Specialization of Néron-Severi class groups of normal varieties

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December 19, 2025

Abstract

The Néron-Severi class group (resp. Néron-Severi group) $\mathrm{Cl}^{\mathrm{ns}} X$ (resp. $\mathrm{NS}\, X$) of a normal variety X is the group of Weil divisors (resp. Cartier divisors) on X modulo algebraic equivalence. Let $f : X \to B$ be a family of normal proper varieties over a field k of characteristic 0. Let $\eta \in B$ be the generic point. We show the existence of a $\mathrm{Cl}^{\mathrm{ns}}$-generic point, i.e., a closed point $b \in |B|$ such that there is a specialization isomorphism $\mathrm{Cl}^{\mathrm{ns}} X_{\bar{\eta}} \xrightarrow{\sim} \mathrm{Cl}^{\mathrm{ns}} X_b$. In fact, when k is finitely generated over $\mathbb{Q}$, we can show that the complement of $\mathrm{Cl}^{\mathrm{ns}}$-generic points is sparse. Additionally, for general k, we prove the potential density of points that are both NS-generic and $\mathrm{Cl}^{\mathrm{ns}}$-generic.

1 Introduction

Let k be an algebraically closed field. Let X be an algebraic variety over k. Two Weil divisors D_1, D_2 on X are called *algebraically equivalent*, if there is a smooth integral scheme T of finite type over k, a Weil divisor E on $X \times_k T$ and points $t_1, t_2 \in T(k)$ with $D_1 - D_2 = E_{t_1} - E_{t_2}$ in the Weil divisor class group $\mathrm{Cl}(X)$. Here for $t \in T(k)$, $E_t \in \mathrm{Cl}(X)$ is the specialization of E at t in the sense of [Ful98, p.176]. Let $\mathrm{Cl}^{\mathrm{ns}} X$ be the group of Weil divisors on X modulo algebraic equivalence. Let $f : X \to B$ be a morphism of algebraic varieties over k with geometrically integer fibers. Assume that B is irreducible with generic point η. For every closed point $b \in |B|$, there is a specialization morphism $\mathrm{Cl}^{\mathrm{ns}} X_{\bar{\eta}} \to \mathrm{Cl}^{\mathrm{ns}} X_{\bar{b}}$ (Lemma 4.2). When the morphism is an isomorphism, the closed point b is called $\mathrm{Cl}^{\mathrm{ns}}$*-generic*. A morphism of algebraic varieties is *normal*, if it is flat with normal geometric fibers.

Theorem 1.1. *Let k be an algebraically closed field of characteristic* 0*. Let $f : X \to B$ be a normal proper morphism of algebraic varieties over k with geometrically irreducible fibers. Assume that B is irreducible. Then the set of* $\mathrm{Cl}^{\mathrm{ns}}$*-generic points is Zariski dense in B.*

Remark 1.2. If $f : X \to B$ is moreover *smooth*, then Theorem 1.1 is proved by André [And96, Théorème 5.2 3)], Maulik and Poonen [MP12, Theorem 1.1]. We recall their results.

Two line bundles L, L' on X are called *algebraically equivalent* if there is a connected scheme T of finite type over k, a line bundle $M \in \mathrm{Pic}(X \times_k T)$ and points $t, t' \in T(k)$ such that $M|_{X \times t} \cong L$ and $M|_{X \times t'} \cong L'$. The isomorphism classes of line bundles algebraically equivalent to O_X form a subgroup of $\mathrm{Pic}(X)$, and the quotient $\mathrm{NS}\, X$ is called the *Néron-Severi group* of X. For every $b \in B$, there is a specialization morphism $\mathrm{NS}\, X_{\bar{\eta}} \to \mathrm{NS}\, X_{\bar{b}}$. André establishes the existence of *NS-generic* point $b \in B(k)$, i.e., for which the specialization is an isomorphism. More precisely, he proves that the set of non NS-generic points is sparse. On the other hand, Maulik and Poonen show that for a prime p where $f : X \to B$ has good reduction, this set is p-adically nowhere dense. Additionally, Ambrosi [Amb23, Corollary 1.7.1.5] extends André's result to positive characteristic.

There is a natural morphism $\mathrm{NS}\, X \to \mathrm{Cl}^{\mathrm{ns}} X$. It is an isomorphism when X is smooth over k, so Theorem 1.1 with smooth $f : X \to B$ recovers the existence of NS-generic points. However, for a normal projective X, $\mathrm{NS}\, X \to \mathrm{Cl}^{\mathrm{ns}} X$ may not be injective nor surjective.

Remark 1.3. Suppose that k is not algebraic over any finite field. Let $f : X \to B$ be a family of normal proper *surfaces*. Using the result of André, Maulik-Poonen and Ambrosi, in arbitrary characteristic Ji [Ji24, Cor. 4.12] proves that there is $b \in B(k)$ such that the specialization morphism $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{\eta}}) \to \mathrm{Cl}^{\mathrm{ns}}(X_b)$ is an isomorphism *up to torsion*. In characteristic 0, Theorem 1.1 extends the result to arbitrary relative dimensions, and takes torsion into consideration.

In fact, we give two results both stronger than Theorem 1.1. For a normal morphism, Theorem 1.4 shows the sparsity of the jumping locus for $\mathrm{Cl}^{\mathrm{ns}}$. Let B be an irreducible variety over a finitely generated extension F of $\mathbb{Q}$. A subset S of the set of closed points $|B|$ is *sparse* if there is a dominant and generically finite morphism $\pi : B' \to B$ of irreducible varieties over F, such that for every $s \in S$, $\pi^{-1}(s)$ is either empty or contains more than one closed point.

Theorem 1.4. *Let $F/\mathbb{Q}$ be a finitely generated extension. Let $f : X \to B$ be a normal proper morphism of algebraic varieties over F with geometrically irreducible fibers. Assume that B is irreducible. Then the subset of $|B|$ consisting of non $\mathrm{Cl}^{\mathrm{ns}}$-generic points is sparse.*

If $f : X \to B$ is moreover smooth projective, then Theorem 1.4 reduces to André's theorem (see, e.g., [MP12, Theorem 8.3]). By the Hilbert irreducibility theorem (see, e.g., [MP12, Proposition 8.5 (e)]), one deduces Theorem 1.1 from Theorem 1.4.

In [Liu25, Theorem 6.5 (b)], we generalize André-Maulik-Poonen's existence of NS-generic points from smooth proper morphisms to normal proper morphisms. We prove the potential density of closed points that are NS-generic and $\mathrm{Cl}^{\mathrm{ns}}$-generic at the same time.

Theorem 1.5. *Let k be a field of characteristic 0. Let B be an irreducible variety over k with generic point η. Then there is an explicit constant $d > 1$ with the following property. For every normal proper morphism $f : X \to B$ with* *geometrically irreducible fibers, there is a subset $S \subset B^{\leq d}$ Zariski dense in B, such that for every $b \in S$ there exist specialization isomorphisms*

$$\mathrm{NS}(X_{\bar\eta}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar b}), \quad \operatorname{Br} X_{\bar\eta} \xrightarrow{\sim} \operatorname{Br} X_{\bar b}, \quad \mathrm{Cl}^{\mathrm{ns}}(X_{\bar\eta}) \xrightarrow{\sim} \mathrm{Cl}^{\mathrm{ns}}(X_{\bar b})$$

simultaneously.

To prove Theorems 1.4 and 1.5, we reduce the normal case to the smooth case via Hironaka's resolution of singularities. The differences is that in Theorem 1.4, we rely on André's sparsity result, while in Theorem 1.5, we apply Maulik and Poonen's nondensity result to refine the proof of [Liu25, Theorem 6.5 (b)].

Notation

By an *algebraic variety* over a field k, we mean a scheme separated of finite type over k. For an algebraic variety X over k, let $|X|$ be the set of closed points of X. For an integer $n > 0$, let $X^{\leq n} := \{x \in |X| : [k(x) : k] \leq n\}$. Let $\operatorname{Br}(X)$ be the torsion subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_m)$, called the *cohomological Brauer group* of X. Let $\mathrm{CaCl}(X)$ be the group of Cartier divisors on X modulo linear equivalence. Let $\mathrm{Cl}(X)$ be the Weil divisor class group of X, and $\mathrm{Cl}^0(X)$ be the subgroup of divisors algebraically equivalent to 0. The quotient $\mathrm{Cl}^{\mathrm{ns}}(X) := \mathrm{Cl}(X)/\mathrm{Cl}^0(X)$ is called the *Néron-Severi class group* of X. For an integer $i \geq 0$, let Z_iX be the group of i-cycles on X. Let A_iX be the Chow group of i-cycles on X modulo rational equivalence. Let B_iX be A_iX modulo algebraic equivalence. If X is irreducible of dimension d, then by [Kah06, p.396], one has $\mathrm{Cl}(X) = A_{d-1}(X)$ and hence $\mathrm{Cl}^{\mathrm{ns}}(X) = B_{d-1}(X)$. Then from [Kah06, Thm. 3], $\mathrm{Cl}^{\mathrm{ns}}(X)$ is a finitely generated abelian group. Let $\rho^{\mathrm{ns}}(X)$ be its rank. If k is algebraically closed and if X is proper over k, then $\mathrm{NS}\, X$ is also a finitely generated abelian group, whose rank is denoted $\rho(X)$. Given a prime ℓ invertible in k, let $D^b_c(X)$ be the triangulated category of complexes of $\mathbb{Q}_\ell$-étale sheaves on X with bounded constructible cohomology. Let $\mathrm{Perv}(X) \subset D^b_c(X)$ be the full subcategory of perverse sheaves on X. A *trait* is the spectrum of a discrete valuation ring.

2 Geometric invariance

Given an algebraically closed field k and an algebraic variety X over k, we show in two ways that $\mathrm{Cl}^{\mathrm{ns}}(X)$ remains invariant under extensions of k. The first way proves a stronger result (Lemma 2.3) for algebraic cycles of arbitrary dimension. The second proof of Lemma 2.5 involving the Chow variety may be of independent interest.

Let X be a reduced scheme of finite type over a field k. By [EGA IV 4, Prop. 21.3.4 b)], the canonical morphism $\mathrm{CaCl}\, X \to \mathrm{Pic}(X)$ is an isomorphism. Assume further that X is irreducible. Let $\mathrm{cyc} : \mathrm{CaCl}\, X \to \mathrm{Cl}(X)$ be the canonical morphism constructed in [EGA IV 4, 21.6.7]. The resulting morphism $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ coincides with $c_1(\cdot) \cap [X]$ in [Ful98, p.41].

Lemma 2.1. *Let X be a integral scheme of finite type over an algebraically closed field k. Then the canonical morphism* $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ *descends to a morphism* $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$.

Proof. Let L be a line bundle on X that is algebraically equivalent to 0. Then there is a smooth integral variety T over k, a line bundle M on $X \times_k T$ and points $t_1, t_2 \in T(k)$ such that $[L] = [i_{t_1}^* M] - [i_{t_2}^* M]$ in $\mathrm{Pic}(X)$. As k is algebraically closed, $X \times_k T$ is integral. Let C (resp. D) be the Cartier divisor on X (resp. $X \times_k T$) corresponding to L (resp. M). From [Sta25, Tag 067P], as the projection $X \times_k T \to T$ is flat, for every section $t : \mathrm{Spec}\, k \to T$, its base change $i_t : X \to X \times_k T$ is a regular immersion. Then by [Ful98, Rk. 6.2.1], one has

$$[X \times_k T]_t = i_t^! [X \times_k T] = [X].$$

Therefore, by [Ful98, Prop. 6.3], the square

$$\begin{tikzcd}
\mathrm{Pic}(X \times T) \arrow[r, "i_t^*"] \arrow[d, "c_1(\cdot)\cap[X\times T]"'] & \mathrm{Pic}(X) \arrow[d, "c_1(\cdot)\cap[X]"] \\
\mathrm{Cl}(X \times T) \arrow[r, "t^!"] & \mathrm{Cl}(X)
\end{tikzcd}$$

is commutative. One has

$$\begin{aligned} C = c_1(L) \cap [X] &= c_1([i_{t_1}^* M] - [i_{t_2}^* M]) \cap [X] \\ = &t_1^!(c_1(M) \cap [X \times T]) - t_2^!(c_1(M) \cap [X \times T]) = D_{t_1} - D_{t_2}, \end{aligned}$$

so C is algebraically equivalent to 0. Therefore, the morphism $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ induces a morphism $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$. □

In Lemma 2.2, we consider two cases where the Chow groups commute with direct limits.

Lemma 2.2. (a) *Let K/k be a field extension. Let X be a scheme of finite type over k. Then for every integer $i \geq 0$, the natural morphisms*

$$\mathrm{colim}_{k'} Z_i(X_{k'}) \to Z_i(X_K), \quad \mathrm{colim}_{k'} A_i(X_{k'}) \to A_i(X_K)$$

are isomorphisms, where k' runs through finitely generated subextensions.

(b) *Let S be an integral variety of dimension m over a field k. Let η be the generic point of S. Let $f : X \to S$ be a finite type morphism of schemes. Then there are natural isomorphisms*

$$\mathrm{colim}_U Z_{i+m}(X_U) \to Z_i(X_\eta), \quad \mathrm{colim}_U A_{i+m}(X_U) \to A_i(X_\eta),$$

where U runs through nonempty open subsets of S.

