FOURIER-MUKAI TRANSFORM FOR SHEAVES WITH CONNECTION ON COMPLEX TORI

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Abstract. Let A, B be dual abelian varieties. Let $B^\natural$ be the universal vectorial extension of B. Laumon and Rothstein independently lift the Fourier-Mukai transform to D_A-modules. They prove that this defines an equivalence of triangulated categories from the derived category $D^b_{\mathrm{coh}}(D_A)$ of coherent D_A-modules to the derived category $D^b_{\mathrm{coh}}(O_{B^\natural})$ of coherent sheaves on $B^\natural$. We extend their results to complex tori. As a replacement of coherent algebraic D-modules, we use Kashiwara's good analytic D-modules. Moreover, to get an equivalence we have to replace $O_{B^\natural}$ by a commutative O_B-algebra, which is locally a polynomial algebra over O_B. As an application, we recover the Matsushima-Morimoto theorem that on a complex torus, a vector bundle admits a connection if and only if it is translation invariant, and in this case, it admits an integrable connection. Moreover, we use the Laumon-Rothstein transform to show that the derived category $D^b_h(D_A)$ of holonomic D-modules is a rigid symmetric monoidal triangulated category under the convolution.

1. Introduction

1.1. **Fourier-Mukai transform on abelian varieties.** Mukai [Muk81, Sec. 2] introduces an analog of the Fourier transform for sheaves of modules on abelian varieties, known as the *Fourier-Mukai transform*. Laumon [Lau96] and Rothstein [Rot96] study independently its lift to sheaves with connection (integrable or not). They both prove the Fourier inversion formula for the lift. Laumon [Lau96, Thm. 6.3.3] applies it to investigate generalized 1-motives. Meanwhile, as an application, Rothstein [Rot96, Thm. 3.2] recovers Matsushima's theorem [Mat59]: every vector bundle on an abelian variety admitting a connection is translation invariant. Schnell's work [Sch15] about holonomic D-modules on abelian varieties also relies upon the lift of the Fourier-Mukai transform.

Let k be an algebraically closed field. For an algebraic variety V over k, let $\mathrm{Mod}(O_V)$ be the abelian category of O_V-modules, let $D(O_V)$ (resp. $D^b(O_V)$) be the unbounded (resp. bounded) derived category of $\mathrm{Mod}(O_V)$. Denote by $D_{\mathrm{qc}}(O_V) \subset D(O_V)$ (resp. $D^b_{\mathrm{coh}}(O_V) \subset D^b(O_V)$) the full subcategory of objects whose cohomologies are quasi-coherent (resp. coherent) O_V-modules. Let A, B be abelian varieties over k dual to each other. Set $g := \dim A$. Let $p_A : A \times B \to A$ and $p_B : A \times B \to B$ denote the projections. Let $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times B$. We adopt the following sign convention for the Fourier-Mukai

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transform:

$$(1)\qquad \begin{aligned} \mathcal{F} &= Rp_{B*}(\mathcal{P}^{-1} \otimes^L p_A^* -) : D(O_A) \to D(O_B), \\ \mathcal{F}' &= Rp_{A*}(\mathcal{P} \otimes^L p_B^* -) : D(O_B) \to D(O_A). \end{aligned}$$

For a triangulated category, let T denote the degree shift automorphism. Mukai [Muk81, Thm. 2.2] establishes an analog of the Fourier inversion formula.

Fact 1.1.1 (Mukai)**.** (i) *There are isomorphisms of functors*

$$\mathcal{F}' \circ \mathcal{F} \cong T^{-g} : D_{\mathrm{qc}}(O_A) \to D_{\mathrm{qc}}(O_A),$$
$$\mathcal{F} \circ \mathcal{F}' \cong T^{-g} : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_B).$$

In particular, $\mathcal{F}' : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_A)$ is an equivalence of triangulated categories, with a quasi-inverse $T^g\mathcal{F}$.

(ii) *The functor $\mathcal{F}' : D(O_B) \to D(O_A)$ restricts to an equivalence $D^b_{\mathrm{coh}}(O_B) \to D^b_{\mathrm{coh}}(O_A)$.*

Let $0 \to H^0(A, \Omega^1_A) \to B^\natural \xrightarrow{\pi} B \to 0$ be the universal vectorial extension of B (constructed in [Ros58, Prop. 11]). For an algebraic variety V, denote by $\mathrm{for}_V : D(D_V) \to D(O_V)$ the forgetful functor. Let $D_{\mathrm{qc}}(D_A) \subset D(D_A)$ (resp. $D^b_{\mathrm{coh}}(D_A) \subset D^b(D_A)$) be the full subcategory of objects whose cohomologies are quasi-coherent O_A-modules (resp. coherent D_A-modules). Laumon and Rothstein lift the Fourier-transform to D-modules and establish a duality result similar to Fact 1.1.1.

Fact 1.1.2 (Laumon, Rothstein)**.** *Let A and B be dual abelian varieties of dimension g.*

(i) *There exist triangulated functors*

$$\tilde{\mathcal{F}}^\natural : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(D_A), \quad \tilde{\mathcal{F}} : D_{\mathrm{qc}}(D_A) \to D_{\mathrm{qc}}(O_{B^\natural})$$

such that there exist isomorphisms of functors

$$R\pi_*\tilde{\mathcal{F}} \cong \mathcal{F}\mathrm{for}_A : D_{\mathrm{qc}}(D_A) \to D_{\mathrm{qc}}(O_B), \quad \mathrm{for}_A\tilde{\mathcal{F}}^\natural \cong \mathcal{F}' R\pi_* : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(O_A).$$

(ii) ([Lau96, Thm. 3.2.1], [Rot96, Thm. 4.5]) *There are isomorphisms of functors*

$$\tilde{\mathcal{F}}^\natural\tilde{\mathcal{F}} \cong T^{-g} : D_{\mathrm{qc}}(D_A) \to D_{\mathrm{qc}}(D_A),$$
$$\tilde{\mathcal{F}}\tilde{\mathcal{F}}^\natural \cong T^{-g} : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(O_{B^\natural}).$$

Thus, $\tilde{\mathcal{F}}^\natural : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(D_A)$ is an equivalence of triangulated categories.

(iii) ([Lau96, Cor. 3.1.3]) *The functor $\tilde{\mathcal{F}}^\natural : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(D_A)$ restricts to an equivalence $\tilde{\mathcal{F}}^\natural : D^b_{\mathrm{coh}}(O_{B^\natural}) \to D^b_{\mathrm{coh}}(D_A)$.*

Rothstein [Rot96, Thm. 6.2] formulates Theorem 1.1.2 (iii) in a slightly different way, which we recall. Because $\pi : B^\natural \to B$ is an affine morphism of finite type of schemes, $\mathcal{A}_B := \pi_*O_{B^\natural}$ is a quasi-coherent O_B-algebra of finite type. By [EGA I, Prop. 9.6.1], an $\mathcal{A}_B$-module F is quasi-coherent if and only if F is quasi-coherent over O_B. Let $\mathrm{Qch}(\mathcal{A}_B)$ (resp. $\mathrm{Coh}(\mathcal{A}_B)$) be the category of quasi-coherent (resp. coherent) $\mathcal{A}_B$-modules. Then

$$\pi_* : \mathrm{Qch}(O_{B^\natural}) \to \mathrm{Qch}(\mathcal{A}_B), \quad \pi_* : \mathrm{Coh}(O_{B^\natural}) \to \mathrm{Coh}(\mathcal{A}_B)$$

are equivalences of abelian categories. Let $D_{\mathrm{qc}}(\mathcal{A}_B) \subset D(\mathrm{Mod}(\mathcal{A}_B))$ (resp. $D^b_{\mathrm{coh}}(\mathcal{A}_B) \subset D^b(\mathrm{Mod}(\mathcal{A}_B)))$ be the full subcategory of complexes whose cohomologies are quasi-coherent (resp. coherent) over $\mathcal{A}_B$. Then

$$R\pi_* : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(\mathcal{A}_B), \quad R\pi_* : D^b_{\mathrm{coh}}(O_{B^\natural}) \to D^b_{\mathrm{coh}}(\mathcal{A}_B)$$

are equivalences of triangulated categories. Thus, one can rewrite Theorem 1.1.2 in terms of $\mathcal{A}_B$ instead of $O_{B^\natural}$.

1.2. **Fourier-Mukai transform on complex tori.** Let X and Y be complex tori dual to each other of dimension g. Define the analytic Fourier-Mukai transform $\mathcal{F}' : D(O_X) \to D(O_Y)$ and $\mathcal{F} : D(O_Y) \to D(O_X)$ as in (1). In [BBP07, Thm. 2.1], a result similar to Fact 1.1.1 is established for complex tori. As [Liu26, Thm. 5.0.1] illustrates, one should replace the quasi-coherent sheaves in the algebraic setting with good sheaves [Kas03, Def. 4.22] in the analytic setting. For a complex manifold Z, let $D_{\mathrm{good}}(O_Z) \subset D(O_Z)$ be the full subcategory of objects whose cohomologies are *good* O_Z-modules.

Fact 1.2.1 (Mukai, Ben-Bassat, Block, Pantev)**.** *There are isomorphisms of functors*

$$\mathcal{F}'\mathcal{F} \cong T^{-g} : D_{\mathrm{good}}(O_Y) \to D_{\mathrm{good}}(O_Y),$$
$$\mathcal{F}\mathcal{F}' \cong T^{-g} : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_X).$$

In particular, $\mathcal{F}' : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_Y)$ (resp. $\mathcal{F}' : D^b_{\mathrm{coh}}(O_X) \to D^b_{\mathrm{coh}}(O_Y)$) is an equivalence of triangulated categories with a quasi-inverse $D_{\mathrm{good}}(O_Y) \to D_{\mathrm{good}}(O_X)$ (resp. $D^b_{\mathrm{coh}}(O_Y) \to D^b_{\mathrm{coh}}(O_X)$) defined by $T^g\mathcal{F}$.

We lift the analytic Fourier-Mukai transform to D-modules, and give an analog of Fact 1.1.2 for complex tori. However, instead of working with the universal vectorial extension $\pi : X^\natural \to X$ directly as in the algebraic case, we have to substitute $O_{X^\natural}$ with its variant $\mathcal{A}_X$. In fact, the proof of Theorem 1.1.2 (ii) applies the flat base change theorem to the cartesian square of algebraic varieties

$$\begin{tikzcd} B^\natural \times A \times B^\natural \arrow[r, "p_{12}"] \arrow[d, "p_{23}"'] \arrow[dr, phantom, "\square"] & B^\natural \times A \arrow[d] \\ A \times B^\natural \arrow[r] & A. \end{tikzcd}$$

By contrast, [Liu26, Rk. 4.2.9] shows that the analytic flat base change theorem [Liu26, Thm. 1.3.2] needs an extra properness assumption. Whenever $g > 0$, $X^\natural$ is not compact, so the projection $X^\natural \times Y \to Y$ is not proper. This is why we replace $O_{X^\natural}$ with an O_X-algebra on the *compact* complex manifold X. Additionally, the O_X-algebra $\pi_*O_{X^\natural}$ is not of finite type, and it is unclear whether it is a good sheaf on X, while $\mathcal{A}_X \subset \pi_*O_{X^\natural}$ is a subalgebra of finite type and good over O_X.

A coherent *algebraic* D-module admits a global good filtration, which fails in the analytic setup. For this reason, it is unclear whether $\tilde{\mathcal{F}}$ sends a coherent *analytic* D_Y-module into $D^b_{\mathrm{coh}}(\mathcal{A}_X)$. A coherent analytic D-module is *good* if it admits a good filtration over every relatively compact open subset. Similarly, one can define good $\mathcal{A}_X$-modules. For a complex manifold Z and a (possibly noncommutative) O_Z-algebra $\mathcal{R}$, let $D_{O-\mathrm{good}}(\mathcal{R}) \subset D(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b(\mathcal{R})$) be the full subcategory of objects whose cohomologies are good over O_Z (resp. $\mathcal{R}$). Let $\mathrm{for}_X : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)$ be the forgetful functor.

Theorem 1.2.2. *Let X and Y be dual complex tori of dimension g.*

(i) (Prop. 5.1.3) *There is a commutative O_X-algebra of finite type $\mathcal{A}_X$, and triangulated functors*

$$\tilde{\mathcal{F}}\colon D(D_Y) \to D(\mathcal{A}_X), \quad \tilde{\mathcal{F}}^\natural : D(\mathcal{A}_X) \to D(D_Y)$$

such that there exist isomorphisms of functors

$$\mathrm{for}_X \tilde{\mathcal{F}} \cong \mathcal{F}\mathrm{for}_Y \colon D(D_Y) \to D(O_X), \quad \mathrm{for}_Y \tilde{\mathcal{F}}^\natural \cong \mathcal{F}'\mathrm{for}_X \colon D(A_X) \to D(O_Y).$$

(ii) (Thm. 5.1.4) *There are isomorphisms of functors*

$$\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}} \cong T^{-g} : D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(D_Y),$$
$$\tilde{\mathcal{F}}\tilde{\mathcal{F}}^\natural \cong T^{-g} : D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(\mathcal{A}_X).$$

Thus, $\tilde{\mathcal{F}}^\natural \colon D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y)$ is an equivalence of triangulated categories.

(iii) (Thm. 6.3.1) *The functors $\tilde{\mathcal{F}}^\natural \colon D(\mathcal{A}_X) \to D(D_Y)$ restricts to an equivalence $\tilde{\mathcal{F}}^\natural : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$.*

Notation and conventions. For a sheaf F on a topological space, let $\mathrm{Supp}F$ be its support. For a (not necessarily commutative) ringed space $(X, \mathcal{R})$, let $\mathrm{Mod}(\mathcal{R})$ be the category of left $\mathcal{R}$-modules. Let $\mathrm{Coh}(\mathcal{R}) \subset \mathrm{Mod}(\mathcal{R})$ be the full subcategory of coherent $\mathcal{R}$-modules in the sense of [Sta26, Tag 01BV]. Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*(\mathcal{R})$ refers to the unbounded/bounded below/bounded above/bounded derived category of the abelian category $\mathrm{Mod}(\mathcal{R})$ in order. Let $D^*_{\mathrm{coh}}(\mathcal{R}) \subset D^*(\mathcal{R})$ be the full subcategory of objects whose cohomologies are coherent $\mathcal{R}$-modules.

Let k be an algebraically closed field. An algebraic variety refers to an integral scheme of finite type and separated over k. For a complex manifold Z and $z \in Z$, let $i_z : \{z\} \hookrightarrow Z$ be the inclusion. Set $\mathbb{C}_{(z)} := (i_z)_*\mathbb{C}$, which is a coherent O_Z-module supported at z. Let X and Y be complex tori dual to each other and of dimension g. Set $\mathfrak{g} := H^1(X, O_X)$.

2. Review of Rothstein's splittings and connections

For the convenience of the reader, we review [Rot97, Sec. 2.1] and adapt some results in the algebraic setup for the analytic setup.

2.1. Categories of splittings.

By [Har77, III, Prop. 6.3 (c)], for a complex manifold Z and a (holomorphic) vector bundle M on Z, one has $H^1(Z, M) = \mathrm{Ext}^1(O_Z, M)$. Thus, every $\alpha \in H^1(Z, M)$ determines a short exact sequence in $\mathrm{Mod}(O_Z)$

$$0 \to M \to \mathcal{E}_\alpha \stackrel{\mu_\alpha}{\to} O_Z \to 0. \tag{2}$$

By [Sta26, Tag 05NJ], since O_Z is flat over itself, for every $F \in \mathrm{Mod}(O_Z)$, the sequence (2) tensored with F

$$0 \to M \otimes_{O_Z} F \to \mathcal{E}_\alpha \otimes_{O_Z} F \stackrel{\mu_\alpha \otimes \mathrm{Id}_F}{\to} F \to 0 \tag{3}$$

remains exact.

Definition 2.1.1. Define a category $\mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$ as follows: the objects are pairs (F, ψ), where F is an O_Z-module, and $\psi : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ is an α-splitting on F, *i.e.*, an O_Z-linear splitting of $\mu_\alpha \otimes \mathrm{Id}_F : \mathcal{E}_\alpha \otimes F \to F$. The morphisms in $\mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$ are O_Z-linear and required to be compatible with the splittings.

Example 2.1.2. If $M = \Omega^1_Z$, $\alpha = 0$ in $H^1(Z, \Omega^1_Z)$, and F is a vector bundle on Z, then an α-splitting $\psi : F \to \mathcal{E}_0 \otimes_{O_Z} F$ is exactly a holomorphic 1-form on Z with values in $\mathcal{E}nd(F)$. The pair (F, ψ) is a Higgs bundle (in the sense of [Sim92, p.6]) if and only if $[\psi, \psi] = 0$.

Lemma 2.1.3. *Fix $\alpha \in H^1(Z, M)$. For an O_Z-module F, there is an α-splitting on F if and only if $i_* : H^1(Z, M) \to H^1(Z, M \otimes_{O_Z} \mathcal{E}nd(F))$ sends α to 0. In that case, the set of α-splittings on F has a natural simple transitive action of the abelian group* $\mathrm{Hom}_{O_Z}(F, M \otimes_{O_Z} F)$.

