B \mathcal{X} \xrightarrow{i \times \text{id}} \mathbf{P}^5 \times \mathcal{X} \\
 \searrow \pi \circ \text{pr}_1 & & \downarrow \pi \times \pi \\
 & & B
 \end{array}$$

and the restriction maps for cohomology induced by the two inclusions. We have the following lemma, where all the cohomology groups are of rational coefficients.

Lemma 6.6 *Let L^\bullet be the Leray filtrations corresponding to the above Leray spectral sequences.*

(i) *The Künneth isomorphisms induce canonical isomorphisms*

$$L^1 H^8(\mathcal{X} \times \mathbf{P}^5) = \bigoplus_{i+j=8} L^1 H^i(\mathcal{X}) \otimes H^j(\mathbf{P}^5),$$

$$L^1 H^8(\mathbf{P}^5 \times \mathcal{X}) = \bigoplus_{i+j=8} H^i(\mathbf{P}^5) \otimes L^1 H^j(\mathcal{X}).$$

(ii) *The natural restriction map*

$$L^1 H^8(\mathcal{X} \times \mathbf{P}^5) \oplus L^1 H^8(\mathbf{P}^5 \times \mathcal{X}) \rightarrow L^1 H^8(\mathcal{X} \times_B \mathcal{X})$$

is surjective.

Proof By snake lemma (or five lemma) and induction, it suffices to prove the corresponding results for the graded pieces.

(i) We only prove the first isomorphism, the second one is similar. For any $p \geq 1$, the isomorphism

$$\text{Gr}_L^p H^8(\mathcal{X} \times \mathbf{P}^5) = \bigoplus_{i+j=8} \text{Gr}_L^p H^i(\mathcal{X}) \otimes H^j(\mathbf{P}^5)$$

by Deligne’ theorem is equivalent to

$$H^p(B, R^{8-p}(\pi \circ \text{pr}_1)_* \mathbf{Q}) = H^p\left(B, \bigoplus_{i+j=8} (R^{i-p} \pi_* \mathbf{Q}) \otimes_{\mathbf{Q}} H^j(\mathbf{P}^5)\right).$$

However, $R^{8-p}(\pi \circ \text{pr}_1)_* \mathbf{Q}$ is a local system with fiber $H^{8-p}(X_b \times \mathbf{P}^5, \mathbf{Q})$, which is by Künneth formula isomorphic to $\bigoplus_{i+j=8} H^{i-p}(X_b) \otimes H^j(\mathbf{P}^5)$, which is exactly the fiber of the local system $\bigoplus_{i+j=8} (R^{i-p} \pi_* \mathbf{Q}) \otimes_{\mathbf{Q}} H^j(\mathbf{P}^5)$. Thus (i) is a consequence of the relative Künneth formula.

(ii) Using (i), for any $p \geq 1$, the surjectivity of $\text{Gr}_L^p H^8(\mathcal{X} \times \mathbf{P}^5) \oplus \text{Gr}_L^p H^8(\mathbf{P}^5 \times \mathcal{X}) \rightarrow \text{Gr}_L^p H^8(\mathcal{X} \times_B \mathcal{X})$ is by Deligne’s theorem equivalent to the surjectivity of

$$H^p\left(B, \bigoplus_{i+j=8} (R^{i-p} \pi_* \mathbf{Q}) \otimes_{\mathbf{Q}} H^j(\mathbf{P}^5)\right) \oplus H^p\left(B, \bigoplus_{i+j=8} H^i(\mathbf{P}^5) \otimes_{\mathbf{Q}} (R^{j-p} \pi_* \mathbf{Q})\right) \\ \rightarrow H^p(B, R^{8-p}(\pi \times \pi)_* \mathbf{Q}).$$

By relative Künneth isomorphism, $R^{8-p}(\pi \times \pi)_* \mathbf{Q} = \bigoplus_{k+l=8-p} R^k \pi_* \mathbf{Q} \otimes R^l \pi_* \mathbf{Q}$. Since $8 - p \leq 7$, either $k < 4$ or $l < 4$. Recall that $H^{\text{odd}}(X_b) = 0$ and the restriction map $H^{2i}(\mathbf{P}^5) \rightarrow H^{2i}(X_b)$ is an isomorphism for $i = 0, 1, 3, 4$, thus $R^{8-p}(\pi \times \pi)_* \mathbf{Q}$ is a direct summand of the local system $\bigoplus_{k+l=8-p} R^k \pi_* \mathbf{Q} \otimes_{\mathbf{Q}} H^l(\mathbf{P}^5) \oplus \bigoplus_{k+l=8-p} H^k(\mathbf{P}^5) \otimes_{\mathbf{Q}} R^l \pi_* \mathbf{Q}$. Therefore the above displayed morphism is induced by the projection of a local system to its direct summand, which is of course surjective on cohomology. \square

