Specialization of Néron-Severi groups of singular varieties

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December 10, 2025

Abstract

Let k be an algebraically closed field of characteristic 0. Let B be an irreducible variety over k with generic point η. Given a normal proper morphism $f : X \to B$, we show the existence of $b \in B(k)$ whose Picard number $\rho(X_b)$ equals $\rho(X_{\bar{\eta}})$. This generalizes the smooth case proved by André (using complex analytic methods), Maulik and Poonen (using p-adic methods). Our approach combines both aspects. We use the Gauss-Manin connection on Hartshorne's relative de Rham cohomology sheaves for singular morphisms, Lawrence-Venkatesh's observation on the corresponding horizontal equation, as well as Mantova and Zannier's weak approximation theorem. A crucial result that may be of independent interest is a variational Lefschetz $(1,1)$-theorem for infinitesimal cohomology in the singular setting.

1 Introduction

Given an algebraic variety X over an algebraically closed field k, two line bundles L, $L' \in \operatorname{Pic}(X)$ are *algebraically equivalent* if there is a connected scheme T of finite type over k, a line bundle $M \in \operatorname{Pic}(X \times_k T)$, and points t_0, $t_1 \in T(k)$ such that $M|_{X\times t_0} \cong L$ and $M|_{X\times t_1} \cong L'$. This defines an equivalence relation on the Picard group $\operatorname{Pic}(X)$. Let $\operatorname{NS} X$ be the group of algebraic equivalence classes of line bundles on X, called the *Néron-Severi group* of X. If X is proper over k, then $\operatorname{NS} X$ is a finitely generated abelian group, and its rank $\rho(X)$ is called the *Picard number* of X.

Our main result concerns the Néron-Severi groups of a family of normal proper varieties. A flat morphism $f : X \to B$ of finite type of schemes is *normal*, if every fiber X_b is geometrically normal over $k(b)$.

Theorem 1.1 (Theorem 6.5)**.** *Let k be an algebraically closed field of characteristic 0. Let B be an irreducible variety over k with generic point η. Let $f : X \to B$ be a proper morphism. Assume either $f : X \to B$ is normal or all geometric fibers are integral. Then the set of $b \in B(k)$ with $\rho(X_{\bar{\eta}}) = \rho(X_b)$ is Zariski dense in B.*

Normal varieties constitute a large class of algebraic varieties allowing singularities, and this class is stable under more operations than the class of smooth varieties. For example, a symmetric power of a smooth projective surface may not be smooth, while a symmetric power of a normal variety is normal. In Theorem 1.1, if the condition that $f : X \to B$ is normal is strengthened to smoothness, then it is first proved by André [And96, Théorème 5.2 3)] (for $f : X \to B$ projective), Maulik and Poonen [MP12, Thm. 1.1]. Theorem 1.1 is less precise than Theorem 6.5, where we show that in the normal case, there is a specialization isomorphism $\mathrm{NS}(X_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{NS}(X_b)$; in the integral case, we give a similar result ignoring the torsion subgroup of the Néron-Severi group.

Remark 1.2. A closed point $b \in |B|$ is called *NS-generic* if there is a specialization isomorphism $\mathrm{NS}\, X_{\bar{\eta}} \xrightarrow{\sim} \mathrm{NS}\, X_{\bar{b}}$. If $f : X \to B$ is moreover smooth, then the jumping locus

$$\{b \in B(k) : \rho(X_b) > \rho(X_{\bar{\eta}})\}$$

is a countable union of strict closed subvarieties of B (see, e.g., [MP12, Corollary 3.9]), so a very general closed point $b \in |B|$ is NS-generic. In particular, when k is moreover uncountable, the existence of NS-generic k-points is relatively elementary.

When $f : X \to B$ is normal, even if $k = \mathbb{C}$, we do not know how to prove Theorem 1.1 using such cardinality consideration. In fact, in the *smooth* case, given a specialization of points $t \rightsquigarrow s$ on B, there is a specialization morphism $\mathrm{NS}\, X_{\bar{t}} \to \mathrm{NS}\, X_{\bar{s}}$ ([MP12, Proposition 3.6]). This construction for an arbitrary specialization $t \rightsquigarrow s$ is important in the proof of [MP12, Corollary 3.9]. When smoothness of $X \to B$ is relaxed to normality, however, we do not know how to construct the specialization of Néron-Severi groups for $t \rightsquigarrow s$. Instead, we can only construct it for a specialization of points $\eta \rightsquigarrow s$ starting from the generic point (Lemma 2.4 (a)). This prevents us from generalizing [MP12, Corollary 3.9] to the normal case directly.

André's proof is complex analytic, Maulik and Poonen's proof is p-adic, while our method has both complex analytic aspects and p-adic aspects. Both proofs of André and of Maulik-Poonen use certain versions of "variational Lefschetz $(1,1)$ theorem". The proof of Theorem 1.1 also leans on a singular version of it, which is an analog of [BO83, Theorem 3.8] in characteristic 0. As a replacement of the crystalline cohomology, we adopt Grothendieck's [Gro68] infinitesimal cohomology.

Theorem 1.3 (Theorem 4.5)**.** *Let V be a Noetherian complete local ring of characteristic 0. Let X be a scheme proper over V. Let L be a line bundle on the special fiber X_0. Assume that $H^1(X, O_X) \to H^1(X_0, O_{X_0})$ is surjective. Then L lifts to $\mathrm{Pic}\, X$ if and only if the canonical morphism $\tau_{\mathrm{inf}} : H^2_{\mathrm{inf}}(X_0/\operatorname{Spf} V) \to H^2_{\mathrm{dR}}(X/V)$ maps the first Chern class in infinitesimal cohomology $c_{1,\mathrm{inf}}(L)$ into $F^1 H^2_{\mathrm{dR}}(X/V)$.*

As a byproduct of our proof of Theorem 1.1, we show that a family of projective varieties is generically projective.

Theorem 1.4. *Let B be an algebraic variety over a field k of characteristic 0. Let $f : X \to B$ be a proper morphism such that all closed fibers are projective. Then for every $b \in B$, X_b is projective over $k(b)$. In particular, there is a Zariski dense open subset V of B, such that $X_V \to V$ is projective.*

Remark 1.5. In Theorem 1.4, when $f : X \to B$ is moreover smooth, Maulik and Poonen [MP12, Theorem 7.1] deduce the result from the smooth case of Theorem 1.1. We prove Theorem 1.4 by the proof of Theorem 1.1 instead of the statement, so that we do not need $f : X \to B$ to be normal. As [MP12, Example 7.5] shows, the analog of Theorem 1.4 over $k = \bar{\mathbb{F}}_p$ fails.

Proof strategy

The proof of Theorem 1.1 involves three important ingredients: Lawrence and Venkatesh's result [LV20, Sec. 3.3] on convergence of power series associated with Gauss-Manin connection, a p-adic result [MP12, Proposition 5.1] on the linear independence of values of linearly independent power series, Mantova and Zannier's [MZ14] weak approximation theorem on algebraic varieties.

First, locally in the p-adic analytic topology, we construct convergent power series such that the jumping locus for $f : X \to B$ is contained in a countable union of their zero loci (Lemma 6.4). The construction makes use of the Gauss-Manin connection for singular morphisms. In fact, for a morphism $X \to B$ of algebraic varieties over a field of characteristic 0, the infinitesimal cohomology group $H^*_{\mathrm{inf}}(X/B)$ is canonically isomorphic to Hartshorne's de Rham cohomology group $H^*_{\mathrm{DR}}(X/B)$ ([Liu25a, Theorem 1.2]), and Hartshorne [Har75, p.75] introduces the Gauss-Manin connection on his relative de Rham cohomology sheaves $\mathcal{H}^i_{\mathrm{DR}}(X/B)$. Given $b \in B(k)$ and a line bundle $L \in \operatorname{Pic} X_b$, we extend $c_{1,\mathrm{inf}}(L) \in H^2_{\mathrm{DR}}(X_b/k)$ to a formal local section of $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ which is horizontal relative to the Gauss-Manin connection. In local coordinates, this formal section is expressed as formal power series. Given a place v of k, Lawrence and Venkatesh observe that in the *smooth* case, the corresponding formal power series converge locally in the analytic topology associated with v. We extend their observation to the *singular* case, to show the convergence of the constructed power series at archimedean places as well as a suitably chosen finite place p. Given a point $b \in B(k)$ outside the so defined union of zero loci, to prove that b is outside the jumping locus, we apply Theorem 1.3 to the projection $X \times_B \operatorname{Spec} \hat{O}_{B,b} \to \operatorname{Spec} \hat{O}_{B,b}$.

To finish the proof of Theorem 1.1, given a proper morphism $f : X \to B$, we show the existence of a closed point $b \in |B|$ outside a countable union of zero loci of power series (Lemma 6.1). We choose a prime p such that the de Rham cohomology sheaf $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ and its Gauss-Manin connection are defined over $\mathbb{Z}_p$. Under the complex Riemann-Hilbert correspondence, up to shrinking B, $\mathcal{H}^i_{\mathrm{DR}}(X/B)$ with the Gauss-Manin connection corresponds to the local system $R^i f^{\mathrm{an}}_* \mathbb{C}$ on $B(\mathbb{C})$. We choose a small complex analytic open subset $\Omega \subset B(\mathbb{C})$ trvializing $R^2 f^{\mathrm{an}}_* \mathbb{C}$. Using the underlying $\mathbb{Q}$-local system $R^2 f^{\mathrm{an}}_* \mathbb{Q}$, we get a finite dimensional vector space of convergent power series. Applying [MP12, Proposition 5.1] to this space, we obtain a p-adic analytic open subset

$U \subset B(\mathbb{C}_p)$ outside the zero loci. Finally, the weak approximation theorem shows the existence of $b_0 \in B(k)$ lying in Ω and U simultaneously.

Comparison with proofs of André, Maulik and Poonen

André (see, e.g., [MP12, Theorem 8.3]) shows that the set of closed points $b \in |B|$ with $\mathrm{NS}(X_{\bar{b}})$ not isomorphic to $\mathrm{NS}(X_{\bar{\eta}})$ is sparse, which implies the existence of NS-generic points. Inspired by André's strategy, Ambrosi [Amb23, Corollary 1.7.1.3] gives an analog for a *smooth* proper morphism $f : X \to B$ in positive characteristic. André's proof is Hodge theoretical as it relies on the fixed part theorem [Del71, Théorème 4.1.1] and the Lefschetz $(1, 1)$ theorem.

These two arguments do not extend obviously to the singular case. For one thing, when X is a normal projective variety over $\mathbb{C}$, the natural map $H^2(X^{\mathrm{an}}, \mathbb{C}) \to H^2_{\mathrm{dR}}(X/\mathbb{C})$ may not send the Hodge filtration $F^1H^2(X^{\mathrm{an}}, \mathbb{C})$ of the mixed Hodge structure inside the naïve Hodge filtration $F^1H^2_{\mathrm{dR}}(X/\mathbb{C})$. For another, the generalization of the Lefschetz $(1, 1)$ theorem for normal projective varieties X [BS00, Theorem 1.1] necessitates the subgroup of $H^2(X^{\mathrm{an}}, \mathbb{Z})$ of *Zariski locally trivial* cohomology classes, a condition we do not know how to verify in this case.

Maulik and Poonen give a completely new p-adic proof, which inspires ours. In the case of good reduction, they [MP12, Theorem 1.7] strengthen the existence of NS-generic points to p-adic nowhere density of jumping locus. To this end, an intermediate result is [MP12, Lemma 4.2], where they construct the power series cutting out the jumping locus using convergent isocrystals and Ogus's relative comparison isomorphism between the crystalline cohomology and the de Rham cohomology. There are three aspects of its proof we do not know how to extend to the singular case. For a scheme $\mathcal{X}$ normal proper (but not necessarily smooth) over $\mathbb{Z}_p$ with generic fiber X_η and special fiber X_s, there may not be an isomorphism between $H^*_{\mathrm{dR}}(X_\eta/\mathbb{Q}_p)$ and the crystalline cohomology $H^*_{\mathrm{cris}}(X_s/\mathbb{Z}_p)[1/p]$. As there may be no specialization morphism $\mathrm{NS}(X_{\bar{\eta}}) \to \mathrm{NS}(X_{\bar{s}})$, we cannot take the reduction of a line bundle as in the proof of [MP12, Lemma 4.2]. Given a singular proper morphism $f : X \to B$ of algebraic varieties, a priori we do not know whether a very general point is NS-generic neither. Additionally, the way we apply Theorem 1.3 is also different from how Maulik and Poonen use its p-adic inspiration [BO83, Theorem 3.8].

Remark 1.6. Slightly more general than the mere existence in Theorem 1.1, we use the weak approximation theorem to prove the potential Zariski density of the set of NS-generic points (Remark 6.7). Potential density of NS-generic points is weaker than both sparsity and p-adic nowhere density of the jumping locus. Due to our use of the complex Riemann-Hilbert correspondence in the proof of Lemma 6.1, we do not prove p-adic nowhere density in the singular case. In fact, we rely on the local constancy of the complex $\mathbb{Q}$-local system $R^2f^{\mathrm{an}}_*\mathbb{Q}$ to construct a *finite dimensional* vector spaces of power series to which [MP12, Prop. 5.1] can be applied. By contrast, on a smooth rigid analytic variety V over $\mathbb{Q}_p$, a p-adic étale lisse sheaf is not necessarily locally constant in the analytic topology.

Notation

For a local ring R with maximal ideal m, let $\hat{R}$ be the m-adic completion of R. By an *algebraic variety* over a field k, we mean a scheme separated of finite type over k. The spectrum of a discrete valuation ring is called a *trait.* Given a finite locally free module E over a scheme X and a point $x \in X$, let $E(x) := E_x \otimes_{O_{X,x}} k(x)$ be the fiber of E over x.

For a scheme X of finite type over a field k, let $|X|$ be the set of closed points of X. For an integer $d > 0$, let $X^{\leq d} := \{x \in |X| : [k(x) : k] \leq d\}$. Let $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ be the group of line bundles on X modulo algebraic equivalence, known as the *Néron-Severi group* of X. Let $\mathrm{Br}(X)$ be the torsion subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_m)$, called the *cohomological Brauer group* of X. By [Sta25, Tag 0A2J], if X is a quasi-projective variety over k, then $\mathrm{Br}(X)$ is the group of equivalence classes of Azumaya O_X-algebras. For an integer $i \geq 0$ and a morphism of schemes $X \to S$, let $H^i_{\mathrm{dR}}(X/S) := H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$ be the i-th *de Rham cohomology* group. For an integer $q \geq 0$, let $F^q H^i_{\mathrm{dR}}(X/S) := \mathrm{Im}(H^i(X, \sigma_{\geq q}\Omega^\bullet_{X/S}) \to H^i_{\mathrm{dR}}(X/S))$ be the q-th Hodge filtration.

Let X be a scheme proper over an algebraically closed field k. Two line bundles $L, L' \in \mathrm{Pic}(X)$ are *numerically equivalent* if for every closed integral curve $C \subset X$ over k, one has $\deg(L|_C) = \deg(L'|_C)$. Let $\mathrm{Num}\, X$ be the group of line bundles on X modulo numerical equivalence. The first statement in Fact 1.7, known as the theorem of the base, is a special case of [SGA 6, XIII, Thm. 5.1]. The second follows from [SGA 6, XIII, Thm. 4.6].

Fact 1.7. *Let X be a scheme proper over an algebraically closed field k. Then $\mathrm{NS}(X)$ is a finitely generated abelian group. The quotient of $\mathrm{NS}(X)$ by the torsion subgroup is $\mathrm{Num}\, X$.*

In Fact 1.7, the rank $\rho(X)$ of $\mathrm{NS}(X)$ is called the *Picard number* of X. The proof of [MP12, Prop. 3.1] proves Fact 1.8.

Fact 1.8. *Let K/k be an extension of algebraically closed fields. Let X be a proper scheme over k. Then the natural morphisms $\mathrm{NS}\, X \to \mathrm{NS}(X_K)$ and $\mathrm{Num}\, X \to \mathrm{Num}(X_K)$ are isomorphisms.*

2 Specialization of Néron-Severi groups

Given a proper morphism $f : X \to B$, we show that up to shrinking B, one can define specialization of the Num group of the geometric fibers. If $f : X \to B$ is furthermore normal, then one can also define specialization of the Néron-Severi groups. Different from the smooth case, we can only construct such morphisms starting from the geometric generic fiber. Using ℓ-adic étale cohomology, we show that the specialization morphisms are injective.

We say that algebraic (resp. numerical) equivalence *spreads universally* for a morphism $f : X \to B$ of schemes, if the following holds. Let B' be an irreducible locally Noetherian scheme with generic point η. Let $\bar{\eta}$ be a geometric point over η. For every line bundle L on $X \times_B B'$ and every geometric point $\bar{b}$ on B', if $L|_{X_{\bar\eta}} \in \operatorname{Pic}(X_{\bar\eta})$ is algebraically (resp. numerically) equivalent to 0, then $L|_{X_{\bar b}} \in \operatorname{Pic}(X_{\bar b})$ is also algebraically (resp. numerically) equivalent to 0.

Lemma 2.1. *Let B be an irreducible, locally Noetherian scheme. Let $f : X \to B$ be a proper morphism of schemes.*

(a) *If $f : X \to B$ is normal, then there is a nonempty open subset U of B such that algebraic equivalence spreads universally for $X \times_B U \to U$.*

(b) *Assume that $f : X \to B$ is flat, locally projective with integral geometric fibers. Then numerical equivalence spreads universally for $f : X \to B$.*

(c) *If every geometric fiber of $f : X \to B$ is integral, then there is a nonempty open subset V of B such that numerical equivalence spreads universally for $X \times_B V \to V$.*

Proof. Let Pic_X be the absolute Picard functor in the sense of [FGI05, Def. 9.2.1]. Let $\operatorname{Pic}_X \to \operatorname{Pic}_{(X/B)(\mathrm{fppf})}$ be the sheafification morphism in the fppf topology. By Fact 1.8, one may replace B by the reduced induced scheme B_{red}. So one may assume that B is integral.

(a) By Grothendieck's generic representability theorem [FGI05, Thm. 9.4.18.2], as $f : X \to B$ is proper, shrinking B, one may assume that the relative Picard scheme $\operatorname{Pic}_{X/B}$ exists and represents the relative Picard functor $\operatorname{Pic}_{(X/B)(\mathrm{fppf})}$. Let $\operatorname{Pic}^0_{X/B} := \cup_{s\in B} \operatorname{Pic}^0_{X_s/k(s)}$. From [Gro62, Cor. 2.3], as $f : X \to B$ is a normal morphism, $\operatorname{Pic}^0_{X/B}$ is a closed subset of $\operatorname{Pic}_{X/B}$. By [FGI05, Exercise 9.4.4], the formation of $\operatorname{Pic}_{X/B}$ commutes with base change in B. By [FGI05, Lem. 9.5.1], so does the formation of $\operatorname{Pic}^0_{X/B}$.

(b) By Grothendieck's theorem [FGI05, Theorem 9.4.8 (1)], under the assumption, the Picard scheme $\operatorname{Pic}_{X/B}$ exists. Set $\operatorname{Pic}^\tau_{X/B} := \cup_{n>0} n^{-1}(\operatorname{Pic}^0_{X/B})$. Then by [FGI05, Theorem 9.6.16], it is an open and closed group subscheme of $\operatorname{Pic}_{X/S}$, and its formation commutes with base change in B.