Proof. Both are special cases of [Sta25, Tag 0FVQ].

(a) One has $K = \operatorname{colim} k'$.

(b) By the Nagata-Zariski theorem (see, *e.g.*, [Sta25, Tag 07R5 (1)]), the regular locus $\operatorname{Reg}(S)$ of S is open. It contains η. Shrinking S to $\operatorname{Reg}(S)$, one may assume that S is regular. We verify the conditions of [Sta25, Tag 0FVQ]. By [Sta25, Tag 0F91], now S is Cohen-Macaulay, so the map

$$\delta : S \to \mathbb{Z}, \quad s \mapsto m - \dim O_{S,s}$$

is a dimension function. By [Har77, II, Exercise 3.20 (d)], one has $\delta(s) = 0$ for all closed points $s \in S$ and $\delta(\eta) = m$. From [Sta25, Tag 02JB], S is universally catenary, so (S, δ) satisfies the situation [Sta25, Tag 02QL]. Moreover, from [Sta25, Tag 02J6 (1)], S is Jacobson. Then from [Sta25, Tag 02QO], for every integral closed subscheme Z of X, one has $\dim Z = \dim_\delta Z$.

As S is integral, the morphism $\operatorname{Spec} k(\eta) \to S$ is flat. Let $\delta'(\eta) = m$. Then (S, δ) and (η, δ') satisfy the situation [Sta25, Tag 0FVG]. Let $(S_j)_{j \in I}$ be the directed system of nonempty *affine* open subsets of S. For every $j \in I$, let $X_j := X \times_S S_j \to S_j$ be the projection. Since $(S_j)_{j \in I}$ is cofinal in the inverse system of all nonempty open subsets of S, one has

$$\operatorname{colim}_{j \in I} A_{i+m}(X_j) = \operatorname{colim}_U A_{i+m}(X_U).$$

Since S is separated over k, for every $j \in I$, the open immersion $S_j \to S$ is affine. One has $\lim_{j \in I} S_j = \operatorname{Spec} k(\eta)$ and $\lim_{j \in I} X_j = X_\eta$. Therefore, by [Sta25, Tag 0FVQ], the natural morphisms

$$\operatorname{colim}_j Z_{i+m}(X_j) \to Z_i(X_\eta), \quad \operatorname{colim}_j A_{i+m}(X_j) \to A_i(X_\eta)$$

are isomorphisms.

□

We prove that the group of algebraic cycles modulo algebraic equivalence is invariant under extensions of an algebraically closed field.

Lemma 2.3. *Let K/k be an extension of fields, with k algebraically closed. Let X be a scheme of finite type over k. Then for every integer $i \geq 0$, the morphism*

$$\phi : B_i(X) \to B_i(X_K), \quad a \mapsto a_K$$

is an isomorphism.

Proof. • To prove injectivity, replacing K by an algebraic closure of it one may assume that K is algebraically closed. For every $a = \sum_j n_j Z_j \in \ker(\phi)$ with $n_j \in \mathbb{Z}$ and $Z_j \in Z_i(X)$, we prove $a = 0$.

Step 1 We descend algebraic equivalence to a finitely generated subextension.

Since $\phi(a) = 0$, there is a smooth integral algebraic variety T over K, a cycle $\alpha \in Z_{i+m}(X \times_k T)$ with $m = \dim T$, and points $t_1, t_2 \in T(K)$ such that $a = \alpha_{t_1} - \alpha_{t_2}$ in $A_i(X_K)$. One has $K = \operatorname{colim} k'$, where k' runs through finitely generated subextensions of K/k. Then by [Sta25, Tag 01ZM (1)], there is a finitely generated subextension k' and a scheme T' of finite presentation over k' such that $T' \otimes_{k'} K \cong T$ as schemes over K. From [Sta25, Tag 01ZQ], one may assume that T' is separated over k'. By [Sta25, Tag 0C0C], one may assume that T' is smooth over k'. Similarly, one may assume that there exist points $t'_1, t'_2 \in T'(k')$ that are over t_1, t_2 respectively under the map $T'(k') \to T(K)$. Then both $t'_i : \operatorname{Spec} k' \to T'$ are regular immersions of codimension m. From Lemma 2.2 (a), one may assume that α has a preimage α' under the morphism

$$A_{i+m}(X \times_k T') \to A_{i+m}(A \times_k T).$$

From [Ful98, Thm. 6.2 (b) and (c)], one has $(\alpha'_{t'_i})_K = \alpha_{t_i}$ in $A_i(X_K)$. Thus, $a_{k'} - \alpha'_{t'_1} + \alpha'_{t'_2}$ is in the kernel of $A_i(X_{k'}) \to A_i(X_K)$. By Lemma 2.2 (a), enlarging k' one may assume $a_{k'} = \alpha'_{t'_1} - \alpha'_{t'_2}$ in $A_i(X_{k'})$.

Step 2 We use a spreading out argument.
By [FJ08, Cor. 2.6.5 (c)], because k is algebraically closed, the extension k'/k is regular. Then from [Con06, p.37], there is a smooth integral variety Y over k with function field k'. Since $\operatorname{Spec} k'$ is the limit of nonempty affine open subsets of Y, shrinking Y one may find a separated smooth morphism $\mathcal{T} \to Y$ of finite type with generic fiber T', two sections $\tau_1, \tau_2 : Y \to \mathcal{T}$ that are preimages of t'_1, t'_2 under the map $\operatorname{Mor}_Y(Y, \mathcal{T}) \to T'(k')$. Let $n = \dim Y$. By Lemma 2.2 (b), because the generic fiber of $X \times_k \mathcal{T} \to Y$ is $X \times_k T'$, shrinking Y one may find $\tilde{\alpha} \in A_{i+m+n}(X \times_k \mathcal{T})$ lying over $\alpha' \in A_{i+m}(X \times_k T')$. By [EGA IV 4, Thm. 17.12.1], because Y and $\mathcal{T}$ are smooth over k, both $\tau_j : Y \to \mathcal{T}$ are regular immersions of codimension m. By [Ful98, Thm. 6.2 (b) and (c)], for every nonempty open subset Y_0 of Y, letting $\mathcal{T}_0 = \mathcal{T} \times_Y Y_0$, one has

$$(\tau_j^! \tilde{\alpha})|_{X \times_k Y_0} = (t'_j)^!(\tilde{\alpha}|_{X \times_k \mathcal{T}_0})$$

in $A_{i+n}(X \times_k Y_0)$. Then the image of $\tau_j^! \tilde{\alpha}$ under the map $A_{i+n}(X \times_k Y) \to A_i(X_{k'})$ is $\alpha'_{t'_j}$. Let

$$a \times_k Y := \sum_j n_j [Z_j \times_k Y] \in A_{i+n}(X \times_k Y).$$

Thus, $a \times_k Y - \tau_1^! \tilde{\alpha} + \tau_2^! \tilde{\alpha}$ is in the kernel of $A_{i+n}(X \times_k Y) \to A_i(X_{k'})$. By Lemma 2.2 (b), shrinking Y one may assume $a \times_k Y = \tau_1^! \tilde{\alpha} - \tau_2^! \tilde{\alpha}$ in $A_{i+n}(X \times_k Y)$.

Step 3 We specialize back to k.

By [Sta25, Tag 055G], shrinking Y, one may assume that $\mathcal{T} \to Y$ has geometrically connected fibers. Because k is algebraically closed, there is $y \in Y(k)$. Then the fiber T_y of $\mathcal{T} \to Y$ over y is a smooth integral variety over k. Let z_j ($j = 1, 2$) be the image of τ_j under the map $\mathrm{Mor}_Y(Y, \mathcal{T}) \to T_y(k)$. Because Y is smooth over k, the section $y : \operatorname{Spec} k \to Y$ is a regular immersion of codimension n. Let

$$\beta := y^! \tilde{\alpha} \in A_{i+m}(X \times_k T_y).$$

By [Ful98, Thm. 6.4], because $\tau_j : Y \to \mathcal{T}$ and $y : \operatorname{Spec} k \to Y$ are regular immersions of codimension m and n respectively, one has $\beta_{z_j} = y^! \tau_j^! \tilde{\alpha}$. Therefore, one has $a = \beta_{z_1} - \beta_{z_2}$ in $A_i(X)$ and hence $a = 0$ in $B_i(X)$.

- We prove surjectivity. For every algebraic cycle Z on X_K of dimension i, there is a finitely generated subextension k' such that Z is defined on k'. Using Lemma 2.2 (b), one may spread out again, so that there is a smooth integral variety Y over k of dimension n with function field k', an algebraic cycle Γ on $X \times_k Y$ of dimension $i + n$ whose image under $Z_{i+n}(X \times_k Y) \to Z_i(X_{k'})$ is Z. As k is algebraically closed, there is $y \in Y(k)$. Then $\Gamma_y := y^! \Gamma$ is in $A_i(X)$.

  We prove that $(\Gamma_y)_K$ is algebraically equivalent to Z. As k is algebraically closed and Y is integral, Y_K is a smooth integral variety over K. Let t_1 and t_2 be the elements of $Y_K(K)$ corresponding to y and the generic point of Y. Let $\alpha \in A_{i+n}(X_K \times_K Y_K)$ be the class of the algebraic cycle Γ_K on $(X \times_k Y) \otimes_k K$. Then $\alpha_{t_1} = (\Gamma_y)_K$ and $\alpha_{t_2} = Z$ in $A_i(X_K)$.

□

We mention the invariance of the Griffiths groups under extensions of algebraically closed fields, although we do not use it.

Corollary 2.4. *Let K/k be an extension of algebraically closed fields. Let X be a scheme smooth proper over k. Let ℓ be a prime invertible in k. Consider ℓ-adic étale cohomology. For every integer $i \geq 0$, let $C_i(X)$ be the quotient of algebraic cycles of dimension i on X modulo homological equivalence. Let* $\mathrm{Griff}_i(X)$ *be the Griffiths group, i.e., the kernel of $B_i(X) \to C_i(X)$. Then the morphisms $C_i(X) \to C_i(X_K)$ and* $\mathrm{Griff}_i(X) \to \mathrm{Griff}(X_K)$ *are isomorphisms.*

Proof. Let $\mathrm{cl} : A_i(X) \to H^{2i}(X)$ be the cycle class morphism, whose image is $C_i(X)$. By Lemma 2.3, $C_i(X) \to C_i(X_K)$ is surjective. From [Mil80, VI, Corollary 4.3], $H^i(X) \to H^i(X_K)$ is an isomorphism, so $C_i(X) \to C_i(X)$ is injective. □

The proof of [Ji24, Lem. 2.4] is incomplete. Lemma 2.5 extends the result from normal proper varieties to integral varieties.

Lemma 2.5. *Let K/k be an extension of algebraically closed fields. Let X be an integral scheme, separated of finite type over k. Then the natural morphism $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)$ is an isomorphism of groups.*

Proof. Let $\dim X = n$. Because X is integral, one has $\mathrm{Cl}^{\mathrm{ns}}(X) = B_{n-1}(X)$. By Lemma 2.3, as k is algebraically closed, the map $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)$ is injective. It remains to prove surjectivity. We proceed in steps.

(a) Assume that X is projective over k. Choose a closed immersion $i : X \to \mathbf{P}^N_k$ over k for some integer $N \geq 0$. Let

$$\mathrm{Chow}^1(X, i) := \sqcup_{d \geq 0} C_{n-1,d}(X, i)$$

be the Chow variety of divisors. By [Fri91, Prop. 1.8], the addition of Weil divisors on X determines the structure of a commutative monoid on $\mathrm{Chow}^1(X, i)$, and as k (resp. K) is algebraically closed, the group completion of the discrete monoid $\pi_0(\mathrm{Chow}^1(X, i))$ (resp. $\pi_0(\mathrm{Chow}^1(X_K, i_K))$) is $\mathrm{Cl}^{\mathrm{ns}}(X)$ (resp. $\mathrm{Cl}^{\mathrm{ns}}(X_K)$). From [Fri91, Prop. 1.1], one has

$$\mathrm{Chow}^1(X_K, i_K) = \mathrm{Chow}^1(X, i) \otimes_k K.$$

Then by [Sta25, Tag 0363], the map

$$\pi_0(\mathrm{Chow}^1(X_K, i_K)) \to \pi_0(\mathrm{Chow}^1(X, i))$$

is bijective, so $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)$ is an isomorphism.

