Fourier-Mukai transform on complex tori, revisited

Haohao Liu*

March 11, 2026

Abstract

We study the Fourier-Mukai transform on complex tori. A Fourier inversion formula is given for good sheaves in the sense of Kashiwara. Good sheaves serve as a complex-analytic analog of quasi-coherent sheaves on algebraic varieties. We explain why goodness is necessary for the Fourier inversion formula in this context. We also prove a flat base change theorem for good sheaves on complex analytic spaces, which may be of independent interest.

Keywords— Fourier-Mukai transform, complex torus, derived category, coherent sheaves

1 Introduction

1.1 Review of Fourier-Mukai transform

Mukai [Muk81, Sec. 2] introduces an analog of the Fourier transform for sheaves on abelian varieties. Let X be an abelian variety over a field k, $\hat{X}$ be its dual abelian variety, $p_X : X \times_k \hat{X} \to X$ and $p_{\hat{X}} : X \times_k \hat{X} \to \hat{X}$ be the projections. Let $\mathcal{P}$ be the normalized Poincaré bundle on $X \times_k \hat{X}$, where "normalized" means that both pullback *modules* $\mathcal{P}|_{X\times 0}$ and $\mathcal{P}|_{0\times\hat{X}}$ are trivial. Mukai defines a pair of triangulated functors

$$\hat{\mathcal{F}} : D(X) \to D(\hat{X}), \quad F \mapsto Rp_{\hat{X}*}(\mathcal{P} \otimes p_X^* F),$$
$$\mathcal{F} : D(\hat{X}) \to D(X), \quad G \mapsto Rp_{X*}(\mathcal{P} \otimes p_{\hat{X}}^* G),$$

known as the *Fourier-Mukai transform*, for which Mukai [Muk81, Thm. 2.2] gives an analog of the Fourier inversion formula. Mukai claims that $\mathcal{F} : D(\hat{X}) \to D(X)$ is an equivalence of categories.

Complex tori are generalizations of complex abelian varieties. Given complex tori X and $\hat{X}$ dual to each other, Ben-Bassat, Block and Pantev similarly define the Fourier-Mukai transform. In this analytic context, [BBP07, Thm. 2.1] claims that "$\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories". However, we show that both claims are true only when $\dim X = 0$.

*kyung@mail.ustc.edu.cn, IRMA, Université de Strasbourg, 7 Rue René Descartes, 67000 Strasbourg, France

Lemma 1.1.1 (Lemma 2.0.7)**.** For dual complex tori (or abelian varieties) X and $\hat{X}$, $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories if and only if $\dim X = 0$.

The claims require an additional quasi-coherence hypothesis, which is standard in the algebraic setting but subtle in the analytic one. In fact, two ingredients of Mukai's proof, namely the projection formula and the flat base change theorem, both are stated for quasi-coherent sheaves on algebraic varieties. A corrected version of Mukai's theorem is that, $\mathcal{F} : D(\hat{X}) \to D(X)$ restricts to an equivalence $D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(X)$, where $D_{\mathrm{qc}}(X)$ denotes the full subcategory of $D(X)$ of objects with quasi-coherent cohomologies, and $D_{\mathrm{qc}}(\hat{X})$ is similarly defined. As a remarkable generalization, [BBP07, Thm. 5.2] extends the equivalence further to non-commutative deformations of complex tori. This important theorem gives a new kind of dualities via deformations.

1.2 Good sheaves and an application

As Lemma 1.1.1 suggests, to state a correct version of [BBP07, Thm. 2.1], one needs a notion of complex analytic quasi-coherence. Several different definitions ([RR74], [EP96], [CS22], *etc.*) of complex analytic quasi-coherent sheaves are present in the literature, and none of them seems definitive. We discuss the relationships among some definitions in Section A.6. On an algebraic variety M, a quasi-coherent sheaf is a filtered colimit of coherent subsheaves, and it is acyclic on affine open subsets of M. On a complex manifold X, however, a quasi-coherent sheaf in the sense of [EGA I] may not be a filtered colimit of coherent subsheaves, nor be acyclic on Stein open subsets of X. Kashiwara's notion of good sheaves [Kas03, Def. 4.22] addresses the first issue.

Definition 1.2.1. Let (X, O_X) be a ringed space such that the underlying topology is locally compact Hausdorff. An O_X-module F is called *good* if for every relatively compact open subset U of X, there exists a directed family $\{G_i\}_{i\in I}$ of coherent O_U-submodules of $F|_U$ with $F|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ being a *directed family* means that for any $i, i' \in I$, there is $i'' \in I$ with $G_i + G_{i'} \subset G_{i''}$ (and hence $F|_U = \mathrm{colim}_{i\in I} G_i$). The full subcategory of $\mathrm{Mod}(O_X)$ of good O_X-modules is denoted by $\mathrm{Good}(X)$.

For a complex analytic space X, let $D_{\mathrm{gd}}(X) \subset D(X)$ be the full subcategory consisting of complexes whose cohomology sheaves are good. As a clue that good sheaves are analogous to algebraic quasi-coherent sheaves, we recall a result of GAGA type. Given a proper algebraic variety M over $\mathbb{C}$, one has a natural equivalence $D_{\mathrm{qc}}(M) \to D_{\mathrm{gd}}(M^{\mathrm{an}})$ ([Liu25a, Theorem 4.2]). As another illustration, we show that a complex analytic variant of Mukai's theorem holds for good sheaves, essentially due to Mukai, Ben-Bassat, Block and Pantev.

Theorem 1.2.2 (Theorem 5.0.1)**.** *Let X and $\hat{X}$ be dual complex tori of dimension g. The functor $\hat{\mathcal{F}} : D(X) \to D(\hat{X})$ (resp. $\mathcal{F} : D(\hat{X}) \to D(X)$) restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(\hat{X})$ (resp. $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X)$). Moreover, there are isomorphisms of functors*

$$\mathcal{F} \circ \hat{\mathcal{F}} \cong [-1]_X^*[-g] : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X),$$
$$\hat{\mathcal{F}} \circ \mathcal{F} \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(\hat{X}),$$

where $[-g]$ denotes degree shift. In particular, $\mathcal{F} : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X)$ and its restriction $\mathcal{F} : D^b_{\mathrm{coh}}(\hat{X}) \to D^b_{\mathrm{coh}}(X)$ are equivalences of triangulated categories.

The Fourier-Mukai transform has many applications in algebraic geometry: the Künneth decomposition for Chow motives [DM91], the study of stable bundles on elliptic surfaces [Bri98], a new proof of Torelli's theorem [BP01], regularity of sheaves on abelian varieties [PP03], *etc.* The Ben-Bassat-Block-Pantev theorem for complex tori (see, *e.g.*, [PPS17, Thm. 13.1]) is applied to study GV-sheaves on compact Kähler manifolds. Matsushima [Mat59] and Morimoto [Mor59] classify homogeneous vector bundles on complex tori, which is extended to abelian varieties over arbitrary fields by Miyanishi [Miy73, Thm. 2.3]. Mukai [Muk81, p.159] applies the Fourier-Mukai transform to recover Miyanishi's result. The basic building blocks are unipotent vector bundles, twisted by line bundles of degree 0. Unipotent vector bundles are extensions of the structure sheaf. Back to complex tori, we apply Theorem 1.2.2 to recover Matsushima-Morimoto's original classification of homogeneous vector bundles.

Theorem 1.2.3 (Theorem 6.3.2)**.** *A vector bundle F on a complex torus X is translation invariant if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \dots, U_n$ on X and $L_1, \dots, L_n \in \mathrm{Pic}^0(X)$ with $F \cong \oplus_{i=1}^n (L_i \otimes U_i)$.*

To prove the analogous classification on abelian varieties, Mukai [Muk81, Lem. 3.3] uses the fact that every positive dimensional algebraic variety contains a curve. By contrast, this already fails for analytic K3 surfaces ([MS08, p.134]). Instead of reducing to a curve, we construct an auxiliary morphism in Lemma 6.3.5 and pullback along it.

1.3 Complex analytic flat base change theorem

For dual complex tori X and $\hat{X}$, to prove that the Fourier-Mukai transform restricts to a functor $\hat{\mathcal{F}} : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(\hat{X})$, we extend Grauert's direct image theorem for coherent sheaves to good sheaves.

Theorem 1.3.1 (Theorem 3.2.1)**.** *Let $f : X \to S$ be a proper morphism of complex analytic spaces. Then $Rf_* : D(X) \to D(Y)$ restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S)$.*

Our proof of Theorem 1.2.2 follows Mukai's strategy. It relies on Spaltenstein's projection formula [Spa88, Prop. 6.18], which does not require any analytic quasi-coherence assumption. As another ingredient, we need a complex analytic analog of the flat base change theorem in algebraic geometry.

Theorem 1.3.2 (Theorem 4.2.7)**.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of complex analytic spaces, with $f : X \to S$ proper and $g : S' \to S$ flat. Then the natural transformation $g^* Rf_* \to Rf'_* g'^*$ restricts to an isomorphism of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$.

As Lemma 1.1.1 shows, analytic quasi-coherence such as goodness is necessary in Theorem 1.3.2. It is unclear whether other notions of "complex analytic quasi-coherence" guarantee similar flat base change theorems, but Clausen and Scholze [CS22] establish a base change result for "derived fiber products" of complex analytic spaces (Remark 5.2.1).

Notation and conventions

For a topological space M, the category of abelian sheaves on M is denoted by $\mathbf{Ab}(M)$. The category of ringed spaces is denoted by $\mathbf{RS}$. For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the category of O_X-modules. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of quasi-coherent (resp. coherent) O_X-modules in the sense of Definition A.1.1 (3) (resp. (6)) is denoted by $\mathrm{Qch}(X)$ (resp. $\mathrm{Coh}(X)$). For a closed subset $Z \subset X$, let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of modules with support contained in Z.

Given a symbol $* \in \{\emptyset, +, -, b\}$, let $\mathrm{Ch}^*(O_X)$ (resp. $D^*(X)$) denote the category of chain complexes (resp. derived category) of $\mathrm{Mod}(O_X)$ which is unbounded/bounded below/bounded above/bounded in order. The full subcategory of $D^*(X)$ consisting of the complexes whose cohomologies are coherent (resp. quasi-coherent) is denoted by $D^*_{\mathrm{coh}}(X)$ (resp. $D^*_{\mathrm{qc}}(X)$). Denote by $R\mathcal{H}om_X : D(X)^{\mathrm{op}} \times D(X) \to D(X)$ the internal hom bifunctor constructed in [Sta26, Tag 08DH].

For a locally ringed space X and $x \in X$, let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. For an $O_{X,x}$-module M, the O_X-module $(i_x)_*M$ is denoted by $M_{(x)}$.

All complex analytic spaces (in the sense of [KK83, Def. 43.2]) are assumed to be paracompact. Let $\mathbf{An}$ be the category of complex analytic spaces. The dimension of a complex manifold always refers to the complex dimension, which is assumed to be finite.

A scheme of finite type and separated over a field is called an algebraic variety. For an abelian variety (resp. a complex torus) X, its dual abelian variety (resp. complex torus) is denoted by $\hat{X}$. The normalized Poincaré bundle on $X \times \hat{X}$ is denoted by $\mathcal{P}$. For $y \in \hat{X}$ (resp. $x \in X$), let P_y (resp. P_x) denote the line bundle $\mathcal{P}|_{X \times y}$ (resp. $\mathcal{P}|_{x \times \hat{X}}$).

2 Fourier-Mukai transform

The Fourier-Mukai transform on abelian varieties proposed by Mukai [Muk81] is an analog of the classical Fourier transform. It is a particular example of integral transforms. For two algebraic varieties M and N, let $p_M : M \times N \to M$ and $p_N : M \times N \to N$ be the projections. For an object $K \in D(M \times N)$, the *integral transform* with integral kernel K is

$$\phi_K^{[M \to N]} : D(M) \to D(N), \quad F \mapsto Rp_{N,*}(K \otimes^L p_M^* F). \tag{1}$$

For complex analytic spaces, one can similarly define integral transforms.

Remark 2.0.1. In (1), one can write Lp_M^* instead of p_M^*. In fact, $p_M : M \times N \to M$ is flat, so $p_M^* : \mathrm{Mod}(O_M) \to \mathrm{Mod}(O_{M \times N})$ is an exact functor, and hence an isomorphism of functors $p_M^* \cong Lp_M^* : D(M) \to D(M \times N)$. It also holds in the complex analytic setting by [CD94, Cor. 2.7, p.111].

Let X be an abelian variety over a field k (resp. a complex torus) of dimension g. Write $\mathcal{F}$ and $\hat{\mathcal{F}}$ for $\phi_{\mathcal{P}}^{[\hat{X} \to X]}$ and $\phi_{\mathcal{P}}^{[X \to \hat{X}]}$ respectively. The pair $(\mathcal{F}, \hat{\mathcal{F}})$ is called the Fourier-Mukai transform of X. The functor $\mathcal{F}$ (resp. $\hat{\mathcal{F}}$) restricts to a functor $D^b(\hat{X}) \to D^b(X)$ (resp. $D^b(X) \to D^b(\hat{X})$).