In case (a), shrink B as above. Let $B' \to B$ be a base change morphism. To prove that algebraic equivalence (in case (a)) or numerical equivalence (in cases (b) and (c)) spreads for the projection $X \times_B B' \to B'$, one may assume that $B' \to B$ is Id_B. A line bundle L on X corresponds to an element $[L] \in \operatorname{Pic}_X(B)$. Let $s \in \operatorname{Pic}_{X/B}(B)$ be its image. Denote by $\eta \in B$ the generic point. Let $s_\eta \in \operatorname{Pic}_{X_\eta/k(\eta)}(k(\eta))$ and $s_{\bar\eta} \in \operatorname{Pic}_{X_\eta/k(\eta)}(k(\bar\eta))$ be the restrictions of the section $s : B \to \operatorname{Pic}_{X/B}$. From [FGI05, Exercise 9.2.3], as $k(\bar\eta)$ is algebraically closed, the morphism

$$\operatorname{Pic}(X_{\bar\eta}) \xrightarrow{\sim} \operatorname{Pic}_{X_\eta/k(\eta)}(k(\bar\eta))$$

is an isomorphism, which sends $[L|_{X_{\bar\eta}}]$ to $s_{\bar\eta}$.

(a) Assume that $L|_{X_{\bar\eta}} = 0$ in $\operatorname{NS} X_{\bar\eta}$. Then by [FGI05, Prop. 9.5.10], one has $s_{\bar\eta} \in \operatorname{Pic}^0_{X_\eta/k(\eta)}(k(\bar\eta))$ and hence $s_\eta \in \operatorname{Pic}^0_{X_\eta/k(\eta)}(k(\eta))$. Because $s : B \to \operatorname{Pic}_{X/B}$ is continuous, $s^{-1}(\operatorname{Pic}^0_{X/B})$ is a closed subset of B containing

η. This implies $s(B) \subset \mathrm{Pic}^0_{X/B}$ and hence $s_{\bar b} \in \mathrm{Pic}^0_{X_b/k(b)}(k(\bar b))$ for all geometric points $\bar b$ on B. By [FGI05, Prop. 9.5.10], $L|_{X_{\bar b}}$ is algebraically equivalent to 0.

(b) The proof is similar to the case (a), except that we replace $\mathrm{Pic}^0_{X/B}$ by $\mathrm{Pic}^\tau_{X/B}$, and use [SGA 6, XIII, Thm. 4.6] in stead of [FGI05, Prop. 9.5.10].

(c) By Chow's lemma, shrinking B, one may assume that there is a proper surjective morphism $\pi : X' \to X$, such that $f\pi : X' \to B$ is projective. As $f : X \to B$ has integral geometric fibers, shrinking B one may assume that $f\pi : X' \to B$ has integral geometric fibers. From generic flatness, as B is integral, shrinking B one may assume that $f\pi : X' \to B$ is flat. Assume that $L|_{X_{\bar\eta}} = 0$ in $\mathrm{Num}\, X_{\bar\eta}$. By [SGA 6, XIII, Proposition 4.5 (i)], one has $(\pi^*L)|_{X'_{\bar\eta}} = 0$ in $\mathrm{Num}\, X'_{\bar\eta}$. From Part (b), for every geometric point $\bar b$ on B, $(\pi^*L)|_{X'_{\bar b}} = 0$ in $\mathrm{Num}\, X'_{\bar b}$. From [SGA 6, XIII, Proposition 4.5 (ii)], as $X'_{\bar b} \to X_{\bar b}$ is surjective, one has $L|_{X_{\bar b}} = 0$ in $\mathrm{Num}\, X_{\bar b}$.

□

Lemma 2.2. *Let B be a Noetherian Japanese scheme with generic point η. Choose an algebraic closure $k(\bar\eta)$ of $k(\eta)$ and let $\bar\eta$ be the corresponding geometric point over η. Let $f : X \to B$ be a proper morphism.*

(a) *Assume that $f : X \to B$ is normal. Then there is a dense open subset U of B such that for every geometric point $\bar b$ on U, one may choose a specialization morphism $\mathrm{sp}_{\bar\eta,\bar b} : \mathrm{NS}\, X_{\bar\eta} \to \mathrm{NS}\, X_{\bar b}$ fitting into a commutative diagram*

$$\begin{tikzcd} & \mathrm{Pic}(X) \arrow[dl] \arrow[dr] & \\ \mathrm{NS}\, X_{\bar\eta} \arrow[rr, dashed, "\mathrm{sp}_{\bar\eta,\bar b}"] & & \mathrm{NS}\, X_{\bar b}. \end{tikzcd} \tag{1}$$

(b) *Assume that $f : X \to B$ has integral geometric fibers. Then similar result holds for* Num *instead of* NS.

(c) *Assume that $f : X \to B$ is normal with integral geometric fibers. Then there is a dense open subset $U \subset B$ with the following properties. For every $b \in U$, one may choose specializations making the square*

$$\begin{tikzcd} \mathrm{NS}\, X_{\bar\eta} \arrow[r, "\mathrm{sp}"] \arrow[d, two heads] & \mathrm{NS}\, X_{\bar b} \arrow[d, two heads] \\ \mathrm{Num}\, X_{\bar\eta} \arrow[r, "\mathrm{sp}"] & \mathrm{Num}\, X_{\bar b} \end{tikzcd}$$

commutative. Let $p \geq 1$ be the characteristic exponent of $k(b)$. Then $\mathrm{sp}[1/p] : \mathrm{NS}\, X_{\bar\eta}[1/p] \to \mathrm{NS}\, X_{\bar b}[1/p]$ *is an isomorphism if and only* $\mathrm{sp}[1/p] : \mathrm{Num}\, X_{\bar\eta}[1/p] \to \mathrm{Num}\, X_{\bar b}[1/p]$ *is an isomorphism.*

Proof. We prove Part (a), and in the resp. case in parentheses we indicate modifications to prove Part (b). First, we prove that there is an integral scheme V' with generic point η', a dominant morphism $V' \to B$ of finite type, such that the morphism

$$\operatorname{Pic}(X \times_B V') \to \operatorname{NS}(X_{\bar\eta'}), \quad E \mapsto [E|_{X_{\bar\eta'}}] \tag{2}$$

is surjective, and that algebraic equivalence (resp. numerical equivalence) spreads universally for $X \times_B V' \to V'$.

By Fact 1.7, as $X_{\bar\eta}$ is proper over $k(\bar\eta)$, one may choose a finite set $\{L_1, \dots, L_n\}$ of line bundles on $X_{\bar\eta}$ generating $\operatorname{NS}(X_{\bar\eta})$. This set also generates $\operatorname{Num} X_{\bar\eta}$. By [Sta25, Tag 0B8W (2)], for every line bundle L on $X_{\bar\eta}$, there is a finite subextension $k(\eta') \subset k(\bar\eta)$ and a line bundle L' on $X_\eta \otimes_{k(\eta)} k(\eta')$ whose scalar extension is L. One may choose $k(\eta')$ such that the finitely many L_i are defined over $k(\eta')$. Let $\nu : B' \to B$ be the normalization of B in $k(\eta')$. As B is Noetherian integral and Japanese, $\nu : B' \to B$ is finite. By [GW20, Prop. 12.43 (1) and (2)], B' is an integral normal scheme with function field $k(\eta')$, and $\nu : B' \to B$ is surjective.

By [Sta25, Tag 0B8W (2)], there is a nonempty open subset V' of B' and line bundles $\{L_1', \dots, L_n'\}$ on $X \times_B V'$ whose restrictions to $X_{\eta'}$ are $\{L_1, \dots, L_n\}$ in order. From Lemma 2.1 (a) (resp. (c)), shrinking V' one may assume that algebraic equivalence (resp. numerical equivalence) spreads universally for $X \times_B V' \to V'$. By Fact 1.8, as X_η is proper over $k(\eta)$, $\operatorname{NS}(X_{\bar\eta}) \to \operatorname{NS}(X_{\bar\eta'})$ is an isomorphism. Then the image of (2) contains a generating set, so (2) is surjective. The composition $V' \hookrightarrow B' \xrightarrow{\nu} B$ is separated dominant quasi-finite and of finite type.

Then $\nu(B' \setminus V')$ is closed in B and

$$\dim \nu(B' \setminus V') \le \dim B' \setminus V' < \dim B' = \dim B.$$

Let $U := B \setminus \nu(B' \setminus V')$, which is a dense open subset of B. For every $b \in U$, we define a morphism $\operatorname{sp}_{\bar\eta,\bar b} : \operatorname{NS}(X_{\bar\eta}) \to \operatorname{NS}(X_{\bar b})$ (resp. $\operatorname{sp} : \operatorname{Num} X_{\bar\eta} \to \operatorname{Num} X_{\bar b}$) as follows. Choose $b' \in B'$ with $\nu(b') = b$. As $b \notin \nu(B' \setminus V')$, one has $b' \in V'$. For every $L \in \operatorname{Pic}(X_{\bar\eta})$, choose a preimage L' of $[L]$ under the surjective composition

$$\operatorname{Pic}(X \times_B V') \overset{(2)}{\twoheadrightarrow} \operatorname{NS}(X_{\bar\eta'}) \xrightarrow{\sim} \operatorname{NS}(X_{\bar\eta}).$$

Let $\operatorname{sp}_{\bar\eta,\bar b}(L)$ be the preimage of $[L'|_{X_{\bar b'}}]$ under the isomorphism $\operatorname{NS}(X_{\bar b}) \xrightarrow{\sim} \operatorname{NS}(X_{\bar b'})$ (resp. $\operatorname{Num}(X_{\bar b}) \xrightarrow{\sim} \operatorname{Num}(X_{\bar b'})$) in Fact 1.8. By Lemma 2.1, the construction of specialization is independent of the choice of L', and $\operatorname{sp}_{\bar\eta,\bar b}$ is a group morphism fitting into the diagram (1) (resp. the similar diagram with Num instead of NS).

Now we prove Part (c). The first statement follows from the proof of Parts (a) and (b). The second follows from Lemma 2.4 (c). $\square$

Lemma 2.3 shows that on proper varieties, algebraic equivalence coincides with homological equivalence for Cartier divisors.

Lemma 2.3. *Let X be an algebraic variety over an algebraically closed field k. Fix a prime ℓ invertible in k.*

(a) *Let $c_1 : \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H^2(X,\mathbb{Z}_\ell(1))$ be the map induced by the ℓ-adic first Chern class map. Then it factors through a morphism $c_1 : \mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H^2(X,\mathbb{Z}_\ell(1))$ and induces a morphism $c_1 : \mathrm{Num}\, X\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^2(X,\mathbb{Q}_\ell(1))$.*

(b) *Assume that $\mathrm{NS}(X)$ is finitely generated. Let $T_\ell \mathrm{Br}(X) := \lim_n H^2_{ét}(X,\mathbb{G}_m)[\ell^n]$. Then there is a short exact sequence*

$$0 \to \mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \xrightarrow{c_1} H^2(X,\mathbb{Z}_\ell(1)) \to T_\ell \mathrm{Br}(X) \to 0.$$

In particular, $c_1 : \mathrm{Num}\, X\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^2(X,\mathbb{Q}_\ell(1))$ is injective and hence $\rho(X) \le \dim_{\mathbb{Q}_\ell} H^2(X,\mathbb{Q}_\ell)$.

Proof. (a) Let L be line bundle on X which is algebraically equivalent to 0, and we prove $c_1(L) = 0$. By [MRM74, Lemma, p.56], there is a connected smooth proper curve C over k, a line bundle M on $X\times_k C$, points $a, b \in C(k)$ such that $[L] = [i_a^* M] - [i_b^* M]$ in $\mathrm{Pic}(X)$. Here for $x \in C(k)$, $i_x : X \to X\times_k C$ denotes the product of $x : \mathrm{Spec}\, k \to C$ with Id_X. By the Künneth formula (see, e.g., [Mil13, Thm. 22.4]), since $H^i(C,\mathbb{Z}_\ell)$ is free over $\mathbb{Z}_\ell$ for all integers i, there is a canonical isomorphism

$$\sum_{r+s=2} H^r(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell} H^s(C,\mathbb{Z}_\ell) \xrightarrow{\sim} H^2(X\times_k C,\mathbb{Z}_\ell).$$

Then $i_x^* : H^2(X\times_k C,\mathbb{Z}_\ell) \to H^2(X,\mathbb{Z}_\ell)$ is the composition of the projection

$$H^2(X\times_k C,\mathbb{Z}_\ell) \to H^2(X,\mathbb{Z}_\ell)\otimes H^0(C,\mathbb{Z}_\ell)$$

followed by

$$\mathrm{Id}_{H^2(X,\mathbb{Z}_\ell)}\otimes x^* : H^2(X,\mathbb{Z}_\ell)\otimes H^0(C,\mathbb{Z}_\ell) \to H^2(X,\mathbb{Z}_\ell).$$

As C is connected, one has $a^* = b^*$ as maps $H^0(C,\mathbb{Z}_\ell) \to \mathbb{Z}_\ell$. It implies $i_a^* = i_b^*$ as maps $H^2(X\times_k C,\mathbb{Z}_\ell) \to H^2(X,\mathbb{Z}_\ell)$. One has

$$c_1(L) = c_1(i_a^* M) - c_1(i_b^* M) = i_a^* c_1(M) - i_b^* c_1(M) = 0,$$

where the second equality uses [SGA5, VII, Prop. 3.4 (ii)]. Therefore, $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}_\ell)$ factors through $\mathrm{NS}(X)$.

(b) For every integer $m > 0$ invertible in k, let

$$0 \to \mu_m \to \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \to 0$$

be the Kummer exact sequence [Tam94, Thm. 4.4.1] of étale sheaves on X. It induces an exact sequence of étale cohomology groups

$$H^1(X,\mathbb{G}_m) \xrightarrow{m} H^1(X,\mathbb{G}_m) \xrightarrow{\partial} H^2(X,\mu_m) \to H^2(X,\mathbb{G}_m) \xrightarrow{m} H^2(X,\mathbb{G}_m),$$

and hence a short exact sequence

$$0 \to \operatorname{Pic}(X)/m \xrightarrow{\partial} H^2(X, \mu_m) \to \operatorname{Br}(X)[m] \to 0. \tag{3}$$

By definition of the ℓ-adic Chern class [SGA5, VII, Proposition 3.4 (i) and Section 3.9], for every $n \geq 0$, the diagram

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r, "c_1"] \arrow[d, two heads] & H^2(X, \mathbb{Z}_\ell(1)) \arrow[d] \\ \operatorname{Pic} X/\ell^n \arrow[r, hook, "\partial"] & H^2(X, \mu_{\ell^n}) \end{tikzcd}$$

is commutative. So $\operatorname{Pic} X/\ell^n \to \operatorname{NS} X/\ell^n$ is an isomorphism. By the Chinese remainder theorem,

$$\operatorname{Pic} X/m \xrightarrow{\sim} \operatorname{NS} X/m \tag{4}$$

is an isomorphism. As $\{\operatorname{NS} X/\ell^n\}_{n\geq 0}$ is a surjective system, taking limit of (3) one gets an exact sequence

$$0 \to \lim_n \operatorname{NS} X/\ell^n \xrightarrow{c_1} H^2(X, \mathbb{Z}_\ell(1)) \to T_\ell \operatorname{Br}(X) \to 0.$$

Now assume that $\operatorname{NS}(X)$ is finitely generated. Then $\operatorname{NS} X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \lim_n \operatorname{NS} X/\ell^n$ is an isomorphism.

□

Let B, η and $\bar\eta$ be as in Lemma 2.2. Let $f : X \to B$ be a proper morphism. By generic local acyclicity [Org03, Théorème 3.3 a)], shrinking B to a dense open subset, one may assume that for every finite set E, $f : X \to B$ is universally locally acyclic relative to the constant sheaf $\underline{E}_X$. Then by [SGA 4 1/2, Arcata, V, Théorème 3.1], as $f : X \to B$ is proper, for every prime ℓ invertible on B and any $n, q \geq 0$, $R^q f_* \mathbb{Z}/\ell^n$ is locally constant. Therefore, $R^q f_* \mathbb{Z}_\ell$ is a lisse sheaf on B.

Lemma 2.4. *Consider the above setting.*

(a) *Assume further that $f : X \to B$ is normal (resp. has integral geometric fibers). Then there is a dense open subset U of B with the following property. For every Henselian trait $S = \operatorname{Spec} R$ with generic point ξ and closed point s, every dominant morphism $\phi : S \to U$ and every prime ℓ invertible on B, the specialization $\operatorname{sp}_\phi : H^2(X_{\bar s}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^2(X_{\bar\xi}, \mathbb{Z}_\ell)$ is an isomorphism fitting into a commutative square*

$$\begin{tikzcd} \operatorname{NS}(X_{\bar\eta}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r, hook, "c_1"] \arrow[d, "\operatorname{sp}_{\bar\eta,\bar b}"] & H^2(X_{\bar\eta}, \mathbb{Z}_\ell(1)) \\ \operatorname{NS}(X_{\bar b}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r, hook, "c_1"] & H^2(X_{\bar b}, \mathbb{Z}_\ell(1)), \arrow[u, "\operatorname{sp}_\phi"'] \end{tikzcd} \quad (\textit{resp.} \quad \begin{tikzcd} \operatorname{Num} X_{\bar\eta} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \arrow[r, hook, "c_1"] \arrow[d, "\operatorname{sp}"] & H^2(X_{\bar\eta}, \mathbb{Q}_\ell(1)) \\ \operatorname{Num} X_{\bar b} \otimes \mathbb{Q}_\ell \arrow[r, hook, "c_1"] & H^2(X_{\bar b}, \mathbb{Q}_\ell(1)), \arrow[u, "\operatorname{sp}_\phi"'] \end{tikzcd})$$

where $b := \phi(s) \in B$.

(b) *Assume that $f : X \to B$ is normal. Let U be as in Part* (a). *For every $b \in U$ with* $\mathrm{char} k(b) = 0$, *there is a surjective specialization of Brauer groups* $\mathrm{Br}(X_{\bar\eta}) \twoheadrightarrow \mathrm{Br}(X_{\bar b})$ *such that for every $m > 0$, the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{NS}\, X_{\bar\eta}/m \arrow[r, "\partial"] \arrow[d, "\mathrm{sp}_{\bar\eta,\bar b}"] & H^2(X_{\bar\eta}, \mu_m) \arrow[r] & \mathrm{Br}(X_{\bar\eta})[m] \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathrm{NS}\, X_{\bar b}/m \arrow[r, "\partial"] & H^2(X_{\bar b}, \mu_m) \arrow[r] \arrow[u, "\cong"', dashed] & \mathrm{Br}(X_{\bar b})[m] \arrow[r] & 0 \end{tikzcd}$$

is commutative and has exact rows. In particular, if $\mathrm{sp}_{\bar\eta,\bar b} : \mathrm{NS}\, X_{\bar\eta} \to \mathrm{NS}\, X_{\bar b}$ *is an isomorphism, then so is* $\mathrm{Br}(X_{\bar\eta}) \to \mathrm{Br}(X_{\bar b})$.

(c) *Let $p \geq 1$ be the characteristic exponent of $k(b)$. Let U be as in Part* (a).

1) *Assume that $f : X \to B$ is normal. Then for every $b \in U$,* $\mathrm{sp}[1/p] : \mathrm{NS}(X_{\bar\eta})[1/p] \to \mathrm{NS}(X_{\bar b})[1/p]$ *is injective with cokernel torsion-free over* $\mathbb{Z}[1/p]$.