Proof. The stated map $i_* \colon H^1(Z, M) \to H^1(Z, M \otimes_{O_Z} \mathcal{E}nd(F))$ is induced by the natural morphism of O_Z-modules

$$i : M \to \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F), \quad i(m)(f) = m \otimes f.$$

There is a canonical evaluation morphism

$$\mathrm{ev} : \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F) \otimes F \to M \otimes_{O_Z} F, \quad \phi \otimes f \mapsto \phi(f).$$

By construction of i, the left part of the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1(O_Z, \mathcal{H}om(F, M \otimes_{O_Z} F)) \arrow[r, "(-)\otimes F"] & \mathrm{Ext}^1(F, \mathcal{H}om(F, M \otimes_{O_Z} F) \otimes F) \arrow[r, "\mathrm{Ext}^1(F,\mathrm{ev})"] & \mathrm{Ext}^1(F, M \otimes_{O_Z} F) \\
\mathrm{Ext}^1(O_Z, M) \arrow[u, "i_*"] \arrow[rr, "(-)\otimes F"'] & & \mathrm{Ext}^1(F, M \otimes_{O_Z} F) \arrow[u, "\mathrm{Id}"'] \arrow[ul, "\mathrm{Ext}^1(F, i\otimes \mathrm{Id}_F)"']
\end{tikzcd}$$

is commutative. One has

$$\mathrm{ev} \circ (i \otimes \mathrm{Id}_F)(m \otimes f) = \mathrm{ev}(i(m) \otimes f) = i(m)(f) = m \otimes f,$$

so $\mathrm{ev} \circ (i \otimes \mathrm{Id}_F) \colon M \otimes_{O_Z} F \to M \otimes_{O_Z} F$ is the identity. Therefore, the triangle on the right of the diagram is also commutative.

By definition, F admits an α-splitting if and only if the exact sequence (3) splits. The extension class of (3) is $\alpha \otimes F \in \mathrm{Ext}^1(F, M \otimes F)$. By the five-term exact sequence associated with the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i(O_Z, \mathcal{E}xt^j(F, M \otimes_{O_Z} F)) \Rightarrow \mathrm{Ext}^{i+j}(F, M \otimes_{O_Z} F),$$

the composition of the two morphisms on the top of the diagram is injective. This injective map sends $i_*(\alpha)$ to $\alpha \otimes F$, so $\alpha \otimes F = 0$ is equivalent to $i_*(\alpha) = 0$.

For two α-splittings ψ, ψ' on F, $\psi' - \psi \colon F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ is an O_Z-linear morphism. Because of

$$(\mu_\alpha \otimes \mathrm{Id}_F)\psi = (\mu_\alpha \otimes \mathrm{Id}_F)\psi' = \mathrm{Id}_F,$$

$\psi' - \psi$ factors through $\ker(\mu_\alpha \otimes \mathrm{Id}_F) = M \otimes_{O_Z} F$, hence $\psi' - \psi \in \mathrm{Hom}_{O_Z}(F, M \otimes_{O_Z} F)$. For every O_Z-linear morphism $\phi \colon F \to M \otimes_{O_Z} F$, $\psi + \phi$ is an α-splitting on F. Therefore, the action of $\mathrm{Hom}_{O_Z}(F, M \otimes_{O_Z} F)$ on the set of α-splittings on F is simple transitive. □

To each object $(F, \psi) \in \mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$, we assign an element

$$[\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes_{O_Z} \mathcal{E}nd(F)) \tag{4}$$

as follows. The sequence (2) induces a short exact sequence

$$0 \to \wedge^2 M \to \wedge^2 \mathcal{E}_\alpha \overset{\omega_\alpha}{\to} M \to 0,$$

where

$$\omega_\alpha(\rho_1 \wedge \rho_2) = \mu_\alpha(\rho_1)\rho_2 - \mu_\alpha(\rho_2)\rho_1$$

for any local sections $\rho_i \in \mathcal{E}_\alpha$ $(i = 1, 2)$. The flatness of M ensures the exactness when tensoring with F:

$$0 \to (\wedge^2 M) \otimes F \to (\wedge^2 \mathcal{E}_\alpha) \otimes F \overset{\omega_\alpha \otimes \mathrm{Id}_F}{\to} M \otimes_{O_Z} F \to 0. \tag{5}$$

Let $a : \mathcal{E}_\alpha \otimes \mathcal{E}_\alpha \to \wedge^2 \mathcal{E}_\alpha$ be the morphism defined by $e \otimes e' \mapsto e \wedge e'$. Let ψ^1 be the O_Z-linear composition

$$\mathcal{E}_\alpha \otimes F \overset{\mathrm{Id}_{\mathcal{E}_\alpha} \otimes \psi}{\to} \mathcal{E}_\alpha \otimes (\mathcal{E}_\alpha \otimes F) \xrightarrow{\sim} (\mathcal{E}_\alpha \otimes \mathcal{E}_\alpha) \otimes F \overset{a \otimes \mathrm{Id}_F}{\to} (\wedge^2 \mathcal{E}_\alpha) \otimes F,$$

where the isomorphism in the middle is from the associativity of tensor product.

Lemma 2.1.4. *For every $(F, \psi) \in \mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$, the composition $(\omega_\alpha \otimes \mathrm{Id}_F)\psi^1\psi \colon F \to M \otimes_{O_Z} F$ is zero.*

Proof. Locally, the vector bundle $\mathcal{E}_\alpha$ has a (holomorphic) frame $\{e_1, \ldots, e_r\}$. For a local section $f \in F$, write $\psi(f) = \sum_{i=1}^r e_i \otimes f_i$, where the f_i are local sections of F. For every $1 \le i \le r$, write $\psi(f_i) = \sum_{j=1}^r e_j \otimes f_j^{(i)}$, where the $f_j^{(i)}$ are local sections of F. As $\psi : F \to \mathcal{E}_\alpha \otimes F$ is a section to $\mu_\alpha \otimes \mathrm{Id}_F : \mathcal{E}_\alpha \otimes F \to F$, one has

$$f = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f) = \sum_{i=1}^r \mu_\alpha(e_i) f_i; \tag{6}$$

$$f_i = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f_i) = \sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)}. \tag{7}$$

From (6), one has

$$\psi(f) = \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i). \tag{8}$$

By construction, one has $\psi^1\psi(f) = \sum_{i,j=1}^r (e_i \wedge e_j) \otimes f_j^{(i)}$. Then

$$\begin{aligned}
(\omega_\alpha \otimes \mathrm{Id}_F)\psi^1\psi(f) &= \sum_{i,j=1}^r \left(\mu_\alpha(e_i)e_j - \mu_\alpha(e_j)e_i\right) \otimes f_j^{(i)} \\
&= \sum_{i=1}^r \mu_\alpha(e_i) \sum_{j=1}^r e_j \otimes f_j^{(i)} - \sum_{i=1}^r e_i \otimes \left(\sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)} \right) \\
&\overset{\text{(a)}}{=} \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i) - \sum_{i=1}^r e_i \otimes f_i \\
&\overset{\text{(b)}}{=} \psi(f) - \psi(f) = 0,
\end{aligned}$$

where (a) and (b) use (7) and (8) respectively. $\square$

From Lemma 2.1.4 and (5), $\psi^1\psi \colon F \to (\wedge^2 \mathcal{E}_\alpha) \otimes F$ factors through $(\wedge^2 M) \otimes F$, which induces an element $[\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes_{O_Z} \mathcal{E}nd(F))$.

Example 2.1.5. Let X be a complex torus, and set $\mathfrak{g} := H^1(X, O_X)$. One has

$$H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = \mathfrak{g}^* \otimes_{\mathbb{C}} \mathfrak{g} = \mathrm{End}(\mathfrak{g}).$$

Hence, we assign an extension $\mathcal{E}_T$ of O_X by $\mathfrak{g}^* \otimes_{\mathbb{C}} O_X$ and a category $\mathrm{Mod}(O_X)_{T-\mathrm{sp}}$ to each $T \in \mathrm{End}(\mathfrak{g})$. The identity element $1 \in \mathrm{End}(\mathfrak{g})$ corresponds to the tautological exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X \to \mathcal{E} \xrightarrow{\mu} O_X \to 0. \tag{9}$$

We also write $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ for $\mathrm{Mod}(O_X)_{1-\mathrm{sp}}$. For every $(F, \psi) \in \mathrm{Mod}(O_X)_{\mathrm{sp}}$, $[\psi, \psi]$ lies in

$$\Gamma(X, \wedge^2 \mathfrak{g}^* \otimes_{\mathbb{C}} O_X \otimes_{O_X} \mathcal{E}nd(F)) = \wedge^2 \mathfrak{g}^* \otimes_{\mathbb{C}} \mathrm{End}(F),$$

and we recover [Rot96, (4.8)]. Similarly, as $H^1(X \times X, \mathfrak{g}^* \otimes O_{X \times X}) = \mathrm{End}(\mathfrak{g}) \oplus \mathrm{End}(\mathfrak{g})$, for every pair $T_1, T_2 \in \mathrm{End}(\mathfrak{g})$, the category $\mathrm{Mod}(O_{X\times X})_{(T_1,T_2)-\mathrm{sp}}$ is defined.

2.2. **Categories of twisted connections.** We review the twisted (relative) connections introduced in [Rot97, p.206]. For a submersion of complex manifolds $f : Z \to S$, let $\Omega^1_{Z/S}$ be the relative cotangent sheaf, which is a vector bundle on Z. Let $d_f : O_Z \to \Omega^1_{Z/S}$ denote the differential relative to $f : Z \to S$. An element $\alpha \in H^1(Z, \Omega^1_{Z/S})$ determines an extension

$$0 \to \Omega^1_{Z/S} \to \mathcal{E}_\alpha \xrightarrow{\mu_\alpha} O_Z \to 0.$$

Definition 2.2.1. On an O_Z-module F, an *α-connection* is an $f^{-1}(O_S)$-linear splitting $\nabla : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ to $\mu_\alpha \otimes \mathrm{Id}_F : \mathcal{E}_\alpha \otimes_{O_Z} F \to F$, satisfying the Leibniz rule

$$\nabla(h\phi) = h\nabla(\phi) + d_f(h) \otimes \phi, \tag{10}$$

where h and ϕ are local sections of O_Z and F respectively. Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ be the category of pairs (F, ∇), where F is an O_Z-module and ∇ is an α-connection on F. The morphisms in $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ are O_Z-linear and required to be compatible with the α-connections.

Example 2.2.2. If $\alpha = 0$ in $H^1(Z, \Omega^1_{Z/S})$, then the α-connections are exactly the f-relative connections. Let $\Theta_{Z/S}$ be the relative tangent sheaf for $f : Z \to S$, which is the vector bundle dual to $\Omega^1_{Z/S}$. Let $D_{Z/S}$ be the sheaf of rings of relative differential operators on Z/S defined in [SS94, p.9]. Define a sheaf $\tilde{D}_{Z/S}$ of O_Z-algebras as follows, of which $D_{Z/S}$ is naturally a quotient. Roughly, different from the definition of $D_{Z/S}$, when defining $\tilde{D}_{Z/S}$ we omit the commutation relations between vector fields in order to describe non-flat connections.

Explicitly, let $\{U_i\}_{i\in I}$ be an open cover of Z, and for every $i \in I$, let $\{\xi^{(i)}_1, \dots, \xi^{(i)}_n\}$ be a local frame of $\Theta_{Z/S}$ on U_i. Endow $\mathcal{F}_i := O_{U_i}\{\xi^{(i)}_1, \dots, \xi^{(i)}_n\}$ with the multiplication law satisfying the commutation relation $[\xi^{(i)}_q, h] = \xi^{(i)}_q(h)$ for any $1 \le q \le n$ and local section h of O_{U_i}. For each $i, j, k \in I$, set $U_{ij} := U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. For $1 \le q \le n$, there exists a unique tuple $(f_{qr})^n_{r=1}$ in $O_Z(U_{ij})$ such that $\xi^{(i)}_q|_{U_{ij}} = \sum^n_{r=1} f_{qr}\xi^{(j)}_r$. Because for every local section h of $O_{U_{ij}}$, one has

$$[\sum_{r=1}^n f_{qr}\xi^{(j)}_r, h] = \sum_{r=1}^n f_{qr}[\xi^{(j)}_r, h] = \sum_{r=1}^n f_{qr}\xi^{(j)}_r(h) = \xi^{(i)}_q(h) = [\xi^{(i)}_q, h],$$

there is a unique morphism $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ of $O_{U_{ij}}$-algebras mapping the $\xi^{(i)}_q$ to $\sum^n_{r=1} f_{qr}\xi^{(j)}_r$. Then $\varphi_{ii} = \mathrm{Id}_{\mathcal{F}_i}$. By construction, the cocycle relation $\varphi_{ik} =$

$\varphi_{jk}\varphi_{ij}$ holds on U_{ijk}. In particular, $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$ is an isomorphism of $O_{U_{ij}}$-algebras. By gluing the sheaves $\{\mathcal{F}_i\}_{i\in I}$ via the isomorphisms $\{\varphi_{ij}\}_{i,j\in I}$, one obtains $\tilde{D}_{Z/S}$. For every $i \in I$, the kernel of the surjection $\tilde{D}_{Z/S}|_{U_i} \twoheadrightarrow D_{Z/S}|_{U_i}$ is the bilateral ideal generated by the sections $[\xi_q^{(i)}, \xi_r^{(i)}]$ with $1 \le q, r \le n$.

Then $\mathrm{Mod}(Z)_{f,0-\mathrm{cxn}} = \mathrm{Mod}(\tilde{D}_{Z/S})$. The category $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$ is denoted by $\mathrm{Mod}(O_Z)_{\mathrm{cxn}}$ when $f : Z \to S$ is the structure morphism $Z \to \mathrm{Specan}(\mathbb{C})$.

Remark 2.2.3. Given $\alpha \in H^1(Z, \Omega^1_{Z/S})$, we explain why an α-connection is a particular kind of splittings. Let $\mathcal{E}_{\alpha'} := \mathcal{E}_\alpha$ as an abelian sheaf on Z, but with the following O_Z-module structure: For any local sections $h \in O_Z$ and $s \in \mathcal{E}_\alpha$, we define $h \bullet s \in \mathcal{E}_\alpha$ by $hs + \mu_\alpha(s) d_f h$. One has

$$\mu_\alpha(h \bullet s) = h\mu_\alpha(s). \tag{11}$$

For another local section $h' \in O_Z$, one has

$$\begin{aligned} & h' \bullet (h \bullet s) = h' \bullet (hs + \mu_\alpha(s) d_f h) \\ =& h'(hs + \mu_\alpha(s) d_f h) + \mu_\alpha(h \bullet s) d_f(h') \\ \overset{(a)}{=}& h'hs + h'\mu_\alpha(s) d_f h + h\mu_\alpha(s) d_f(h') \\ =& h'hs + \mu_\alpha(s) d_f(h'h) = (h'h) \bullet s, \end{aligned}$$

where (a) uses (11). Therefore, this construction is indeed an O_Z-module structure on $\mathcal{E}_{\alpha'}$.

By (11), $\mu_\alpha \colon \mathcal{E}_{\alpha'} \to O_Z$ is O_Z-linear, so it gives rise to an extension of O_Z-modules

$$0 \to \Omega^1_{Z/S} \to \mathcal{E}_{\alpha'} \to O_Z \to 0. \tag{12}$$

Let $\alpha' \in \mathrm{Ext}^1(O_Z, \Omega^1_{Z/S})$ be the extension class of (12). The passage from α to α' is to turn $f^{-1}O_S$-linear splittings to O_Z-linear splittings. More precisely, for every O_Z-module F, giving an α-connection $\nabla : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ is equivalent to giving an α'-splitting $\psi : F \to \mathcal{E}_{\alpha'} \otimes_{O_Z} F$. Hence, we get an equivalence of categories

$$\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{\alpha'-\mathrm{sp}}.$$

Let $P^1_{Z/S}$ be the sheaf of principal parts of first order of $f : Z \to S$ ([EGA IV 4, Déf. 16.3.1]). If $\alpha = 0$, then (12) is the prolongation sequence

$$0 \to \Omega^1_{Z/S} \to P^1_{Z/S} \to O_Z \to 0.$$

Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn,int}}$ be the full subcategory of $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ of objects whose connections are *integrable* in the sense of [Rot97, Remark, p.206]. Then $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn,int}}$ coincides with $\mathrm{MIC}(f)$ defined in [ABC20, 4.3.7], which is further equivalent to $\mathrm{Mod}(D_{Z/S})$.

Example 2.2.4. Let X and Y be dual complex tori, and let $p_X : X \times Y \to X$ be the projection. Since $\Omega^1_{X\times Y/X} = p_X^*(\mathfrak{g}^* \otimes_{\mathbb{C}} O_X)$, there is a natural morphism

$$p_X^* : \mathrm{End}(\mathfrak{g}) = H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \to H^1(X \times Y, \Omega^1_{X\times Y/X}).$$

For every $T \in \mathrm{End}(\mathfrak{g})$, the category $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^*T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^*T-\mathrm{cxn,int}}$) is also written as $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn,int}}$).

By [Rot97, pp.206–207], the Poincaré bundle $\mathcal{P}$ is naturally an object of $\mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn},\mathrm{int}}$. In local coordinates, the $p_X^*(-1)$-connection on $\mathcal{P}$ is explained in [Rot96, (1.10) and p.575*ff.*], except that we use a Stein open cover of X instead of Rothestein's affine open cover.