(b) Assume that X is proper over k and normal. By Chow's lemma (see, *e.g.*, [EGA II, Cor. 5.6.2]), there is an integral scheme X' projective over k with $\dim X' = n$, and a surjective birational proper morphism $f : X' \to X$ over k. From [Sta25, Tag 0BXR (4)], taking normalization of X', one may assume that X' is normal. Let $\{E_i\}_{i \in I}$ be the exceptional divisors of $f : X' \to X$. Then by [Kol18, Lem. 18], as X, X' are normal, the pushforward by $f : X' \to X$ gives an isomorphism

$$\mathrm{Cl}^{\mathrm{ns}}(X') / \oplus_i \mathbb{Z}[E_i] \cong \mathrm{Cl}^{\mathrm{ns}}(X).$$

By [Sta25, Tag 038O], X'_K, X_K are normal, so similarly $f_K : X'_K \to X_K$ induces an isomorphism

$$\mathrm{Cl}^{\mathrm{ns}}(X'_K) / \oplus_i \mathbb{Z}[E_{i,K}] \cong \mathrm{Cl}^{\mathrm{ns}}(X_K).$$

Because X' is projective, the morphism $\mathrm{Cl}^{\mathrm{ns}}(X') \to \mathrm{Cl}^{\mathrm{ns}}(X'_K)$ is an isomorphism. Thus, the right vertical morphism in the commutative diagram

$$\begin{tikzcd}
\oplus \mathbb{Z}[E_i] \arrow[r] \arrow[d] & \mathrm{Cl}^{\mathrm{ns}}(X') \arrow[r, "f_*"] \arrow[d] & \mathrm{Cl}^{\mathrm{ns}}(X) \arrow[r] \arrow[d] & 0 \\
\oplus_i \mathbb{Z}[E_{i,K}] \arrow[r] & \mathrm{Cl}(X'_K) \arrow[r, "f_{K,*}"] & \mathrm{Cl}^{\mathrm{ns}}(X_K) \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism.

(c) Assume that X is normal. By [Sta25, Tag 0F41], there is an irreducible scheme $\bar{X}$ proper over k with an open immersion $j : X \to \bar{X}$ over k. Taking normalization of $\bar{X}$, one may assume further that $\bar{X}$ is normal. Then the morphism $\mathrm{Cl}^{\mathrm{ns}}(\bar{X}) \to \mathrm{Cl}^{\mathrm{ns}}(\bar{X}_K)$ is an isomorphism. By Lemma 2.6, the morphism $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)$ is an isomorphism.

(d) Consider general X. By [Sta25, Tag 07R5 (1)], the regular locus U of X is open. Because X is integral, U contains the generic point of X. Then by Lemma 2.6, the morphism $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)$ is surjective.

□

Lemma 2.6. *Let K/k be an extension of algebraically closed fields. Let X be an integral, separated finite type scheme over k. Let U be a nonempty open subscheme of X. Then the restriction map*

$$\ker(\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)) \twoheadrightarrow \ker(\mathrm{Cl}^{\mathrm{ns}}(U) \to \mathrm{Cl}^{\mathrm{ns}}(U_K))$$

is surjective, and

$$\operatorname{coker}(\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X_K)) \to \operatorname{coker}(\mathrm{Cl}^{\mathrm{ns}}(U) \xrightarrow{\sim} \mathrm{Cl}^{\mathrm{ns}}(U_K))$$

is an isomorphism.

Proof. Let Z be the reduced complement of U in X. Let C (resp. C_K) be the kernel of $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(U)$ (resp. $\mathrm{Cl}^{\mathrm{ns}}(X_K) \to \mathrm{Cl}^{\mathrm{ns}}(U_K)$). Because X is integral, $\mathrm{Cl}^0(X) \to \mathrm{Cl}^0(U)$ is surjective. Then by [Ful98, Prop. 1.8], the morphism $Z_{n-1}(Z) \to C$ is surjective. Similarly, as X_K is integral, $Z_{n-1}(Z_K) \to C_K$ is surjective. Since $Z_{n-1}(Z)$ is the free abelian group on irreducible components of Z of dimension $n-1$, the morphism $Z_{n-1}(Z) \to Z_{n-1}(Z_K)$ is surjective. Then the morphism $C \to C_K$ is surjective. The snake lemma applied to

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d] & \mathrm{Cl}^{\mathrm{ns}}(X) \arrow[r] \arrow[d] & \mathrm{Cl}^{\mathrm{ns}}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_K \arrow[r] & \mathrm{Cl}^{\mathrm{ns}}(X_K) \arrow[r] & \mathrm{Cl}^{\mathrm{ns}}(U_K) \arrow[r] & 0,
\end{tikzcd}$$

finishes the proof. □

3 Cycle class maps for divisors

We review the cycle class maps for Weil divisors. In Theorem 3.2, given a normal variety X of dimension d, we show that the cycle class map

$$\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H_{2d-2}(X, \mathbb{Q}_\ell)$$

is injective with image $\mathrm{IH}^2(X, \mathbb{Q}_\ell)$. Then using cycle class maps, we get finiteness of the kernel of the natural morphism $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$ when X is also proper (Lemma 3.10).

Let X be a scheme of dimension d, of finite type over an algebraically closed field k. Consider étale homology groups. By [Lau76, Thm. 6.1 et 6.3], for every prime ℓ invertible in k, there is a natural homological cycle class map

$$\mathrm{cl}_X : Z_{d-1}X \to H_{2d-2}(X, \mathbb{Z}_\ell(1-d))$$

which factors through $B_{d-1}(X)$.

Let $[X] \in H_{2d}(X, \mathbb{Z}_\ell)(-d)$ be the fundamental class of X in the sense of [Lau76, Def. 3.1]. By [Lau76, Prop. 6.4], if X is smooth and equidimensional, then there is a natural cohomological cycle class map

$$\mathrm{cl}^X : Z_{d-1}X \to H^2(X, \mathbb{Z}_\ell(1)),$$

such that its composition with the isomorphism

$$(-) \cap [X] : H^2(X, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} H_{2d-2}(X, \mathbb{Z}_\ell(1-d))$$

is $\mathrm{cl}_X : Z_{d-1}X \to H_{2d-2}(X, \mathbb{Z}_\ell(1-d))$.

Lemma 3.1 shows that on proper integral varieties, algebraic equivalence coincides with homological equivalence for Cartier divisors.

Lemma 3.1. *Let X be an integral variety of dimension d over an algebraically closed field k. Fix a prime ℓ invertible in k. Then the diagram*

$$\begin{tikzcd}
\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r, "c_1"] \arrow[d, "c_1(\cdot)\cap X"] & H^2(X, \mathbb{Z}_\ell(1)) \arrow[d, "\cap[X]"] \\
\mathrm{Cl}^{\mathrm{ns}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r, "\mathrm{cl}_X"'] & H_{2d-2}(X, \mathbb{Z}_\ell(1-d))
\end{tikzcd}$$

is commutative. If X is smooth over k, then the cycle class map

$$\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow H_{2d-2}(X, \mathbb{Z}_\ell(1-d))$$

is injective.

Proof. The commutativity of the diagram follows from [Li13, Thm. 6.1]. Now assume that X is smooth. Then the map $c_1(\cdot) \cap X : \mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$ is an isomorphism. By [Lau76, Prop. 3.2], as X is smooth the map

$$\cap[X] : H^2(X, \mathbb{Z}_\ell(1) \to H_{2d-2}(X, \mathbb{Z}_\ell(1-d))$$

is an isomorphism. By [Kah06, Thm. 3], $\mathrm{Cl}^{\mathrm{ns}}(X)$ is a finitely generated abelian group, so is $\mathrm{NS}(X)$. Therefore, the injectivity follows from [Liu25, Lemma 2.3 (b)]. $\square$

For an integral variety X over a field k with regular locus U, a *resolution of singularities* refers to a regular integral variety $\tilde{X}$ with a proper birational morphism $\mu : \tilde{X} \to X$ over k, such that $\mu|_{\mu^{-1}(U)} : \mu^{-1}(U) \to U$ is an isomorphism.

Let X be a geometrically integral algebraic variety over a field k of dimension n. By [Sta25, Tag 056V], there is a dense open subset U of X that is smooth over k. Let $j : U \to X$ be the open immersion. Fix a prime ℓ invertible in k. The perverse sheaf $j_{!*}(\mathbb{Q}_{\ell,U}[n])$ on X is called the *intersection complex* of X. Let $\mathrm{IC}_X := j_{!*}(\mathbb{Q}_{\ell,U}[n])[-n]$. From [HS06, (1.1.1)], there is a unique morphism $\gamma_X : \mathbb{Q}_{\ell,X} \to \mathrm{IC}_X$ in $D_c^b(X)$ whose restriction to U is the $\mathrm{Id}_{\mathbb{Q}_{\ell,U}}$. There is an exact triangle

$$\mathcal{K} \to \mathbb{Q}_{\ell,X}[n] \xrightarrow{\gamma_X} \mathrm{IC}_X[n] \xrightarrow{+1} \tag{1}$$

in $D_c^b(X)$. Since $\gamma_X|_U$ is an isomorphism, one has $\mathcal{K}|_U = 0$.

Theorem 3.2. *Let X be a normal integral variety over an algebraically closed field k of characteristic* 0. *Let $d := \dim X$. Then for every prime ℓ, the cycle class map*

$$\mathrm{cl}_X \otimes \mathbb{Q}_\ell : \mathrm{Cl}^{\mathrm{ns}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H_{2d-2}(X, \mathbb{Q}_\ell(1-d))$$

is injective. Its image is contained in $\mathrm{IH}^2(X, \mathbb{Q}_\ell(1))$. *In particular,* $\rho^{\mathrm{ns}}(X) \leq \dim_{\mathbb{Q}_\ell} \mathrm{IH}^2(X, \mathbb{Q}_\ell)$.

Proof. Let $j : U \hookrightarrow X$ be the regular locus of X. Since k is perfect, U is smooth over k. Let $Z := X \setminus U$ be endowed with the reduced induced closed subscheme structure. As X is normal, one has $\dim Z \leq d - 2$. By Hironaka's resolution of singularities [Hir64, p.132, Main Theorem 1], because k has characteristic 0, there is a resolution of singularities $\mu : \tilde{X} \to X$. From [Lau76, (4.0)], as $\mu : \tilde{X} \to X$ is proper, it induces a morphism

$$H_{2d-2}(\mu) : H_{2d-2}(\tilde{X}, \mathbb{Q}_\ell) \to H_{2d-2}(X, \mathbb{Q}_\ell).$$

By Lemma 3.7, as X is normal, one has $\mathrm{im} H_{2d-2}(\mu) = \mathrm{IH}^2(X, \mathbb{Q}_\ell)(d)$. Then one has

$$\begin{aligned} \dim\ker H_{2d-2}(\mu) &= \dim H_{2d-2}(\tilde{X}, \mathbb{Q}_\ell) - \dim \mathrm{IH}^2(X, \mathbb{Q}_\ell) \\ &= \dim H_c^{2d-2}(\tilde{X}, \mathbb{Q}_\ell) - \dim \mathrm{IH}_c^{2d-2}(X, \mathbb{Q}_\ell), \end{aligned} \tag{2}$$

where the last equality is from the Poincaré duality.

By [HS06, Thm. 2], there is a morphism $\lambda : \mathrm{IC}_X \to R\mu_*\mathbb{Q}_{\ell,\tilde{X}}$ in $D_c^b(X)$ fitting into a commutative diagram

$$\begin{tikzcd} \mathbb{Q}_{\ell,X} \arrow[d, "\gamma_X"'] \arrow[dr] & \\ \mathrm{IC}_X \arrow[r, dashed, "\lambda"] & R\mu_*\mathbb{Q}_{\ell,\tilde{X}}. \end{tikzcd}$$

Thus, $\lambda|_U : \mathbb{Q}_{\ell,U} \to R\mu_*\mathbb{Q}_{\ell,\mu^{-1}(U)}$ is an isomorphism in $D_c^b(U)$. By Lemma 3.8, there is $M \in D_c^b(X)$ and an isomorphism

$$\phi : R\mu_*\mathbb{Q}_{\ell,\tilde{X}} \xrightarrow{\sim} \mathrm{IC}_X \oplus M,$$

such that λ is the inclusion of a direct summand. Let $\iota : Z \to X$ be the closed immersion. Then M is supported in Z and hence $H_c^{2d-2}(X, M) = H_c^{2d-2}(Z, \iota^* M)$. As $\mu : \tilde{X} \to X$ is proper, ϕ induces an isomorphism

$$H_c^{2d-2}(\tilde{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{IH}_c^{2d-2}(X, \mathbb{Q}_\ell) \oplus H_c^{2d-2}(X, M).$$

Hence, (2) implies

$$\dim \ker H_{2d-2}(\mu) = \dim H_c^{2d-2}(Z, \iota^* M). \tag{3}$$

Let $\mu' : \mu^{-1}(Z) \to Z$ be the base change of $\mu : \tilde{X} \to X$ along $\iota : Z \to X$. By the proper base change theorem, as $\mu : \tilde{X} \to X$ is proper, the natural morphism

$$\iota^* R\mu_* \mathbb{Q}_{\ell, \tilde{X}} \xrightarrow{\sim} R\mu'_* \mathbb{Q}_{\ell, \mu^{-1}(Z)}$$

is an isomorphism in $D_c^b(Z)$. It induces an isomorphism

$$H_c^{2d-2}(Z, \iota^* R\mu_* \mathbb{Q}_{\ell, \tilde{X}}) \xrightarrow{\sim} H_c^{2d-2}(\mu^{-1}(Z), \mathbb{Q}_\ell). \tag{4}$$

The isomorphism ϕ induces another isomorphism

$$H_c^{2d-2}(Z, \iota^* R\mu_* \mathbb{Q}_{\ell, \tilde{X}}) \xrightarrow{\sim} H_c^{2d-2}(Z, \iota^* \mathrm{IC}_X) \oplus H_c^{2d-2}(Z, \iota^* M). \tag{5}$$

By [Bei+82, Cor. 1.4.24], because of $\mathrm{IC}_X[d] = j_{!*} \mathbb{Q}_{\ell, U}[d] \in \mathrm{Perv}(X)$, $\iota^* \mathrm{IC}_X$ lies in ${}^p D^{\leq d-1}(Z)$. Then from [Bei+82, 4.2.4], one has

$$H_c^{2d-2}(Z, \iota^* \mathrm{IC}_X) = 0. \tag{6}$$

Let $\{E_i\}_{i \in I}$ be the exceptional divisors (in the sense of [SR13, Def., p.119]) of $\mu : \tilde{X} \to X$, which coincide with the irreducible components of dimension $d-1$ of $\mu^{-1}(Z)$. One has

$$\dim \ker H_{2d-2}(\mu) \overset{\text{(a)}}{=} \dim H_c^{2d-2}(\mu^{-1}(Z), \mathbb{Q}_\ell) = \dim H_{2d-2}(\mu^{-1}(Z), \mathbb{Q}_\ell) \overset{\text{(b)}}{=} \# I,$$

where (a) follows from (3), (5), (6), (4), and (b) uses [Lau76, Prop. 2.4 (iii)].