2) *Suppose that $f : X \to B$ has integral geometric fibers. Then for every $b \in U$,* $\mathrm{sp}[1/p] : \mathrm{Num}(X_{\bar\eta})[1/p] \to \mathrm{Num}(X_{\bar b})[1/p]$ *is injective.*

In particular, in both cases $\rho(X_{\bar\eta}) \leq \rho(X_{\bar b})$.

Proof. (a) By [FK88, I, Lem. 8.12], as we assume universal local acyclicity for $f : X \to B$, the specialization of cohomology groups is an isomorphism. From Lemma 2.2 (a) (resp. (b)), one may choose U over which one can define the specialization morphism of NS (resp. Num). Choose an algebraic closure $k(\bar\xi)$ of $k(\xi)$, and let $\bar S$ be the normalization of S in it. As S is Henselian, $\bar S$ is the spectrum of a (possibly non Noetherian) local domain of Krull dimension 1. Let $\bar\xi$ be the generic point of $\bar S$, which is identified with a geometric generic point of S. Let $\bar s$ be the unique closed point of $\bar S$. By the Krull-Akizuki theorem [Sta25, Tag 00PG], for every finite subextension K_α of $k(\bar\xi)$, the normalization S_α of S in K_α is Noetherian. As S is a Henselian trait, S_α is a trait.

Let T be a trait with generic point τ and closed point t. By [SGA5, VII, Prop. 3.4 (ii)], for every scheme Y proper over T, the diagram

$$\begin{tikzcd} \mathrm{Pic}(Y_\tau) \arrow[d, "c_1"] & \mathrm{Pic}(Y) \arrow[l] \arrow[r] \arrow[d, "c_1"] & \mathrm{Pic}(Y_t) \arrow[d, "c_1"] \\ H^2(Y_\tau, \mathbb{Z}_\ell(1)) & H^2(Y, \mathbb{Z}_\ell(1)) \arrow[l] \arrow[r] & H^2(Y_t, \mathbb{Z}_\ell(1)) \end{tikzcd} \tag{5}$$

is commutative. Given a scheme Z proper over S, the diagrams (5) for the proper morphisms $Z \times_S S_\alpha \to S_\alpha$ form an inductive system. By the proof of [Sta25, Tag 0CDT], the Picard group functor turns the limit of schemes to colimit of groups. From [SGA4II, VII, Cor. 5.8], taking inductive limit of the diagrams, one gets a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(Z_{\bar{\xi}}) \arrow[d, "c_1"] & \mathrm{Pic}(Z \times_S \bar{S}) \arrow[l] \arrow[r] & \mathrm{Pic}(Z_{\bar{s}}) \arrow[d, "c_1"] \\
H^2(Z_{\bar{\xi}}, \mathbb{Z}_\ell(1)) & & H^2(Z_{\bar{s}}, \mathbb{Z}_\ell(1)). \arrow[ll, "\mathrm{sp}"']
\end{tikzcd}$$

Then the commutativity of the stated square follows from the construction of specialization morphism.

(b) By $\mathrm{char} k(s) = 0$, (3) and (4), the rows in the stated diagram are exact. By the proof of [Sta25, Tag 0C0N], one can choose a strictly Henselian trait for the specialization $\eta \rightsquigarrow b$ on U. Then from Part (a), as $f : X \to B$ is normal, the left square is commutative. Therefore, there is a unique morphism $\mathrm{sp}[m] : \mathrm{Br}(X_{\bar{\eta}})[m] \to \mathrm{Br}(X_{\bar{b}})[m]$ keeping the diagram commutative. By definition, the cohomological Brauer group is torsion, so there is a unique morphism $\mathrm{Br}(X_{\bar{\eta}}) \to \mathrm{Br}(X_{\bar{b}})$ restricting to the morphisms $\mathrm{sp}[m] : \mathrm{Br}(X_{\bar{\eta}})[m] \to \mathrm{Br}(X_{\bar{b}})[m]$ for all $m > 0$.

(c) In both cases, the injectivity follows from Part (a) and Lemma 2.3 (b). Consider only case *1*). From Fact 1.7, $G := \mathrm{coker}(\mathrm{sp} : \mathrm{NS}\, X_{\bar{\eta}} \to \mathrm{NS}\, X_{\bar{b}})$ is a finitely generated abelian group. By the four lemma, for every prime $\ell \neq p$, because the commutative diagram

$$\begin{tikzcd}
 & \mathrm{NS}\, X_{\bar{\eta}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r, "\mathrm{sp}"] \arrow[d, equal] & \mathrm{NS}\, X_{\bar{b}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r] \arrow[d, hook, "c_1"] & G \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{NS}\, X_{\bar{\eta}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \arrow[r] & H^2(X_{\bar{\eta}}, \mathbb{Z}_\ell) \arrow[r] & T_\ell \mathrm{Br}(X_{\bar{\eta}}) \arrow[r] & 0
\end{tikzcd}$$

has exact rows, the right vertical morphism is injective. By construction, $T_\ell \mathrm{Br}(X_{\bar{\eta}})$ is a torsion-free $\mathbb{Z}_\ell$-module. Therefore, the ℓ-primary torsion $G[\ell^\infty]$ is zero. Then the finite $\mathbb{Z}[1/p]$-module $G[1/p]$ is torsion-free.

□

Remark 2.5. In the setting of Lemma 2.4 (c) *1*) (resp. *2*)), up to shrinking U, the cokernel of $\mathrm{sp}[1/p] : \mathrm{NS}(X_{\bar{\eta}})[1/p] \to \mathrm{NS}(X_{\bar{b}})[1/p]$ (resp. of $\mathrm{sp} : \mathrm{Num}\, X_{\bar{\eta}} \to \mathrm{Num}\, X_{\bar{b}}$) is independent of choice.

We only prove the case of *1*), and the case of *2*) is similar. By [EGA IV 2, Corollaire 6.13.3], as B is Japanese, its normal locus is open in B. Thus, shrinking U one may require further that U is normal. Then by [Sta25, Tag 0BQM], $\pi_1(\eta) \to \pi_1(U, \bar{\eta})$ is surjective. By Lemma 2.3 (b), for every prime $\ell \neq p$, as $X_{\bar{\eta}}$ is proper over $k(\bar{\eta})$, the $\pi_1(\eta)$-equivariant morphism $c_1 : \mathrm{NS}(X_{\bar{\eta}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow H^2(X_{\bar{\eta}}, \mathbb{Z}_\ell(1))$ is injective. Because $R^2 f_* \mathbb{Z}_\ell$ is a lisse sheaf on B, the Galois action of $\pi_1(\eta)$ on $\mathrm{NS}(X_{\bar{\eta}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ factors through $\pi_1(U, \bar{\eta})$. To prove independence, let $\phi' : S' \to U$ be another morphism from a Henselian trait for the specialization $\eta \rightsquigarrow b$. Then $\mathrm{sp}_{\phi'} \mathrm{sp}_\phi^{-1} : H^2(X_{\bar{\eta}}, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} H^2(X_{\bar{\eta}}, \mathbb{Z}_\ell(1))$ is the action of an element of $\pi_1(B,\bar{\eta})$, so it preserves $\mathrm{NS}(X_{\bar{\eta}})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$. Hence, the images of the morphisms

$$\mathrm{sp}_{\phi'}\mathrm{sp}_{\phi}^{-1}c_1,\, c_1 : \mathrm{NS}(X_{\bar{\eta}})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to H^2(X_{\bar{\eta}},\mathbb{Z}_\ell(1))$$

coincide. By Lemma 2.4 (a), the two specializations $\mathrm{NS}(X_{\bar{\eta}})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \to \mathrm{NS}(X_{\bar{b}})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ have the same image. In case *2)*, as $\mathrm{Num}\, X_{\bar{b}}$ is finite free over $\mathbb{Z}$, the two specializations $\mathrm{Num}\, X_{\bar{\eta}} \to \mathrm{Num}\, X_{\bar{b}}$ have the same image.

3 Review of infinitesimal cohomology

We recall the infinitesimal cohomology and the construction of infinitesimal first Chen class. Let $X \to S$ be a morphism of schemes. Let $\mathrm{Inf}(X/S)$ and $(X/S)_{\mathrm{inf}}$ be the associated infinitesimal site and infinitesimal topos (introduced in [Gro68]) respectively. Let $O_{X/S}$ be the infinitesimal structure sheaf on $\mathrm{Inf}(X/S)$. Let $J_{X/S} \subset O_{X/S}$ be the infinitesimal ideal sheaf. Let $u_{X/S} : (X/S)_{\mathrm{inf}} \to X_{\mathrm{Zar}}$ be the projection from the infinitesimal topos to the Zariski topos. Let $i_{X/S} : X_{\mathrm{Zar}} \to (X/S)_{\mathrm{inf}}$ be the immersion of the Zariski topos into the infinitesimal topos. We refer the reader to [Liu25a, Sections 2–4] for fundamentals of infinitesimal cohomology.

3.1 Infinitesimal first Chen class

For every object $U \to T$ of $\mathrm{Inf}(X/S)$, $(J_{X/S})_{(U,T)} \in \mathrm{Mod}(O_T)$ is the ideal sheaf corresponding to the finite order thickening $U \to T$. In particular, $J^n_{X/S,(U,T)} = 0$ for some integer $n > 0$. Therefore, the short exact sequence

$$0 \to J_{X/S} \to O_{X/S} \to i_{X/S*}O_X \to 0$$

induces a multiplicative exact sequence

$$0 \to 1 + J_{X/S} \to O^*_{X/S} \to i_{X/S*}O^*_X \to 0 \tag{6}$$

in $\mathrm{Ab}(\mathrm{Inf}(X/S))$ and hence a morphism $\partial : i_{X/S*}O^*_X[-1] \to (1 + J_{X/S})$ in $D^b(\mathrm{Ab}(\mathrm{Inf}(X/S)))$.

Assume that X is of characteristic 0. Then one may define a morphism

$$\log : 1 + J_{X/S} \to J_{X/S}, \quad \log(1+x) := \sum_{m>0}(-1)^{m-1}x^m/m. \tag{7}$$

The series is a finite sum as $J_{X/S}$ is a sheaf of nilideals. Similarly, one may define a morphism by a finite sum

$$\exp : J_{X/S} \to 1 + J_{X/S}, \quad \exp(x) = \sum_{m\geq 0}\frac{x^m}{m!}. \tag{8}$$

The logarithm morphism (7) and the exponential morphism (8) are isomorphisms inverse to each other. Denote by $c_{1,\mathrm{inf}}$ the composition

$$\begin{aligned} \operatorname{Pic}(X) = H^1(X, O_X^*) = H^1_{\mathrm{inf}}(X/S, i_{X/S*}O_X^*) \\ \xrightarrow{\partial} H^2_{\mathrm{inf}}(X/S, 1 + J_{X/S}) \xrightarrow{\log} H^2_{\mathrm{inf}}(X/S, J_{X/S}) \to H^2_{\mathrm{inf}}(X/S). \end{aligned}$$

The group morphism $c_{1,\mathrm{inf}} : \operatorname{Pic}(X) \to H^2_{\mathrm{inf}}(X/S)$ is called the *infinitesimal first Chen class map.* It takes values in the first infinitesimal filtration $F^1H^2_{\mathrm{inf}}(X/S)$.

Lemma 3.1. *Let k be an algebraically closed field of characteristic 0. Let X be a scheme separated of finite type over k. Then $c_{1,\mathrm{inf}} : \operatorname{Pic}(X) \to H^2_{\mathrm{inf}}(X/k)$ factors through $\operatorname{NS}(X)$.*

Proof. As in the proof of [Kle68, Prop 1.2.1], the result follows from the Künneth formula for infinitesimal cohomology ([HJ14, Proposition 7.29] and [Liu25a, Remark 11.5]). ☐

For line bundles on a complex algebraic variety, we show that the infinitesimal first Chern class is compatible with the topological first Chern class.

Lemma 3.2. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then the isomorphism from [Liu25a, Theorem 14.1] fits into a commutative square*

$$\begin{tikzcd} \operatorname{Pic}(X) \arrow[r] \arrow[d, "c_{1,\mathrm{inf}}"] & \operatorname{Pic}(X^{\mathrm{an}}) \arrow[d, "2\pi i \cdot c_1"] \\ H^2_{\mathrm{inf}}(X/\mathbb{C}) \arrow[r, equal] & H^2(X^{\mathrm{an}}, \mathbb{C}). \end{tikzcd}$$

Proof. By [BI70, Proposition 2.3], the left triangle in the diagram

$$\begin{tikzcd} H^2_{\mathrm{inf}}(X/\mathbb{C}) \arrow[rrr, "\cong"] \arrow[dd] & & & H^2(X^{\mathrm{an}}, \mathbb{C}) \arrow[dd, hook] \\ & \operatorname{Pic}(X) \arrow[r] \arrow[ul, "c_{1,\mathrm{inf}}"'] \arrow[dl, "c_{1,\mathrm{dR}}"] & \operatorname{Pic}(X^{\mathrm{an}}) \arrow[ur, "2\pi i \cdot c_1"] \arrow[dr, "c_{1,\mathrm{dR}}"] & \\ H^2_{\mathrm{dR}}(X/\mathbb{C}) \arrow[rrr] & & & H^2_{\mathrm{dR}}(X^{\mathrm{an}}) \end{tikzcd}$$

is commutative. By construction, the lower trapezoid is commutative. From [Huy05, Proposition 4.4.12] and the proof of [Del71, (2.2.5)], the right triangle is commutative. From [Liu25a, Theorem 14.1], the outer rectangular is commutative. From the Bloom-Herrera theorem [BH69, Theorem 3.11] (see also [SGA4II, Vbis, Proposition 5.2.8]), since X is locally of finite type over $\mathbb{C}$, $H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^i_{\mathrm{dR}}(X^{\mathrm{an}})$ is injective for every $i \geq 0$. Therefore, the upper trapezoid is also commutative. ☐

3.2 Stratifying topos and linearization functor

For a morphism $X \to Y$ of schemes over S, let $Y\,\mathrm{Strat}(X/S)$ (resp. $(X/S)_{Y\,\mathrm{Strat}}$) be the stratifying site (resp. stratifying topos) in the sense of [Liu25a, Definition 4.1]. Let $j : (X/S)_{Y\,\mathrm{Strat}} \to (X/S)_{\mathrm{inf}}$ be the natural morphism of topoi [Liu25a, (5)]. Denote by $u'_{X/S}$ the composition

$$(X/S)_{Y\,\mathrm{Strat}} \xrightarrow{j} (X/S)_{\mathrm{inf}} \xrightarrow{u_{X/S}} X_{\mathrm{Zar}}.$$

Let $\mathrm{Diff}(Y/S)$ be the category of O_Y-modules with differential operators of finite order relative to S as morphisms. Let $L : \mathrm{Diff}(Y/S) \to \mathrm{Mod}(O_{X/S})$ be the linearization functor ([Liu25a, Definition 8.1]). Let K be the kernel of the natural morphism $L(O_Y) \to j^* i_{X/S*} O_X$ from [Liu25a, Lemma 8.6]. Lemma 3.3 is an infinitesimal analog of [Liu25b, (19)]. Different from the crystalline case, $L(O_Y)$ may not be a crystal nor locally quasi-coherent. For $(U,T) \in Y\,\mathrm{Strat}(X/S)$, we do not know whether the ideal sheaf $K_{(U,T)}$ of $L(O_Y)_{(U,T)}$ is locally nilpotent on T.

Lemma 3.3. *Let $X \to Y$ be a morphism of schemes over a scheme S. Assume that X is of characteristic 0. Then there is a canonical exact sequence $0 \to 1 + K \to L(O_Y)^* \to j^* i_{X/S*} O_X^* \to 0$ of abelian sheaves on $Y\,\mathrm{Strat}(X/S)$.*

Proof. For every $n \geq 0$, let $P^n_{Y/S}$ be the sheaf of principal parts of order n on Y in the sense of [EGA IV 4, Définition 16.3.1]. Let H_n be the kernel of the natural projection $P^n_{Y/S} \to O_Y$. For an object $(U,T) \in Y\,\mathrm{Strat}(X/S)$, choose a morphism $h : T \to Y$ with $h|_U = g|_U$. By [Liu25a, (44)], for as the $(n+1)$-th power of H_n vanishes, $K_{n,(U,T)}$ is a nilpotent ideal of $h^* P^n_{Y/S}$. Therefore, the sequence

$$0 \to 1 + K_{n,(U,T)} \to (h^* P^n_{Y/S})^* \to O_U^* \to 0 \tag{9}$$

of abelian sheaves on T is exact. As X is of characteristic 0, so is T. Then one can define a logarithmic isomorphism

$$\log_{n,(U,T)} : 1 + K_{n,(U,T)} \xrightarrow{\sim} K_{n,(U,T)}.$$

The isomorphisms $(\log_{n,(U,T)})_{n\geq 0}$ fit to an isomorphism of inverse systems of abelian sheaves on T. By [Liu25a, Example 4.9 (b)], because $(K_{n,(U,T)})_{n\geq 0}$ is a surjective system of quasi-coherent O_T-modules, the system satisfies condition (*). Then by [Sta25, Tag 0BKS], one has

$$R^1 \lim_n (1 + K_{n,(U,T)}) = 0$$

in $\mathrm{Ab}(T)$. From [Liu25a, (35)], one has $L(O_Y)^*_{(U,T)} = \lim_n (h^* P^n_{Y/S})^*$. Thus, taking limit of the sequences (9), one obtains that the sequence

$$0 \to 1 + K_{(U,T)} \to L(O_Y)^*_{(U,T)} \to O_U^* \to 0$$

is exact. □

3.3 Cohomology of linearization of differential complex

We prove Lemma 3.7, an analog of [Liu25b, Lemma 3.1.5] in infinitesimal cohomology.

Lemma 3.4. *Let $i : X \hookrightarrow Y$ be an immersion of schemes over a scheme S. Let $M^\bullet$ be a graded* quasi-coherent *O_Y-module bounded below. For every integer i, let $d^i : M^i \to M^{i+1}$ be a differential operator relative to S. Suppose that $L(M^\bullet) := (L(M^i), L(d^i))_i$ is a* complex *of $O_{X/S}$-modules on $Y\,\mathrm{Strat}(X/S)$. Then $\lim_n(P^n_X(Y) \otimes_{O_Y} M^\bullet)$ is naturally a complex over the abelian category $\mathrm{Mod}(X, O_S)$. There is a canonical isomorphism*

$$\lim_n (P^n_X(Y) \otimes_{O_Y} M^\bullet) \cong Ru'_{X/S*} L(M^\bullet)$$

in $D^+(X, O_S)$. It induces an isomorphism

$$R\Gamma((X/S)_{Y\,\mathrm{Strat}}, L(M^\bullet)) \cong R\Gamma(X, \lim_n (P^n_X(Y) \otimes_{O_Y} M^\bullet))$$

in $D^+(O_S(S))$. In particular, for every quasi-coherent O_Y-module E, $L(E)$ is right acyclic for $u'_{X/S*}$.

Proof. By [Liu25a, (32)], for every i, the differential operator $d^i : M^i \to M^{i+1}$ induces a morphism

$$\lim_n (P^n_X(Y) \otimes_{O_Y} M^i) \to \lim_n (P^n_X(Y) \otimes_{O_Y} M^{i+1})$$

in $\mathrm{Mod}(X, O_S)$. For every sheaf E on $Y\,\mathrm{Strat}(X/S)$, let $\mathrm{CA}^\bullet_Y(E)$ be the Čech-Alexander complex of E on X ([Liu25a, Sec. 4]). From [Liu25a, Lemma 10.4], as $i : X \to Y$ is an immersion, for every integer k, there is a natural resolution

$$\lim_n (P^n_X(Y) \otimes_{O_Y} M^k) \to \mathrm{CA}^\bullet_Y(L(M^k)),$$

and the resolutions are compatible with the differential operators of $M^\bullet$.