For abelian varieties, Mukai [Muk81, (3.1) (resp. (3.7))] shows that the usual exchange of translation and time shifting (resp. multiplication and convolution) of the

Fourier transform finds analog for the Fourier-Mukai transform, namely the exchange of translation and line bundle twisting (resp. tensor product and Pontryagin product). Moreover, Mukai proves a duality theorem similar to the classical Fourier inversion formula, which is generalized to arbitrary abelian schemes in [GW23, Thm. 27.243].

Fact 2.0.2. Let X be an abelian variety over a field k of dimension g. Let $\hat{X}$ be the abelian variety dual to X. Then there are isomorphisms of functors

$$\mathcal{F} \circ \hat{\mathcal{F}} \cong [-1]_X^*[-g] : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X);$$
$$\hat{\mathcal{F}} \circ \mathcal{F} \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(\hat{X}).$$

In particular, $\mathcal{F} : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(X)$ is an equivalence of triangulated categories, with a quasi-inverse $[-1]_{\hat{X}}^* \circ \hat{\mathcal{F}}[g] : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(\hat{X})$.

Example 2.0.3 ([Muk81, Example 2.6])**.** For every $y \in \hat{X}(k)$, one has $\mathcal{F}(k_{(y)}) = P_y$ and $\hat{\mathcal{F}}(P_y) = k_{(-y)}[-g]$.

Remark 2.0.4. Rothstein's version [Rot96, Mukai's Theorem, p.569] is stated for $D^b(\mathrm{Qch}(-))$ instead of $D_{\mathrm{qc}}(-)$. In [Rot96, p.571], the quasi-coherence condition is essential for the use of Čech resolution with respect to affine covers. We sketch how to deduce Rothstein's version from Fact 2.0.2. By [BN93, Cor. 5.5], for an algebraic variety M, the natural functor $D(\mathrm{Qch}(M)) \to D_{\mathrm{qc}}(M)$ is an equivalence of triangulated categories. By [TT90, Cor. B.9], for a morphism of algebraic varieties $f : M \to N$, the direct image $Rf_* : D(M) \to D(N)$ restricts to a functor $D^b_{\mathrm{qc}}(M) \to D^b_{\mathrm{qc}}(N)$, and the restriction is identified with the right derived functor of $f_* : \mathrm{Qch}(M) \to \mathrm{Qch}(N)$ *via* the equivalence. By contrast, from Bondal's result, for a generic complex torus X of dimension $g > 2$, the natural functor $D(\mathrm{Good}(X)) \to D_{\mathrm{gd}}(X)$ is not an equivalence (see, e.g., [Liu25a, Remark 5.12]).

The proof of Fact 2.0.2 uses the projection formula and the flat base change theorem (see, *e.g.*, [Lip09, Propositions 3.9.4 and 3.9.5]). Compared with Fact 2.0.2, the original statement of [Muk81, Thm. 2.2] has no quasi-coherence restriction.

Statement 2.0.5. The functor $\mathcal{F}$ gives an equivalence of categories between $D(\hat{X})$ and $D(X)$, and its quasi-inverse is $[-1]_{\hat{X}}^* \circ \hat{\mathcal{F}}[g]$.

Every complex abelian variety is naturally a complex torus. Every complex torus of dimension 1 is an abelian variety. By contrast, for every integer $g \geq 2$, a very general complex torus of dimension g is not an abelian variety (see, *e.g.*, [BZ23, p.21]). In [BBP07, Thm. 2.1], an assertion similar to Statement 2.0.5 is made for complex tori.

Statement 2.0.6. Let X be a complex torus. Then the integral transform $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence of triangulated categories.

However, Lemma 2.0.7 shows that Statements 2.0.5 and 2.0.6 hold if and only if $g = 0$. In the algebraic case, the argument is sketched in [t3suji]. We adapt this strategy for the analytic case, where we have to study the sections of direct sums of infinitely many sheaves on complex manifolds (Section A.5).

Lemma 2.0.7. Let X be an abelian variety or a complex torus. Then the functor $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories exactly when $\dim X = 0$.

Proof. If $\dim X = 0$, then $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence. Conversely, assume that $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence. When X is a complex torus, let $k = \mathbb{C}$. In both cases, let $F = k_{(0)}^{\mathbb{N}}$ be the product of a countable infinite family of $k_{(0)}$ in $\mathrm{Mod}(O_{\hat{X}})$. Since $k^{\mathbb{N}} \cong k^{\oplus I}$ as k-module for some index set I, the direct sum sheaf $k_{(0)}^{\oplus I}$ is isomorphic to F. Therefore, by [Sta26, Tag 07D9 (2)], F is the direct sum of I copies of $k_{(0)}$ in $D^b(\hat{X})$.

We claim that F is also the product of $\mathbb{N}$ copies of $k_{(0)}$ in $D^b(\hat{X})$. By [Gro57, p.129], the abelian category $\mathrm{Mod}(O_{\hat{X},0})$ satisfies the AB 4*) axiom, i.e. it has products and products are exact. From [Sta26, Tag 07KC (2)], the inclusion $\mathrm{Mod}(O_{\hat{X},0}) \to D^b(\mathrm{Mod}(O_{\hat{X},0}))$ commutes with countable products. Let $i : 0 \to \hat{X}$ be the closed immersion. Since $i_* : \mathrm{Mod}(O_{\hat{X},0}) \to \mathrm{Mod}(O_{\hat{X}})$ is exact, there is a commutative square

$$\begin{tikzcd} \mathrm{Mod}(O_{\hat{X},0}) \arrow[r, "i_*"] \arrow[d, hook] & \mathrm{Mod}(O_{\hat{X}}) \arrow[d, hook] \\ D^b(\mathrm{Mod}(O_{\hat{X},0})) \arrow[r, "Ri_*"] & D^b(\hat{X}). \end{tikzcd}$$

Since $Ri_* : D^b(\mathrm{Mod}(O_{\hat{X},0})) \to D^b(\hat{X})$ has a left adjoint, it commutes with products. As $F = i_*(k^{\mathbb{N}})$, the claim is proved.

As $\mathcal{F} : D^b(\hat{X}) \to D^b(X)$ is an equivalence, $\mathcal{F}(F)$ is the direct sum of I copies of $\mathcal{F}(k_{(0)})$ in $D^b(X)$, as well as the product of $\mathbb{N}$ copies of $\mathcal{F}(k_{(0)})$. By Example 2.0.3 (when X is an abelian variety) and Lemma 2.0.9 (when X is a complex torus), one has $\mathcal{F}(k_{(0)}) = O_X$. Therefore, $\mathcal{F}(F)$ is isomorphic to $O_X^{\oplus I}$ and to $O_X^{\mathbb{N}}$ in $\mathrm{Mod}(O_X)$.

Assume to the contrary $\dim X > 0$. Then there is a nonempty subset V of X which is *connected* and open in the algebraic or analytic topology respectively, such that $O_X(V)$ is an integral domain but not a field. In particular, the ring $O_X(V)$ is not Artinian. By [Har77, II, Exercise 1.11] (when X is an abelian variety) and Corollary A.5.4 (when X is a complex torus), the $O_X(V)$-module $\Gamma(V, \mathcal{F}(F))$ is isomorphic to $O_X(V)^{\oplus I}$ and to $O_X(V)^{\mathbb{N}}$. However, this contradicts Fact 2.0.8. $\square$

Fact 2.0.8 ([Len68, Thm., p.211])**.** If A is a commutative ring such that $A^{\mathbb{N}}$ is a free A-module, then A is Artinian.

Lemma 2.0.9 computes the integral transform of a skyscraper sheaf.

Lemma 2.0.9. Let X, Y be two complex analytic spaces, let $K \in D(X \times Y)$, and let $x \in X$. Consider the closed embedding $h_x : Y \hookrightarrow X \times Y, \quad y \mapsto (x, y)$. Then $\phi_K^{[X \to Y]}(\mathbb{C}_{(x)}) = Lh_x^* K$.

Proof. Let $p : X \times Y \to X$, $q : X \times Y \to Y$ be the two projections. Let $j_x : x \to X$ be the closed embedding of complex analytic spaces. The cartesian square

$$\begin{tikzcd} Y \arrow[r, "p_0"] \arrow[d, "h_x"'] \arrow[dr, phantom, "\square"] & x \arrow[d, "j_x"] \\ X \times Y \arrow[r, "p"] & X \end{tikzcd}$$

induces a natural morphism $\phi : p^* \mathbb{C}_{(x)} \to h_{x,*} O_Y = R h_{x,*} O_Y$ in $\mathrm{Mod}(O_{X \times Y})$.

We show that $\phi : p^* \mathbb{C}_{(x)} \to h_{x,*} O_Y$ is an isomorphism. It suffices to prove the isomorphism near $(x, y) \in X \times Y$ for every $y \in Y$. One may shrink X and Y to Stein open neighborhoods of x and y respectively. Thus, one may assume that X and Y are Stein spaces. Then by [DV74, I, p.30],

$$\Gamma(X \times Y, \phi) : \Gamma(X \times Y, p^*\mathbb{C}_{(x)}) \xrightarrow{\sim} \Gamma(X \times Y, h_{x,*}O_Y)$$

is an isomorphism. From [For67, p.383], because $p^*\mathbb{C}_{(x)}$ and $h_{x,*}O_Y$ are coherent sheaves on the Stein space $X \times Y$, $\phi : p^*\mathbb{C}_{(x)} \xrightarrow{\sim} h_{x,*}O_Y$ is also an isomorphism. (That ϕ is an isomorphism also follows from Theorem 1.3.2.)

Back to the case with X, Y not necessarily Stein. By [Sta26, Tag 0B55], the natural morphism $(Rh_{x,*}O_Y) \otimes^L K \to Rh_{x,*}(Lh_x^*K)$ is an isomorphism. Then one has

$$\phi_K^{[X\to Y]}(\mathbb{C}_{(x)}) = Rq_*(p^*\mathbb{C}_{(x)} \otimes^L K) \xrightarrow{\sim} Rq_*\Big((Rh_{x,*}O_Y) \otimes^L K\Big)$$

$$\xrightarrow{\sim} Rq_*Rh_{x,*}(Lh_x^*K) \overset{\text{(a)}}{\cong} R(q \circ h_x)_*(Lh_x^*K) = Lh_x^*K,$$

where (a) uses [Spa88, Prop. 6.7 (b)]. □

The minor issue with Statement 2.0.5 occurs in the proof of [Muk81, Prop. 1.3], when the flat base change theorem [Har66, II, Prop. 5.12] stated for objects of $D_{\mathrm{qc}}(-)$ is applied to objects in $D^-(-)$. Similarly, the minor issue with Statement 2.0.6 originates from a lack of certain analytic quasi-coherence in Statement 2.0.10 ([BBP07, p.427]), which is a counterpart of [Muk81, Prop. 1.3]. A modification of Statement 2.0.10 is Proposition 5.1.2.

Statement 2.0.10. If M, N, and P are compact complex manifolds and $K \in D^b(M\times N)$ and $L \in D^b(N \times P)$, then one has a natural isomorphism of functors from $D^b(M)$ to $D^b(P)$:

$$\phi_L^{[N\to P]} \circ \phi_K^{[M\to N]} \cong \phi_{K*L}^{[M\to P]},$$

where

$$K * L = Rp_{M\times P*}(p_{M\times N}^*K \otimes^L p_{N\times P}^*L) \in D^b(M \times P),$$

and $p_{M\times N}$, $p_{M\times P}$, $p_{N\times P}$ are the natural projections $M \times N \times P \to M \times N$, *etc.*

3 Good modules and functoriality

As Section 2 explains, to obtain an analytic analogue of Fact 2.0.2, it is necessary to find a substitute for quasi-coherence on complex manifolds. We show that goodness introduced by Kashiwara and reviewed in Section A.4 can be used as such. In Corollary 3.2.11, we prove that goodness is preserved by integral transforms. To prove this, we show that goodness is preserved by the operations involved in (1).

3.1 Derived pullback and derived tensor product

Lemma 3.1.1 ([Har66, Example 1., p.68])**.** Let $f : X \to Y$ be a morphism of ringed spaces. Then the derived pullback $Lf^* : D(Y) \to D(X)$ constructed in [Spa88, Prop. 6.7 (a)] is bounded above in the sense of [Lip09, 1.11.1], and $Rf_* : D(X) \to D(Y)$ is bounded below.

Proposition 3.1.2. *Let $f : X \to Y$ be a morphism of complex analytic spaces. Then $Lf^* : D(Y) \to D(X)$ restricts to*

(1) *a functor $D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ when Y is a complex manifold or $f : X \to Y$ is flat;*

(2) *and a functor $D_{\mathrm{gd}}(Y) \to D_{\mathrm{gd}}(X)$.*

For a commutative ring R, its *weak dimension* $\mathrm{wgld}(R) \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is the supremum of the flat dimensions of all R-modules.

Lemma 3.1.3 (Serre)**.** Let R be a commutative, Noetherian, regular local ring. Then $\mathrm{wgld}(R)$ coincides with the Krull dimension $\dim(R)$, which is finite.

Proof. From [Osb12, Cor. 4.21], $\mathrm{wgld}(R)$ coincides with the global dimension of R. By Serre's theorem [Ser55, Thm. 1], as R is regular, its global dimension equals $\dim(R)$, which is finite. □

For a morphism $f : X \to Y$ of ringed spaces, its *tor-dimension* is the lower dimension in the sense of [Lip09, 1.11.1] of the functor $Lf^* : D^-(Y) \to D(X)$. Flat morphisms have tor-dimension 0. If $f : X \to Y$ has finite tor-dimension, then $Lf^* : D^-(Y) \to D(X)$ restricts to a functor $D^b(Y) \to D^b(X)$. For a ringed space (X, O_X), its *weak dimension* $\mathrm{wgld}(X) \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is $\sup_{x \in X} \mathrm{wgld}(O_{X,x})$.

