MOTIVES OF MODULI SPACES OF BUNDLES ON CURVES VIA VARIATION OF STABILITY AND FLIPS

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Abstract

We study the rational Chow motives of certain moduli spaces of vector bundles on a smooth projective curve with additional structure (such as a parabolic structure or Higgs field). In the parabolic case, these moduli spaces depend on a choice of stability condition given by weights; our approach is to use explicit descriptions of variation of this stability condition in terms of simple birational transformations (standard flips/flops and Mukai flops) for which we understand the variation of the Chow motives. For moduli spaces of parabolic vector bundles, we describe the change in motive under wall-crossings, and for moduli spaces of parabolic Higgs bundles, we show the motive does not change under wall-crossings. Furthermore, we prove a motivic analogue of a classical theorem of Harder and Narasimhan relating the rational cohomology of moduli spaces of vector bundles with and without fixed determinant. For rank 2 vector bundles of odd degree, we obtain formulas for the rational Chow motives of moduli spaces of semistable vector bundles, moduli spaces of Higgs bundles and moduli spaces of parabolic (Higgs) bundles that are semistable with respect to a generic weight (all with and without fixed determinant).

Contents

1. INTRODUCTION

Let C be a smooth projective geometrically connected curve of genus g over a field k. Let $\mathcal{N} = \mathcal{N}_C(n, d)$ denote the moduli space of semistable vector bundles of rank n and degree d on C. When n and d are coprime, $\mathcal{N}$ is a smooth projective variety of dimension $n^2(g-1)+1$.

There has been a long history of work on the cohomological invariants of $\mathcal{N}$. Inductive formulas for the Betti numbers of $\mathcal{N}$ were obtained by Harder and Narasimhan [36] using the Weil conjectures and point counting over finite fields, by Atiyah and Bott [4] over $k = \mathbb{C}$ using gauge theory, and by Bifet, Ghione and Letizia [13] using more algebro-geometric methods. All three approaches essentially involve first describing the cohomology of the stack $\mathrm{Bun}_{n,d}$ of all vector bundles on C and then inductively computing the cohomology of $\mathcal{N}$ by performing a Harder–Narasimhan recursion. The approach of [13] lead to a closed formula for the class of $\mathrm{Bun}_{n,d}$ in a dimensional completion of the Grothendieck ring of varieties [11] and in Voevodsky's

triangulated category of motives over k with $\mathbb{Q}$-coefficients [39]. Furthermore, the ideas in [13] were used by del Baño to show that the Chow motive of $\mathcal{N}$ lies in the tensor subcategory generated by the motive of the curve [25].

More generally, for a smooth projective variety X and a moduli space of sheaves of some kind (possibly with some additional structure) on X, there are several examples in which the motive of this moduli space lies in the tensor subcategory generated by the motive of X. This holds for the moduli space of stable Higgs bundles on C of coprime rank and degree [40] and for certain moduli spaces of semistable sheaves on K3 and abelian surfaces (as well as for closely related spaces such as crepant resolutions, twisted and non-commutative analogues) [19, 28].

1.1. **Qualitative results in arbitrary rank.** Our motivation for this paper was to provide more explicit descriptions of the Chow motives of certain moduli spaces of vector bundles on a smooth projective curve with additional structure. Our approach is to use concrete geometric descriptions of the birational transformations between moduli spaces under variation of stability [16, 67] together with recent descriptions giving the change in Chow motives of smooth projective varieties under standard flips and flops and Mukai flops [47, 55]. In fact, since we also want to work with moduli spaces of Higgs bundles on C, which are non-proper, but smooth if n and d are coprime, we first need to extend the last of these results to smooth quasi-projective varieties (*cf.* Theorem 3.6) using a local-to-global trick explained in Appendix A.

One can simplify the geometry by fixing a degree d line bundle $\mathcal{L}$ and studying the moduli space $\mathcal{N}_{\mathcal{L}} = \mathcal{N}_{\mathcal{L},C}(n,d)$ of semistable vector bundles with determinant isomorphic to $\mathcal{L}$. When n and d are coprime, $\mathcal{N}_{\mathcal{L}}$ is also smooth and projective and, moreover, is Fano [27] and rational [52]. Quite remarkably, Harder and Narasimhan [36] showed that, for any prime ℓ different from the characteristic of k, the ℓ-adic cohomology of $\mathcal{N}$ is the tensor product of that of $\mathcal{N}_{\mathcal{L}}$ and $\mathrm{Jac}(C)$. Our first result gives a motivic refinement of this classical theorem (see Theorem 4.2).

Theorem 1.1. *Let n and d be coprime; then there is an isomorphism of rational Chow motives*
$$\mathfrak{h}(\mathcal{N}) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \otimes \mathfrak{h}(\mathrm{Jac}(C)).$$

The proof involves first checking that $\mathfrak{h}(\mathcal{N})$ and $\mathfrak{h}(\mathcal{N}_{\mathcal{L}})$ lie in the tensor subcategory generated by $\mathfrak{h}(C)$, and thus are abelian motives; for $\mathcal{N}$ this is a theorem of del Baño [25] and for $\mathcal{N}_{\mathcal{L}}$, see Proposition 4.1, which uses an argument of Bülles [19] and Beauville [10]. We reduce to the case of a field of characteristic zero, then use the theorem of Harder and Narasimhan and the fact that the ℓ-adic realisation is conservative on abelian geometric motives in characteristic 0 [71].

In the case of parabolic vector bundles, which are vector bundles with flags of a specified type at a finite number of parabolic points $D = \{p_1, \dots, p_N\}$ on C, there are various notions of stability, encoded by a set of parabolic weights α, which give rise to different moduli spaces $\mathcal{N}^{\alpha} = \mathcal{N}_{C,D}^{\alpha}(n,d)$. For generic weights (i.e. where semistability and stability coincide), these moduli spaces are smooth and projective. The space of weights admits a wall and chamber decomposition by considering how the notion of (semi)stability changes and the birational transformation between moduli spaces separated by a wall can be explicitly described [16, 18, 67]. In the nicest cases, the wall-crossing transformation is a standard flip or flop whose centres can be explicitly described as certain projective bundles over a product of smaller moduli spaces and the dimensions of the projective space fibres can be computed in terms of the dimension of a certain Ext group. Consequently, for such walls, one obtains an explicit description of how the Chow motive varies with α (Corollary 5.16). We also provide a motivic description of flag degeneration (Corollary 5.19). Furthermore, for coprime rank and degree and sufficiently small weights α, there is a forgetful map $\mathcal{N}^{\alpha} \to \mathcal{N}$ which is an iterated flag bundle [16, 18]. Hence, it suffices to know $\mathfrak{h}(\mathcal{N})$ to compute the Chow motives of $\mathfrak{h}(\mathcal{N}^{\alpha})$ for generic α via wall-crossing; for some explicit formulas in rank $n = 2$, see §1.2. In particular, the motive $\mathfrak{h}(\mathcal{N}^{\alpha})$ depends on α, as was already known for the Poincaré polynomial [8, 38]. As a corollary of these motivic formulas, we obtain descriptions of the Chow groups of 1-cycles (see Corollaries 5.18 and 5.21) strengthening and slightly correcting [20, 21] and for fixed i, a stabilisation result for $\mathrm{CH}^i(\mathcal{N}^{\alpha})$ (see Corollary 5.22).

The algebraic symplectic analogues of moduli spaces of (parabolic) vector bundles are the moduli spaces of Higgs bundles $\mathcal{M}$ and parabolic Higgs bundles $\mathcal{M}^\alpha$, which are no longer proper, but are smooth when semistability and stability coincide. There is a $\mathbb{G}_m$-action on $\mathcal{M}$ given by scaling the Higgs field [37, 65] which gives an associated Białynicki-Birula decomposition [12] that enables one to see that the cohomology of $\mathcal{M}$ is nonetheless pure. In [40, Corollary 6.9], the second and third authors show that the Voevodsky motive of $\mathcal{M}$ is a Chow motive by this method. The same holds for moduli spaces of parabolic Higgs bundles which are semistable with respect to a generic weight (*cf.* Lemma 6.6). Moreover, via the Białynicki-Birula decomposition, the Chow motive of $\mathcal{M}$ (resp. $\mathcal{M}^\alpha$) can be expressed in terms of $\mathcal{N}$ (resp. $\mathcal{N}^\alpha$) and moduli spaces of chains (resp. parabolic chains); see [40] for the non-parabolic case.

There are several surprising changes as we pass from (parabolic) vector bundles to (parabolic) Higgs bundles. First, the analogue of Theorem 1.1 does not hold if we replace $\mathcal{N}$ by $\mathcal{M}$; this was already seen on cohomology in rank $n = 2$ by Hitchin [37]. Moreover, for a general curve C of genus at least 2, the motive of $\mathcal{M}_\mathcal{L}$ is not contained in the subcategory generated by $\mathfrak{h}(C)$ (see Proposition 6.3). Second, although the Chow motive of $\mathcal{N}^\alpha$ depends on α, this is no longer the case for $\mathcal{M}^\alpha$.

Theorem 1.2. *(Corollary 6.7) For a generic weight α, the integral Chow motive of the moduli space $\mathcal{M}^\alpha$ of α-semistable parabolic Higgs bundles of rank n and degree d is independent of α.*

Our proof relies on Thaddeus' description [68] of wall-crossings as Mukai flops, which can be thought of as algebraic symplectic versions of standard flops and do not alter the motive by Theorem 3.6.

1.2. **Explicit formulas in rank 2 and odd degree.** In order to compute the Chow motives of these moduli spaces, the basic building block is $\mathfrak{h}(\mathcal{N})$. By Theorem 1.1, it suffices to know $\mathfrak{h}(\mathcal{N}_\mathcal{L})$. In rank $n = 2$ and odd degree d, we compute this by using variation of stability for pairs consisting of a vector bundle and a section studied by Thaddeus [66, 67] and improving work of del Baño in [26] on the motive of $\mathcal{N}_\mathcal{L}$ in a semisimple category of motives by using the fact that $\mathfrak{h}(\mathcal{N}_\mathcal{L})$ is abelian (*cf.* Proposition 4.1) and hence Kimura finite-dimensional. Consequently, we obtain corresponding decompositions of the Chow groups of these moduli spaces (see §4.3.1 for corollaries and comparisons with previous results for $\mathcal{N}_\mathcal{L}$). We denote by $\mathrm{Pic}^d(C)$ the Picard scheme of line bundles of degree d on C.

Theorem 1.3. *For $n = 2$, d odd and $\mathcal{L} \in \mathrm{Pic}^d(C)(k)$, the rational Chow motive of the moduli space $\mathcal{N}_\mathcal{L}$ of semistable bundles with determinant $\mathcal{L}$ is*

$$\mathfrak{h}(\mathcal{N}_\mathcal{L}) \simeq \mathfrak{h}(\mathrm{Sym}^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} \mathfrak{h}(\mathrm{Sym}^i(C)) \otimes (\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i)) \,.$$

We obtain the following formulas for the integral Chow motives of the other bundle moduli spaces in terms of $\mathfrak{h}(\mathcal{N})$ or $\mathfrak{h}(\mathcal{N}_\mathcal{L})$. For parabolic vector bundles, this comes from studying the wall-crossing flips/flops. For (parabolic) Higgs bundles, this come from using Białynicki-Birula decompositions associated to $\mathbb{G}_m$-actions scaling the Higgs field and Theorem 1.2.

Theorem 1.4. *For $n = 2$, d odd and $\mathcal{L} \in \mathrm{Pic}^d(C)(k)$, we have the following formulas for the integral Chow motives of associated bundle moduli spaces.*

(i) *(Theorem 5.32) For the moduli space $\mathcal{N}^\alpha$ of parabolic bundles on C with flags at N parabolic points which are semistable with respect to a generic weight α, we have*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g+j)^{\oplus b_j(\alpha)}.$$

where the exponents[1] $b_j(\alpha)$ are given in Definition 5.31.

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[1]If M is a Chow motive, then the notation $M(i)^{\oplus n}$ denotes the direct sum of n copies of $M(i)$; this is not ambiguous since this is also canonically isomorphic to $M \otimes (\mathbb{Q}(i)^{\oplus n})$.

(ii) *(Corollary 5.34) For the moduli space $\mathcal{N}_{\mathcal{L}}^{\alpha}$ of parabolic bundles on C with determinant $\mathcal{L}$ and flags at N parabolic points which are semistable with respect to a generic weight α, we have*

$$\mathfrak{h}(\mathcal{N}_{\mathcal{L}}^{\alpha}) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))(g+j)^{\oplus b_j(\alpha)}$$

where the exponents $b_j(\alpha)$ are as above.

(iii) *(Theorem 6.1) For the moduli space $\mathcal{M}$ of semistable Higgs bundles, we have*

$$\mathfrak{h}(\mathcal{M}) \simeq \mathfrak{h}(\mathcal{N}(2,d)) \oplus \bigoplus_{j=1}^{g-1} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j-1}(C))(3g-2j-2),$$

where $a_{d,j} = g - j + (d-1)/2$.

(iv) *(Proposition 6.2) For the moduli space $\mathcal{M}_{\mathcal{L}}$ of semistable Higgs bundles with determinant $\mathcal{L}$, we have*

$$\mathfrak{h}(\mathcal{M}_{\mathcal{L}}) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \oplus \bigoplus_{j=1}^{g-1} \mathfrak{h}(\widetilde{\mathrm{Sym}}^{2j-1}(C))(3g-2j-2)$$

where $\widetilde{\mathrm{Sym}}^{i}(C) \to \mathrm{Sym}^i(C)$ is the pull back of the multiplication-by-2 map on $\mathrm{Jac}(C)$.

(v) *(Theorem 6.8) For the moduli space $\mathcal{M}^{\alpha}$ of parabolic Higgs bundles on C with flags at N parabolic points which are semistable with respect to a generic weight α, we have*

$$\mathfrak{h}(\mathcal{M}^{\alpha}) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \le l \le N \\ \frac{l+1-N}{2} \le j \le g-1}} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j+N-l-1}(C))(3g-2j+l-2)^{\oplus \binom{N}{l}}$$

where $a_{d,j} := g - j + (d-1)/2$. As in Theorem 1.2, this is independent of α.

(vi) *(Proposition 6.9) For the moduli space $\mathcal{M}_{\mathcal{L}}^{\alpha}$ of parabolic Higgs bundles on C with determinant $\mathcal{L}$ and flags at N parabolic points which are semistable with respect to a generic weight α, we have*

$$\mathfrak{h}(\mathcal{M}_{\mathcal{L}}^{\alpha}) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \le l \le N \\ \frac{l+1-N}{2} \le j \le g-1}} \mathfrak{h}(\widetilde{\mathrm{Sym}}^{2j+N-l-1}(C))(3g-2j+l-2)^{\oplus \binom{N}{l}}$$

where $\widetilde{\mathrm{Sym}}^{i}(C) \to \mathrm{Sym}^i(C)$ is the pull back of the multiplication-by-2 map on $\mathrm{Jac}(C)$.

1.3. Relation to other works.

Let us mention here some previous and subsequent works that are closely related to above results:

- The consequences on the Betti numbers of moduli spaces of parabolic vector bundles in Theorem 1.4 (i) and (ii) were obtained by Bauer [8] in genus zero and by Holla [38] in a recursive form in general (for arbitrary rank).
- For the moduli spaces of parabolic Higgs bundles over curves, the phenomenon of independence of the parabolic data in Theorem 1.2 was proved on the level of Betti numbers over $k = \mathbb{C}$ [17], then later for the diffeomorphic type in [60], and recently on the level of Betti numbers for any reductive Lie group in [15].
- As is mentioned before, our approach to Theorem 1.3 (iii) and (iv) using the $\mathbb{G}_m$-action and the associated Białynicki-Birula decomposition goes back to ideas of Hitchin [37] and Simpson [65]. Motivic computations following this approach already appeared in [33], [34], and [40].
- (Rank 3) Theorem 1.3 and Theorem 1.4 (iii) and (iv) are extended in our subsequent paper [30] to obtain formulas for the motives of moduli spaces of semistable vector bundles and Higgs bundles of rank 3 and coprime degree. The analogue of Theorem 1.4 (v) and (vi) in rank 3 has not been treated yet, but the Betti numbers have been worked out in the non-parabolic case in [35] and in parabolic case in [32] using Morse-theoretic methods.

2. Background on motives

Let k be a field and R be a commutative ring.

2.1. **Chow motives.** We write $\mathrm{CHM}(k, R)$ for the category of Chow motives over k with coefficients in R. This is of course a very classical object, but since the definition is short and there are two possible conventions in the literature, we provide some details. Objects in $\mathrm{CHM}(k, R)$ are triples (X, p, n), where X is a smooth projective k-variety, p is a projector in the ring $\mathrm{CH}^*(X \times X, R)$ of correspondences up to rational equivalence (with product given by composition of correspondences) and $n \in \mathbb{Z}$. We adopt the standard notations: $\mathfrak{h}(X) = (X, \mathrm{id}_X, 0)$ and $R(n) = (\mathrm{Spec}(k), \mathrm{id}_{\mathrm{Spec}(k)}, n)$. The category $\mathrm{CHM}(k, R)$ admits a symmetric monoidal structure coming from products in SmProj_k. Morphisms in $\mathrm{CHM}(k, R)$ are given, following an homological convention, by

$$\mathrm{CHM}(k, R)((X, p, m), (Y, q, n)) = q \circ \mathrm{CH}^{\dim(Y)+n-m}(X \times Y, R) \circ p$$

(where $\dim(Y)$ has to be interpreted as a locally constant function if Y is reducible). With this convention, there is a covariant functor $\mathfrak{h} \colon \mathrm{SmProj}_k \to \mathrm{CHM}(k, R)$. For a morphism $f \colon X \to Y$ between smooth projective varieties, $\mathfrak{h}(f)$ is denoted by $f_* \colon \mathfrak{h}(X) \to \mathfrak{h}(Y)$; that is, the morphism given by the graph of f. Using Poincaré duality $\mathfrak{h}(X)^\vee \cong \mathfrak{h}(X)(-\dim X)$, we have the Gysin morphism $f^* \colon \mathfrak{h}(Y) \to \mathfrak{h}(X)(\dim Y - \dim X)$.

By definition, we get the following, which we state for later reference.

Lemma 2.1. *Let X be a smooth projective variety. Then $\mathrm{CH}^i(X, R) \simeq \mathrm{Hom}(\mathfrak{h}(X), R(i))$, which thus vanishes if $i < 0$ or $i > \dim(X)$.*

A Chow motive $M \in \mathrm{CHM}(k, \mathbb{Q})$ is Kimura finite-dimensional, in the sense of [51] (see also [2]), if there exists a decomposition $M = M_{\mathrm{even}} \oplus M_{\mathrm{odd}}$ and $n_{\mathrm{even}}, n_{\mathrm{odd}} \in \mathbb{N}$ such that $\Lambda^{n_{\mathrm{even}}}(M_{\mathrm{even}}) = 0$ and $\mathrm{Sym}^{n_{\mathrm{odd}}}(M_{\mathrm{odd}}) = 0$. Let $\mathrm{CHM}(k, \mathbb{Q})^{\mathrm{Kim}}$ denote the full subcategory of Kimura finite-dimensional motives. Kimura finite-dimensionality has some very pleasant categorical consequences, one of which we now recall.

Proposition 2.2 (see [2]). *The category $\mathrm{CHM}(k, \mathbb{Q})^{\mathrm{Kim}}$ is closed under taking direct sums, tensor products, direct summands and duals. The functor $\mathrm{CHM}(k, \mathbb{Q})^{\mathrm{Kim}} \to M_{\mathrm{num}}(k, \mathbb{Q})$ to the category of motives for numerical equivalence is full and conservative.*

In particular, because $M_{\mathrm{num}}(k, \mathbb{Q})$ is abelian and semisimple by [46], $\mathrm{CHM}(k, \mathbb{Q})^{\mathrm{Kim}}$ satisfies cancellation: if there exists an isomorphism $M \oplus P \simeq N \oplus P$, then $M \simeq N$.

Kimura and O'Sullivan conjectured that every Chow motive is Kimura finite-dimensional. While this is wide open in general, we still have the following result for abelian motives which turns out to cover all the motives encountered in this paper. The category $\mathrm{CHM}(k, \mathbb{Q})^{\mathrm{ab}}$ of abelian motives is the thick subcategory generated by motives of abelian varieties, or equivalently, the thick tensor subcategory generated by motives of curves (see [62, Proposition 4.5, Theorem 5.2]).

Proposition 2.3 ([2, Théorème 2.8]). *Abelian motives are Kimura finite-dimensional.*

2.2. **Voevodsky motives.** Some of the moduli spaces we are interested in, namely moduli spaces of (parabolic) Higgs bundles, are not smooth projective varieties, but only smooth and quasi-projective. A natural way to associate a motive to them is via the triangulated category $\mathrm{DM}(k, R)$ of Voevodsky motives over k with coefficients in R. It turns out that, in all the cases considered in this paper, the resulting motive is actually a Chow motive, where we identify $\mathrm{CHM}(k, R)$ with a full subcategory of $\mathrm{DM}(k, R)$ via the fundamental embedding theorem of Voevodsky [69], which we recall now in the form we need.

Theorem 2.4. *Assume that k is a perfect field or that the characteristic of k is invertible in R. Then the "motive" functor*

$$M \colon \mathrm{SmProj}_k \to \mathrm{DM}(k, R)$$

factors through the category of Chow motives and induces a fully faithful embedding

$$\mathrm{CHM}(k, R) \hookrightarrow \mathrm{DM}(k, R).$$

Proof. This result follows directly (by passing to idempotent completions) from the following formula for Hom groups in $\mathrm{DM}(k, R)$. Let X, Y be smooth varieties of pure dimensions d, e with Y projective. There is an isomorphism

$$(1) \qquad\qquad \mathrm{DM}(k, R)(M(X), M(Y)) \simeq \mathrm{CH}^e(X \times_k Y, R)$$

which is natural in X and Y. This is a standard result but we could not find it precisely in this form in the literature, so we explain how to deduce it from existing references.

Let us assume first that k is perfect. The formula then follows from the combination of the following results, which all only require k perfect: duality for smooth projective varieties in the category $\mathrm{DM}^{\mathrm{eff}}(k, R)$ of effective Voevodsky motives (see [41, Appendix B] for a reference which only requires k perfect), the representability of Chow groups in $\mathrm{DM}^{\mathrm{eff}}(k, R)$ [57, Proposition 14.16 and Corollary 19.2], and Voevodsky's cancellation theorem saying that $\mathrm{DM}^{\mathrm{eff}}(k, R) \to \mathrm{DM}(k, R)$ is fully faithful [70].

Let us now assume that the characteristic of k is invertible in R. Let k^{perf}/k be a perfect closure of k. Under the assumption on R, the base change functor $\mathrm{DM}(k, R) \to \mathrm{DM}(k^{\mathrm{perf}}, R)$ is an equivalence of categories [23, Proposition 8.1], while the Chow groups also do not change under purely inseparable extensions. This implies the general case. $\square$

We also recall the following useful result.

Lemma 2.5. *Let K/k be any field extension. Then the base change functor*

$$(-)_K : \mathrm{DM}_c(k, \mathbb{Q}) \to \mathrm{DM}_c(K, \mathbb{Q})$$

on constructible Voevodsky motives with rational coefficients is conservative. In particular, the base change functor

$$(-)_K : \mathrm{CHM}(k, \mathbb{Q}) \to \mathrm{CHM}(K, \mathbb{Q})$$

is conservative.

Proof. When working with rational coefficients, the category $\mathrm{DM}(k, \mathbb{Q})$ is equivalent to the category $\mathrm{DM}_B(k, \mathbb{Q})$ of Beilinson motives [24, Corollary 16.1.4] (note that we use a Roman B instead of a cyrilic letter in our notation). Our claim is then a special case of the following much stronger result: if $f : X \to Y$ be any finite type surjective morphism, then $f^* : \mathrm{DM}_B(Y) \to \mathrm{DM}_B(X)$ is conservative [24, Theorem 14.3.3, Definition 2.1.7].