As $L(M^\bullet)$ is a complex, $\mathrm{CA}^\bullet_Y\, L(M^\bullet)$ is a bicomplex that is a natural resolution of the complex $\lim_n(P^n_X(Y) \otimes_{O_Y} M^\bullet)$. Thus, there is a quasi-isomorphism

$$\lim_n (P^n_X(Y) \otimes_{O_Y} M^\bullet) \to \mathrm{CA}^\bullet_Y(L(M^\bullet)).$$

By Lemma 3.5 (b), for any integers $v \geq 0$ and k, as the O_Y-module M^k is quasi-coherent, the inverse system $(L(M^k)_{(X, \Delta^i_X(Y^{v+1}))})_i$ satisfies condition (*). Then by [Liu25a, Lemma 4.6], as $M^\bullet$ is bounded below, there is a canonical isomorphism

$$Ru'_{X/S*} L(M^\bullet) \cong \mathrm{CA}^\bullet_Y(L(M^\bullet)) \tag{10}$$

in $D^+(X, u'_{X/S*} O_{X/S})$. Let $f : X \to S$ be the structure morphism. By [Liu25a, Remark 2.3], there is a canonical morphism $u^*_{X/S} f^{-1} O_S \to O_{X/S}$. By adjunction, the composition

$$u^*_{X/S} f^{-1} O_S \to O_{X/S} \to j_* j^{-1} O_{X/S}$$

induces a morphism $f^{-1}O_S \to u'_{X/S*}O_{X/S}$ of sheaves of rings on X. Therefore, (10) is also an isomorphism in $D^+(X, f^{-1}O_S)$. $\square$

Lemma 3.5. *Let $i : X \to Y$ be an immersion of schemes over S. Let M be a* quasi-coherent *O_Y-module.*

(a) *Let $(U, T) \in Y\,\mathrm{Strat}(X/S)$. Regard $L(M)_{(U,T)}$ as an abelian sheaf on U. Then for every integer $q > 0$ and every affine open subset U_0 of U, one has $H^q(U_0, L(M)_{(U,T)}) = 0$.*

(b) *For every integer $v \geq 0$, the system $(F_i) := (L(M)_{(X,\Delta^i_X(Y^{v+1}))})_{i\geq 0}$ in* $\mathrm{Ab}(X)$ *satisfies condition (*).*

Proof. (a) Choose a morphism $h : T \to Y$ with $h|_U = i|_U$ as a morphism $U \to Y$. By [Liu25a, (35)], as M is quasi-coherent, the O_T-module $L(M)_{(U,T)}$ is the inverse limit of the surjective system $\{h^*(P^n_{Y/S} \otimes_{O_Y} M)\}_{n\geq 0}$ of quasi-coherent sheaves on T. Then by [Sta25, Tag 0BKS], for every affine open subset T_0 of T and every integer $q > 0$, one has $H^q(T_0, L(M)_{(U,T)}) = 0$. Let T_1 be the open subscheme of T with underlying set U_0. From [Sta25, Tag 06AD], as U_0 is affine, so is T_1. Thus, one has $H^q(U_0, L(M)_{(U,T)}) = H^q(T_1, L(M)_{(U,T)}) = 0$.

(b) Let $\mathcal{B}$ be the set of affine open subsets U of X, such that $i(U)$ is contained in an affine open subset of Y. Then $\mathcal{B}$ is base for X. By Part (a), it remains to prove that for every $U \in \mathcal{B}$, the system $(F_i(U))$ is surjective, i.e., for every $i \geq 0$, the morphism $F_{i+1}(U) \to F_i(U)$ is surjective.

By [Sta25, Tag 06AD], as $U \to \Delta^i_U(Y^{v+1})$ is a finite order thickening, $\Delta^i_U(Y^{v+1}) = \operatorname{Spec} B_i$ is affine. Moreover, $\Delta^i_U(Y^{v+1})$ is the open subscheme of $\Delta^i_X(Y^{v+1})$ with underlying set U. Let $h : \Delta^i_X(Y^{v+1}) \to Y$ be the morphism induced by the last projection $Y^{v+1}_S (= Y \times_S \times \cdots \times_S Y) \to Y$. Then $h|_X = i$ as morphism $X \to Y$. Choose an affine open subset $V = \operatorname{Spec} A$ of Y containing $g(U)$. Then V contains $h(\operatorname{Spec} B_i)$. Let $A \to B_i$ be the ring morphism induced by $h : \Delta^i_U(Y^{v+1}) \to V$. One has

$$\begin{aligned}
&\Gamma(U, F_i) = \Gamma(\operatorname{Spec} B_i, L(M)_{(X,\Delta^i_X(Y^{v+1}))}) \\
&= \lim_n \Gamma(\operatorname{Spec} B_i, h^*(P^n_{Y/S} \otimes_{O_Y} M)) \\
&= \lim_n \left(\Gamma(V, P^n_{Y/S} \otimes_{O_Y} M) \otimes_A B_i\right) \\
&= \lim_n \left(\Gamma(V, P^n_{Y/S}) \otimes_A \Gamma(V, M) \otimes_A B_i\right).
\end{aligned}$$

As V is affine, and $(P^n_{Y/S})_{n\geq 0}$ is a surjective system of quasi-coherent sheaves on Y, the system $\{\Gamma(V, P^n_{Y/S})\}_{n\geq 0}$ is surjective. The closed immersion $\Delta^i_U(Y^{v+1}) \to \Delta^{i+1}_U(Y^{v+1})$ over V induces a surjective morphism $B_{i+1} \to B_i$ of A-algebras. By Lemma 3.6, for every $n \geq 0$,

$$\Gamma(V, P^n_{Y/S}) \otimes_A \Gamma(V, M) \otimes_A B_{i+1} \to \Gamma(V, P^n_{Y/S}) \otimes_A \Gamma(V, M) \otimes_A B_i$$

is surjective, so the morphism of A-modules $F_{i+1}(U) \to F_i(U)$ is also surjective.

$\square$

Lemma 3.6. *Let A be a commutative ring. Let $(M_i)_{i\geq 0}$ be an inverse system of A-modules that is satisfying the Mittag-Leffler condition. Then for every surjective morphism $F \twoheadrightarrow G$ of A-modules, the induced morphism $\lim_i(M_i \otimes_A F) \to \lim_i(M \otimes_A G)$ is surjective.*

Proof. Let $K = \ker(F \to G)$. The functor $(\cdot)\otimes_A K : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is right exact, so the system $(M_i \otimes_A K)_{i\geq 0}$ also satisfies the Mittag-Leffler condition. For every $i \geq 0$, the sequence

$$M_i \otimes K \to M_i \otimes F \to M_i \otimes G \to 0$$

is exact, so $M_i \otimes K \twoheadrightarrow \ker(M_i \otimes F \to M_i \otimes G)$ is surjective. Thus, the system $\{\ker(M_i \otimes F \to M_i \otimes G)\}_i$ satisfies the Mittag-Leffler condition. Then from [Sta25, Tag 03CA], the sequence

$$0 \to \lim_i \ker(M_i \otimes F \to M_i \otimes G) \to \lim_i(M_i \otimes F) \to \lim_i(M_i \otimes G) \to 0$$

is exact. $\square$

Using Lemma 3.3 and Lemma 3.4, one can prove Lemma 3.7 similarly to [Liu25b, Lemma 3.1.5].

Lemma 3.7. *Let S be a scheme. Let $S_0 \to S$ be a nilpotent thickening. Let $f : X \to S$ be a morphism of schemes. Assume that X is of characteristic 0. Let $X_0 = X \times_S S_0$. Then the natural morphism $Rf_{X_0/S*}O_{X_0/S} \to Rf_*\Omega^\bullet_{X/S}$ from [Liu25a, (77)] fits into a commutative diagram*

$$\begin{tikzcd}
Rf_{X_0/S*}i_{X_0/S*}O^*_{X_0}[-1] \arrow[r, "\partial"] \arrow[d, equal] & Rf_{X_0/S*}(1+J_{X_0/S}) \arrow[r, "\log"] & Rf_{X_0/S*}J_{X_0/S} \arrow[r] & Rf_{X_0/S*}O_{X_0/S} \arrow[d] \\
Rf_*O^*_{X_0}[-1] \arrow[r, "\partial"] & Rf_*K^\times_{X/S} \arrow[r, "\log^\bullet"] & Rf_*I^\bullet_{X/S} \arrow[r] & Rf_*\Omega^\bullet_{X/S}
\end{tikzcd}$$

in $D^+(\mathrm{Ab}(S))$.

3.4 Infinitesimal cohomology over an adic ring

By [Liu25a, Remark 11.6], for an integer $q \geq 0$ and a finite type morphism of Noetherian schemes $f : X \to S$, $\mathcal{H}^q_{\mathrm{DR}}(X/S)$ coincides with the higher direct image of the infinitesimal structure sheaf $R^q f_{X/S*}O_{X/S}$. Compared with Hartshorne's de Rham cohomology, infinitesimal cohomology has the extra flexibility which provides a characteristic 0 analog of the crystalline cohomology over an adic ring.

Setup 3.8. Let A be a Noetherian ring with an ideal I. Assume that A is I-adically complete and separated, hence an adic ring. For every integer $n \geq 0$, let $A_n = A/I^{n+1}$, $S_n = \operatorname{Spec} A_n$, $S = \operatorname{Spec} A$ and $\hat{S} = \operatorname{Spf} A$.

Let X be a scheme over S_0. For any integers $0 \leq m \leq n$, the inclusion

$$u_{mn} : \operatorname{Inf}(X/S_m) \to \operatorname{Inf}(X/S_n)$$

is a continuous and cocontinuous functor. It defines a morphism of sites f_{mn} : $\operatorname{Inf}(X/S_n) \to \operatorname{Inf}(X/S_m)$ and hence a morphism of topoi

$$\mu_{mn} : (X/S_n)_{\mathrm{inf}} \to (X/S_m)_{\mathrm{inf}}.$$

From [Sta25, Tag 00XO], u_{mn} defines another morphism of topoi

$$i_{mn} : (X/S_m)_{\mathrm{inf}} \to (X/S_n)_{\mathrm{inf}}.$$

For every $(U, T) \in \operatorname{Inf}(X/S_n)$, let h_T be the corresponding representable sheaf on $\operatorname{Inf}(X/S_n)$. By [Sta25, Tag 00XR (1)], for every $(V, Z) \in \operatorname{Inf}(X/S_m)$, one has

$$(i_{mn}^* h_T)(V, Z) = \operatorname{Hom}_{\operatorname{Inf}(X/S_n)}((V, Z), (U, T)) = \operatorname{Hom}_{\mathrm{Id}_X}(Z, T) = (\mathrm{Id}_X^* T)(V, Z).$$

Therefore, by [Liu25a, Theorem 3.3], $i_{mn} = (\mathrm{Id}_X)_{\mathrm{inf}}$ is the morphism induced by functoriality. By [Sta25, Tag 03L5], the morphisms $(i_{mn})_{0 \leq m \leq n}$ are compatible with composition.

Remark 3.9. The functor

$$v_{mn} : \operatorname{Inf}(X/S_n) \to \operatorname{Inf}(X/S_m), \quad (U, T) \mapsto (U, T \times_{S_n} S_m)$$

is continuous. It induces a morphism of sites $\iota_{mn} : \operatorname{Inf}(X/S_m) \to \operatorname{Inf}(X/S_n)$. The corresponding morphism of topoi is i_{mn}.

Definition 3.10. Define the infinitesimal site $\operatorname{Inf}(X/\hat{S})$ of X over $\hat{S}$ as follows. An object of $\operatorname{Inf}(X/\hat{S})$ is a finite order thickening $U \to T$ over S, where U is an open subset of X and $I^{n+1} \cdot O_T = 0$ for some integer $n \geq 0$. A morphism $(U, T) \to (U', T')$ in $\operatorname{Inf}(X/\hat{S})$ is a morphism in $\operatorname{Inf}(X/S_n)$ for some n such that both pairs belong to $\operatorname{Inf}(X/S_n)$. A family of morphisms $\{(U_i, T_i) \to (U, T)\}_{i \in I}$ in $\operatorname{Inf}(X/\hat{S})$ is a *covering*, if $\{T_i \to T\}_{i \in I}$ is a Zariski open covering of T. Define the infinitesimal structure sheaf on $\operatorname{Inf}(X/\hat{S})$ as a sheaf of rings $O_{X/\hat{S}} : (U, T) \mapsto O_T(T)$.

Then $\operatorname{Inf}(X/\hat{S})$ is a site, and $(\operatorname{Inf}(X/\hat{S}), O_{X/\hat{S}})$ is a ringed site. Similar to the infinitesimal sites, a sheaf on $\operatorname{Inf}(X/\hat{S})$ can be described as a compatible family of Zariski sheaves on objects. Let $(X/\hat{S})_{\mathrm{inf}}$ be the corresponding topos.

Remark 3.11. Define a category $\operatorname{Inf}(X/S_\bullet)$ as follows. An object of it is a triple (n, U, T), where $n \geq 0$ is an integer, and $(U, T) \in \operatorname{Inf}(X/S_n)$. A morphism $(m, U', T') \to (n, U, T)$ refers to an Id_X-morphism $T' \to T$ in the sense of

[Liu25a, Definition 3.1]. The existence of such a morphism implies $m \le n$ and $U' \subset U$. View $\mathbb{N}$ as a small category, where $\mathrm{Hom}_{\mathbb{N}}(m,n)$ is a singleton when $m \le n$, and is empty otherwise. The functor

$$p : \mathrm{Inf}(X/S_\bullet) \to \mathbb{N}^{\mathrm{op}}, \quad (n,U,T) \mapsto n$$

defines a fibred site in the sense of [SGA4II, VI, 7.2.1]. For every morphism $m \to n$ in $\mathbb{N}$, the inverse image functor is $u_{mn} : \mathrm{Inf}(X/S_m) \to \mathrm{Inf}(X/S_n)$. Then $\mathrm{Inf}(X/\hat{S})$ is the projective limit of this fibred site in the sense of [SGA4II, VI, Définition 8.2.5]. From [SGA4II, VI, Théorème 8.2.3 2)], $(X/\hat{S})_{\mathrm{inf}}$ is the projective limit of the inverse system of topoi $((X/S_n)_{\mathrm{inf}}, \mu_{mn})$ in the sense of [SGA4II, VI, Définition 8.1.1]. Let Tot be the total site associated with the fibred site $p : \mathrm{Inf}(X/S_\bullet) \to \mathbb{N}^{\mathrm{op}}$ in the sense of [SGA4II, Définition 7.4.1]. Then by [SGA4II, Exercice 7.4.15], there is a canonical isomorphism of functors

$$R\Gamma(\mathrm{Tot}, \cdot) \to R\lim_{n\in\mathbb{N}} R\Gamma((X/S_n)_{\mathrm{inf}}, (\cdot)|_{\mathrm{Inf}(X/S_n)}) : \quad D^+(\mathrm{Ab}(\mathrm{Tot})) \to D^+(\mathrm{Ab}).$$

For every integer $n \ge 0$, the inclusion functor $u_n : \mathrm{Inf}(X/S_n) \to \mathrm{Inf}(X/\hat{S})$ is continuous, cocontinuous and fully faithful. Let

$$(u_n)_p : \mathrm{PSh}(\mathrm{Inf}(X/S_n)) \to \mathrm{PSh}(\mathrm{Inf}(X/\hat{S}))$$

be the functor defined in [Sta25, Tag 00VC]. Then for every $G \in \mathrm{PSh}(\mathrm{Inf}(X/S_n))$ and every $(V,Z) \in \mathrm{Inf}(X/\hat{S})$, one has

$$((u_n)_p G)(V,Z) = \mathrm{colim}_{\substack{(V,Z)\to(U,T),\\ (U,T)\in\mathrm{Inf}(X/S_n)}} G(U,T).$$

If $I^{n+1}O_Z \neq 0$, then the colimit is over an empty diagram, so $((u_n)_p G)(V,Z) = \emptyset$. If $I^{n+1}O_Z = 0$, then $(V,Z) \in \mathrm{Inf}(X/S_n)$ and the diagram underlying the colimit has a final object $\mathrm{Id}_{(V,Z)} : (V,Z) \to (V,Z)$. Thus, one has $((u_n)_p G)(V,Z) = G(V,Z)$. In particular, $(u_n)_p$ restricts to an exact functor

$$(u_n)_s : \mathrm{Sh}(\mathrm{Inf}(X/S_n)) \to \mathrm{Sh}(\mathrm{Inf}(X/\hat{S})).$$

Therefore, u_n defines a morphism of sites $f_n : \mathrm{Inf}(X/\hat{S}) \to \mathrm{Inf}(X/S_n)$. Let

$$\mu_n : (X/\hat{S})_{\mathrm{inf}} \to (X/S_n)_{\mathrm{inf}}$$

be the induced morphism of topoi.

For every $n \ge 0$ and every $(U,T) \in \mathrm{Inf}(X/\hat{S})$, the nilpotent thickening $U \to T$ factors through $T \times_{\hat{S}} S_n$, and $U \to T \times_{\hat{S}} S_n$ is also a nilpotent thickening. Thus, $(U, T \times_{\hat{S}} S_n) \in \mathrm{Inf}(X/S_n)$. The functor

$$v_n : \mathrm{Inf}(X/\hat{S}) \to \mathrm{Inf}(X/S_n), \quad (U,T) \mapsto (U, T \times_{\hat{S}} S_n)$$

is continuous. The induced functor

$$(v_n)_p : \mathrm{PSh}(\mathrm{Inf}(X/\hat{S})) \to \mathrm{PSh}(\mathrm{Inf}(X/S_n))$$

sends $G \in \mathrm{PSh}(\mathrm{Inf}(X/\hat{S}))$ to the presheaf

$$(V, Z) \mapsto G(V, Z).$$

In particular, $(v_n)_p$ restricts to an exact functor

$$(v_n)_s : \mathrm{Sh}(\mathrm{Inf}(X/\hat{S})) \to \mathrm{Sh}(\mathrm{Inf}(X/S_n)).$$

Thus, v_n induces a morphism of sites $\iota_n : \mathrm{Inf}(X/S_n) \to \mathrm{Inf}(X/\hat{S})$. Since $v_n \circ u_n = \mathrm{Id}_{\mathrm{Inf}(X/S_n)}$, one has $f_n \circ \iota_n = \mathrm{Id}_{\mathrm{Inf}(X/S_n)}$ as morphism of sites. Let

$$i_n : (X/S_n)_{\mathrm{inf}} \to (X/\hat{S})_{\mathrm{inf}}$$

be the morphism of topoi induced by ι_n. Then $i_n^{-1} O_{X/\hat{S}} = O_{X/S_n}$, $i_n^{-1} J_{X/\hat{S}} = J_{X/S_n}$ and i_n is naturally a morphism of ringed topoi.