The formation of cycle class map is functorial for proper morphisms [Lau76, Thm. 6.1], so the $\mathrm{cl}_X(E_i) \in \ker H_{2d-2}(\mu)$ for all $i \in I$. We prove that the family $\{\mathrm{cl}_X(E_i)\}_{i \in I}$ is linearly independent in the $\mathbb{Q}_\ell$-vector space $\ker H_{2d-2}(\mu)$. If a finite $\mathbb{Q}_\ell$-linear combination $\sum_{i \in I} a_i \, \mathrm{cl}_X(E_i)$ is 0 in $H_{2d-2}(\tilde{X}, \mathbb{Q}_\ell(1-d))$, we prove $a_i = 0$ for all $i \in I$. Taking a multiple, one may assume $a_i \in \mathbb{Z}_\ell$ for all $i \in I$. Then $\mathrm{cl}_X(\sum_{i \in I} a_i E_i)$ is a torsion element of $H_{2d-2}(\tilde{X}, \mathbb{Z}_\ell(1-d))$. Taking a multiple again, one may assume

$$\mathrm{cl}_X(\sum_{i \in I} a_i E_i) = 0$$

in $H_{2d-2}(\tilde{X}, \mathbb{Z}_\ell(1-d))$. By Lemma 3.1, as $\tilde{X}$ is smooth over k, one has $\sum_{i \in I} a_i [E_i] = 0$ in $\mathrm{Cl}^{\mathrm{ns}}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. From [Kol18, Lem. 18], as $\mu : \tilde{X} \to X$ is birational, one has $a_i = 0$ for all $i \in I$.

Therefore, the map $\mathrm{cl}_X : \oplus_{i \in I} \mathbb{Q}_\ell [E_i] \to \ker H_{2d-2}(\mu)$ is an isomorphism. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{i\in I}\mathbb{Q}_\ell[E_i] \arrow[r] \arrow[d, "\mathrm{cl}_{\tilde{X}}"] & \mathrm{NS}(\tilde{X})\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[r] \arrow[d, hook, "\mathrm{cl}_{\tilde{X}}"] & \mathrm{Cl}^{\mathrm{ns}}(X)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \arrow[r] \arrow[d, "\mathrm{cl}_X"] & 0 \\
0 \arrow[r] & \ker H_{2d-2}(\mu) \arrow[r] & H_{2d-2}(\tilde{X},\mathbb{Q}_\ell(1-d)) \arrow[r, "H_{2d-2}(\mu)"] & H_{2d-2}(X,\mathbb{Q}_\ell(1-d)) &
\end{tikzcd}$$

with exact rows. The stated injectivity follows. The image of $\mathrm{cl}_X\otimes\mathbb{Q}_\ell$: $\mathrm{Cl}^{\mathrm{ns}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H_{2d-2}(X,\mathbb{Q}_\ell(1-d))$ is contained in $\mathrm{im}H_{2d-2}(\mu)$. □

Example 3.3. We give an example to show that in Theorem 3.2, the cycle class map

$$\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H_{2d-2}(X,\mathbb{Z}_\ell(1-d))$$

may not be injective. Let

$$X := \mathrm{Proj}\,\mathbb{C}[x_0,x_1,x_2,x_3]/(x_1x_2-x_3^2)$$

be a closed subvariety of $\mathbf{P}^3_{\mathbb{C}}$. Then X is a normal integral complex projective surface. It is the projective cone over the curve $C = \mathrm{Proj}\,\mathbb{C}[y_0,y_1,y_2]/(y_0y_1 - y_2^2) \subset \mathbf{P}^2$.

By [Har92, p.128], X is isomorphic to the weighted projective space $\mathbf{P}(1,1,2)$. Then from [Kaw73, Thm. 1], one has

$$H^2(X^{\mathrm{an}},\mathbb{Z}) = \mathbb{Z}, \quad H^3(X^{\mathrm{an}},\mathbb{Z}) = 0.$$

From the universal coefficient theorem [Hat05, Theorem 3.2], one has $H_2(X^{\mathrm{an}},\mathbb{Z}) = \mathbb{Z}$. One has $\mathrm{Cl}^{\mathrm{ns}}(X) = \mathbb{Z}\oplus\mathbb{Z}/2$. Therefore, the cycle class map $\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X) \to H_2(X^{\mathrm{an}},\mathbb{Z})$ is not injective. By the comparison theorem [Kat94, p.23], one has $H^3(X,\mathbb{Z}_\ell) = 0$. Then from the perfect pairing in [Lau76, p.170], the $\mathbb{Z}_\ell$-module $H_2(X,\mathbb{Z}_\ell)$ is torsion free, so $\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X) \to H_2(X,\mathbb{Z}_\ell)$ is not injective, neither.

Remark 3.4. Let X be an integral variety of dimension d over $\mathbb{C}$. For $i \geq 0$, let $H_i^{\mathrm{BM}}(X^{\mathrm{an}},\mathbb{Q})$ be the Borel-Moore homology of X. Similar to the proof of Theorem 3.2, if X is normal, then the cycle class map $\mathrm{cl}_X : \mathrm{Cl}^{\mathrm{ns}}(X)\otimes_{\mathbb{Z}}\mathbb{Q} \to H^{\mathrm{BM}}_{2d-2}(X^{\mathrm{an}},\mathbb{Q})$ from [Ful98, p.372] is injective.

Lemma 3.5. *Let X be a geometrically integral variety over a field k. Let ℓ be a prime invertible in k. Then $\mathcal{K} \in {}^pD^{\leq 0}(X)$.*

Proof. Set $n := \dim X$. Let $\phi : {}^pH^0(\mathbb{Q}_{\ell,X}[n]) \to \mathrm{IC}_X[n]$ be the morphism in $\mathrm{Perv}(X)$ induced by $\gamma_X : \mathbb{Q}_{\ell,X} \to \mathrm{IC}_X$. Let U be a nonempty smooth open subset of X. Since $\phi|_U = \mathrm{Id}_{\mathbb{Q}_{\ell,U}[\dim X]}$, ϕ is nonzero. By [Bei+82, Théorème 4.3.1 (ii)], as X is geometrically integral, $\mathrm{IC}_X[n]$ is a simple object of $\mathrm{Perv}(X)$, so ϕ is a surjection. Moreover, (1) induces exact sequences

$$\begin{gathered}
{}^pH^0(\mathbb{Q}_{\ell,X}[n]) \overset{\phi}{\twoheadrightarrow} \mathrm{IC}_X[n] \to {}^pH^1(\mathcal{K}) \to {}^pH^1(\mathbb{Q}_{\ell,X}[n]),\\
{}^pH^i(\mathrm{IC}_X[n]) \to {}^pH^{i+1}(\mathcal{K}) \to {}^pH^{i+1}(\mathbb{Q}_{\ell,X}[n])
\end{gathered}$$

for all integers $i > 0$ in $\mathrm{Perv}(X)$. By [Bei+82, Corollary 4.2.4], one has $\mathbb{Q}_{\ell,X}[n] \in {}^pD^{\leq 0}(X)$. It implies ${}^pH^i(\mathcal{K}) = 0$ for all integers $i > 0$ and hence $\mathcal{K} \in {}^pD^{\leq 0}(X)$. □

Lemma 3.6. *Let X be a normal integral variety of dimension n over an algebraically closed field k. Then for every prime ℓ invertible in k and every integer $i \leq 2$, the canonical morphism $\mathrm{IH}^i(X, \mathbb{Q}_\ell)(n) \to H_{2n-i}(X, \mathbb{Q}_\ell)$ is injective. Equivalently, the canonical morphism $H_c^{2n-i}(X, \mathbb{Q}_\ell) \to \mathrm{IH}_c^{2n-i}(X, \mathbb{Q}_\ell)$ is surjective.*

Proof. Consider the Verdier dual of the exact triangle (1)

$$\mathrm{IC}_X(n)[n] \to (\mathbb{D}_X \mathbb{Q}_{\ell,X})[-n] \to \mathbb{D}_X \mathcal{K} \xrightarrow{+1}$$

in $D_c^b(X)$. It induces an exact sequence

$$H^{i-n-1}(X, \mathbb{D}_X \mathcal{K}) \to \mathrm{IH}^i(X, \mathbb{Q}_\ell)(n) \to H^{i-2n}(X, (\mathbb{D}_X \mathbb{Q}_{\ell,X}))$$

of vector spaces over $\mathbb{Q}_\ell$. One has $H^{i-2n}(X, (\mathbb{D}_X \mathbb{Q}_{\ell,X})) = H_{2n-i}(X, \mathbb{Q}_\ell)$. Since X is normal, there is a closed subvariety $\iota : Z \hookrightarrow X$ with $\mathrm{codim}_X(Z) \geq 2$, such that the complement $X \setminus Z$ is regular. Because k is perfect, $X \setminus Z$ is smooth over k and hence $\mathcal{K}|_{X \setminus Z} = 0$. There is $M \in D_c^b(Z)$ with $\iota_* M = \mathbb{D}_X \mathcal{K}$. By [Bei+82, Corollaire 4.1.3], as $\iota : Z \to X$ is a closed immersion, for every integer $j < 0$, one has $\iota_* {}^p H^j(M) = {}^p H^j(\mathbb{D}_X \mathcal{K}) = 0$ in $\mathrm{Perv}(X)$. Then Lemma 3.5 implies ${}^p H^j(M) = 0$ and hence $M \in {}^p D^{\geq 0}(Z)$. One has

$$H^{i-n-1}(X, \mathbb{D}_X \mathcal{K}) = H^{i-n-1}(Z, M) = 0,$$

where the second the equality uses [Bei+82, 4.2.4]. The last statement follows from $(H_{2n-i}(X, \mathbb{Q}_\ell))^\vee = H_c^{2n-i}(X, \mathbb{Q}_\ell)$ and the Poincaré duality $\mathrm{IH}^i(X, \mathbb{Q}_\ell)(n) = \left(\mathrm{IH}_c^{2n-i}(X, \mathbb{Q}_\ell)\right)^\vee$. $\square$

Lemma 3.7. *Let $f : X \to Y$ be a proper surjective morphism of integral varieties over an algebraically closed field k. Suppose that Y is normal and X is smooth. Let $n := \dim Y$. Then for every prime ℓ invertible in k and every integer $i \leq 2$, the image of the morphism $H_{2n-i}(f) : H_{2n-i}(X, \mathbb{Q}_\ell) \to H_{2n-i}(Y, \mathbb{Q}_\ell)$ is $\mathrm{IH}^i(Y, \mathbb{Q}_\ell)(n)$.*

Proof. As $f : X \to Y$ is proper, it induces a natural morphism $H_c^{2n-i}(Y, \mathbb{Q}_\ell) \to H_c^{2n-i}(X, \mathbb{Q}_\ell)$. By the Poincaré duality $\mathrm{IH}^i(Y, \mathbb{Q}_\ell)^\vee = \mathrm{IH}_c^{2n-i}(Y, \mathbb{Q}_\ell)$, it suffices to prove the natural morphism $H_c^{2n-i}(Y, \mathbb{Q}_\ell) \to H_c^{2n-i}(X, \mathbb{Q}_\ell)$ descends to an injection $\psi : \mathrm{IH}_c^{2n-i}(Y, \mathbb{Q}_\ell) \to H_c^{2n-i}(X, \mathbb{Q}_\ell)$. By [HS06, Thm. 2], as k is algebraically closed, there is a morphism $\lambda : \mathrm{IC}_Y \to Rf_* \mathrm{IC}_X$ (which may not be unique) fitting to a commutative square

$$\begin{tikzcd}
\mathbb{Q}_{\ell,Y} \arrow[r] \arrow[d, "\gamma_Y"] & Rf_* \mathbb{Q}_{\ell,X} \arrow[d, "Rf_* \gamma_X"] \\
\mathrm{IC}_Y \arrow[r, dashed, "\lambda"] & Rf_* \mathrm{IC}_X
\end{tikzcd}$$

in $D_c^b(Y)$. It induces a commutative square

$$\begin{tikzcd}
H^{2n-i}_c(Y,\mathbb{Q}_\ell) \arrow[r] \arrow[d, two heads] & H^{2n-i}_c(X,\mathbb{Q}_\ell) \arrow[d, "\cong"] \\
\mathrm{IH}^{2n-i}_c(Y,\mathbb{Q}_\ell) \arrow[r, hook, dashed, "\psi"] & \mathrm{IH}^{2n-i}_c(X,\mathbb{Q}_\ell)
\end{tikzcd}$$

of vector spaces. From Lemma 3.6, as Y is normal and $i \leq 2$, the left vertical morphism is surjective. Then every linear map $\mathrm{IH}^{2n-i}_c(Y,\mathbb{Q}_\ell) \to \mathrm{IH}^{2n-i}_c(X,\mathbb{Q}_\ell)$ fitting into such a commutative square must be unique. By Lemma 3.8, ψ is injective. As X is smooth, the right vertical morphism is an isomorphism. This ψ is identified with an injection $\mathrm{IH}^{2n-i}_c(Y,\mathbb{Q}_\ell) \hookrightarrow H^{2n-i}_c(X,\mathbb{Q}_\ell)$. $\square$

Lemma 3.8 shows that for a surjective proper morphism $f : X \to Y$, IC_Y is a direct summand of $Rf_* \mathrm{IC}_X$.