Lemma 3.1.4. Let Y be a complex manifold. Then $\mathrm{wgld}(Y) \leq \dim Y$. In particular, every morphism $f : X \to Y$ of complex analytic spaces has finite tor-dimension.

Proof. Because Y is a complex manifold, for every $y \in Y$, $O_{Y,y}$ is Noetherian and regular. By Lemma 3.1.3, $\mathrm{wgld}(O_{Y,y})$ coincides with the Krull dimension $\dim(O_{Y,y})$, which equals the dimension of the complex manifold Y near y. This implies $\mathrm{wgld}(Y) \leq \dim Y$. In particular, for every $x \in X$, the flat dimension of the $O_{Y,f(x)}$-module $O_{X,x}$ is at most $\dim Y$. Hence, the second statement follows from [Lip09, (2.7.6.4)]. □

Proof of Proposition 3.1.2. (1) By Lemma 3.1.4, because Y is smooth or $f : X \to Y$ is flat, $f : X \to Y$ has finite tor-dimension. Thus, $Lf^* : D(Y) \to D(X)$ restricts to a functor $D^b(Y) \to D^b(X)$. The result follows from Lemma A.3.3.

(2) (1) First, we consider the case with $G \in D^-_{\mathrm{gd}}(Y)$. By Lemma 3.1.1, Lemma A.4.2 (3) and a dual of [Har66, I, Prop. 7.3 (ii)], to prove $Lf^*G \in D_{\mathrm{gd}}(X)$, one may assume $G \in \mathrm{Good}(Y)$. Let U be a relatively compact open subset of X. As $f(\bar{U})$ is a compact subset of Y, it is contained in a relatively compact open subset V of Y. Since G is good, one can write $G|_V = \sum_{i \in I} G_i$ for some directed family $(G_i)_{i \in I}$ of coherent O_V-submodules of $G|_V$. From [Sta26, Tag 0949], one has $G|_V = \mathrm{hocolim}_{i \in I} G_i$ in $D(V)$. As direct sums in $D(Y)$ are given by direct sums of representative complexes, $Lf^* : D(Y) \to D(X)$ commutes with direct sums. Let $g : f^{-1}(V) \to V$ be the restriction of $f : X \to Y$. By definition of homotopy colimits, $Lg^* : D(V) \to D(f^{-1}(V))$ also commutes with homotopy colimits. Thus, one has
$$(Lf^*G)|_{f^{-1}(V)} = \mathrm{hocolim}_{i \in I} Lg^* G_i.$$
For every integer n, one has
$$\begin{aligned}(L^n f^*G)|_{f^{-1}(V)} &= H^n((Lf^*G)|_{f^{-1}(V)}) \\ &= H^n(\mathrm{hocolim}_{i \in I} Lg^* G_i) \overset{\text{(a)}}{=} \mathrm{colim}_{i \in I} L^n g^*(G_i),\end{aligned}$$

where (a) uses [Sta26, Tag 0CRK]. From Lemma A.3.3, for every $i \in I$, since G_i is coherent, $L^n g^*(G_i)$ is coherent over $O_{f^{-1}(V)}$. By Lemma A.4.2 (3), $(L^n f^* G)|_{f^{-1}(V)}$ is a good sheaf on $f^{-1}(V)$. Since $\bar{U}$ is a compact subset of $f^{-1}(V)$, U is relatively compact in $f^{-1}(V)$. Hence, $(L^n f^* G)|_U$ is the sum of certain directed family of coherent submodules. Thus, one has $Lf^* G \in D_{\mathrm{gd}}(X)$.

(2) Then consider the general case with $C \in D_{\mathrm{gd}}(Y)$. For every integer $m \geq 0$, the m-th canonical truncation in the sense of [Sta26, Tag 0118 (4)] $C_m := \tau^{\leq m} C$ is in $D^-_{\mathrm{gd}}(Y)$. From the proof of [Lip09, Prop. 2.5.5], there is a bounded above complex of flat O_Y-modules $Q_m \in \mathrm{Ch}^-(O_Y)$ with a quasi-isomorphism $Q_m \to C_m$ that is functorial in C_m. Moreover, the termwise colimit complex $Q := \mathrm{colim}_m Q_m \in \mathrm{Ch}(O_Y)$ is K-flat in the sense of [Spa88, Def. 5.1], and the canonical morphism $Q \to C$ is a quasi-isomorphism. Let $\mathrm{colim}_m f^* Q_m \in \mathrm{Ch}(O_X)$ be the termwise colimit of the system of complexes $(f^* Q_m)_{m \geq 0}$. Then the resulting morphisms

$$\mathrm{colim}_m f^* Q_m \to f^* Q \leftarrow Lf^* Q \to Lf^* C$$

are all isomorphisms in $D(X)$. By [Lip09, Exercise, p.132], for every integer n, the natural morphism

$$\mathrm{colim}_m H^n(f^* Q_m) \to H^n(\mathrm{colim}_m f^* Q_m)$$

is an isomorphism in $\mathrm{Mod}(O_X)$. Thus, one obtains an isomorphism

$$L^n f^* C \cong \mathrm{colim}_m H^n(f^* Q_m) \xrightarrow{\sim} \mathrm{colim}_m L^n f^*(C_m)$$

in $\mathrm{Mod}(O_X)$. Since $C_m \in D^-_{\mathrm{gd}}(Y)$, $L^n f^*(C_m)$ is a good sheaf on X. By Lemma A.4.2 (3), $L^n f^* C$ is a good sheaf on X. $\square$

Proposition 3.1.5. *Let X be a complex analytic space. Then the bifunctor* (constructed in [Spa88, Thm. A (ii)]) $(-) \otimes^L (+) : D(X) \times D(X) \to D(X)$ *restricts to*

(1) *a bifunctor $D^b(X) \times D^b(X) \to D^b(X)$ (resp. $D^b_{\mathrm{coh}}(X) \times D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$) when X is a complex manifold,*

(2) *and a bifunctor $D_{\mathrm{gd}}(X) \times D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X)$.*

Proof.

(1) The statement for $D^b(X)$ follows from Lemma 3.1.4 and [HT07, (C.2.20)]. Consider any $F, G \in D^b_{\mathrm{coh}}(X)$. By [Har66, I, Prop. 7.3 (i)], to prove $F \otimes^L G \in D^b_{\mathrm{coh}}(X)$, one may assume $F, G \in \mathrm{Coh}(X)$. Then the conclusion follows from [GH78, 4., p.700].

(2) Take $F, G \in D_{\mathrm{gd}}(X)$. As in the proof of Proposition 3.1.2 (2), to prove $F \otimes^L G \in D_{\mathrm{gd}}(X)$, one may assume $F, G \in D^-_{\mathrm{gd}}(X)$. By a dual of [Har66, I, Prop. 7.3 (ii)], one may further assume $F, G \in \mathrm{Good}(X)$.

Let U be a relatively compact open subset of X. For every integer n, we claim that $H^n(F \otimes^L_{O_X} G)|_U$ is a good sheaf on U. By assumption, $F|_U = \sum_{i \in I} F_i$ and $G|_U = \sum_{j \in J} G_j$ can be written as sums of directed families of coherent submodules. The functor $(-) \otimes^L_{O_U} (G|_U) : D(U) \to D(U)$ commutes with direct sums, so it also commutes with homotopy colimits. One has

$$(F \otimes^L_{O_X} G)|_U = \mathrm{hocolim}_{i \in I} \left(F_i \otimes^L (G|_U) \right).$$

Then by [Sta26, Tag 0CRK], for every integer n, the canonical morphism

$$\operatorname{colim}_{i\in I} H^n(F_i \otimes^L_{O_U} G|_U) \xrightarrow{\sim} H^n(F \otimes^L_{O_X} G)|_U$$

is an isomorphism. Because $\mathrm{Good}(U)$ is closed under colimits in $\mathrm{Mod}(O_U)$ by Lemma A.4.2 (3), one may assume that $F|_U$ is coherent. Similarly, one may assume further that $G|_U$ is coherent. In this case, the claim follows from Lemma A.3.4. $\square$

3.2 Derived direct image

Theorem 3.2.1 shows that derived direct images along proper morphisms preserve good sheaves, which may be of independent interest.

Theorem 3.2.1. *Let $f : X \to Y$ be a* proper *morphism of complex analytic spaces. Then $Rf_* : D(X) \to D(Y)$ restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y)$. If moreover* $\dim X$ *is finite, then it also restricts to a functor $D^b_{\mathrm{gd}}(X) \to D^b_{\mathrm{gd}}(Y)$.*

The proof of Theorem 3.2.1 relies essentially on Grauert's direct image theorem, see *e.g.*, [CAS, p.207].

Fact 3.2.2. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then $Rf_* : D(X) \to D(Y)$ restricts to a functor $D^b_{\mathrm{coh}}(X) \to D^+_{\mathrm{coh}}(Y)$.

The proof of Fact 3.2.3 is similar to that of [KS90, Prop. 3.2.2].

Fact 3.2.3. Let X be a Hausdorff, locally compact topological space which is countable at infinity. Suppose that there is an integer $n \geq 0$, such that every point of X has an open neighborhood homeomorphic to a locally closed subset of $\mathbb{R}^n$. Then for every abelian sheaf F on X and every integer $j > n$, one has $H^j(X, F) = 0$.

From Fact 3.2.3, one gets Fact 3.2.4 as a special case of [Spa88, Prop. 6.18].

Fact 3.2.4 (Projection formula)**.** Let $f : X \to Y$ be a morphism of complex analytic spaces. Then there is a canonical isomorphism of bifunctors

$$(Rf_!-) \otimes^L_{O_Y} (+) \to Rf_!(- \otimes^L_{O_X} Lf^*+) : D(X) \times D(Y) \to D(Y).$$

We divide the proof of Theorem 3.2.1 into a series of lemmas.

Lemma 3.2.5. Let $f : X \to Y$ be a *proper* map between Hausdorff, locally compact spaces. Then for every integer $n \geq 0$, the functor $R^n f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ commutes with filtrant colimits.

Proof. Let $(F_i, f_{ij})_{i\in I}$ be a filtrant inductive system with colimit F in $\mathbf{Ab}(X)$. Since the abelian category $\mathbf{Ab}(Y)$ is Grothendieck, the filtrant colimit $G := \operatorname{colim}_{i\in I} R^n f_* F_i$ exists and there is a canonical morphism $\phi : G \to R^n f_* F$ in $\mathbf{Ab}(Y)$. For every $y \in Y$, the functor $\mathbf{Ab}(Y) \to \mathbf{Ab}$ taking the stalk at y commutes with colimits, so $G_y = \operatorname{colim}_{i\in I}(R^n f_* F_i)_y$. By the topological proper base change theorem (see, *e.g.*, [Mil13, Thm. 17.2]), for every $i \in I$, $(R^n f_* F_i)_y$ is canonically identified with $H^n(X_y, F_i|_{X_y})$. Then by [God58, Thm. 4.12.1], the morphism $\phi_y : G_y \to (R^n f_* F)_y$ is an isomorphism. Therefore, $\phi : G \xrightarrow{\sim} R^n f_* F$ is an isomorphism. $\square$

Lemma 3.2.6 is a derived version of [Har77, III, Prop. 8.1].

Lemma 3.2.6. Let $f : X \to Y$ be a continuous map of topological spaces. Then for every integer i and every $F \in D(\mathbf{Ab}(X))$, $R^i f_* F$ is the sheafification of the abelian presheaf $V \mapsto H^i(f^{-1}(V), F)$.

Proof. By [Spa88, Thm. D], there is a quasi-isomorphism $F \to I^\bullet$, where $I^\bullet$ is a K-injective complex of abelian sheaves on X. Then the canonical morphism $Rf_*F \to f_*I^\bullet$ is an isomorphism in $D(\mathbf{Ab}(Y))$. By [Mur06, Lem. 3], the sheaf associated with the presheaf

$$V \mapsto H^i\big(\Gamma(V, f_*I^\bullet)\big) = H^i\big(\Gamma(f^{-1}(V), I^\bullet)\big) = H^i(f^{-1}(V), F)$$

is $R^i f_* F$. $\square$

To handle unbounded complexes in the proof of Theorem 3.2.1, we need Lemma 3.2.7 to replace an unbounded complex with a bounded one when computing higher direct images.

Lemma 3.2.7. Let X be a ringed space as in Fact 3.2.3. Let $F : \mathrm{Mod}(O_X) \to \mathbf{Ab}$ be an additive functor. Assume that F commutes with countable products, and there is an integer $N \geq 0$ with $R^m F(M) = 0$ for every integer $m \geq N$ and every $M \in \mathrm{Mod}(O_X)$. Then for any integers $i \geq j$, the natural transformation

$$R^i F \xrightarrow{\sim} R^i F \tau^{\geq j-N+1}(-) : D(X) \to \mathbf{Ab}$$

is an isomorphism of functors.