By Remark 3.11 and [SGA4II, VI, (8.1.4.1)], passing to limit, the inverse system of morphisms of topoi $(u_{X/S_n} : (X/S_n)_{\mathrm{inf}} \to X_{\mathrm{Zar}})_{n \geq 0}$ induces a morphism of topoi

$$u_{X/\hat{S}} : (X/\hat{S})_{\mathrm{inf}} \to X_{\mathrm{Zar}}.$$

The diagram of topoi

$$\begin{tikzcd}
(X/S_n)_{\mathrm{inf}} \arrow[r, "i_n"] \arrow[dr, equal] & (X/\hat{S})_{\mathrm{inf}} \arrow[r, "u_{X/\hat{S}}"] \arrow[d, dashed, "\mu_n"] & X_{\mathrm{Zar}} \\
& (X/S_n)_{\mathrm{inf}} \arrow[ur, dashed, "u_{X/S_n}"'] &
\end{tikzcd}$$

is commutative. Similarly, the inverse system of morphisms of ringed topoi $(i_{X/S_n} : X_{\mathrm{Zar}} \to (X/S_n)_{\mathrm{inf}})_{n \geq 0}$ induces a morphism of ringed topoi

$$i_{X/\hat{S}} : X_{\mathrm{Zar}} \to (X/\hat{S})_{\mathrm{inf}}$$

satisfying $u_{X/\hat{S}} \circ i_{X/\hat{S}} = \mathrm{Id}_{X_{\mathrm{Zar}}}$.

Using Remark 3.11, one can prove Lemma 3.12 similarly to [BO78, Proposition 7.22].

Lemma 3.12. *Let E be an $O_{X/\hat{S}}$-module. Assume that for every $(U, T) \in \mathrm{Inf}(X/\hat{S})$, $E_{(U,T)}$ is a quasi-coherent O_T-module. Then there are canonical isomorphisms*

$$Ru_{X/\hat{S}*} E \xrightarrow{\sim} R \lim_n Ru_{X/S_n*} i_n^* E, \quad R\Gamma(X/\hat{S}, E) \xrightarrow{\sim} R \lim_n R\Gamma(X/S_n, i_n^* E).$$

Let $J_{X/\hat{S}}$ be the kernel of the canonical morphism $O_{X/\hat{S}} \to i_{X/\hat{S}*} O_X$. Then for every object $(U, T) \in \mathrm{Inf}(X/\hat{S})$, $J_{X/\hat{S},(U,T)}$ is the ideal sheaf of the closed immersion $U \to T$. In particular, it is nilpotent. Similar to (6), there is a short exact sequence

$$0 \to 1 + J_{X/\hat{S}} \to O^*_{X/\hat{S}} \to i_{X/\hat{S}*} O^*_X \to 0.$$

Suppose that X is of characteristic 0. Then similar to (7), one can define an isomorphism of abelian sheaves

$$\log : 1 + J_{X/\hat{S}} \to J_{X/\hat{S}}, \quad 1 + x \mapsto \sum_{m>0} (-1)^{m-1} x^m/m.$$

As every $J_{X/\hat{S},(U,T)}$ is nilpotent, the series is a finite sum.

Applying $Ru_{X/\hat{S}*} : D^+(\mathrm{Ab}(\mathrm{Inf}(X/\hat{S}))) \to D^+(\mathrm{Ab}(X))$ to the composition

$$i_{X/\hat{S}*} O_X^*[-1] \xrightarrow{\partial} 1 + J_{X/\hat{S}} \xrightarrow{\log} J_{X/\hat{S}} \hookrightarrow O_{X/\hat{S}},$$

one gets a morphism $O_X^*[-1] \to Ru_{X/\hat{S}*} O_{X/\hat{S}}$ in $D^+(\mathrm{Ab}(X))$. Taking cohomology groups, one gets a morphism

$$c_{1,\mathrm{inf}} : \mathrm{Pic}(X) \to H^2_{\mathrm{inf}}(X/\hat{S}),$$

called the *first Chern class morphism* of X over $\hat{S}$.

4 Cohomological criterion of lifting line bundle

Given a complete local ring A of characteristic 0 with residue field k, a scheme X proper over A and a line bundle L on the special fiber X_s, we prove a cohomological criterion of when L extends to X. The condition is that the infinitesimal first Chern class $c_{1,\mathrm{inf}}(L) \in H^2_{\mathrm{inf}}(X_s/k)$ maps inside $F^1 H^2_{\mathrm{dR}}(X/A)$ under the comparison morphism $H^2_{\mathrm{inf}}(X_0/k) \to H^2_{\mathrm{dR}}(X/A)$. This criterion is a characteristic 0 analog of [BO83, Theorem 3.8]. We first establish it in the context of formal schemes (Theorem 4.4). Using Grothendieck's algebraization theorem, we deduce the case with usual schemes (Theorem 4.5). In the smooth case, the results reduce to [LP19, Theorem 6.9] and the local case of the variational Hodge conjecture (stated in [Mor19, p.3469] without proof) respectively when $A = k[[x_1, \dots, x_n]]$ is a formal power series ring. We draw a consequence (Lemma 4.12) of this criterion, which is what we need to prove Theorem 1.1.

4.1 Proof of Theorem 1.3

Let $\hat{S}$ be as in Setup 3.8. We prove Theorem 4.5 which extends Theorem 1.3 to the more general base $\hat{S}$. For every formal scheme $\mathfrak{X}$ proper over $\hat{S}$ and every $n \geq 0$, let $X_n := \mathfrak{X} \times_{\hat{S}} S_n$. Lemma 4.1 is a counterpart of [Liu25b, Lemma 3.2.2]. It follows from [Liu25a, Corollary 11.9] and Lemma 3.12.

Lemma 4.1. *Let $\hat{S}$ be as in Setup 3.8. Let $f : \mathfrak{X} \to \hat{S}$ be a proper morphism of Noetherian formal schemes. Then there is a natural morphism*

$$\tau_{\mathrm{inf}} : R\Gamma_{\mathrm{inf}}(X_0/\hat{S}) \to R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S}). \tag{11}$$

For every integer $i \geq 0$, (11) induces a morphism $H^i_{\mathrm{inf}}(X_0/\hat{S}) \to H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$. They are isomorphisms if $f : \mathfrak{X} \to \hat{S}$ is smooth and if $\mathfrak{X}$ is of characteristic 0.

Remark 4.2. Let $f : \mathfrak{X} \to \hat{S}$ be a proper morphism of Noetherian formal schemes. Assume that S is of characteristic 0. Then the ideal I of A admits a unique divided power structure. It gives rise to a logarithm morphism

$$\log : 1 + I \cdot O_{\mathfrak{X}} \to I \cdot O_{\mathfrak{X}} \tag{12}$$

similar to [Liu25b, (31)]. In this case, the logarithm morphism is an isomorphism, with an inverse given by the exponential morphism.

Lemma 4.3 is parallel to [Liu25b, Lemma 3.2.7], which can be deduced similarly from Lemma 3.7.

Lemma 4.3. *Let $f : \mathfrak{X} \to \hat{S}$ be a proper morphism of Noetherian formal schemes. Let $\Omega^{\times}_{\mathfrak{X}/\hat{S}}$ be the multiplicative de Rham complex*

$$O^*_{\mathfrak{X}} \overset{d\log}{\to} \Omega^1_{\mathfrak{X}/\hat{S}} \overset{d}{\to} \Omega^2_{\mathfrak{X}/\hat{S}} \cdots$$

Let $K^{\times}_{\mathfrak{X}/S}$ (resp. $I^{\bullet}_{\mathfrak{X}/\hat{S}}$) be the kernel of $\Omega^{\times}_{\mathfrak{X}/\hat{S}} \to O^*_{X_0}$ (resp. $\Omega^{\bullet}_{\mathfrak{X}/\hat{S}} \to O_{X_0}$). Then the diagram

$$\begin{tikzcd}
\mathrm{Pic}(X_0) \arrow[rrr, "c_{1,\mathrm{inf}}"] \arrow[d, "\cong"] & & & H^2_{\mathrm{inf}}(X_0/\hat{S}) \arrow[d, "\tau_{\mathrm{inf}}"] \\
H^1(\mathfrak{X}, O^*_{X_0}) \arrow[r, "\partial"] \arrow[dr, "\partial"'] & H^2(\mathfrak{X}, K^{\times}_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^{\bullet}"] \arrow[d] & H^2(\mathfrak{X}, I^{\bullet}_{\mathfrak{X}/\hat{S}}) \arrow[r] \arrow[d] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[d] \\
 & H^2(\mathfrak{X}, K^{\times,0}_{\mathfrak{X}/\hat{S}}) \arrow[r, "(12)"] & H^2(\mathfrak{X}, I^0_{\mathfrak{X}/\hat{S}}) \arrow[r, "i"] & H^2(\mathfrak{X}, O_{\mathfrak{X}}).
\end{tikzcd}$$

is commutative.

Let $\hat{S}$ be as in Setup 3.8. For an integer $q \geq 0$ and a proper morphism $f : \mathfrak{X} \to \hat{S}$ of Noetherian formal schemes, we say f satisfies *(CF) in degree* q, if the map $H^q(\mathfrak{X}, O_{\mathfrak{X}}) \to H^q(X_0, O_{X_0})$ is surjective. From [EGA III 1, Corollaire 3.4.4], as A is Noetherian and $f : \mathfrak{X} \to \hat{S}$ is proper, (CF) in degree q is equivalent to that for every integer $n \geq 0$, the canonical map $H^q(X_n, O_{X_n}) \to H^q(X_0, O_{X_0})$ is surjective.

Theorem 4.4. *Let $\hat{S}$ be as in Setup 3.8. Assume that S is of characteristic* 0. *Let $f : \mathfrak{X} \to \hat{S}$ be a proper morphism of Noetherian formal schemes satisfying (CF) in degree* 1. *Let $L_0 \in \mathrm{Pic}\, X_0$. Then L_0 lifts to $\mathrm{Pic}\, \mathfrak{X}$ if and only if the image of $c_{1,\mathrm{inf}}(L_0)$ under the morphism $\tau_{\mathrm{inf}} : H^2_{\mathrm{inf}}(X_0/\hat{S}) \to H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$ is in the Hodge filtration $F^1 H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$. Equivalently, there is an exact sequence*

$$\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X_0) \to H^2(\mathfrak{X}, O_{\mathfrak{X}}),$$

where the second morphism is the composition

$$\mathrm{Pic}(X_0) \overset{c_{1,\mathrm{inf}}}{\to} H^2_{\mathrm{inf}}(X_0/\hat{S}) \overset{\tau_{\mathrm{inf}}}{\to} H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \to H^2(\mathfrak{X}, O_{\mathfrak{X}}).$$

Proof. We only prove the "if" part. The short exact sequence $0 \to I \cdot O_{\mathfrak{X}} \to O_{\mathfrak{X}} \to O_{X_0} \to 0$ of $O_{\mathfrak{X}}$-modules induces an exact sequence

$$H^1(\mathfrak{X}, O_{\mathfrak{X}}) \to H^1(X_0, O_{X_0}) \to H^2(\mathfrak{X}, I \cdot O_{\mathfrak{X}}) \xrightarrow{i} H^2(\mathfrak{X}, O_{\mathfrak{X}}).$$

From Remark 4.2, the logarithm isomorphism (12) induces an isomorphism

$$\log : H^2(\mathfrak{X}, 1 + I \cdot O_{\mathfrak{X}}) \xrightarrow{\sim} H^2(\mathfrak{X}, I \cdot O_{\mathfrak{X}}).$$

By [Liu25b, Lemma 3.2.5], there is an exact sequence

$$H^1(\mathfrak{X}, O_{\mathfrak{X}}^*) \to H^1(\mathfrak{X}, O_{X_0}^*) \xrightarrow{\partial} H^2(\mathfrak{X}, 1 + I \cdot O_{\mathfrak{X}}).$$

Then by Lemma 4.3, $\tau_{\mathrm{inf}}(c_{1,\mathrm{inf}}(L_0)) \in F^1 H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$ implies $i \circ \log \circ \partial(L_0) = 0$ in $H^2(\mathfrak{X}, O_{\mathfrak{X}})$. By (CF) condition in degree 1, the map $i : H^2(\mathfrak{X}, I \cdot O_{\mathfrak{X}}) \to H^2(\mathfrak{X}, O_{\mathfrak{X}})$ is injective. Thus, one has $\partial(L_0) = 0$ in $H^2(\mathfrak{X}, 1 + I \cdot O_{\mathfrak{X}})$, so L_0 lifts to a line bundle on $\mathfrak{X}$. □

Theorem 4.5. *Let S be as in Setup 3.8. Assume that S is of characteristic 0. Let $f : X \to S$ be a proper morphism of schemes. Assume that the map*

$$H^1(X, O_X) \twoheadrightarrow H^1(X_0, O_{X_0}) \tag{13}$$

is surjective. Let $L_0 \in \mathrm{Pic}(X_0)$. Then L_0 lifts to $\mathrm{Pic}\, X$ if and only if the image of $c_{1,\mathrm{inf}}(L_0)$ under the morphism $\tau_{\mathrm{inf}} : H^2_{\mathrm{inf}}(X_0/\hat{S}) \to H^2_{\mathrm{dR}}(X/S)$ is in the Hodge filtration $F^1 H^2_{\mathrm{dR}}(X/S)$.

Proof. It follows from Theorem 4.4 and [Liu25b, Lemma 3.2.1 *ii*)]. □

Remark 4.6. The assumption that (13) is surjective is necessary in Theorem 4.5.

4.2 Lifting line bundles for cohomologically flat morphism

As an application of Theorem 4.5, we deduce the key ingredient Lemma 4.12 used in the proof of Theorem 1.1. Roughly speaking, it gives a criterion of when a line bundle on a closed fiber of a cohomologically flat morphism of algebraic varieties admits a global lift. We first review basic properties of cohomological flatness, and explain how it is related to (CF) condition that we introduce.

Definition 4.7. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Let F be a coherent sheaf on X. For an integer $q \geq 0$, F is *cohomologically flat* in degree q if the formation of $R^q f_* F$ commutes with base change in Y. If O_X is cohomologically flat in degree q, then $f : X \to Y$ is called *cohomologically flat* in degree q.

By definition, if $f : X \to Y$ is cohomologically flat in degree q, then for every locally Noetherian scheme Y' and every morphism $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is also cohomologically flat in degree q.

Remark 4.8. Let S be as in Setup 3.8. Fix an integer $q \geq 0$. Let $f : X \to S$ be a proper morphism cohomologically flat in degree q. Then for every integer $n \geq 0$, the canonical map $H^q(X, O_X) \otimes_A A/I^{n+1} \to H^q(X_n, O_{X_n})$ is an isomorphism, which implies surjectivity of (13) when $q = 1$. Let $\hat{f} : \mathfrak{X} \to \hat{S}$ be the induced morphism of formal schemes. Then by [EGA III 1, Théorème 4.1.5], it satisfies (CF) in degree q.

Remark 4.9. Let $f : X \to Y$ be a *smooth* proper morphism of schemes of characteristic 0. Then by Deligne's theorem [Del68, Théorème 5.5 (i)], $f : X \to Y$ is cohomological flat in all degrees. Similarly, every smooth proper morphism $g : \mathfrak{X} \to \hat{S}$ of formal schemes satisfies (CF) in all degrees.

Lemma 4.10 shows that generic cohomological flatness holds.

Lemma 4.10. *Let $f : X \to Y$ be a proper morphism of schemes, with Y reduced and Noetherian. Let F be a coherent sheaf on X. Then there is an open dense subset U of Y, such that $F|_{X_U}$ is cohomologically flat over U in all degrees and flat over U.*

Proof. By [Sta25, Tag 08E2], as $f : X \to Y$ is proper and Y is Noetherian, one has $Rf_*F \in D^b_{\mathrm{coh}}(O_Y)$. From generic flatness (see, e.g., [Sta25, Tag 052B]), as Y is reduced, shrinking Y one may assume that F is flat over Y, and that for every $q \geq 0$, $R^q f_* F$ is flat over O_Y.

Shrinking Y, one may assume that $Y = \operatorname{Spec} A$ is affine. Then by [MRM74, Theorem, p.46], as A is Noetherian, there is a finite complex $K^\bullet$ of finite projective A-modules, such that for every $q \geq 0$, there is an isomorphism

$$H^q(X \times_Y \operatorname{Spec} B, F \otimes_A B) \xrightarrow{\sim} H^q(K^\bullet \otimes_A B)$$

of functors on the category of A-algebras B. In particular, one has $H^q(K^\bullet) \cong H^q(X, F)$. As Y is affine, one has $H^q(X, F) = \Gamma(Y, R^q f_* F)$. Therefore, $H^q(K^\bullet)$ is flat over A. Every projective module is flat, so $K^\bullet$ is a complex of flat A-modules. Then by Lemma 4.11, one has an isomorphism

$$H^q(K^\bullet) \otimes_A B \xrightarrow{\sim} H^q(K^\bullet \otimes_A B)$$

and hence an isomorphism

$$H^q(X, F) \otimes_A B \xrightarrow{\sim} H^q(X \times_Y \operatorname{Spec} B, F \otimes_A B). \tag{14}$$

To prove that F is cohomologically flat over Y in degree q, let $g : Y' \to Y$ be a morphism of schemes . Consider the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

and we show that the base change morphism $g^* R^q f_* F \to R^q f'_* g'^* F$ is an isomorphism. This property is local in Y', so one may assume that $Y' = \operatorname{Spec} B$ is affine. Both sheaves are quasi-coherent over Y'. One has

$$\Gamma(Y', g^* R^q f_* F) = H^q(X, F) \otimes_A B, \quad \Gamma(Y', R^q f'_* g'^* F) = H^q(X \times_Y \operatorname{Spec} B, F \otimes_A B),$$

so (14) proves the result. □

Lemma 4.11. *Let R be a commutative ring. Let $K^\bullet \in \mathrm{Ch}^+(R)$ be a bounded above complex of flat R-modules. Suppose that $H^q(K)$ is flat over R for every integer q. Then for every R-module M, the natural morphism $H^i(K) \otimes_R M \to H^i(K \otimes_R M)$ is an isomorphism for every integer i.*

Proof. As each term of $K^\bullet$ is flat, one has $K^\bullet \otimes^L M = K^\bullet \otimes M$. Similarly, as every $H^q(K)$ is flat, one has $H^q(K) \otimes^L M = H^q(K) \otimes M$. By [Sta25, Tag 0662], as $K^\bullet$ is bounded above, there is a spectral sequence

$$E_2^{p,q} = H^p(H^q(K) \otimes^L M) \Rightarrow H^{p+q}(K \otimes^L L).$$

Since $E_2^{p,q} = 0$ whenever $p \neq 0$, the spectral sequence degenerates at page E_2. It implies $H^i(K \otimes M) = H^0(H^i(K) \otimes^L M) = H^i(K) \otimes M$. □

Lemma 4.12. *Let $f : X \to B$ be a proper morphism of algebraic varieties over a field k of characteristic 0. Assume that $f : X \to B$ is cohomologically flat in degree 1. Fix a closed point $0 \in B$. Let $L \in \operatorname{Pic}(X_0)$. Set $A := \hat{O}_{B,0}$. Let $\hat{S} := \operatorname{Spf}(A)$ and $\hat{X} := X \times_B \hat{S}$. Suppose that $c_{1,\mathrm{inf}}(L)$ is in the kernel of the composition*

$$H^2_{\mathrm{inf}}(X_0/\hat{S}) \overset{\tau_{\mathrm{inf}}}{\to} H^2_{\mathrm{dR}}(\hat{X}/\hat{S}) \to H^2(\hat{X}, O_{\hat{X}}).$$

Then there is a connected open neighborhood B_0 of 0 in B, a quasi-finite étale morphism $\pi : U \to B_0$ with U connected, a point $u \in U$ over 0 with $k(0) \xrightarrow{\sim} k(u)$, such that letting $X_U := X \times_B U$, there is $L' \in \operatorname{Pic}(X_U)$ with $L|_{X_u} = L'|_{X_u}$ in $\operatorname{Pic}(X_u)$.