Combining Lemmas 6.5 and 6.6, we can decompose the cohomology class $[\mathcal{Z}]$ as follows:

$$[\mathcal{Z}] = \sum_{i=0}^4 \text{pr}_1^* A_i \cdot \text{pr}_2^*[h]^{4-i} + \sum_{j=0}^4 \text{pr}_1^*[h]^{4-j} \cdot \text{pr}_2^* B_j \in H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}), \quad (35)$$

where $\text{pr}_i : \mathcal{X} \times_B \mathcal{X} \rightarrow \mathcal{X}$ is the i -th projection, $A_i \in H^{2i}(\mathcal{X}, \mathbf{Q})$, $B_j \in H^{2j}(\mathcal{X}, \mathbf{Q})$ and $h \in \text{CH}^1(\mathcal{X})$ is the pull back by the natural morphism $\mathcal{X} \rightarrow \mathbf{P}^5$ of the hyperplane divisor $c_1(\mathcal{O}_{\mathbf{P}^5}(1))$.

We remark that A_i and B_j must be *algebraic*, that is, they are the cohomology classes of algebraic cycles of \mathcal{X} . The reason is very simple: $[\mathcal{Z}]$ being algebraic, so is

$$\text{pr}_{1,*} \left([\mathcal{Z}] \cdot \text{pr}_2^*[h]^i \right) = 3A_i + (\text{a rational number}) [h]^i,$$

thus A_i is algebraic. The algebraicity of B_j is similar. We denote still by $A_i \in \text{CH}^i(\mathcal{X})_{\mathbf{Q}}$ and $B_j \in \text{CH}^j(\mathcal{X})_{\mathbf{Q}}$ for the algebraic cycles with the respective cohomology classes. Therefore (35) becomes

$$[\mathcal{Z}] = \sum_{i=0}^4 [\text{pr}_1^* A_i \cdot \text{pr}_2^* h^{4-i}] + \sum_{j=0}^4 [\text{pr}_1^* h^{4-j} \cdot \text{pr}_2^* B_j] \in H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}).$$

In other words, there exist algebraic cycles

$$\begin{aligned} \mathcal{Z}' &:= \sum_{i=0}^4 \text{pr}_1^* A_i \cdot \text{pr}_2^* h^{4-i} \in \text{Im} \left(\text{CH}^4(\mathcal{X} \times \mathbf{P}^5)_{\mathbf{Q}} \rightarrow \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}} \right), \quad \text{and} \\ \mathcal{Z}'' &:= \sum_{j=0}^4 \text{pr}_1^* h^{4-j} \cdot \text{pr}_2^* B_j \in \text{Im} \left(\text{CH}^4(\mathbf{P}^5 \times \mathcal{X})_{\mathbf{Q}} \rightarrow \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}} \right), \end{aligned}$$

such that

$$[\mathcal{Z}] = [\mathcal{Z}' + \mathcal{Z}''] \in H^8(\mathcal{X} \times_B \mathcal{X}, \mathbf{Q}). \tag{36}$$

This is an equality up to homological equivalence. Now enters the result of Sect. 5: thanks to Proposition 5.1, up to shrinking B to a dense open subset (still denoted by B), the cohomological decomposition (36) in fact implies the following equality up to *rational equivalence* in Chow groups:

$$\mathcal{Z} = \mathcal{Z}'|_{\mathcal{X} \times_B \mathcal{X}} + \mathcal{Z}''|_{\mathcal{X} \times_B \mathcal{X}} \in \text{CH}^4(\mathcal{X} \times_B \mathcal{X})_{\mathbf{Q}}.$$

Combining this with (29), (31), Proposition 6.3 and (32), we obtain a decomposition of the projector (34) announced in Proposition 6.4.

Now we can deduce Theorem 4.3 from this decomposition as follows. For any $b \in B$ (thus general in \bar{B}), taking the fiber of (34) over b , we get an equality in $\text{CH}^4(X_b \times X_b)_{\mathbf{Q}}$.

$$\Delta_{X_b} = \frac{1}{n} (\Delta_{X_b} + \Gamma_f + \dots + \Gamma_{f^{n-1}}) + \Gamma'_b + Z'_b|_{X_b \times X_b} + Z''_b|_{X_b \times X_b}, \tag{37}$$

where we still write f for f_b the restriction of the action on fiber X_b , Γ'_b is supported on $Y_b \times Y_b$ with Y_b a closed algebraic subset of codimension 2 in X_b , and Z'_b (*resp.* Z''_b) is a cycle of $X_b \times \mathbf{P}^5$ (*resp.* $\mathbf{P}^5 \times X_b$) with rational coefficients.