Lemma 3.8. *Let $f : X \to Y$ be a surjective proper morphism of integral varieties over an algebraically closed field k. Then for every morphism $\lambda : \mathrm{IC}_Y \to Rf_* \mathrm{IC}_X$ in $D^b_c(Y)$ fitting into a commutative square*

$$\begin{tikzcd}
\mathbb{Q}_{\ell,Y} \arrow[r] \arrow[d, "\gamma_Y"] & Rf_*\mathbb{Q}_{\ell,X} \arrow[d, "Rf_*\gamma_X"] \\
\mathrm{IC}_Y \arrow[r, dashed, "\lambda"] & Rf_* \mathrm{IC}_X,
\end{tikzcd}$$

there is a morphism $r : Rf_* \mathrm{IC}_X \to \mathrm{IC}_Y$ in $D^b_c(Y)$ with $r \circ \lambda = \mathrm{Id}_{\mathrm{IC}_Y}$. In particular, for every integer j, the morphisms

$$\mathrm{IH}^j_c(Y,\mathbb{Q}_\ell) \to \mathrm{IH}^j_c(X,\mathbb{Q}_\ell), \quad \mathrm{IH}^j(Y,\mathbb{Q}_\ell) \to \mathrm{IH}^j(X,\mathbb{Q}_\ell)$$

induced by λ are injective.

Proof. Let $i : Z \to X$ be as in Lemma 3.9. Denote by ψ the morphism in $D^b_c(Y)$

$$R(fi)_*\gamma_Z : R(fi)_*\mathbb{Q}_{\ell,Z} \to R(fi)_* \mathrm{IC}_Z .$$

By [HS06, Thm. 2], as k is algebraically closed, there exist morphisms $\alpha : \mathrm{IC}_Y \to R(fi)_* \mathrm{IC}_Z$ and $\beta : \mathrm{IC}_X \to Ri_* \mathrm{IC}_Z$ fitting into commutative diagrams

$$\begin{tikzcd}
\mathbb{Q}_{\ell,Y} \arrow[r] \arrow[d, "\gamma_Y"] & R(fi)_*\mathbb{Q}_{\ell,Z} \arrow[d, "\psi"] \\
\mathrm{IC}_Y \arrow[r, dashed, "\alpha"] & R(fi)_* \mathrm{IC}_Z,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}_{\ell,X} \arrow[r] \arrow[d, "\gamma_X"] & i_*\mathbb{Q}_{\ell,Z} \arrow[d] \\
\mathrm{IC}_X \arrow[r, dashed, "\beta"] & Ri_* \mathrm{IC}_Z .
\end{tikzcd}$$

Let $m := \dim Y$. As $fi : Z \to Y$ is proper, $\mathbb{D}_Y\,\psi$ is a morphism

$$R(fi)_* \mathrm{IC}_Z(m)[2m] \to R(fi)_* \mathbb{D}_Z\, \mathbb{Q}_{\ell,Z}.$$

One has a commutative diagram in $D^b_c(Y)$

$$\begin{tikzcd}
\mathbb{Q}_{\ell,Y} \arrow[r] \arrow[d, "\gamma_Y"] & Rf_*\mathbb{Q}_{\ell,X} \arrow[r] \arrow[d, "Rf_*\gamma_X"] & R(fi)_*\mathbb{Q}_{\ell,Z} \arrow[d, "\psi"'] & \\
\mathrm{IC}_Y \arrow[r, "\lambda"] & Rf_*\,\mathrm{IC}_X \arrow[r, dashed, "Rf_*\beta"] & R(fi)_*\,\mathrm{IC}_Z \arrow[r, dashed, "(\mathbb{D}_Y\,\alpha)(-m)[-2m]"] \arrow[d, "(\mathbb{D}_Y\,\psi)(-m)[-2m]"'] & \mathrm{IC}_Y \arrow[d, "\mathbb{D}_Y\,\gamma_Y(-m)[-2m]"] \\
& & (R(fi)_*\,\mathbb{D}_Z\,\mathbb{Q}_{\ell,Z})(-m)[-2m] \arrow[r] & (\mathbb{D}_Y\,\mathbb{Q}_{\ell,Y})(-m)[-2m].
\end{tikzcd} \tag{7}$$

Let η, ξ be the generic points of Y and Z respectively. By generic smoothness [Sta25, Tag 056V] for Y, there is a nonempty open subset V of Y that is smooth over k. Thus,

$$(\gamma_Y)|_V : \mathbb{Q}_{\ell,V} \xrightarrow{\sim} \mathrm{IC}_Y|_V \tag{8}$$

is an isomorphism. By the proper base change theorem, $(fi)^{-1}(\eta) = \xi$ and generic smoothness for Z, the $\mathbb{Q}_\ell$-linear map

$$\psi_{\bar\eta} : (R(fi)_*\mathbb{Q}_{\ell,Z})_{\bar\eta} \xrightarrow{\sim} (R(fi)_*\,\mathrm{IC}_Z)_{\bar\eta} \tag{9}$$

is an isomorphism.

Let $d := [k(Z) : k(Y)]$ be the degree of the generically finite morphism $fi : Z \to Y$. Then the canonical morphism

$$(\mathbb{Q}_{\ell,Y})_{\bar\eta} \to (R(fi)_*\mathbb{Q}_{\ell,Z})_{\bar\eta}$$

can be identified with the diagonal inclusion $\mathbb{Q}_\ell \to \mathbb{Q}_\ell^d$. By the Verdier duality, its dual

$$(Rf_*\,\mathbb{D}_Z\,\mathbb{Q}_{\ell,Z})_{\bar\eta}(-m)[-2m] \to (\mathbb{D}_Y\,\mathbb{Q}_{\ell,Y})_{\bar\eta}(-m)[-2m]$$

can be identified with the sum morphism $\mathbb{Q}_\ell^d \to \mathbb{Q}_\ell$. Let

$$t : \mathbb{Q}_{\ell,Y} \to (\mathbb{D}_Y\,\mathbb{Q}_{\ell,Y})(-m)[-2m]$$

be the morphism resulting from Diagram (7). Then by (9),

$$t_{\bar\eta} : \mathbb{Q}_\ell \to (\mathbb{D}_Y\,\mathbb{Q}_{\ell,Y})_{\bar\eta}(-m)[-2m]$$

is multiplication by d. Since

$$(\mathbb{D}_Y\,\mathbb{Q}_{\ell,Y})|_V(-m)[-2m] = \mathbb{Q}_{\ell,V},$$

$t|_V : \mathbb{Q}_{\ell,V} \to \mathbb{Q}_{\ell,V}$ is also multiplication by d. Let $r : Rf_*\,\mathrm{IC}_X \to \mathrm{IC}_Y$ be the composition

$$\frac{1}{d}(\mathbb{D}_Y\,\alpha) \circ Rf_*\beta(-m)[-2m].$$

From (8), the restriction $\mathbb{Q}_{\ell,V} \to \mathbb{Q}_{\ell,V}$ of $r \circ \lambda : \mathrm{IC}_Y \to \mathrm{IC}_Y$ to V is the identity. By the continuation principle [KW01, III, Cor. 5.11], one has $r \circ \lambda = \mathrm{Id}_{\mathrm{IC}_Y}$. $\square$

Lemma 3.9. *Let $f : X \to Y$ be a dominant morphism of integral varieties over a field k. Let η be the generic point of Y. Then there is a closed integral subvariety $i : Z \to X$ with $\dim Z = \dim Y$, such that the set $(fi)^{-1}(\eta)$ consists of the generic point of Z, and the extension $k(Y) \subset k(Z)$ is finite.*

Proof. By the relative Noether normalization lemma [Die74, 4.4, Prop. 14], there is a nonempty open subset W of Y, a nonempty open subset U of $f^{-1}(W)$, an integer $n \geq 0$ and a factorization $U \xrightarrow{g} W \times_k \mathbf{A}_k^n \to W$ of $f|_U : U \to W$, where g is finite surjective. Let $S = g^{-1}(W \times 0)$, which is a closed subscheme of U. Let Z' be the Zariski closure of S in X. Since $f(Z')$ contains $f(S) = W$, it is dense in Y. Since Y is irreducible, there is an irreducible component Z of Z' with $f(Z)$ dense in Y. Consider the commutative diagram

$$\begin{tikzcd}
 & Z \cap S \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\
W \arrow[d, hook] \arrow[dr, phantom, "\square"] & S \arrow[l] \arrow[r, hook] \arrow[d, hook] & Z' \arrow[d, hook] \\
\mathbf{A}_W^n \arrow[d] & U \arrow[l, two heads, "g"'] \arrow[r, hook] & X \arrow[d, "f"] \\
W \arrow[rr, hook] & & Y.
\end{tikzcd}$$

Endow Z with the reduced induced closed subscheme structure. Then $fi : Z \to Y$ is dominant. As $S \subset X$ is locally closed, S is open in Z'. Then $Z \cap S := Z \times_X S$ is an open subscheme of Z. The composition $Z \cap S \hookrightarrow S \xrightarrow{f} W$ is finite, which implies the second equality in

$$\dim Z = \dim Z \cap S = \dim W = \dim Y.$$

By [Sta25, Tag 02NX], $[k(Z) : k(Y)]$ is finite. $\square$

Lemma 3.10 shows that up to torsion, algebraic equivalence of line bundles agree with that of Weil divisors. A normal variety over a field is called $\mathbb{Q}$-*factorial*, if every Weil divisor has a nonzero multiple that is a Cartier divisor.

Lemma 3.10. *Let X be a normal integral scheme of dimension d, proper over an algebraically closed field k of characteristic 0. Then the kernel of the morphism $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$ from Lemma 2.1 is finite. In particular, one has $\rho(X) \leq \rho^{\mathrm{ns}}(X)$. If X is $\mathbb{Q}$-factorial, then $\mathrm{NS}(X)_{\mathbb{Q}} \to \mathrm{Cl}^{\mathrm{ns}}(X)_{\mathbb{Q}}$ is an isomorphism.*

Proof. By [Sta25, Tag 01ZM (2)], there is a subfield $F \subset k$ that is finitely generated over $\mathbb{Q}$, a variety X_F over F such that $X_F \otimes_F k$ is isomorphic to X. From the fpqc descent of properness [Sta25, Tag 0F4J], X_F is proper over F. By [Sta25, Tag 038H], X_F is irreducible. From [Sta25, Tag 038P], X_F is geometrically normal over F. Then by Lemma 2.5 and [Liu25, Fact 1.8], one may assume that k is the algebraic closure of F. Thus, k admits an embedding into $\mathbb{C}$. Using Lemma 2.5 and [Liu25, Fact 1.8], one may assume $k = \mathbb{C}$.

We consider singular cohomology groups. As X is proper over $\mathbb{C}$, $V_{\mathbb{Q}} := H^2(X, \mathbb{Q})$ is a mixed $\mathbb{Q}$-Hodge structure of weight ≤ 2. Let $c_1 : \mathrm{NS}(X) \to H^2(X, \mathbb{Z})$ be the first Chern class map. Then $c_1(\mathrm{NS}(X)) \subset F^1 H^2(X, \mathbb{C})$ and hence the image of $c_1 : \mathrm{NS}(X) \to H^2(X, \mathbb{Q})$ lies in $\ker(V_{\mathbb{Q}} \to \mathrm{Gr}_0^F V_{\mathbb{C}})$. From [Del74, Théorème 8.2.4 (iii)], one has $\mathrm{Gr}_F^i V_{\mathbb{C}} = 0$ for every integer $i \notin \{0, 1, 2\}$. Then by [PP25, Lemma 4.8], the natural map

$$\ker(V_{\mathbb{Q}} \to \mathrm{Gr}_0^F V_{\mathbb{C}}) \hookrightarrow \mathrm{Gr}_2^W V_{\mathbb{Q}}$$

is injective.