*Proof.*Step 1 We extend L over A.

The Noetherian local k-algebra A is complete. Let $S := \operatorname{Spec} A$, $X_S := X \times_B S$. Because $f : X \to B$ is cohomologically flat in degree 1, so is the projection $f_S : X_S \to S$. Then by Remark 4.8, $H^1(X_S, O_{X_S}) \to H^1(X_0, O_{X_0})$ is surjective. Therefore, by Theorem 4.5 and the supposition, as $\operatorname{char} k = 0$ and $f_S : X_S \to S$ is proper, there is $L_1 \in \operatorname{Pic}(X_S)$ with $L_1|_{X_0} \cong L$.

Step 2 We extend L over the henselization $O^h_{B,0}$ of $O_{B,0}$.

By [Sta25, Tag 07QU], $O_{B,0}$ is an excellent ring. Then by Greco's theorem [Sta25, Tag 0AH3], $O^h_{B,0}$ is a G-ring. By [Sta25, Tag 03QM], the inclusion $O^h_{B,0} \to A$ is the completion map, so it is a regular map of rings. Consequently, by Popescu's theorem [Sta25, Tag 07GC], one may write

$$A = \operatorname{colim}_{i \in I} A_i$$

as a filtered colimit of smooth $O^h_{B,0}$-algebras. For every $i \in I$, let $X_i = X \times_B \operatorname{Spec} A_i$. Then $X_S = \lim_i X_i$. By [Sta25, Tag 0B8W (2)], there is $i_0 \in I$ and a line bundle $L_2 \in \operatorname{Pic}(X_{i_0})$ with $L_2|_{X_S} \cong L_1$.

From [Mil80, Exercise 4.13], as $O^h_{B,0} \to A_{i_0}$ is smooth and $O^h_{B,0}$ is Henselian, there is a morphism $\sigma : A_{i_0} \to O^h_{B,0}$ of $O^h_{B,0}$-algebras fitting into a commutative diagram

$$\begin{tikzcd}
A_{i_0} \arrow[r, dashed, "\sigma"] \arrow[d] & O^h_{B,0} \arrow[d, two heads] \\
A \arrow[r, two heads] & k(0).
\end{tikzcd}$$

Let $X^h := X \times_B \operatorname{Spec} O^h_{B,0}$. Then σ induces a morphism $\phi_\sigma : X^h \to X_{i_0}$ of schemes, depicted in the commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] \arrow[rrr, hook, bend left] & X_S \arrow[r] \arrow[d, "f_S"'] & X^h \arrow[r, dashed, "\phi_\sigma"'] \arrow[d] & X_{i_0} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} k(0) \arrow[r, hook] & S \arrow[r] & \operatorname{Spec} O^h_{B,0} \arrow[r, dashed] & \operatorname{Spec} A_{i_0} \arrow[r] & B,
\end{tikzcd}$$

such that $\phi_\sigma|_{X_0} : X_0 \to X_{i_0}$ is the inclusion. Therefore, $L_3 := \phi_\sigma^* L_2 \in \operatorname{Pic}(X^h)$ satisfies $L_3|_{X_0} \cong L$.

Step 3 We extend L_3 over an étale neighborhood of 0 in B.

From [EGA I, Lemme 6.1.8], shrinking B to an open neighborhood of 0, one may assume that every open neighborhood of 0 in B is connected. Let $\mathcal{N}$ be the category of elementary étale neighborhoods $(V, v) \to (B, 0)$ with V affine. By the proof of [Sta25, Tag 05KS], one has

$$O^h_{B,0} = \operatorname{colim}_{(V,v) \in \mathcal{N}^{\mathrm{op}}} O(V).$$

Then $\operatorname{Spec} O^h_{B,0} = \lim_{(V,v) \in \mathcal{N}} V$. Therefore, there is an elementary étale neighborhood $\pi : (U, u) \to (B, 0)$, a line bundle $L' \in \operatorname{Pic}(X_U)$ with $L'|_{X^h} \cong L_3$. Then $L'|_{X_u} = L|_{X_u}$. Shrinking B and U to open neighborhoods of 0 and u respectively, one may assume that every open neighborhood of u in U is connected and that $\pi : U \to B$ is of finite type. Then by [Sta25, Tag 03WS], the étale morphism $U \to B$ is quasi-finite.

□

5 Initial value problem and Gauss-Manin connection

5.1 Sections horizontal for an integrable connection

Recall that our strategy to Theorem 1.1 is using the Gauss-Manin connection to construct the power series defining the jumping locus. Given a smooth proper family of algebraic varieties with good reduction, Lawrence and Venkatesh [LV20, Sec. 3.3] study the formal solutions to the equation induced by the Gauss-Manin connection. We slightly extend their work to integrable connections not coming from a smooth morphism.

Let R be a discrete valuation ring (DVR) or a field. Let K be the fraction field of R. Let k be the residue field of R. Let $\phi : \mathcal{Y} \to \mathbf{A}_R^m$ be an étale morphism of schemes over R. It induces a morphism $R[Z_1, \ldots, Z_m] \to \Gamma(\mathcal{Y}, O_{\mathcal{Y}})$ of R-algebras. Let $z_i \in \Gamma(\mathcal{Y}, O_{\mathcal{Y}})$ be the image of Z_i ($i = 1, \ldots, m$). By [Sta25, Tags 06B6 and 02H9], one has $\Omega^1_{\mathcal{Y}/R} = \oplus_{i=1}^m O_{\mathcal{Y}} dz_i$. Let $y_0 \in \mathcal{Y}(R)$ be a preimage of $0 \in \mathbf{A}_R^m(R)$. Let $\overline{y_0} \in \mathcal{Y}(k)$ be its reduction. Let Y be the generic fiber of $\mathcal{Y} \to \operatorname{Spec} R$. Let $\tilde{y}_0 \in Y(K)$ be the point corresponding to y_0.

Lemma 5.1. *There is a natural injective morphism $O_{\mathcal{Y},\overline{y_0}} \hookrightarrow R[[z_1, \ldots, z_m]]$ of R-algebras.*

Proof. The image of $\overline{y_0}$ under $\phi : \mathcal{Y} \to \mathbf{A}_R^m$ is $\bar{0} \in \mathbf{A}_R^m(k)$. The morphism of residue fields $k \to \kappa(\overline{y_0})$ is an isomorphism. From [Sta25, Tag 039N], as R is Noetherian, $O_{\mathbf{A}_R^m,\bar{0}} \to O_{\mathcal{Y},\overline{y_0}}$ is an étale homomorphism of local rings. By [Sta25, Tag 039M], as it is essentially of finite type, it induces an isomorphism on completions $\hat{O}_{\mathbf{A}_R^m,\bar{0}} \xrightarrow{\sim} \hat{O}_{\mathcal{Y},\overline{y_0}}$. Then $\hat{O}_{\mathbf{A}_R^m,\bar{0}} = \hat{R}[[Z_1, \ldots, Z_m]]$. Similarly, one has $\hat{O}_{Y,\tilde{y}_0} = K[[z_1, \ldots, z_m]]$. Since $\overline{y_0}$ is a specialization of $\tilde{y}_0$ on $\mathcal{Y}$, there is a localization $O_{\mathcal{Y},\overline{y_0}} \to O_{\mathcal{Y},\tilde{y}_0}$. As $\mathcal{Y}$ is smooth over R, it is regular. By [Sta25, Tag 00NP], $O_{\mathcal{Y},\overline{y_0}}$ is a domain. Therefore, $O_{\mathcal{Y},\overline{y_0}} \to O_{\mathcal{Y},\tilde{y}_0}$ is injective. Because R is a DVR, Y is open in $\mathcal{Y}$ and hence $O_{Y,\tilde{y}_0} = O_{\mathcal{Y},\tilde{y}_0}$. From $K \cap \hat{R} = R$, one has $O_{\mathcal{Y},\overline{y_0}} \subset \hat{R}[[z_1, \ldots, z_m]] \cap K[[z_1, \ldots, z_m]] = R[[z_1, \ldots, z_m]]$. □

Let $\mathcal{H}$ be a finite free $O_{\mathcal{Y}}$-module. Choose a basis $\{v_1, \ldots, v_r\} \subset \Gamma(\mathcal{Y}, \mathcal{H})$ for it. Let $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{O_{\mathcal{Y}}} \Omega^1_{\mathcal{Y}/R}$ be an integrable connection relative to R. For every $1 \leq i \leq r$, write $\nabla v_i = \sum_{j=1}^r v_j \otimes A_i^j$, where $A_i^j \in \Gamma(\mathcal{Y}, \Omega^1_{\mathcal{Y}/R})$. A local section $s := \sum_{i=1}^r f^i v_i$ of $\mathcal{H}$ (with f^i local sections of $O_{\mathcal{Y}}$) is horizontal exactly when

$$df^i + \sum_{j=1}^r A_j^i f^j = 0 \text{ for all } 1 \leq i \leq r. \tag{15}$$

Lemma 5.2. *Assume that $R = K$ is a field of characteristic 0. Then for every $s_0 = \sum_{i=1}^r f_0^i v_i(y_0)$ in the K-vector space $\mathcal{H}(y_0)$, there exists a unique formal solution $f^1, \ldots, f^r \in K[[z_1, \ldots, z_m]]$ to* (15) *with initial condition $f^i(0) = f_0^i$ for all $1 \leq i \leq r$. Let $M := \mathcal{H}(\tilde{y}_0) \otimes_{O_{\mathcal{Y},\tilde{y}_0}} \hat{O}_{\mathcal{Y},\tilde{y}_0}$. Thus, one gets an injective K-linear map $\mathcal{H}(y_0) \hookrightarrow M$ which is independent of the choice of the basis $\{v_1, \ldots, v_r\}$.*

Proof. The $K[[z_1,\dots,z_m]]$-module M is finite free and equipped with an *integrable* connection for the continuous K-derivations on $K[[z_1,\dots,z_m]]$. Let M^∇ be the set of horizontal elements of M. By [Kat70, Proposition 8.9], as K is of characteristic 0, the composition

$$M^\nabla \hookrightarrow M \twoheadrightarrow M \otimes_{K[[z_1,\dots,z_m]]} K = \mathcal{H}(y_0)$$

is an isomorphism. Therefore, there exists a unique $s \in M^\nabla$ lying over s_0. As $\mathcal{H}(\tilde{y}_0) = \oplus_{i=1}^r O_{\mathcal{Y},\tilde{y}_0} v_i$, one has $M = \oplus_{i=1}^r K[[z_1,\dots,z_m]]v_i$. Then one can write s as $\sum_{i=1}^r f^i v_i$ in a unique way. □

Let $\hat{K}$ be the fraction field of $\hat{R}$, which is a complete DVF. Let $|\cdot| : \hat{K} \to [0,+\infty)$ be an associated nonarchimedean absolute value.

Lemma 5.3. *Suppose that R is a DVR of mixed characteristic $(0,p)$. Then the formal solution in Lemma 5.2 for $\phi_K : Y \to A_K^m$ absolutely converges on the open polydisc $P(0,|p|^{1/(p-1)}) := \{(z_1,\dots,z_m) \in \hat{K}^m : |z_1|,\dots,|z_m| < |p|^{1/(p-1)}\}$.*

Proof. Set $C := \max_{1\le i\le r} |f_0^i|$. For a multi-index $I = (i_1,\dots,i_m) \in \mathbb{Z}_{\ge 0}^m$, denote

$$|I| := \sum_{j=1}^m i_j, \quad I! := i_1!\dots i_m!, \quad z^I := z_1^{i_1}\dots z_m^{i_m}.$$

We write $f^i = \sum_{I\in\mathbb{Z}_{\ge0}^m} f_I^i z^I$ with $f_I^i \in K$. Write $A_i^j = \sum_{s=1}^m a_i^{j,s} dz_s$ with $a_i^{j,s} \in \Gamma(\mathcal{Y}, O_\mathcal{Y})$. By Lemma 5.1, one has $(a_i^{j,s})_{\overline{y_0}} \in R[[z_1,\dots,z_m]]$. Then write $(a_i^{j,s})_{\overline{y_0}} = \sum_{I\in\mathbb{Z}_{\ge0}^m} (a_i^{j,s})_I z^I$ with $(a_i^{j,s})_I \in R$.

By induction, we prove $\max_{1\le i\le r} |f_I^i| \le \frac{C}{|I!|}$ for all $I \in \mathbb{Z}_{\ge0}^m$. By choice of C, it holds when $I = \vec{0}$. Fix $I = (i_1,\dots,i_m)$ and assume the inequalities for all multi-indices $J \le I$. Given $1 \le s \le m$, we proceed to show the inequality for $I + 1_s := (i_1,\dots,i_{s-1},i_s+1,i_{s+1},\dots,i_m)$. Taking the coefficient of $z^I dz_s$ in (15), one has

$$(1+i_s) f_{I+1_s}^i + \sum_{j=1}^r \sum_{\substack{J+L=I,\\ J,L\in\mathbb{Z}_{\ge0}^m}} (a_j^{i,s})_J f_L^j = 0$$

for every $1 \le i \le r$. For each $L \le I$, one has $L!|I!$, so $|I!| \le |L!|$. By the inductive hypothesis, one has

$$|(a_j^{i,s})_J f_L^j| \le |f_L^j| \le \frac{C}{|L!|} \le \frac{C}{|I!|}.$$

As $(\hat{K}, |\cdot|)$ is non-archimedean, it implies

$$|f_{I+1_s}^i| \le \frac{C}{|I!|\cdot|i_s+1|} = \frac{C}{|(I+1_s)!|}.$$

The induction is completed. By Legendre's formula, for every integer $n > 0$, one has $|n!| > |p|^{\frac{n}{p-1}}$, so $\frac{1}{|I!|} < |p|^{-\frac{|I|}{p-1}}$. For every point $(z_1, \ldots, z_m)$ in the disk and every $1 \leq i \leq r$, one has

$$|f_I^i z^I| < C\left(\frac{\max_{1\leq i\leq m}|z_i|}{|p|^{\frac{1}{p-1}}}\right)^{|I|}.$$

It implies $|f_I^i z^I| \to 0$ when $|I| \to +\infty$. Because $\hat{K}$ is a complete valued field, the result follows. □

Remark 5.4. Assume $R = \mathbb{C}$. By the Riemann-Hilbert correspondence [Del70, Théorème 2.17], there is an open neighborhood Ω of y_0 in Y^{an}, such that for every initial value $s_0 \in \mathcal{H}(y_0)$, the formal solution in Lemma 5.2 converges absolutely on Ω and defines an element of $H^0_{\mathrm{DR}}(\Omega, (\mathcal{H}, \nabla)^{\mathrm{an}})$.

5.2 Gauss-Manin connection for singular morphisms

We review Hartshorne's de Rham cohomology sheaf and the Gauss-Manin connection on it. Given an integer $i \geq 0$ and a morphism $f : X \to B$ of finite type of Noetherian schemes of characteristic 0, set

$$\mathcal{H}^i_{\mathrm{dR}}(X/B) := R^i f_* \Omega^\bullet_{X/B}.$$

Hartshorne [Har75, Definition, p.74] introduces the relative de Rham cohomology sheaf $\mathcal{H}^i_{\mathrm{DR}}(X/B)$, which is an O_B-module admitting a natural O_B-linear morphism $\mathcal{H}^i_{\mathrm{DR}}(X/B) \to \mathcal{H}^i_{\mathrm{dR}}(X/B)$. For an embeddable $f : X \to B$, i.e., when there is a closed immersion $i : X \hookrightarrow Y$ over B with Y a scheme smooth over B, denote by $\hat{\Omega}^\bullet_{Y/B}$ the formal completion of the de Rham complex $\Omega^\bullet_{Y/B}$ along X. In this case, $\mathcal{H}^i_{\mathrm{DR}}(X/B)$ is defined as $R^i f_* \hat{\Omega}^\bullet_{Y/B}$. In particular, if $f : X \to B$ is smooth, then $\mathcal{H}^i_{\mathrm{DR}}(X/B) \to \mathcal{H}^i_{\mathrm{dR}}(X/B)$ is an isomorphism. For a general $f : X \to B$, the definition of $\mathcal{H}^i_{\mathrm{DR}}(X/B)$ can be globalized by a Čech process similar to [Har75, pp.28–29]. By [GD71, 10.8.7], formal completion along X coincides with that along X_{red}. Therefore, $\mathcal{H}^i_{\mathrm{DR}}(X/B) \to \mathcal{H}^i_{\mathrm{DR}}(X_{\mathrm{red}}/B)$ is an isomorphism, so one can remove the reducedness assumption on X in [Har75, III, Theorem 5.1] to get Fact 5.5.

Fact 5.5. *Let k be a field of characteristic 0, B be a reduced scheme of finite type over k, and $f : X \to B$ be a finite type morphism. Then there is a dense open subset $U \subset B$ such that every $\mathcal{H}^i_{\mathrm{DR}}(X/B)$ is finite locally free over U.*

If B is smooth of finite type over a field k of characteristic 0, even if $f : X \to B$ is singular, Hartshorne [Har75, p.75] defines a canonical Gauss-Manin connection $\nabla : \mathcal{H}^i_{\mathrm{DR}}(X/B) \to \mathcal{H}^i_{\mathrm{DR}}(X/B) \otimes_{O_B} \Omega^1_{B/k}$ which is integrable.

Lemma 5.6. *Let $f : X \to B$ be a morphism of algebraic varieties over $\mathbb{C}$, with B smooth over $\mathbb{C}$. Fix an integer $n \geq 0$, and assume that $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is coherent over O_B.*

(a) *Then $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is locally free, and its Gauss-Manin connection ∇ has regular singularities.*

(b) *If $R^n f^{\mathrm{an}}_* \mathbb{C}$ is a local system on B^{an}, then it corresponds to $(\mathcal{H}^n_{\mathrm{DR}}(X/B), \nabla)$ under Deligne's Riemann-Hilbert correspondence. In particular, the canonical morphism*

$$H^*_{\mathrm{DR}}(B, \mathcal{H}^n_{\mathrm{DR}}(X/B), \nabla) \to H^*(B^{\mathrm{an}}, R^n f^{\mathrm{an}}_* \mathbb{C})$$

is an isomorphism.