For any homologically trivial 1-cycle $\gamma \in \text{CH}_1(X_b)_{\mathbf{Q}, \text{hom}}$, let both sides of (37) act on it by correspondences. We have in $\text{CH}_1(X_b)_{\mathbf{Q}}$:

- $\Delta_{X_b}^*(\gamma) = \gamma$;
- $\frac{1}{n} (\Delta_{X_b} + \Gamma_f + \dots + \Gamma_{f^{n-1}})^*(\gamma) = \frac{1}{n} (\gamma + f^*\gamma + \dots + (f^*)^{n-1}\gamma)$;
- $\Gamma_b'^*(\gamma) = 0$ because the support of Γ_b' has the projection to the first coordinate codimension 2;

- $(Z'_b|_{X_b \times X_b})^*(\gamma) = (Z''_b|_{X_b \times X_b})^*(\gamma) = 0$, since they both factorizes through $\text{CH}^*(\mathbf{P}^5)_{\mathbf{Q}, \text{hom}} = 0$.

As a result, we have in $\text{CH}_1(X_b)_{\mathbf{Q}}$,

$$\gamma = \frac{1}{n} \left(\gamma + f^* \gamma + \dots + (f^*)^{n-1} \gamma \right),$$

where the right hand side is obviously invariant by f^* , hence so is the left hand side. In other words, f^* acts on $\text{CH}_1(X_b)_{\mathbf{Q}, \text{hom}}$ as identity. Finally, since $H^6(X_b, \mathbf{Q})$ is 1-dimensional with f^* acting trivially, we have for any $\gamma \in \text{CH}_1(X_b)_{\mathbf{Q}}$,

$$\pi^{\text{inv},*}(\gamma) - \gamma \in \text{CH}_1(X_b)_{\mathbf{Q}, \text{hom}},$$

where $\pi^{\text{inv},*} = \frac{1}{n} (\text{id} + f^* + \dots + (f^*)^{n-1})$. Therefore by what we just obtained,

$$f^* \left(\pi^{\text{inv},*}(\gamma) - \gamma \right) = \pi^{\text{inv},*}(\gamma) - \gamma.$$

As $\pi^{\text{inv},*}(\gamma)$ is obviously f^* -invariant, we have $f^*(\gamma) = \gamma$ in $\text{CH}_1(X_b)_{\mathbf{Q}}$. Theorem 4.3, as well as the main Theorem 1.5, is proved.

7 A remark

In the main Theorem 1.5, we assumed that the automorphism of the Fano variety of lines is induced from an automorphism of the cubic fourfold itself. In this section we want to reformulate this hypothesis.

Proposition 7.1 *Let $X \subset \mathbf{P}^5$ be a (smooth) cubic fourfold, and $(F(X), \mathcal{L})$ be its Fano variety of lines equipped with the Plücker polarization induced from the ambient Grassmannian $\text{Gr}(\mathbf{P}^1, \mathbf{P}^5)$. An automorphism ψ of $F(X)$ comes from an automorphism of X if and only if it is polarized (i.e. $\psi^* \mathcal{L} \simeq \mathcal{L}$).*

Proof The following proof is taken from [9, Proposition 4]. Consider the projective embedding of $F(X)$ determined by the Plücker polarization \mathcal{L} :

$$F(X) =: F \subset \text{Gr}(\mathbf{P}^1, \mathbf{P}^5) =: G \subset \mathbf{P}(\wedge^2 H^0(\mathbf{P}^5, \mathcal{O}(1))^\vee) =: \mathbf{P}^{14}.$$

If ψ is induced from an automorphism f of X , which must be an automorphism of \mathbf{P}^5 , then it is clear that ψ is the restriction of a linear automorphism of \mathbf{P}^{14} , thus the Plücker polarization is preserved.

Conversely, if the automorphism ψ preserves the polarization, it is then the restriction of a projective automorphism of \mathbf{P}^{14} , which we denote still by ψ . It is proved in [1, 1.16 (iii)] that G is the intersection of all the quadrics containing F . It follows that ψ is an automorphism of G , because ψ sends any quadric containing F to a quadric containing F . However any automorphism of G is induced by a projective automorphism f of \mathbf{P}^5 (cf. [10]). As a result, f sends a line contained in X to a line contained in X , thus f preserves X and ψ is induced from f . □

Define $\text{Aut}^{\text{pol}}(F(X))$ to be the group of polarized automorphisms of $F(X)$ (equipped with Plücker polarization \mathcal{L}). Then the previous proposition says that we have an isomorphism

$$\begin{aligned} \text{Aut}(X) &\xrightarrow{\simeq} \text{Aut}^{\text{pol}}(F(X)) \\ f &\mapsto \hat{f} \end{aligned}$$

This gives us the last statement in main Theorem 1.5: any polarized symplectic automorphism of $F(X)$ acts as identity on $\text{CH}_0(F(X))$.