By [Web04, Theorem 9.2], the image of the natural map $H^2(X, \mathbb{Q}) \to \mathrm{IH}^2(X, \mathbb{Q})$ is $\mathrm{Gr}_2^W V_{\mathbb{Q}}$. By the complex analog of Lemma 3.6, as X is normal, the map $\mathrm{IH}^2(X, \mathbb{Q}) \to H_{2d-2}(X, \mathbb{Q})$ is injective. Then one gets a commutative square

$$\begin{tikzcd}
\mathrm{NS}(X) \arrow[r, "c_1"] \arrow[d] & \ker(V_{\mathbb{Q}} \to \mathrm{Gr}_0^F V_{\mathbb{C}}) \arrow[r, hook] & \mathrm{Gr}_2^W V_{\mathbb{Q}} \\
\mathrm{Cl}^{\mathrm{ns}}(X) \arrow[r, "\mathrm{cl}"] & H_{2d-2}(X, \mathbb{Q}) & \mathrm{IH}^2(X, \mathbb{Q}). \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

So for every L in the kernel of $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$, one has $c_1(L) = 0$ in $H^2(X, \mathbb{Q})$. There is an integer $n > 0$ with $c_1(L^{\otimes n}) = 0$ in $H^2(X, \mathbb{Z})$. By [BVRS09, p.261], $L^{\otimes n} = 0$ in $\mathrm{NS}(X)$. By the theorem of the base [SGA 6, XIII, Thm. 5.1], as X is proper over k, $\mathrm{NS}(X)$ is finitely generated. As the kernel of $\mathrm{NS}(X) \to \mathrm{Cl}^{\mathrm{ns}}(X)$ is torsion, it is finite. $\square$

4 Specialization of Néron-Severi class groups

In Lemma 4.2, given a morphism $f : X \to B$ of algebraic varieties and a specialization of points $t \rightsquigarrow s$ of B, we construct a specialization morphism $\mathrm{Cl}^{\mathrm{ns}} X_{\bar{t}} \to \mathrm{Cl}^{\mathrm{ns}} X_{\bar{s}}$ of Néron-Severi class groups.

Lemma 4.1. *Let R be a Henselian discrete valuation ring with fraction field K and residue field k. Let $f : X \to \operatorname{Spec} R$ be a separated morphism of finite type of schemes. Let $\bar{K}$ be an algebraic closure of K. Let $\bar{k}$ be the residue field of the integral closure $\bar{R}$ of R in $\bar{K}$.*

(a) *Then for every integer $n \geq 0$, there is a natural morphism $\sigma_n : B_n(X_{\bar{K}}) \to B_n(X_{\bar{k}})$.*

(b) *Assume that X is proper over R. Let ℓ be a prime invertible in R. Assume that $\mathbb{Z}_{\ell,X}$ is universally locally acyclic relative to $X \to \operatorname{Spec} R$. Then for every integer $n \geq 0$, there is a natural commutative diagram*

$$\begin{tikzcd} B_n(X_{\bar{K}}) \arrow[r, "\sigma_n"] \arrow[d, "\mathrm{cl}_{X_{\bar{K}}}"'] & B_n(X_{\bar{k}}) \arrow[d, "\mathrm{cl}_{X_{\bar{k}}}"] \\ H_{2n}(X_{\bar{K}}, \mathbb{Z}_\ell)(-n) & H_{2n}(X_{\bar{k}}, \mathbb{Z}_\ell)(-n), \arrow[l, "\mathrm{sp}"'] \end{tikzcd}$$

whose bottom is a specialization isomorphism.

(c) *If X_k is locally equidimensional, then for every $n \geq 0$, there is a natural specialization morphism $\sigma^n : B^n(X_{\bar{K}}) \to B^n(X_{\bar{k}})$.*

For an integer $n \geq 0$ and a scheme X of finite type over a field k, let $\mathrm{Alg}_n(X)$ be the subgroup of $Z_n X$ of cycles algebraically equivalent to zero. For a regular Noetherian scheme S and a scheme X separated of finite type over S, let $A_n(X/S)$ be the group of rational equivalence classes of n-cycles on X in the sense of [Ful98, p.394].

Proof. From [Art71, Proposition 1.4], as R is Henselian, $\bar{R}$ is a local ring. By construction, $\bar{k}$ is an algebraic closure of k.

(a) From [Ful98, Example 20.3.5], because R is Henselian, there is a natural specialization morphism $\sigma_n : A_n(X_{\bar{K}}) \to A_n(X_{\bar{k}})$. It remains to prove

$$\sigma_n(\mathrm{Alg}_n(X_{\bar{K}})/\operatorname{Rat}_n(X_{\bar{K}})) \subset \mathrm{Alg}_n(X_{\bar{k}})/\operatorname{Rat}_n(X_{\bar{k}}). \tag{10}$$

By [Ful98, Example 10.3.2] (see also [ACMV19, Thm. 1]), for every $a \in \mathrm{Alg}_n(X_{\bar{K}})$, there is a connected smooth projective curve $C_{\bar{K}}$ over $\bar{K}$, two points $c_1, c_2 \in C_{\bar{K}}(\bar{K})$, and finitely many n-dimensional integral closed subvarieties $V_1, \ldots, V_r$ of $X_{\bar{K}} \times_{\bar{K}} C_{\bar{K}}$ flat over $C_{\bar{K}}$ with

$$a = \sum_{j=1}^{r} [(V_j)_{c_1}] - [(V_j)_{c_2}]$$

in $A_n(X_{\bar{K}})$. There is a subextension $K' \subset \bar{K}$ finite over K, such that a is in the image of $A_n(X_{K'}) \to A_n(X_{\bar{K}})$, and C, c_1, c_2 as well as the V_i can be defined over K'. Replacing R by its integral closure in K', which is again a Henselian discrete valuation ring, and K by K', one may assume $K = K'$. Let C_K be a smooth, projective, geometrically connected curve over K with $C_K \times_K \bar{K}$ isomorphic to $C_{\bar{K}}$. By [Sta25, Tag 0FD2], the map $K \to H^0(C_K, O_{C_K})$ is an isomorphism. Then by [Sta25, Tag 0C2W], there is a regular integral scheme C flat proper over R with generic fiber C_K. From [Sta25, Tag 0AY8], the special fiber C_k is geometrically connected. By [EGA IV 2, Prop. 2.8.5], for every $1 \leq i \leq r$, there is a closed subscheme $\mathcal{V}_i$ of $X \times_R C$, flat over R with geometric generic fiber V_i.

Let $s \in \operatorname{Spec} R$ be the closed point. As C is proper over R, the map $C(R) \to C_K(K)$ is bijective. For $i = 1, 2$, let $x_i \in C(R)$ be the preimage of c_i. Then $x_i : \operatorname{Spec} R \to C$ is an immersion. By [EGA IV 4, Prop. 19.1.1], because C is regular at $x_i(s)$, $x_i : \operatorname{Spec} R \to C$ is a regular immersion at s.

From [EGA IV 4, Thm. 17.12.1, c)⇒a)], $C \to \operatorname{Spec} R$ is smooth at $x_i(s)$. Then $C_k \to \operatorname{Spec} k$ is also smooth at $x_i(s)$.

By [Ful98, p.399], one has $\sigma(a) = \sum_{i=1}^{r}[(\mathcal{V}_{i,\bar{k}})_{c_1'}] - [(\mathcal{V}_{i,\bar{k}})_{c_2'}]$ in $A_n(X_{\bar{k}})$. By [Wei54, Lem. 9], because C_k is smooth at $x_1(s)$ and $x_2(s)$, one has $\sigma_n(a) \in \operatorname{Alg}_n(X_{\bar{k}})/\operatorname{Rat}_n(X_{\bar{k}})$, which proves (10). Thus, σ_n descends to a morphism $B_n(X_{\bar{K}}) \to B_n(X_{\bar{k}})$.

(b) Let $j : X_K \to X$ and $\iota : X_k \to X$ be the open immersion and closed immersion respectively. As $\mathbb{Z}_\ell$ is universally locally acyclic relative to $X \to \operatorname{Spec} R$, the restriction of $\mathbb{D}_{X/R}\,\mathbb{Z}_\ell = Rf^!\mathbb{Z}_\ell$ to X_k is $\mathbb{D}_{X_k}\,\mathbb{Z}_\ell = Rf_s^!\mathbb{Z}_\ell$. By [Sta25, Tag 0A3S], as R is a Henselian local ring and X is proper over R, the morphism

$$\iota^* : H^{-2n}(X, \mathbb{D}_{X/R}\,\mathbb{Z}_\ell) \to H^{-2n}(X_k, \mathbb{D}_{X_k}\,\mathbb{Z}_\ell)$$

is an isomorphism.

Let $A_n'(X/R)$ be the subgroup of $A_n(X/R)$ generated by integral closed subschemes $i : V \to X$ such that $g = f \circ i : V \to \operatorname{Spec} R$ is flat of relative dimension n. For such a V, the trace morphism in [SGA4III, XVIII, Théorème 2.9] induces a canonical morphism $t_g : \mathbb{Z}_{\ell,V} \to Rg^!\mathbb{Z}_\ell\langle -n\rangle$ in $D_c^b(V)$, which corresponds to the ℓ-adic cycle class $c_\ell(V/\operatorname{Spec} R) \in H_{2n}(V/R, \mathbb{Z}_\ell)(-n)$ of V introduced by [Li13, p.81].

The closed immersion $i : V \hookrightarrow X$ is proper, so one has push-forward $i_* : H_{2n}(V/R, \mathbb{Z}_\ell) \to H_{2n}(X/R, \mathbb{Z}_\ell)$ in the sense of [Li13, Definition 4.1]. Set $\operatorname{cl}_{X/R}(V) := i_* c_\ell(V/\operatorname{Spec} R) \in H_{2n}(X/R, \mathbb{Z}_\ell)(-n)$. Thus, one gets a group morphism

$$\operatorname{cl}_{X/R} : A_n'(X/R) \to H_{2n}(X/R, \mathbb{Z}_\ell)(-n) = H^{-2n}(X, Rf^!\mathbb{Z}_\ell)(-n).$$

By construction, the left square in the diagram

$$\begin{tikzcd}
 & A_n'(X/R) \arrow[dl, "j^*"'] \arrow[dr, "\iota^!"] \arrow[dd, "\operatorname{cl}_{X/R}"] & \\
A_n X_K \arrow[dd, "\operatorname{cl}_{X_K}"'] & & A_n X_k \arrow[dd, "\operatorname{cl}_{X_k}"] \\
 & H^{-2n}(X, \mathbb{D}_{X/R}\,\mathbb{Z}_\ell)(-n) \arrow[dl, "j^*"'] \arrow[dr, "\iota^*"] & \\
H^{-2n}(X_K, \mathbb{D}_{X_K}\,\mathbb{Z}_\ell)(-n) & & H^{-2n}(X_k, \mathbb{D}_{X_k}\,\mathbb{Z}_\ell)(-n) \arrow[ll, dashed, "\mathrm{sp}:=j^*(\iota^*)^{-1}"]
\end{tikzcd} \tag{11}$$

is commutative.

The composition

$$\mathbb{Z}_{\ell,X} \xrightarrow{\delta_i} Ri_*\mathbb{Z}_{\ell,V} \xrightarrow{Ri_*(t_g)} Ri_*Rg^!\mathbb{Z}_\ell\langle -n\rangle = Ri_*Ri^!Rf^!\mathbb{Z}_\ell\langle -n\rangle \xrightarrow{\theta_i} Rf^!\mathbb{Z}_\ell\langle -n\rangle$$

in $D^b_c(X)$ corresponds to $i_* c_\ell(V/\operatorname{Spec} R)$, where δ_i and θ_i are natural morphisms from adjunction. Restricting it to X_k one has the first row of the diagram

$$\begin{tikzcd}
\mathbb{Z}_{\ell,X}|_{X_k} \arrow[r, "\delta_i|_{X_k}"] \arrow[d, equal] & (i_*\mathbb{Z}_{\ell,V})|_{X_k} \arrow[r, "(Ri_*t_g)|_{X_k}"] \arrow[d, "\phi"] & (i_*Rg^!\mathbb{Z}_\ell)|_{X_k}\langle -n\rangle \arrow[r, equal] \arrow[d, "\psi"] & (i_*Ri^!Rf^!\mathbb{Z}_\ell)|_{X_k}\langle -n\rangle \arrow[r, "\theta_i|_{X_k}"] \arrow[d] & (Rf^!\mathbb{Z}_{\ell,X})|_{X_k}\langle -n\rangle \arrow[d, "\xi"] \\
\mathbb{Z}_{\ell,X_k} \arrow[r, "\delta_{i_s}"] & i_{s*}\mathbb{Z}_{\ell,V_s} \arrow[r, "Ri_{s*}t_{g_s}"] & i_{s*}Rg_s^!\mathbb{Z}_\ell\langle -n\rangle \arrow[r, equal] & i_{s*}Ri_s^!Rf_s^!\mathbb{Z}_\ell\langle -n\rangle \arrow[r, "\theta_{i_s}"] & Rf_s^!\mathbb{Z}_\ell\langle -n\rangle,
\end{tikzcd}$$

where ϕ is from the proper base change theorem, ψ and ξ are the natural morphisms from [SGA4III, XVIII, (3.1.14.2)], and the composition in the second row corresponds to $\mathrm{cl}_{X_k}(V_s) = i_{s*}c_\ell(V_s/k) \in H_{2n}(X_k, \mathbb{Z}_\ell)(-n)$. The vertical morphisms are natural, so the diagram is commutative. As $\mathbb{Z}_{\ell,X}$ is universally locally acyclic relative to $X \to \operatorname{Spec} R$, the last vertical morphism ξ is an isomorphism. This shows that the right square in the diagram (11) is commutative.