The statement about Chow motives is then obtained from the one on constructible Voevodsky motives by using Theorem 2.4. $\square$

3. Motives under flips and flops

In the context of the Minimal Model Program (see [53]), flips and flops are among the basic building blocks for birational transforms between algebraic varieties of dimension greater than 2. We recall here the geometry of two particularly elementary instances, namely standard flips and flops, and Mukai flops[2], and present some results on the corresponding change of motives, which will be applied later to birational transformations between various moduli spaces of stable vector bundles on curves.

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[2]In the literature, standard flips and flops are also called elementary, ordinary or Atiyah flips and flops; Mukai flops also go under the name of elementary transforms, especially in the context of hyper-Kähler geometry.

3.1. **Standard flips and flops.**

Definition 3.1 (Standard flips and flops)**.** Let S be a smooth k-variety. Let m and l be two positive integers. Let V be a vector bundle on S of rank $m + 1$ and $\varpi : Z = \mathbb{P}(V) \to S$ the associated projective bundle. Let X be a smooth variety containing Z as a closed subvariety such that the restriction of the normal bundle $N_{Z/X}$ to each fibre of ϖ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(-1)^{\oplus(l+1)}$, or equivalently (see [55, Lemma 1.1]), $N_{Z/X} \simeq \mathcal{O}_{\varpi}(-1) \otimes \varpi^*(V')$ for some vector bundle V' of rank $l + 1$ on S.

Let $\tau \colon \widetilde{X} \to X$ be the blow-up of X along the smooth center Z; then the exceptional divisor E is isomorphic to $\mathbb{P}(V) \times_S \mathbb{P}(V')$ with normal bundle $N_{E/\widetilde{X}} \simeq \mathcal{O}(-1, -1)$. Therefore we can contract E along the other direction, $E \to Z' := \mathbb{P}(V')$, which identifies E as the projectivisation of the vector bundle $\varpi'^*(V')$, where $\varpi' \colon Z' \to S$ is the natural projection. The contraction of E to Z' amounts to blowing down $\widetilde{X}$ to a new (smooth) variety X', and the normal bundle becomes $N_{Z'/X'} \simeq \mathcal{O}_{\varpi'}(-1) \otimes \varpi'^*V$. We call the induced birational transform $\phi : X \dashrightarrow X'$ a *standard flip of type* (m, l) *with centre* S. When $m = l$, it is called a *standard flop*. In terms of the K-partial order $\geq_K$ on the birational class of X (as in [50, Definition 1.1]), for a standard flip as above, we have $X \geq_K X'$ if and only if $m \geq l$.

In this paper, we assume that X is quasi-projective. Note that the flipped variety X' could fail to be quasi-projective in general. We assume further that there exists a small extremal contraction (in the sense of the Minimal Model Program [53]) $X \to \bar{X}$ whose restriction to Z is ϖ. Then X' is quasi-projective (see [49, Proposition 4.2] and [55, Proposition 1.3]). We summarise the situation in the following commutative diagram.

(2)

$$
\begin{array}{c}
E = Z \times_S Z' \\
\downarrow{\scriptstyle j} \\
\widetilde{X}
\end{array}
$$

If all the varieties are projective, the relation between the Chow motives of X and X' are established by Lee–Lin–Wang [55, Theorem 2.1] in the case of flops ($m = l$) and by Jiang [47, Theorem 3.4.(3), Corollary 3.8] in general.

Theorem 3.2. *Let* $\phi \colon X \dashrightarrow X'$ *be a standard flip of type* (m, l) *with centre* S *between smooth projective varieties, as in Definition 3.1. Suppose* $m \geq l$, *then there is an isomorphism of integral Chow motives:*

$$
(\tau_* \tau'^*, \sum_j i_* \circ (c_1(\mathcal{O}_{\varpi}(1))^j \cdot \_) \circ \varpi^*) \colon \mathfrak{h}(X') \oplus \bigoplus_{j=l+1}^{m} \mathfrak{h}(S)(j) \xrightarrow{\simeq} \mathfrak{h}(X).
$$

In particular, when ϕ *is a standard flop* ($m = l$), *we have an isomorphism* $\tau'_* \tau^* \colon \mathfrak{h}(X) \xrightarrow{\simeq} \mathfrak{h}(X')$, *with inverse* $\tau_* \tau'^*$.

3.2. **Mukai flops.** The so-called Mukai flop was discovered by Mukai [59] as a typical instance of birational transforms between hyper-Kähler manifolds (see also [42, §3]). The following definition is a mild generalisation of the classical one[3], which is enough for the purpose of this paper.

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[3]Our definition is called a twisted Mukai flop in [55].

Definition 3.3 (Mukai flops)**.** Let S be a smooth variety and let m be a positive integer. Let V be a vector bundle on S of rank $m + 1$ and $\varpi : Z = \mathbb{P}(V) \to S$ the associated projective bundle. Let X be a smooth variety containing Z as a closed subvariety such that

$$N_{Z/X} \simeq \Omega_{Z/S} \otimes \varpi^*(L),$$

for some line bundle L on S, where $\Omega_{Z/X}$ is the relative cotangent bundle of $\varpi : Z \to S$.

Let $\tau : \widetilde{X} \to X$ be the blow-up of X along the smooth center Z, then the exceptional divisor $E = \mathbb{P}(\Omega_{Z/S})$, which is isomorphic to the incidence hypersurface inside $\mathbb{P}(V) \times_S \mathbb{P}(V^\vee)$, has normal bundle $N_{E/\widetilde{X}} \simeq \mathcal{O}(-1, -1)$. By contracting E along the other direction $E \to Z' := \mathbb{P}(V^\vee)$, we blow down $\widetilde{X}$ to a new (smooth) variety X'. We call the induced birational transform $\phi : X \dashrightarrow X'$ a *Mukai flop* of type m with centre S. A significant difference compared to standard flops is that the fibre product $X \times_{\bar{X}} X'$ has two irreducible components of the same dimension, namely, $\widetilde{X}$ and $Z \times_S Z'$, intersecting along E.

As in Definition 3.1, in order to stay in the category of quasi-projective varieities, we assume in this paper that there exists a small extremal contraction in the sense of the Minimal Model Program $X \to \bar{X}$ whose restriction to Z is ϖ. The situation is summarised in the following diagram.

(3)

In the projective situation, Lee–Lin–Wang [55, Theorem 6.3] proved the invariance of Chow motives under a Mukai flop; see also [29] for a generalisation to *stratified Mukai flops*.

Theorem 3.4. *Let $\phi: X \dashrightarrow X''$ be a Mukai flop between smooth projective varieties, as in Definition 3.3, then the cycle given by the fibre product $[X \times_{\bar{X}} X'] = [\widetilde{X}] + [Z \times_S Z'] \in$ $\mathrm{CH}^{\dim X}(X \times X')$ induces an isomorphism of integral Chow motives:*

$$\mathfrak{h}(X) \xrightarrow{\simeq} \mathfrak{h}(X'),$$

whose inverse is given by (the transpose of) the same cycle.

Note that the isomorphism in the previous theorem is nothing but

$$\tau'_* \tau^* + i'_* \operatorname{pr}_{2,*} \operatorname{pr}_1^* i^* : \mathfrak{h}(X) \to \mathfrak{h}(X'),$$

where the notation for functoriality of Chow motives is explained in §2.1.

Remark 3.5. In fact, the argument in [55] proves that Theorem 3.4 holds in a slightly different situation: if the smooth variety S is not necessarily proper, while X and X' are both smooth varieties which are defined over S and proper and smooth over S (in particular requiring that in Diagram (3) there is an extra proper morphism $\bar{X} \to S$ making the inclusion $S \hookrightarrow \bar{X}$ a section), then the proof in [55] still shows that the cycle $[X \times_{\bar{X}} X'] = [\widetilde{X}] + [Z \times_S Z'] \in$

$\mathrm{CH}_{\dim X}(X \times_S X')$ induces an isomorphism of Voevodsky motives $M(X) \xrightarrow{\simeq} M(X')$, and, in particular, an isomorphism of Chow groups $\mathrm{CH}^*(X) \xrightarrow{\simeq} \mathrm{CH}^*(X')$.

As such a claim is not explicitly stated in [55], let us give a simple argument in a special case that is sufficient for our application in the proof of Theorem 3.6: let all varieties be defined and projective over S, and assume that Z, Z', X, X' and the Mukai flop $X \dashrightarrow X'$ are all *Zariski locally trivial* over S. Then, because X and X' are smooth projective over S, the cycle $[X \times_{\bar{X}} X']$ induces a morphism of *relative* Voevodsky motives $M_S(X) \to M_S(X')$ over S (in the sense of [24, Definition 11.1.1], and using [24, Example 11.2.3, §11.3.8] to relate correspondances to morphisms of relative motives). Relative Voevodsky motives satisfy Zariski descent (as a special case of Nisnevich descent, see [24, §11.1.2]), so it is enough to show this morphism is an isomorphism after passing to a Zariski open cover. By assumption and smooth base change of relative motives (applied here to open immersions) [24, §11.1.2], this reduces us to the case where the whole flopping situation is obtained from Diagram (3) with $S = \mathrm{Spec}(k)$ by a base change $- \times_k S$. The latter case follows directly from Theorem 3.4 for $S = \mathrm{Spec}(k)$ and base change. We conclude that $[X \times_{\bar{X}} X']$ induces an isomorphism $M_S(X) \xrightarrow{\simeq} M_S(X')$, and we obtain the isomorphism $M(X) \xrightarrow{\simeq} M(X')$ by applying the functor $p_\sharp : \mathrm{DM}(S, R) \to \mathrm{DM}(k, R)$ where $p : S \to \mathrm{Spec}(k)$ is the structure morphism [24, §11.1.2].

Later when dealing with parabolic Higgs bundles in §6.2, we will need the following slight generalisation of Theorem 3.4 to treat certain smooth quasi-projective varieties.

Theorem 3.6. *Let $\phi \colon X \dashrightarrow X'$ be a Mukai flop between smooth varieties. Using the notation in Diagram (3), there is an isomorphism of Chow groups*

$$\tau'_* \tau^* + i'_* \mathrm{pr}_{2,*} \mathrm{pr}_1^* i^* \colon \mathrm{CH}^*(X) \xrightarrow{\simeq} \mathrm{CH}^*(X'),$$

whose inverse is given by $\tau_* \tau'^* + i_* \mathrm{pr}_{1,*} \mathrm{pr}_2^* i'^*$. Here all the pull-backs are Gysin homomorphisms as in [31]. Assuming that the Voevodsky motives of X and X' are Chow motives, i.e. are in the essential image of the embedding from Theorem 2.4 (in particular k is perfect or we work with coefficients in a ring R where the characteristic is invertible), then these isomorphisms give isomorphisms between $M(X)$ and $M(X')$.

Proof. Let $X_{\mathrm{loc}} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}_Z) \xrightarrow{\pi} Z$ and $X'_{\mathrm{loc}} = \mathbb{P}_{Z'}(N_{Z'/X'} \oplus \mathcal{O}_{Z'}) \xrightarrow{\pi'} Z'$ which contain Z and Z' respectively via the zero sections. Then there is a Mukai flop $\phi_{\mathrm{loc}} \colon X_{\mathrm{loc}} \dashrightarrow X'_{\mathrm{loc}}$ as in Definition 3.3, which should be considered as the local model of ϕ. Let ι and ι' be the inclusions of E into X_{loc} and X'_{loc} as the infinite parts respectively (see Appendix A). Consider the following diagram

(4)
$$
\begin{array}{ccccc}
\mathrm{CH}^{l-1}(E) & \xrightarrow{(j_*, -\iota_*)} & \mathrm{CH}^l(\widetilde{X}) \oplus \mathrm{CH}^l(X_{\mathrm{loc}}) & \xrightarrow{(j^*, -\iota^*)} & \mathrm{CH}^k(E) \\
\Big\| {\scriptstyle\mathrm{id}} & & \Big\downarrow {\scriptstyle(\mathrm{id}, \mathcal{F}_{\mathrm{loc}})} & & \Big\| {\scriptstyle\mathrm{id}} \\
\mathrm{CH}^{l-1}(E) & \xrightarrow{(j_*, -\iota'_*)} & \mathrm{CH}^l(\widetilde{X}) \oplus \mathrm{CH}^l(X'_{\mathrm{loc}}) & \xrightarrow{(j^*, -\iota'^*)} & \mathrm{CH}^l(E),
\end{array}
$$

where $\mathcal{F}_{\mathrm{loc}}$ is the isomorphism obtained by applying Theorem 3.4 to the Mukai flop ϕ_{loc}, or rather, by applying Remark 3.5, as S is not necessarily proper, but X_{loc} and X'_{loc} are both defined and proper smooth over S. (Note that Z, Z', X_{loc}, X'_{loc} and the Mukai flop $X_{\mathrm{loc}} \dashrightarrow X'_{\mathrm{loc}}$ are all Zariski locally trivial over S.)

Diagram (4) is indeed commutative, as $\mathcal{F}_{\mathrm{loc}}$ restricts to identity outside of Z and Z'. By the local-to-global trick recalled in Proposition A.1, $\mathrm{CH}^l(X)$ (resp. $\mathrm{CH}^l(X')$) is isomorphic, via an explicit correspondence, to the middle cohomology of the top (resp. bottom) row of Diagram (4). Hence, we conclude that the composition

$$\mathcal{F} := (\tau'_*, i'_* \pi'_*) \circ (\mathrm{id}, \mathcal{F}_{\mathrm{loc}}) \circ (\tau^*, \pi^* i^*) = \tau'_* \tau^* + i'_* \mathrm{pr}_{2,*} \mathrm{pr}_1^* i^*$$

induces an isomorphism between $\mathrm{CH}^*(X)$ and $\mathrm{CH}^*(X')$, with inverse given by the composition

$$\mathcal{F}^{-1} = (\tau_*, i_* \pi_*) \circ (\mathrm{id}, \mathcal{F}_{\mathrm{loc}}^{-1}) \circ (\tau'^*, \pi'^* i'^*) = \tau_* \tau'^* + i_* \mathrm{pr}_{1,*} \mathrm{pr}_2^* i'^*.$$

Let R be any coefficient ring such that the characteristic of k is invertible if k is not perfect. Assume that the Voevodsky motives of X and X' in $\mathrm{DM}(k, R)$ are Chow motives, i.e., that there exist Chow motives $\mathfrak{h}_X$ and $\mathfrak{h}_{X'}$ in $\mathrm{CHM}(k, R)$ which are sent to $M(X)$ and $M(X')$ via the full embedding $\mathrm{CHM}(k, R) \hookrightarrow \mathrm{DM}(k, R)$ of Theorem 2.4. By the Yoneda lemma, to show that $M(X) \simeq M(X')$, or equivalently that $\mathfrak{h}_X \simeq \mathfrak{h}_{X'}$, it suffices to show that for any Chow motive $M \in \mathrm{CHM}(k, R)$, there is a bijection

$$\mathrm{Hom}_{\mathrm{CHM}(k,R)}(\mathfrak{h}_X, M) \simeq \mathrm{Hom}_{\mathrm{CHM}(k,R)}(\mathfrak{h}_{X'}, M)$$

which is natural in M. It is enough to do this for $M = \mathfrak{h}(T)(n)$ where T is a smooth projective k-variety and $n \in \mathbb{Z}$, as any Chow motive is a direct factor of such a motive. It is also easy to see that we can assume T is connected, and thus equidimensional of some dimension d (as the general case involves distinguishing connected components and just introduces more notation, but no additional difficulty). By definition of $\mathfrak{h}_X$ and the isomorphism (1) in the proof of Theorem 2.4 (which applies because X is smooth and T is smooth and projective), we have

$$\mathrm{Hom}_{\mathrm{CHM}(k,R)}(\mathfrak{h}_X, \mathfrak{h}(T)(n)) \simeq \mathrm{Hom}_{\mathrm{DM}(k,R)}(M(X), M(T)(n)) \simeq \mathrm{CH}^{d+n}(X \times_k T, R).$$

The product $\phi \times \mathrm{id} \colon X \times_k T \dashrightarrow X' \times_k T$ is again a Mukai flop, so the first part of the proof (which works just as well with coefficients in the ring R) produces a bijection

$$\mathrm{CH}^{d+n}(X \times_k T, R) \simeq \mathrm{CH}^{d+n}(X' \times_k T, R).$$

As Diagram (3) for $\phi \times \mathrm{id}$ is simply the product of the diagram for ϕ with T, this isomorphism is induced by a correspondence of the form $\alpha \times \Delta_T \in \mathrm{CH}^*(X \times_k X' \times_k T \times_k T, R)$ for some $\alpha \in \mathrm{CH}^*(X \times_k X', R)$.

It remains to show that this isomorphism is natural in $\mathfrak{h}(T)(n)$. Let T' be another smooth projective variety of equal dimension d' and $n' \in \mathbb{Z}$. A morphism $M(T)(n) \to M(T')(n')$ is given by a correspondence $\beta \in \mathrm{CH}^*(T \times_k T', R)$, and the corresponding morphism $\mathrm{CH}^{d+n}(X \times_k T, R) \to \mathrm{CH}^{d'+n'}(X \times_k T', R)$ is induced by the correspondence $\Delta_X \times \beta$. There is an equality of composition of correspondences

$$(\alpha \times \Delta_{T'}) \circ (\Delta_X \times \beta) = \alpha \times \beta = (\Delta_{X'} \times \beta) \circ (\alpha \times \Delta_T),$$

which thus implies the claimed naturality. $\square$

4. Motives of moduli spaces of stable vector bundles

Let $\mathcal{N} = \mathcal{N}_C(n, d)$ denote the moduli space of semistable vector bundles of rank n and degree d on a smooth projective geometrically connected genus g curve C over k. We recall that a vector bundle E is *semistable* if for all proper subbundles $F \subset E$, we have $\mu(F) \leq \mu(E)$, where $\mu(E) := \deg(E)/\mathrm{rk}(E)$. It is stable if this equality is strict, and geometrically stable[4] if it is stable after all field extensions. The moduli space $\mathcal{N}$ is a projective k-variety constructed by GIT [63] containing the moduli space $\mathcal{N}^s$ of geometrically stable bundles as an open subset.

In this section, we assume that n and d are coprime, so that semistability, stability and geometric stability for rank n degree d vector bundles coincide and so $\mathcal{N}$ is a smooth projective k-variety of dimension $n^2(g - 1) + 1$. For $g = 0$, this moduli space is empty unless $n = 1$ (in which case it is a point) and for $g = 1$, we have $\mathcal{N} \simeq \mathrm{Jac}(C)$ when $k = \bar{k}$ [3, Theorem 7]. The moduli space $\mathcal{N}$ admits a universal family $\mathcal{E}$, as n and d are coprime [61]; that is, $\mathcal{E}$ is a vector bundle on $C \times \mathcal{N}$ such that for every geometric point p of $\mathcal{N}$, the bundle $\mathcal{E}_{|C \times \{p\}}$ is a stable vector bundle in the isomorphism class specified by p.

Let $\mathcal{L} \in \mathrm{Pic}^d(C)(k)$ and consider the moduli space $\mathcal{N}_{\mathcal{L}} = \mathcal{N}_{\mathcal{L}}(n)$ of semistable vector bundles with determinant isomorphic to $\mathcal{L}$. As n and d are coprime, $\mathcal{N}_{\mathcal{L}}$ is a smooth projective variety admitting a universal family. In this section, we study the Chow motives $\mathfrak{h}(\mathcal{N})$ and $\mathfrak{h}(\mathcal{N}_{\mathcal{L}})$.

---

[4]Over an algebraically closed field k, the notions of stability and geometric stability coincide.

4.1. **Beauville's diagonal trick.** Given a moduli space $\mathcal{M}$ of sheaves of some kind on a smooth projective variety X, there are several cases in which the motive of $\mathcal{M}$ lies in the tensor subcategory generated by the motive of X. In the case of the Chow motive of $\mathcal{N}$, this was shown by del Baño in [25]. Other cases where this general principle is known to hold include the moduli space of stable Higgs bundles on C [40] and certain moduli spaces of (semistable) sheaves on K3 and abelian surfaces (and closely related spaces: crepant resolutions, twisted and non-commutative analogues) [19, 28]. However, in §6.1.2, we will see that this is not the case for moduli spaces of stable Higgs bundles with fixed determinant on a general curve C, and that motives of certain finite covers of the curve are necessary to generate the motive of the moduli space in that case (Proposition 6.3).

Based on an idea of Beauville [10] using Chern classes of the universal family to describe the diagonal of $\mathcal{N}$, Bülles [19] observed that one can give a short proof of del Baño's theorem. We review the argument and check that it gives a similar result for $\mathcal{N}_{\mathcal{L}}$.

Proposition 4.1. *Assume that n and d are coprime and $\mathcal{L}$ is a line bundle of degree d on C. Then the rational Chow motives $\mathfrak{h}(\mathcal{N})$ and $\mathfrak{h}(\mathcal{N}_{\mathcal{L}})$ both lie in the tensor subcategory of $\mathrm{CHM}(k, \mathbb{Q})$ generated by $\mathfrak{h}(C)$ (i.e. appear as direct summands of the Chow motive of a large enough power of C).*

Proof. Since the argument is the same in both cases, we give the proof for $\mathcal{N}_{\mathcal{L}}$ as this case has not explicitly appeared in the literature. We follow closely the exposition of [19, Theorem 0.1]. First, consider the natural projections

$$
\begin{array}{ccc}
 & \mathcal{N}_{\mathcal{L}} \times C \times \mathcal{N}_{\mathcal{L}} & \\
{\scriptstyle \pi_{12}} \swarrow & {\scriptstyle \pi} \downarrow & \searrow {\scriptstyle \pi_{23}} \\
\mathcal{N}_{\mathcal{L}} \times C & \mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}} & C \times \mathcal{N}_{\mathcal{L}}.
\end{array}
$$

Since n and d are coprime, there is a universal family on $\mathcal{N}_{\mathcal{L}}$ given by a vector bundle $\mathcal{E}$ on $C \times \mathcal{N}_{\mathcal{L}}$. Following Beauville [10], let us define a K-theory class

$$
[\mathcal{E}\mathrm{xt}^!_\pi] := \sum_i (-1)^i [R^i \pi_* \mathcal{H}om(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})] \in K_0(\mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}}).
$$

As C is smooth and 1-dimensional, this sum has only two potentially non-zero terms, for $i = 0$ or 1, and it is possible to represent this class by a two-term complex $u : K^0 \to K^1$ of locally free sheaves on $\mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}}$. For stable vector bundles E and F on C, the homomorphism group $\mathrm{Hom}(E, F)$ vanishes unless $E \simeq F$, and is one-dimensional if $E \simeq F$. From this, Beauville deduces that the diagonal $\Delta_{\mathcal{N}_{\mathcal{L}}}$ is equal to the degeneracy locus of u (at least set-theoretically, which is enough to carry out the rest of the proof) [10, p.28]. This degeneracy locus is a determinantal subvariety of the expected codimension and by applying Porteous's formula [31, Theorem 14.4] one obtains

$$
(5) \qquad c_N(-[\mathcal{E}\mathrm{xt}^!_\pi]) = [\Delta_{\mathcal{N}_{\mathcal{L}}}] \in \mathrm{CH}^N(\mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}})_{\mathbb{Q}},
$$

where $N = \dim \mathcal{N}_{\mathcal{L}} = (n^2 - 1)(g - 1)$.

The Chow group $\mathrm{CH}^*(\mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}})_{\mathbb{Q}}$ has a ring structure given by convolution of cycles, and the class $[\Delta_{\mathcal{N}_{\mathcal{L}}}]$ of the diagonal is a two-sided unit. Following Bülles, we introduce

$$
I := \langle \beta \circ \alpha \, | \, \alpha \in \mathrm{CH}^*(\mathcal{N}_{\mathcal{L}} \times C^k)_{\mathbb{Q}}, \beta \in \mathrm{CH}^*(C^k \times \mathcal{N}_{\mathcal{L}})_{\mathbb{Q}}, k \geq 1 \rangle
$$

which is clearly a two-sided ideal of $\mathrm{CH}^*(\mathcal{N}_{\mathcal{L}} \times \mathcal{N}_{\mathcal{L}})_{\mathbb{Q}}$. The same computation as in [19, Proof of Theorem 0.1] shows that I is also closed under intersection products.