Proof. By [Spa88, Thm. D], every complex in $\mathrm{Mod}(O_X)$ has a quasi-isomorphism towards a K-injective complex. Then by [Sta26, Tag 070K], the right derived functor $RF : D(X) \to D(\mathbf{Ab})$ exists. For every integer m and every $E \in D(X)$, set $E_m := \tau^{\geq -m} E$. Then $\{E_m\}_{m \in \mathbb{Z}}$ forms an inverse system in $D(X)$. Let n be as in Fact 3.2.3. Then for every open subset U of X, any integers $p (> n)$ and q, one has $H^p(U, H^q(E)) = 0$. Then by [Sta26, Tag 0D64], the canonical morphism $E \to R\lim_m E_m$ is an isomorphism in $D(X)$. From [Sta26, Tag 08U1], since F commutes with countable products, in $D(\mathbf{Ab})$ one has $RF(E) \xrightarrow{\sim} R\lim_m RF(E_m)$. By [Sta26, Tag 08U5], for every integer i, there is a short exact sequence

$$0 \to R^1 \lim_m R^{i-1} F(E_m) \to R^i F(E) \to \lim_m R^i F(E_m) \to 0 \tag{2}$$

in the category $\mathbf{Ab}$.

We claim $R^1 \lim_m R^{i-1} F(E_m) = 0$. By [Sta26, Tag 08J5], for every integer $m \geq N - i$, there is an exact triangle

$$H^{-m}(E)[m] \to E_m \to E_{m-1} \xrightarrow{+1} H^{-m}(E)[m+1] \tag{3}$$

in $D(X)$. By assumption, one has

$$R^i F\left(H^{-m}(E)[m]\right) = R^{i+m} F\left(H^{-m}(E)\right) = 0;$$
$$R^i F\left(H^{-m}(E)[m+1]\right) = R^{i+m+1} F\left(H^{-m}(E)\right) = 0.$$

Taking the long exact sequence associated with (3), one concludes that the canonical morphism $R^i F(E_m) \to R^i F(E_{m-1})$ in $\mathbf{Ab}$ is an isomorphism. Since the inverse system

$\{R^iF(E_m)\}_{m\geq 1}$ is constant starting from $m = N-i-1$, it satisfies the Mittag-Leffler condition in the sense of [Sta26, Tag 02N0]. From [Sta26, Tag 07KW (3)], one obtains

$$R^1 \lim_m R^iF(E_m) = 0,$$

which proves the claim.

When $i \geq j$, as the inverse system is constant from $m = N - j - 1$, one has $\lim_m R^iF(E_m) = R^iF(E_{N-j-1})$. Then the sequence (2) induces an isomorphism $R^iF(E) \to R^iF(\tau^{\geq j-N+1}E)$. □

Lemma 3.2.8. Let X be a complex analytic space of finite dimension n. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then for an object $E \in D(X)$ with $H^m(E) = 0$ for all integers $m > 0$, one has $R^if_*E = 0$ for every integer $i > 2n$. In particular, $Rf_* : D(X) \to D(Y)$ is bounded.

Proof. From $i > 2n$ and Fact 3.2.3, for every open subset V of Y and every O_X-module M, one has $H^i(f^{-1}(V), M) = 0$. Applying Lemma 3.2.7 to the functor $\Gamma(f^{-1}(V), -) : \mathrm{Mod}(O_X) \to \mathbf{Ab}$, one gets

$$H^i(f^{-1}(V), E) = H^i(f^{-1}(V), \tau^{\geq 1}E) = 0.$$

By Lemma 3.2.6, one has $R^if_*E = 0$. □

Proof of Theorem 3.2.1. We show that for every $F \in D_{\mathrm{gd}}(X)$ and every integer $n \geq 0$, R^nf_*F is a good sheaf on Y. It suffices to show that for every relatively compact open subset $V \subset Y$, $(R^nf_*F)|_V$ is a good sheaf on V. As $f : X \to Y$ is proper, $U := f^{-1}(V)$ is a relatively compact open subset of X. In particular, $\dim U$ is finite. Thus, replacing $f : X \to Y$ by its restriction $g : U \to V$, one may assume further that $\dim X$ is finite. In this case, by [Har66, I, Prop. 7.3 (iii)], Lemmas 3.2.8 and A.4.2 (3), one may assume $F \in \mathrm{Good}(X)$. One can write $F|_U = \mathrm{colim}_{i\in I}F_i$, where $(F_i)_{i\in I}$ is a directed family of coherent O_U-submodules of $F|_U$. By Lemma 3.2.5, in $\mathrm{Mod}(O_V)$ one has

$$(R^nf_*F)|_V = R^ng_*(F|_U) = \mathrm{colim}_{i\in I}R^ng_*F_i.$$

By Fact 3.2.2, as $g : U \to V$ is proper, for every $i \in I$, the O_V-module $R^ng_*F_i$ is coherent. By $\mathrm{Coh}(V) \subset \mathrm{Good}(V)$ and Lemma A.4.2 (3), $(R^nf_*F)|_V$ are good sheaves for all relatively compact open subsets $V \subset Y$. Therefore, R^nf_*F is a good sheaf on Y. By Lemma 3.2.8, when $\dim X$ is finite, $Rf_* : D(X) \to D(Y)$ restricts to a functor $D^b(X) \to D^b(Y)$. □

As Example 3.2.9 shows, the properness assumption of $f : X \to Y$ in Theorem 3.2.1 is necessary.

Example 3.2.9. Let $h : \mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\} \hookrightarrow \mathbf{P}^1$ be the open inclusion of algebraic varieties over $\mathbb{C}$. Let $j : X \hookrightarrow Y$ be its analytification. We show that j_*O_X is not a good sheaf on Y.

Let $\varphi : O_Y \to j_*O_X$ be the morphism induced by $\mathrm{Id} : j^*O_Y \xrightarrow{\sim} O_X$ and adjunction. Let $\tau : O_Y \to j_*O_X$ be the morphism induced by $\sin(t\pi) \in \Gamma(X, O_X) = \Gamma(Y, j_*O_X)$. Then for every $x \in X$, $\tau_x : O_{Y,x} \to O_{Y,x}$ is surjective if and only if $\sin(x\pi) \neq 0$, i.e., $x \notin \mathbb{Z}$. In particular, the support of $\mathrm{coker}(\tau)|_X$ is $\mathbb{Z}$. Assume the contrary that j_*O_X is a good sheaf. Then by [Liu25a, Proposition 3.2], there is a quasi-coherent sheaf F on $\mathbf{P}^1$ such that the analytification F^{an} is isomorphic to j_*O_X, and two morphisms $\psi, \sigma : O_{\mathbf{P}^1} \to F$ on $\mathbf{P}^1$ whose analytifications are identified with φ, τ :

$O_Y \to j_*O_X$ respectively. From [SGA 1, XII, Thm. 1.1], for every $x \in \mathbf{A}^1(\mathbb{C})$, the local homomorphism $O_{\mathbf{A}^1,x} \to O_{X,x}$ is faithfully flat. Thus, $\psi|_{\mathbf{A}^1} : O_{\mathbf{A}^1} \to F|_{\mathbf{A}^1}$ is an isomorphism because its analytification is $\varphi|_X = \mathrm{Id}_{O_X}$. The analytification of

$$(\psi|_{\mathbf{A}^1})^{-1} \circ \sigma|_{\mathbf{A}^1} : O_{\mathbf{A}^1} \to O_{\mathbf{A}^1}$$

is identified with the nonzero morphism $\tau|_X : O_X \to O_X$. Since every nonzero element of $\Gamma(\mathbf{A}^1, O_{\mathbf{A}^1}) = \mathbb{C}[t]$ has only finitely many roots, the support of $\mathrm{coker}(\tau|_X)$ is finite, which is a contradiction.

Remark 3.2.10. In Theorem 3.2.1, if X is of infinite dimension, Rf_*F may not lie in $D^b(Y)$ for vector bundles F on X. For every integer $m \geq 1$, let X_m be a complex torus of dimension m, and let $f_m : X_m \to \operatorname{Specan}\mathbb{C}$ be the canonical morphism. Let $f : X \to Y$ be $\sqcup_{m\geq 1} f_m : \sqcup_{m\geq 1} X_m \to \sqcup_{m\geq 1} \operatorname{Specan}\mathbb{C}$, which is a proper morphism. Then for every integer $q \geq 1$, one has $R^q f_* O_X \neq 0$.

Corollary 3.2.11. Let X, Y be complex manifolds (resp. complex analytic spaces), with X compact. Let F be an object of $D^b_{\mathrm{coh}}(X \times Y)$ (resp. $D_{\mathrm{gd}}(X \times Y)$). Then $\phi_F^{[X\to Y]} : D(X) \to D(Y)$ restricts to a functor $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ (resp. $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y)$).

Proof. Because X is compact, the projection $X \times Y \to Y$ is proper. The result is a combination of Proposition 3.1.2 (1) (resp. (2)), Proposition 3.1.5 (1) (resp. (2)), Fact 3.2.2 and Lemma 3.2.8 (resp. Theorem 3.2.1). $\square$

Remark 3.2.12. Although we don't need the functor $f^!$, it is interesting to know whether it preserves goodness.

For a complex algebraic variety X, let $\psi_X : X^{\mathrm{an}} \to X$ be the complex analytification morphism. With quasi-coherence condition, the algebraic and analytic integral transforms are compatible.

Corollary 3.2.13. Let X, Y be complex algebraic varieties, with X *proper* over $\mathbb{C}$. Then for every $K \in D_{\mathrm{qc}}(X \times Y)$, the natural transformation $\psi_Y^* \phi_K^{[X\to Y]} \to \phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}}\to Y^{\mathrm{an}}]} \psi_X^* : D(X) \to D(Y^{\mathrm{an}})$ restricts to an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$.

Proof. From [Sta26, Tags 08DW (1), 08DX (1) and 08D5 (1)], $\phi_K^{[X\to Y]}$ restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$. By Corollary 3.2.11 and compactness of X^{an}, $\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}}\to Y^{\mathrm{an}}]}$ restricts to a functor $D_{\mathrm{gd}}(X^{\mathrm{an}}) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$. By [Liu25a, Lem. 2.5], ψ_X^* (resp. ψ_Y^*) restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(X^{\mathrm{an}})$ (resp. $D_{\mathrm{qc}}(Y) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$). By [Sta26, Tag 0D5S (resp. Tag 079U)], analytification commutes with derived pullback (resp. derived tensor product). As X is proper over $\mathbb{C}$, the projection $p_Y : X \times Y \to Y$ is proper. Then by [Liu25a, Prop. 3.1], analytification commutes with $Rp_{Y*} : D_{\mathrm{qc}}(X \times Y) \to D_{\mathrm{qc}}(Y)$. The result follows. $\square$

4 Flat base change theorem for complex analytic spaces

As a replacement for the algebraic flat base change theorem used in Mukai's proof of Fact 2.0.2, we give an analytic flat base change theorem. Consider a cartesian square in the category **An**:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S. \end{tikzcd} \tag{4}$$

Then [Sta26, Tag 08HY] gives a natural transformation of functors

$$Lg^* Rf_* \to Rf'_* Lg'^* : D(X) \to D(S'). \tag{5}$$

It comes from the adjunction in [Sta26, Tag 079W], and is called the Beck-Chevalley map.

4.1 Product case

We say that a morphism of complex analytic spaces $g : S' \to S$ satisfies property $\mathcal{Q}_S$ if for every *proper* morphism $f : X \to S$ of complex analytic spaces, every coherent O_X-module F and every integer $i \geq 0$, the base change morphism $g^* R^i f_* F \to R^i f'_*(g'^* F)$ induced by (4) is an isomorphism in $\mathrm{Mod}(O_{S'})$. Lemma 4.1.1 shows that the property $\mathcal{Q}$ is local on the source and the target, whose proof is omitted.

Lemma 4.1.1. Let $g : S' \to S$ and be a morphism of complex analytic spaces.

(1) Let $h : S'' \to S'$ be another morphism of complex analytic spaces. If $g : S' \to S$ and $h : S'' \to S'$ satisfy $\mathcal{Q}_S$ and $\mathcal{Q}_{S'}$ respectively, then $gh : S'' \to S$ satisfies $\mathcal{Q}_S$.

(2) If $g : S' \to S$ is an open embedding of complex analytic spaces, then it satisfies $\mathcal{Q}_S$.

(3) Assume that $\{S'_i\}_{i \in I}$ (resp. $\{S_j\}_{j \in J}$) is an open covering of S' (resp. S) such that for every $i \in I$ (resp. $j \in J$), the morphism $g|_{S'_i} : S'_i \to S$ (resp. $g^{-1}(S_j) \to S_j$) satisfies $\mathcal{Q}_S$ (resp. $\mathcal{Q}_{S_j}$). Then $g : S' \to S$ satisfies $\mathcal{Q}_S$.

Lemma 4.1.2. Let S, Z be two complex analytic spaces. Then the projection $g : S \times Z \to S$ satisfies $\mathcal{Q}_S$.

For two (possibly non Hausdorff) locally convex topological vector spaces E and F over $\mathbb{C}$, the completed projective topological tensor product $E \hat{\otimes}_{\mathbb{C}} F = E \hat{\otimes} F$ is defined in [Gro55, I, Déf. 2, p.32], which is a complete locally convex space.