(c) *Up to shrinking B to a dense open subset, the formation of $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ commutes with any base change in B.*

Proof. (a) and (c) By [Kat70, Proposition 8.8], as $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is coherent equipped with an integrable connection, it is locally free. As B is smooth, one may assume that B is irreducible. For every dense open subset U of B, the morphism $\pi_1(U^{\mathrm{an}}) \to \pi_1(B^{\mathrm{an}})$ is surjective. So to determine the local system corresponding to $(\mathcal{H}^n_{\mathrm{DR}}(X/B), \nabla)$, one may shrink B to a dense open subset. From [Del70, II, Proposition 4.6 (iii)], to prove regularity, one may also shrink B. Let $\eta \in B$ be the generic point. Let $n := \dim X_\eta$. The isomorphism $\mathcal{H}^n_{\mathrm{DR}}(X/B) \xrightarrow{\sim} \mathcal{H}^n_{\mathrm{DR}}(X_{\mathrm{red}}/B)$ is compatible with the Gauss-Manin connection. Let f_{red} be the composition $X_{\mathrm{red}} \to X \to B$. As $X^{\mathrm{an}}_{\mathrm{red}} \to X^{\mathrm{an}}$ is a homeomorphism, $R^n f^{\mathrm{an}}_* \mathbb{C} = R^n f^{\mathrm{an}}_{\mathrm{red}*} \mathbb{C}$ as sheaves on B^{an}. Therefore, one may assume that X is reduced. From [Sta25, Tag 054Z], as X is reduced, so is X_η. By [Gui+88, I, Théorème 2.15] (see also [PS08, Theorem 5.26]), since $k(\eta)$ is of characteristic 0 and X_η is reduced, separated of finite type over $k(\eta)$, there is an $(n+1)$-cubical hyperresolution (in the sense of [Gui+88, I, Définition 2.12]) (X_I) of X_η with $\dim X_I \leq n + 1 - |I|$ for all I.

Let $\epsilon_\eta : X_{\eta,\bullet} \to X_\eta$ be the associated semi-simplicial resolution. By cohomological descent [Gui+88, III, Théorème 1.3], for every $i \geq 0$, the canonical morphism $H^i_{\mathrm{DR}}(X_\eta/\eta) \to H^i_{\mathrm{DR}}(X_{\eta,\bullet}/\eta)$ is an isomorphism. By definition, every $X_{\eta,p}$ is proper over X_η and smooth over $k(\eta)$. There are only finitely many $p \geq 0$ with $X_{\eta,p}$ nonempty. Therefore, by shrinking B and spreading out, one may assume that there is a cubical variety $X_\bullet$ over $\mathbb{C}$ with an augmentation $\epsilon : X_\bullet \to X$, such that

1) there are only finitely many p with X_p nonempty;

2) for every $p \geq 0$, $X_p \to X$ is proper, and the structure morphism $f_p : X_p \to B$ is smooth;

3) the pullback of $\epsilon : X_\bullet \to X$ along $X_\eta \hookrightarrow X$ is $\epsilon_\eta : X_{\eta,\bullet} \to X_\eta$;

4) for every $s \in B(\mathbb{C})$, $X_\bullet \times_B s \to X_s$ is a cubical hyperresolution.

From [Del70, II, Théorème 6.13 and 7.9], by Assumptions *1*) and *2*), shrinking B one may assume that for all $p, q \geq 0$,

- $R^q f^{\mathrm{an}}_{p*}\mathbb{C}$ is a local system on B^{an}, and its formation commutes with any base change in B;
- $\mathcal{H}^q_{\mathrm{DR}}(X_p/B)$ is finite locally free, and its formation commutes with any base change in B;
- the Gauss-Manin connection on $\mathcal{H}^q_{\mathrm{DR}}(X_p/B)$ is regular, and the associated local system is $R^q f^{\mathrm{an}}_{p*}\mathbb{C}$.

By the spectral sequence

$$E_1^{p,q} = \mathcal{H}^q_{\mathrm{DR}}(X_p/B) \Rightarrow \mathcal{H}^{p+q}_{\mathrm{DR}}(X_\bullet/B)$$

of O_B-modules, $\mathcal{H}^{p+q}_{\mathrm{DR}}(X_\bullet/B)$ is coherent over O_B, and its formation commutes with any base change in B. Let $\mathrm{MIC}(B/\mathbb{C})$ be the category of O_B-modules with integrable connection. From [ABC20, 4.3.8], as B is smooth over $\mathbb{C}$, $\mathrm{MIC}(B/\mathbb{C})$ is an abelian category. By construction, the differential morphisms are horizontal for the Gauss-Manin connections, so it is a spectral sequence in the abelian category $\mathrm{MIC}(B/\mathbb{C})$. Then by [HT07, Proposition 5.3.4], for every $n \geq 0$, $\mathcal{H}^n_{\mathrm{DR}}(X_\bullet/B)$ is locally free and its Gauss-Manin connection is regular. From the spectral sequence

$$E_1'^{p,q} = R^q f^{\mathrm{an}}_{p*}\mathbb{C} \Rightarrow R^{p+q} f^{\mathrm{an}}_{\bullet*}\mathbb{C}_\bullet,$$

the local system associated with $\mathcal{H}^n_{\mathrm{DR}}(X_\bullet/B)$ is $R^n f^{\mathrm{an}}_{\bullet*}\mathbb{C}_\bullet$, and the formation of $R^n f^{\mathrm{an}}_{\bullet*}\mathbb{C}_\bullet$ commutes with any base change in B.

For every $n \geq 0$, the canonical morphism $\phi : \mathcal{H}^n_{\mathrm{DR}}(X/B) \to \mathcal{H}^n_{\mathrm{DR}}(X_\bullet/B)$ is O_B-linear and horizontal for Gauss-Manin connection. By Fact 5.5 and [Har75, III, Proposition 5.2], the generic fiber of $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is $H^n_{\mathrm{DR}}(X_\eta/\eta)$. Similarly, by Assumption *3*), the generic fiber of $\mathcal{H}^n_{\mathrm{DR}}(X_\bullet/B)$ is $H^n_{\mathrm{DR}}(X_{\eta,\bullet}/\eta)$. Therefore, ϕ restricts to an isomorphism of generic fibers. Let $\mathrm{Conn}(B)$ be the category of finite locally free O_B-module with an integrable connection. By [Sta25, Tag 01B9], shrinking B one may assume that ϕ is an isomorphism in $\mathrm{Conn}(B)$. Therefore, Part (c) holds, and the Gauss-Manin connection on $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is also regular.

(b) Up to shrinking B,

$$\phi : (\mathcal{H}^n_{\mathrm{DR}}(X/B), \nabla) \to (\mathcal{H}^n_{\mathrm{DR}}(X_\bullet/B), \nabla)$$

is an isomorphism in $\mathrm{Conn}(B)$. Therefore, the local system associated with $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ is also $R^n f^{\mathrm{an}}_{\bullet*}\mathbb{C}_\bullet$. It remains to prove that up to shrinking B to a dense Zariski open subset, the natural morphism

$$\psi : R^n f^{\mathrm{an}}_{*}\mathbb{C} \to R^n f^{\mathrm{an}}_{\bullet*}\mathbb{C}_\bullet$$

of sheaves on B^{an} is an isomorphism. From Deligne's generic base change theorem for complex varieties (see, e.g., [Cat09, Theorem 3.16]), shrinking

B one may assume that the formation of $R^n f^{\mathrm{an}}_* \mathbb{C}$ commutes with arbitrary base change. Then for every $s \in B(\mathbb{C})$,

$$\psi_s : (R^n f^{\mathrm{an}}_* \mathbb{C})_s \to (R^n f^{\mathrm{an}}_{\bullet *} \mathbb{C}_\bullet)_s$$

is identified with $H^n(X^{\mathrm{an}}_s, \mathbb{C}) \to H^n((X_\bullet \times_B s)^{\mathrm{an}}, \mathbb{C})$. By Assumption *4*) and cohomological descent [Gui+88, I, Théorème 6.9], it is an isomorphism. Therefore, ψ is also an isomorphism. Then from Part (a) and the comparison theorem [Del70, II, (6.1.1) and Théorème 6.2], $H^*_{\mathrm{DR}}(B, \mathcal{H}^n_{\mathrm{DR}}(X/B)) \to H^*(B^{\mathrm{an}}, R^n f^{\mathrm{an}}_* \mathbb{C})$ is an isomorphism.

□

Remark 5.7. In fact, Lemma 5.6 (c) holds over general fields. Let k be a field of characteristic 0. Let $f : X \to B$ be a morphism of algebraic varieties over k, with B reduced. Then for every $n \geq 0$, up to shrinking B to a dense open subset, the formation of $\mathcal{H}^n_{\mathrm{DR}}(X/B)$ commutes with any base change $B' \to B$ which is a morphism of algebraic varieties over k. We only sketch a proof. First, assume that $f : X \to B$ is smooth, so that $\mathcal{H}^n_{\mathrm{DR}}(X/B) = \mathcal{H}^n_{\mathrm{dR}}(X/B)$. Then the result follows from the generic base change theorem [ABC20, Theorem 26.2.1]. One can prove the general case using cohomological descent similarly to Lemma 5.6 (c).

6 Existence of NS-generic points

Given a normal proper morphism $f : X \to B$ over an algebraically closed field k of characteristic 0, we show the existence of an NS-generic point $b \in B(k)$. First, in Lemma 6.1 we show the existence of a point outside the zero loci of the convergent power series defining the jumping locus. Such power series are constructed from the Gauss-Manin connection on Hartshorne's relative de Rham cohomology sheaves. Then we show that such point are NS-generic (Lemma 6.4). The proof is a mixture of complex analytic and p-adic methods.

Let k be a field of characteristic 0. Let $f : X \to B$ be a proper morphism of algebraic varieties over k, with B is smooth over k and irreducible. By Fact 5.5, shrinking B one may assume that $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ is a finite free O_B-module. From Remark 5.7, shrinking B one may assume that the formation of $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ commutes with any base change in B. In particular, for every $b \in B$, the fiber $\mathcal{H}^2_{\mathrm{DR}}(X/B)(b)$ over b is $H^2_{\mathrm{DR}}(X_b/k(b))$.

By [Sta25, Tag 0FLY], as $f : X \to B$ is proper, $\mathcal{H}^i_{\mathrm{dR}}(X/B)$ is coherent over O_B. Let $F^1\mathcal{H}^2_{\mathrm{dR}}(X/B)$ (resp. $\mathcal{H}^{02}(X/B)$) be the image (resp. cokernel) of the morphism $R^2 f_* \Omega^{\geq 1}_{X/B} \to \mathcal{H}^2_{\mathrm{dR}}(X/B)$ of O_B-modules. Then both are coherent. The composition $\Omega^{\geq 1}_{X/B} \hookrightarrow \Omega^\bullet_{X/B} \to \Omega^\bullet_{X/B} \to O_X$ vanishes, so the natural morphism $\mathcal{H}^2_{\mathrm{dR}}(X/B) \to R^2 f_* O_X$ factors through $\mathcal{H}^{02}(X/B)$. By generic freeness, as B is integral, shrinking B one may assume that $\mathcal{H}^{02}(X/B)$ is finite free over O_B. Choose bases $\{v_1, \dots, v_r\} \subset \Gamma(B, \mathcal{H}^2_{\mathrm{DR}}(X/B))$ and $\{w_1, \dots, w_t\} \subset \Gamma(B, \mathcal{H}^{02}(X/B))$ for $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ and $\mathcal{H}^{02}(X/B)$ respectively.

For every $1 \le i \le r$, write $\sum_{j=1}^t g_i^j w_j$ for the image of v_i in $\Gamma(B, \mathcal{H}^{02}(X/B))$, with $g_i^j \in \Gamma(B, O_B)$.

From [Sta25, Tag 054L], for every $b \in B(k)$, as B is smooth at b, there is an affine open neighborhood B_0 of b in B and an étale morphism $(z_1, \dots, z_m) : B_0 \to \mathbf{A}_k^m$ mapping b to 0. We refer to such a choice as an *étale coordinate* centered at b. By $\mathrm{char} k = 0$ and Lemma 5.2, for every initial condition $s_0 \in H^2_{\mathrm{DR}}(X_b/k)$, there is a unique formal solution $(f^1, \dots, f^r)$ in $k[[z_1, \dots, z_m]]$ such that the formal section $\sum_{i=1}^r f^i v_i$ of $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ is horizontal for the Gauss-Manin connection and takes value s_0 at b. Define

$$\lambda^j := \sum_{i=1}^r f^i (g_i^j)_b \in k[[z_1, \dots, z_m]].$$

Then $\sum_{i=1}^r f^i v_i$ is mapped to the "formal" section $\sum_{j=1}^t \lambda^j w_j$ of $\mathcal{H}^{02}(X/B)$. By [Liu25a, Remark 11.5], Hartshorne's de Rham cohomology is identified with the infinitesimal cohomology.

Now assume that k is algebraically closed. For $b \in B(k)$, let

$$\Lambda_b \subset k[[z_1, \dots, z_m]]^t \cong \mathcal{H}^{02}(X/B) \otimes_{O_B} \hat{O}_{B,b}$$

be the subgroup of tuples $\lambda = (\lambda^1, \dots, \lambda^t)$ determined by initial conditions s_0 in the image of $c_{1,\inf} : \mathrm{Pic}(X_b) \to H^2_{\inf}(X_b/k)$. In Lemma 6.1, we find a point $b_0 \in B(k)$ such that for every $L \in \mathrm{Pic}(X_{b_0})$, the formal section of $\mathcal{H}^2_{\mathrm{dR}}(X/B)$ determined by the initial condition $c_{1,\inf}(L) \in H^2_{\mathrm{DR}}(X_{b_0}/k)$ is a formal section of $F^1\mathcal{H}^2_{\mathrm{dR}}(X/B)$.

Lemma 6.1. *Let k be an algebraically closed field of characteristic* 0. *Let B be a smooth irreducible variety over k with generic point η. Let $f : X \to B$ be a proper morphism of schemes. Then there is $b_0 \in B(k)$ with the following property. For every étale coordinate $(\xi_1, \dots, \xi_m) : B_0 \to \mathbf{A}_k^m$ centered at b_0, one has $\Lambda_{b_0} = 0$ in $k[[\xi_1, \dots, \xi_m]]^t$.*

Proof. Shrinking B, one may assume that B is affine. Choose a prime ℓ. Shrinking B, one may assume that $R^2 f_* \mathbb{Q}_\ell$ is a lisse sheaf on B. By generic flatness and Katz's generic base change theorem [Kat70, Thm 8.0], as $f : X \to B$ is proper, shrinking B, one may assume that $\mathcal{H}^2_{\mathrm{dR}}(X/B)$ and $\mathcal{H}^{02}(X/B)$ are finite free over O_B and their formation commutes with base change in B. In particular, for every $b \in B$, there is a short exact sequence

$$0 \to F^1 H^2_{\mathrm{dR}}(X_b/k(b)) \to H^2_{\mathrm{dR}}(X_b/k(b)) \to \mathcal{H}^{02}(X/B)(b) \to 0. \tag{16}$$

Step 1 We find an integral model of the Gauss-Manin connection.

As k is algebraically closed, one may choose $b_1 \in B(k)$. Then choose an étale coordinate $\phi = (z_1, \dots, z_m) : B \to \mathbf{A}_k^m$ centered at b_1. Choose a finitely generated $\mathbb{Z}$-subalgebra A of k, such that

- B is the base change along $A \hookrightarrow k$ of an irreducible scheme B_A smooth separated of finite type over A;

- $f : X \to B$ and $\phi : B \to \mathbf{A}_k^m$ are the base change of a proper morphism $X_A \to B_A$ and an étale morphism $B_A \to \mathbf{A}_A^k$ over A respectively;
- b_1 is also defined over A, and that there is a finite free O_{B_A}-module $\mathcal{H}$ with an integrable connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{O_{B_A}} \Omega^1_{B_A/A}$, whose pullback to B is $\mathcal{H}^2_{\mathrm{DR}}(X/B)$ with the Gauss-Manin connection.

Then the fraction field F of A is a finitely generated extension of $\mathbb{Q}$. Choose a basis $\{v_1, \dots, v_r\}$ of $\mathcal{H}$ and $\{w_1, \dots, w_t\}$ of $\mathcal{H}^{02}(X/B)$.

Step 2 We consider a complex place and find a complex analytic open neighborhood Ω of b_1 trivializing the Gauss-Manin connection.

Let $\bar{F}$ be the algebraic closure of F in k. Given a point $b \in B(\bar{F})$, let $b' \in B(k)$ be the corresponding point. By Lemma 3.1 and Fact 1.8, one has $\Lambda_b = \Lambda_{b'}$. Therefore, one may shrink k to $\bar{F}$. Then there is a field embedding $\iota : k \hookrightarrow \mathbb{C}$. From Remark 6.2, as $f : X \to B$ is proper, $R^2 f_*^{\mathrm{an}}\mathbb{Q}$ is a local system on B^{an}. Then by Lemma 5.6 (b), the local system $R^2 f_*^{\mathrm{an}}\mathbb{C}$ corresponds to $(\mathcal{H}^2_{\mathrm{DR}}(X/B), \nabla)$ under Deligne's Riemann-Hilbert correspondence [Del70, Théorème 5.9]. By the proper base change theorem, as $f : X \to B$ is proper, one has $(R^2 f_*^{\mathrm{an}}\mathbb{C})_b = H^2(X_b^{\mathrm{an}}, \mathbb{C})$ for every $b \in B(\mathbb{C})$.

By [SGA1, XII, Remarque 3.3 a)], because $(z_1, \dots, z_m) : B \to \mathbf{A}_k^m$ is an étale morphism, there is an open neighborhood Ω of b_1 in B^{an} such that $(\Omega, z_1, \dots, z_m)$ is a holomorphic chart in the complex manifold B^{an}. Shrinking Ω one may assume that it is identified with a polydisc in $\mathbb{C}^m$ with center 0. In particular, Ω is connected and simply connected. Thus, for every $b \in \Omega$,

$$H^0_{\mathrm{DR}}(\Omega, \mathcal{H}^2_{\mathrm{DR}}(X/B)^{\mathrm{an}}, \nabla) = \Gamma(\Omega, R^2 f_*^{\mathrm{an}}\mathbb{C}) \xrightarrow{\sim} H^2(X_b, \mathbb{C})$$

is an isomorphism. Every $s_0 \in H^2_{\mathrm{DR}}(X_b/\mathbb{C}) = H^2(X_b, \mathbb{C})$ extends to a unique horizontal section $s \in H^0_{\mathrm{DR}}(\Omega, \mathcal{H}^2_{\mathrm{DR}}(X/B)^{\mathrm{an}})$. Write $s = \sum_{i=1}^r f^i v_i$, where every $f^i : \Omega \to \mathbb{C}$ is a holomorphic function. Let $\sum_{j=1}^t \lambda^j w_j$ be the formal section of $\mathcal{H}^{02}(X/B)$ determined by the initial condition s_0. Then for every $1 \leq j \leq t$, $\lambda^j \in \mathbb{C}[[z_1, \dots, z_m]]$ is the Taylor expansion of the holomorphic function $\sum_{i=1}^r f^i g_i^j : \Omega \to \mathbb{C}$ at b. When $b = b_1$, as Ω is a polydisc with center b_1, λ^j converges to $\sum_{i=1}^r f^i g_i^j$ at every point of Ω.

Step 3 We add periods defined by X_{b_1} into k, to get a field containing the coefficients of the formal solutions with initial value in the Betti cohomology.