Let us show that (5) implies that $[\Delta_{\mathcal{N}_{\mathcal{L}}}] \in I$. As in [19, Proof of Theorem 0.1], the Grothendieck–Riemann–Roch Theorem implies that

$$
\mathrm{ch}(-[\mathcal{E}\mathrm{xt}^!_\pi]) = -\pi_* (\pi_{12}^* \, \mathrm{ch}(\mathcal{E}^\vee) \cdot \pi_{23}^*(\mathcal{E}) \cdot \pi_2^* \, \mathrm{td}(C))
$$

with $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O})$. Let $\alpha := \mathrm{ch}(\mathcal{E}^\vee) \cdot \pi_2^* \sqrt{\mathrm{td}\, C}$ and $\beta := \mathrm{ch}(\mathcal{E}) \cdot \pi_2^* \sqrt{\mathrm{td}\, C}$. Then, for $a \in \mathbb{N}$, by considering the graded part of the previous equation in codimension a, we see that

$$\mathrm{ch}_a(-[\mathcal{E}\mathrm{xt}^!_\pi]) = -\sum_{i+j=a+2} \alpha^i \circ \beta^j \in I.$$

An induction on a then implies that $c_a(-[\mathcal{E}\mathrm{xt}^!_\pi])$ also lies in I. By (5), we see that $[\Delta_{\mathcal{N}_{\mathcal{L}}}] \in I$. By the argument at the end of [19, Proof of Theorem 0.1], we then see that $\mathfrak{h}(\mathcal{N}_{\mathcal{L}})$ can be realised as a direct summand of a motive of the form $\bigoplus_i \mathfrak{h}(C^{k_i})(n_i)$. This last motive is in the tensor subcategory of $\mathrm{CHM}(k, \mathbb{Q})$ generated by $\mathfrak{h}(C)$, and can also be embedded as a direct summand of the motive of a large enough power of C. This completes the proof. $\square$

4.2. **Motives of moduli spaces of vector bundles with and without fixed determinant.**

Theorem 4.2. *Assume that n and d are coprime and $\mathcal{L} \in \mathrm{Pic}^d(C)(k)$. In $\mathrm{CHM}(k, \mathbb{Q})$, we have*

$$\mathfrak{h}(\mathcal{N}) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \otimes \mathfrak{h}(\mathrm{Jac}(C)).$$

Proof. Since n and d are coprime, the moduli spaces $\mathcal{N}$ and $\mathcal{N}_{\mathcal{L}}$ are fine moduli spaces and we fix universal families $\mathcal{E}$ (resp. $\mathcal{E}'$) on $\mathcal{N} \times C$ (resp. $\mathcal{N}_{\mathcal{L}} \times C$). One can choose $\mathcal{E}'$ such that the induced morphism $\iota : \mathcal{N}_{\mathcal{L}} \to \mathcal{N}$ satisfies $(\iota \times \mathrm{id})^*(\mathcal{E}) \simeq \mathcal{E}'$. The morphism ι is easily seen to be a proper monomorphism (for instance by checking the valuative criterion of properness), hence a closed immersion. Combining this with the action of $\mathrm{Jac}(C)$ on $\mathcal{N}$ by tensor product, we get a morphism

$$\phi : \mathcal{N}_{\mathcal{L}} \times \mathrm{Jac}(C) \to \mathcal{N}, \quad (E, L) \mapsto \iota(E) \otimes L.$$

We will show that the map of Chow motives induced by this last morphism is an isomorphism.

First, let us show that we can reduce to the case of a field k of characteristic zero. Let $B = \mathrm{Spec}(R)$ be a complete local trait with R a discrete valuation ring with fraction field K of characteristic 0 and residue field k. Since C is a smooth projective curve, there exists a smooth projective curve $\mathcal{C}/B$ which lifts C [1, Exposé III Théorème 7.3]. Moreover, since C has dimension 1, we have $H^2(C, \mathcal{O}_C) = 0$; hence there exists a line bundle Λ on $\mathcal{C}$ which lifts $\mathcal{L}$ [44, Corollary 5.6.(a)]. There is a relative moduli space $\mathcal{N}_{\mathcal{C}/B}$ (resp. $\mathcal{N}_{\mathcal{C}/B, \Lambda}$) of vector bundles of rank n and degree d (resp. determinant Λ) over $\mathcal{C}/B$, which are smooth projective schemes over B whose generic and special fibres are the corresponding moduli spaces over K and k (see [43, Theorem 4.3.7] for a very general construction of moduli spaces of sheaves in a relative setting). In particular, we have a relative Jacobian $\mathrm{Jac}(\mathcal{C}/B) = \mathcal{N}_{\mathcal{C}/B}(1, 0)$. The construction of the previous paragraph works in this generality and produces a B-morphism

$$\phi_{\mathcal{C}} : \mathcal{N}_{\mathcal{C}/B, \Lambda} \times_B \mathrm{Jac}(\mathcal{C}/B) \to \mathcal{N}_{\mathcal{C}/B}.$$

Assume now that we know that, over the generic fibre, this induces a isomorphism of Chow motives $\mathfrak{h}(\phi_{\mathcal{C}_K}) : \mathfrak{h}(\mathcal{N}_{\mathcal{C}_K, \Lambda_K} \times \mathrm{Jac}(\mathcal{C}_K)) \to \mathfrak{h}(\mathcal{N}_{\mathcal{C}_K})$. By applying the specialisation morphisms for Chow groups in smooth projective families over B [31, §10.1], which are compatible with composition of correspondences, we see that this implies the same claim over k.

We can thus assume that k is of characteristic 0. Let L/k be a finite field extension. The base change functor $\mathrm{CHM}(k, \mathbb{Q}) \to \mathrm{CHM}(L, \mathbb{Q})$ is conservative by Lemma 2.5. Since the characteristic of k is 0, by replacing k by a finite extension, we can assume that the group scheme $\mathrm{Jac}(C)[n]$ of n-torsion points in the Jacobian is the constant finite étale group scheme associated to $\Gamma_n := \mathrm{Jac}(C)[n](k)$, and we identify the two.

The group Γ_n acts on $\mathcal{N}_{\mathcal{L}}$ via $M \cdot E := E \otimes M^{-1}$ and on $\mathrm{Jac}(C)$ by translation; we let $\mathcal{N}_{\mathcal{L}} \times^{\Gamma_n} \mathrm{Jac}(C)$ denote the quotient by the diagonal Γ_n-action. The morphism ϕ factors through this quotient and induces a morphism

$$\overline{\phi} : \mathcal{N}_{\mathcal{L}} \times^{\Gamma_n} \mathrm{Jac}(C) \to \mathcal{N}.$$

It is well known that this morphism is an isomorphism, but we could not find a reference in this generality; let us sketch the argument. It is enough to prove that this is an isomorphism after extending the field k, hence we can assume that k is algebraically closed. We construct an inverse in the other direction as follows. Let S be a k-scheme and $[\mathcal{F} \to C \times S] \in \mathcal{N}(S)$ be

the class of a family of rank n degree d stable vector bundles parametrised by S (recall that $\mathcal{F}$ and $\mathcal{F} \otimes \pi_S^* \Lambda$, where Λ is a line bundle on S, determine the same S-point of $\mathcal{N}$). Then $\det(\mathcal{F}) \otimes \pi_C^* \mathcal{L}^{-1} \to C \times S$ is a family of line bundles of degree 0 and in particular defines an S-point of $\mathrm{Jac}(C)$. This family of line bundles does not necessarily admit an n-th root, but it does after passing to the finite étale cover $S' \to S$ defined by the pullback square

$$\begin{array}{ccc} S' & \xrightarrow{\quad M \quad} & \mathrm{Jac}(C) \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle [n]} \\ S & \xrightarrow{\det(\mathcal{F}) \otimes \mathcal{L}^{-1}} & \mathrm{Jac}(C). \end{array}$$

By construction, S' is a Γ_n-torsor over S and the induced map $S' \to \mathrm{Jac}(C)$ gives us a family M of degree 0 line bundles on C parametrised by S' (this is the point where it is useful to assume k is algebraically closed), uniquely determined up to the pullback of a line bundle on S'. We have $\det(p_C^* \mathcal{F} \otimes M^{-1}) \simeq p_C^* \det(\mathcal{F}) \otimes M^{\otimes -n} \simeq \pi_C^* \mathcal{L} \otimes \pi_{S'}^* \Lambda$ for $p_C := \mathrm{Id}_C \times p$ and $\pi_C : C \times S' \to C$, and Λ a line bundle on S'. Hence, the pair $(p_C^* \mathcal{F} \otimes M^{-1}, M)$ defines a morphism $S' \to \mathcal{N}_L \times \mathrm{Jac}(C)$, which is Γ_n-equivariant and descends to a morphism $S \to \mathcal{N}_L \times^{\Gamma_n} \mathrm{Jac}(C)$ between the Γ_n-quotients. One checks that the resulting morphism does not depend on the choice of $\mathcal{F}$ and M in their equivalence class. The whole construction is functorial in S, and defines a morphism $\mathcal{N} \to \mathcal{N}_L \times^{\Gamma_n} \mathrm{Jac}(C)$, which one can show is an inverse to $\overline{\phi}$.

Since we are working with Chow motives with rational coefficients, we deduce that

$$\mathfrak{h}(\mathcal{N}) \simeq \mathfrak{h}(\mathcal{N}_\mathcal{L} \times^{\Gamma_n} \mathrm{Jac}(C)) \simeq \mathfrak{h}(\mathcal{N}_\mathcal{L} \times \mathrm{Jac}(C))^{\Gamma_n} \simeq (\mathfrak{h}(\mathcal{N}_\mathcal{L}) \otimes \mathfrak{h}(\mathrm{Jac}(C)))^{\Gamma_n}.$$

Let us write $\mathfrak{h}(\mathrm{Jac}(C)) = \mathfrak{h}(\mathrm{Jac}(C))^{\Gamma_n} \oplus R$. Since the morphism $[n] : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ is a finite étale Γ_n-torsor, it induces an isomorphism of motives $[n]^* : \mathfrak{h}(\mathrm{Jac}(C)) \simeq \mathfrak{h}(\mathrm{Jac}(C))^{\Gamma_n}$. Moreover, $\mathfrak{h}(\mathrm{Jac}(C))^{\Gamma_n}$ is Kimura finite-dimensional, as it is a direct factor of $\mathfrak{h}(\mathrm{Jac}(C))$; thus, by Proposition 2.2, we can cancel on both sides and deduce that $R \simeq 0$. We deduce that the Γ_n-action on $\mathfrak{h}(\mathrm{Jac}(C))$ is trivial (see [48, Lemma 2.1] for a different argument using results of Beauville on Chow groups of abelian varieties [9]). We deduce that

$$\mathfrak{h}(\mathcal{N}) \simeq (\mathfrak{h}(\mathcal{N}_\mathcal{L}))^{\Gamma_n} \otimes \mathfrak{h}(\mathrm{Jac}(C)).$$

Finally, it remains to show that the action of Γ_n on $\mathfrak{h}(\mathcal{N}_\mathcal{L})$ is trivial. A classical theorem of Harder–Narasimhan [36] shows that this is the case for the Γ_n-action on the ℓ-adic cohomology $H^*(\mathcal{N}_\mathcal{L}, \mathbb{Q}_\ell)$ for any prime ℓ different from the characteristic of k. In other words, the ℓ-adic realisation of the morphism $\mathfrak{h}(\mathcal{N}_\mathcal{L})^{\Gamma_n} \to \mathfrak{h}(\mathcal{N}_\mathcal{L})$ is an isomorphism. By Proposition 4.1, this is a morphism between abelian Chow motives. Since k is now a field of characteristic 0, the ℓ-adic realisation functor $R_\ell : \mathrm{DM}(k, \mathbb{Q}) \to D(\mathbb{Q}_\ell)$ is conservative when restricted to abelian geometric motives by [71, Theorem 1.12]. We deduce that $\mathfrak{h}(\mathcal{N}_\mathcal{L})^{\Gamma_n} \to \mathfrak{h}(\mathcal{N}_\mathcal{L})$ is an isomorphism, which concludes the proof. $\qquad \square$

4.3. A closed formula in rank 2 using wall-crossing for stable pairs.
In this section, we obtain a formula for the rational Chow motive of the moduli space of stable vector bundles of rank two and odd degree d from a result of del Bãno [26]. Del Bãno's proof relies in turn on work of Thaddeus [66] involving variation of stability for so-called (Bradlow) stable pairs, consisting of a vector bundle with a non-zero section. The notion of (semi)stability for such pairs depends on a stability parameter $\sigma \in \mathbb{Q}_{>0}$ and involves checking an inequality for all subbundles. For each σ, there is an associated moduli space of σ-semistable pairs $\mathcal{P}^\sigma = \mathcal{P}_C^\sigma(2, d)$, which is a projective variety constructed via GIT [66]. The space of stability parameters $\mathbb{Q}_{>0}$ admits a wall and chamber decomposition by considering how the notion of (semi)stability changes as σ varies. The walls correspond to critical values of σ for which semistability and stability do not coincide (i.e., there is a subbundle violating stability) and in the chambers, semistability and stability coincide and the corresponding moduli space is smooth. For moduli spaces $\mathcal{P}_\mathcal{L}^\sigma = \mathcal{P}_{C,\mathcal{L}}^\sigma(2, d)$ of pairs with fixed determinant, Thaddeus showed (i) for σ in the minimal chamber, there is a forgetful map $\mathcal{P}_\mathcal{L}^\sigma \to \mathcal{N}_\mathcal{L}$ (which is a projective bundle if $d \geq 4g - 3$); (ii) each wall-crossing corresponds to standard flip (or flop) between the $\mathcal{P}_\mathcal{L}^\sigma$'s on both sides with centre a symmetric

power of C and iii) for σ in the maximal non-empty chamber, the moduli space of stable pairs is a projective space.

In [66], the field k is assumed to be the complex numbers. We claim that this assumption is not necessary. First of all, the constructions in [66] are purely algebro-geometric and work at the very least over any algebraically closed field. The key observation to pass to a general base field is then the following: when $k = \bar{k}$, for a given stability parameter σ and a pair (E, s), Thaddeus shows that there is at most one destabilising line subbundle $L \subset E$ with $s \in H^0(L)$ and one subbundle $M \subset E$ with $s \notin H^0(M)$ [66, (1.4)]. The same uniqueness statement for those two types of line subbundles then holds over any field k, and in fact for families of pairs over any base scheme over k (defined in the obvious way), by passing to geometric points and using the statement over an algebraically closed field. Moreover, when σ is not on a wall, the line subbundle L (resp. M) exists precisely for the unstable pairs which become stable after passing to the right (resp. to the left) chamber [66, Proofs of (3.2), (3.3)].

When $k = \bar{k}$, this is how Thaddeus defines the flipping loci that relate the $\mathcal{P}_{\mathcal{L}}^{\sigma}$'s. For a general base field k, the uniqueness statement above applied to a general base k-scheme implies that the same flipping loci are defined over k and implies a modular description of those loci (as the image of the moduli spaces of triples (E, s, L) (resp. (E, s, M)) with L (resp. M) line subbundle with $s \in H^0(L)$ (resp. $s \notin H^0(M)$) via the map forgetting L (resp. M), which is a closed immersion because of the uniqueness statement). Using that modular description, all the arguments of [66, §3] identifying these flipping loci and proving that the resulting birational maps are standard flips go through over an arbitrary field k. Alternatively, one can adapt the proof in [43, Theorem 1.3.7] that the uniqueness of the Harder–Narasimhan filtration for sheaves implies this filtration is stable under base field extension to the setting of pairs. Indeed, if k is perfect, one can also argue by Galois descent and check that the whole diagram of flips over a fixed algebraic closure $\bar{k}$ and the isomorphisms described by [66, §3] are stable under the action of $\mathrm{Gal}(\bar{k}/k)$.

Theorem 4.3. *Let $\mathcal{L}$ be a line bundle on C of odd degree d. Then the rational Chow motive of the moduli space $\mathcal{N}_{\mathcal{L}} = \mathcal{N}_{C,\mathcal{L}}(2, d)$ of stable rank 2 bundles with fixed determinant $\mathcal{L}$ is given by*[5]

$$\mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \simeq \mathfrak{h}(\mathrm{Sym}^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} \mathfrak{h}(\mathrm{Sym}^i(C)) \otimes (\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i))$$

and the rational Chow motive of the moduli space $\mathcal{N} = \mathcal{N}_C(2, d)$ of stable rank 2 degree d vector bundles is given by

$$\mathfrak{h}(\mathcal{N}) \simeq \mathfrak{h}(\mathrm{Jac}(C)) \otimes \left(\mathfrak{h}(\mathrm{Sym}^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} \mathfrak{h}(\mathrm{Sym}^i(C)) \otimes (\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i)) \right).$$

Proof. For $g = 0$, both moduli spaces are empty and both formulas hold. For $g = 1$, the determinant induces a morphism $\mathcal{N} \to \mathrm{Pic}^d(C) \simeq \mathrm{Jac}(C)$ (where we use $\mathrm{Pic}^d(C)(k) \neq \emptyset$). We claim that the induced morphism of Chow motives is an isomorphism. By Lemma 2.5, it suffices to check this after passing to the algebraic closure $k = \bar{k}$. In that case, the determinant morphism is actually an isomorphism and $\mathcal{N}_{\mathcal{L}}$ is a point by [3, Theorem 7]. Since we are working with rational coefficients, we can use a $\mathbb{Q}$-divisor of degree 1 to get an isomorphism $\mathfrak{h}(C) \simeq \mathfrak{h}(\mathrm{Jac}(C))$ in $\mathrm{CHM}(k, \mathbb{Q})$ (even when $C(k) = \emptyset$).

By Theorem 4.2, it suffices to prove the formula for $\mathfrak{h}(\mathcal{N}_{\mathcal{L}})$. By assumption, C has a degree d line bundle $\mathcal{L}$ and tensoring with $\mathcal{L}^{\otimes e}$ induces an isomorphism $\mathcal{N}_C(2, d) \cong \mathcal{N}_C(2, (2e+1)d)$; thus we can assume without loss of generality that $d \geq 4g - 3$ and $g \geq 2$. Let us write $d = 4g - 3 + 2\delta$ with $\delta \geq 0$.

Since $d \geq 4g - 3$, Thaddeus' work [66] shows there are $\frac{d-1}{2} = 2g - 2 + \delta$ walls and $2g - 1 + \delta$ chambers for pairs with corresponding smooth pairs moduli spaces $\mathcal{P}_{\mathcal{L}}^{\sigma_i}$ for $0 \leq i \leq 2g - 2 + \delta$

---

[5]The formula holds for all $g \geq 0$, with the convention that $\mathrm{Sym}^{-1}(C) = \emptyset$.

with $\sigma_0 > \cdots > \sigma_{2g-2+\delta} > 0$ which fit into a diagram

$$
\begin{array}{ccccccccc}
\mathcal{P}_{\mathcal{L}}^{\sigma_0} & \dashrightarrow & \mathcal{P}_{\mathcal{L}}^{\sigma_1} & \dashrightarrow & \mathcal{P}_{\mathcal{L}}^{\sigma_2} & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{P}_{\mathcal{L}}^{\sigma_{2g-3+\delta}} & \xrightarrow{\;(2g-3+\delta,g-1+2\delta)\;} & \mathcal{P}_{\mathcal{L}}^{\sigma_{2g-2+\delta}} \\
\end{array}
$$

with the horizontal maps labelled $\frac{(0,5g-7+2\delta)}{\text{centre } C}$, $\frac{(1,5g-9+2\delta)}{\substack{\text{centre}\\ \mathrm{Sym}^2(C)}}$, and $\frac{(2g-3+\delta,g-1+2\delta)}{\substack{\text{centre}\\ \mathrm{Sym}^{2g-2+\delta}(C)}}$, with vertical maps $\simeq$ down to $\mathbb{P}^{5g-5+2\delta}$ and π down to $\mathcal{N}_{\mathcal{L}}$,

where the horizontal maps are standard flips (or flops) with the given type and centre and π is the forgetful map, which is a $\mathbb{P}^{2g-2+2\delta}$-bundle.

Let us first assume that C admits a line bundle of degree 1 and that $d = 4g - 3$ (which in this case can be achieved by tensoring by that line bundle). Under this assumption, using the above sequence of flips with $\delta = 0$, del Baño computes the class of the motive of $\mathcal{N}_{\mathcal{L}}(2, d)$ in the completion $\mathcal{K}$ of the ring $K_0(\mathrm{CHM}_k^{\mathrm{eff}})$ along the ideal generated by the Lefschetz motive $\mathbb{L} := \mathbb{Q}(1)$ [26, Theorem 2.6] in terms of $\mathfrak{h}^1(C)$.

In general, suppose that C does not admit a line bundle of degree 1 and that δ is not necessarily 0. We claim that the proof of [26, Theorem 2.6] still applies in this case with some minor modifications. The assumption that C has a line bundle of degree 1 is not used in the proof, besides the reduction to degree $d = 4g - 3$ (in particular, as del Baño observes in [26, §1.2.4], the formula for motives with rational coefficients of high enough symmetric powers in terms of the motive of the Jacobian holds without any assumption on C, despite the fact that those symmetric powers are not projective bundles over the Jacobian). The computation for $\delta > 0$ is then completely parallel to the computation for $\delta = 0$ in the proof of [26, Theorem 2.6] and shows that the class of the motive of $\mathcal{N}_{\mathcal{L}}$ in the ring $\mathcal{K}$ is given by applying the homomorphism $\mathcal{K}[[T]] \to \mathcal{K}, T \mapsto \mathbb{L}$ to the expression (generalising the penultimate line on page 8 of [26], which only deals with the case $\delta = 0$)

$$
\frac{1}{1 - T^{2g-1+2\delta}}\left[\frac{(1 - T^{2g-1+2\delta})(1 + T)^{\mathfrak{h}^1(C)}}{(1 - T)(1 - T^2)} - \frac{(1 + 1)^{\mathfrak{h}^1(C)}}{(1 - T)}\left(\frac{T^{2g-1+\delta} - T^{2g-1+\delta}}{1 - T} - \frac{T^{3g-1+2\delta} - T^g}{1 - T^2}\right)\right]
$$

which simplifies to

$$
\frac{(1 + T)^{\mathfrak{h}^1(C)} - (1 + 1)^{\mathfrak{h}^1(C)}}{(1 - T)(1 - T^2)}
$$

and is thus independent of δ.

At this point, we have the formula above in the ring $\mathcal{K}$. Del Baño then claims in [26, Corollary 2.7] that this implies that we have

$$
(6) \qquad \mathfrak{h}_{\mathrm{num}}(\mathcal{N}_{\mathcal{L}}) \simeq \bigoplus_{i=0}^{g}\left(\Lambda^i \mathfrak{h}_{\mathrm{num}}^1(C) \otimes \left(\bigoplus_{j=0}^{g-i-1} \mathbb{Q}(j)\right) \otimes \left(\bigoplus_{l=0}^{g-i-1} \mathbb{Q}(2j)\right) \otimes \mathbb{Q}(i)\right)
$$

in the category $M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q})$ of effective numerical motives with rational coefficients, where we have used the decomposition $\mathfrak{h}(C) = \mathbb{Q}(0) \oplus \mathfrak{h}^1(C) \oplus \mathbb{Q}(1)$ of the motive of C. Since he does not give a complete proof and this is a crucial step in our argument, let us fill in the details.

The first part of the proof of [26, Corollary 2.7] is complete and shows that formula (6) holds in $\mathcal{K}$. Write $\mathcal{K}_{\mathrm{num}}$ for the completion of $K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q}))$ along the ideal generated by $\mathbb{L}$. Formula (6) in $\mathcal{K}$ implies the same formula in $\mathcal{K}_{\mathrm{num}}$.