2.3. **Functors between categories of splittings and categories of twisted connections.** Recall that the Fourier-Mukai transform (1) is the composition of the pullback, the tensor product with $\mathcal{P}$ as well as the derived direct image. Rothstein's lift to modules with connection keeps an extra track of the splittings and the connections.

Remark 2.3.1. Combining [Rot97, (2.21)] with the fact that α-connections are kinds of splittings (Remark 2.2.3), the categories under consideration ($\mathrm{Mod}(O_X)_{\mathrm{sp}}$, $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$, *etc.*) are equivalent to categories of modules over sheaves of certain noncommutative *flat* O-algebras. In particular, each of them is a Grothendieck abelian category. By [Sta26, Tag 079P], each category has enough K-injectives. From [HT07, Lem. 1.5.2 (ii)], each category has enough objects flat over O. By [Sta26, Tags 070K, 079P], all the (left exact) direct image functors involved below admit right derived functors on the unbounded derived categories.

2.3.1. *Functor turning splittings to connections.* Given $T \in \mathrm{End}(\mathfrak{g})$ and $(F,\psi) \in \mathrm{Mod}(O_X)_{T-\mathrm{sp}}$, the induced morphism

$$p_X^{-1}\psi : p_X^{-1}F \to p_X^{-1}\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F$$

is $p_X^{-1}O_X$-linear. By Example 2.2.4, there is a short exact sequence

$$0 \to \Omega^1_{X\times Y/X} \to p_X^*\mathcal{E}_T \to O_{X\times Y} \to 0$$

in $\mathrm{Mod}(O_{X\times Y})$. Its extension class is $p_X^*T \in H^1(X\times Y, \Omega^1_{X\times Y/X})$. Define another $p_X^{-1}O_X$-linear morphism

$$\begin{aligned}\nabla_\psi : p_X^*F(= O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}F) \to p_X^*\mathcal{E}_T \otimes_{O_{X\times Y}} p_X^*F(=\\ p_X^*\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F = O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F)\end{aligned}$$

by

$$\nabla_\psi(h\otimes s) = d_{p_X}(h)\otimes s + h\otimes((p_X^{-1}\psi)(s)),$$

where h (resp. s) is a local section of $O_{X\times Y}$ (resp. $p_X^{-1}F$). By construction, ∇_ψ satisfies the Leibniz rule (10). So it is a p_X^*T-connection on p_X^*F. Thus, we get the *exact* functor in [Rot97, (2.5)]:

$$p_X^* : \mathrm{Mod}(O_X)_{T-\mathrm{sp}} \to \mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}. \tag{13}$$

2.3.2. *Tensoring with Poincaré bundle.* By [Rot97, (2.10)], the functor

$$(-)\otimes_{O_{X\times Y}} \mathcal{P} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \tag{14}$$

restricts to $\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn},\mathrm{int}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn},\mathrm{int}}(\cong \mathrm{Mod}(D_{X\times Y/X}))$. The functor (14) is an equivalence of abelian categories, with a quasi-inverse

$$(-)\otimes_{O_{X\times Y}} \mathcal{P}^{-1} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}}.$$

2.3.3. *Functor turning connections to splittings.* For every $(F,\nabla)\in \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}}$, the morphism

$$\nabla : F \to p_X^*\mathcal{E}\otimes_{O_{X\times Y}} F (= p_X^{-1}\mathcal{E}\otimes_{p_X^{-1}O_X} F)$$

is a $p_X^{-1}O_X$-linear splitting to $(p_X^{-1}\mu_1)\otimes \mathrm{Id}_F$. By the projection formula (see *e.g*, [KS90, Prop. 2.6.6]), the induced morphism

$$p_{X*}\nabla : p_{X*}F \to \mathcal{E}\otimes_{O_X} p_{X*}F$$

is an O_X-linear splitting to $\mu_1\otimes_{O_X}\mathrm{Id}_{p_{X*}F}$. Hence $(p_{X*}F, p_{X*}\nabla)\in \mathrm{Mod}(O_X)_{\mathrm{sp}}$. Thus, one has a left exact functor

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_X)_{\mathrm{sp}}. \tag{15}$$

If (F,∇) is integrable, then $[p_{X*}\nabla, p_{X*}\nabla]$ defined in (4) is zero.

2.3.4. *Functors between the categories of connections.* We define the inverse image and the direct image of relative connections on changing bases. Consider a cartesian square of complex manifolds

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "f"] \\ T \arrow[r, "g"] & S, \end{tikzcd} \tag{16}$$

where $f : Z\to S$ is a submersion. By Theorem 2.2.3, for every $(F,\nabla)\in \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$, the relative connection $\nabla : F \to \Omega^1_{Z/S}\otimes_{O_Z} F$ is equivalent to an O_Z-linear splitting $\psi : F\to P^1_{Z/S}\otimes_{O_Z} F$ to the natural projection $P^1_{Z/S}\otimes_{O_Z} F\to F$. Then $g'^*\psi : g'^*F \to P^1_{W/T}\otimes_{O_W} g'^*F$ is an O_W-linear splitting to the natural projection $P^1_{W/T}\otimes_{O_W} g'^*F\to g'^*F$, or equivalently, a relative connection $g'^*F\to \Omega^1_{W/T}\otimes_{O_W} g'^*F$. Thus, one gets an inverse image functor

$$g'^* : \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}} \to \mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}}. \tag{17}$$

It is right exact. By [ABC20, Sec. 5.1], the connection induced by ∇ is integrable if ∇ is so.

Fix $\alpha\in H^1(Z,\Omega^1_{Z/S})$. By the projection formula (see *e.g*, [Har77, II, Ex. 5.1 (d)]), for every $(F,\nabla)\in \mathrm{Mod}(O_W)_{f',g'^*\alpha-\mathrm{cxn}}$, one has

$$g'_*(F\otimes_{O_W} g'^*\mathcal{E}_\alpha) = (g'_*F)\otimes_{O_Z}\mathcal{E}_\alpha.$$

Then the induced morphism

$$g'_*\nabla : g'_*F \to (g'_*F)\otimes_{O_Z}\mathcal{E}_\alpha$$

is $f^{-1}(O_S)$-linear. Since $d_{f'} : O_W\to \Omega^1_{f'}$ and $d_f : O_Z \to \Omega^1_{Z/S}$ are related by $g'^*d_f = d_{f'}$, the induced map $g'_*\nabla$ satisfies the Leibniz rule (10) and hence $(g'_*F, g'_*\nabla)\in \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$. In this manner, we get a left exact functor

$$g'_* : \mathrm{Mod}(O_W)_{f',g'^*\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}. \tag{18}$$

When $\alpha = 0$, the functor (18) restricts to $\mathrm{MIC}(f')\to \mathrm{MIC}(f)$.

Example 2.3.2. Take (16) to be

$$\begin{tikzcd} X \times Y \arrow[r, "p_Y"] \arrow[d, "p_X"] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Specan}(\mathbb{C}). \end{tikzcd}$$

Then $p_Y^* : \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}$ sits on the left of the diagram [Rot97, (2.15)] and

(19) $$p_{Y*} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y)_{\mathrm{cxn}}$$

is [Rot97, (2.12)]. They restrict respectively to functors

(20) $$p_Y^* : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_{X\times Y/X}),$$

(21) $$p_{Y*} : \mathrm{Mod}(D_{X\times Y/X}) \to \mathrm{Mod}(D_Y).$$

Remark 2.3.3. Take $\alpha = 0 \in H^1(Z, \Omega^1_{Z/S})$. From another point of view, the morphism $O_Z \to g'_* O_W$ between sheaves of rings extends to a morphism $\tilde{D}_{Z/S} \to g'_* \tilde{D}_{W/T}$. Then (17) and (18) are respectively the pullback and the pushforward along the induced morphism $(W, \tilde{D}_{W/T}) \to (Z, \tilde{D}_{Z/S})$ of ringed spaces. By [Sta26, Tag 0096], the functor (17) is the left adjoint to (18). Then from [Sta26, Tag 09T5], the derived functor

$$Lg'^* : D(\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}})$$

is the left adjoint to

$$Rg'_* : D(\mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}).$$

3. Rothstein transform on modules with connection

For dual abelian varieties A, B over an algebraically closed field k, Rothstein [Rot96, p.571] introduces a pair of triangulated functors which interchanges sheaves with splittings on B and sheaves with connections on A. He [Rot96, Thm. 2.2] proves that this pair gives rise to an equivalence between the bounded derived categories of such sheaves. We extend Rothstein's transform to the analytic setup. Let X and Y be dual complex tori. Let $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ be the category of O_X-modules F equipped with a 1-splitting $\psi : F \to \mathcal{E} \otimes_{O_X} F$. Let $\mathrm{Mod}(O_Y)_{\mathrm{cxn}}$ be the category of O_Y-modules E equipped with a connection $\nabla : E \to E \otimes_{O_Y} \Omega^1_Y$.

3.1. Compatibility of Rothstein transform and Fourier-Mukai transform.

The exact functor (13) induces a triangulated functor $p_X^* \colon D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}})$. Let

$$Rp_{X*} \colon D(\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_X)_{\mathrm{sp}}),$$
$$Rp_{Y*}^{(\mathrm{cxn})} \colon D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$$

be the right derived functors of (15) and (19) respectively. We also write Rp_{Y*} for $Rp_{Y*}^{(\mathrm{cxn})}$. In Theorem 3.1.1, the passage between the category of connections and the category of splittings originates from the functors p_X^* and Rp_{X*}.

Definition 3.1.1. For dual complex tori X and Y, define triangulated functors

$$\mathfrak{F} := Rp_{X*}(\mathcal{P}^{-1} \otimes_{O_{X\times Y}} p_Y^* -) \colon D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D(\mathrm{Mod}(O_X)_{\mathrm{sp}}),$$
$$\mathfrak{F}' := Rp_{Y*}(\mathcal{P} \otimes_{O_{X\times Y}} p_X^* -) \colon D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}).$$

The pair $(\mathfrak{F}, \mathfrak{F}')$ is called the *Rothstein transform* for (X, Y).

Let $D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}) \subset D(\mathrm{Mod}(O_Y)_{\text{cxn}})$ and $D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}) \subset D(\mathrm{Mod}(O_X)_{\text{sp}})$ be the full subcategories of objects whose cohomologies are good O-modules. Proposition 3.1.2 shows that the Rothstein transform is compatible with the Fourier-Mukai transform. Let

$$\mathrm{for}_X : \mathrm{Mod}(O_X)_{\text{sp}} \to \mathrm{Mod}(O_X), \quad \mathrm{for}_Y : \mathrm{Mod}(O_Y)_{\text{cxn}} \to \mathrm{Mod}(O_Y)$$

be the forgetful functors.

Proposition 3.1.2. *For dual complex tori X and Y, there are isomorphisms of functors*

$$\mathrm{for}_Y \mathfrak{F}' \cong \mathcal{F}' \mathrm{for}_X : D(\mathrm{Mod}(O_X)_{\text{sp}}) \to D(O_Y),$$
$$\mathrm{for}_X \mathfrak{F} \cong \mathcal{F} \mathrm{for}_Y : D(\mathrm{Mod}(O_Y)_{\text{cxn}}) \to D(O_X).$$

In particular, the Rothstein transform restricts to functors

$$\mathfrak{F} : D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}),$$
$$\mathfrak{F}' : D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}).$$

Proof. Let $\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}} \to \mathrm{Mod}(O_{X\times Y})$ be the forgetful functor. All the functors (13), (14),

$$p_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X\times Y}), \quad \mathcal{P} \otimes_{O_{X\times Y}} (-) : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_{X\times Y})$$

as well as the forgetful functors involved in the proof are exact.

To prove the first isomorphism, it remains to show that there is an isomorphism of functors

$$\mathrm{for}_Y Rp_{Y*}^{(\text{cxn})} \cong Rp_{Y*} \mathrm{for}_{X\times Y} : D(\mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}}) \to D(O_Y).$$

Since $\mathrm{for}_Y : D(\mathrm{Mod}(O_Y)_{\text{cxn}}) \to D(O_Y)$ is exact, $\mathrm{for}_Y Rp_{Y*}^{(\text{cxn})} : D(\mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}}) \to D(O_Y)$ is the right derived functor of

$$\mathrm{for}_Y \circ p_{Y*}^{(\text{cxn})} : \mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}} \to \mathrm{Mod}(O_Y).$$

From Remark 2.3.1, [Sta26, Tags 0096 and 08BJ], $\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}} \to \mathrm{Mod}(O_{X\times Y})$ preserves K-injective complexes. Therefore, $Rp_{Y*}\mathrm{for}_{X\times Y} : D(\mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}}) \to D(O_Y)$ is the right derived functor of

$$p_{Y*}\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}} \to \mathrm{Mod}(O_Y).$$

Since there is an isomorphism of functors

$$\mathrm{for}_Y \circ p_{Y*}^{(\text{cxn})} \cong p_{Y*} \circ \mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\text{cxn}} \to \mathrm{Mod}(O_Y),$$

one obtains the first stated isomorphism of functors. Together with [Liu26, Cor. 3.2.11], it shows that $\mathfrak{F}'$ preserves O-goodness. The other half about $\mathfrak{F}$ is similar. □

3.2. **Rothstein's theorem.**

Theorem 3.2.1 (Rothstein)**.** *Let X be a complex torus of dimension g. Let Y be the dual complex torus. Then there are isomorphisms of functors*

$$\mathfrak{F}'\mathfrak{F} \cong T^{-g} : D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}),$$
$$\mathfrak{F}\mathfrak{F}' \cong T^{-g} : D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}).$$

In particular, $\mathfrak{F} : D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}})$ is an equivalence of triangulated categories, with a quasi-inverse $T^g\mathfrak{F}' : D_{O-\text{good}}(\mathrm{Mod}(O_X)_{\text{sp}}) \to D_{O-\text{good}}(\mathrm{Mod}(O_Y)_{\text{cxn}})$.

We begin the proof of Theorem 3.2.1 with Lemma 3.2.2, which is a direct adaption of [Rot97, Propositions 2.4, 2.5 and 3.1]. For a complex torus X, define a morphism of complex tori $\epsilon_X : X\times X \to X, \quad (x_1, x_2) \mapsto x_2 - x_1$. Let $\Delta_X \subset X\times X$ be the diagonal.

Lemma 3.2.2. *Let X and Y be dual complex tori of dimension g.*

(i) *Let $p_{12} : X \times X \times Y \to X \times X$ be the projection to the first two factors. Then there is an isomorphism $Rp_{12*}(\epsilon_X \times \mathrm{Id}_Y)^*\mathcal{P} \cong T^{-g}O_{\Delta_X}$ in $D^b(\mathrm{Mod}(O_{X\times X})_{(1,-1)-\mathrm{sp}})$.*
(ii) *Let $p_{23} : X \times Y \times Y \to Y \times Y$ be the projection to the last two factors. Then there is an isomorphism $Rp_{23*}(\mathrm{Id}_X \times \epsilon_Y)^*\mathcal{P} \cong T^{-g}O_{\Delta_Y}$ in $D^b(\mathrm{Mod}(O_{Y\times Y})_{(\epsilon_Y,0)-\mathrm{cxn}})$.*
(iii) *There exist isomorphisms of functors*

$$Rp_{1*}(O_{\Delta_X} \otimes^L p_2^*-) \cong \mathrm{Id}\colon D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(\mathrm{Mod}(O_X)_{\mathrm{sp}}),$$
$$Rp_{1*}(O_{\Delta_Y} \otimes^L p_2^*-) \cong \mathrm{Id}\colon D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}).$$

Proof. (i) The identification $Rp_{X*}\mathcal{P} \cong T^{-g}\mathbb{C}_{(0)}$ in $D^b(O_X)$ from [Kem91, Thm. 3.15] can be lifted to an isomorphism in $D^b(\mathrm{Mod}(O_X)_{-1-\mathrm{sp}})$. As stated in the last sentence of the proof of [Vig21, Prop. 2.1.21], a morphism of modules with splittings (or connections) is an isomorphism whenever the underlying morphism of O-modules is so. By [Liu26, Thm. 1.3.2] applied to the cartesian square

$$\begin{tikzcd} X\times X\times Y \arrow[r, "\epsilon_X\times 1_Y"] \arrow[d, "p_{12}"] & X\times Y \arrow[d, "p_X"] \\ X\times X \arrow[r, "\epsilon_X"] & X, \end{tikzcd}$$

the natural morphism $\epsilon_X^* Rp_{X*}\mathcal{P} \to Rp_{12*}(\epsilon_X\times 1_Y)^*\mathcal{P}$ in $D^b(\mathrm{Mod}(O_{X\times X})_{(1,-1)-\mathrm{sp}})$ is mapped to an isomorphism in $D^b_{\mathrm{coh}}(O_{X\times X})$, so itself is an isomorphism.
(ii) The proof is similar to that of Part (i).
(iii) The proof is similar to that of [Liu26, Lem. 5.1.1]. □

Proof of Theorem 3.2.1. We shall use the projection formula and the flat base change theorem for modules equipped with connections or splittings. The comparison morphisms are constructed from the adjunction between the derived inverse image and the derived direct image in Remark 2.3.3. Similar to Proposition 3.1.2, one can prove that these comparison morphisms are compatible with the comparison morphisms of the underlying O-modules. Thus, ignoring the connections and splittings, one needs to prove that the comparison morphisms of O-modules are isomorphisms. (This type of arguments can also be found in the proof of [Vig21, Prop. 2.1.21, Thm. 2.1.33].)