Proof. By Lemma 4.1.1, we may assume that S, Z are Stein spaces. Fix a proper morphism $X \to S$ of complex analytic spaces and a coherent O_X-module F. In the notation of the cartesian square

$$\begin{tikzcd} X \times Z \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S \times Z \arrow[r, "g"] & S, \end{tikzcd}$$

we prove that $g^* R^i f_* F \to R^i f'_* g'^* F$ is an isomorphism for every $i \geq 0$. By Fact 3.2.2, as $f : X \to S$ is proper, $R^i f_* F$ and $R^i f'_* g'^* F$ are coherent sheaves. By [Gra60, Satz 5, p.248], as S and Z are Stein,

$$H^i(X, F) \xrightarrow{\sim} \Gamma(S, R^i f_* F), \quad H^i(X \times Z, g'^* F) \xrightarrow{\sim} \Gamma(S \times Z, R^i f'_* g'^* F)$$

are isomorphisms. Then by Example 4.2.1, $H^i(X,F)$ is a naturally a Fréchet space. One has

$$\Gamma(S\times Z, g^*R^if_*F) \overset{\text{(a)}}{=} O(Z)\hat{\otimes}_{\mathbb{C}}\Gamma(S, R^if_*F) = O(Z)\hat{\otimes}_{\mathbb{C}}H^i(X,F),$$

$$\Gamma(S\times Z, R^if'_*g'^*F) = H^i(X\times Z, g'^*F) \overset{\text{(b)}}{=} O(Z)\hat{\otimes}_{\mathbb{C}}H^i(X,F)$$

where (a) uses [DV74, I, Thm. p.30], and (b) uses [Dem12, IX, Cor. 5.22 a)] and that the $H^q(X,F)$ are Hausdorff. Therefore, $\Gamma(S\times Z, g^*R^if_*F)\to\Gamma(S\times Z, R^if'_*g'^*F)$ is an isomorphism. By [For67, Satz 2.1], since $S\times Z$ is a Stein space, $g^*R^if_*F\to R^if'_*g'^*F$ is also an isomorphism. □

Lemma 4.1.3. Consider a cartesian square of complex analytic spaces

$$\begin{tikzcd} X\times Z \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S\times Z \arrow[r, "g"] & S, \end{tikzcd}$$

where $g: S\times Z\to S$ is the projection, and $f: X\to S$ is proper. Then $g^*Rf_*\to Rf'_*g'^*$ restricts to an isomorphism of functors $D_{\mathrm{gd}}(X)\to D_{\mathrm{gd}}(S\times Z)$.

Proof. For every $K\in D_{\mathrm{gd}}(X)$, we prove that the base change morphism $g^*Rf_*K\to Rf'_*g'^*K$ in $D(S\times Z)$ is an isomorphism. As $g: S\times Z\to S$ is flat, it suffices to prove that for every integer q, $g^*R^qf_*K\to R^qf'_*g'^*K$ is an isomorphism of sheaves on $S\times Z$. By the local nature of sheaves, it remains to prove that every $(s,z)\in S\times Z$ has an open neighborhood over which

$$g^*R^qf_*K\to R^qf'_*g'^*K$$

are isomorphisms for all integers q. Thus, one can shrink S and Z to relatively compact open neighborhoods S_0 and Z_0 of s and z respectively. Then $f^{-1}(S_0)$ is a relatively compact open subset of X. Thus, one may assume further that X and Z are finite dimensional. By Lemma 3.2.8, because X and $X\times Z$ are finite dimensional, $Rf_*: D(X)\to D(S)$ and $Rf'_*: D(X\times Z)\to D(S\times Z)$ are bounded. From [Har66, I, Prop. 7.1 (iii)] and Lemma A.4.2 (3), one may assume $K\in\mathrm{Good}(X)$.

Every $s\in S$ has a relatively compact open neighborhood V. Consider the cartesian square

$$\begin{tikzcd} f^{-1}(V)\times Z \arrow[r, "v'"] \arrow[d, "u'"] \arrow[dr, phantom, "\square"] & f^{-1}(V) \arrow[d, "u"] \\ V\times Z \arrow[r, "v"] & V. \end{tikzcd}$$

As $f: X\to S$ is proper, $f^{-1}(V)$ is a relatively compact open subset of X. As K is good, one can write $K|_{f^{-1}(V)} = \mathrm{colim}_{i\in I}G_i$, where $\{G_i\}_{i\in I}$ is a directed family of coherent submodules of $K|_{f^{-1}(V)}$. By Lemma 3.2.5, for every integer q, the natural morphism

$$(g^*R^qf_*K)|_{g^{-1}(V)}\to R^qf'_*(g'^*K)|_{g^{-1}(V)} \tag{6}$$

in $\mathrm{Mod}(O_{g^{-1}(V)})$ is the colimit of the directed family of morphisms

$$(v^*R^qu_*G_i\to R^qu'_*v'^*G_i)_{i\in I}.$$

By Lemma 4.1.2, it is a family of isomorphisms. Then (6) is an isomorphism. Consequently, $g^*R^qf_*K\to R^qf'_*g'^*K$ is also an isomorphism. □

4.2 Tôr-independent case

Recktenwald [Rec19] extends Spaltenstein's flat base change theorem [Spa88] to cartesian squares of ringed spaces that are tor-independent, a condition expressed in terms of algebraic Tor functors. As Example 4.2.11 illustrates, a fiber product of complex analytic spaces need not be the fiber product of the underlying ringed spaces. Roughly, for ringed spaces, the structure sheaf of a fiber product is the algebraic tensor product of the individual structure sheaves, while it involves a completed tensor product for complex analytic spaces. For this difference, to state a base change theorem (Theorem 4.2.7) for cartesian squares of complex analytic spaces, we require a complex analytic modification of the tor-independence hypothesis. Intuitively, the "complex analytic tor-independence" condition ensures that the "derived fiber product" of usual complex analytic spaces is again a usual complex analytic space. We use topological Tor functors instead of algebraic Tor functors to express this condition.

A $\hat{\otimes}$-algebra is a Hausdorff locally convex space A over $\mathbb{C}$ endowed with a *jointly* continuous bilinear map $A \times A \to A$ making A a commutative $\mathbb{C}$-algebra. Given a $\hat{\otimes}$-algebra A, a $\hat{\otimes}$-module over A is a Hausdorff locally convex space M over $\mathbb{C}$, with a jointly continuous bilinear map $A \times M \to M$ making M an A-module. A *Fréchet algebra* is a $\hat{\otimes}$-algebra such that the underlying space is a Fréchet space. Given a Fréchet algebra A, a *Fréchet A-module* M is a $\hat{\otimes}$-module over A such that the underlying space is a Fréchet space. For nuclear topological vector spaces [Gro55, II, Déf. 4], several frequently used topological tensor products agree. If A (resp. M) is moreover nuclear as a topological vector space, then it is called an *FN algebra* (resp. *FN A-module*). The strong dual of a nuclear Fréchet space is called a *DFN space*. Similarly one defines DFN algebras and DFN modules.

Example 4.2.1. By [DV74, I, Prop. 2 et II, p.33], for every complex analytic space X, $O(X)$ is naturally an FN algebra, and for every coherent sheaf F on X, $F(X)$ has a natural topology τ_F making it an FN $O(X)$-module. From [GR09, VIII, Sec. A, Thm. 7], for every open subset $U \subset X$, $F(X) \to F(U)$ is continuous, and for a morphism $\psi : F \to G$ of coherent O_X-modules, $F(X) \to G(X)$ is continuous, hence a morphism of FN $O(X)$-modules. By [GR09, VIII, Sec. A, Thm. 9], if U is relatively compact in X, then $F(X) \to F(U)$ is a compact map. For every compact subset $K \subset X$, $O(K)$ is a DFN algebra, and $F(K)$ is a DFN $O(K)$-module (Lemma A.6.5 (1)).

Assume further that $\psi : F \hookrightarrow G$ is injective, and we prove that the continuous map $(F(X), \tau_F) \hookrightarrow (G(X), \tau_G)$ is a topological closed embedding. Let $H := G/F$, which is coherent over O_X. As $F(X)$ is the kernel of the continuous linear map $(G(X), \tau_G) \to (H(X), \tau_H)$, it is closed in $(G(X), \tau_G)$, and hence the subspace topology $(F(X), \tau_G)$ is a Fréchet space. By the open mapping theorem (see, *e.g.*, [Gro73, Thm. 9, p.39]), $\mathrm{Id}_{F(X)} : (F(X), \tau_F) \to (F(X), \tau_G)$ is an isomorphism of Fréchet spaces.

Definition 4.2.2. Given a $\hat{\otimes}$-algebra A and $\hat{\otimes}$-modules M and N over A, the construction of the *Bar complex* $B^A_\bullet(M, N)$ is as follows. For every $n \geq 0$, set $B^A_n(M, N) := M \hat{\otimes} A^{\hat{\otimes} n} \hat{\otimes} N$, and the differential $d_{n+1} : B^A_{n+1}(M, N) \to B^A_n(M, N)$ is defined by

$$\begin{aligned} e \otimes a_0 \otimes \cdots \otimes a_n \otimes f \mapsto & a_0 e \otimes \cdots \otimes a_n \otimes f + (-1)^{n+1} e \otimes a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n f \\ & + \sum_{j=1}^{n} (-1)^j e \otimes a_0 \otimes \cdots \otimes a_{j-1} a_j \otimes \cdots \otimes a_n \otimes f. \end{aligned}$$

The topological Tor functor is $\hat{\mathrm{Tor}}_n^A(M,N) := H_n(B_\bullet^A(M,N))$, and the topological tensor product is $M\hat{\otimes}_A N := \hat{\mathrm{Tor}}_0^A(M,N)$.

In Definition 4.2.2, if M or N has a resolution by finite free A-modules, then one has $\hat{\mathrm{Tor}}_n^A(M,N) = \mathrm{Tor}_n^A(M,N)$ for all $n \geq 0$.

Definition 4.2.3. Let A be a Fréchet algebra, M and N be Fréchet modules over A. If the topology of $M\hat{\otimes}_A N$ is Hausdorff, and $\hat{\mathrm{Tor}}_n^A(M,N) = 0$ for all $n > 0$, then M and N are called *transversal* over A .

Let $f : X \to S$ and $g : Y \to S$ be two morphisms of complex analytic spaces. If for every triple $(x, y, s) \in X \times Y \times S$ with $f(x) = g(y) = s$, one has $\hat{\mathrm{Tor}}_q^{O_{S,s}}(O_{X,x}, O_{Y,y}) = 0$ for all $q > 0$, then we say that X and Y are *transversal* over S. Given a coherent O_X-module F and a coherent O_Y-module G, we refer the reader to [DV74, II, Déf. 7] for the definition of transversality of F and G over S. From [DV74, II, Remarques 2)], if F is flat over O_S, then F and G are transverse over S. If for every such triple of points (x, y, s), there exist fundamental systems of open neighborhoods $(X_i)_{i>0}$, $(Y_i)_{i>0}$ and $(S_i)_{i>0}$ of $x \in X$, $y \in Y$ and $s \in S$ respectively, such that for every integer $i > 0$,

- X_i, Y_i and S_i are Stein spaces,
- $f(X_i) \cup g(Y_i) \subset S_i$,
- and $O(X_i)$ and $O(Y_i)$ are transversal over $O(S_i)$,

then we write $X \perp_S Y$.

Lemma 4.2.4. Let $f : X \to S$ and $g : Y \to S$ be two morphisms of complex analytic spaces. Then X and Y are transversal over S if and only if $X \perp_S Y$.

To prove Lemma 4.2.4, we give a condition under which the topological Tor functor commutes with inductive limits of rings. Given a direct family $(E_i)_{i\in I}$ of locally convex spaces over $\mathbb{C}$, the finest locally convex topology on $E := \mathrm{colim}_i E_i$ for which the $E_i \to E$ are continuous is called the *inductive limit topology* of the (E_i).

Lemma 4.2.5. Let I be a countable directed set. Let $(A_i)_{i\in I}$ be a direct system of FN algebras. For every $i \in I$, let M_i and N_i be FN A_i-modules. For any $i \leq j$, let $f_{ij} : M_i \to M_j$ and $g_{ij} : N_i \to N_j$ be continuous linear maps compatible with $\alpha_{ij} : A_i \to A_j$. Assume that the inductive limit topologies on $A := \mathrm{colim}_i A_i$, $M := \mathrm{colim}_i M_i$ and $N := \mathrm{colim}_i N_i$ are Hausdorff. Fix $q \geq 0$. Suppose that for every $0 \leq n \leq q$ and every $i \in I$, $\hat{\mathrm{Tor}}_n^{A_i}(M_i, N_i)$ is Hausdorff. Then there is a natural A-linear isomorphism $\mathrm{colim}_{i\in I} \hat{\mathrm{Tor}}_q^{A_i}(M_i, N_i) \xrightarrow{\sim} \hat{\mathrm{Tor}}_q^A(M,N)$ of Hausdorff topological vector spaces.