We claim that the map $K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q})) \to \mathcal{K}_{\mathrm{num}}$ is injective. For this, it suffices to show that the filtration $(\mathbb{L}^n)_{n \geq 0}$ is separated, i.e. $\cap_{n \geq 0}(\mathbb{L}^n) = 0$. By Jannsen's theorem [46], $M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q})$ is a semisimple abelian category; thus $K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q}))$ is the free abelian group generated by isomorphism classes of simple objects of $M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q})$. Let us write I for this set of isomorphism classes of simple objects, and choose a simple representative $M_i \in M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q})$ for every $i \in I$. Let $\alpha \in (\mathbb{L}^n)$, i.e. $\alpha = \mathbb{L}^n \cdot \beta$ with $\beta \in K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k, \mathbb{Q}))$; we write these elements uniquely as

$$
\alpha = \sum_{i \in I} m_i[M_i] \quad \text{and} \quad \beta = \sum_{i \in I} n_i[M_i].
$$

Since the functor $M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q}) \to M_{\mathrm{num}}(k,\mathbb{Q})$ is fully faithful, there are no $\mathbb{L}$-torsion objects in $M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$. In particular, multiplication by $\mathbb{L}$ induces an injective map from I to I. By comparing the expansions of α and $\mathbb{L}^n \cdot \beta$, we see that whenever $m_i \neq 0$, we must have that M_i is divisible by $\mathbb{L}^n$, i.e. $M_i \simeq \mathbb{L}^n \otimes M_i'$ for some $M_i' \in M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$. Let now $\alpha \neq 0$ be in $\cap_{n \geq 0}(\mathbb{L}^n)$ and choose $i \in I$ such that $m_i \neq 0$; the previous argument shows that M_i is also divisible by $\mathbb{L}^n$ for all $n \geq 0$. It is thus enough to show that if $M \in M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$ is such that $[M] \in \cap_{n \geq 0}(\mathbb{L}^n)$, then $M = 0$. By Yoneda, it suffices to show that for any $N \in M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$, we have $\mathrm{Hom}(N,M) = 0$. Writing $M = M' \otimes \mathbb{L}^n$, we see that this group vanishes for n large enough for dimension reason. This finishes the proof of the separatedness of the filtration, and thus the injectivity of $K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})) \to \mathcal{K}_{\mathrm{num}}$.

Since both sides of (6) lie in $M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$, we now deduce from this injectivity statement that formula (6) holds in $K_0(M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q}))$. Finally, since $M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$ is semisimple, the formula of [26, Corollary 2.7] holds for isomorphism classes of objects in $M_{\mathrm{num}}^{\mathrm{eff}}(k,\mathbb{Q})$.

Recall that the motive $\mathfrak{h}(C)$ decomposes as $\mathfrak{h}(C) = \mathbb{Q}(0) \oplus \mathfrak{h}^1(C) \oplus \mathbb{Q}(1)$. By using this decomposition and expanding the formula in our Theorem, we see that it is equivalent to del Baño's formula. We have thus proven our theorem in $M_{\mathrm{num}}(k,\mathbb{Q})$.

By Theorem 4.1, both sides of the formula are abelian Chow motives. By Propositions 2.3 and 2.2, we deduce that the formula also holds in $\mathrm{CHM}(k,\mathbb{Q})$. This concludes the proof. $\square$

We note that as we used Kimura finite-dimensionality, the above isomorphism is not explicit.

4.3.1. *Corollaries and comparisons with previous results.* From Theorem 4.3, one obtains the following description of the Chow groups of $\mathcal{N}_{\mathcal{L}}$ for $n = 2$ and d odd, when $\mathrm{Pic}^1(C)(k) \neq \emptyset$. We assume that $g \geq 2$ for simplicity and to avoid some case distinctions.

Corollary 4.4. *Let $a \in \mathbb{N}$. There are isomorphisms*

$$\mathrm{CH}^a(\mathcal{N}_{\mathcal{L}})_{\mathbb{Q}} \simeq \mathrm{CH}^{a+1-g}(\mathrm{Sym}^{g-1}(C))_{\mathbb{Q}} \oplus \bigoplus_{i=0}^{g-2} \left(\mathrm{CH}^{a-i}(\mathrm{Sym}^i(C))_{\mathbb{Q}} \oplus \mathrm{CH}^{a-3g+3+2i}(\mathrm{Sym}^i(C))_{\mathbb{Q}} \right)$$

and

$$\mathrm{CH}_a(\mathcal{N}_{\mathcal{L}})_{\mathbb{Q}} \simeq \mathrm{CH}_{a+1-g}(\mathrm{Sym}^{g-1}(C))_{\mathbb{Q}} \oplus \bigoplus_{i=0}^{g-2} \left(\mathrm{CH}_{a-3g+3+2i}(\mathrm{Sym}^i(C))_{\mathbb{Q}} \oplus \mathrm{CH}_{a-i}(\mathrm{Sym}^i(C))_{\mathbb{Q}} \right).$$

For a small or close to $3g - 3$, many of these terms vanish for dimensional reasons (Lemma 2.1). Consequently, we can recover several descriptions of rational Chow groups in the literature (although our isomorphism is non-explicit).

Corollary 4.5.

(i) [61] $\mathrm{CH}^1(\mathcal{N}_{\mathcal{L}}) \simeq \mathbb{Z}$.

(ii) [7] $\mathrm{CH}^2(\mathcal{N}_{\mathcal{L}})_{\mathbb{Q}} \simeq \begin{cases} \mathrm{CH}_0(C)_{\mathbb{Q}} & \text{if } g = 2, \\ \mathrm{CH}_0(C)_{\mathbb{Q}} \oplus \mathbb{Q} & \text{if } g > 2. \end{cases}$

(iii) [22] $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathbb{Q}} \simeq \mathrm{CH}_0(C)_{\mathbb{Q}}$.

Proof. These are special cases of Corollary 4.4 and Lemma 2.1, with the additional remark that $\mathrm{CH}^1(\mathcal{N}_{\mathcal{L}})$ is torsion free because $\mathcal{N}_{\mathcal{L}}$ is a smooth projective Fano variety. $\square$

In fact, by examining the flip sequence in the proof of Theorem 4.3 and using Theorem 3.2, we can be more precise and compute some integral Chow groups of $\mathcal{N}_{\mathcal{L}}$; for example, one can deduce that $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}} \simeq \mathrm{Jac}(C) \simeq \mathrm{CH}^2(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}}$ (see [7, 22, 56]).

More interestingly, our formula enables us to deduce simple descriptions of Chow groups which were previously unknown. Let us give the first examples.

Corollary 4.6. *We have*

$$\mathrm{CH}_2(\mathcal{N}_{\mathcal{L}})_{\mathbb{Q}} \simeq \begin{cases} \mathbb{Q} & \text{if } g = 2, \\ \mathbb{Q} \oplus \mathrm{CH}_0(\mathrm{Sym}^2(C))_{\mathbb{Q}} & \text{if } g \geq 3 \end{cases}$$

and

$$\mathrm{CH}^3(\mathcal{N}_\mathcal{L})_\mathbb{Q} \simeq \begin{cases} \mathbb{Q} & \text{if } g = 2, \\ \mathbb{Q} \oplus \mathrm{Pic}(\mathrm{Jac}(C))_\mathbb{Q} & \text{if } g = 3, \\ \mathbb{Q}^{\oplus 2} \oplus \mathrm{Pic}(\mathrm{Jac}(C))_\mathbb{Q} & \text{if } g \geq 4. \end{cases}$$

Proof. This follows from Corollary 4.4 and Lemma 2.1 together with the fact that

$$\mathrm{Pic}(\mathrm{Sym}^2(C))_\mathbb{Q} \simeq \mathbb{Q} \oplus \mathrm{Pic}(\mathrm{Jac}(C))_\mathbb{Q}.$$

which follows as $\mathfrak{h}(C) = \mathbb{Q}(0) \oplus \mathfrak{h}^1(C) \oplus \mathbb{Q}(1)$. $\qquad\square$

For each value of a, the descriptions of $\mathrm{CH}^a(\mathcal{N}_\mathcal{L})_\mathbb{Q}$ and $\mathrm{CH}_a(\mathcal{N}_\mathcal{L})_\mathbb{Q}$ vary in low genus but stabilise in higher genus to the following uniform formulas.

Corollary 4.7. *Let $a \in \mathbb{N}$. For $g \geq a + 1$, we have*

$$\mathrm{CH}^a(\mathcal{N}_\mathcal{L})_\mathbb{Q} \simeq \bigoplus_{i=\lceil \frac{a}{2} \rceil}^{a} \mathrm{CH}^{a-i}(\mathrm{Sym}^i(C))_\mathbb{Q};$$

$$\mathrm{CH}_a(\mathcal{N}_\mathcal{L})_\mathbb{Q} \simeq \bigoplus_{i=\lceil \frac{a}{2} \rceil}^{a} \mathrm{CH}_{a-i}(\mathrm{Sym}^i(C))_\mathbb{Q}.$$

Proof. These follow from Corollary 4.4 by determining the terms which are not forced to vanish by Lemma 2.1 and the condition $g \geq a + 1$. $\qquad\square$

5. MOTIVES OF MODULI SPACES OF PARABOLIC VECTOR BUNDLES

Moduli spaces of parabolic vector bundles were introduced by Mehta and Seshadri [58], where one of the key differences with moduli of vector bundles is that in their GIT construction there is a choice of linearisations of the action giving rise to various notions of stability, encoded by a set of parabolic weights. Several authors have studied the geometry of the birational transformations between these moduli spaces for different weights [16, 18, 67]. The Betti numbers and Poincaré polynomials of these moduli spaces over the complex numbers have been computed by Holla [38] using a gauge theoretic approach à la Atiyah–Bott. For $C = \mathbb{P}^1$ and rank $n = 2$, the Poincaré polynomials have been studied using variation of parabolic stability [8]. We will use the explicit wall-crossing descriptions to give formulae for the Chow motives of these moduli spaces in rank $n = 2$.

5.1. Moduli spaces of parabolic vector bundles.
Throughout this section, we fix a set $D = \{p_1, \ldots, p_N\}$ of distinct k-rational points on C, which we refer to as the *parabolic points*. We limit ourselves to the case of rational points, mostly for simplicity; it is likely that very similar formulas hold for closed points.

5.1.1. (Quasi)-Parabolic vector bundles.

Definition 5.1. A *quasi-parabolic vector bundle* $E_* = (E, E_{i,j})$ on (C, D) is a vector bundle E on C with a *quasi-parabolic structure*, consisting of flags $E_{i,j}$ in the fibres at each point $p_i \in D$

$$E_{p_i} = E_{i,1} \supsetneqq E_{i,2} \supsetneqq \cdots \supsetneqq E_{i,l_i} \supsetneqq E_{i,l_i+1} = 0.$$

A *parabolic vector bundle* is a quasi-parabolic vector bundle E_* with weights $\alpha = (\alpha_{i,j})$ satisfying

$$0 \leq \alpha_{i,1} < \cdots < \alpha_{i,l_i} < 1 \quad \text{for each } 1 \leq i \leq N.$$

The *discrete invariants* of E_* are given by the rank and degree of E and for each $p_i \in D$, the *length* $l_i := l(E_{i,j})$ and *flag type* $n_{i,j} := \dim E_{i,j}$ (or equivalently, the *multiplicity* $m(E_*) = (m_{i,j})$ defined by $m_{i,j} := n_{i,j} - n_{i,j+1}$ for $1 \leq i \leq N, 1 \leq j \leq l_i$). We write these invariants as a tuple $\eta(E_*) = (\mathrm{rk}(E), \deg(E), m(E_*))$. The flags are *full* if $m_{i,j} = 1$ for all i, j.

For us, we will be interested in varying the weights for quasi-parabolic bundles and we will later think of these weights as defining a notion of stability for quasi-parabolic vector bundles and the variation of weights gives different notions of stability and moduli spaces.

Remark 5.2. One can equivalently think of a quasi-parabolic vector bundle $E_* = (E, E_{i,j})$ on (C, D) as a vector bundle with a sheaf filtration (by locally free sheaves) for each $p_i \in D$

$$E = E_1^i \supsetneq E_2^i \supsetneq \cdots \supsetneq E_{l_i}^i \supsetneq E_{l_i+1}^i = E(-p_i)$$

where E_j^i is the kernel of the sheaf homomorphism $E \twoheadrightarrow (E_{p_i}/E_{i,j}) \otimes \mathcal{O}_{p_i}$. If (E_*, α) is a parabolic vector bundle, then the weights α determine, for each $p_i \in D$, a filtered sheaf $\mathcal{E}_x^i$ indexed by $x \in \mathbb{R}$ (cf. [64] and [18, §5]): for $\alpha_{i,j-1} < x \leq \alpha_{i,j}$ set $\mathcal{E}_x^i := E_j^i$ (where we define $\alpha_{i,0} = 0$ and $\alpha_{i,l_{i+1}} = 1$), which defines $\mathcal{E}_x^i$ for $x \in (0, 1]$ and then set $\mathcal{E}_{x+m}^i := \mathcal{E}_x^i(-mp_i)$ for $m \in \mathbb{Z}$.

Definition 5.3. Let (E_*, α) and (F_*, β) be parabolic vector bundles on (C, D). A homomorphism $\phi : E \to F$ is said to be *parabolic* (resp. *strongly parabolic*) if $\alpha_{i,j} > \beta_{i,k}$ (resp. $\alpha_{i,j} \geq \beta_{i,k}$) implies $\phi(E_{i,k}) \subset F_{i,k+1}$ for all i, j, k.

If E_* and F_* have the same flag lengths and $\alpha = \beta$, then a homomorphism $\phi : E \to F$ is parabolic if $\phi(E_{i,j}) \subset F_{i,j}$ (resp. strongly parabolic if $\phi(E_{i,j}) \subset F_{i,j+1}$) for all i, j.

There is a subsheaf of parabolic homomorphisms $\mathcal{P}ar\mathcal{H}om((E_*, \alpha), (F_*, \beta)) \subset \mathcal{H}om(E, F)$ with torsion quotient supported on D. Consequently, the Euler characteristic of this sheaf is

$$\chi(\mathcal{P}ar\mathcal{H}om((E_*, \alpha), (F_*, \beta)) = \chi(\mathcal{H}om(E, F)) - \sum_{p_i \in D} (\mathrm{rk}(E)\,\mathrm{rk}(F) - P_i((E_*, \alpha), (F_*, \beta))) ,$$

where $P_i((E_*, \alpha), (F_*, \beta)) := \dim \mathrm{ParHom}((E_{i,*}, \alpha_{i,*}), (F_{i,*}, \beta_{i,*}))$; see [73, §4] and [16, Lemma 2.4]. As this Euler characteristic only depends on the underlying discrete invariants and weights, we write

$$\chi_{\mathrm{par}}((\eta(E_*), \alpha)^\vee \otimes (\eta(F_*), \beta)) := \chi(\mathcal{P}ar\mathcal{H}om((E_*, \alpha), (F_*, \beta)).$$

Remark 5.4. If E_* and F_* have the same flag lengths and $\alpha = \beta$, we drop the weights from the notation and we have

$$\chi_{\mathrm{par}}(\eta(E_*)^\vee \otimes \eta(F_*)) = -n_F d_E + n_E d_F + n_E n_F (1 - g) - \sum_{i=1}^{N} \sum_{j>k} m_{i,j}(E_*) m_{i,k}(F_*).$$

5.1.2. *Parabolic weights and stability.* For a fixed rank n, the space of weights for rank n parabolic vector bundles on (C, D) is

$$\mathcal{A} := \{\widetilde{\alpha} = (\widetilde{\alpha}_{i,j}) \in \mathbb{R}^{Nn} : 0 \leq \widetilde{\alpha}_{i,1} \leq \widetilde{\alpha}_{i,2} \leq \cdots \leq \widetilde{\alpha}_{i,n} < 1 \text{ for } 1 \leq i \leq N\}.$$

Note that in the space of weights $\mathcal{A}$, we allow $\widetilde{\alpha}_{i,j} = \widetilde{\alpha}_{i,j+1}$.

Definition 5.5. For a weight $\widetilde{\alpha} \in \mathcal{A}$ and $1 \leq i \leq N$, let $l_i(\widetilde{\alpha})$ denote the number of distinct weights in $\widetilde{\alpha}_{i,1} \leq \widetilde{\alpha}_{i,2} \leq \cdots \leq \widetilde{\alpha}_{i,n}$ and for $1 \leq j \leq l_i(\widetilde{\alpha})$, we let $m_{i,j}(\widetilde{\alpha})$ denote the multiplicities of these distinct weights in increasing order. We refer to $l(\widetilde{\alpha}) = (l_i(\widetilde{\alpha}))_i$ and $m(\widetilde{\alpha}) = (m_{i,j}(\widetilde{\alpha}))_{i,j}$ as the *length* and *multiplicity* of $\widetilde{\alpha}$. There is a decomposition

$$\mathcal{A} = \bigsqcup_m \mathcal{A}_m$$

by multiplicities and the ordering on multiplicities by successive refinement defines an ordering on the set of weights of fixed multiplicity: $\mathcal{A}_m > \mathcal{A}_{m'}$ if $\mathcal{A}_{m'}$ is a proper face contained in the closure of $\mathcal{A}_m$. For $\widetilde{\alpha}$ of length l and multiplicity m, the *collapsed weight* α is obtained by deleting for each i repeated instances of weights:

$$\alpha := (\alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i})_{1 \leq i \leq N} \text{ where } \alpha_{i,j} := \widetilde{\alpha}_{i,1+\sum_{k<j} m_{i,k}}.$$

When the rank n and multiplicity m are fixed, we can freely go back and forth between extended weights $\widetilde{\alpha}$ and collapsed weights α by deleting or inserting repeated weights.

Definition 5.6. Let E_* be a quasi-parabolic vector bundle on (C, D) of rank n and with multiplicity $m = (m_{i,j})$. For $\widetilde{\alpha} \in \mathcal{A}_m$, we define the α-*slope* of E_* by

$$\mu_\alpha(E_*) = \frac{\deg_\alpha(E_*)}{\mathrm{rk}(E)}, \quad \text{where } \deg_\alpha(E_*) = \deg(E) + \sum_{p_i \in D} \sum_{j=1}^{n} \widetilde{\alpha}_{i,j} = \deg(E) + \sum_{p_i \in D} \sum_{j=1}^{l_i} \alpha_{i,j} m_{i,j}.$$

For a subbundle $E' \subset E$, we can intersect E'_{p_i} with the flags $E_{i,j}$ in E_{p_i}. If we do not delete repeated subspaces in this flag in E'_{p_i}, we obtain a length l_i flag in E'_{p_i} with *multiplicity* defined (slightly unconventionally, but as in [18]) by setting for $1 \leq j \leq l_i$

$$m'_{i,j} := \dim(E'_{p_i} \cap E_{i,j}) - \dim(E'_{p_i} \cap E_{i,j+1})$$

which may now also be zero. Then the α-slope of this subbundle is defined by

$$\mu_\alpha(E'_*) := \frac{\deg_\alpha(E'_*)}{\mathrm{rk}(E')}, \quad \text{where } \deg_\alpha(E'_*) = \deg(E') + \sum_{p_i \in D} \sum_{j=1}^{l_i} \alpha_{i,j} m'_{i,j}.$$

If instead we delete repeated subspaces in the flag in E'_{p_i}, we obtain a flag of length $l'_i \leq l_i$

$$E'_{p_i} = E'_{i,1} \supsetneqq E'_{i,2} \supsetneqq \cdots \supsetneqq E'_{i,l'_i} \supsetneqq E'_{i,l'_i+1} = 0,$$

giving a quasi-parabolic structure[6] E'_*. We endow this with a subset of the weights

(7) $$(\alpha'_{i,k}) = (\alpha_{i,j_1} < \alpha_{i,j_2} < \cdots < \alpha_{i,j_{l'_i}})_{1 \leq i \leq N}, \quad \text{where } j_k := \max\{j : E'_{i,k} \subseteq E_{i,j}\}.$$

Then $\deg_\alpha(E'_*) = \deg(E') + \sum_{p_i \in D} \sum_{k=1}^{l'_i} \alpha_{i,j_k}(\dim E'_{i,k} - \dim E'_{i,k+1})$.

Remark 5.7. Given a short exact sequence $E' \hookrightarrow E \twoheadrightarrow E''$ with quasi-parabolic structure E_* on E, one can similarly give E'' a quasi-parabolic structure and define its multiplicity m'' analogously. The advantage of our unconventional definition of multiplicities m' and m'' of sub- and quotient bundles is that then $m = m' + m''$. If E has weights α, then we also have subsets α' and α'' of the weights as described above which satisfy $\widetilde{\alpha} = \widetilde{\alpha}' + \widetilde{\alpha}''$ for the associated extended weights. Moreover $(E'_*, \alpha') \hookrightarrow (E_*, \alpha) \twoheadrightarrow (E''_*, \alpha'')$ are parabolic homomorphisms.

Definition 5.8. A quasi-parabolic bundle E_* with $m(E_*) = m(\alpha)$ is α-*semistable* (resp. α-*stable*) if for all subbundles $E' \subsetneq E$, we have $\mu_\alpha(E'_*) \leq \mu_\alpha(E_*)$ (resp. $\mu_\alpha(E'_*) < \mu_\alpha(E_*)$).

Remark 5.9. We note that the notion of stability is invariant under the shift

$$(\alpha_{i,j}) \mapsto (\alpha_{i,j} + C_i)$$

for constants C_i for $1 \leq i \leq N$. Hence, we may assume $\alpha_{i,1} = 0$ without loss of generality.

Fix discrete invariants $\eta = (n, d, m)$. For each $\alpha \in \mathcal{A}_m$, there is a moduli space $\mathcal{N}^\alpha_{C,D}(\eta)$ of α-semistable parabolic bundles with invariants η, which is a normal projective variety [58]. If it is non-empty, by the deformation theory of parabolic bundles [73, Theorem 5.1], it has dimension

$$\dim \mathcal{N}^\alpha_{C,D}(\eta) = \dim \mathcal{N}_C(n,d) + \sum_{i=1}^N \dim \mathcal{F}(m_i) = n^2(g-1) + 1 + \sum_{i=1}^N \sum_{j>k} m_{i,j} m_{i,k}$$

where $\mathcal{F}(m_i)$ is the variety of flags of type $n = \sum_{j=1}^{l_i} m_{i,j} > \sum_{j=2}^{l_i} m_{i,j} > \cdots > m_{i,l_i} > 0$. The moduli space of α-stable parabolic bundles $\mathcal{N}^{\alpha-s}_{C,D}(\eta)$ is a smooth open subvariety of $\mathcal{N}^\alpha_{C,D}(\eta)$.

5.1.3. *Geometric description of variation of stability.* Fix the discrete invariants $\eta = (n, d, m)$ given by the rank n, degree d and multiplicity m. For simplicity, we write $\mathcal{N}^\alpha := \mathcal{N}^\alpha_{C,D}(\eta)$ and similarly $\mathcal{N}^{\alpha-s}$ for the open subset of α-stable parabolic bundles.

We can divide the space of weights $\mathcal{A}$ into walls and chambers such that stability is constant on the chambers and changes over the walls. More precisely, on the walls there are strictly semistable bundles, i.e., we have $\mu_\alpha(E'_*) = \mu_\alpha(E_*)$ for a subbundle $E' \subset E$. To describe the walls it therefore suffices to consider the possible discrete invariants η' of subbundles.

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[6]Usually the multiplicity of E'_* is defined as the differences $\dim E'_{i,k} - \dim E'_{i,k+1} > 0$ in the collapsed flag.