Let p_i $(i = 1, 2)$ be the projections $X \times X \to X$. Let p_{ij} be the projections of $X \times X \times Y$. By the universal property of the Poincaré bundle, there is an isomorphism

$$p_{13}^*\mathcal{P}^{-1} \otimes p_{23}^*\mathcal{P} \cong (\epsilon_X \times \mathrm{Id}_Y)^*\mathcal{P} \tag{22}$$

in $\mathrm{Mod}(X\times X\times Y)_{(1,-1)-\mathrm{cxn}}$. Now we adapt the proof of [Rot97, Thm. 3.2] to the analytic case and add more details. There are isomorphisms

$$\begin{aligned}
\mathfrak{F}\mathfrak{F}' =& Rp_{X*}(\mathcal{P}^{-1}\otimes p_Y^* Rp_{Y*}(\mathcal{P}\otimes p_X^* -)) \\
\overset{(a)}{\xrightarrow{\sim}}& Rp_{X*}(\mathcal{P}^{-1}\otimes Rp_{13*}p_{23}^*(\mathcal{P}\otimes p_X^* -)) \\
\overset{(b)}{\xrightarrow{\sim}}& Rp_{X*}Rp_{13*}(p_{13}^*\mathcal{P}^{-1}\otimes p_{23}^*(\mathcal{P}\otimes p_X^* -)) \\
\cong& Rp_{1*}Rp_{12*}((p_{13}^*\mathcal{P}^{-1}\otimes p_{23}^*\mathcal{P})\otimes p_{23}^*p_X^* -) \\
\overset{(c)}{\cong}& Rp_{1*}Rp_{12*}((\epsilon_X\times \mathrm{Id}_Y)^*\mathcal{P}\otimes p_{12}^*p_2^* -) \\
\overset{(d)}{\xleftarrow{\sim}}& Rp_{1*}(Rp_{12*}(\epsilon_X\times \mathrm{Id}_Y)^*\mathcal{P}\otimes^L p_2^* -) \\
\overset{(e)}{\cong}& Rp_{1*}(T^{-g}O_{\Delta_X}\otimes p_2^* -)\overset{(f)}{\cong} T^{-g}
\end{aligned}$$

of functors $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})\to D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})$. Here, (a) uses the goodness of $\mathcal{P}\otimes p_X^*(-)$ over $O_{X\times Y}$ and applies the flat base change theorem [Liu26, Thm. 1.3.2] to the cartesian square

$$\begin{array}{ccc}
X\times X\times Y & \xrightarrow{p_{23}} & X\times Y \\
{\scriptstyle p_{13}}\downarrow & \square & \downarrow{\scriptstyle p_Y} \\
X\times Y & \xrightarrow[p_Y]{} & Y.
\end{array}$$

The isomorphisms (b) and (d) rely on the projection formula [Spa88, Prop. 6.18]. Finally, (c), (e) and (f) are from (22), Theorem 3.2.2 (i) and (iii) respectively.

Thus, the second stated isomorphism is proved. Similarly, using Theorem 3.2.2 (ii), one can prove the first stated isomorphism. □

3.3. **Matsushima's theorem.** For a complex torus Y and a point $y\in Y$, let $T_y : Y\to Y$ be the translation by y. A vector bundle E on Y is *homogeneous* if $T_y^*E\cong E$ for every $y\in Y$. Matsushima [Mat59, Thm. 1] shows that a holomorphic principal bundle on Y admits a holomorphic connection if and only if it is homogeneous. Matsushima's proof of the "only if" part utilizes the Lie algebra of holomorphic vector fields. Rothstein [Rot96, Thm. 3.2] uses the Fourier-Mukai transform to prove a parallel result in the algebraic setting. He shows that over an algebraically closed field k of characteristic 0, a vector bundle on an abelian variety admitting a connection is homogeneous. Inspired by Rothstein's method, for a vector bundle with a connection (E,∇) on Y, we use the Rothstein transform to prove a slightly stronger result that the pair (E,∇) is translation invariant. Conversely, we shall use the Laumon-Rothstein transform to show that every homogeneous vector bundle on Y admits an integrable connection in Theorem 5.2.1 below.

Proposition 3.3.1 (Matsushima)**.** *Let E be a coherent sheaf on a complex torus Y with a holomorphic connection $\nabla : E\to E\otimes_{O_Y}\Omega_Y^1$. Then for every $y\in Y$, $T_y^*(E,\nabla)$ is isomorphic to (E,∇). In particular, E is a homogeneous vector bundle.*

The proof of Theorem 3.3.1 relies on Lemma 3.3.2, which is a complex analytic analog of [Rot96, Lem. 3.1]. Rothstein's proof relies on the fact that every positive dimensional projective variety contains a curve. By contrast, a positive dimensional complex torus may not contain any 1-dimensional analytic subset. For this reason, we have to replace Rothstein's argument with [Liu26, Lem. 6.3.5]. Let F be an O_M-module on a complex manifold M. For every $x \in M$, let $T(F_x)$ be the set of $s_x \in F_x$ such that there is a nonzero $g_x \in O_{M,x}$ with $g_x s_x = 0$. Then $T(F) := \cup_{x\in M} T(F_x)$ is an O_M-submodule of F, called the *torsion* of F.

Lemma 3.3.2. *Let F be a* coherent *sheaf admitting a 1-splitting on a complex torus X. Then F is finitely supported.*

Proof. Suppose to the contrary that $\mathrm{Supp}(F)$ is infinite. By [CAS, p.76], $\mathrm{Supp}(F)$ is an analytic set in X. Then $\dim \mathrm{Supp}(F) \geq 1$. Let C be an irreducible component of $\mathrm{Supp}(F)$ of maximal dimension. Write $i : C \to X$ for the inclusion.

By [Liu26, Lem. 6.3.5], there is a connected compact Kähler manifold Z and a morphism $h : Z \to X$ such that $h(Z) = C$ and $F'' := F'/T(F')$ is a *vector bundle* on Z of positive rank r, where $F' = h^*F$. As X is a complex torus, its Albanese is X itself. The morphism of Albanese tori $\mathrm{alb}(h) : \mathrm{Alb}(Z) \to X$ contains a translate of C in its image. Since $\dim C > 0$, $\mathrm{alb}(h) : \mathrm{Alb}(Z) \to X$ is nonzero. Therefore, the dual morphism of complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is also nonzero. However, we claim that its tangent map at origin $h^* : \mathfrak{g} \to H^1(Z, O_Z)$ is zero.

Let $\mathcal{E}' = h^*\mathcal{E}$. By [Sta26, Tag 05NJ], because O_X is flat over itself, pulling back (9) to Z and tensoring with F'', one has a short exact sequence of O_Z-modules

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} F'' \to \mathcal{E}' \otimes_{O_Z} F'' \to F'' \to 0. \tag{23}$$

Since $\mathcal{E}'$ is a vector bundle on Z, one has

$$\frac{\mathcal{E}' \otimes F'}{T(\mathcal{E}' \otimes F')} = \mathcal{E}' \otimes F''.$$

Then a 1-splitting $\psi \colon F \to \mathcal{E} \otimes_{O_X} F$ induces a splitting $\psi' \colon F'' \to \mathcal{E}' \otimes_{O_Z} F''$ of (23).

Let $\beta : O_Z \to \mathcal{E}nd(F'')$ be the natural morphism. By Lemma 2.1.3, the composition

$$\mathrm{End}(\mathfrak{g}) \stackrel{\mathrm{Id}_{\mathfrak{g}^*}\otimes h^*}{\to} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, O_Z) \stackrel{\mathrm{Id}_{\mathfrak{g}^*}\otimes H^1(Z,\beta)}{\to} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, \mathcal{E}nd(F''))$$

sends $1 \in \mathrm{End}(\mathfrak{g})$ to 0. Therefore, the map $H^1(Z, \beta) \circ h^* : \mathfrak{g} \to H^1(Z, \mathcal{E}nd(F''))$ is zero. The morphism $\tau : \mathcal{E}nd(F'') \to O_Z$ which takes trace satisfies $\tau\beta = r \cdot \mathrm{Id}_{O_Z}$. Then $h^* = \frac{1}{r}\tau_* H^1(Z, \beta) h^* = 0$ as maps $\mathfrak{g} \to H^1(Z, O_Z)$. The claim follows. This gives a contradiction. $\square$

Proof of Proposition 3.3.1. By Proposition 3.1.2, $\mathrm{for}_X : D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(O_X)$ sends $\mathfrak{F}(E, \nabla)$ to an object isomorphic to $\mathcal{F}(E)$. From [Liu26, Cor. 3.2.11], $\mathcal{F}(E)$ lies in $D^b_{\mathrm{coh}}(O_X)$. Then by Lemma 3.3.2, for every integer i, the support of $H^i\mathcal{F}(E)$ is finite. As there exist only finitely many $i \in \mathbb{Z}$ with $H^i\mathcal{F}(E) \neq 0$, there is an open subset $U \subset X$ with $P_y|_U \cong O_U$ which contains the supports of all $H^i\mathcal{F}(E)$. By Lemma 7.5.1 below, there is an isomorphism $\mathfrak{F}(T_y^*(E, \nabla)) \cong \mathfrak{F}(E, \nabla) \otimes_{O_X} P_y$ in $D(\mathrm{Mod}(O_X)_{\mathrm{sp}})$. By the choice of U, $\mathfrak{F}(E, \nabla) \otimes_{O_X} P_y$ is isomorphic to $\mathfrak{F}(E, \nabla)$. Then by Theorem 3.2.1, $T_y^*(E, \nabla)$ is isomorphic to (E, ∇) in $\mathrm{Mod}(O_Y)_{\mathrm{cxn}}$. $\square$

4. Laumon-Rothstein sheaf of algebras

Let X, Y be complex tori dual to each other of dimension g. As a difference between the algebraic and the analytic case, when lifting the Fourier-Mukai transform from O_Y-modules to D_Y-modules, we have to replace $O_{X^\natural}$ with an O_X-algebra $\mathcal{A}_X$ to ensure that the lift remains an equivalence. In the algebraic case, Rothstein [Rot96, p.576] explains how $O_{X^\natural}$ and $\mathcal{A}_X$ give rise to equivalent derived categories, which fails in the analytic setup. Let $\mathcal{E}$ be the tautological extension of O_X by $\mathfrak{g}^* \otimes_{\mathbb{C}} O_X$ in (9).

4.1. **Construction of the O_X-algebra $\mathcal{A}_X$.** We recall the sheaf $\mathcal{A}_X$ from [Rot96, p.576]. In the notation of (9), fix a $\mathbb{C}$-basis $\{\omega^1, \dots, \omega^g\}$ of the $\mathbb{C}$-vector space

$$H^0(Y, \Omega_Y^1) = \mathfrak{g}^* = \Gamma(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \subset \Gamma(X, \mathcal{E}).$$

By Cartan's Theorem B (see, *e.g.*, [KK83, Sec. 52, Thm. B]), for each Stein open subset $U \subset X$, one has $H^1(U, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = 0$. Thence (9) induces a short exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U) \to \mathcal{E}(U) \xrightarrow{\mu} O_X(U) \to 0.$$

Whence, there is $\rho \in \mathcal{E}(U)$ with $\mu(\rho) = 1 \in O_X(U)$. For two such pairs (U, ρ) and $(\tilde{U}, \tilde{\rho})$ with $U \cap \tilde{U} \neq \emptyset$, one has $\mu(\tilde{\rho} - \rho) = 0 \in O_X(U \cap \tilde{U})$, so $\tilde{\rho} - \rho \in \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U \cap \tilde{U})$. There exists a unique tuple $f_1, \dots, f_g \in O_X(U \cap \tilde{U})$ such that

$$\tilde{\rho} - \rho = \sum_{i=1}^{g} \omega^i \otimes f_i$$

in $\mathcal{E}(U \cap \tilde{U})$.

Definition 4.1.1. For each chosen pair (U, ρ) as above, introduce independent variables $x_1^\rho, \dots, x_g^\rho$ and put

$$\mathcal{A}_X|_U = O_U[x_1^\rho, \dots, x_g^\rho].$$

For another choice $(\tilde{U}, \tilde{\rho})$ with the tuple $(f_1, \dots, f_g)$ as above, we glue $\mathcal{A}_X|_U$ and $\mathcal{A}_X|_{\tilde{U}}$ by the rule

$$(x_i^\rho - x_i^{\tilde{\rho}})|_{U \cap \tilde{U}} = f_i. \tag{24}$$

The resulting sheaf $\mathcal{A}_X$ is a sheaf of commutative O_X-algebras of finite type. As an O_X-module, $\mathcal{A}_X$ is good and locally free.

Let

$$0 \to \mathfrak{g}^* \to X^\natural \xrightarrow{\pi} X \to 0$$

be the universal vectorial extension of X constructed in [Liu23, (22)]. Then $X^\natural$ is the moduli space of flat line bundles on Y. In coordinate-free terms, $\mathcal{A}_X \subset \pi_* O_{X^\natural}$ is the O_X-subalgebra of sections whose restriction to each fiber of $\pi : X^\natural \to X$ is a polynomial on $\mathfrak{g}^*$. For every integer $m \geq 0$, let $O_{X^\natural}(m) \subset O_{X^\natural}$ denote the subsheaf of sections whose restriction to the fibers of π are homogeneous polynomials of degree m. Similar to [Bjö93, Def 1.6.1], there exists a sheaf of graded rings $O_{[X^\natural]} := \oplus_{m \geq 0} O_{X^\natural}(m) (\subset O_{X^\natural})$ on $X^\natural$. Then $\mathcal{A}_X = \pi_* O_{[X^\natural]}$ and $\Gamma(X, \mathcal{A}_X) = \mathbb{C}$.

Remark 4.1.2. Different from the analytic case, if X is an abelian variety, then the notation $\mathcal{A}_X$ in [Rot96, p.576] refers to the *algebraic* direct image $\pi_* O_{X^\natural}$. Morally, this difference also lies between algebraic and analytic D-modules. For a complex manifold or a smooth algebraic variety V, let $p : T^*V \to V$ be the natural projection of the cotangent bundle. Denote by GD_V the associated graded ring of the degree filtration on D_V. From [HT07, p.57], in the algebraic case GD_V is $p_*O_{T^*V}$. By contrast, in the analytic case, GD_V is the O_V-submodule of $p_*O_{T^*V}$ of sections whose restriction to each fiber of $p : T^*V \to V$ is a polynomial.

Remark 4.1.3. The sheaf of rings $\mathcal{A}_X$ is functorial in X in the following sense. Let $\phi : X' \to X$ be a morphism of complex tori. Let $\hat{\phi} : Y \to Y'$ be its dual morphism of complex tori. By [Liu23, Prop. 5.4.7], it induces a morphism $\phi^\natural : X'^\natural \to X^\natural$ of complex Lie groups fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(Y', \Omega^1_{Y'}) \arrow[r] \arrow[d, "\hat{\phi}^*"] & X'^\natural \arrow[r, "\pi'"] \arrow[d, "\phi^\natural"] & X' \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & H^0(Y, \Omega^1_Y) \arrow[r] & X^\natural \arrow[r, "\pi"] & X \arrow[r] & 0. \end{tikzcd}$$

For each local section $h \in O_{[X^\natural]}$, the local section $(\phi^\natural)^*h \in O_{X'^\natural}$ restricts to a polynomial on each fiber of $\pi' \colon X'^\natural \to X'$. Indeed, this restriction is the $\hat{\phi}^*$-pullback of the restriction of h to a fiber of $\pi \colon X^\natural \to X$. Therefore, the natural morphism $O_{X^\natural} \to \phi^\natural_* O_{X'^\natural}$ restricts to a morphism $O_{[X^\natural]} \to \phi^\natural_* O_{[X'^\natural]}$. The resulting morphism of ringed spaces $(X'^\natural, O_{[X'^\natural]}) \to (X^\natural, O_{[X^\natural]})$ descends to another morphism of ringed spaces

$$\tilde{\phi} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X), \tag{25}$$

which is compatible with the morphism $\phi : (X', O_{X'}) \to (X, O_X)$ of ringed spaces. In particular, there is an isomorphism of functors $\mathrm{for}_X R\tilde{\phi}_* \to R\phi_* \mathrm{for}_{X'} : D(\mathcal{A}_{X'}) \to D(O_X)$. If M is an O_X-module, then

$$L\tilde{\phi}^*(\mathcal{A}_X \otimes_{O_X} M) = \mathcal{A}_{X'} \otimes_{O_{X'}} L\phi^* M. \tag{26}$$

4.2. **Properties of $\mathcal{A}_X$.** Notice that $\mathcal{A}_X$ has a natural increasing degree filtration $\{\mathcal{A}_X(m)\}_{m \geq 0}$, where

$$\mathcal{A}_X(m) = \pi_*(\oplus_{j=0}^m O_{X^\natural}(j))$$

is the O_X-submodule of $\mathcal{A}_X$ of polynomials of degree at most m. Then $\mathcal{A}_X(0) = O_X$, $\mathcal{A}_X(1) = \mathcal{E}^\vee$, and every $\mathcal{A}_X(m)$ is a locally free O_X-module of finite rank. Moreover, for any integers $m, n \geq 0$, one has

$$\mathcal{A}_X(n)\mathcal{A}_X(m) = \mathcal{A}_X(n+m).$$

Thus, $\mathcal{A}_X$ is a sheaf of positively filtered rings (in the sense of [Bjö93, p.459, p.464]) on the complex torus X.