By [Ful98, p.399], $\sigma_n : A_n(X_K) \to A_n(X_k)$ is the unique morphism fitting into a commutative triangle

$$\begin{tikzcd}
& A_n(X/R) \arrow[dl, "j^*"'] \arrow[dr, "\iota^!"] & \\
A_n(X_K) \arrow[rr, dashed, "\sigma_n"] & & A_nX_k.
\end{tikzcd}$$

Then from (11), the square

$$\begin{tikzcd}
A_nX_K \arrow[r, "\sigma_n"] \arrow[d, "\mathrm{cl}_{X_K}"'] & A_nX_k \arrow[d, "\mathrm{cl}_{X_k}"] \\
H_{2n}(X_K, \mathbb{Z}_\ell)(-n) & H_{2n}(X_k, \mathbb{Z}_\ell)(-n) \arrow[l, "\mathrm{sp}"']
\end{tikzcd}$$

is commutative. Passing to limit, we get a commutative square

$$\begin{tikzcd}
A_nX_{\bar K} \arrow[r, "\sigma_n"] \arrow[d, "\mathrm{cl}_{X_{\bar K}}"'] & A_nX_{\bar k} \arrow[d, "\mathrm{cl}_{X_{\bar k}}"] \\
H_{2n}(X_{\bar K}, \mathbb{Z}_\ell)(-n) & H_{2n}(X_{\bar k}, \mathbb{Z}_\ell)(-n), \arrow[l, "\mathrm{sp}"']
\end{tikzcd}$$

where $\mathrm{sp} : H_{2n}(X_{\bar k}, \mathbb{Z}_\ell) \to H_{2n}(X_{\bar K}, \mathbb{Z}_\ell)$ is the specialization of the étale sheaf $R^{-2n} f_* \mathbb{D}_{X/R}\mathbb{Z}_{\ell,X}$ on $\operatorname{Spec} R$. From [Sta25, Tag 0GJW], it is an isomorphism. By [Lau76, Thm. 6.3], this square induces the commutative square in the statement.

(c) It follows from Part (a) and [SGA 6, X, App., (7.14.1 bis)].

$\square$

Lemma 4.2. *Let $X \to B$ be a separated finite type morphism of locally Noetherian schemes. Let $s, t \in B$ be such that s is a specialization of t. Choose geometric points $\bar{s}, \bar{t}$ over s and t respectively. Assume that X_t and X_s are geometrically integral. Then there is a (noncanonical) specialization morphism* $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{t}}) \to \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{s}})$.

Proof. As B is locally Noetherian, one can a morphism $\operatorname{Spec} R \to B$ from a Henselian trait sending the generic point and the closed point of $\operatorname{Spec} R$ to t and s respectively. Let $\bar{K}$ and $\bar{k}$ be as in Lemma 4.1. By Lemma 4.1 (c), as $X_{\bar{s}}$ is irreducible, there is a specialization morphism $\sigma : \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{K}}) \to \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{k}})$. From Lemma 2.5, as X_t and X_s are geometrically integral, there exist isomorphisms $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{t}}) \xrightarrow{\sim} \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{K}})$ and $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{s}}) \xrightarrow{\sim} \mathrm{Cl}(X_{\bar{k}})$. Thus, one gets a morphism $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{t}}) \to \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{s}})$. $\square$

Remark 4.3. Let B be an irreducible variety with generic point η over a field k of characteristic 0. Let $f : X \to B$ be a normal proper morphism of schemes with irreducible geometric fibers. Using Theorem 3.2 and Lemma 4.1 (b), one can prove that up to shrinking B, for every $b \in B$ the specialization morphism $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{\eta}}) \to \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{b}})$ has finite kernel.

Remark 4.4. In Lemma 4.2, let $\pi : X' \to X$ be a proper morphism of schemes over B, such that $X'_{\bar{t}}$ and $X'_{\bar{s}}$ are also integral. By construction and [Ful98, Prop. 20.3 (a)], one may choose specialization morphisms fitting into a commutative square

$$\begin{array}{ccc} \mathrm{Cl}^{\mathrm{ns}}(X'_{\bar{t}}) & \longrightarrow & \mathrm{Cl}^{\mathrm{ns}}(X'_{\bar{s}}) \\ \downarrow{\scriptstyle \pi_{\bar{t},*}} & & \downarrow{\scriptstyle \pi_{\bar{s},*}} \\ \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{t}}) & \longrightarrow & \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{s}}). \end{array}$$

In Lemma 4.5, we prove that the specializations for Weil divisors and Cartier divisors are compatible.

Lemma 4.5. *Let B be a Noetherian Japanese scheme with generic point η. Let $f : X \to B$ be a normal proper morphism with irreducible geometric fibers. Then there is a dense open subset $U \subset B$, such that for every geometric point $\bar{b}$ on U, one can* choose *specializations fitting into a commutative square*

$$\begin{array}{ccc} \mathrm{NS}(X_{\bar{\eta}}) & \xrightarrow{\mathrm{sp}_{\bar{\eta},\bar{b}}} & \mathrm{NS}(X_{\bar{b}}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{\eta}}) & \xrightarrow{\sigma} & \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{b}}). \end{array}$$

Proof. By [Liu25, Lemma 2.2 (a)], as $f : X \to B$ is normal, there is a dense open subset $U \subset B$ such that for every $b \in U$, one can choose a specialization $\mathrm{sp} : \mathrm{NS}\, X_{\bar{\eta}} \to \mathrm{NS}\, X_{\bar{b}}$. One may choose a Henselian trait $S = \operatorname{Spec} R$ and a morphism $\phi : S \to U$ sending the generic (resp. closed) points to η (resp. b). Let $\bar{S} := \operatorname{Spec} \bar{R}$. As $f : X \to B$ is normal, so is its base change $X \times_B \bar{S} \to \bar{S}$. Then $X \times_B \bar{S}$ is a normal integral scheme, and all fibers of $X \times_B \bar{S} \to \bar{S}$ are geometrically integral. For every finite extension K_α/K, let S_α be the normalization of S in K_α. As every S_α is a trait, the result now follows from Lemma 4.6. ☐

Lemma 4.6. *Let $f : X \to S$ be a flat, separated, finite type morphism from a reduced scheme to a trait. Assume the fibers of f are integral. Let $i : X_s \to X$ (resp. $j : X_\eta \to X$) be the inclusion of the special (resp. generic) fiber. Then the specialization morphism $\sigma : \mathrm{Cl}(X_\eta) \to \mathrm{Cl}(X_s)$ fits into a commutative diagram*

$$\begin{tikzcd}
\mathrm{Pic}(X_\eta) \arrow[d] & \mathrm{Pic}(X) \arrow[l, "j^*"'] \arrow[r, "i^*"] & \mathrm{Pic}(X_s) \arrow[d] \\
\mathrm{Cl}(X_\eta) \arrow[rr, "\sigma"] & & \mathrm{Cl}(X_s).
\end{tikzcd}$$

Proof. As the fibers are reduced, the vertical morphisms exist. As the fibers are irreducible, the specialization morphism $A^1 X_\eta \to A^1 X_s$ from [Ful98, p.399] is identified with $\sigma : \mathrm{Cl}(X_\eta) \to \mathrm{Cl}(X_s)$.

We prove that X_s is a principal, effective Cartier divisor on X. As S is a trait, i.e., $R := O_S(S)$ is a discrete valuation ring, one can choose a uniformizer $\pi \in R$. For every $x \in X_s$ and every affine neighborhood $U = \operatorname{Spec} A$ of x in X, one has $U \cap X_s = U_s = \operatorname{Spec} A/\pi$. As $f : X \to S$ is flat, the R-module A is flat, so π is a nonzerodivisor of A. By [Sta25, Tag 01WS], X_s is an effective Cartier divisor on X and $X_s = \operatorname{div}_X(\pi)$. In particular, $i : X_s \to X$ is a regular immersion.

As X is reduced, for every $L \in \mathrm{Pic}(X)$, there is a Cartier divisor D on X with $O_X(D) \cong L$. Then D_η is a Cartier divisor on X_η with $O_{X_\eta}(D_\eta) \cong j^*L$. Let $\iota : s \to S$ be the inclusion. One has

$$\sigma(D_\eta) \overset{\text{(a)}}{=} \iota^! D \overset{\text{(b)}}{=} i^* D \overset{\text{(c)}}{=} X_s \cdot [D] \overset{\text{(d)}}{=} D \cdot [X_s],$$

where (a), (b), (c) and (d) are from [Ful98, p.399, Rk. 6.2.1, (2) on p.33, Thm. 2.4] respectively. By [Ful98, p.41], it is the image of i^*L under the morphism $\mathrm{Pic}(X_s) \to \mathrm{Cl}(X_s)$. ☐

5 Existence of $\mathrm{Cl}^{\mathrm{ns}}$-generic and NS-generic points

We prove Theorems 1.4 and 1.5. Both rely on Hironaka's resolution of singularities. In Lemma 5.1, Part (a) constructs a simultaneous resolution of singularities. Part (b) shows that the exceptional divisors for this family of birational morphisms are locally constant.

Lemma 5.1. *Let $f : X \to B$ be a separated morphism of finite type of locally Noetherian schemes. Assume that B is integral with generic point η, and $\mathrm{char} k(\eta) = 0$. Suppose that X_η is geometrically integral.*

(a) *Then up to shrinking B to a dense open subset, there is a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & B \end{tikzcd}$$

of schemes, where $f' : X' \to B$ is smooth with geometrically integral fibers, $\pi : X' \to X$ is proper, and there is an open subset $U \subset X$ with $U \to B$ surjective, such that for every $b \in B$, $\pi_b^{-1}(U_b) \xrightarrow{\sim} U_b$ is an isomorphism. In particular, $\pi_b : X'_b \to X_b$ is birational.

(b) *Assume further that $f : X \to B$ is a normal morphism of reduced varieties over a field k of characteristic 0. Then there is a smooth irreducible variety B' over k and a quasi-finite étale morphism $B' \to B$ over k, such that up to replacing B by B', in addition to the diagram as in Part* (a)*, there exist finitely many reduced closed subschemes $\{E_i\}_{i\in I}$ of X' with all $E_i \to B$ flat surjective, such that for every geometric point $\bar{b}$ on B, $\{E_{i,\bar{b}}\}_{i\in I}$ is the set of exceptional divisors of $\pi_{\bar{b}} : X'_{\bar{b}} \to X_{\bar{b}}$.*

Proof. (a) By Hironaka's theorem [Hir64, Main Theorem I], because $\operatorname{char} k(\eta) = 0$ and X_η is separated, geometrically integral of finite type over $k(\eta)$, there is a smooth, geometrically integral variety X'_η and a proper birational morphism $\pi_\eta : X'_\eta \to X_\eta$ over $k(\eta)$. By [Sta25, Tag 01ZM (1) and (2)], shrinking B, one may assume that there is a morphism of finite presentation $f' : X' \to B$ with generic fiber X'_η, and a morphism $\pi : X' \to X$ over B whose base change along $\eta \to B$ is $\pi_\eta : X'_\eta \to X_\eta$. As X is locally Noetherian, B is integral and X'_η is smooth, shrinking B one may assume that $f' : X' \to B$ is smooth. By [Sta25, Tag 08K1], as $\pi_\eta : X'_\eta \to X_\eta$ is proper, shrinking B one may assume that $\pi : X' \to X$ is proper.

From [Sta25, Tag 0559, Tag 0578], as X_η, X'_η are geometrically integral, shrinking B one may assume the fibers of $f : X \to B$ and $f' : X' \to B$ are geometrically integral. By [Sta25, Tag 0BAJ], as $\pi_\eta : X'_\eta \to X_\eta$ is birational, there is a dense open subset U_η of X_η such that $\pi_\eta^{-1}(U_\eta) \to U_\eta$ is an isomorphism. There is a nonempty open subset U of X with $U \cap X_\eta = U_\eta$. From [Sta25, Tag 05F5], shrinking B one may assume $f|_U : U \to B$ is surjective. By [Sta25, Tag 081E], since the generic fiber of $\pi^{-1}(U) \to B$ is $\pi_\eta^{-1}(U_\eta)$, shrinking B one may assume that $\pi^{-1}(U) \to U$ is an isomorphism. Thus, $\pi : X' \to X$ is birational. For every $b \in B$, $\pi_b^{-1}(U_b) \to U_b$ is an isomorphism. As U_b is nonempty, $\pi_b : X'_b \to X_b$ is a birational morphism.