8 A consequence: action on $\text{CH}_2(F)_{\mathbf{Q},\text{hom}}$

As an application of Theorem 1.5, we study in this section the induced action on the Chow group of 2-cycles. The conclusion is Corollary 1.7 in the introduction.

Proof of Corollary 1.7 As showed in the previous section, the polarized automorphism \hat{f} on $F(X)$ comes from an automorphism f of finite order n of the smooth cubic fourfold X . We consider again the projector $\Gamma := \Delta_F - \pi^{\text{inv}} \in \text{CH}_4(F \times F)_{\mathbf{Q}}$, where

$$\pi^{\text{inv}} = \frac{\Delta_F + \Gamma_{\hat{f}} + \dots + \Gamma_{\hat{f}^{n-1}}}{n} \in \text{CH}_4(F \times F)_{\mathbf{Q}}.$$

We remark that $\Gamma = {}^t\Gamma$ since $\hat{f}^{-1} = \hat{f}^{n-1}$. Our main result Theorem 1.5 says in particular that the action of Γ on $\text{CH}_0(F)_{\mathbf{Q}}$ is zero:

$$\Gamma_* = 0 : \text{CH}_0(F)_{\mathbf{Q}} \rightarrow \text{CH}_0(F)_{\mathbf{Q}}.$$

Equivalently speaking, the restriction of Γ to each fiber is zero:

$$\Gamma|_{\{t\} \times F} = 0 \in \text{CH}_0(F)_{\mathbf{Q}}, \quad \forall t \in F.$$

By the argument of Bloch–Srinivas (cf. [8, 21, §10.2]), there exist an effective reduced divisor $D \subsetneq X$, a resolution of singularities $\tau : \tilde{D} \rightarrow D$ and an algebraic cycle $\Gamma' \in \text{CH}_4(\tilde{D} \times F)_{\mathbf{Q}}$ such that $\Gamma = (\tilde{\tau} \times \text{id}_F)_* \Gamma'$, where $\tilde{\tau}$ is the composition of τ and the inclusion of D into X . Consequently, the action of $\Gamma = {}^t\Gamma$ on $\text{CH}_2(F)_{\mathbf{Q}}$ factorises as:

$$\begin{array}{ccc} \text{CH}_2(F)_{\mathbf{Q}} & \xrightarrow{\Gamma_* = \Gamma'^*} & \text{CH}_2(F)_{\mathbf{Q}} \\ & \searrow \Gamma'^* & \nearrow \tilde{\tau}_* \\ & \text{CH}^1(\tilde{D})_{\mathbf{Q}} & \end{array}$$

Since these correspondences preserve the homological equivalence as well as the Abel–Jacobi equivalence (cf. [21, Chapter 9]), we have in fact the following factorization:

$$\begin{array}{ccc} \text{CH}_2(F)_{\mathbf{Q},AJ} & \xrightarrow{\Gamma_* = \Gamma'^*} & \text{CH}_2(F)_{\mathbf{Q},AJ} \\ & \searrow \Gamma'^* & \nearrow \tilde{\tau}_* \\ & \text{CH}^1(\tilde{D})_{\mathbf{Q},AJ} & \end{array}$$

where AJ means the Abel–Jacobi kernels. However, it is well-known that for divisors the Abel–Jacobi map is an isomorphism. Hence

$$\text{CH}^1(\tilde{D})_{\mathbf{Q},AJ} = 0.$$

Now the factorization implies that Γ acts as zero on $\text{CH}_2(F)_{\mathbf{Q},AJ}$, thus \hat{f} acts as identity on $\text{CH}_2(F)_{\mathbf{Q},AJ}$. To conclude, it suffices to remark that from the vanishing $H^3(F) = 0$ (because

F is deformation equivalent to the second punctual Hilbert scheme of K3 surfaces [4], which has vanishing odd cohomology), the Abel–Jacobi kernel

$$\text{CH}_2(F)_{\mathbf{Q},AJ} := \ker \left(\text{CH}_2(F)_{\mathbf{Q},hom} \rightarrow J^3(F) := \frac{H^{0,3}(F) \oplus H^{1,2}(F)}{H^3(F, \mathbf{Z})} = 0 \right)$$

is equal to $\text{CH}_2(F)_{\mathbf{Q},hom}$. \square

Remark 8.1 We want to remark that Corollary 1.7 is also predicted by Bloch–Beilinson conjecture (a more general version than Conjecture 1.4, cf. [21, Chapter 11], [2, Chapitre 11]).

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