We review some terminology from [Bjö93, A:III]. A coherent sheaf of rings on a locally compact Hausdorff space is called *noetherian* if every increasing sequence of ideal sheaves is stationary over relatively compact subsets ([Bjö93, 2.24, p.470]). Let R be a commutative filtered ring. If the subring $\oplus_{v\in\mathbb{Z}} R_v T^v$ of $R[T, T^{-1}]$ is a noetherian ring, then R is called a *noetherian filtered ring*.

Definition 4.2.1 ([Bjö93, A.III, 1.7, Def. 1.11, 1.19])**.** A filtration on an R-module M is a family of additive subgroups $\{M_v\}_{v\in\mathbb{Z}}$ such that

$$M_v \subset M_{v+1}; \quad R_k M_v \subset M_{k+v}; \quad \cup_v M_v = M.$$

This filtration is called *separated* if $\cap_{v\in\mathbb{Z}} M_v = 0$, and called *good* if $\oplus_{v\in\mathbb{Z}} M_v T^v$ is a finitely generated $\oplus_{v\in\mathbb{Z}} R_v T^v$-module.

A *zariskian filtered ring* is a noetherian filtered ring such that all the good filtrations on every finitely generated module are separated. A filtered sheaf of rings is called *stalkwise zariskian* if every stalk is a zariskian filtered ring ([Bjö93, Def. 2.6, p.465]).

Lemma 4.2.2. *For a complex torus X, the sheaf of rings $\mathcal{A}_X$ is coherent and noetherian. The sheaf of filtered rings $\mathcal{A}_X$ is stalkwise zariskian.*

Proof. By (24), the graded ring associated with the degree filtration of $\mathcal{A}_X$ is

$$G\mathcal{A}_X := \oplus_{m\geq 0}\mathcal{A}_X(m)/\mathcal{A}_X(m-1) = \mathrm{Sym}(\mathfrak{g})\otimes_{\mathbb{C}} O_X = O_X[x_1,\dots,x_g]. \tag{27}$$

Here for each chosen pair (U,ρ) as in Section 4.1, $x_i|_U \in \Gamma(U,\mathcal{A}_X(1)/\mathcal{A}_X(0)) \subset \Gamma(U, G\mathcal{A}_X)$ is the image of $x_i^\rho \in \Gamma(U,\mathcal{A}_X(1))$. From [Bjö79, Thm. 1.26, p.460], $\mathcal{A}_X$ is stalkwise zariskian. The other part follows from [Bjö79, Prop. 1.27, p.460, Thm. 2.7, p.465]. (See also the proof of [Bjö93, Thm. 1.2.5].) □

In view of the difference mentioned in Remark 4.1.2, the statement of [Rot96, Prop. 4.4] is slightly modified as Fact 4.2.3. For every $\mathcal{A}_X$-module F and every chosen pair (U,ρ) as in Section 4.1, define $\psi_U^\rho : F(U)\to \mathcal{E}(U)\otimes_{O_X(U)} F(U)$ by

$$\psi_U^\rho(s) = \rho\otimes s + \sum_{i=1}^g \omega^i|_U \otimes (x_i^\rho s).$$

Then $(\mu\otimes \mathrm{Id}_F)(\psi_U^\rho(s)) = s$. In light of (24), the family $\{\psi_U^\rho\}_{(U,\rho)}$ glues to a 1-splitting $\psi_F : F\to \mathcal{E}\otimes_{O_X} F$. One has $\psi_F = \psi_{\mathcal{A}_X}\otimes_{\mathcal{A}_X}\mathrm{Id}_F$. By commutativity of $\mathcal{A}_X$ and [Rot96, (4.9)], one has $[\psi_F,\psi_F]=0$.

Fact 4.2.3. *The resulting functor $\mathrm{Mod}(\mathcal{A}_X)\to \mathrm{Mod}(O_X)_{\mathrm{sp}}$ induces an equivalence of abelian categories from $\mathrm{Mod}(\mathcal{A}_X)$ to the full subcategory of $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ of objects (F,ψ) with $[\psi,\psi]=0$.*

From Fact 4.2.3 and the proof of [Rot96, Prop. 4.1], the functor (13) restricts to an exact functor $p_X^* : \mathrm{Mod}(\mathcal{A}_X)\to \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,int}}$. Similarly by [Rot96, Prop. 4.2], the functor (15) restricts to a left exact functor

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,int}} \to \mathrm{Mod}(\mathcal{A}_X). \tag{28}$$

5. Laumon-Rothstein transform

For dual complex tori X and Y, we lift the Fourier-Mukai transform on O_Y-modules to a functor on D_Y-modules, which is an analytic analog of the transform introduced by Laumon [Lau96] and Rothstein [Rot96]. As an application, we reprove Morimoto's theorem that every homogeneous vector bundle on Y admits an integrable connection.

5.1. **Properties of Laumon-Rothstein transform.**

Definition 5.1.1. Define functors

$$\tilde{\mathcal{F}} = Rp_{X*}(\mathcal{P}^{-1}\otimes^L_{O_{X\times Y}} p_Y^*-) : D(D_Y)\to D(\mathcal{A}_X),$$
$$\tilde{\mathcal{F}}^\natural = Rp_{Y*}(\mathcal{P}\otimes^L_{O_{X\times Y}} p_X^*-) : D(\mathcal{A}_X)\to D(D_Y),$$

where $Rp_{Y*} : D(D_{X\times Y/X})\to D(D_Y)$ (resp. $Rp_{X*} : D(\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,int}})\to D(\mathcal{A}_X)$) is the right derived functor of (21) (resp. (28)), and $p_Y^* : D(D_Y)\to D(D_{X\times Y/X})$ is induced by the exact functor (20). The pair $(\tilde{\mathcal{F}},\tilde{\mathcal{F}}^\natural)$ is called the *Laumon-Rothstein transform* for (X,Y).

Example 5.1.2. Let L be a line bundle on Y equipped with an integrable connection ∇. Let $z \in X^\natural$ be the corresponding point. The $\pi_* O_{X^\natural}$-module $\pi_*\mathbb{C}_{(z)}$ is naturally an $\mathcal{A}_X$-module. Its underlying O_X-module is $\mathbb{C}_{\pi(z)}$. One has $\tilde{\mathcal{F}}^\natural(\pi_*\mathbb{C}_{(z)}) = (L, \nabla)$.

Proposition 5.1.3. *For dual complex tori X and Y, let*

$$\mathrm{for}_X : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X), \quad \mathrm{for}_Y : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(O_Y)$$

be the forgetful functors. Then there exist isomorphisms of functors

$$\mathrm{for}_Y \tilde{\mathcal{F}}^\natural \cong \mathcal{F}' \mathrm{for}_X \colon D(\mathcal{A}_X) \to D(O_Y),$$
$$\mathrm{for}_X \tilde{\mathcal{F}} \cong \mathcal{F} \mathrm{for}_Y \colon D(D_Y) \to D(O_X).$$

In particular, the Laumon-Rothstein transform restricts to functors

$$\tilde{\mathcal{F}} \colon D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X), \quad \tilde{\mathcal{F}}^\natural \colon D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y).$$

Proof. The proof is similar to that of Proposition 3.1.2, as $\mathcal{A}_X$ (resp. D_Y) is flat over O_X (resp. O_Y). □

Theorem 5.1.4 (Laumon, Rothstein)**.** *There are isomorphisms of functors*

$$\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}} \cong T^{-g} \colon D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(D_Y),$$
$$\tilde{\mathcal{F}} \tilde{\mathcal{F}}^\natural \cong T^{-g} \colon D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(\mathcal{A}_X).$$

In particular, $\tilde{\mathcal{F}} \colon D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X)$ is an equivalence of triangulated categories, with a quasi-inverse $T^g \tilde{\mathcal{F}}^\natural \colon D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y)$.

Proof. With Proposition 5.1.3, the proof is similar to that of Theorem 3.2.1. □

Proposition 5.1.5. *There are isomorphisms of functors*

$$\tilde{\mathcal{F}}(D_Y \otimes^L_{O_Y} -) \cong \mathcal{A}_X \otimes^L_{O_X} \mathcal{F}(-) : D_{\mathrm{good}}(O_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X);$$
$$\tilde{\mathcal{F}}^\natural(\mathcal{A}_X \otimes^L_{O_X} -) \cong D_Y \otimes^L_{O_Y} \mathcal{F}'(-) : D_{\mathrm{good}}(O_X) \to D_{O-\mathrm{good}}(D_Y).$$

Proof. The result follows from Proposition 5.1.3, Theorem 5.1.4 and Fact 1.1.1 (i) as in the proof of [Rot96, Thm. 6.1], *cf.* [Lau96, Prop. 3.1.2, Cor. 3.2.4]. □

For $x \in X$ (resp. $y \in Y$), let $P_x = \mathcal{P}|_{x \times Y}$ (resp. $P_y = \mathcal{P}|_{X \times y}$) be the pullback line bundle on Y (resp. X). For a complex manifold Z and a point $z \in Z$, let $i_z : z \to Z$ be the inclusion. Then $i_{z+}\mathbb{C}$ is a D_Z-module.

Corollary 5.1.6. *For any $x \in X$ and $y \in Y$, one has*

$$\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} \mathbb{C}_{(y)}) = \mathcal{A}_X \otimes_{O_X} P_{-y}, \quad T^g \tilde{\mathcal{F}}(D_Y \otimes_{O_Y} P_x) = \mathcal{A}_X \otimes_{O_X} \mathbb{C}_{(x)},$$
$$T^g \tilde{\mathcal{F}}^\natural(\mathcal{A}_X \otimes_{O_X} P_{-y}) = D_Y \otimes_{O_Y} \mathbb{C}_{(y)} = i_{y+}\mathbb{C}, \quad \tilde{\mathcal{F}}^\natural(\mathcal{A}_X \otimes_{O_X} \mathbb{C}_{(x)}) = D_Y \otimes_{O_Y} P_x.$$

Proof. By [HT07, Example 1.6.4], one has $D_Y \otimes_{O_Y} \mathbb{C}_{(y)} = i_{y+}\mathbb{C}$. The result follows from Theorem 5.1.4, Proposition 5.1.5, Fact 1.2.1 and [Liu26, Lem. 2.0.9]. □

5.2. **Application of the Laumon-Rothstein theorem.** Theorem 5.2.1, due to Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], is a converse to Proposition 3.3.1. For abelian varieties, Nakayashiki [Nak94, Prop. 5.9] gives a proof using the Fourier-Mukai transform and the existence of ample line bundles.

Theorem 5.2.1. *A homogeneous vector bundle on a complex torus admits an integrable connection.*

The proof of Theorem 5.2.1 needs Lemma 5.2.2, a converse to Lemma 3.3.2.

Lemma 5.2.2. *Let F be an O_X-module with finite support on a complex torus X. Then F admits a structure of $\mathcal{A}_X$-module.*

Proof. As $\pi\colon X^\natural \to X$ is a submersion and $\mathrm{Supp}(F)$ is finite, there exists an open neighborhood $U \subset X$ of $\mathrm{Supp}(F)$ and a morphism of complex manifolds $s : U \to X^\natural$ that is a local section to $\pi : X^\natural \to X$. Let $\iota : U \hookrightarrow X$ be the inclusion. Applying π_* to the morphism of sheaves of rings $O_{X^\natural} \to s_*O_U$, one gets a morphism $\pi_*O_{X^\natural} \to \iota_*O_U$. As $\mathcal{A}_X$ is an O_X-subalgebra of $\pi_*O_{X^\natural}$, this endows ι_*O_U with an $\mathcal{A}_X$-module structure.[1] Since $\mathrm{Id}_F \otimes \iota^\# : F \to F \otimes_{O_X} \iota_*O_U$ is an isomorphism of O_X-modules, F also obtains an $\mathcal{A}_X$-module structure. □

Proof of Theorem 5.2.1. Let E be a homogeneous vector bundle on Y. Set $\hat{E} := H^g\mathcal{F}(E) \in \mathrm{Mod}(O_X)$. According to [Liu26, Prop. 6.3.3] and Fact 1.1.1, one has $E = H^0\mathcal{F}'(\hat{E})$ and $\mathrm{Supp}(\hat{E})$ is a finite subset of X. By Lemma 5.2.2, $\hat{E}$ has an $\mathcal{A}_X$-module structure. By Proposition 5.1.3, the O_Y-module underlying $H^0\tilde{\mathcal{F}}^\natural(\hat{E}) \in \mathrm{Mod}(D_Y)$ is isomorphic to E, so E carries an integrable connection. □

We recover a slight generalization of Morimoto's theorem [Mor59, Thm. 2, p.91].

Corollary 5.2.3 (Morimoto)**.** *Let Y be a complex torus. Let E be a coherent sheaf on Y admitting a connection. Then E is a vector bundle admitting an integrable connection.*

Proof. It follow from Proposition 3.3.1 and Theorem 5.2.1. □

6. Good modules

In the algebraic case, Laumon and Rothstein independently prove that the Fourier-Mukai transform lifts to an equivalence of the bounded derived category of *coherent* D-modules. For dual complex tori X and Y, however, it is unclear whether the Laumon-Rothstein transform $\tilde{\mathcal{F}}$ restricts to a functor $D^b_{\mathrm{coh}}(D_Y) \to D^b_{\mathrm{coh}}(\mathcal{A}_X)$. Nevertheless, we can show that $\tilde{\mathcal{F}}$ restricts to an equivalence for smaller categories of good $\mathcal{A}_X$-modules and of good D_Y-modules. Every holonomic D_Y-module is good, so the bounded derived category $D^b_{\mathrm{good}}(D_Y)$ of good D_Y-modules is sufficiently large for our application.

6.1. **Good $\mathcal{A}_X$-modules and good D_Y-modules.** We define good $\mathcal{A}_X$-modules. We also review several definitions of good D-modules in the literature, and show that they are equivalent.

[1] This example shows that the coherence hypothesis in Lemma 3.3.2 is necessary.

Definition 6.1.1. Let $\mathcal{R}$ be a positively filtered sheaf of rings on a complex manifold Z such that the associated graded ring $G\mathcal{R}$ is coherent. Let M be a coherent left $\mathcal{R}$-module. A filtration on M is an increasing sequence of subsheaves $\{M_v\}_{v\in\mathbb{Z}}$ satisfying $\cup_{v\in\mathbb{Z}}M_v = M$ and $\mathcal{R}_k M_v \subset M_{k+v}$ for all integers $k \geq 0$ and v. This filtration is called

- *B-good* ([Bjö93, Remark 2.16, p.467]) if for every $x \in Z$, there exists an open neighborhood U, a finite set $\{m_1, \dots, m_s\} \subset \Gamma(U, M)$ and integers $k_1, \dots, k_s$ such that $M_v|_U = \sum_{i=1}^{s} \mathcal{R}_{v-k_i} m_i$ for all integers v.
- *locally good* ([Meb89, Prop. 2.1.12 (i)]) if every M_v is coherent over O_Z, and if for every $x \in Z$, there is an open neighborhood U of x and an integer $k_0 \geq 0$ such that $\mathcal{R}_m M_{k_0} = M_{m+k_0}$ on U for all integers $m \geq 0$.

Similar to [HT07, Prop. 2.1.1], using (27) one can show that a filtration $M_\bullet$ on a coherent $\mathcal{A}_X$-module M is B-good exactly when it is locally good. In that case, we call $M_\bullet$ a *good filtration* on M. From [HT07, Thm. 2.1.3 (i)], on a complex smooth algebraic variety V, a coherent *algebraic* D_V-module admits a globally defined good filtration. By contrast, on the complex manifold $\mathbb{C}^* \times \mathbb{CP}^1$, Malgrange [Mal04, p.405] constructs a coherent *analytic* D-module that does not admit any global good filtration.

Definition 6.1.2. On a complex manifold Z, an O_Z-module F is called

- *countably quasi-good* ([KS97, p.942]) if every compact subset of Z has an open neighborhood U such that $F|_U$ is the union of an increasing sequence of coherent O_U-submodules.
- *quasi-good* ([KS16, p.12]) if for every relatively compact open subset $U \subset Z$, the restriction $F|_U$ is a sum of coherent O_U-submodules.

A D_Z-module M is called

- *good coherent* if for every relatively compact open subset U of Z, there is a finite filtration $\{M_k\}_{k\in\mathbb{Z}}$ of $M|_U$ such that each quotient M_k/M_{k-1} is a coherent D_U-module admitting a good filtration ([Sai89, p.369], [SS94, p.10] and [KS96, p.43]).
- *S-quasi-good* ([KS96, p.43]) if for every relatively compact open subset $U \subset Z$, the restriction $M|_U$ admits a filtration $\{M_v\}_{v\in\mathbb{Z}}$ by coherent D_U-submodules such that each quotient M_v/M_{v-1} admits a good filtration and $M_v = 0$ for $v \ll 0$.