(b) Shrink B so that one gets a diagram as in Part (a). By [Sta25, Tag 037A], X and X' are integral. From [Sta25, Tag 056V], shrinking B one may assume that B is smooth. From [EGA IV 2, Prop. 6.14.1], as B is normal, so is X. Since $\pi^{-1}(U) \to U$ is an isomorphism, $\pi : X' \to X$ is birational. Then by Zariski's main theorem (see, *e.g.*, [Deb01, Sec. 1.40]), as X is integral and *normal*, the exceptional locus $E := \mathrm{Exc}(\pi)$ of $\pi : X' \to X$ is $\{x' \in X' | \dim X_{\pi(x')} > 0\}$. Similarly, for every geometric point $\bar{b}$ on B, as $X_{\bar{b}}$ is integral and *normal*, one has

$$\mathrm{Exc}(\pi_{\bar{b}}) = \{x_1 \in X'_{\bar{b}} | \dim \pi_{\bar{b}}^{-1}(\pi_{\bar{b}}(x_1)) > 0\}.$$

For $x_1 \in X'_{\bar{b}}$, let x' be its image in X'. Then $\pi_{\bar{b}}^{-1}(\pi_{\bar{b}}(x_1))$ is a scalar extension of $X'_{\pi(x')}$, so

$$\dim \pi_{\bar{b}}^{-1}(\pi_{\bar{b}}(x_1)) = \dim X'_{\pi(x')}.$$

Consequently, one has $\mathrm{Exc}(\pi_{\bar{b}}) = E_{\bar{b}}$ set theoretically.

Endow E with the reduced induced structure. By [Sta25, Tag 0551], replacing B by a finite étale cover of a dense open subset, one may assume further that all irreducible fibers of E_η are geometrically irreducible. Let $E_\eta = \cup_{i \in I'} E_{i,\eta}$ be the decomposition into irreducible components. For every $i \in I'$, let E_i be the Zariski closure of $E_{i,\eta}$ in E, endowed with the reduced induced structure. Shrinking B one may assume that the morphisms $E_i \to B$ are surjective flat. From the last paragraph of the proof of [Sta25, Tag 055A], shrinking B one may assume that for every $b \in B$, $E_b = \cup_{i \in I'} E_{i,b}$ is the decomposition into geometrically irreducible components. By flatness of $E_i \to B$ and dimension formula [Har77, III, Prop. 9.5], one has

$$\mathrm{codim}_{X'_b} E_{i,b} = \mathrm{codim}_{X'_\eta} E_{i,\eta}.$$

Let

$$I := \{i \in I' \mid \mathrm{codim}_{X'_\eta} E_{i,\eta} = 1\}.$$

Then $\{E_{i,\bar{b}}\}_{i \in I}$ is the set of exceptional divisors of $\pi_{\bar{b}} : X'_{\bar{b}} \to X_{\bar{b}}$.

□

Remark 5.2. In some cases, we restrict our attention to normal morphisms with *geometrically integral* fibers. Let $f : X \to B$ be a normal morphism of finite type of Noetherian schemes. Assume that B is irreducible with generic point η. By [Sta25, Tag 0551], as $f : X \to B$ is of finite type, there is a nonempty open subset $B_0 \subset B$, an irreducible scheme B_1 and a surjective finite étale morphism $B_1 \to B_0$, such that all irreducible components of the generic fiber X_{η_1} of $f_1 : X \times_B B_1 \to B_1$ are geometrically irreducible. Then by the proof of [Sta25, Tag 055A], as X_{η_1} is geometrically normal over $k(\eta_1)$, there is a nonempty open subset $B_2 \subset B_1$, such that one can write $X \times_B B_2 = \sqcup_{i=1}^n X_i$ as a scheme over B_2, where each $X_i \to B_2$ is a normal morphism with geometrically integral fibers.

Proof of Theorem 1.4. We do several elementary reductions. By Lemma 2.5, given an irreducible variety B' with a dominant morphism $B' \to B$ over F, one may replace B by B'. Replacing B by B_{red}, one may assume that B is integral. As in Lemma 5.1 (b), one replaces B by a quasi-finite étale cover, takes a morphism $\pi : X' \to X$ and a family of exceptional divisors $\{E_i\}_{i\in I}$.

We prove that the jumping locus for $f : X \to B$ is contained in that of $f' : X' \to B$. Let $\eta \in B$ be the generic point. By André's theorem [MP12, Theorem 8.3, Remarks 1.3 and 8.4], as $f' : X' \to B$ is smooth proper and F is finitely generated over $\mathbb{Q}$, the subset of $b \in |B|$ such that the specialization $\mathrm{sp}' : \mathrm{NS}(X'_{\bar\eta}) \to \mathrm{NS}(X'_{\bar b})$ is not an isomorphism is sparse. For every $b \in |B|$ outside this subset, we prove that $\sigma : \mathrm{Cl}^{\mathrm{ns}}(X_{\bar\eta}) \to \mathrm{Cl}^{\mathrm{ns}}(X_b)$ is an isomorphism.

From Remark 4.4, one may choose specialization morphisms fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{i\in I}\mathbb{Z}[E_{i,\bar\eta}] \arrow[r] \arrow[d, dashed] & \mathrm{NS}(X'_{\bar\eta}) \arrow[r, "\pi_{\bar\eta,*}"] \arrow[d, "\sigma'"] & \mathrm{Cl}^{\mathrm{ns}}(X_{\bar\eta}) \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & \oplus_{i\in I}\mathbb{Z}[E_{i,b}] \arrow[r] & \mathrm{NS}(X'_{b}) \arrow[r, "\pi_{b,*}"] & \mathrm{Cl}^{\mathrm{ns}}(X_b) \arrow[r] & 0.
\end{tikzcd}$$

By [Kol18, Lem. 18], as the fibers of $f : X \to B$ are geometrically normal, both rows are exact. From the construction of specialization in [Ful98, p.399], flatness of $E_i \to B$ and uniqueness in [EGA IV 2, Prop. 2.8.5], one has $\sigma'(E_{i,\bar\eta}) = E_{i,b}$ for all $i \in I$. Then by the five lemma, the right vertical morphism is an isomorphism. $\square$

Corollary 5.3. *Let $F/\mathbb{Q}$ be a finitely generated extension of fields. Let B be an irreducible variety over F with generic point η. Let $f : X \to B$ be a normal proper morphism of schemes over F with irreducible geometric fibers. Assume that $X_{\bar\eta}$ is $\mathbb{Q}$-factorial. Then there is a sparse subset $Z \subset |B|$, such that for every $b \in |B| \setminus Z$, there is a specialization isomorphism* $\mathrm{sp} : \mathrm{NS}\, X_{\bar\eta} \xrightarrow{\sim} \mathrm{NS}\, X_{\bar b}$.

Proof. One may assume that B is integral. From [MP12, Proposition 8.5 (a) and (b)], one can shrink B to a dense open subset. Thus, one can assume that $f : X \to B$ satisfies the result of Lemma 4.5. By [Liu25, Lemma 2.4 *1)*], as $f : X \to B$ is normal proper, shrinking B one can assume that for every $b \in B$, the specialization $\mathrm{sp} : \mathrm{NS}(X_{\bar\eta}) \to \mathrm{NS}(X_{\bar b})$ is injective with torsion-free cokernel. We prove that for every $\mathrm{Cl}^{\mathrm{ns}}$-generic closed point $b \in |B|$ is also NS-generic. By Lemma 3.10, since $X_{\bar\eta}$ is $\mathbb{Q}$-factorial, the left (resp. right) vertical morphism in the commutative square

$$\begin{tikzcd}
\mathrm{NS}(X_{\bar\eta})_{\mathbb{Q}} \arrow[r, hook, "\mathrm{sp}"] \arrow[d, "\cong"] & \mathrm{NS}(X_{\bar b})_{\mathbb{Q}} \arrow[d, hook] \\
\mathrm{Cl}^{\mathrm{ns}}(X_{\bar\eta})_{\mathbb{Q}} \arrow[r, "\sigma"] & \mathrm{Cl}^{\mathrm{ns}}(X_{\bar b})_{\mathbb{Q}}.
\end{tikzcd}$$

is bijective (resp. injective). As $\sigma : \mathrm{Cl}^{\mathrm{ns}}(X_{\bar\eta}) \to \mathrm{Cl}^{\mathrm{ns}}(X_{\bar b})$ is an isomorphism, so is $\mathrm{sp} : \mathrm{NS}(X_{\bar\eta}) \to \mathrm{NS}(X_{\bar b})$. The result follows from Theorem 1.4. $\square$

Proof of Theorem 1.5. Enlarging k to certain finite extension k' and choose an irreducible component of $B_{k'}$, one may assume that B is geometrically irreducible. Replacing B by B_{red}, one may assume that B is reduced. Thus, shrinking B one can assume that B is smooth over k. As in the proof of Theorem 1.4, because $f : X \to B$ is normal, one may assume that there is a proper morphism $\pi : X' \to X$ satisfying the results of Lemma 5.1. In particular, $f' = f\pi : X' \to B$ is smooth proper, and every NS-generic point for $f' : X' \to B$ is $\mathrm{Cl}^{\mathrm{ns}}$-generic for $f : X \to B$.

We modify the proof of [Liu25, Lemma 6.1]. We can choose an explicit constant $d > 1$ depending only on B with $B^{\leq d}$ Zariski dense in B. Choose a point $b_1 \in B^{\leq d}$. Replacing k by $k(b_1)$, we may assume that $b_1 \in B(k)$. One can choose an étale coordinate $\phi : B \to \mathbf{A}_k^m$ centered at b_1. Let $\mathcal{H}^2_{\mathrm{dR}}(X/B)$ (resp. $\mathcal{H}^2_{\mathrm{DR}}(X/B)$) be the second relative de Rham cohomology sheaf (resp. Hartshorne's de Rham cohomology sheaf) of $f : X \to B$. Similar to [Liu25, Step 1 of the proof of Lemma 6.1], we choose a finitely generated $\mathbb{Z}$-subalgebra A of k such that

- everything is defined over A. More precisely, there is an irreducible scheme B_A smooth separated of finite type over A with $B = B_A \otimes_A k$. Additionally, $f : X \to B$, $\pi : X' \to X$, b_1 and $\phi : B \to \mathbf{A}_k^m$ are all defined over A as a proper morphism $f_A : X_A \to B_A$, a proper morphism $\pi_A : X'_A \to X_A$ with $f'_A = f_A \circ \pi'_A X'_A \to B_A$ smooth, $\beta_1 \in B_A(A)$ and an étale morphism $B_A \to \mathbf{A}_A^k$ respectively;
- there is a finite free O_{B_A}-module $\mathcal{H}$ with an integrable connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{O_{B_A}} \Omega^1_{B_A/A}$, whose pullback to B is $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ with the Gauss-Manin connection.

Thus, as in [Liu25, Step 4 of the proof of Lemma 6.1], we choose a prime p, an embedding $A \hookrightarrow \mathbb{Z}_p$ and a nonempty open subset U of the p-adic manifold $B(O_{\mathbb{C}_p})$. Then by [MP12, Theorem 1.7], as $f'_A : X'_A \to B_A$ is smooth proper, shrinking U one may assume further that for every $u \in U$, there is a specialization isomorphism $\mathrm{NS}(X'_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{NS}(X'_{\bar{u}})$. Therefore, $\mathrm{Cl}^{\mathrm{ns}}(X_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{Cl}^{\mathrm{ns}}(X_{\bar{u}})$ is an isomorphism.

For every $b \in |B|$, let $c_{1,\mathrm{inf}} : \mathrm{Pic}(X_{\bar{b}}) \to H^2_{\mathrm{inf}}(X_{\bar{b}}/\bar{k})$ be the infinitesimal first Chern class map. If for every initial conditions in the image, the formal section of $\mathcal{H}^2_{\mathrm{dR}}(X_{\bar{k}}/B_{\bar{k}})$ determined by solving the horizontal equation of the Gauss-Manin connection on $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ lies in the first Hodge filtration $F^1\mathcal{H}^2_{\mathrm{dR}}(X_{\bar{k}}/B_{\bar{k}})$, then we write $\Lambda_{\bar{b}} = 0$. (We refer the reader to [Liu25, Sec. 5] for more details.) By the weak approximation theorem [MZ14] (see also [Liu25, Remark 6.7]), enlarging d one may assume that $\mathcal{G} := \{b \in B^{\leq d} | \Lambda_{\bar{b}} = 0,\, b \in U\}$ is Zariski dense in B. By the proof of [Liu25, Theorem 6.5], for every $b \in \mathcal{G}$, there also exist specialization isomorphisms $\mathrm{NS}(X_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar{b}})$ and $\mathrm{Br}\, X_{\bar{\eta}} \xrightarrow{\sim} \mathrm{Br}\, X_{\bar{b}}$. □

Acknowledgments

I would like to thank Emiliano Ambrosi for having introduced me to this problem and for his kind help. I thank Lena Ji for answering my questions on her work,

and for indicating a proof of Lemma 2.5. I am grateful to Bruno Kahn for telling me a variant of the proof of Lemma 2.3.

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