Since $H^2(X_{b_1}^{\mathrm{an}}, \mathbb{Q})$ is of finite dimension over $\mathbb{Q}$, we may choose a basis $\{\gamma_1, \dots, \gamma_d\}$ of it. Using the comparison isomorphism [Har75, III, Proposition 5.2 and IV, Theorem 1.1]

$$H^2_{\mathrm{DR}}(X_{b_1}/k) \otimes_k \mathbb{C} \xrightarrow{\sim} H^2_{\mathrm{DR}}(X_{b_1}/\mathbb{C}) \cong H^2(X_{b_1}^{\mathrm{an}}, \mathbb{C}),$$

we write each γ_i as a finite sum $\sum_{j=1}^{N_i} \delta_{ij} \otimes a_{ij}$ with $\delta_{ij} \in H^2_{\mathrm{DR}}(X_{b_1}/k)$ and $a_{ij} \in \mathbb{C}$. Let $A' \subset \mathbb{C}$ be a finitely generated $\mathbb{Z}$-subalgebra containing A and all the a_{ij}. Let $F' \supset F$ be its fraction field. Let $k' \supset k$ be the algebraic closure of F' in $\mathbb{C}$. The inclusion $H^2(X^{\mathrm{an}}_{b_1}, \mathbb{Q}) \hookrightarrow H^2(X^{\mathrm{an}}_{b_1}, \mathbb{C})$ maps the γ_i inside $H^2_{\mathrm{DR}}(X_{b_1}/k') \subset H^2_{\mathrm{DR}}(X_{b_1}/k') \otimes_{k'} \mathbb{C}$. From $\mathbb{Q} \subset k'$, one has $H^2(X^{\mathrm{an}}_{b_1}, \mathbb{Q}) \subset H^2_{\mathrm{DR}}(X_{b_1}/k')$. Let

$$\Lambda \subset k'[[z_1, \ldots, z_m]]^t$$

be the subset consisting of $\lambda = (\lambda^1, \ldots, \lambda^t)$ determined by initial values $s_0 \in H^2(X^{\mathrm{an}}_{b_1}, \mathbb{Q})$. Then Λ is a finite dimensional $\mathbb{Q}$-vector subspace.

Step 4 We consider a nonarchimedean place and find a p-adic analytic open subset outside the jumping locus.

By [Cas86, Ch. 5, Theorem 1.1], because $F'/\mathbb{Q}$ is a finitely generated extension, there is a prime p and a field embedding $F' \hookrightarrow \mathbb{Q}_p$ which restricts to a morphism $A' \hookrightarrow \mathbb{Z}_p$ of rings. It extends to a field embedding $k' \hookrightarrow \bar{\mathbb{Q}}_p$. By the implicit function theorem [Igu00, Theorem 2.1.1], as $\mathbb{C}_p$ is complete, one may choose $\epsilon \in (0, |p|^{1/(p-1)})$ such that the closed and open polydisc

$$D := \{(z_1, \ldots, z_m) \in \mathbb{C}_p^m : |z_i| \leq \epsilon\}$$

is analytically isomorphic to an open neighborhood of b_1 in $B(\mathbb{C}_p)$ via $\phi : B \to \mathbf{A}^m_k$, and that for every $g^j_i \in \Gamma(B, O_B)$, the local expansion of the corresponding analytic function $g^j_i : B(\mathbb{C}_p) \to \mathbb{C}_p$ at b_1 converges on D. By Lemma 5.3, as $(\mathcal{H}, \nabla)$ is defined over A, for every initial condition $s_0 \in H^2_{\mathrm{DR}}(X_{b_1}/\mathbb{Q}_p)$, the corresponding formal solutions $f^1, \ldots, f^r \in \mathbb{Q}_p[[z_1, \ldots, z_m]]$ converge on $\{(z_1, \ldots, z_m) \in \mathbb{C}_p^m : |z_i| < |p|^{1/(p-1)}\}$. Because of $\epsilon < |p|^{1/(p-1)}$, for every $1 \leq j \leq t$, the formal power series $\lambda^j := \sum_{i=1}^r f^i g^j_i \in \mathbb{Q}_p[[z_1, \ldots, z_m]]$ converges on D.

Let Λ_p be the $\mathbb{Q}_p$-vector subspace generated by $\Lambda \subset \bar{\mathbb{Q}}_p[[z_1, \ldots, z_m]]^t$. It is of finite dimension over $\mathbb{Q}_p$. Then by [MP12, Proposition 5.1], there is a nonempty open subset U of D such that for every $u \in U$, the evaluation map

$$\mathrm{ev}_u : \Lambda_p \to \mathbb{C}_p^t$$

is injective. From [MP12, Lemma 6.1], because $\bar{\mathbb{Q}}$ is dense in $\mathbb{C}_p$, and because U is identified with a nonempty analytic open subset of $B(\mathbb{C}_p)$, $U \cap B(k)$ is nonempty.

Step 5 We show that a k-point in Ω and in U satisfies the result.

From Fact 6.3 (b), as k is algebraically closed and B is quasi-projective and irreducible, there is $b_0 \in B(k)$ which lies in Ω and U simultaneously. For every initial condition $s_0 \in H^2(X_{b_0}, \mathbb{Q})$ such that the corresponding power series $\lambda \in \mathbb{C}[[\xi_1, \ldots, \xi_m]]^t$ satisfies $\lambda(0) = 0$, we prove $\lambda = 0$.

Because of $b_0 \in \Omega$, the map $\Gamma(\Omega, R^2 f_*^{\mathrm{an}}\mathbb{Q}) \xrightarrow{\sim} H^2(X_{b_0}, \mathbb{Q})$ is an isomorphism. Let s be the preimage of s_0 under this map. Let σ be the image of s under the map

$$\Gamma(\Omega, R^2 f_*^{\mathrm{an}}\mathbb{Q}) \hookrightarrow H^0_{\mathrm{DR}}(\Omega, \mathcal{H}^2_{\mathrm{DR}}(X/B)^{\mathrm{an}}) \to \Gamma(\Omega, \mathcal{H}^{02}(X/B)^{\mathrm{an}}).$$

Then the assumption $\lambda(0) = 0$ is equivalent to $\sigma(b_0) = 0$ in the fiber $\mathcal{H}^{02}(X/B)(b_0) = H^{02}(X_{b_0}/\mathbb{C})$. Let s_1 be the image of s under $\Gamma(\Omega, R^2 f_*^{\mathrm{an}}\mathbb{Q}) \xrightarrow{\sim} H^2(X_{b_1}, \mathbb{Q})$. Let $\lambda_1 \in \Lambda$ be the element determined by the initial condition s_1. Then each $\lambda_1^j \in k'[[z_1, \dots, z_m]]$ converges to a holomorphic function $\lambda_1^j : \Omega \to \mathbb{C}$ and $\sigma = \sum_{j=1}^t \lambda_1^j w_j$. It implies $\lambda_1(b_0) = 0$ in $\mathbb{C}^t$. By $b_0 \in U$, one has $\lambda_1 = 0$ in Λ. For every $1 \le j \le t$, because $\lambda^j \in \mathbb{C}[[\xi_1, \dots, \xi_m]]$ is the Taylor expansion of λ_1^j at b_0, one has $\lambda^j = 0$.

Consider the diagram

$$\begin{tikzcd}
 & & \Lambda_{b_0} \arrow[d, hook] & \\
\mathrm{Pic}(X_{b_0}) \arrow[r, "c_{1,\mathrm{inf}}"'] \arrow[d, "c_1"] \arrow[urr, dashed, two heads] & H^2_{\mathrm{DR}}(X_{b_0}/k) \arrow[r] \arrow[d, hook, "1/(2\pi i)"] & k[[\xi_1, \dots, \xi_m]]^t \arrow[r, "\mathrm{ev}_0"] \arrow[d, hook, "1/(2\pi i)"] & H^{02}(X/B)(b_0) = \oplus_{j=1}^t k \cdot w_j(b_0) \arrow[d, hook] \\
H^2(X_{b_0}^{\mathrm{an}}, \mathbb{Q}) \arrow[r, hook] & H^2(X_{b_0}^{\mathrm{an}}, \mathbb{C}) \arrow[r] & \mathbb{C}[[\xi_1, \dots, \xi_m]]^t \arrow[r, "\mathrm{ev}_0"] & H^{02}(X_{\mathbb{C}}/B_{\mathbb{C}})(b_0) = \oplus_{j=1}^t \mathbb{C} \cdot w_j(b_0),
\end{tikzcd}$$

where the horizontal morphisms in the middle take the corresponding formal solutions λ with given initial values. By Lemma 3.2, the square on the left is commutative. By [BI70, Proposition 2.3], for every $L \in \mathrm{Pic}(X_{b_0})$, the map $H^2_{\mathrm{DR}}(X_{b_0}/k) \to H^2_{\mathrm{dR}}(X_{b_0}/k)$ maps $c_{1,\mathrm{inf}}(L)$ to $c_{1,\mathrm{dR}}(L) \in F^1 H^2_{\mathrm{dR}}(X_{b_0}/k)$. Therefore, by (16), the corresponding element $\lambda_L \in \Lambda_{b_0}$ satisfies $\lambda_L(0) = 0$. From the commutative diagram, one has $\lambda_L = 0$.

□

Remark 6.2. Let $f : X \to B$ be a proper morphism of algebraic varieties over $\mathbb{C}$. Let ℓ be a prime. Let $i \ge 0$ be an integer such that $R^i f_* \mathbb{Q}_\ell$ is a lisse sheaf on B. We recall why $R^i f_*^{\mathrm{an}}\mathbb{Q}$ is a local system on B^{an}. For a scheme S of finite type over $\mathbb{C}$, let S_{cl} be the topos associated with the topological space S^{an}. Let $\epsilon : S_{\mathrm{cl}} \to S_{\text{ét}}$ be the morphism of topoi. Then ϵ^* restricts to a functor from the category of $\mathbb{Q}_\ell$-lisse sheaves on S to the category of $\mathbb{Q}_\ell$-local systems on S^{an}. By [Ver76, Corollaire 2.7.6], the analytification of $R^i f_* \mathbb{Q}_\ell$ is $R^i f_*^{\mathrm{an}} \mathbb{Q}_\ell$, so it is locally constant. From the proper base change theorem and the universal coefficient theorem, as $f : X \to B$ is proper, $(R^i f_*^{\mathrm{an}}\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to R^i f_*^{\mathrm{an}} \mathbb{Q}_\ell$ is an isomorphism of sheaves on B^{an}. Therefore, $R^i f_*^{\mathrm{an}}\mathbb{Q}$ is also locally constant.

Fact 6.3 from [MZ14, Theorem 1.2 and p.2955] is an Artin-Whaples approximation theorem for algebraic varieties. The authors consider projective varieties, which implies the case with quasi-projective varieties. An equivalence class of absolute values on a field k is called a *place* of k.

Fact 6.3. *Let X be a quasi-projective, geometrically irreducible variety over a field k of dimension n. Let $\bar{k}$ be an algebraic closure of k. Let S be a finite set of places of k. For every $v \in S$, extend it in some way to $\bar{k}$, and let k_v (resp. $\bar{k}_v$) be the completion of k (resp. $\bar{k}$) at v.*

(a) *Choose a geometrically irreducible closed subvariety $\bar{X} \subset \mathbf{P}^N_k$ of degree d containing X as an open subscheme. For every $v \in S$, fix $Q_v \in X(k_v)$ and let $\Omega_v \subset X(\bar{k}_v)$ be an analytic open neighborhood of Q_v. Then there is $P \in X(\bar{k})$ with*

$$[k(P) : k] \leq d^{\#S\cdot(\min(2,n)+1)-1}$$

such that $P \in \Omega_v$ for all $v \in S$.

(b) *In particular, if k is algebraically closed, then the diagonal inclusion $X(k) \hookrightarrow \prod_{v\in S} X(k_v)$ has dense image.*

Lemma 6.4. *Let k be a field of characteristic 0. Let B be a smooth, geometrically irreducible variety over k. Let $f : X \to B$ be a proper morphism that is cohomologically flat in degree 1. Let $b_0 \in B(k)$ be such that a geometric point $\bar{b}_0 \in B(\bar{k})$ over it satisfies the result of Lemma 6.1.*

(a) *Then for every $L \in \operatorname{Pic}(X_{b_0})$, there is a smooth integral variety U, a quasi-finite étale morphism $U \to B$ over k and a point $u \in U(k)$ over b_0, such that letting $X_U := X \times_B U$, there is $L' \in \operatorname{Pic}(X_U)$ with $L|_{X_u} = L'|_{X_u}$.*

(b) *Let η be the generic point of B. Suppose that $f : X \to B$ is normal (resp. has integral geometric fibers) and satisfies the results of Lemma 2.2* (a) *(resp.* (b)*) and Lemma 2.4* (c) 1*) (resp.* 2*)). Then there is a specialization isomorphism*

$$\operatorname{NS} X_{\bar{\eta}} \xrightarrow{\sim} \operatorname{NS} X_{\bar{b}_0} \text{ (resp. } \operatorname{Num} X_{\bar{\eta}} \xrightarrow{\sim} \operatorname{Num} X_{\bar{b}_0}).$$

Proof. (a) Let

$$S := \operatorname{Spec} \hat{O}_{B,b_0}, \quad \hat{S} := \operatorname{Spf} \hat{O}_{B,b_0}, \quad X_S := X \times_B S, \quad \hat{X} := X \times_B \hat{S}.$$

Consider the commutative diagram

$$\begin{tikzcd}
H^2_{\mathrm{inf}}(X_{b_0}/k) \arrow[r] \arrow[d, "\mathrm{GM}"] & H^2_{\mathrm{inf}}(X_{b_0}/\hat{S}) \arrow[r, "\tau_{\mathrm{inf}}"] & H^2_{\mathrm{dR}}(\hat{X}/\hat{S}) \arrow[r] & H^2(X_S, O_{X_S}) \\
\mathcal{H}^2_{\mathrm{DR}}(X/B) \otimes_{O_B} \hat{O}_{B,b_0} \arrow[r] & \mathcal{H}^2_{\mathrm{dR}}(X/B) \otimes_{O_B} \hat{O}_{B,b_0} \arrow[r] & \mathcal{H}^{02}(X/B) \otimes_{O_B} \hat{O}_{B,b_0} \arrow[r] & (R^2 f_* O_X) \otimes_{O_B} \hat{O}_{B,b_0}, \arrow[u]
\end{tikzcd}$$

where the

$$\mathrm{GM} : H^2_{\mathrm{inf}}(X_{b_0}/k) \to \mathcal{H}^2_{\mathrm{DR}}(X/B) \otimes_{O_B} \hat{O}_{B,b_0}$$

is from Lemma 5.2. Because of $\Lambda_{b_0} = 0$, $c_{1,\mathrm{inf}}(L) \in H^2_{\mathrm{inf}}(X_{b_0}/k)$ maps to 0 in $\mathcal{H}^{02}(X/B) \otimes_{O_B} \hat{O}_{B,b_0}$ and hence also in $H^2(X_S, O_{X_S})$. As $\operatorname{char} k = 0$ and $f : X \to B$ is cohomologically flat in degree 1, the result follows from Lemma 4.12.

(b) By assumption, the specialization map is defined and injective. As $\mathrm{char} k = 0$, it remains to prove that for every $L \in \mathrm{Pic}(X_{\bar{b}_0})$, $[L]$ lies in the image of the specialization morphism. There is a finite extension k'/k with L defined on $X_{b_0} \otimes_k k'$. Base changing along $k \hookrightarrow k'$, one may assume $L \in \mathrm{Pic}(X_{b_0})$ so that one can take $U \to B$, $u \in U(k)$ and L' as in Part (a). Let $\eta' \in U$ be the generic point, which is over η. By construction of the specialization, one has $\mathrm{sp}(L'|_{X_{\bar{\eta}'}}) = [L'|_{X_{\bar{u}}}] = [L]$.

□

It is relatively formal that an NS-generic point also admits a specialization isomorphism of Brauer groups.

Theorem 6.5. *Let k be an algebraically closed field of characteristic 0. Let B be an irreducible variety over k with generic point η. Let $f : X \to B$ be a proper morphism.*

(a) *If $f : X \to B$ has integral geometric fibers, then the set of $b \in B(k)$ with a specialization isomorphism $\mathrm{Num}(X_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{Num}(X_b)$ is Zariski dense in B.*

(b) *If $f : X \to B$ is normal, then the set of $b \in B(k)$ with specialization isomorphisms*

$$\mathrm{NS}(X_{\bar{\eta}}) \xrightarrow{\sim} \mathrm{NS}(X_b), \quad \mathrm{Br}\, X_{\bar{\eta}} \xrightarrow{\sim} \mathrm{Br}\, X_b$$

is Zariski dense in B.

Proof. Let U be a nonempty open subset of B, and we prove that it contains such points. Shrinking B to U, one may assume $B = U$. By Fact 1.8, replacing B by B_{red} one may assume further that B is reduced. Then shrinking B, one may assume that B is smooth over k. From Lemma 4.10, as $f : X \to B$ is proper, shrinking B one may assume that $f : X \to B$ is cohomologically flat in degree 1. Shrinking B one may assume that it satisfies the results in Lemmas 2.2 and 2.4. The result follows from Lemmas 6.1 and 6.4 (b). □

Remark 6.6. Assume $k = \mathbb{C}$ in Theorem 6.5. As we only construct specialization of Néron-Severi groups for specializations of the generic point (Lemma 2.4 (a)), a priori we do not know whether the jumping locus

$$B(k)_{\mathrm{jumping}} := \{b \in B(k) : \rho(X_b) > \rho(X_{\bar{\eta}})\}$$

is contained in a countable union of strict closed analytic subsets of B^{an}. By contrast, in the smooth case over general k, the jumping locus is a countable union of strict closed algebraic subvarieties (see, e.g., [MP12, Corollary 3.9]).

Remark 6.7. The set of NS-generic points is potentially Zariski dense in the following sense. Let k be a field of characteristic 0. Consider the setting of Lemma 6.1 over k. Fact 6.3 (a) can be used to show that there exists an explicit constant $d > 1$ depending only on B, such that $\{b \in B^{\leq d} | \Lambda_{\bar{b}} = 0\}$ is Zariski dense in B. Then the result of Theorem 6.5 remains true if one requires $b \in B^{\leq d}$ instead of $b \in B(\bar{k})$.

Proof of Theorem 1.4. One may assume that B is irreducible. Let η be the generic point of B. We only prove that the generic fiber X_η is projective. One may base extend to a finite extension k'/k and select an irreducible component of $B_{k'}$, thus one may assume that B is geometrically irreducible. By [Sta25, Tag 0BDG], replacing B by B_{red} one may assume that B is reduced. Then shrinking B, one may assume that B is smooth over k. Shrinking B, one may assume that $f : X \to B$ is cohomologically flat in degree 1. By Lemma 6.1, one may choose $b_0 \in B(\bar{k})$ with $\Lambda_{b_0} = 0$. As X_{b_0} is projective over $\bar{k}$, it admits an ample line bundle L. From Lemma 6.4 (a), there is a smooth integral variety U over $\bar{k}$, a quasi-finite étale morphism $U \to B_{\bar{k}}$ over $\bar{k}$, a point $u \in U(\bar{k})$ over b_0 and a line bundle $L' \in \mathrm{Pic}(X_U)$ with $L|_{X_u} = L'|_{X_u}$. Then from [EGA III 1, Théorème 4.7.1], shrinking U to an open neighborhood of u, one may assume that L' is ample relative to $X_U \to U$. Let $\xi \in U$ be the generic point of U, which is over η. Then $L'|_{X_\xi}$ is ample, hence X_ξ is projective over $k(\xi)$. By [Sta25, Tag 0BDG], X_η is projective over $k(\eta)$. $\square$

Acknowledgments

This note owes special thanks to Emiliano Ambrosi. He had the vision that the theorem of André, Maulik and Poonen may extend to the singular case. He generously shared with me the insight that the infinitesimal cohomology, especially Theorem 1.3, should help to solve the problem. This work resulted from our multiple discussions and was polished upon his useful suggestions, yet he declined to be named a coauthor. I thank Cédric Pépin for discussion on his work. I benefited from helpful discussions with Zhenghui Li, Christopher Nicol and Hui Zhang. I express gratitude to Xinyu Shao and Yicheng Zhou for answering a lot of questions.

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