Definition 5.10. For $\eta' := (n', d', m')$ such that $0 < n' < n$, $d' \in \mathbb{Z}$ and $m' = (m'_{i,j})$ are non-negative integers with $m'_{i,j} \leq m_{i,j}$ and $\sum_{j=1}^{l_i} m'_{i,j} = n'$, we define the corresponding *wall*

$$W_{m,\eta'} = \left\{ \alpha \in \mathcal{A}_m : \mu_\alpha(\eta') = \mu_\alpha(\eta) \right\} = \mathcal{A}_m \cap W_{\eta'}$$

where for $\alpha \in \mathcal{A}_m$:

$$\mu_\alpha(\eta') = \mu_\alpha(\eta) \iff \frac{d' + \sum_{i=1}^N \sum_{j=1}^{l_i} \alpha_{i,j} m'_{i,j}}{n'} = \frac{d + \sum_{i=1}^N \sum_{j=1}^{l_i} \alpha_{i,j} m_{i,j}}{n}.$$

The complement of $\mathbb{R}^{Nn} \setminus W_{\eta'}$ of each wall is two half-spaces $H_{\eta'}^\pm$ such that for $\alpha \in H_{\eta'}^\pm$ we have $\pm(\mu_\alpha(\eta') - \mu_\alpha(\eta)) > 0$. The connected components of $\mathcal{A}_m \setminus \cup_\eta W_{m,\eta}$ are called *chambers* and we refer to weights $\alpha \in \mathcal{A}_m \setminus \cup_\eta W_{m,\eta}$ as being *generic*.

Note that there are only finitely many walls in $\mathcal{A}_m$. The invariants $\eta' := (n', d', m')$ have complementary invariants $\eta'' = (n'', d'', m'')$ satisfying $n' + n'' = n$, $d' + d'' = d$, $m' + m'' = m$ and determine the same wall[7]. We say a wall $W_{m,\eta'}$ is *good* if there are no discrete invariants other than η' and η'' which define this wall. All walls are good, if we take multiplicities corresponding to full flags (see [68, (2.4)]).

By [18, Proposition 3.2], $\mathcal{A}_m$ contains a generic weight if and only if the degree d and multiplicities $m_{i,j}$ have greatest common divisor 1. In this case, for any generic weight, semistability coincides with stability and corresponding the moduli space of stable parabolic vector bundles is a fine moduli space and a smooth projective variety.

The geometric description of the birational tranformation between parabolic moduli spaces for two generic weights separated by a good wall is described by Boden–Hu [16], Boden–Yokogawa [18] and Thaddeus [67]. These papers assume that k is algebraically closed. However, when crossing a good wall, their proofs show that, for a parabolic bundle in the exceptional locus, there is a unique subbundle which does not violate stability on one side of the wall but does on the other side. A descent argument similar to that in the second and third paragraphs of §4.3, then implies that the description of the wall-crossing holds over a general field k.

Theorem 5.11 (Boden–Hu, Boden–Yokogawa and Thaddeus)**.** *Consider a line segment which joins two adjacent chambers and passes through a weight α on a single good wall $W_{m,\eta'}$. Let $\alpha(\pm) \in H_{\eta'}^\pm$ be weights in adjacent chambers which satisfy*

$$\pm \left(\mu_{\alpha(\pm)}(\eta') - \mu_{\alpha(\pm)}(\eta) \right) > 0.$$

Then there is a standard flip

$$\mathcal{N}^{\alpha(-)} \dashrightarrow_{\substack{(n_-, n_+) \\ \text{centre } \mathcal{N}^{\alpha-sss}}} \mathcal{N}^{\alpha(+)}$$
$$\searrow \qquad \swarrow$$
$$\mathcal{N}^\alpha$$

with centre isomorphic to a product of smooth projective moduli spaces

$$\mathcal{N}^{\alpha-sss} := \mathcal{N}^\alpha \setminus \mathcal{N}^{\alpha-s} \cong \mathcal{N}_{C,D}^{\alpha'}(\eta') \times \mathcal{N}_{C,D}^{\alpha''}(\eta'')$$

with smaller invariants η' and complementary invariants η'' (the weights α' are defined at (7) and similarly for α''). Over the centre the fibres are projective spaces of dimensions

$$n_- := n_-(\eta') = - \chi_{\mathrm{par}}((\eta'', \alpha'')^\vee \otimes (\eta', \alpha')) - 1$$
$$n_+ := n_+(\eta') = - \chi_{\mathrm{par}}((\eta', \alpha')^\vee \otimes (\eta'', \alpha'')) - 1.$$

The closest formulation in the literature to the above statement is [18, Theorem 4.1], whose proof computes the flip type. Our (slightly unconventional) approach to multiplicities of sub-

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[7]If $\eta' = \eta(E'_*)$ for a subbundle $E'_* \subset E_*$, then η'' is the discrete invariants of the quotient bundle E''_*.

and quotient bundles allows us to easily compute the dimensions of the fibres using Remark 5.4:

$$(8) \qquad \begin{aligned} n_-(\eta') &= n'd'' - n''d' + n'n''(g-1) - 1 + \sum_{i=1}^{N} \sum_{j>k} m_{i,j}'' m_{i,k}' \\ n_+(\eta') &= n''d' - n'd'' + n''n'(g-1) - 1 + \sum_{i=1}^{N} \sum_{j>k} m_{i,j}' m_{i,k}'', \end{aligned}$$

which satisfy $n_- + n_+ + 1 = \mathrm{codim}(\mathcal{N}^{\alpha-sss}, \mathcal{N}^{\alpha})$ as stated in [18, Theorem 4.1].

5.1.4. *Geometric description of flag degeneration.* For weights with different multiplicities that are not separated by walls $W_{\eta'}$, one can consider the flag degeneration giving by forgetting part of the flags. The geometric description of this is due to Boden–Hu [16] and Boden–Yokogawa [18].

Theorem 5.12. [18, Theorem 4.2] *Let $\mathcal{A}_{m'} > \mathcal{A}_m$ and suppose $\alpha \in \mathcal{A}_{m'}$ and $\beta \in \mathcal{A}_m$ are generic weights which are not separated by any walls $W_{\eta'} \subset \mathcal{A}$. Then there is a forgetful map*

$$\mathcal{N}^{\alpha} \to \mathcal{N}^{\beta}$$

which is an iterated Zariski locally trivial flag bundle.

Remark 5.13. In fact, one can factor this forgetful map into a tower of Grassmannian bundles as follows. Consider a chain $\mathcal{A}_{m'} = \mathcal{A}_{m(0)} > \mathcal{A}_{m(1)} > \cdots > \mathcal{A}_{m(t)} = \mathcal{A}_m$ such that for each $0 \le k < t$ there exists a unique i_k such that the length of l_{i_k} of $m(k)$ and $m(k+1)$ differ by 1. Take generic weights $\alpha(k) \in \mathcal{A}_{m(k)}$ not separated by walls with $\alpha(0) = \alpha$ and $\alpha(t) = \beta$, then there is a unique j_k such that the morphism $\mathcal{N}^{\alpha(k)} \to \mathcal{N}^{\alpha(k+1)}$ is given by forgetting the subspace E_{i_k, j_k} in the flag of $E_{p_{i_k}}$. This map is a Grassmannian bundle with fibre $\mathrm{Gr}(m(k)_{i_k, j_k}, m(k)_{i_k, j_k} + m(k)_{i_k, j_k - 1})$. Hence, we obtain a factorisation into a tower of Grassmannian bundles

$$\mathcal{N}^{\alpha} = \mathcal{N}^{\alpha(0)} \to \mathcal{N}^{\alpha(1)} \to \cdots \to \mathcal{N}^{\alpha(t)} = \mathcal{N}^{\beta}.$$

Similarly using flag degenerations, one can relate moduli spaces of parabolic vector bundles to moduli spaces of vector bundles (viewed as parabolic vector bundles with trivial flags) provided one takes sufficiently small weights; see [18, Proposition 5.3].

5.1.5. *Hecke modifications of parabolic vector bundles.* One can compare moduli spaces of vector bundles of rank n for different values of d using Hecke modifications of parabolic vector bundles. This useful observation was noted at the end of [58] in rank $n = 2$ and then expanded upon in [18, §5]. Let us introduce Hecke modifications as isomorphisms between moduli spaces of parabolic vector bundles for the same rank but different degrees, multiplicities and weights. The name Hecke modification reflects the fact that on the underlying vector bundle, we perform a Hecke modification at one of the parabolic points $p_i \in D$ using part of the flag at that point.

Definition 5.14. Fix invariants $\eta = (n, d, m)$ and a weight $\alpha \in \mathcal{A}_m$. For $p_i \in D$ and $1 \le j < l_i$ and $\alpha_{i,j} < \beta \le \alpha_{i,j+1}$, we define new invariants $\mathcal{H}_{i,j}(\eta)$ and weights $\mathcal{H}_{i,j,\beta}(\alpha)$ by

(1) $\mathcal{H}_{i,j}(\eta) = (n, d - \sum_{k=1}^{j} m_{i,k}, \hat{m})$ with cyclically permuted multiplicities

$$(\hat{m}_{i,1}, \dots \hat{m}_{i,l_i}) = (m_{i,j+1}, \dots, m_{i,l_i}, m_{i,1}, \dots, m_{i,j})$$

and $\hat{m}_{i',j} = m_{i',j}$ for all $i' \ne i$ and $1 \le j \le l_{i'}$,

(2) $\hat{\alpha} := \mathcal{H}_{i,j,\beta}(\alpha)$ is given by

$$(\hat{\alpha}_{i,1}, \dots \hat{\alpha}_{i,l_i}) = (\alpha_{i,j+1} - \beta, \alpha_{i,j+2} - \beta, \dots, \alpha_{i,l_i} - \beta, 1 + \alpha_{i,1} - \beta, \dots, 1 + \alpha_{i,j} - \beta)$$

and $\hat{\alpha}_{i',j} = \alpha_{i',j}$ for all $i' \ne i$ and $1 \le j \le l_{i'}$.

Then the *Hecke modification* at p_i with respect to j and $\alpha_{i,j} < \beta \leq \alpha_{i,j+1}$ is the isomorphism

$$\mathcal{H}_{i,j,\beta} : \mathcal{N}^\alpha(\eta) \to \mathcal{N}^{\mathcal{H}_{i,j,\beta}(\alpha)}(\mathcal{H}_{i,j}(\eta))$$

sending E_* to E'_* with $E' := \ker(E \twoheadrightarrow E_{p_i}/E_{i,j+1} \otimes \mathcal{O}_{p_i})$ which inherits a quasi-parabolic structure from E with multiplicities as specified above (see [18, §5]).

Hecke modifications are isomorphisms with inverses given by Hecke modifications. Moreover, Hecke modifications at different points commute in the obvious sense.

The Hecke modification $\mathcal{H}_{i,j,\beta}$ is most simply understood in terms of the N-tuple of $\mathbb{R}$-indexed filtered sheaves $\mathcal{E}_x^{i'}$ associated to the parabolic vector bundle (E_*, α) (see Remark 5.2) as performing a shift by β in the $\mathbb{R}$-indexed filtration at the parabolic point p_i (see [18, §5]):

$$\mathcal{H}_{i,j,\beta}(\mathcal{E}_x^{i'}) = \hat{\mathcal{E}}_x^{i'} \quad \text{where} \quad \hat{\mathcal{E}}_x^{i'} := \begin{cases} \mathcal{E}_{x+\beta}^i & \text{if } i' = i \\ \mathcal{E}_x^{i'} & \text{if } i' \neq i. \end{cases}$$

Example 5.15. Let us consider the case for $n = 2$. For a parabolic point p_i at which the multiplicity m specifies a full flag (i.e. $l_i = 2$), we get a Hecke modification for $j = 1$. This decreases the degree d by 1 and as $(m_{i,1}, m_{i,2}) = (1,1)$ permuting these multiplicities does not change m, thus $\mathcal{H}_{i,1}(\eta) = (n, d-1, m)$. For $\alpha_{i,1} < \beta \leq \alpha_{i,2}$, the new weight $\hat{\alpha} = \mathcal{H}_{i,1,\beta}(\alpha)$ is given by $(\hat{\alpha}_{i,1}, \hat{\alpha}_{i,2}) = (\alpha_{i,2} - \beta, 1 + \alpha_{i,1} - \beta)$ and is unchanged for $i' \neq i$. When we take the maximal value $\beta = \alpha_{i,2}$, we write the Hecke modification $\mathcal{H}_{i,1,\alpha_{i,2}}$ simply as

$$\mathcal{H}_{p_i} : \mathcal{N}^\alpha(2, d, m) \to \mathcal{N}^{\alpha(p_i)}(2, d-1, m)$$

with $\alpha(p_i)_{i,j} = (0, 1 + \alpha_{i,1} - \alpha_{i,2})$ and $\alpha(p_i)_{i',j} = \alpha_{i',j}$ for all $i' \neq i$. More generally, given a subset $D' \subset D$ with $l_i = 2$ for all $p_i \in D'$, we let

$$\mathcal{H}_{D'} : \mathcal{N}^\alpha(2, d, m) \to \mathcal{N}^{\alpha(D')}(2, d - |D'|, m)$$

denote the composition (in any order) of the Hecke modifications $\mathcal{H}_{p_i}$ for $p_i \in D'$.

For degrees d and d' which are coprime to n, one can relate the vector bundle moduli spaces $\mathcal{N}(n, d)$ and $\mathcal{N}(n, d')$ using moduli spaces of parabolic vector bundles, Hecke modifications and a sequence of wall-crossing flips, together with degeneration of flag structures (see [58, 18]).

5.2. Motivic consequences.

5.2.1. *Motivic variation of stability for moduli of parabolic vector bundles.* By combining Theorem 5.11 and Theorem 3.2, we obtain the following result describing the Chow motives of moduli spaces of parabolic vector bundles for different weights.

Corollary 5.16. *For a line segment which joins two adjacent chambers and passes through a weight α on a single good wall $W_{m,\eta'}$, let $\alpha(\pm) \in \mathcal{H}_{\eta'}^\pm$ be weights in adjacent chambers satisfying*

$$\pm \big(\mu_{\alpha(\pm)}(\eta') - \mu_{\alpha(\pm)}(\eta) \big) > 0.$$

Then there is an explicit isomorphism of integral Chow motives

$$(9) \qquad \mathfrak{h}(\mathcal{N}^{\alpha(+)}) \oplus \bigoplus_{n_-(\eta) < j \leq n_+(\eta)} \mathfrak{h}(\mathcal{N}^{\alpha-sss})(j) \simeq \mathfrak{h}(\mathcal{N}^{\alpha(-)}) \oplus \bigoplus_{n_+(\eta) < j \leq n_-(\eta)} \mathfrak{h}(\mathcal{N}^{\alpha-sss})(j),$$

where $n_\pm(\eta')$ are specified in (8).

Note that in the above isomorphism at least one of the big direct sums on either side is empty. In the case when $n_-(\eta') = n_+(\eta')$, this wall-crossing is a flop and both big direct sums are empty, so the Chow motives of these moduli spaces are isomorphic.

Remark 5.17. For fixed invariants $\eta = (n, d, m)$, the space $\mathcal{A}_m$ of corresponding weights is cut into chambers by finitely many walls $W_{m,\eta'}$, which do not depend on the genus g of C (see Definition 5.10). However, the types of the flips at these walls, given by $n_\pm(\eta')$ in Equation (8), grow linearly with g. For low genus, there may be some chambers which give rise to empty moduli spaces and there may be some wall-crossings which are simply blow-ups. However, for

fixed invariants $\eta = (n, d, m)$, if g is sufficiently large, then we have $n_\pm(\eta') > 0$ for all invariants η' specifying walls in $\mathcal{A}_m$. For example, for rank $n = 2$ with full flags, it suffices to take $g \geq 2$ (see the bounds on the flip type given in Proposition 5.24).

As an application, we slightly correct and strengthen the result of Chakraborty [20, Theorem 1.1] to integral coefficients.

Corollary 5.18. *Fix invariants $\eta = (n, d, m)$ with m corresponding to full flags. There exists an integer g_0, depending only on η, such that as long as $g \geq g_0$, we have isomorphisms between the Chow groups of 1-cycles and the third homology H_3 for two generic weights α and β:*

$$\mathrm{CH}_1(\mathcal{N}^\alpha) \simeq \mathrm{CH}_1(\mathcal{N}^\beta),$$

$$H_3(\mathcal{N}^\alpha) \simeq H_3(\mathcal{N}^\beta),$$

where H_3 denotes either singular homology equipped with its Hodge structure if $k = \mathbb{C}$, or the ℓ-adic homology over $\bar{k}$ (in the sense of [54], say) equipped with its Galois action if ℓ is coprime to the characteristic of k. In particular, the intermediate Jacobians for 1-cycles are isomorphic.

Proof. As the flags are full, every wall is good (i.e. corresponds uniquely to a pair of complementary invariants (η', η'') with $\eta = \eta' + \eta''$). By wall-crossing, it suffices to show the isomorphism for α and β in adjacent chambers. As in Remark 5.17, we take g sufficiently large so that $n_\pm(\eta') > 0$ for all invariants η' specifying walls in $\mathcal{A}_m$. We see that in (9), the index j, when appearing, is at least 2. Hence the big direct sums do not contribute to CH_1, nor to H_3 (once we apply either the Betti [5] or ℓ-adic realisation [6] of Voevodsky motives). $\square$

5.2.2. *Motivic descriptions of flag degenerations.* We have the following motivic consequence of Theorem 5.12.

Corollary 5.19. *Let $\mathcal{A}_{m'} > \mathcal{A}_m$ and suppose $\alpha \in \mathcal{A}_{m'}, \beta \in \mathcal{A}_m$ are generic weights which are not separated by any walls $W_{\eta'} \subset \mathcal{A}$. Then there is an explicit isomorphism of integral Chow motives*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}^\beta) \otimes T$$

where $T \in \mathrm{CHM}(k, \mathbb{Z})$ is a tensor product of motives of Grassmannians as appearing in Remark 5.13; thus T is a pure Tate motive.

Proof. By Theorem 5.12, the forgetful map $\mathcal{N}^\alpha \to \mathcal{N}^\beta$ is an iterated flag bundle. More precisely, as in Remark 5.13, we can factor this forgetful map into a tower Grassmannian bundles

$$\mathcal{N}^\alpha = \mathcal{N}^{\alpha(0)} \to \mathcal{N}^{\alpha(1)} \to \cdots \to \mathcal{N}^{\alpha(t)} = \mathcal{N}^\beta$$

with $\mathcal{N}^{\alpha(k)} \to \mathcal{N}^{\alpha(k+1)}$ having fibre a Grassmannian $\mathrm{Gr}_k := \mathrm{Gr}(m(k)_{i_k,j_k}, m(k)_{i_k,j_k} + m(k)_{i_k,j_k-1})$. Then we have explicit isomorphisms

$$\mathfrak{h}(\mathcal{N}^{\alpha(k)}) \simeq \mathfrak{h}(\mathcal{N}^{\alpha(k+1)}) \otimes \mathfrak{h}(\mathrm{Gr}_k)$$

which together give

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}^\beta) \otimes T$$

where T is a tensor product of motives $\mathfrak{h}(\mathrm{Gr}_k)$ of Grassmannians for $0 \leq k < t$. $\square$

In the case when we forget all of the flags at all of the parabolic points, we obtain simply a vector bundle. Suppose that n and d are coprime, so that semistability and stability coincide for rank n degree d vector bundles and $\mathcal{N} = \mathcal{N}(n, d)$ is smooth and projective. In this case the weight given by the origin $\underline{0} \in \mathcal{A}$ is a generic weight and corresponds to parabolic vector bundles with trivial flags (i.e. vector bundles) and we have $\mathcal{N} = \mathcal{N}^{\underline{0}}$. By [18, Proposition 5.3], if $\alpha \in \mathcal{A}_m$ is a generic weight which is sufficiently small (i.e in a chamber whose closure contains the origin $\underline{0}$), then α and $\underline{0}$ are not separated by any wall and consequently we obtain the following special case of the above result.

Corollary 5.20. *Let n and d be coprime. Then for a sufficiently small generic weight $\alpha \in \mathcal{A}_m$ as above, we have an explicit isomorphism of integral Chow motives*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes T, \quad where \quad T = \bigotimes_{i=1}^N \mathfrak{h}(\mathcal{F}(m_i))$$

and $\mathcal{F}(m_i)$ is the variety of flags $n = \sum_{j=1}^{l_i} m_{i,j} > \sum_{j=2}^{l_i} m_{i,j} > \cdots > m_{i,l_i} > 0$.

As an application, we compute the intermediate Jacobian of 1-cycles for the moduli spaces $\mathcal{N}_{\mathcal{L}}^\alpha$ of parabolic vector bundles with fixed determinant[8] $\mathcal{L}$ for a generic weight α; this strengthens the results of Chakraborty [20, Theorem 1.2] and [21, Proposition 3.4] to integral coefficients.

Corollary 5.21. *Fix invariants $\eta = (n, d, m)$ with m corresponding to full flags. Fix a degree d line bundle $\mathcal{L}$. There exists an integer $g_0 > 2$, depending only on η, such that as along as $g \geq g_0$, for a generic weight α, we have an isomorphism of Chow groups of 1-cycles,*

$$\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha)_{\mathrm{hom}} \simeq \mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}}$$

and if $k = \mathbb{C}$, we also have an isomorphism of abelian varieties:

$$J\big(H_3(\mathcal{N}_{\mathcal{L}}^\alpha)\big) \simeq \mathrm{Jac}(C),$$

where J denotes the functor that associates to a level 1 Hodge structure its Jacobian.

For rank $n = 2$ with full flags at N parabolic points, and always $g \geq g_0$ and α generic, we have $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha) \simeq \mathrm{CH}_0(C) \oplus \mathbb{Z}^{\oplus N}$ and, if $k = \mathbb{C}$, we have

$$\mathrm{AJ} \colon \mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha)_{\mathrm{hom}} \xrightarrow{\cong} J\big(H_3(\mathcal{N}_{\mathcal{L}}^\alpha)\big) \simeq \mathrm{Jac}(C),$$

where AJ is the Abel–Jacobi map.

Proof. As in Remark 5.17, we take g sufficiently large so that $n_\pm(\eta') > 0$ for all invariants η' specifying walls in $\mathcal{A}_m$. By Corollary 5.18, the CH_1 and the intermediate Jacobian do not vary for generic weights. Therefore, we can assume that α is sufficiently small. In this case, by Corollary 5.20, we have isomorphisms of Chow groups and Hodge structure (for $k = \mathbb{C}$)

$$\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha)_{\mathrm{hom}} \simeq \mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}} \oplus \mathrm{CH}_0(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}}^{\oplus m},$$
$$H_3(\mathcal{N}_{\mathcal{L}}^\alpha, \mathbb{Z}) \simeq H_3(\mathcal{N}_{\mathcal{L}}, \mathbb{Z}) \oplus H_1(\mathcal{N}_{\mathcal{L}}, \mathbb{Z})(1)^{\oplus m},$$

where m is the number of summands $\mathbb{Z}(1)$ in the Tate motive T. However, $H_1(\mathcal{N}_{\mathcal{L}}, \mathbb{Z}) = 0$ and $\mathrm{CH}_0(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}} = 0$, as $\mathcal{N}_{\mathcal{L}}$ is rational [52]. Therefore,

$$\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha)_{\mathrm{hom}} \simeq \mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}} \quad \text{and} \quad H_3(\mathcal{N}_{\mathcal{L}}^\alpha, \mathbb{Z}) \simeq H_3(\mathcal{N}_{\mathcal{L}}, \mathbb{Z}),$$

and they are compatible with respect to the Abel–Jacobi maps. When $g > 2$, by [45, Theorem 2.1], $H_3(\mathcal{N}_{\mathcal{L}}, \mathbb{Z}) \simeq H_1(C, \mathbb{Z})(1)$ as Hodge structures; thus we obtain the claimed result for the intermediate Jacobian.

Now suppose $n = 2$ and we take full flags at N parabolic points. By the recent work [56], we have $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}) \simeq \mathrm{CH}_0(C)$ and as $\mathcal{N}_{\mathcal{L}}$ is rational, we also have $\mathrm{CH}_0(\mathcal{N}_{\mathcal{L}}) = \mathbb{Z}$. By applying Corollary 5.20, where the Tate motive T is the motive of $(\mathbb{P}^1)^N$, we deduce $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}}^\alpha) \simeq \mathrm{CH}_0(C) \oplus \mathbb{Z}^{\oplus N}$. Over $k = \mathbb{C}$, we have $\mathrm{CH}_1(\mathcal{N}_{\mathcal{L}})_{\mathrm{hom}} \simeq \mathrm{Jac}(C)$, which implies the claimed Abel–Jacobi isomorphism with the intermediate Jacobian. □

Similarly to Corollary 4.7 in §4.3.1, we have the following stabilisation result.