Proof. As every multiplication $A_i \times A_i \to A_i$ is continuous, one gets a continuous map $A \times A \to A$ which makes A a commutative $\mathbb{C}$-algebra. As A is Hausdorff, it is a $\hat{\otimes}$-algebra. Similarly, one can prove that M, N are $\hat{\otimes}$-modules over A. Thus, $\hat{\mathrm{Tor}}_q^A(M,N)$ is well defined. First, we prove the case with $q = 0$. For every $i \in I$, there is a natural continuous $\mathbb{C}$-linear map $M_i\hat{\otimes}_{A_i} N_i \to M\hat{\otimes}_A N$ that is compatible with $\alpha_i : A_i \to A$. Thus, there is a natural continuous A-linear map

$$\tau : \mathrm{colim}_i(M_i\hat{\otimes}_{A_i} N_i) \to M\hat{\otimes}_A N.$$

From [Gro55, I, Prop. 2 3.], because A, M, N are Hausdorff, $M\hat{\otimes}N$ and $M\hat{\otimes}A\hat{\otimes}N$ are also Hausdorff. By [JT83, Thm. 2.8], because I is countable, the canonical morphisms

$$\mathrm{colim}_i(M_i\hat{\otimes}N_i) \xrightarrow{\sim} M\hat{\otimes}N, \quad \mathrm{colim}_i(M_i\hat{\otimes}A_i\hat{\otimes}N_i) \xrightarrow{\sim} M\hat{\otimes}A\hat{\otimes}N$$

are isomorphisms. From [JT83, Lemma 2.12], because every A_i is an FN algebra, M_i, N_i are FN A_i-modules, and every $M_i\hat{\otimes}_{A_i}N_i$ is Hausdorff, we deduce that $M\hat{\otimes}_A N$ is Hausdorff and the natural morphism $\mathrm{colim}_i(M_i\hat{\otimes}_{A_i}N_i) \to M\hat{\otimes}_A N$ is an isomorphism.

By [Gro55, I, Prop. 5 1. et II, Thm. 9 3.], as A_i, M_i and N_i are FN spaces, so is $B_n^{A_i}(M_i, N_i)$ for every $n \geq 0$. Similarly, every $B_n^A(M, N)$ is Hausdorff. The natural maps $B_n^{A_i}(M_i, N_i) \to B_n^A(M, N)$ are continuous and compatible with $\alpha_i : A_i \to A$. They are compatible with the differentials, so they define a morphism of complexes $B_\bullet^{A_i}(M_i, N_i) \to B_\bullet^A(M, N)$. By the proved case with $q = 0$, there is a natural isomorphism

$$\mathrm{colim}_i B_n^{A_i}(M_i, N_i) \xrightarrow{\sim} B_n^A(M, N)$$

of topological vector spaces. Therefore, the termwise colimit of the direct system $(B_\bullet^{A_i}(M_i, N_i))_{i\in I}$ of complexes is $B_\bullet^A(M, N)$. From [JT83, Lemma 2.12], every cohomology of the truncated complex $B_{\leq q+1}^{A_i}(M_i, N_i)$ is Hausdorff, so one has

$$\mathrm{colim}_{i\in I}\, \hat{\mathrm{Tor}}_q^{A_i}(M_i, N_i) \xrightarrow{\sim} \hat{\mathrm{Tor}}_q^A(M, N)$$

and $\hat{\mathrm{Tor}}_q^A(M, N)$ is Hausdorff. □

Proof of Lemma 4.2.4. The "only if" part is [KV71, Bemerkung 4.3.1]. We prove the "if" part. Let $(X_i)_i$, (Y_i) and (S_i) be as in the definition of $X \perp_S Y$. By [JT80, p.297], the inductive limit topologies on

$$\mathrm{colim}_i O(S_i) = O_{S,s}, \quad \mathrm{colim}_i O(X_i) = O_{X,x}, \quad \mathrm{colim}_i O(Y_i) = O_{Y,y}$$

are Hausdorff. Then from Lemma 4.2.5, for every $q > 0$, as $\hat{\mathrm{Tor}}_q^{O(S_i)}(O(X_i), O(Y_i)) = 0$, one has $\hat{\mathrm{Tor}}_q^{O_{S,s}}(O_{X,x}, O_{Y,y}) = 0$. □

Fact 4.2.6 is from [DV74, II, Sec. 4, Thm.]. It proves $X \perp_S Y$ whenever $f : X \to S$ or $g : Y \to S$ is flat. Together with Lemma 4.2.4, it proves that the converse of [EP96, Open problem 4.5.12] holds.

Fact 4.2.6. Let $f : X \to S$ and $g : Y \to S$ be two morphisms of complex analytic spaces. Assume that O_X and O_Y are transverse over S. Then for any *Stein* open subsets $U \subset X$, $V \subset Y$, $W \subset S$ with $f(U) \cup g(V) \subset W$, $O_X(U)$ and $O_Y(V)$ are transversal over $O_S(W)$, and one has $O_X(U)\hat{\otimes}_{O_S(W)}O_Y(V) = O_{X\times_S Y}(U \times_W V)$. In particular, one has $X \perp_S Y$.

By Fact 4.2.6, Theorem 1.3.2 is a special case of Theorem 4.2.7, which may be of independent interest even for coherent sheaves.

Theorem 4.2.7. *In the cartesian square* (4)*, assume further that X and S' are transversal over S and $f : X \to S$ is proper. Then $Lg^*Rf_* \to Rf'_*Lg'^*$ restricts to an isomorphism of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$.*

The proof of Lemma 4.2.8 is similar to that of Lemma 4.2.5.

Lemma 4.2.8. Let I be a directed set, $(A_i)_{i\in I}$ be a direct system of commutative rings. For every $i \in I$, let M_i and N_i be A_i-modules. For any $i \leq j$ in I, let $f_{ij} : M_i \to M_j$ and $g_{ij} : N_i \to N_j$ be maps compatible with $\alpha_{ij} : A_i \to A_j$. Let $A := \mathrm{colim}_i A_i$, $M := \mathrm{colim}_i M_i$ and $N := \mathrm{colim}_i N_i$. Then for every $q \geq 0$, there is a natural isomorphism $\mathrm{colim}_{i\in I} \mathrm{Tor}_q^{A_i}(M_i, N_i) \xrightarrow{\sim} \mathrm{Tor}_q^A(M, N)$ of A-modules.

Proof. First, we prove that case with $q = 0$. For every $i \in I$, there is a natural morphism $M_i \otimes_{A_i} N_i \to M \otimes_A N$ that is compatible with $\alpha_i : A_i \to A$. Thus, one gets a natural morphism

$$\mathrm{colim}_i(M_i \otimes_{A_i} N_i) \to M \otimes_A N \tag{7}$$

of A-modules. For every $j \in I$, the composition

$$M_j \times N_j \to M_j \otimes_{A_j} N_j \to \mathrm{colim}_i(M_i \otimes_{A_i} N_i)$$

is A_j-bilinear. Thus, it induces an A-bilinear map $M \times N \to \mathrm{colim}_i(M_i \otimes_{A_i} N_i)$, which induces a morphism

$$M \otimes_A N \to \mathrm{colim}_i(M_i \otimes_{A_i} N_i) \tag{8}$$

of A-modules. By universal properties, (7) and (8) are isomorphisms inverse to each other.

We construct a free resolution $P_\bullet(A, M) \to M \to 0$ inductively as follows. Let $P_{-1}(A, M) = M$, $P_0(A, M) := \oplus_{m\in M} A \cdot e_m$. Let $\epsilon : P_0(A, M) \to M$ be the canonical surjective morphism of A-modules defined by $e_m \mapsto m$. If

$$P_n(A, M) \to \cdots \to P_0(A, M) \to M \to 0$$

is constructed for some $n \geq 0$, then we set

$$K_n(A, M) := \ker\left(P_n(A, M) \to P_{n-1}(A, M)\right), \quad P_{n+1}(A, M) := P_0(A, K_n(A, M)),$$

and take $P_{n+1}(A, M) \to P_n(A, M)$ to be the composition of the canonical surjection $P_0(A, K_n(A, M)) \twoheadrightarrow K_n(A, M)$ with the inclusion $K_n(A, M) \hookrightarrow P_n(A, M)$. For any $i \leq j$ in I, there is a natural morphism of chain complexes $P_\bullet(A_i, M_i) \to P_\bullet(A_j, M_j)$ which is compatible with $\alpha_{ij} : A_i \to A_j$ and $f_{ij} : M_i \to M_j$.

By induction on $n \geq 0$, we prove that $\mathrm{colim}_i P_n(A_i, M_i) \to P_n(A, M)$ is an isomorphism. For every $\sum_{m\in M} a_m \cdot e_m \in P_0(A, M)$ with $a_m \in A$, there are only finitely many $m \in M$ with $a_m \neq 0$. Thus, there is $i_0 \in I$ such that whenever $a_m \neq 0$, m is in the image of $f_{i_0} : M_{i_0} \to M$, and a_m is in the image of $\alpha_{i_0} : A_{i_0} \to A$. Therefore, $\sum_{m\in M} a_m \cdot e_m$ lies in the image of $P_0(A_{i_0}, M_{i_0}) \to P_0(A, M)$. Hence, $\mathrm{colim}_i P_0(A_i, M_i) \to P_0(A, M)$ is surjective. For every $i \in I$ and every $\sum_{x\in M_i} a_x \cdot e_x$ in the kernel of $P_0(A_i, M_i) \to P_0(A, M)$ with $a_x \in A_i$, one has

$$\sum_{y\in M} \Big(\sum_{x\in f_i^{-1}(y)} \alpha_i(a_x)\Big) e_y = 0.$$

Therefore, for every $y \in M$, one has $\sum_{x\in f_i^{-1}(y)} \alpha_i(a_x) = 0$ in A. There are only finitely many $x \in M_i$ with $a_x \neq 0$, so there exist only finitely many $y \in M$ such that there is $x \in f_i^{-1}(y)$ with $a_x \neq 0$. Thus, there is $j \geq i$ in I such that for any such $y \in M$, $f_{ij}(\{x \in f_i^{-1}(y) \mid a_x \neq 0\})$ is a singleton, and $\alpha_{ij}(\sum_{x\in f_i^{-1}(y)} a_x) = 0$ in A_j. Then $\sum_{x\in M_i} a_x \cdot e_x$ maps to 0 in $P_0(A_j, M_j)$ and hence also in $\mathrm{colim}_i P_0(A_i, M_i)$. Therefore, $\mathrm{colim}_i P_0(A_i, M_i) \to P_0(A, M)$ is injective. The statement with $n = 0$ is proved.

Assume that for some $n_0 \geq 0$, the statement holds for all $0 \leq n \leq n_0$. The inductive hypothesis implies $\mathrm{colim}_i K_n(A_i, M_i) = K_n(A, M)$ for all $0 \leq n \leq n_0$. Thus, one has

$$\mathrm{colim}_i P_{n_0+1}(A_i, M_i) = \mathrm{colim}_i P_0(A_i, K_{n_0}(A_i, M_i)) = P_0(A, K_{n_0}(A, M)) = P_{n_0+1}(A, M).$$

The induction is completed. Similarly, one can prove that the isomorphisms are compatible with differential maps, so the termwise colimit of the directed family of complexes $(P_\bullet(A_i, M_i) \to M_i \to 0)_{i\in I}$ is $P_\bullet(A, M) \to M \to 0$. By construction, one has $\mathrm{Tor}_q^{A_i}(M_i, N_i) = H_q(P_\bullet(A_i, M_i) \otimes_{A_i} N_i)$. As filtered colimits are exact in $\mathrm{Mod}(A)$, one has

$$\begin{aligned}\mathrm{colim}_{i\in I}\,\mathrm{Tor}_q^{A_i}(M_i, N_i) &= H_q\left(\mathrm{colim}_i(P_\bullet(A_i, M_i) \otimes_{A_i} N_i)\right)\\ &\overset{(a)}{=} H_q(P_\bullet(A, M) \otimes_A N) = \mathrm{Tor}_q^A(M, N),\end{aligned}$$

where (a) uses the proved $q = 0$ case. □

Proof of Theorem 4.2.7. The morphism $f' : X' \to S'$ is a base change of $f : X \to S$, hence a proper morphism. By Theorem 3.2.1 and Proposition 3.1.2 (2), Lg^*Rf_* and $Rf'_*Lg'^*$ restrict to functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$. Consider the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "i'"'] \arrow[rr, bend left, "g'"] \arrow[d, "f'"] & S' \times X \arrow[r, "p'"'] \arrow[d, "\mathrm{Id}_{S'} \times f"] & X \arrow[d, "f"] \\ S' \arrow[r, "i"] \arrow[rr, bend right, "g"'] & S' \times S \arrow[r, "p"] & S, \end{tikzcd}$$

where the morphism $i : S' \to S' \times S$ is defined by $i(s') = (s', g(s'))$, and $p : S' \times S \to S$ is the projection. Then $i : S' \to S' \times S$ is a closed embedding of complex analytic spaces, and both squares are cartesian in **An**. By Remark 2.0.1 and Lemma 4.1.3, the natural transformation of functors

$$Lp^*Rf_* \to R(\mathrm{Id}_{S'} \times f)_*Lp'^* : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S' \times S)$$

is an isomorphism.