Proposition 6.1.3. *Let M be a coherent D_Z-module on a complex manifold Z. Then the following conditions are equivalent.*

(i) *For every relatively compact open subset U of Z, there is a coherent O_U-submodule $F \subset M|_U$ with $D_U \cdot F = M|_U$.*
(ii) *For every relatively compact open subset U of Z, the D_U-module $M|_U$ admits a good filtration.*
(iii) *The D_Z-module M is good coherent.*
(iv) *The D_Z-module M is S-quasi-good.*
(v) *The O_Z-module M is countably quasi-good.*
(vi) *The O_Z-module M is good.*
(vii) *The O_Z-module M is quasi-good.*

Proof. The implications (ii)⇒(iii), (v)⇒(vi) and (vi)⇒(vii) are direct. See [Bjö93, 1.4.10] for (i)⇒(ii).

(iii)⇒(iv) For every relatively compact open subset U of Z, consider the filtration $\{M_k\}$ in the definition. By induction on k, one proves that each M_k is D_U-coherent.

(iv)⇒(v) By [Bjö93, Cor. 1.4.6], as every quotient M_v/M_{v-1} admits a good filtration, it is countably quasi-good. By induction on v and using [KS97, Lem. 2.1.1], one proves that every M_v is countably quasi-good. Therefore, for every integer v, there is an increasing sequence $\{M_v^k\}_{k\geq 1}$ of coherent O_U-submodules of M_v with $M_v = \cup_{k\geq 1} M_v^k$. For every integer $k \geq 1$, let $M^k := \sum_{i\leq k, v\leq k} M_v^i$. By [Sta26, Tag 01BY], M^k is a coherent O_U-submodule of M_k. Then

$$M = \cup_{v\in\mathbb{Z}} M_v = \cup_{v\in\mathbb{Z}} \cup_{i\geq 1} M_v^i = \cup_{k\geq 1} M^k,$$

so M is countably quasi-good.

(vii)⇒(i) Let U be a relatively compact open subset of Z. Because M is a finite type D_Z-module, for every $x \in \bar{U}$, there is a relatively compact open neighborhood $U(x) \subset Z$ of x, an integer $n(x) \geq 1$ and sections

$$\{s_i^x\}_{1\leq i\leq n(x)} \subset \Gamma(U(x), M)$$

generating the $D_{U(x)}$-module $M|_{U(x)}$. By compactness of $\bar{U}$, the open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{1\leq j\leq r}$. Then $V = \cup_{j=1}^r U(x_j)$ is a relatively compact open subset of Z containing U. By Condition (vii), one may write $M|_V = \sum_{\alpha\in I} G_\alpha$, where I is an index set, and each G_α is a coherent O_V-submodule of $M|_V$.

For every $x \in \bar{U}$, there is an open neighborhood $V(x) \subset U(x)$ of x, such that for each $1 \leq i \leq n(x)$, $s_i^x|_{V(x)}$ is in $\Gamma(V(x), G_{\alpha(x,i)})$ for some index $\alpha(x, i) \in I$. By compactness of $\bar{U}$ again, the open cover $\{V(x)\}_{x\in\bar{U}}$ has a finite subcover $\{V(x'_k)\}_{1\leq k\leq m}$. Then

$$F := \sum_{1\leq k\leq m, 1\leq i\leq n(x'_k)} G_{\alpha(x'_k,i)}$$

is a finite type O_V-submodule of $M|_V$. By Lemma 6.2.4 below, it is coherent over O_V. Moreover, one has $D_U \cdot F|_U = M|_U$. □

Proposition 6.1.4. *Let M be a coherent $\mathcal{A}_X$-module on a complex torus X. Then M is good over O_X if and only if there is a coherent O_X-submodule $F \subset M$ with $\mathcal{A}_X \cdot F = M$.*

Proof. The proof is similar to that of Proposition 6.1.3. □

Let the sheaf of rings $\mathcal{R}$ be either D_Z on a complex manifold Z or $\mathcal{A}_X$ on the fixed complex torus X. A coherent $\mathcal{R}$-module is *good* if the underlying O-module is good. For example, by Lemma 4.2.2 and [Bjö93, Thm. 1.2.5], the left $\mathcal{R}$-module $\mathcal{R}$ is good. It is unclear whether every coherent analytic D-module on a compact complex manifold is good. Let $\mathrm{Good}(\mathcal{R}) \subset \mathrm{Coh}(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b_{O-\mathrm{good}}(\mathcal{R})$) be the full subcategory of good $\mathcal{R}$-modules (resp. objects whose cohomologies are good $\mathcal{R}$-modules). By Proposition 6.1.3, the category $D^b_{\mathrm{good}}(D_Z)$ is what Björk denotes by $D^b_{\mathrm{coh}}(D_Z)_f$ in [Bjö93, p.119].

On a complex manifold Z, a coherent D_Z-module is *holonomic* if its characteristic variety is of (minimal) dimension $\dim Z$. Malgrange ([Mal94, p.35], [Mal96, p.367], see also [Sab11, Thm. 4.3.4 (2)]) proves that every holonomic D_Z-module M is generated by a coherent O_Z-submodule, so M is a good D_Z-module. Let $D^b_h(D_Z) \subset D^b(D_Z)$ be the full subcategory of objects with holonomic cohomologies. The good D_Y-module D_Y is not holonomic whenever $g > 0$.

6.2. **Basic properties of good modules.** Let $\mathcal{R}$ be either D_Z on a complex manifold Z or $\mathcal{A}_X$ on the fixed complex torus X.

Lemma 6.2.1. *The functor $\mathcal{R} \otimes_{O_Z} (-) : \mathrm{Mod}(O_Z) \to \mathrm{Mod}(\mathcal{R})$ is exact. It restricts to a functor $\mathcal{R} \otimes_{O_Z} (-) : \mathrm{Coh}(O_Z) \to \mathrm{Good}(\mathcal{R})$, and induces a t-exact functor $\mathcal{R} \otimes^L_{O_Z} (-) : D^b_{\mathrm{coh}}(O_Z) \to D^b_{\mathrm{good}}(\mathcal{R})$.*

Proof. It follows from [Liu26, Prop. 3.1.5 (2)], [Bjö93, Thm. 1.2.5] and Lemma 4.2.2. □

Lemma 6.2.2. *The category $\mathrm{Good}(\mathcal{R})$ is a weak Serre subcategory of $\mathrm{Mod}(\mathcal{R})$. In particular, $D^b_{\mathrm{good}}(\mathcal{R})$ is a triangulated subcategory of $D^b(\mathcal{R})$.*

Proof. The first half is a combination of [Kas03, Prop. 4.23], [Sta26, Tag 01BY] and [Sta26, Tag 0754]. The second half follows from [Yek19, Prop. 7.4.5]. □

For a morphism of complex manifolds $f : M \to N$, let $f_+ : D(D_M) \to D(D_N)$ denote the direct image functor of D-modules ([Bjö93, 2.3.12]).

Lemma 6.2.3. *Let $f : W \to Z$ be a proper morphism of complex manifolds. Then $f_+ : D(D_W) \to D(D_Z)$ restricts to a functor $D_{O-\mathrm{good}}(D_W) \to D_{O-\mathrm{good}}(D_Z)$.*

Proof. Take $M \in D_{O-\mathrm{good}}(D_W)$. By [Sab11, Remark 3.3.4 (4)], $f_+ : D(D_W) \to D(D_Z)$ has finite cohomological dimension. So by [Har66, I, Prop. 7.3 (iii)], to prove $f_+M \in D_{O-\mathrm{good}}(D_Z)$, one may assume $M \in \mathrm{Mod}(D_W)$. Define a morphism $i : W \to W \times Z, \quad w \mapsto (w, f(w))$, which is a closed embedding. Let $q : W \times Z \to Z$ be the projection. By [Sab11, Thm. 3.3.6 (1)], one has $f_+ = q_+ i_+$. By [Bjö93, Prop. 2.4.8], one has $f_+M = Rq_* \mathrm{DR}_{W\times Z/Z}(i_+M)[\dim Z]$. By [Liu26, Thm. 3.2.1], as each term of the relative de Rham complex $\mathrm{DR}_{W\times Z/Z}(i_+M)$ is $O_{W\times Z}$-good and supported on W, one has $Rq_* \mathrm{DR}_{W\times Z/Z}(i_+M) \in D_{\mathrm{good}}(O_Z)$. □

On a complex manifold Z, an O_Z-module F is *pseudo-coherent* if for every open subset U of X, every finite type O_U-submodule of $F|_U$ is of finite presentation ([Kas03, Def. A.5]).

Lemma 6.2.4. *If M is a coherent $\mathcal{R}$-module, then M is pseudo-coherent over O_Z.*

Proof. By [Meb89, Prop. 2.1.9] and its analog for $\mathcal{A}_X$, for every point $x \in Z$, there exists an open neighborhood U of x in Z and a good filtration on $M|_U$. By [Bjö93, Cor. 1.4.6], $M|_U$ is the sum of an increasing sequence of coherent O_U-submodules. Hence $M|_U$ is good over O_U. By [Liu26, Lem. A.4.1 (1)], the O_U-module $M|_U$ is pseudo-coherent. As pseudo-coherence is a local property, M is pseudo-coherent over O_Z. □

Lemma 6.2.5. *Let M be a good $\mathcal{R}$-module. Let N be a finite type $\mathcal{R}$-submodule of M. Then N is good over $\mathcal{R}$.*

Proof. By [Sta26, Tag 01BY (1)], N is coherent over $\mathcal{R}$. For every relatively compact open subset U of X and every $x \in \bar{U}$, there is an open neighborhood $U(x) \subset X$ of x, an integer $n(x) > 0$ and sections $\{s_i(x)\}_{i=1}^{n(x)} \subset \Gamma(U(x), N)$ generating the $\mathcal{R}|_{U(x)}$-module $N|_{U(x)}$. The open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover

$\{U(x_j)\}_{j=1}^m$. Let N_0 be the O_U-submodule of $N|_U$ generated by the finitely many local sections

$$\{s_i(x_j)\}_{1\leq j\leq m, 1\leq i\leq n(x_j)}.$$

Then N_0 is a finite type O_U-module. By Lemma 6.2.4, because $M|_U$ is good over $\mathcal{R}|_U$, the O_U-module N_0 is coherent. By construction, one has $\mathcal{R}|_U \cdot N_0 = N|_U$. Therefore, the $\mathcal{R}$-module N is good by Propositions 6.1.3 (in the case $\mathcal{R} = D_Z$) and 6.1.4 (in the case $\mathcal{R} = \mathcal{A}_X$). □

6.3. **Laumon-Rothstein transform preserves goodness.**

Theorem 6.3.1. *For dual complex tori X and Y of dimension g, $\tilde{\mathcal{F}}^\natural : D(\mathcal{A}_X) \to D(D_Y)$ restricts to an equivalence $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$, with a quasi-inverse $T^g\tilde{\mathcal{F}} : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$.*

Proof. (i) For every coherent O_Y-module F, we prove $\tilde{\mathcal{F}}(D_Y \otimes^L_{O_Y} F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

By Proposition 5.1.5, one has $\tilde{\mathcal{F}}(D_Y \otimes^L_{O_Y} F) = \mathcal{A}_X \otimes^L_{O_X} \mathcal{F}(F)$. By Fact 1.2.1, one has $\mathcal{F}(F) \in D^b_{\mathrm{coh}}(O_X)$. From Lemma 6.2.1, one gets $\mathcal{A}_X \otimes^L_{O_X} \mathcal{F}(F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

(ii) We prove that for every good D_Y-module M and every integer i, the $\mathcal{A}_X$-module $H^i\tilde{\mathcal{F}}(M)$ is good.

We use the descending induction on $i \in \mathbb{Z}$. By Theorem 5.1.3, the O_X-module underlying $H^i\tilde{\mathcal{F}}(M)$ is isomorphic to $H^i\mathcal{F}(M)$. By Lemma 6.3.2, if $i > 2g$, then $H^i\mathcal{F}(M) = 0$ and hence $H^i\tilde{\mathcal{F}}(M) = 0$.

Assume the statement for $i + 1$. By Proposition 6.1.3, there is a coherent O_Y-submodule $F \subset M$ with $D_Y \cdot F = M$. Let M' be the kernel of the natural epimorphism $D_Y \otimes_{O_Y} F \to M$, so that

$$0 \to M' \to D_Y \otimes_{O_Y} F \to M \to 0 \tag{29}$$

is a short exact sequence in $\mathrm{Mod}(D_Y)$. By Lemma 6.2.1, the D_Y-module $D_Y \otimes_{O_Y} F$ is good. By Lemma 6.2.2, so is M'. From (29), one has an exact sequence

$$\begin{aligned} &H^i\tilde{\mathcal{F}}(M') \to H^i\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F) \to H^i\tilde{\mathcal{F}}(M) \\ \to &H^{i+1}\tilde{\mathcal{F}}(M') \to H^{i+1}\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F) \end{aligned} \tag{30}$$

in $\mathrm{Mod}(\mathcal{A}_X)$. By (i), the $\mathcal{A}_X$-module $H^j\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F)$ is good for $j \in \{i, i+1\}$. By the inductive hypothesis, so is $H^{i+1}\tilde{\mathcal{F}}(M')$.

Let $G := \ker\left(H^{i+1}\tilde{\mathcal{F}}(M') \to H^{i+1}\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F)\right)$. By Lemma 6.2.2, the $\mathcal{A}_X$-module G is good (hence of finite type). The sequence (30) yields an exact sequence

$$H^i\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F) \to H^i\tilde{\mathcal{F}}(M) \to G \to 0,$$

so $H^i\tilde{\mathcal{F}}(M)$ is a finite type $\mathcal{A}_X$-module for every good D_Y-module M. In particular, as M' is good over D_Y, $H^i\tilde{\mathcal{F}}(M')$ is a finite type $\mathcal{A}_X$-module.

Let $N := \mathrm{im}\left(H^i\tilde{\mathcal{F}}(M') \to H^i\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F)\right)$. It is a finite type $\mathcal{A}_X$-submodule of the good $\mathcal{A}_X$-module $H^i\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F)$. By Lemma 6.2.5, the $\mathcal{A}_X$-module N is good. The sequence (30) yields an exact sequence

$$0 \to N \to H^i\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F) \to H^i\tilde{\mathcal{F}}(M) \to H^{i+1}\tilde{\mathcal{F}}(M') \to H^{i+1}\tilde{\mathcal{F}}(D_Y \otimes_{O_Y} F).$$

By Lemma 6.2.2, the $\mathcal{A}_X$-module $H^i\tilde{\mathcal{F}}(M)$ is good. The induction is completed.

From (ii), Lemma 6.2.2 and [Har66, I, Prop. 7.3 (i)], $\mathcal{F} \colon D(D_Y) \to D(\mathcal{A}_X)$ restricts to a functor $D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$. Similarly, using Proposition 6.1.4, one can prove that $\tilde{\mathcal{F}}^\natural$ restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$. By Theorem 5.1.4, the restrictions are equivalences. □

The proof of Theorem 6.3.1 needs a cohomological dimension estimation for the Fourier-Mukai transform.

Lemma 6.3.2. *Let X and Y be dual complex tori of dimension g. Then for every O_X-module F, one has $\mathcal{F}'(F) \in D^{[0,2g]}(O_Y)$. Similarly, for an O_Y-module G, one has $\mathcal{F}(G) \in D^{[0,2g]}(O_X)$.*

Proof. By left exactness of the functor $p_{Y*} : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_Y)$, one has $H^i\mathcal{F}'(F) = 0$ for every integer $i < 0$. For every $y \in Y$, let M be the restriction of $\mathcal{P} \otimes_{O_{X\times Y}} p_X^* F$ to $X \times y$ as a sheaf. By the proper base change theorem (see *e.g.*, [Mil13, Thm. 17.2]), for every integer j, one has $H^j\mathcal{F}'(F)_y = H^j(X \times y, M)$. By [KS90, Prop. 3.2.2 (iv)], if $j > 2g$, then $H^j(X\times y, M) = 0$. Therefore, $H^j\mathcal{F}'(F) = 0$. The other part is similar. □

We outline another application of the Laumon-Rothstein transform. For a complex torus X of dimension g, let $\Pi(X) := \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ be the group of characters of $\pi_1(X)$. It is naturally a linear algebraic group over $\mathbb{C}$ isomorphic to $\mathbb{G}_m^{2g}$. For every $\chi \in \Pi(A)$, let L_χ be the corresponding local system on X. One can recover a special case of the generic vanishing theorem [BSS18, Theorem 1.1] for complex tori. For complex abelian varieties, the result is due to Krämer, Weissauer [KW15, Theorem 1.1] and Schnell [Sch15, Proposition 18.2] independently.

Theorem 6.3.3. *Let X be a complex torus. Let $K \in D^b_c(X, \mathbb{C}_X)$ be a perverse sheaf on X. Then there is a dense Zariski open subset $U \subset \Pi(X)$, such that for every $\chi \in U$ and every $i \neq 0$, one has $H^i(X, K \otimes_{\mathbb{C}} L_\chi) = 0$.*

Proof. Using Theorem 6.3.1 and Malgrange's theorem [Mal94, p.35], one can prove the result by following Schnell's strategy of [Sch15, Proposition 18.2] in the algebraic case. □

7. Relations with other functors

The properties [Muk81, (3.1), (3.4), (3.8)] of the Fourier-Mukai transform have analogs for the Laumon-Rothstein transform.