Corollary 5.22. *Fix invariants $\eta = (n, d, m)$ with m corresponding to full flags. For $i \in \mathbb{N}$, there is an integer g_0, depending only on η, such that for C of genus $g \geq g_0$ and any generic weight α, the Chow groups $\mathrm{CH}^i(\mathcal{N}^\alpha)$ and $\mathrm{CH}_i(\mathcal{N}^\alpha)$ are both independent of α and can be computed in terms of Chow groups of $\mathcal{N}$ by the formula $\mathfrak{h}(\mathcal{N}^\alpha) = \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathcal{F})^{\otimes N}$, where $\mathcal{F}$ is the full flag variety[9] and N is the number of parabolic points.*

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[8]Fixing the determinant does not change the type of the wall-crossing flips, only their centres.

[9]The motive of the full flag variety $\mathcal{F}$ in a D-dimensional vector space is pure Tate; by [14, Theorem 5.1], we have $\mathfrak{h}(\mathcal{F}) \simeq \bigoplus_{i=0}^{D(D-1)/2} \mathbb{Q}(i)^{l_i}$, where l_i is the number of elements of length i in the symmetric group S_D seen as a Coxeter group.

Proof. Recall from Remark 5.17 that there are finitely many walls $W_{m,\eta'}$, which are all good and do not depend on the genus g of C, but the flip types $n_{\pm}(\eta')$ of these walls grow linearly with g as in Equation (8). In particular, for fixed i, if we take g sufficiently large, then $n_{\pm}(\eta') \geq i$ for all walls. In Corollary 5.16, the index j in (9) then satisfies $j > i$ and thus the big direct sums do not contribute to CH^i nor CH_i, as $i - j < 0$. Consequently $\mathrm{CH}^i(\mathcal{N}^\alpha)$ and $\mathrm{CH}_i(\mathcal{N}^\alpha)$ are both independent of α. Thus we can take α sufficiently small to apply Corollary 5.20, where $T = \mathfrak{h}(\mathcal{F})^{\otimes N}$ and $\mathcal{F}$ is the full flag variety, and express $\mathrm{CH}^i(\mathcal{N}^\alpha)$ and $\mathrm{CH}_i(\mathcal{N}^\alpha)$ in terms of Chow groups of $\mathcal{N}$. $\square$

5.3. Closed formulas for the motive in rank 2.
Let us consider rank $n = 2$ vector bundles of degree d with N parabolic points. By the shifting trick described in Remark 5.9, we can without loss of generality set the first parabolic weight at each parabolic point to be zero. Therefore, we assume $\alpha_{i,1} = 0$ for $i = 1, \ldots, N$ and let us simply write $\alpha_i := \alpha_{i,2}$; then we consider weights lying in the half-open hypercube

$$\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathcal{A}_N := [0, 1)^N.$$

Weights in the open hypercube $(0, 1)^N$ correspond to the multiplicity $m = (m_{i,j})$ with $m_{i,j} = 1$ for all i, j (i.e. full flags at each parabolic point). We fix invariants $\eta = (2, d, m)$ for rank 2 degree d parabolic vector bundles with full flags at each of the N parabolic points.

Unlike for the moduli space $\mathcal{N} = \mathcal{N}(2, d)$ of rank 2 vector bundles, where we had to restrict to d odd to describe the Chow motive (as for $g > 2$ and d even, $\mathcal{N}$ is singular), for parabolic vector bundles we can consider even degrees provided we choose a generic weight. In rank $n = 2$, we will compute the Chow motives of the associated parabolic moduli spaces for all d and all choices of generic weights. In fact, the moduli spaces of rank 2 parabolic vector bundles for odd and even degrees are related by Hecke modifications and so it suffices to compute these Chow motives for odd degrees (see Remark 5.35, for how to compute them in even degrees using a Hecke modification).

5.3.1. *Description of the walls.* We recall that the walls correspond to subbundles of the same slope. In rank 2, the invariants of a subbundle have the form $\eta' = (1, d', m')$, where for each $1 \leq i \leq N$, we have $m'_{i,1} + m'_{i,2} = 1$. In particular, $m'_{i,1}$ is determined by $m'_{i,2} \in \{0, 1\}$ and so we simply write $m'_i := m'_{i,2}$. The associated wall in $\mathcal{A}_N$ is given by the equation

$$2d' - d = \sum_{i=1}^{N} \alpha_i (1 - 2m'_i).$$

Note that the integer $2d' - d$ has the same parity as d. Since the complementary invariants $\eta'' = (n'' = 1, d'', m'')$ define the same wall, we can assume without loss of generality that $2d' - d \geq 0$. Furthermore, the multiplicity m' can be encoded in a subset $\mathcal{I} \subset \{1, \ldots, N\}$ given by $\mathcal{I} = \{1 \leq i \leq N : m'_i = 0\}$. Then the corresponding wall has the equation

$$0 \leq 2d' - d = \sum_{i \in \mathcal{I}} \alpha_i - \sum_{i \in \mathcal{I}^c} \alpha_i$$

where $\mathcal{I}^c := \{1, \ldots, N\} \setminus \mathcal{I}$. This wall has non-empty intersection with $\mathcal{A}_N$ precisely when $|\mathcal{I}| > 2d' - d$. This proves the following description of the walls.

Lemma 5.23. *The walls in $\mathcal{A}_N = [0, 1)^N$ can be enumerated as follows.*

(1) For odd d, for each $s \in \mathbb{N}$ with $2s + 1 < N$ and for each $\mathcal{I} \subset \{1, \ldots, N\}$ with cardinality $|\mathcal{I}| > 2s + 1$, we have the wall $W_{s, \mathcal{I}}$ defined by

$$2s + 1 = \sum_{i \in \mathcal{I}} \alpha_i - \sum_{i \in \mathcal{I}^c} \alpha_i.$$

This wall is determined by the discrete invariants $\eta'(s, \mathcal{I}) = (1, d'(s), m'(\mathcal{I}))$, where $d'(s) = (2s + 1 + d)/2$ and $m'(\mathcal{I})_{i,1} = 1$ (and $m'(\mathcal{I})_{i,2} = 0$) if and only if $i \in \mathcal{I}$.

(2) *For even d, for each $s \in \mathbb{N}$ with $2s < N$ and $\mathcal{I} \subset \{1, \ldots, N\}$ with cardinality $|\mathcal{I}| > 2s$, we have the wall $W_{k,\mathcal{I}}$ defined by*

$$2s = \sum_{i \in \mathcal{I}} \alpha_i - \sum_{i \in \mathcal{I}^c} \alpha_i.$$

This wall is determined by the discrete invariants $\eta'(s,\mathcal{I}) = (1, d'(s), m'(\mathcal{I}))$, where $d'(s) = (2s + d)/2$ and $m'(\mathcal{I})_{i,1} = 1$ (and $m'(\mathcal{I})_{i,2} = 0$) if and only if $i \in \mathcal{I}$.

Let us refer to a vertex of the closed hypercube $[0,1]^N$ as *even* (resp. *odd*) if it has an even (resp. odd) number of 1's. The only vertex which lies in $\mathcal{A}_N = [0,1)^N$ is the origin, which is even. For d odd (resp. even), we see that precisely the odd (resp. even) vertices lie on the walls. In particular, for d odd, the origin $\underline{0} \in \mathcal{A}_N$ does not lie on any wall, whereas for d even, the origin lies on the intersection of all the walls indexed by $s = 0$.

Let us focus on the odd degree case for now and later describe the even degree case by performing a Hecke modification (see Example 5.15 and Remark 5.35).

Proposition 5.24. *For odd d, consider the walls $W_{s,\mathcal{I}} \subset \mathcal{A}_N = [0,1)^N$ in Lemma 5.23.*

(1) *The centre of the hypercube $\frac{1}{2} \in \mathcal{A}_N$ lies on the wall $W_{s,\mathcal{I}}$ if and only if $|\mathcal{I}| = 2s + 1 + \frac{N}{2}$. In particular, for odd N, the centre does not lie on any wall.*

(2) *The centre of the hypercube $\frac{1}{2} \in \mathcal{A}_N$ lies on $W_{s,\mathcal{I}}$ if and only if this is a flopping wall.*

(3) *Crossing the wall $W_{s,\mathcal{I}}$ is a flip with centre $\mathrm{Jac}(C)^2$ and type $(n_-(s,\mathcal{I}), n_+(s,\mathcal{I}))$ where*

(10) $n_-(s,\mathcal{I}) = g + |\mathcal{I}| - 2s - 3 \quad and \quad n_+(s,\mathcal{I}) = g + N - |\mathcal{I}| + 2s - 1.$

In particular, the types $n_\pm(s,\mathcal{I})$ (i.e. the dimensions of the projective space appearing over the centre) satisfy $g - 1 \leq n_\pm(s,\mathcal{I}) \leq g + N - 3$.

(4) *For any wall $W_{s,\mathcal{I}}$ not containing $\frac{1}{2}$, crossing this wall in the direction from the centre of the hypercube is a flip which decreases the canonical divisor. More precisely, if $\beta, \gamma \in \mathcal{A}_N$ are weights in chambers on either side of $W_{k,\mathcal{I}}$ and we suppose that the line joining β and γ meets $W_{k,\mathcal{I}}$ in a point α which lies on no other wall, then if β denotes the weight nearest to $\frac{1}{2}$ we have a wall-crossing flip $\mathcal{N}^\beta >_K \mathcal{N}^\gamma$.*

Proof. The first statement is immediate from the explicit equations of the walls given in Lemma 5.23. For the third statement, we use Theorem 5.11 to describe the flip for crossing the wall $W_{s,\mathcal{I}}$: the centre of this flip is a product of two moduli spaces for rank 1 parabolic vector bundles, thus is a product of Jacobians, and the type of the flip is given by $n_\pm(s,\mathcal{I}) := n_\pm(\eta'(s,\mathcal{I}))$, which we can compute using (8) to give the formula in (10). For the final part of the third statement, since $2 \leq |\mathcal{I}| - 2s \leq N$, we obtain inequalities

$$g - 1 \leq n_\pm(s,\mathcal{I}) \leq g + N - 3.$$

Note that $n_-(s,\mathcal{I}) = n_+(s,\mathcal{I})$ if and only if $|\mathcal{I}| = 2s + 1 + \frac{N}{2}$, which proves the second statement.

The final statement will follow from the claim that for each wall $W_{s,\mathcal{I}}$, we have

$$4\left(\mu_{\frac{1}{2}}(\eta'(s,\mathcal{I})) - \mu_{\frac{1}{2}}(\eta)\right) = n_+(s,\mathcal{I}) - n_-(s,\mathcal{I}) = 2(2s + 1) + N - 2|\mathcal{I}|.$$

Indeed we have

$$\mu_{\frac{1}{2}}(\eta'(s,\mathcal{I})) - \mu_{\frac{1}{2}}(\eta) = \frac{2s + 1 + d}{2} + \sum_{i=1}^N \frac{1}{2} m'(\mathcal{I})_{i,2} - \frac{d + \sum_{i=1}^N \frac{1}{2} m_{i,2}}{2} = \frac{1}{4}\left(2(2s + 1) + N - 2|\mathcal{I}|\right).$$

To conclude the final statement, we then use Theorem 5.11: if $n_+(s,\mathcal{I}) - n_-(s,\mathcal{I}) > 0$, then $\beta = \alpha(+)$ and $\gamma = \alpha(-)$ and if $n_+(s,\mathcal{I}) - n_-(s,\mathcal{I}) < 0$, then $\beta = \alpha(-)$ and $\gamma = \alpha(+)$. In both cases, we obtain $\mathcal{N}^\beta >_K \mathcal{N}^\gamma$ and so crossing a wall away from the centre decreases K.   □

5.3.2. *Group actions on the space of weights.* In this section we exploit certain symmetries of the weight space $\mathcal{A}_N$ by introducing two group actions on $\mathcal{A}_N$ which enable us to significantly reduce the number of wall-crossings we must study. The first group action arises from Hecke modifications and the second group action arises from permutations of the weights.

By §5.1.5 (see Example 5.15), performing a Hecke modification at a subset $D' \subset D$ gives an isomorphism

$$\mathcal{H}_D : \mathcal{N}^\alpha(\eta) \cong \mathcal{N}^{\alpha(D')}(\eta(D'))$$

where $\eta(D') = (2, d - |D'|, m)$ and (recalling we set $\alpha_{i,1} = 0$ and $\alpha_i := \alpha_{i,2}$ for all i)

$$\alpha(D')_i = \begin{cases} \alpha_i & \text{if } p_i \notin D' \\ 1 - \alpha_i & \text{if } p_i \in D'. \end{cases}$$

If we perform a Hecke modification at a subset D' of even cardinality $|D'| = 2e$, then after tensoring by a line bundle of degree e, we obtain an isomorphism

$$\mathcal{N}^\alpha(\eta) \cong \mathcal{N}^{\alpha(D')}(\eta)$$

between moduli spaces for the same invariants η but different weights. This gives a natural action on the weight space $\mathcal{A}_N$ by Hecke modifications at an even number of points: this group is generated by Hecke modifications at the points (p_i, p_{i+1}) for $1 \le i \le N - 1$. Therefore, we consider the group $H \cong (\mathbb{Z}/2\mathbb{Z})^{N-1}$ acting on $\mathcal{A}_N$ by, for $1 \le i \le N - 1$, the involution

$$(\alpha_1, \ldots, \alpha_N) \mapsto (\alpha_1, \ldots, \alpha_{i-1}, 1 - \alpha_i, 1 - \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_N).$$

Note that any weights in the same H-orbit have isomorphic moduli spaces and the H-action sends walls to walls and chambers to chambers.

We also have the natural permutation action of the symmetric group S_N on $\mathcal{A}_N = [0, 1)^N$. Although, the corresponding moduli spaces for weights in the same S_N-orbit are not necessarily isomorphic, this action sends walls to walls and chambers to chambers, and preserves the numerical data of the wall-crossings in the following sense.

Lemma 5.25. *The action of $\sigma \in S_N$ on $\mathcal{A}_N$ sends $W_{s,\mathcal{I}}$ to $W_{s,\sigma(\mathcal{I})}$ and preserves the type of the wall-crossing, in the sense that $n_\pm(\eta'(s, \mathcal{I})) = n_\pm(\eta'(s, \sigma(\mathcal{I})))$.*

Proof. This follows as $n_\pm(\eta'(s, \mathcal{I}))$ in (10) only depend on g, s and the cardinality $|\mathcal{I}|$. □

We can utilise these actions to reduce the number of wall-crossings we must consider.

Proposition 5.26. *Let d be an odd degree. A set of representatives for the walls under the $H \rtimes S_N$-action is given by, for $0 \le l \le N/2 - 1$, the wall $W(l)$ defined by*

$$1 = -\sum_{i=1}^l \alpha_i + \sum_{i=l+1}^N \alpha_i$$

corresponding to the invariants $\eta(l) = (1, (d+1)/2, m(l))$ with $m(l)_{i,1} = 0$ for $i \le l$ and $m(k)_{i,1} = 1$ for $i > l$.

Proof. Let us start with a wall of the form

$$2s + 1 = \sum_{i \in \mathcal{I}} \alpha_i - \sum_{i \in \mathcal{I}^c} \alpha_i$$

of Lemma 5.23. If $s > 0$, then as $|\mathcal{I}| > 2s + 1$, we pick a subset $\mathcal{J} \subset \mathcal{I}$ of cardinality $2s$ and perform a Hecke modification at the set $E = \{p_j : j \in \mathcal{J}\} \subset D$. This wall transforms to

$$1 = \sum_{i \in \mathcal{K}} \alpha_i - \sum_{i \in \mathcal{K}^c} \alpha_i$$

with $\mathcal{K} = \mathcal{I} \setminus \mathcal{J}$ of cardinality $|\mathcal{K}| > 1$. If $|\mathcal{K}^c| \le \frac{N}{2} - 1$, then after a permutation we can put it in the above form. If $|\mathcal{K}^c| > \frac{N}{2} - 1$, then $|\mathcal{K}| < \frac{N}{2} + 1$. After performing a Hecke modification at a set $\mathcal{J} \subset \mathcal{K}$ of cardinality 2 and multiplying the equation by -1, we obtain a wall

$$1 = -\sum_{i \in \mathcal{K} \setminus \mathcal{J}} \alpha_i + \sum_{i \in \mathcal{K}^c \cup \mathcal{J}} \alpha_i$$

with $|\mathcal{K} \setminus \mathcal{J}| < \frac{N}{2} - 1$, and after a permutation we can put it in the above form. $\qquad \square$

Recall that each wall $W(l)$ separates $\mathbb{R}^N \setminus W(l)$ into two half-spaces $H(l)^{\pm}$ such that for $\alpha \in H(l)^{\pm}$, we have $\pm(\mu_\alpha(\eta(l)) - \mu_\alpha(\eta)) > 0$. For odd d, note that $\underline{0} \in H(l)^+$.

Proposition 5.27. *For an odd degree d and $0 \le l \le N/2 - 1$, crossing the wall $W(l)$ at a point on no other wall from $H(l)^+$ to $H(l)^-$ is a flip of type $(g + l - 1, g + N - 3 - l)$ with centre $\mathrm{Jac}(C)^2$. In particular, for $l < N/2 - 1$, this flip increases K and for N even and $l = N/2 - 1$, this wall-crossing is a flop and so preserves K.*

Proof. This follows from Theorem 5.11 and Equation (10) in Proposition 5.24. $\qquad \square$

We recall that in the wall and chamber decomposition of $\mathcal{A}_N$ for an odd degree d, only the odd vertices in $[0, 1]^N$ lie on the walls. Moreover the centre of the hypercube lies on a wall if and only if N is even. We make the following definition of minimal and maximal chambers, whose terminology will be justified in Proposition 5.29 below.

Definition 5.28. *For an odd degree d, a chamber in $\mathcal{A}_N^\circ = (0, 1)^N$ is said to be*

   (1) *minimal if it contains an even vertex in its closure,*
   (2) *maximal if it contains the centre of the hypercube in its closure.*

The minimal chambers are the chambers lying on the exterior of the hypercube. Note that if N is odd, then there is a unique maximal chamber containing the centre of the hypercube.

Proposition 5.29. *Consider the K-ordering on chambers in $\mathcal{A}_N$ given by*

$$C_1 >_K C_2 : \iff \mathcal{N}^{\alpha_1} >_K \mathcal{N}^{\alpha_2} \text{ for } \alpha_i \in C_i.$$

For an odd degree d, the K-minimal (resp. K-maximal) chambers are the minimal (resp. maximal) chambers defined above. Moreover, the moduli spaces $\mathcal{N}^\alpha$ for all weights α in minimal chambers are isomorphic and the moduli spaces $\mathcal{N}^\alpha$ for all weights α in maximal chambers are related by a sequence of flops.

Proof. The first part follows the last statement in Proposition 5.24. Since Hecke modifications can be used to identify all the minimal chambers, their corresponding moduli spaces are isomorphic. The maximal chambers are all related by crossing walls through the centre of the hypercubes, which are precisely the flopping walls by Proposition 5.24. $\qquad \square$

5.3.3. *Explicit formulas for the motive.*

Proposition 5.30. *Let $\alpha \in \mathcal{A}_N^\circ = (0, 1)^N$ be a generic weight and let $\mathfrak{h}(\mathcal{N}^\alpha)$ denote the integral Chow motive of the moduli space $\mathcal{N}^\alpha$ of α-semistable rank 2 and odd degree d parabolic bundles.*

   (1) *If $\alpha = \alpha_{\min}$ is in a minimal chamber, then there is an explicit isomorphism*

$$\mathfrak{h}(\mathcal{N}^{\alpha_{\min}}) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N}.$$

   (2) *If $\alpha = \alpha_{\max}$ is in a maximal chamber, then there is an explicit isomorphism*

$$\mathfrak{h}(\mathcal{N}^{\alpha_{\max}}) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \le s < M \\ 2s \le l < 2M}} \bigoplus_{j=l}^{N-3-l} \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g + j)^{\oplus \binom{N}{l-2s}}$$

      where $M := (N - 2)/4$.

   (3) *For arbitrary generic α, there is an explicit isomorphism*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g + j)^{\oplus b_j}$$

      such that the exponents $b_j \in \mathbb{N}$ can be inductively computed (see Theorem 5.32, for a closed formula). In particular, this Chow motive is a direct factor of $\mathfrak{h}(\mathcal{N}^{\alpha_{\max}})$ and contains $\mathfrak{h}(\mathcal{N}^{\alpha_{\min}})$ as a direct summand.

Proof. Let us first consider the minimal chambers. It suffices to consider a weight α in the minimal chamber containing the origin in its closure, as all other minimal chambers are related by Hecke modifications and so their corresponding moduli spaces are isomorphic (*cf.* Proposition 5.29). By successively degenerating the flag as in Theorem 5.12, we obtain a forgetful morphism

$$\mathcal{N}^\alpha \to \mathcal{N}$$

which is a $(\mathbb{P}^1)^N$-fibration, which gives the above description of $\mathfrak{h}(\mathcal{N}^{\alpha_{\min}})$ (*cf.* Corollary 5.20).

To describe the Chow motives of the moduli spaces for weights in maximal chambers, it suffices to consider one maximal chamber, as the moduli spaces $\mathcal{N}^\alpha$ for all weights α in maximal chambers are related by a sequence of flops (*cf.* Proposition 5.29). We will compute this using a sequence of flips from a minimal chamber to a maximal chamber. The line segment from the origin to the centre of the hypercube crosses the wall $W_{s,\mathcal{I}}$ (given in Lemma 5.23) if and only if $2s + 1 \le |\mathcal{I}| - \frac{N}{2}$, with equality if and only if the wall-crossing happens at the centre $\frac{1}{2}$ (i.e this is a flopping wall by Proposition 5.24). By replacing the centre of the hypercube by a small perturbation of the form $(\frac{1}{2} - \epsilon_1, \dots, \frac{1}{2} - \epsilon_N)$, we can ensure this line segment goes from a minimal chamber to a maximal chamber by crossing one wall at a time and does not cross the flopping walls. Hence this line segment crosses the walls $W_{s,\mathcal{I}}$ with $2s + 1 < |\mathcal{I}| - \frac{N}{2}$ (or equivalently $|\mathcal{I}^c| < \frac{N}{2} - (2s+1)$). Since $|\mathcal{I}| \le N$, it follows that $2s + 1 < \frac{N}{2}$. We can enumerate these walls as follows: for each $0 \le s < M := \frac{N-2}{4}$ and each $0 \le t < \frac{N}{2} - (2k+1)$ and each subsets $\mathcal{I}^c \subset \{1, \dots, n\}$ of cardinality t, we cross the wall $W_{s,\mathcal{I}}$. Each of these wall-crossings is a flip whose type is explicitly described in Proposition 5.24 and so by applying Corollary 5.16, we obtain the above formula. More precisely, any such wall $W_{s,\mathcal{I}}$ is transformed by Hecke modifications and permutations to a standard wall of the form $W(|\mathcal{I}^c| + 2s)$ in Proposition 5.26. Since $0 \le |\mathcal{I}^c| < \frac{N}{2} - (2s + 1)$, we have $2s \le |\mathcal{I}^c| + 2s < 2M$. Thus for $0 \le s < M$ and $2s \le l < 2M$ we have $\binom{N}{l-2s}$ wall-crossings of the type $W(l)$, which is a flip of type $(g - 1, g + N - 3 - l)$ with centre $\mathrm{Jac}(C)^2$ by Proposition 5.27.