We prove that the square on the left is tor-independent, i.e., for every point $(x, s') \in X \times_S S'$ over $s \in S$, one has

$$\mathrm{Tor}_q^{O_{S'\times S,(s',s)}}(O_{S',s'}, O_{S'\times X,(s',x)}) = 0 \tag{9}$$

for all $q > 0$. By Lemma 4.2.4, one has $X \perp_S S'$, i.e., we may find arbitrarily small Stein open neighborhoods $X_0 \subset X$, $S_0 \subset S$ and $S'_0 \subset S'$ of x, s and s' respectively with $f(X_0) \cup g(S'_0) \subset S_0$, such that $O_X(X_0)$ and $O_{S'}(S'_0)$ are transversal over $O_S(S_0)$. In particular, one has

$$\hat{\mathrm{Tor}}_q^{O(S_0)}(O(S'_0), O(X_0)) = 0.$$

From Fact 4.2.6, as $p : S' \times S \to S$ is flat, $O(S'_0 \times S_0)$ and $O(X_0)$ are transversal over $O(S_0)$, and one has $O(S'_0 \times X_0) = O(S'_0 \times S_0)\hat{\otimes}_{O(S_0)}O(X_0)$. Then by [KV71, Bemerkung 1.12], one has

$$\hat{\mathrm{Tor}}_q^{O(S'_0\times S_0)}(O(S'_0), O(S'_0 \times X_0)) = 0. \tag{10}$$

By Lemma A.3.1, shrinking S_0 and S_0', one may assume that $i_*O_{S'}|_{S_0'\times S_0}$ has a resolution by finite free $O_{S_0'\times S_0}$-modules. By Cartan's Theorem B, $\Gamma(S_0'\times S_0, i_*O_{S'}) = O(S_0')$ has a resolution by finite free $O(S_0'\times S_0)$-modules. Therefore, (10) implies

$$\mathrm{Tor}_q^{O(S_0'\times S_0)}(O(S_0'), O(S_0'\times X_0)) = 0.$$

From Lemma 4.2.8, when X_0, S_0 and S_0' run through fundamental systems of Stein neighborhoods of x, s and s' respectively, one obtains (9).

By [Gro60, Remarque 2.10], because $i : S' \to S' \times S$ is a closed embedding, the square on the left is also cartesian in the category **RS**. By the base change theorem for ringed spaces [Rec19, Thm. 4.4.20] (the (†) assumption that Recktenwald requires is verified by Fact 3.2.3), as $\mathrm{Id}_{S'} \times f : S' \times X \to S' \times S$ is proper,

$$Li^*R(\mathrm{Id}_{S'}\times f) \to Rf'_*Li'^* : D(S'\times X) \to D(S')$$

is an isomorphism of functors. Thus, there are isomorphisms of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$

$$\begin{aligned} Lg^*Rf_* \cong Li^*Lp^*Rf_* &\xrightarrow{\sim} Li^*R(\mathrm{Id}_{S'}\times f)_*Lp'^* \\ \xrightarrow{\sim} Rf'_*Li'^*Lp'^* \cong Rf'_*Lg'^*. \end{aligned} \tag{11}$$

By [Sta26, Tag 0E47], the isomorphism (11) is induced by (5). $\square$

Remark 4.2.9. Theorem 1.3.2 would fail without properness of $f : X \to S$. For instance, take $X = S' = \mathbb{C}$ and $S = \mathrm{Specan}\,\mathbb{C}$. Since $f_*O_X = O(X)$, one has $g^*f_*O_X = O(X)\otimes_{\mathbb{C}} O_{S'}$. By Corollary A.5.4, it implies $\Gamma(S', g^*f_*O_X) = O(X)\otimes_{\mathbb{C}} O(S')$. One has $\Gamma(S', f'_*g'^*O_X) = O(X\times S')$.

Let x (resp. y) be the coordinate on X (resp. S'). We show that $e^{xy} \in O(\mathbb{C}^2)$ does not lie in the image of $O(\mathbb{C})\otimes_{\mathbb{C}} O(\mathbb{C}) \to O(\mathbb{C}^2)$. Suppose to the contrary that $\sum_{i=1}^n f_i(x)\otimes g_i(y)$ maps to e^{xy}, where $f_1,\dots,f_n \in O(X)$ and $g_1,\dots,g_n \in O(S')$. For every integer $m \geq 0$, $e^{my} = \sum_{i=1}^n f_i(m)g_i(y)$ lies in the finite dimensional $\mathbb{C}$-span of $g_1,\dots,g_n$ in $O(S')$. In particular, the sequence $(e^{my})_{m\geq 0}$ is linearly dependent over $\mathbb{C}$. There exists an integer $d \geq 0$ and $(a_0,\dots,a_d) \neq 0$ in $\mathbb{C}^{d+1}$, with $\sum_{m=0}^d a_m e^{my} = 0$ in $O(S')$. For every $z_0 \in \mathbb{C}\backslash\{0\}$, there is $y_0 \in \mathbb{C}$ with $e^{y_0} = z_0$. One has $\sum_{m=0}^d a_m z_0^m = 0$, so the nonzero polynomial $\sum_{m=0}^d a_m z^m \in \mathbb{C}[z]$ has infinitely many roots, which is a contradiction. As $\Gamma(S', g^*f_*O_X) \to \Gamma(S', f'_*g'^*O_X)$ is not surjective, $g^*f_*O_X \to f'_*g'^*O_X$ is not an isomorphism of $O_{S'}$-modules.

Remark 4.2.10. In [BBR94], the authors prove that the Fourier-Mukai transform for stable bundles on 2-dimensional complex tori and the Nahm transform for instanton bundles turn out to be essentially equivalent via Donaldson's correspondence. For arbitrary complex tori, they also show an inversion formula [BBR94, Cor. 3] for vector bundles satisfying W.I.T. In the proof of [BBR94, Lem. 5], an analytic flat base change result is applied without further justification. The excellent paper [MS08] surveys results on the automorphism groups of generic complex analytic K3 surfaces, and gives a comprehensive introduction to derived categories of coherent sheaves on complex manifolds. In [MS08, p.153], a flat base change theorem for complex manifolds is stated, referring to [Spa88] for the proof. However, the cited result [Spa88, Prop. 6.20] is for cartesian squares in the category **RS**. In general, a cartesian square in the category of complex manifolds may not be cartesian in **RS**, as Example 4.2.11 shows.

Example 4.2.11. Let **LRS** be the category of locally ringed spaces. The forgetful functors $\mathbf{An} \to \mathbf{LRS}$ and $\mathbf{An} \to \mathbf{RS}$ do not commute with products. For example, $\mathbb{C}^2$

is the product of two copies of $\mathbb{C}$ in **An**, but we shall prove that it is not the product in **LRS** nor **RS**. By contrast, from [Sta26, Tag 01JN], every cartesian square in the category of schemes remains cartesian in **LRS**.

In fact, by [Gil11, Cor. 5], the product E of two copies of $\mathbb{C}$ in **LRS** exists. By the universal property of E, there is a natural morphism $f : \mathbb{C}^2 \to E$ in **LRS** induced by the two projections $p_i : \mathbb{C}^2 \to \mathbb{C}$. Let $o = f(0) \in E$. We claim that the local ring $O_{E,o}$ is not Noetherian.

The local ring $A := O_{\mathbb{C},0} = \mathbb{C}\{z\}$ is the ring of convergent power series. Let $B = A \otimes_{\mathbb{C}} A$. Let $\epsilon : B \to A$ be the surjective diagonal morphism defined by $\epsilon(f \otimes g) = fg$. Set $I := \ker(\epsilon)$. Let $c : A \to \mathbb{C}$ be the ring map taking the constant term. Then $c\epsilon : B \to \mathbb{C}$ is surjective, so $m = \ker(c\epsilon)$ is a maximal ideal of B containing I. Set $S := B \setminus m$. Then $O_{E,o} = S^{-1}B$. From [Tu97, p.367], I/I^2 is a free B/I-module of *infinite* rank. Thus, $S^{-1}(I/I^2) = (S^{-1}I)/(S^{-1}I^2)$ is a free $S^{-1}(B/I) = (S^{-1}B)/(S^{-1}I)$-module of infinite rank. In particular, the ideal $S^{-1}I$ of the ring $S^{-1}B$ is not finitely generated. The claim is proved. By [GH78, p.679], the ring $\mathbb{C}\{x, y\}$ is Noetherian. Thus, the local morphism $f_0^{\#} : O_{E,o} \to O_{\mathbb{C}^2,0} = \mathbb{C}\{x, y\}$ is not an isomorphism. Hence, $f : \mathbb{C}^2 \to E$ is not an isomorphism in **LRS**.

Assume to the contrary that $\mathbb{C}^2$ is the product in **RS**. The two projections $q_i : E \to \mathbb{C}$ induce a morphism $g : E \to \mathbb{C}^2$ in **RS**. By universal properties, one has $fg = \mathrm{Id}_E$ and $gf = \mathrm{Id}_{\mathbb{C}^2}$ in **RS**. Therefore, $g : E \to \mathbb{C}^2$ is an isomorphism in **LRS**, which is a contradiction.

5 Mukai duality for complex tori

Theorem 5.0.1 (Mukai, Ben-Bassat, Block, Pantev)**.** *Let X be a complex torus of dimension g. Let $\hat{X}$ be the dual complex torus. There are isomorphisms of functors*

$$\mathcal{F} \circ \hat{\mathcal{F}} \cong [-1]_X^*[-g] : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X);$$
$$\hat{\mathcal{F}} \circ \mathcal{F} \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(\hat{X}).$$

In particular, $\mathcal{F} : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X)$ (resp. $\mathcal{F} : D^b_{\mathrm{coh}}(\hat{X}) \to D^b_{\mathrm{coh}}(X)$) is an equivalence of triangulated categories, with a quasi-inverse $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(\hat{X})$ (resp. $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(\hat{X})$) defined by $[-1]_{\hat{X}}^*\hat{\mathcal{F}}[g]$.

Remark 5.0.2. Given a compact complex manifold X, Block [Blo10, Thm. 4.3] uses cohesive modules to construct a dg-enhancement of $D^b_{\mathrm{coh}}(X)$, which enables one to study coherent sheaves using global differential geometric constructions. For dual complex tori X and $\hat{X}$, an equivalence $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(\hat{X}))$ similar to Theorem 5.0.1 is stated in [Blo10, p.314]. However, here we should stick to $D^b_{\mathrm{coh}}(-)$. In fact, in general the abelian category $\mathrm{Coh}(X)$ does not have enough injectives, so it is unclear how to define the derived direct image involved in [Blo10, p.314]. We discuss the comparison of $D^b(\mathrm{Coh}(-))$ and $D^b_{\mathrm{coh}}(-)$. For an algebraic variety M, the natural functor $D^b(\mathrm{Coh}(M)) \to D^b_{\mathrm{coh}}(M)$ is always an equivalence of triangulated categories. For a compact complex manifold X, the natural functor $D^b(\mathrm{Coh}(X)) \to D^b_{\mathrm{coh}}(X)$ is an equivalence if X is algebraic ([Hal23]) or if $\dim X \leq 2$ ([BB03, Cor. 5.2.2]). However, recently Prof. Bondal announced[1] that for $g > 2$ and a generic complex torus X of dimension g, $D^b(\mathrm{Coh}(X)) \to D^b_{\mathrm{coh}}(X)$ is *not* an equivalence.

We follow the strategy of [BBP07, Thm. 2.1] to prove Theorem 5.0.1.

[1] `https://www.mathnet.ru/eng/present35371`

5.1 Proof ingredients

Lemma 5.1.1, an analytic analog of [Muk81, Example 1.2], exhibits the derived pullback and derived direct image as particular examples of integral transforms.

Lemma 5.1.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. Let $i : \Gamma_f \hookrightarrow X \times Y$ be the inclusion of its graph. Set $F = i_* O_{\Gamma_f} \in \mathrm{Coh}(X \times Y)$. Then there are canonical isomorphism of functors

$$\phi_F^{[X \to Y]} \xrightarrow{\sim} Rf_* : D(X) \to D(Y); \tag{12}$$

$$\phi_F^{[Y \to X]} \xrightarrow{\sim} Lf^* : D(Y) \to D(X). \tag{13}$$

Proof. Let $g : \Gamma_f \to X$ be the projection. Since it is an isomorphism of complex analytic spaces, one has a canonical isomorphism of functors

$$Lg^* \cong R(g^{-1})_* : D(X) \to D(\Gamma_f). \tag{14}$$

Consider the commutative diagram

$$\begin{tikzcd} \Gamma_f \arrow[r, hook, "i"] \arrow[d, "g"] & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X \arrow[rr, "f"] & & Y. \end{tikzcd}$$

By [Sta26, Tag 0B55], as $i : \Gamma_f \hookrightarrow X \times Y$ is a closed embedding of complex analytic spaces, the natural transformation

$$Ri_* O_{\Gamma_f} \otimes^L Lp_X^*(-) \xrightarrow{\sim} Ri_* Li^* Lp_X^*(-) : D(X) \to D(X \times Y) \tag{15}$$

is an isomorphism of functors. One has

$$\begin{aligned} \phi_F^{[X \to Y]} :=& Rp_{Y*}(F \otimes^L p_X^* -) := Rp_{Y*}(Ri_* O_{\Gamma_f} \otimes^L Lp_X^* -) \\ \overset{(a)}{\xrightarrow{\sim}}& Rp_{Y*} Ri_* Li^* Lp_X^* \overset{(b)}{\xrightarrow{\sim}} Rp_{Y*} Ri_* Lg^* \\ \overset{(c)}{\xrightarrow{\sim}}& Rp_{Y*} Ri_* R(g^{-1})_* \overset{(d)}{\xrightarrow{\sim}} Rf_*, \end{aligned}$$

where (a) (resp. (c)) uses (15) (resp. (14)), and (b), (d) are from [Spa88, Thm. A (iii)]. Thus, (12) is proved. The proof of (13) is similar. □

Proposition 5.1.2 is an ingredient of the proof of Theorem 5.0.1, which expresses the composition of two integral transforms as another integral transform.