7.1. **Duality.** Let Z be a complex manifold. Denote by Δ^{O_Z} the duality functor $R\mathcal{H}om_{O_Z}(\cdot, \omega_Z^{-1})[\dim Z] : D^b_{\mathrm{coh}}(O_Z)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(O_Z)$. The duality functor on D_Z-modules $\Delta^{D_Z} : D(D_Z)^{\mathrm{op}} \to D(D_Z)$ is defined by $\Delta^{D_Z} F = G[\dim Z]$, where G is the complex of left D_Z-modules associated with the complex $R\mathcal{H}om_{D_Z}(F, D_Z)$ of *right* D_Z-modules. By [Bjö93, Def. 2.11.1], Δ^{D_Z} restricts to a functor $D^b_{\mathrm{coh}}(D_Z)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(D_Z)$, and the natural transformation $\mathrm{Id} \to \Delta^{D_Z} \circ \Delta^{D_Z} : D^b_{\mathrm{coh}}(D_Z) \to D^b_{\mathrm{coh}}(D_Z)$ is an isomorphism of functors.

Lemma 7.1.1. *For a complex manifold Z, $\Delta^{D_Z} : D(D_Z)^{\mathrm{op}} \to D(D_Z)$ restricts to a functor $D^b_{\mathrm{good}}(D_Z)^{\mathrm{op}} \to D^b_{\mathrm{good}}(D_Z)$.*

Proof. Let F be a coherent O_Z-module and $N := D_Z \otimes_{O_Z} F$. By [Bjö93, (ii), p.122], there is $G \in D^b_{\mathrm{coh}}(O_Z)$ with $\Delta^{D_Z} N = D_Z \otimes_{O_Z} G$. By Lemma 6.2.1, $\Delta^{D_Z} N \in D^b_{\mathrm{good}}(D_Z)$.

Take $M \in D^b_{\mathrm{good}}(D_Z)$. By [Har66, I, Prop. 7.3 (i)], to prove $\Delta^{D_Z} M \in D^b_{\mathrm{good}}(D_Z)$, one may assume $M \in \mathrm{Good}(D_Z)$. By Proposition 6.1.3 and the exactness of

Spencer's complex [Bjö93, Thm. 1.5.8], for every relatively compact open subset $U \subset Z$, there is a finite length exact sequence in $\mathrm{Mod}(D_U)$:

$$0 \to D_U \otimes_{O_U} F^{-n} \to \cdots \to D_U \otimes_{O_U} F^0 \to M|_U \to 0,$$

where each F^i is a coherent O_U-module. For every i, one has $\Delta^{D_U}(D_U \otimes_{O_U} F^i) \in D^b_{\mathrm{good}}(D_U)$. By Lemma 6.2.2, one has $(\Delta^{D_Z} M)|_U = \Delta^{D_U}(M|_U) \in D^b_{\mathrm{good}}(D_U)$. Hence, $\Delta^{D_Z} M$ lies in $D^b_{\mathrm{good}}(D_Z)$. □

Lemma 7.1.2. *Let Z be a complex manifold.*

(i) *The functor $\Delta^{D_Z} : D^b_h(D_Z)^{\mathrm{op}} \to D^b_h(D_Z)$ is an equivalence of triangulated categories.*
(ii) *Let M be a coherent D_Z-module. Then M is holonomic if and only if $H^i(\Delta^{D_Z} M) = 0$ for all integers $i \neq 0$.*

Proof. For algebraic varieties, analogous results are stated as [HT07, Cor. 2.6.8 (iii), Prop. 3.2.1]. From [HT07, p.101], all the arguments in [HT07, Sec. 2.6] are valid for analytic D-modules. □

For a complex torus Y, ω_Y is isomorphic to O_Y. Therefore, by [Bjö93, (ii), p.122], there is an isomorphism of functors

$$\Delta^{D_Y}(D_Y \otimes^L_{O_Y} -) \cong D_Y \otimes^L_{O_Y} (\Delta^{O_Y} -) \colon D^b_{\mathrm{coh}}(O_Y) \to D^b_{\mathrm{coh}}(D_Y).$$

Define the duality functor $\Delta^{\mathcal{A}_X} : D^b(\mathcal{A}_X)^{\mathrm{op}} \to D^b(\mathcal{A}_X)$ as

$$\Delta^{\mathcal{A}_X} = T^g R\mathcal{H}om_{\mathcal{A}_X}(-, \mathcal{A}_X).$$

It restricts to a functor $D^b_{\mathrm{coh}}(\mathcal{A}_X) \to D^b_{\mathrm{coh}}(\mathcal{A}_X)$. Similar to Lemma 7.1.1, it restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$.

Proposition 7.1.3. *There are isomorphisms of functors*

$$\tilde{\mathcal{F}}\Delta^{D_Y} \cong [-1]^*_X T^{-g} \Delta^{\mathcal{A}_X} \tilde{\mathcal{F}} : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X),$$
$$\Delta^{D_Y} \tilde{\mathcal{F}}^\natural \cong [-1]^*_Y T^g \tilde{\mathcal{F}}^\natural \Delta^{\mathcal{A}_X} : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y).$$

Lemma 7.1.4. *On a complex manifold Z, for any objects $K, L \in D(O_Z)$ and $M \in D^-_{\mathrm{coh}}(O_Z)$, the natural morphism* (provided by [Sta26, Tag 0BYS])

$$K \otimes^L_{O_Z} R\mathcal{H}om_{O_Z}(M, L) \to R\mathcal{H}om_{O_Z}(M, K \otimes^L_{O_Z} L) \tag{31}$$

is an isomorphism in $D(O_Z)$.

Proof. By [Har66, I, Prop. 7.1 (ii)], one may assume $M \in \mathrm{Coh}(O_Z)$. By [Sta26, Tag 08DL] and [GH78, p.696], one may shrink Z such that M admits a globally free resolution $F^\bullet \to M$, where the complex $F^\bullet$ is

$$0 \to O_Z^{k_n} \to \cdots \to O_Z^{k_1} \to O_Z^{k_0} \to 0$$

with $O_Z^{k_i}$ placed in degree $-i$. The morphism (31) is identified with

$$K \otimes^L_{O_Z} \mathcal{H}om_{O_Z}(F^\bullet, L) \to \mathcal{H}om_{O_Z}(F^\bullet, K \otimes^L_{O_Z} L),$$

which is an isomorphism. □

Lemma 7.1.5. *There is an isomorphism of functors*

$$\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} -) \cong \mathcal{A}_X \otimes^L_{O_X} (\Delta^{O_X} -) \colon D^b_{\mathrm{coh}}(O_X)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(\mathcal{A}_X).$$

Proof. By [Rot96, (6.2)], there exist isomorphisms

$$\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} -) := T^g R\mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} -, \mathcal{A}_X) \cong T^g R\mathcal{H}om_{O_X}(-, \mathcal{A}_X)$$

of functors $D^b_{\mathrm{coh}}(O_X)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(\mathcal{A}_X)$. By Lemma 7.1.4, this functor is isomorphic to $T^g R\mathcal{H}om_{O_X}(-, O_X) \otimes^L_{O_X} \mathcal{A}_X \cong \mathcal{A}_X \otimes^L_{O_X} (\Delta^{O_X} -)$. ☐

Proof of Theorem 7.1.3. The result can be deduced from Theorem 7.1.5, Proposition 5.1.5 and [Liu26, Prop. 6.1.5], in the same way that [Rot96, Prop. 6.3] is proved. ☐

Theorem 7.1.6 (Rothstein)**.** *Let X be a complex torus of dimension g. Let $F \in D^b_{\mathrm{good}}(\mathcal{A}_X)$ be an object such that $\tilde{\mathcal{F}}^\natural(F)$ is concentrated in a single degree $i \in \mathbb{Z}$. Then $H^i\tilde{\mathcal{F}}^\natural(F)$ is holonomic if and only if $\tilde{\mathcal{F}}^\natural\Delta^{\mathcal{A}_X}F$ is concentrated in degree $g-i$.*

Proof. The result follows from Proposition 7.1.3 and Theorem 7.1.2 (ii), in the same way how Theorem 6.5 follows from Propositions 6.3 and 6.4 in [Rot96]. ☐

Example 7.1.7. Let $F = T^g\mathcal{A}_X \in D^b_{\mathrm{good}}(\mathcal{A}_X)$. By Corollary 5.1.6, one has $\tilde{\mathcal{F}}^\natural(F) = D_Y \otimes_{O_Y} \mathbb{C}_{(0)}$. One has $\Delta^{\mathcal{A}_X}F = \mathcal{A}_X$, and $\tilde{\mathcal{F}}^\natural\Delta^{\mathcal{A}_X}F$ is concentrated in degree g. Then by Theorem 7.1.6, the D_Y-module $D_Y \otimes_{O_Y} \mathbb{C}_{(0)}$ is holonomic.

7.2. **Pullback and pushforward.**

Proposition 7.2.1. *Let $f : X' \to X$ be a morphism of complex tori, with* $\dim X' = g'$. *Let $\hat{f} : Y \to Y'$ be the morphism dual to $f : X' \to X$. Let $\tilde{f} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X)$ be the induced morphism of ringed spaces (25). Then there are isomorphisms of functors*

$$\begin{aligned} L\hat{f}^*\tilde{\mathcal{F}}^{\natural\prime} &\cong \tilde{\mathcal{F}}^\natural R\tilde{f}_* : D_{O-\mathrm{good}}(\mathcal{A}_{X'}) \to D_{O-\mathrm{good}}(D_Y);\\ R\tilde{f}_*\tilde{\mathcal{F}}' &\cong T^{g-g'}\tilde{\mathcal{F}}L\hat{f}^* : D_{O-\mathrm{good}}(D_{Y'}) \to D_{O-\mathrm{good}}(\mathcal{A}_X),\\ \tilde{\mathcal{F}}'\hat{f}_+ &\cong L\tilde{f}^*\tilde{\mathcal{F}} \colon D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_{X'});\\ \hat{f}_+\tilde{\mathcal{F}}^\natural &\cong T^{g'-g}\tilde{\mathcal{F}}^{\natural\prime}L\tilde{f}^* \colon D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_{Y'}). \end{aligned}$$

Proof. One can prove the first isomorphism using [Liu26, Prop. 3.1.2 (2), Thm. 4.2.7, (19)]. The third isomorphism can be proved using [Bjö93, Thm. 2.8.1, 2.8.7, 2.11.8], (26) and Theorem 6.3.1. By Theorem 5.1.4 (resp. Theorem 6.3.1), the second (resp. last) isomorphism follows from the first (resp. third). ☐

7.3. **External tensor product.** For two complex manifolds U, V, let

$$(-) \boxtimes_O (+) : \mathrm{Mod}(D_U) \times \mathrm{Mod}(D_V) \to \mathrm{Mod}(D_{U\times V}) \tag{32}$$

be the exact external tensor product bifunctor ([Bjö93, 2.4.4]). By exactness, it induces a triangulated bifunctor

$$D(D_U) \times D(D_V) \to D(D_{U\times V}). \tag{33}$$

Remark 7.3.1. By [Bjö93, 2.4.13], the bifunctor (32) restricts to bifunctors

$$\mathrm{Coh}(D_U) \times \mathrm{Coh}(D_V) \to \mathrm{Coh}(D_{U\times V}), \quad \mathrm{Good}(D_U) \times \mathrm{Good}(D_V) \to \mathrm{Good}(D_{U\times V}).$$

Then by [Har66, I, Prop. 7.3 (i)], the bifunctor (33) restricts to bifunctors

$$D^b_{\mathrm{coh}}(D_U)\times D^b_{\mathrm{coh}}(D_V) \to D^b_{\mathrm{coh}}(D_{U\times V}), \quad D^b_{\mathrm{good}}(D_U)\times D^b_{\mathrm{good}}(D_V) \to D^b_{\mathrm{good}}(D_{U\times V}).$$

By [Bjö93, p.139], it also restricts to a bifunctor $D^b_h(D_U) \times D^b_h(D_V) \to D^b_h(D_{U\times V})$.

Lemma 7.3.2.

(i) *Let U, V, Z be complex manifolds. Let $f : U \to V$ be a* proper *morphism. Then the natural transformation*

$$f_+(-) \boxtimes_O (+) \to (f \times \mathrm{Id}_Z)_+(- \boxtimes_O +) : D_{O-\mathrm{good}}(D_U) \times D(D_Z) \to D(D_{V\times Z})$$

is an isomorphism.

(ii) *Let $f_i : U_i \to V_i$ $(i = 1, 2)$ be two* proper *morphisms of complex manifolds. Then the natural transformation*

$$(f_{1+}-) \boxtimes_O (f_{2+}+) \to (f_1 \times f_2)_+(- \boxtimes_O +):$$
$$D_{O-\mathrm{good}}(D_{U_1}) \times D_{O-\mathrm{good}}(D_{U_2}) \to D_{O-\mathrm{good}}(D_{V_1\times V_2})$$

is an isomorphism.

Proof. Using [Liu26, Lem. 6.1.3] (in place of [HT07, Lem. 1.5.31]), Lemma 6.2.3 and [Sab11, Thm. 3.3.6 (1)], one can argue as in [HT07, Prop. 1.5.30] to prove the result. □

For a complex torus X, let $\mathrm{for}_X : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)$ be the forgetful functor. Let X' be another complex torus. Set $X'' = X \times X'$. Write $u : X'' \to X$ and $u' : X'' \to X'$ for the projections. Let Y', Y'' be the dual of X' and X'' respectively. For an $\mathcal{A}_X$-module F and an $\mathcal{A}_{X'}$-module G, denote $\tilde{u}^*F \otimes_{\mathcal{A}_{X''}} \tilde{u'}^* G$ by $F \boxtimes_{\mathcal{A}} G$. By construction, one has

$$\begin{aligned}\mathcal{A}_{X''} =& \mathcal{A}_X \boxtimes_O \mathcal{A}_{X'} = u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'}\\ =& O_{X\times X'} \otimes_{u^{-1}O_X \otimes_{\mathbb{C}} u'^{-1}O_{X'}} (u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}).\end{aligned} \tag{34}$$

Since the $u^{-1}O_X \otimes_{\mathbb{C}} u'^{-1}O_{X'}$-module $O_{X\times X'}$ is flat, so is the $u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}$-module $\mathcal{A}_{X''}$. As

$$F \boxtimes_{\mathcal{A}} G = \mathcal{A}_{X''} \otimes_{u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}} (u^{-1}F \otimes_{\mathbb{C}} u'^{-1}G),$$

the bifunctor

$$- \boxtimes_{\mathcal{A}} + : \mathrm{Mod}(\mathcal{A}_X) \times \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(\mathcal{A}_{X''})$$

is exact in both arguments. Consider the diagonal morphism $\delta : X \to X^2$. There is a canonical isomorphism of bifunctors

$$L\tilde{\delta}^*(- \boxtimes_{\mathcal{A}} +) \cong (-) \otimes^L_{\mathcal{A}_X} (+) : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X). \tag{35}$$

Although the tensor product of two $\mathcal{A}_X$-modules is different from the tensor product of the underlying O_X-module, Lemma 7.3.3 shows that external products do agree. It is used in the proof of Lemma 7.3.4.

Lemma 7.3.3. *For a product of complex tori $X'' = X \times X'$, there is an isomorphism of bifunctors*

$$\mathrm{for}_{X''}(- \boxtimes_{\mathcal{A}} +) \cong (\mathrm{for}_X -) \boxtimes_O (\mathrm{for}_{X'} +) : \mathrm{Mod}(\mathcal{A}_X) \times \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''}).$$

Proof. There are isomorphisms of functors $\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_{X''})$:

$$\begin{aligned}
\mathrm{for}_{X''}\tilde{u}^* :=&(u^{-1}-) \otimes_{u^{-1}\mathcal{A}_X} \mathcal{A}_{X''}\\
\overset{(a)}{=}&(u^{-1}-) \otimes_{u^{-1}\mathcal{A}_X} (u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'})\\
\cong&(u^{-1}-) \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'}\\
\cong&(u^{-1} - \otimes_{u^{-1}O_X} O_{X''}) \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}\\
\cong&(u^*\mathrm{for}_X-) \otimes_{O_{X''}} u'^*\mathcal{A}_{X'},
\end{aligned}$$

where (a) uses (34). Similarly, there is an isomorphism of functors $\mathrm{for}_{X''}\tilde{u'}^* \cong u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}\cdot : \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''})$. One has isomorphisms of bifunctors

$$\begin{aligned}
&\mathrm{for}_{X''}(- \boxtimes_{\mathcal{A}} +) := (\tilde{u}^*-) \otimes_{\mathcal{A}_{X''}} (\tilde{u'}^*+)\\
\cong&(u^*\mathrm{for}_X - \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}) \otimes_{u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}} (u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}+)\\
\cong&(u^*\mathrm{for}_X-) \otimes_{O_{X''}} (u'^*\mathrm{for}_{X'}+)\\
:=&(\mathrm{for}_X-) \boxtimes_O (\mathrm{for}_{X'}+).
\end{aligned}$$

□

Lemma 7.3.4. *Let $(\tilde{\mathcal{F}}', \tilde{\mathcal{F}}^{\natural\prime})$ and $(\tilde{\mathcal{F}}'', \tilde{\mathcal{F}}^{\natural\prime\prime})$ be the Laumon-Rothstein transforms for (X', Y') and (X'', Y'') respectively. Then there are isomorphisms of bifunctors*

$$\tilde{\mathcal{F}}''(- \boxtimes_O +) \cong (\tilde{\mathcal{F}}-) \boxtimes_{\mathcal{A}} (\tilde{\mathcal{F}}'+) \colon D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_{Y'}) \to D_{O-\mathrm{good}}(\mathcal{A}_{X''}),$$
$$\tilde{\mathcal{F}}^{\natural\prime\prime}(- \boxtimes_{\mathcal{A}} +) \cong (\tilde{\mathcal{F}}^{\natural}-) \boxtimes_O (\tilde{\mathcal{F}}^{\natural\prime}+) \colon D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_{X'}) \to D_{O-\mathrm{good}}(D_{Y''}).$$

Proof. It follows from [Liu26, Prop. 6.1.2], Lemma 7.3.3 and Proposition 5.1.3. □

7.4. **Convolution and tensor product.** For the dual complex tori X and Y, let $m : X^2 \to X$ and $\mu : Y^2 \to Y$ be their respective group law. Let $\tilde{m} : (X^2, \mathcal{A}_{X^2}) \to (X, \mathcal{A}_X)$ be the morphism of ringed spaces induced by $m : X^2 \to X$ *via* Remark 4.1.3.