For an arbitrary generic weight α, consider the ray from the centre of the hypercube through α to the edge of the hypercube. By perturbing α, we can assume this ray passes each wall one by one. By Proposition 5.24, we know that moving along this ray from the centre to the edge of the hypercube decreases K. Therefore, we consider the corresponding sequence of flips running backwards along this ray, starting at a weight $\alpha_{\min}$ in a minimal chamber (at the exterior of the cube) and passing through α and then ending at a weight $\alpha_{\max}$ in a maximal chamber. All of these wall-crossings are flips with centre $\mathrm{Jac}(C)^2$ and the types are specified by Proposition 5.24. Hence, by iterative applications of Corollary 5.16, one obtains the final statement. $\qquad\square$

We can more explicitly compute the exponents appearing above using a method inspired by Bauer's computation [8] of the Poincaré polynomials of moduli spaces of parabolic vector bundles on $\mathbb{P}^1$. In fact, for parabolic vector bundles on a curve C of positive genus, the wall-crossing picture and associated combinatorics is the same, except for the fact that rather than the inductive description starting with a chamber where the parabolic moduli space in $g = 0$ is empty, we start with a chamber where the parabolic moduli space is related to the moduli space $\mathcal{N} = \mathcal{N}_C(2, d)$ of semistable vector bundles on C.

Definition 5.31. For a weight α and $0 \le j \le N$, define

$$d_j(\alpha) = \#\left\{\mathcal{I} \subset \{1, \dots, N\} : |\mathcal{I}| \equiv j \mod 2, \text{ and } -1 + j < |\mathcal{I}| + \sum_{i \in \mathcal{I}^c} \alpha_i - \sum_{i \in \mathcal{I}} \alpha_i < 1 + j\right\}$$

and

$$b_j(\alpha) = \sum_{i=0}^{j} \left\lfloor \frac{i+2}{2} \right\rfloor c_{j-i}(\alpha), \text{ where } c_j(\alpha) = \binom{N}{j} - d_j(\alpha).$$

The relationship between the constants $b_j(\alpha)$ and $d_j(\alpha)$ is given by the following equality of polynomials in x

$$(11) \qquad (1-x)(1-x^2) \sum_{j=0}^{N-3} b_j(\alpha)x^j = (1+x)^N + \sum_{j=0}^{N} d_j(\alpha)x^j.$$

Theorem 5.32. *For a generic weight $\alpha \in \mathcal{A}_N^\circ$, the integral Chow motive of the moduli space of rank 2 and odd degree α-semistable parabolic vector bundles is given by*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g+j)^{\oplus b_j(\alpha)}$$

with the exponents $b_j(\alpha)$ given in Definition 5.31.

Proof. By inductively crossing walls, it suffices to check this formula holds for a particular chamber and then show that the formula respects wall-crossings. First consider $\alpha = (\epsilon_1, \ldots, \epsilon_N) \in (0,1)^N$ in the minimal chamber very close to the origin. Then

$$d_j(\alpha) = \# \left\{ \mathcal{I} \subset \{1, \ldots, N\} : |\mathcal{I}| = j \right\} = \binom{N}{j} \text{ and } c_j(\alpha) = b_j(\alpha) = 0,$$

which agrees with the formula for $\mathfrak{h}(\mathcal{N}^{\alpha_{\min}})$ in Proposition 5.30.

Now suppose we have two generic weights $\alpha_\pm$ which are separated by a single wall $W_{s,\mathcal{I}}$ defined by $\eta' := \eta'(s, \mathcal{I})$ as in Lemma 5.23. Suppose that $\alpha_\pm \in H_{s,\mathcal{I}}^\pm$; thus

$$\mu_{\alpha_-}(\eta') - \mu_{\alpha_-}(\eta) < 0 < \mu_{\alpha_+}(\eta') - \mu_{\alpha_+}(\eta)$$

or equivalently

$$2s + 1 + \sum_{i \in \mathcal{I}^c}(\alpha_-)_i - \sum_{i \in \mathcal{I}}(\alpha_-)_i < 0 < 2s + 1 + \sum_{i \in \mathcal{I}^c}(\alpha_+)_i - \sum_{i \in \mathcal{I}}(\alpha_+)_i.$$

This implies

$$\sum_{i \in \mathcal{I}^c}(\alpha_-)_i - \sum_{i \in \mathcal{I}}(\alpha_-)_i + |\mathcal{I}| < |\mathcal{I}| - 2s - 1 < \sum_{i \in \mathcal{I}^c}(\alpha_+)_i - \sum_{i \in \mathcal{I}}(\alpha_+)_i + |\mathcal{I}|$$

and also for $\mathcal{J} = \mathcal{I}^c$ we have

$$\sum_{i \in \mathcal{J}^c}(\alpha_+)_i - \sum_{i \in \mathcal{J}}(\alpha_+)_i + |\mathcal{J}| < |\mathcal{J}| + 2s + 1 = N - |\mathcal{I}| + 2s + 1 < \sum_{i \in \mathcal{J}^c}(\alpha_-)_i - \sum_{i \in \mathcal{J}}(\alpha_-)_i + |\mathcal{J}|.$$

Hence, the only differences between the values of $d_j(\alpha_-)$ and $d_j(\alpha_+)$ are as follows

$$d_{|\mathcal{I}|-2s-2}(\alpha_-) = d_{\mathcal{I}-2s-2}(\alpha_+) + 1 \qquad\qquad d_{|\mathcal{I}|-2s}(\alpha_+) = d_{\mathcal{I}-2s}(\alpha_-) + 1$$
$$d_{N-|\mathcal{I}|+2s}(\alpha_+) = d_{N-|\mathcal{I}|+2s}(\alpha_-) + 1 \qquad d_{N-|\mathcal{I}|+2s+2}(\alpha_-) = d_{N-|\mathcal{I}|+2s+2}(\alpha_+) + 1.$$

From (11), we obtain a corresponding relationship between the $b_j(\alpha_\pm)$

$$(12) \quad (1-x)(1-x^2) \sum_{j=0}^{N-3}(b_j(\alpha_+) - b_j(\alpha_-))x^j = x^{|\mathcal{I}|-2s} - x^{|\mathcal{I}|-2s-2} + x^{N-|\mathcal{I}|+2s} - x^{N-|\mathcal{I}|+2s+2}.$$

By Proposition 5.26, crossing the wall $W_{s,\mathcal{I}}$ from α_- to α_+ is a flip with centre $\mathrm{Jac}(C)^2$ of type $(n_-, n_+) = (g + |\mathcal{I}| - 2s - 3, g + N - |\mathcal{I}| + 2s - 1)$. By taking Poincaré polynomials of the corresponding equation in Corollary 5.16, we obtain

$$P_t(\mathcal{N}^{\alpha_+}) - P_t(\mathcal{N}^{\alpha_-}) = P_t(\mathrm{Jac}(C))^2 \left(P_t(\mathbb{P}^{n_-}) - P_t(\mathbb{P}^{n_+}) \right) = (1+t)^{4g} \frac{t^{2(n_++1)} - t^{2(n_-+1)}}{1-t^2}.$$

Thus to prove the claimed formula respects this wall-crossing, we need to show

$$(1-t^2) \sum_{j=0}^{N-3}(b_j(\alpha_+) - b_j(\alpha_-)t^{2j} = t^{2(N-|\mathcal{I}|+2s)} - t^{2(|\mathcal{I}|-2s-2)}.$$

After multiplying this by $(1-t^4)$, we see this holds by inserting $x = t^2$ in (12). $\qquad\square$

Corollary 5.33. *For a generic weight $\alpha \in \mathcal{A}_N^\circ$, the rational Chow motive of the moduli space $\mathcal{N}^\alpha$ of α-semistable parabolic vector bundles of rank 2 and odd degree d is given by*

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \left(\bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g+j)^{\oplus b_j(\alpha)} \right) \oplus \mathfrak{h}(\mathrm{Jac}(C)) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \otimes \mathfrak{h}(\mathrm{Sym}^{g-1}(C))(g-1)$$

$$\oplus \bigoplus_{i=0}^{g-2} \mathfrak{h}(\mathrm{Jac}(C)) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \otimes \mathfrak{h}(\mathrm{Sym}^i(C)) \otimes (\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i))$$

with the exponents $b_j(\alpha)$ given in Definition 5.31.

For $\mathcal{L} \in \mathrm{Pic}^d(C)$, we have the moduli space $\mathcal{N}_{\mathcal{L}}^\alpha$ of α-semistable rank 2 parabolic vector bundles with determinant $\mathcal{L}$. The wall-crossing picture remains the same except that the centre of each flip/flop between fixed determinant parabolic moduli spaces is only one copy of the Jacobian. Consequently, by the same argument we obtain the following formula.

Corollary 5.34. *For a generic weight $\alpha \in \mathcal{A}_N^\circ$, the integral Chow motive of the moduli space of α-semistable parabolic vector bundles of rank 2 and odd degree determinant $\mathcal{L}$ is given by*

$$\mathfrak{h}(\mathcal{N}_{\mathcal{L}}^\alpha) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} \mathfrak{h}(\mathrm{Jac}(C))(g+j)^{\oplus b_j(\alpha)}$$

with the exponents $b_j(\alpha)$ given in Definition 5.31.

Remark 5.35. For $n = 2$ and an even degree d, one can perform a Hecke modification at a single point p_i to obtain an isomorphism $\mathcal{N}^\alpha(2, d) \cong \mathcal{N}^{\alpha'}(2, d-1)$, where $\alpha' = \alpha(p_i)$ (see Example 5.15). Therefore, for $n = 2$, d even and α generic, we also obtain formulas for the Chow motive of $\mathcal{N}^\alpha(2, d)$.

Example 5.36. Let us explicitly compute some examples for low values of N.

(1) For $N = 2$, we have $\mathcal{A}_2^\circ = (0, 1)^2$ and the following picture shows the wall and chamber decomposition for d odd (on the left) and d even (on the right).


These pictures are related via the Hecke modification $(\alpha_1, \alpha_2) \mapsto (1 - \alpha_1, \alpha_2)$ at p_1. There is one wall (shown in red), which is a flopping wall, as it goes through the center of the hypercube. Thus for any generic α and any d, we have

$$\mathfrak{h}(\mathcal{N}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes 2}.$$

In this case the minimal and maximal chambers coincide.

(2) For $N = 3$, we have $\mathcal{A}_3^\circ = (0, 1)^3$. For d odd, there are 4 walls which cut out a tetrahedron inside this cube through the odd vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ as in the diagram below.
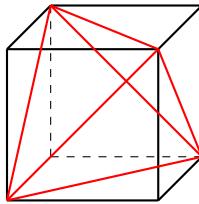
In this case the interior of the tetrahedron is the maximal chamber and there are four exterior chambers, all of which are minimal. The motives of the parabolic moduli spaces

for weights in the minimal and maximal chambers are given by

$$\mathfrak{h}(\mathcal{N}^{\alpha_{\min}}) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes 3}$$

$$\mathfrak{h}(\mathcal{N}^{\alpha_{\max}}) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes 3} \oplus \mathfrak{h}(\mathrm{Jac}(C))^{\otimes 2}(g),$$

where the second formula is obtained from the first formula by crossing the back wall $1 = \alpha_1 + \alpha_2 + \alpha_3$ of the tetrahedron, which is of type $(g-1, g)$ with centre $\mathrm{Jac}(C)^2$.

6. MOTIVES OF MODULI SPACES OF (PARABOLIC) HIGGS BUNDLES

6.1. **Moduli spaces of Higgs bundles.** Let $\omega_C := \Omega^1_{C/k}$ denote the canonical line bundle on C. A *Higgs bundle* on C is a pair (E, Φ) consisting of a vector bundle E and an $\mathcal{O}_C$-linear homomorphism $\Phi\colon E \to E \otimes \omega_C$ called the *Higgs field*. A *Higgs subbundle* of (E, Φ) is a subbundle of E that is invariant under the Higgs field. There is a notion of semistability for Higgs bundles which involves saying that the slope is increasing on Higgs subbundles. Let $\mathcal{M} = \mathcal{M}(n, d)$ denote the moduli space of semistable rank n degree d Higgs bundles on C. The moduli space $\mathcal{M}$ is a quasi-projective variety which contains the cotangent bundle to the moduli space $\mathcal{N} = \mathcal{N}(n, d)$ of semistable vector bundles as a dense open subvariety. Over $k = \mathbb{C}$, $\mathcal{M}$ is a non-compact hyper-Kähler manifold [37]. Moreover, $\mathcal{M}$ contains the moduli space $\mathcal{M}^s$ of (geometrically) stable Higgs bundles as a smooth open subset. In this section, we will assume that n and d are coprime, so semistability and stability coincide and $\mathcal{M} = \mathcal{M}^s$ is a smooth quasi-projective variety.

We can also fix $\mathcal{L} \in \mathrm{Pic}^d(C)(k)$ and consider the moduli space $\mathcal{M}_\mathcal{L} = \mathcal{M}_\mathcal{L}(n, d)$ of semistable rank n degree d Higgs bundles on C with fixed determinant $\mathcal{L}$, which is also quasi-projective and is smooth when n and d are coprime.

6.1.1. *Motives of moduli spaces of Higgs bundles.* Although the smooth quasi-projective moduli spaces $\mathcal{M}$ and $\mathcal{M}_\mathcal{L}$ are not proper, Hitchin [37] and Simpson [65] observed that there is a $\mathbb{G}_m$-action on both moduli spaces given by scaling the Higgs field such that the fixed locus is proper and the limit as $t \in \mathbb{G}_m$ tends to zero exists for all points (such a $\mathbb{G}_m$-action is referred to as being semi-projective, see [40, Definition A.1], and induces an associated Białynicki-Birula decomposition [12]). In particular, the flow under this action is a deformation retract and consequently the Voevodsky motives of $\mathcal{M}$ and $\mathcal{M}_\mathcal{L}$ are pure (see [40, Corollary 6.9] for the case of $\mathcal{M}$) and we can consider them as Chow motives via the embedding of Theorem 2.4. With a slight abuse of notation, we will denote the corresponding Chow motives by $\mathfrak{h}(\mathcal{M})$ and $\mathfrak{h}(\mathcal{M}_\mathcal{L})$. By [40, Thereorem 1.1], the rational Chow motive $\mathfrak{h}(\mathcal{M})$ is contained in the subcategory of $\mathrm{CHM}(k, \mathbb{Q})$ generated by $\mathfrak{h}(C)$ and is a direct summand of the motive of a sufficiently large power of C.

Unlike in the case of the moduli spaces $\mathcal{N}$ and $\mathcal{N}_\mathcal{L}$ of semistable vector bundles without and with fixed determinant whose motives are related by Theorem 4.2, for the Higgs moduli spaces $\mathcal{M}$ and $\mathcal{M}_\mathcal{L}$ we have

$$\mathfrak{h}(\mathcal{M}) \not\simeq \mathfrak{h}(\mathcal{M}_\mathcal{L}) \otimes \mathfrak{h}(\mathrm{Jac}(C)).$$

Indeed, already for rank $n = 2$, Hitchin shows this over $k = \mathbb{C}$ on the level of singular cohomology [37] (similar phenomenon was also observed in rank 3 by Gothen [35]). We will describe the motives of $\mathcal{M}$ and $\mathcal{M}_\mathcal{L}$ for rank $n = 2$ below and show in Proposition 6.3 that the motive of $\mathcal{M}_\mathcal{L}$ is in general *not* contained in the tensor subcategory of $\mathrm{CHM}(k, \mathbb{Q})$ generated by $\mathfrak{h}(C)$.

6.1.2. *Formulas for the motive of the rank 2 Higgs moduli spaces.* In this section, we consider rank 2 Higgs bundles of odd degree d and let $\mathcal{M} = \mathcal{M}(2, d)$ and $\mathcal{M}_\mathcal{L} = \mathcal{M}_\mathcal{L}(2, d)$ denote the moduli spaces of Higgs bundles without and with fixed determinant respectively.

Theorem 6.1. *For an odd integer d, the integral Chow motive of the moduli space of semistable rank 2 degree d Higgs bundles on C is given by an explicit isomorphism*

$$\mathfrak{h}(\mathcal{M}(2, d)) \simeq \mathfrak{h}(\mathcal{N}(2, d)) \oplus \bigoplus_{j=1}^{g-1} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j-1}(C))(3g - 2j - 2),$$

where $a_{d,j} = g - j + (d-1)/2$. When working with rational coefficients, we have a non-explicit isomorphism

$$\mathfrak{h}(\mathcal{M}(2,d)) \simeq \mathfrak{h}(\mathrm{Jac}(C)) \otimes \left(\mathfrak{h}(\mathrm{Sym}^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} \mathfrak{h}(\mathrm{Sym}^i(C)) \otimes (\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i)) \right)$$

$$\oplus \bigoplus_{j=1}^{g-1} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j-1}(C))(3g-2j-2).$$

Proof. We prove the above formula using the motivic Białynicki-Birula decomposition associated to the $\mathbb{G}_m$-action on $\mathcal{M}(2,d)$ given by $t \cdot (E, \Phi) = (E, t\Phi)$. For this, we first need to describe the fixed locus and the associated geometric Białynicki-Birula decomposition, which in rank 2 was studied by Hitchin [37]. We claim that

$$\mathcal{M}(2,d)^{\mathbb{G}_m} \simeq \mathcal{N}(2,1) \sqcup \bigsqcup_{j=1}^{g-1} \mathrm{Sym}^{2j-1}(C) \times \mathrm{Pic}^{a_{d,j}}(C).$$

Indeed if (E, Φ) is a $\mathbb{G}_m$-fixed point, then either $\Phi = 0$ and E is a semistable vector bundle, or $\Phi \neq 0$ and so we have $\mathbb{G}_m \subset \mathrm{Aut}(E)$ giving a weight space decomposition $E = L_0 \oplus L_1$ as a sum of two line bundles such that Φ is given by a homomorphism $L_0 \to L_1 \otimes \omega_C$. In this latter case, let $(e, d-e)$ denote the degrees of (L_0, L_1); then since the Higgs field gives a non-zero section of $L_0^\vee \otimes L_1 \otimes \omega_C$, we have $d - 2e + 2g - 2 \geq 0$. Furthermore, stability of the Higgs bundle means that the Higgs subbundle $L_1 \subset (E, \Phi)$ has slope less than that of E and so we must have $d < 2e$. Therefore, we have $(d+1)/2 \leq e \leq g - 1 + (d-1)/2$. For each e in this range, the Higgs bundle is determined by a degree e line bundle L_0 and an effective divisor of degree $d - 2e + 2g - 2$. After setting $j = g - e + (d-1)/2$, we get the fixed locus components $F_j := \mathrm{Sym}^{2j-1}(C) \times \mathrm{Pic}^{a_{d,j}}(C)$ for $1 \leq j \leq g-1$.

For $1 \leq j \leq g-1$, let $\mathcal{M}(2,d)_j^+$ denote the locus of points whose flow as $t \to 0$ lies in F_j. The rank of the fibration $\mathcal{M}(2,1)_j^+ \to F_j$ is $\frac{1}{2} \dim \mathcal{M}(2,1) = \dim \mathcal{N}(2,1)$, as the downward flow is Lagrangian; thus the codimension c_j^+ of $\mathcal{M}(2,1)_j^+$ is given by $c_j^+ = \dim \mathcal{N}(2,1) - (2j-1+g)$ (see [40, Proposition 2.2] and Equation (2) in *loc. cit.* as well as the references therein). The result then follows from the motivic Białynicki-Birula decomposition (see [40, Theorem A.4], where although the isomorphism is given in Voevodsky's triangulated category, all motives appearing are pure and so we can interpret this in the category of Chow motives) and Theorem 4.3. $\square$

If C admits a degree 1 line bundle, then we have $\mathfrak{h}(\mathrm{Pic}^i(C)) \simeq \mathfrak{h}(\mathrm{Jac}(C))$ for all i, and in Theorem 6.1, we obtain the same formula for all d. One can replace the Chow motives of $\mathrm{Sym}^n(C)$ for $g \leq n \leq 2g-2$ appearing in the formulas in Theorem 6.1 with lower symmetric powers of C and Tate twists of $\mathfrak{h}(\mathrm{Jac}(C))$ using [47, Corollary 5.1].

Similarly to above, one can use the motivic Białynicki-Birula decomposition for the moduli space $\mathcal{M}_{\mathcal{L}}(2,d)$ of Higgs bundles with fixed determinant.

Proposition 6.2. *For d odd and $\mathcal{L} \in \mathrm{Pic}^d(C)$, the integral Chow motive of the moduli space of semistable rank 2 Higgs bundles with determinant $\mathcal{L}$ on C is given by an explicit isomorphism*

$$(13) \qquad \mathfrak{h}(\mathcal{M}_{\mathcal{L}}(2,d)) \simeq \mathfrak{h}(\mathcal{N}_{\mathcal{L}}(2,d)) \oplus \bigoplus_{j=1}^{g-1} \mathfrak{h}(\widetilde{\mathrm{Sym}}^{2j-1}(C))(3g-2j-2)$$

where $\widetilde{\mathrm{Sym}}^{2j-1}(C) \to \mathrm{Sym}^{2j-1}(C)$ is the degree 2^{2g} étale cover given by the base change of the multiplication-by-2 map on $\mathrm{Jac}(C)$.

Proof. The proof is almost the same as in the case without fixed determinant, except now the fixed locus for the $\mathbb{G}_m$-action is

$$\mathcal{M}_{\mathcal{L}}(2,d)^{\mathbb{G}_m} \simeq \mathcal{N}_{\mathcal{L}}(2,d) \sqcup \bigsqcup_{j=1}^{g-1} \widetilde{\mathrm{Sym}}^{2j-1}(C),$$

as the determinant is fixed (see [37]). $\qquad\qquad\square$

By combining this with Theorem 4.3, one obtains a non-explicit isomorphism describing the rational Chow motive of $\mathcal{M}_{\mathcal{L}}(2, d)$ with d odd.

The following result contrasts with Proposition 4.1.

Proposition 6.3. *The rational Chow motive of $\mathcal{M}_{\mathcal{L}}(2, d)$ lies in the tensor subcategory of* $\mathrm{CHM}(k, \mathbb{Q})$ *generated by the motive of the 2^{2g}-cover $\pi\colon \widetilde{C} \to C$ given as the base change of the multiplication-by-2 map on $\mathrm{Jac}(C)$. Furthermore, over $k = \mathbb{C}$, we have:*

(i) $\mathfrak{h}(\mathcal{M}_{\mathcal{L}}(2, d))$ *lies in the tensor subcategory of* $\mathrm{CHM}(k, \mathbb{Q})$ *generated by the motives of the étale double covers of C (or equivalently, C and the Prym varieties associated to the étale double covers of C).*

(ii) *For a general curve C of genus ≥ 2, $\mathfrak{h}(\mathcal{M}_{\mathcal{L}}(2, d))$ does* not *belong to the tensor subcategory of* $\mathrm{CHM}(k, \mathbb{Q})$ *generated by the motive of C.*

Proof. For any $k \in \mathbb{N}$, there is a surjective morphism $\mathrm{Sym}^k(\widetilde{C}) \to \widetilde{\mathrm{Sym}}^k(C)$, thus $\mathfrak{h}(\widetilde{\mathrm{Sym}}^k(C))$ is a summand of $\mathfrak{h}(\mathrm{Sym}^k(\widetilde{C})) \simeq \mathrm{Sym}^k \mathfrak{h}(\widetilde{C})$. Since also $\mathfrak{h}_1(C)$ is a direct factor of $\mathfrak{h}_1(\widetilde{C})$, we deduce the first statement from Propositions 4.1 and 6.2.

Let us now assume that $k = \mathbb{C}$. We thank Salvatore Floccari and Zhi Jiang for their kind help on the proof. Étale double covers (not necessarily connected) of C (or $\mathrm{Jac}(C)$) are in bijection with the following abelian group

$$T := \mathrm{Hom}(H_1(C, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = H^1(C, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}.$$

For any $t \neq 0 \in T$, we denote by $\pi_t\colon C'_t \to C$ the corresponding étale double cover and denote by $P_t := \mathrm{Prym}(C'_t/C)$ the associated $(g-1)$-dimensional Prym variety. By convention, $C'_0 = C \coprod C$ and $P_0 = \mathrm{Jac}(C)$. Note that the rank 2 variation of Hodge structures $\pi_{t,*}\mathbb{Q}_{C'_t}$ splits up as $\mathbb{Q}_C \oplus L_t$. In particular, there is an isomorphism of Hodge structures $H^1(P_t, \mathbb{Q}) \simeq H^1(C, L_t)$.