Definition 7.4.1 (Convolution, [Lau96, p.22])**.** Define bifunctors

$$\begin{aligned}
*_D : D(D_Y) \times D(D_Y) \to D(D_Y), &\quad (-) *_D (+) = \mu_+(- \boxtimes^L_O +),\\
*_{\mathcal{A}} : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X), &\quad (-) *_{\mathcal{A}} (+) = R\tilde{m}_*(- \boxtimes^L_{\mathcal{A}} +).
\end{aligned}$$

By [Bjö93, Thm. 2.8.1, 2.8.7, 3.2.13 (3)], [Sab11, Thm. 4.4.1] and Lemma 6.2.3, as $\mu : Y^2 \to Y$ is proper, $\mu_+ \colon D(D_{Y^2}) \to D(D_Y)$ restricts to functors

$$D^b_{\mathrm{good}}(D_{Y^2}) \to D^b_{\mathrm{good}}(D_Y), \quad D_{O-\mathrm{good}}(D_{Y^2}) \to D_{O-\mathrm{good}}(D_Y) \quad D^b_h(D_{Y^2}) \to D^b_h(D_Y).$$

Together with Remark 7.3.1, this implies that the bifunctor $*_D$ restricts to bifunctors

$$\begin{gathered}
D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_Y),\\
D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(D_Y),\\
D^b_h(D_Y) \times D^b_h(D_Y) \to D^b_h(D_Y).
\end{gathered}$$

As an example of the convolution of D-modules, for every $y \in Y$, there is an isomorphism of functors $(i_{y+}\mathbb{C}) *_D (-) \to T^*_{-y} \colon D(D_Y) \to D(D_Y)$.

Lemma 7.4.2. *The pair $(D(D_Y), *_D)$ is a symmetric tensor triangulated category* (in the sense of [Bal10, Def. 3]) *with unit $D_Y \otimes_{O_Y} \mathbb{C}_{(0)}$.*

Proof. Let $i : \{0\} \hookrightarrow Y$ be the inclusion of the origin. Then $D_Y \otimes_{O_Y} \mathbb{C}_{(0)} = i_+\mathbb{C}$. There are isomorphisms

$$\begin{aligned}(i_+\mathbb{C}) *_D (-) :=& \mu_+((i_+\mathbb{C}) \boxtimes_O -)\\ \overset{\text{(a)}}{\cong}& \mu_+(i \times \mathrm{Id}_Y)_+(\mathbb{C} \boxtimes_O -)\\ \overset{\text{(b)}}{\cong}& \mathrm{Id}_{Y+} = \mathrm{Id}_{D(D_Y)}\end{aligned}$$

of functors $D(D_Y) \to D(D_Y)$, where (a) and (b) use Theorem 7.3.2 (i) and [Sab11, Thm. 3.3.6 (1)] respectively. Therefore, $D_Y \otimes_{O_Y} \mathbb{C}_{(0)}$ is the unit. The other axioms can be verified as in [Wei07, pp. 10-11]. □

Proposition 7.4.3. *For dual complex tori X and Y, $\tilde{\mathcal{F}} : (D^b_{\mathrm{good}}(D_Y), *_D) \to (D^b_{\mathrm{good}}(\mathcal{A}_X), \otimes^L_{\mathcal{A}_X})$ is a strong monoidal functor* (in the sense of [SS03, Def. 3.3]). *In fact, there are canonical isomorphisms of bifunctors*

$$\tilde{\mathcal{F}}(- *_D +) \cong (\tilde{\mathcal{F}}-) \otimes^L_{\mathcal{A}_X} (\tilde{\mathcal{F}}+) : D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X);$$

$$(\tilde{\mathcal{F}}^\natural-) *_D (\tilde{\mathcal{F}}^\natural+) \cong T^{-g}\tilde{\mathcal{F}}^\natural(- \otimes^L_{\mathcal{A}_X} +) : D^b_{\mathrm{good}}(\mathcal{A}_X) \times D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y);$$

$$\tilde{\mathcal{F}}^\natural(- *_{\mathcal{A}} +) \cong (\tilde{\mathcal{F}}^\natural-) \otimes^L_{O_Y} (\tilde{\mathcal{F}}^\natural+) : D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y);$$

$$(\tilde{\mathcal{F}}-) *_{\mathcal{A}} (\tilde{\mathcal{F}}+) \cong T^{-g}\tilde{\mathcal{F}}(- \otimes^L_{O_Y} +) : D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X).$$

Proof. It follows from (35), Theorems 5.1.4 and 6.3.1. □

Theorem 7.4.4 is a complex analytic analog of Weissauer's result [Wei11] for perverse sheaves on abelian varieties.

Theorem 7.4.4. *For a complex torus Y, both $(D^b_{\mathrm{good}}(D_Y), *_D)$ and $(D^b_h(D_Y), *_D)$ are rigid symmetric monoidal triangulated categories. In fact, for every $M \in D^b_{\mathrm{good}}(D_Y)$, the functor $(-)*_D M : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_Y)$ admits a right adjoint $([-1]^*_Y \Delta^{D_Y} M) *_D (-)$.*

Proof. By [Sta26, Tag 08DJ], for every object $N \in D^b_{\mathrm{good}}(\mathcal{A}_X)$, the functor $(-)\otimes^L_{\mathcal{A}_X} N : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$ admits a right adjoint $R\mathcal{H}om_{\mathcal{A}_X}(N, \cdot)$. By [Huy06, p.84], the right adjoint is naturally isomorphic to $T^{-g}\Delta^{\mathcal{A}_X}(N)\otimes^L_{\mathcal{A}_X}(-)$. The result follows from Propositions 7.1.3 and 7.4.3. □

7.5. **Translation and multiplication.** For dual complex tori X and Y, let $p_X : X \times Y \to X$ be the projection. Denote by $p_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(D_{X\times Y/X})$ the pull-back functor of relative D-modules. For every $y \in Y$, there is an isomorphism $T^*_{(0,y)}\mathcal{P} \cong \mathcal{P} \otimes_{O_{X\times Y}} p_X^* P_y$ in $\mathrm{Mod}(X \times Y)_{-1-\mathrm{cxn,int}}$. By [Rot97, (2.10)], there is an equivalence of abelian categories

$$\mathcal{P} \otimes_{O_{X\times Y}} (-) : \mathrm{Mod}(D_{X\times Y/X}) \to \mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn,int}}.$$

Lemma 7.5.1. *There are isomorphisms of functors*

$$\tilde{\mathcal{F}} \circ T^*_y \cong \tilde{\mathcal{F}}(-) \otimes_{O_X} P_y : D(D_Y) \to D(\mathcal{A}_X),$$

$$\tilde{\mathcal{F}}^\natural \circ (- \otimes_{O_X} P_y) \cong T^*_y \circ \tilde{\mathcal{F}}^\natural : D(\mathcal{A}_X) \to D(D_Y).$$

Similar results hold for the Rothstein transform.

Proof. Arguing as in [Muk81, (3.1)], one can deduce the isomorphisms from the projection formula. □

Proposition 7.5.2. *For a line bundle equipped with an integrable connection (L,∇) on Y, let $z \in X^\natural$ be the corresponding point. Then there are isomorphisms of functors*

$$\tilde{\mathcal{F}}(- \otimes_{O_Y} (L,\nabla)) \cong \tilde{\mathcal{F}}(-) *_{\mathcal{A}} \pi_* \mathbb{C}_{(z)} : D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X), \tag{36}$$

$$\tilde{\mathcal{F}}^\natural(-) \otimes_{O_Y} (L,\nabla) \cong \tilde{\mathcal{F}}^\natural(- *_{\mathcal{A}} \pi_* \mathbb{C}_{(z)}) : D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y). \tag{37}$$

Proof. By Example 5.1.2 and Theorem 5.1.4, one has $\tilde{\mathcal{F}}(L,\nabla) = T^{-g}\pi_*\mathbb{C}_{(z)}$. Then (36) follows from the last isomorphism in Theorem 7.4.3. By Theorem 5.1.4, (36) implies (37). □

Appendix A. Another proof of rigidity of convolution

We give another proof of Theorem 7.4.4, which does not use the Laumon-Rothstein transform. This proof is inspired by Weissauer's strategy for perverse sheaves on abelian varieties [Wei11], but it is not a strict translation of Weissauer's proof via the Riemann-Hilbert correspondence, so that we can handle good D-modules that are not holonomic.

Second proof of Theorem 7.4.4. Define an automorphism of complex torus $f : Y^2 \to Y^2$, $\quad (a,b) \mapsto (a+b, -a)$. Then $p_1 f = \mu$, $p_2 f = [-1]_Y p_1$ and $\mu f = p_2$. One has $Lf^* O_{Y^2} = O_{Y^2}$ in $D^b(D_{Y^2})$.

For any objects $F, G \in D^b_{\mathrm{good}}(D_Y)$, there are canonical bijections

$$\begin{aligned}
&\mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(F *_D M, G) := \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(\mu_+(F \boxtimes_O M), G)\\
&\overset{(a)}{=} \mathrm{Hom}_{D(D_{Y^2})}(F \boxtimes_O M, T^g \mu^* G)\\
&\overset{(b)}{=} \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, \Delta^{D_{Y^2}}(F \boxtimes_O M) \otimes^L_{O_{Y^2}} T^g \mu^* G)\\
&\overset{(c)}{=} \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, (\Delta^{D_Y} F) \boxtimes_O (\Delta^{D_Y} M) \otimes^L_{O_{Y^2}} T^g \mu^* G)\\
&:= \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, p_1^*\Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^*\Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G)\\
&= \mathrm{Hom}_{D(D_{Y^2})}\left(f^* O_{Y^2}, f^*\left(p_1^*\Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^*\Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G\right)\right)\\
&= \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, \mu^*\Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_1^*[-1]_Y^*\Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g p_2^* G)\\
&:= \mathrm{Hom}_{D(D_{Y^2})}\left(O_{Y^2}, T^g \mu^*\Delta^{D_Y} F \otimes^L_{O_{Y^2}} ([-1]_Y^*\Delta^{D_Y} M \boxtimes_O G)\right)\\
&\overset{(d)}{=} \mathrm{Hom}_{D(D_{Y^2})}\left(O_{Y^2}, T^g \Delta^{D_Y}(\mu^* F) \otimes^L_{O_{Y^2}} ([-1]_Y^*\Delta^{D_Y} M \boxtimes_O G)\right)\\
&\overset{(e)}{=} \mathrm{Hom}_{D(D_{Y^2})}(\mu^* F, T^g([-1]_Y^*\Delta^{D_Y} M \boxtimes_O G))\\
&\overset{(f)}{=} \mathrm{Hom}_{D(D_Y)}(F, \mu_+([-1]_Y^*\Delta^{D_Y} M \boxtimes_O G))\\
&\overset{(g)}{=} \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(F, ([-1]_Y^*\Delta^{D_Y} M) *_D G).
\end{aligned}$$

Here (a), (c), (d), (f) and (g) use [Bjö93, Thm. 2.11.8], Proposition A.0.1, [Kas03, Thm. 4.12], [Kas03, Thm. 4.40] and Lemma 7.1.1 in order. Both (b) and (e) use [Kas03, (3.13)]. As the bijections are functorial in F and G, the adjunction follows. □

The above proof of Theorem 7.4.4 needs the commutativity of duality with external tensor product for D-modules.

Proposition A.0.1. *Let Z_i $(i = 1, 2)$ be two complex manifolds. Then there is an isomorphism of functors*

$$(\Delta^{D_{Z_1}} -) \boxtimes_O (\Delta^{D_{Z_2}} +) \to \Delta^{D_{Z_1 \times Z_2}} (- \boxtimes_O +) : D^b_{\mathrm{coh}}(D_{Z_1}) \times D^b_{\mathrm{coh}}(D_{Z_2}) \to D^b_{\mathrm{coh}}(D_{Z_1 \times Z_2})^{\mathrm{op}}.$$

Proof. For a complex manifold Z, the sheaf $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$ is naturally a $\mathbb{C}_Z$-algebra, and D_Z is naturally a left $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$-module. By [HT07, p.39], for $N_i \in D(D_{Z_i^{\mathrm{op}}})$, there is a natural isomorphism in $D(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$N_1 \boxtimes_O N_2 = (N_1 \boxtimes_{\mathbb{C}} N_2) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2}. \tag{38}$$

First, we construct the natural transformation. Take $M_i \in D^b_{\mathrm{coh}}(D_{Z_i})$.

Claim A.0.2. Then there is a natural morphism in $D^b((D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes_{\mathbb{C}} R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \\ \to &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}). \end{aligned} \tag{39}$$

Claim A.0.3. There is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \\ \to &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1 \times Z_2}). \end{aligned} \tag{40}$$

Again, there is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1 \times Z_2}) \to R\mathcal{H}om_{D_{Z_1 \times Z_2}}(M_1 \boxtimes_O M_2, D_{Z_1 \times Z_2}), \tag{41}$$

which can be defined by taking a $D_{Z_1 \times Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1 \times Z_2}$-injective resolution of $D_{Z_1 \times Z_2}$.

Composing the morphisms (38), (39), (40) and (41) in order, one gets a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes_O R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \\ \to &R\mathcal{H}om_{D_{Z_1 \times Z_2}}(M_1 \boxtimes_O M_2, D_{Z_1 \times Z_2}). \end{aligned} \tag{42}$$

By [Har66, I, Prop. 7.1 (i)], to show that (42) is an isomorphism, one may assume $M_i \in \mathrm{Coh}(D_{Z_i})$ for both $i = 1, 2$. By shrinking Z_i and using [KS90, Prop. 11.2.6], one may find a bounded resolution of M_i by free D_{Z_i}-modules of finite rank. Thus, one may further assume that $M_i = D_{Z_i}$. By [HT07, Eg. 2.6.3], since $\omega_{Z_1 \times Z_2} = \omega_{Z_1} \boxtimes_O \omega_{Z_2}$ in $\mathrm{Mod}(D^{\mathrm{op}}_{Z_1 \times Z_2})$, in this case (42) is an isomorphism. □

Proof of Claim A.0.2. Take a $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-injective resolution $D_{Z_i} \to I_i^\bullet$. Then $I_1^\bullet \boxtimes_{\mathbb{C}} I_2^\bullet$ is a complex of modules over

$$(D_{Z_1} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1}) \boxtimes_{\mathbb{C}} (D_{Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_2}) = (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}}. \tag{43}$$

By [Sta26, Tag 013K (2)], there exists an injective resolution $I_1^\bullet \boxtimes_{\mathbb{C}} I_2^\bullet \to I^\bullet$ (hence an induced injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to I^\bullet$) over (43). The natural morphism $D_{Z_i} \to D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$ is flat, so every injective $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-module is injective over

D_{Z_i}. Similarly, every term of the complex $I^\bullet$ is injective over $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}$. Then (39) is defined to be the composition of the natural morphisms

$$\begin{aligned}&\mathcal{H}om_{D_{Z_1}}(M_1, I_1^\bullet) \boxtimes_{\mathbb{C}} \mathcal{H}om_{D_{Z_2}}(M_2, I_2^\bullet) \to \mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I_1^\bullet \boxtimes_{\mathbb{C}} I_2^\bullet)\\ &\to \mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I^\bullet).\end{aligned}$$

One can show that it is independent of the choices of injective resolutions. □

Proof of Claim A.0.3. Take an injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to J^\bullet$ over (43). By [Sta26, Tag 013K (2)], over $(D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1 \times Z_2}$ there exists an injective resolution $J^\bullet \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \to K^\bullet$. Then (40) is defined to be the composition of the natural morphisms

$$\begin{aligned}&\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^\bullet) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2}\\ \to&\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^\bullet \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2})\\ \to&\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, K^\bullet).\end{aligned}$$

One can show that it is independent of the choices of injective resolutions. □

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