Since $\pi\colon \widetilde{C} \to C$ is an abelian cover, the variation of Hodge structures $\pi_*\mathbb{Q}_{\widetilde{C}}$ splits into a direct sum of rank one variations of Hodge structures:

$$\pi_*\mathbb{Q}_{\widetilde{C}} \simeq \bigoplus_{t \in T} L_t,$$

where L_t is the local system on C corresponding to the double cover C'_t/C. Therefore, we have isomorphisms of rational Hodge structures:

$$H^1(\widetilde{C}, \mathbb{Q}) \simeq H^1(C, \pi_*\mathbb{Q}_{\widetilde{C}}) \simeq \bigoplus_{t \in T} H^1(C, L_t) \simeq \bigoplus_{t \in T} H^1(P_t, \mathbb{Q}).$$

Therefore we have an isogeny

$$(14) \qquad\qquad \mathrm{Jac}(\widetilde{C}) \simeq_{\mathbb{Q}} \prod_{t \in T} P_t$$

Consequently, the tensor subcategory of $\mathrm{CHM}(k, \mathbb{Q})$ generated by $\mathfrak{h}(\widetilde{C})$ is the same as the tensor subcategory generated by the motives of C and all Prym varieties P_t for $t \in T$. Let us denote this subcategory by $\langle \mathfrak{h}(P_t), t \in T \rangle$. This is also equivalent to the category generated by the motives of the double covers C'_t of C. Since $\mathfrak{h}(\mathrm{Sym}^j(\widetilde{C})) \in \langle \mathfrak{h}(P_t), t \in T \rangle$ for all $j \in \mathbb{N}$ and $\mathfrak{h}(\mathcal{N}_{\mathcal{L}}(2, d)) \in \langle \mathfrak{h}(C) \rangle$ (Proposition 4.1), we deduce statement (i) from Equation (13).

To prove (ii), it suffices to show that for some $t \in T \backslash \{0\}$, the Hodge structure $H^1(P_t, \mathbb{Q})$ does not belong to the tensor subcategory generated by $H^1(C, \mathbb{Q})$. Assuming the contrary, by Tannakian duality, $H^1(P_t, \mathbb{Q})$ is a $(2g-2)$-dimensional representation of the Hodge group of C, which is the symplectic group $\mathrm{Sp}(H^1(C, \mathbb{Q})) \simeq \mathrm{Sp}_{2g}$ when C is general. But this is absurd since a nontrivial representation of the symplectic group Sp_{2g} is of dimension at least $2g$. $\qquad\square$

Remark 6.4. We think that part (i) of this proposition also holds over field $k \neq \mathbb{C}$, as we suspect the isogeny (14) holds in greater generality.

6.2. **Motives of moduli spaces of parabolic Higgs bundles.** Let C be a smooth projective curve of genus g. Let $p_1, \ldots, p_N$ be N distinct k-rational points on C and denote $D = p_1 + \cdots + p_N$. Assume that $2g - 2 + N \geq 0$, i.e. $\omega_C(D)$ is nef.

Definition 6.5. A (quasi) *parabolic Higgs bundle* on (C, D) is a pair (E_*, Φ) consisting of

- a (quasi) parabolic vector bundle E_* (see Definition 5.1) with full flag-type[10] at each marked point p_i, $1 \leq i \leq N$;
- an $\mathcal{O}_C$-linear homomorphism $\Phi \colon E \to E \otimes \omega_C(D)$, called a *(strongly parabolic) Higgs field*, satisfying that $\Phi(E_{i,j}) \subset E_{i,j+1} \otimes \omega_C(D)$, for any $1 \leq i \leq N$ and any $1 \leq j \leq \mathrm{rk}(E)$.

The notion of (semi-)stablity is defined similarly as in Definition 5.8 using α-slopes, except that only parabolic Higgs subbundles need to be considered, i.e. those parabolic subbundles F of E_* that are Φ-invariant: $\Phi(F) \subset F \otimes \omega_C(D)$.

For $n \in \mathbb{N}^*, d \in \mathbb{Z}$ and a weight α, Yokogawa [73, 72] constructed the moduli space of α-semistable parabolic Higgs bundles rank n degree d, which we denote by $\mathcal{M}^\alpha(n, d)$. The (geometrically) stable locus $\mathcal{M}^{\alpha-s}(n, d)$ forms an open subset.

Similarly to the case of parabolic bundles discussed in §5, there is a wall and chamber structure in the weight space, such that the corresponding moduli space of stable parabolic Higgs bundles stays the same when varying the weight within a chamber, and undergoes a birational transform when crossing a wall. However, Boden–Yokogawa [17] observed that moduli spaces of stable parabolic Higgs bundles have the same Betti numbers when crossing a wall, and they conjectured that the diffeomorphism type should also be preserved, which was proved by Nakajima [60] shortly after. Subsequently, Thaddeus [68] gave a more precise geometric picture: when crossing a wall in the weight space, the moduli space of stable parabolic Higgs bundles undergoes a very special birational transform, namely, a *Mukai flop*; see §3.2, for the precise definition.

For a generic weight α, the moduli space $\mathcal{M}^\alpha(n, d) = \mathcal{M}^{\alpha-s}(n, d)$ is a smooth quasi-projective (and over $k = \mathbb{C}$, hyper-Kähler) variety [60], but in general it is non-proper. Neverthless, its Voevodsky motive is pure.

Lemma 6.6. *For α generic (i.e. not on any wall), the Voevodsky motive of $\mathcal{M}^\alpha(n, d)$ lies in the subcategory of Chow motives.*

Proof. The argument is almost identical to the result for (non-parabolic) Higgs moduli spaces when n and d are coprime [40, Corollary 6.9] and so we simply sketch the details. In the parabolic setting, there is also a $\mathbb{G}_m$-action on parabolic Higgs moduli space $\mathcal{M}^\alpha(n, d)$ given by scaling the parabolic Higgs field. For generic α, the parabolic Higgs moduli space $\mathcal{M}^\alpha(n, d)$ is a smooth quasi-projective variety. In this case, the $\mathbb{G}_m$-action on the smooth variety $\mathcal{M}^\alpha(n, d)$ is semi-projective (see [40, Definition A.1]) in the sense that the fixed locus is proper and the limit as $t \in \mathbb{G}_m$ tends to zero exists for all points. Indeed the $\mathbb{G}_m$-fixed points and flow are described in [64, Theorem 8]: the $\mathbb{G}_m$-fixed loci are moduli spaces of chains of parabolic vector bundles for appropriate stability parameters, which are projective varieties by their GIT constructions. In particular, there is an associated Białynicki-Birula decomposition [12] of $\mathcal{M}^\alpha(n, d)$, and as the flow under this $\mathbb{G}_m$-action is a deformation retract and the fixed loci are smooth projective varieties, the Voevodsky motive of $\mathcal{M}^\alpha(n, d)$ is pure (see [40, Appendix A]). $\square$

As a consequence of Theorem 3.6 combined with Thaddeus' aforementioned result [68, 6.2], we obtain the following result.

Corollary 6.7. *Fix (C, D) and n and d. Then for a generic weight α, the integral Chow motive of the moduli space $\mathcal{M}^\alpha_{C,D}(n, d)$ of α-semistable parabolic Higgs bundles of rank n and degree d is independent of α.*

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[10]One can also consider non-full flags, but we make this simplifying assumption so that all walls are good (see §5.1.3) and so all wall-crossings can be explicitly described.

6.2.1. *Closed formula for motives of rank* 2 *parabolic Higgs bundles.* In this section we consider moduli spaces parabolic Higgs bundles of rank $n = 2$ and odd degree d with full flags at N points $p_1, \ldots p_N$. For a generic weight α, we compute the Chow motive of $\mathcal{M}^\alpha = \mathcal{M}^\alpha(2, d)$.

Theorem 6.8. *For a generic weight α, we have an explicit isomorphism of integral Chow motives*

$$\mathfrak{h}(\mathcal{M}^\alpha) \simeq \mathfrak{h}(\mathcal{N}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \leq l \leq N \\ \frac{l+1-N}{2} \leq j \leq g-1}} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j+N-l-1}(C))(3g-2j+l-2)^{\oplus\binom{N}{l}}$$

where $a_{d,j} := g - j + (d-1)/2$ and $\mathcal{N} = \mathcal{N}(2, d)$ is the moduli space of semistable vector bundles.

Proof. We use the $\mathbb{G}_m$-action on $\mathcal{M}^\alpha$ given by $t \cdot (E_*, \Phi) = (E_*, t\Phi)$ and its associated motivic Białynicki-Birula decomposition to prove the above formula. Since $\mathfrak{h}(\mathcal{M}^\alpha)$ is independent of α by Corollary 6.7, we will take a particular choice of α.

By performing a linear shift, we can assume that α has the form $(\alpha_{i,1}, \alpha_{i,2}) = (0, \alpha_i)$ with $\alpha_i > 0$. The fixed points of this $\mathbb{G}_m$-action in rank 2 for trivial parabolic degree and fixed determinant is described in [17] and the fixed locus in our setting is a minor modification of this. If (E_*, Φ) is a $\mathbb{G}_m$-fixed point, then either $\Phi = 0$ and E_* is an α-semistable parabolic vector bundle or $\Phi \neq 0$ and so we have $\mathbb{G}_m \subset \mathrm{Aut}(E)$ giving a weight space decomposition $E = L \oplus M$ as a sum of two line bundles such that Φ is given by a non-zero strongly parabolic homomorphism $\Phi : L \to M \otimes \omega_C$. The parabolic structure and weights on L and M are induced by that of E_* as follows. Let $d' = \deg(L)$ and m' denote the multiplicity of $L \subset E_*$ given by $m'_i = \dim(L_{p_i} \cap E_{i,2}) \in \{0, 1\}$. Then the weight of the induced flags in L_{p_i} is α_i if $m'_i = 1$, and 0 if $m'_i = 0$. Hence $\Phi : L \to M \otimes \omega_C$ being strongly parabolic means $\Phi(L_{p_i}) = 0$ for all i with $m'_i = 1$. Equivalently this means that Φ factors as

$$\Phi : L \to M \otimes \omega_C(D - \sum_{i=1}^N m'_i p_i) \hookrightarrow M \otimes \omega_C(D).$$

In this case $(E_* = L \oplus M, \Phi)$ is specified by $L \in \mathrm{Pic}^{d'}(C)$ and $L \to M \otimes \omega_C(D - \sum_{i=1}^N m'_i p_i)$, which corresponds to an effective divisor of degree $d - 2d' + 2g - 2 + N - |m'| \geq 0$, where $|m'| := \sum_{i=1}^N m'_i$. Since (E_*, Φ) is α-stable, the parabolic Higgs subbundle $M_* \subset E_*$ satisfies $\mu_\alpha(M_*) < \mu_\alpha(E_*)$, or equivalently $\mu_\alpha(L_*) > \mu_\alpha(E_*)$, which gives

$$(15) \qquad\qquad 2d' > d + \sum_{i=1}^N \alpha_i(1 - 2m'_i)$$

Therefore the $\mathbb{G}_m$-fixed locus with non-zero Higgs field is indexed by tuples (m', d') such that $d + \sum_{i=1}^N \alpha_i(1 - 2m'_i) < 2d' \leq 2g - 2 + N + d - |m'|$ and the corresponding fixed locus is $\mathrm{Pic}^{d'}(C) \times \mathrm{Sym}^{2g+d-2d'+N-l-2}(C)$.

Now let us pick α_i very small, so that the moduli space $\mathcal{N}^\alpha$ of α-stable parabolic vector bundles is a $(\mathbb{P}^1)^N$-bundle over the moduli space $\mathcal{N}$ of stable vector bundles (*cf.* Corollary 5.20) and so (15) is equivalent to $2d' \geq d + 1$ for all possible m'. If we set $j = g - d' + (d-1)/2$, then for this α, we have fixed set

$$(\mathcal{M}^\alpha)^{\mathbb{G}_m} = \mathcal{N}^\alpha \sqcup \bigsqcup_{\substack{m' \in \{0,1\}^N \\ \frac{l+1-N}{2} \leq j \leq g-1}} \mathrm{Pic}^{a_{d,j}}(C) \times \mathrm{Sym}^{2j+N-l-1}(C).$$

and the codimension of the Białynicki-Birula stratum indexed by (m', j) is

$$c^+_{m',j} = \dim \mathcal{N}^\alpha - (2j + N - l - 1 + g) = 3g - 2j + l - 2$$

since the downward flow is Lagrangian (as in the proof of Theorem 6.1). For $0 \leq l \leq N$, there are $\binom{N}{l}$ multiplicities m' with $l = \sum_{i=1}^N m'_i$ and so the motivic Białynicki-Birula decomposition

(see [40, Theorem A.4], again interpreted in the category of Chow motives) gives

$$\mathfrak{h}(\mathcal{M}^\alpha) \simeq \mathfrak{h}(\mathcal{N}^\alpha) \oplus \bigoplus_{\substack{0 \leq l \leq N \\ \frac{l+1-N}{2} \leq j \leq g-1}} \mathfrak{h}(\mathrm{Pic}^{a_{d,j}}(C)) \otimes \mathfrak{h}(\mathrm{Sym}^{2j+N-l-1}(C))(3g-2j+l-2)^{\oplus\binom{N}{l}}.$$

To conclude, we use the fact that $\mathcal{N}^\alpha \to \mathcal{N}$ is a $(\mathbb{P}^1)^N$-bundle (*cf.* Corollary 5.20). □

By combining Theorem 6.8 with the formula for the rational Chow motive of $\mathcal{N}(2,d)$ in Theorem 4.3, we obtain a formula for the rational Chow motive of $\mathcal{M}^\alpha(2,d)$ in terms of sums and tensor products of Tate twists of motives of $\mathrm{Pic}^i(C)$ and $\mathrm{Sym}^j(C)$.

For even degree d and generic α, we can also compute the Chow motive of $\mathcal{M}^\alpha(2,d)$ using a Białynicki-Birula decomposition, where the fixed locus $\mathcal{N}^\alpha(2,d)$ is isomorphic to $\mathcal{N}^{\alpha'}(2,d-1)$ by a Hecke modification at a single parabolic point, where $\alpha' := \alpha(p_i)$ (see Example 5.15 and Remark 5.35).

We can also obtain a formula for the Chow motives of moduli space $\mathcal{M}_\mathcal{L}^\alpha(2,d)$ of parabolic Higgs bundles with fixed determinant $\mathcal{L}$. Indeed these are also invariant of α (for α generic), as the Mukai flops for $\mathcal{M}^\alpha(2,d)$ restrict to $\mathcal{M}_\mathcal{L}^\alpha(2,d)$, where the centres are pullbacks of symmetric powers of C under the multiplication-by-2 map on $\mathrm{Jac}(C)$. The proof of the following formula is essentially the same as Theorem 6.8 using a modification as in Proposition 6.2.

Proposition 6.9. *For a generic weight α, the Chow motive of moduli space $\mathcal{M}_\mathcal{L}^\alpha(2,d)$ of parabolic Higgs bundles with fixed determinant $\mathcal{L}$ of odd degree is given by an explicit isomorphism*

$$\mathfrak{h}(\mathcal{M}_\mathcal{L}^\alpha) \simeq \mathfrak{h}(\mathcal{N}_\mathcal{L}) \otimes \mathfrak{h}(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \leq l \leq N \\ \frac{l+1-N}{2} \leq j \leq g-1}} \mathfrak{h}(\widetilde{\mathrm{Sym}}^{2j+N-l-1}(C))(3g-2j+l-2)^{\oplus\binom{N}{l}}$$

where $\widetilde{\mathrm{Sym}}^i(C) \to \mathrm{Sym}^i(C)$ is the degree 2^{2g} étale cover given as the base change of the multiplication-by-2 map on $\mathrm{Jac}(C)$.

APPENDIX A. A LOCAL-TO-GLOBAL TRICK

We present a local-to-global trick employed in [55] and [29], which sometimes allows one to reduce the problem of computing the change of motives or Chow groups under a birational transform to the same problem for a local model.

Let $i \colon Z \hookrightarrow X$ be a closed immersion between smooth varieties. Let $\tau \colon \widetilde{X} \to X$ be the blow-up along Z and $E = \mathbb{P}(N_{Z/X})$ the exceptional divisor. Define $X_{\mathrm{loc}} := \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}_Z)$, the compactification of the total space of the vector bundle $N_{Z/X}$, with the infinite part E. We think of the inclusion by zero section $Z \hookrightarrow X_{\mathrm{loc}}$ as the local projective model of i. We summarise the situation in the following diagram.

(16)
$$\begin{array}{ccccc}
X_{\mathrm{loc}} & \xleftarrow{\;\iota\;} & E & \xhookrightarrow{\;j\;} & \widetilde{X} \\
& \searrow{\scriptstyle\pi} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle\tau} \\
& & Z & \xhookrightarrow{\;i\;} & X
\end{array}$$

Proposition A.1. *With the above notation, we have the following isomorphism*

(17) $\quad (\tau^*, \pi^* i^*) \colon \mathrm{CH}^k(X) \xrightarrow{\;\simeq\;} \dfrac{\ker\left(\mathrm{CH}^k(\widetilde{X}) \oplus \mathrm{CH}^k(X_{\mathrm{loc}}) \xrightarrow{(j^*,-\iota^*)} \mathrm{CH}^k(E)\right)}{\mathrm{Im}\left(\mathrm{CH}^{k-1}(E) \xrightarrow{(j_*,-\iota_*)} \mathrm{CH}^k(\widetilde{X}) \oplus \mathrm{CH}^k(X_{\mathrm{loc}})\right)},$

whose inverse is given by $(\tau_*, i_* \pi_*)$.

Proof. The statement and the argument are essentially contained in [55, §4]. The surjectivity of $(j^*, -\iota^*)$ follows from the surjectivity of ι^*, and the injectivity of ι_* implies the injectivity of $(j_*, -\iota_*)$.

Denote by $\xi = c_1(\mathcal{O}_p(1)) \in \mathrm{CH}^1(E)$. Since $N_{E/\widetilde{X}} \simeq \mathcal{O}_p(-1)$, the composition $j^* j_*$ is multiplication by $-\xi$. Since $N_{E/X_{\mathrm{loc}}} \simeq \mathcal{O}_p(1)$, the composition $\iota^* \iota_*$ is multiplication by ξ. Therefore, $(j^*, -\iota^*) \circ (j_*, -\iota_*) = 0$, i.e. $\mathrm{Im}(j_*, -\iota_*) \subset \ker(j^*, -\iota^*)$, and so the right-hand side of (17) makes sense.

To see that $(\tau^*, \pi^* i^*)$ is well-defined: $(j^*, -\iota^*) \circ (\tau^*, \pi^* i^*) = j^* \tau^* - \iota^* \pi^* i^* = 0$, by the commutativity of Diagram (16). Similarly, one checks that $(\tau_*, i_* \pi_*)$ is well-defined.

The composition $(\tau_*, i_* \pi_*) \circ (\tau^*, \pi^* i^*) = \tau_* \tau^* + i_* \pi_* \pi^* i^* = \mathrm{id}$, thanks to the projection formula. It remains to show that $(\tau_*, i_* \pi_*)$ is injective. More explicitly, for any $a \in \mathrm{CH}^l(\widetilde{X})$ and $b \in \mathrm{CH}^l(X_{\mathrm{loc}})$ satisfying

$$(18) \qquad\qquad\qquad\qquad j^*(a) = \iota^*(b),$$

$$(19) \qquad\qquad\qquad\qquad \tau_*(a) = -i_* \pi_*(b),$$

we want to show the existence of $\gamma \in \mathrm{CH}^{l-1}(E)$, such that $a = j_*(\gamma)$ and $b = -\iota_*(\gamma)$.

To this end, by abuse of notation, let ξ denote both $c_1(\mathcal{O}_\pi(1))$ and $c_1(\mathcal{O}_p(1))$ (note that the former does restrict to the latter). By the blow-up formula and the projective bundle formula, we can write

$$(20) \qquad\qquad a = \tau^*(a_0) + j_*(p^*(a_1) + \cdots + p^*(a_{e-1})\xi^{e-2});$$

$$(21) \qquad\qquad b = \pi^*(b_0) + \pi^*(b_1)\xi + \cdots + \pi^*(b_e)\xi^e,$$

for some $a_0 \in \mathrm{CH}^l(X)$, $a_r \in \mathrm{CH}^{l-r}(Z)$ and $b_r \in \mathrm{CH}^{l-r}(Z)$, where $e := \mathrm{codim}(Z \subset X)$.

Using (18), (20), (21), one obtains the following relations:

$$(22) \qquad\qquad\qquad i^*(a_0) = b_0 - b_e c_e(N);$$

$$(23) \qquad\qquad\qquad a_r + b_r = b_e c_{e-r}(N), \quad 1 \le r \le e-1,$$

where $N := N_{Z/X}$ is of rank e and, for $E = \mathbb{P}(N)$, we used the following identity in $\mathrm{CH}^*(E)$:

$$\xi^e + p^* c_1(N)\xi^{e-1} + \cdots + p^* c_e(N) = 0.$$

By (19), (20), (21), we have

$$(24) \qquad\qquad\qquad\qquad a_0 = -i_*(b_e),$$

which implies that $i^*(a_0) = -i^* i_*(b_e) = -b_e c_e(N)$. Combining with (22), we get $b_0 = 0$. Using again (24) and the excess intersection formula [31, §6.3],

$$(25) \qquad\qquad\qquad \tau^*(a_0) = -j_*(p^*(b_e)c_{e-1}(\mathcal{E})),$$

where $\mathcal{E} := p^*(N)/\mathcal{O}_p(-1)$ is the excess normal bundle of the blow-up square in Diagram (16). Putting (25) into (20), we obtain that $a = j_*(\gamma)$ with

$$\gamma := -p^*(b_e)c_{e-1}(\mathcal{E}) + p^*(a_1) + p^*(a_2)\xi + \cdots + p^*(a_{e-1})\xi^{e-2}.$$

Therefore, it remains to show that $-\iota_*(\gamma) = b$. To this end,

$$\begin{aligned}
-\iota_*(\gamma) &= \iota_*(p^*(b_e)c_{e-1}(\mathcal{E})) - \iota_*(p^*(a_1) + \cdots + p^*(a_{e-1})\xi^{e-2}), \\
&= \iota_*(p^*(b_e)c_{e-1}(\mathcal{E})) + \iota_*(p^*(b_1 - b_e c_{e-1}(N))) + \cdots + \iota_*(p^*(b_{e-1} - b_e c_1(N))\xi^{e-2}), \\
&= \pi^*(b_e)\iota_* c_{e-1}(\mathcal{E}) + \pi^*(b_1 - b_e c_{e-1}(N))\xi + \cdots + \pi^*(b_{e-1} - b_e c_1(N))\xi^{e-1}, \\
&= b + \pi^*(b_e)\left(\iota_* c_{e-1}(\mathcal{E}) - \pi^*(c_{e-1}(N))\xi - \cdots - \pi^*(c_1(N))\xi^{e-1} - \xi^e\right), \\
&= b,
\end{aligned}$$

where the second equality uses (23), the third equality uses the projection formula (note that $p = \pi \circ \iota$) and the fact that $[E] = \xi \in \mathrm{CH}^1(X_{\mathrm{loc}})$, the fourth equality uses (21) and that $b_0 = 0$, and the last equality uses the equality

$$c_{e-1}(\mathcal{E}) = p^*(c_{e-1}(N)) + \cdots + p^*(c_1(N))\xi^{e-2} + \xi^{e-1},$$

which can be easily deduced from $c_t(N) = c_t(\mathcal{E})c_t(\mathcal{O}_p(-1)) = c_t(\mathcal{E})(1 - t\xi)$. $\square$

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