Proposition 5.1.2. *Let M, N, P be complex analytic spaces, with M, N* compact. *Let p_{ij} be the projections of the product $M \times N \times P$. For $K \in D_{\mathrm{gd}}(M \times N)$ and $L \in D(N \times P)$, set*

$$H := Rp_{13*}(p_{12}^* K \otimes^L p_{23}^* L) (\in D(M \times P)).$$

Then there is a natural isomorphism of functors

$$\phi_L^{[N \to P]} \circ \phi_K^{[M \to N]} \cong \phi_H^{[M \to P]} : D_{\mathrm{gd}}(M) \to D(P).$$

Proof. Let

$$\begin{aligned} a &: M\times N\to M, & b &: N\times P\to P,\\ p &: M\times N\to N, & q &: N\times P\to N,\\ u &: M\times P\to M, & v &: M\times P\to P \end{aligned}$$

be the projections.

By Propositions 3.1.2 (2) and 3.1.5 (2), $K\otimes^L a^*(-) : D(M)\to D(M\times N)$ restricts to a functor $D_{\mathrm{gd}}(M)\to D_{\mathrm{gd}}(M\times N)$. As M is compact, $p : M\times N\to N$ is proper. Then one can apply Theorem 1.3.2 to the cartesian square

$$\begin{tikzcd} M\times N\times P \arrow[r, "p_{12}"] \arrow[d, "p_{23}"'] \arrow[dr, phantom, "\square"] & M\times N \arrow[d, "p"] \\ N\times P \arrow[r, "q"'] & N, \end{tikzcd}$$

to get an isomorphism of functors

$$q^*Rp_*(K\otimes^L a^*-)\xrightarrow{\sim} Rp_{23*}p_{12}^*(K\otimes^L a^*-) : D_{\mathrm{gd}}(M)\to D_{\mathrm{gd}}(N\times P). \tag{16}$$

Thus, one has isomorphisms

$$\begin{aligned} \phi_L^{[N\to P]}\phi_K^{[M\to N]} :=& Rb_*\left(L\otimes^L q^*Rp_*(K\otimes^L a^*-)\right)\\ \overset{\text{(a)}}{\xrightarrow{\sim}}& Rb_*\left(L\otimes^L Rp_{23*}p_{12}^*(K\otimes^L a^*-)\right)\\ \overset{\text{(b)}}{\xrightarrow{\sim}}& Rb_*Rp_{23*}\left(p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*-)\right)\\ \cong& Rp_{3*}\left(p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*-)\right)\\ \cong& Rv_*Rp_{13*}(p_{12}^*K\otimes^L p_{23}^*L\otimes^L p_1^*-)\\ \overset{\text{(c)}}{\xleftarrow{\sim}}& Rv_*(H\otimes^L u^*-) =: \phi_H^{[M\to P]} \end{aligned}$$

of functors $D_{\mathrm{gd}}(M)\to D(P)$, where (a) uses (16), and (b) (resp. (c)) is from the compactness of M (resp. N) and Fact 3.2.4. $\square$

Fact 5.1.3 calculates the cohomology of the Poincaré bundle.

Fact 5.1.3 ([Kem91, Thm. 3.15])**.** Let X and $\hat{X}$ be complex tori of dimension g dual to each other. Let $p_X : X\times\hat{X}\to X$, $p_{\hat{X}} : X\times\hat{X}\to\hat{X}$ be the two projections. Then for the normalized Poincaré bundle $\mathcal{P}$, one has

$$Rp_{X*}\mathcal{P}=\mathbb{C}_{(0)}[-g]\in D^b_{\mathrm{coh}}(X),\quad Rp_{\hat{X}*}\mathcal{P}=\mathbb{C}_{(0)}[-g]\in D^b_{\mathrm{coh}}(\hat{X}).$$

5.2 Proof of Theorem 5.0.1

By Corollary 3.2.11, the functor $\mathcal{F}$ (resp. $\hat{\mathcal{F}}$) restricts to a functor $D_{\mathrm{gd}}(\hat{X})\to D_{\mathrm{gd}}(X)$ (resp. $D_{\mathrm{gd}}(X)\to D_{\mathrm{gd}}(\hat{X})$). Let p_{ij} be the projections of $X\times X\times\hat{X}$. Set

$$H := Rp_{12,*}(p_{13}^*\mathcal{P}\otimes^L p_{23}^*\mathcal{P}).$$

By Propositions 3.1.2 (1) and 3.1.5 (1), Fact 3.2.2 and Lemma 3.2.8, one has $H \in D^b_{\mathrm{coh}}(X \times X)$. By Proposition 5.1.2, one has an isomorphism of functors

$$\mathcal{F} \circ \hat{\mathcal{F}} \cong \phi_H^{[X \to X]} : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X).$$

Let $m : X \times X \to X$ be the group law. Since the $O_{X \times X \times \hat{X}}$-module $p_{13}^*\mathcal{P}$ is flat, one has $p_{13}^*\mathcal{P} \otimes^L p_{23}^*\mathcal{P} = p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}$. By [BL04, Lem. 14.1.7],[2] one has

$$p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P} \cong (m \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}.$$

It implies $H \cong Rp_{12,*}(m \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}$.

Applying Theorem 1.3.2 to the cartesian square

$$\begin{tikzcd}
X \times X \times \hat{X} \arrow[r, "m \times \mathrm{Id}_{\hat{X}}"] \arrow[d, "p_{12}"'] \arrow[dr, phantom, "\square"] & X \times \hat{X} \arrow[d, "p_X"] \\
X \times X \arrow[r, "m"'] & X,
\end{tikzcd}$$

one has $m^*Rp_{X,*}\mathcal{P} \cong H$. Let $i : \Gamma_{[-1]} \to X \times X$ be the inclusion of the graph of $[-1]_X : X \to X$. One deduces

$$H \overset{(a)}{\cong} m^*\mathbb{C}_{(0)}[-g] \overset{(b)}{\xrightarrow{\sim}} i_* O_{\Gamma_{[-1]}}[-g],$$

where (a) uses Fact 5.1.3, and (b) is applying Theorem 1.3.2 to the cartesian square

$$\begin{tikzcd}
\Gamma_{[-1]} \arrow[r] \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & 0 \arrow[d, hook] \\
X \times X \arrow[r, "m"] & X.
\end{tikzcd}$$

By Lemma 5.1.1, there is an isomorphism of functors $\phi_H^{[X \to X]} \xrightarrow{\sim} [-1]_X^*[-g] : D(X) \to D(X)$, which shows the isomorphism of functors

$$\mathcal{F} \circ \hat{\mathcal{F}} \cong [-1]_X^*[-g] : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X).$$

The proof of the second isomorphism is similar. That $\mathcal{F} : D^b_{\mathrm{coh}}(\hat{X}) \to D^b_{\mathrm{coh}}(X)$ is an equivalence follows from Corollary 3.2.11.

Remark 5.2.1. We discuss the Fourier-Mukai transform for liquid quasi-coherent sheaves. Using condensed mathematics, Clausen and Scholze [CS22, Def. 11.14] introduce the notion of generalized complex analytic spaces, generalizing the usual complex analytic spaces. To every generalized complex analytic space X, they [CS22, p.110] assign a closed symmetric monoidal stable ∞-category $\mathcal{D}_{\mathrm{Liq}}(X)$ of "derived liquid quasi-coherent sheaves" on X. This assignment supports a six functor formalism. For example, for every morphism $f : X \to Y$ of generalized complex analytic spaces, the pullback $f^* : \mathcal{D}_{\mathrm{Liq}}(Y) \to \mathcal{D}_{\mathrm{Liq}}(X)$ is a symmetric monoidal functor. It admits a right adjoint $f_* : \mathcal{D}_{\mathrm{Liq}}(X) \to \mathcal{D}_{\mathrm{Liq}}(Y)$ ([CS22, Prop. 12.10]). If $f : X \to Y$ is proper, then one has the projection formula

$$G \otimes f_*F \xrightarrow{\sim} f_*((f^*G) \otimes F)$$

[2]It is stated for abelian varieties, but its proof works for complex tori.

for all $F \in \mathcal{D}_{\mathrm{Liq}}(X)$ and $G \in \mathcal{D}_{\mathrm{Liq}}(Y)$, and the formation of $f_* : X \to Y$ commutes with arbitrary base change in Y ([CS22, Thm. 12.11]).

Now let X be a complex torus of dimension g, $\hat{X}$ be its dual, $\mathcal{P}$ be the normalized Poincaré bundle on $X \times \hat{X}$. Then $\mathcal{P}$ is an object of $\mathcal{D}_{\mathrm{Liq}}(X \times \hat{X})$. One can define an ∞-functor

$$\mathrm{FM}_X : \mathcal{D}_{\mathrm{Liq}}(X) \to \mathcal{D}_{\mathrm{Liq}}(\hat{X}), \quad F \mapsto p_{\hat{X}*}(\mathcal{P} \otimes p_X^* F).$$

It is likely that one can similarly prove the following liquid analog of Theorem 5.0.1: $\mathrm{FM}_{\hat{X}} \circ \mathrm{FM}_X$ is equivalent to $[-1]_X^*[-g]$, and $\mathrm{FM}_X \circ \mathrm{FM}_{\hat{X}}$ is equivalent to $[-1]_{\hat{X}}^*[-g]$, so that $\mathrm{FM}_X : \mathcal{D}_{\mathrm{Liq}}(X) \to \mathcal{D}_{\mathrm{Liq}}(\hat{X})$ is a categorical equivalence of ∞-categories.

6 Properties of Fourier-Mukai transform

For later reference purposes, we study basic properties of the Fourier-Mukai transform. We also check that each result starting from Theorem 2.2 to (3.12') in [Muk81] has a complex analytic version, and we only indicate the necessary modifications in statements and proofs.

For a complex torus X, let g_X be its dimension. Let $(\mathcal{F}_X, \hat{\mathcal{F}}_X)$ be the Fourier-Mukai transform of X. The subscripts are omitted when there is only one complex torus in context. Let $p_X : X \times \hat{X} \to X$, $p_{\hat{X}} : X \times \hat{X} \to \hat{X}$ be the projections. For a morphism $\phi : X \to Y$ of complex tori, let $\hat{\phi} : \hat{Y} \to \hat{X}$ be the dual morphism.

6.1 Functoriality

Exchange of the direct image and the inverse image

The Fourier-Mukai transform is functorial in the complex torus.

Proposition 6.1.1. *For a morphism $\phi : Y \to X$ of complex tori, there are isomorphisms of functors*

$$L\phi^* \circ \mathcal{F}_X \cong \mathcal{F}_Y \circ R\hat{\phi}_* : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y), \tag{17}$$

$$R\phi_* \circ \mathcal{F}_Y \cong \mathcal{F}_X \circ L\hat{\phi}^*(-)[g_X - g_Y] : D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(X). \tag{18}$$

Proof. The isomorphism (18) follows from (17) as follows. There are isomorphisms

$$\begin{aligned} R\phi_*\mathcal{F}_Y &\overset{\text{(a)}}{\cong} [-1]_X^* \mathcal{F}_X \hat{\mathcal{F}}_X R\phi_* \mathcal{F}_Y(-)[g_X] \\ &\overset{\text{(b)}}{\cong} [-1]_X^* \mathcal{F}_X L\hat{\phi}^* \hat{\mathcal{F}}_Y \mathcal{F}_Y(-)[g_X] \\ &\overset{\text{(c)}}{\cong} [-1]_X^* \mathcal{F}_X L\hat{\phi}^* [-1]_Y^*(-)[g_X - g_Y] \\ &= \mathcal{F}_X L\hat{\phi}^*(-)[g_X - g_Y] \end{aligned}$$

of functors $D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(X)$, where both (a) and (c) rely on Theorem 5.0.1, and (b) uses (17).

To prove (17), we show that there is an isomorphism of line bundles

$$(\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \cong (\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y. \tag{19}$$

Set $L := (\phi \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}_X \otimes_{O_{Y\times\hat{X}}} (\mathrm{Id}_Y \times \hat{\phi})^*\mathcal{P}_Y^{-1}$. By definition, on the one hand for every $\hat{x} \in \hat{X}$, one has $L|_{Y\times\hat{x}} \cong \phi^* P_{\hat{x}} \otimes P^{-1}_{\hat{\phi}(\hat{x})} \cong O_Y$; on the other hand, one has $L|_{0\times\hat{X}} \cong \hat{\phi}^* O_{\hat{Y}} \cong O_{\hat{X}}$. By [BL04, Cor. A.9], these imply $L \cong O_{Y\times\hat{X}}$ and hence (19).

Applying Theorem 1.3.2 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "p_2"] \arrow[d, "\mathrm{Id}_Y \times \hat{\phi}"'] \arrow[dr, phantom, "\Box"] & \hat{X} \arrow[d, "\hat{\phi}"] \\ Y \times \hat{Y} \arrow[r, "p_{\hat{Y}}"'] & \hat{Y}, \end{tikzcd}$$

one has an isomorphism of functors

$$p_{\hat{Y}}^* R\hat{\phi}_* \xrightarrow{\sim} R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y \times \hat{Y}). \tag{20}$$

By Propositions 3.1.2 (2) and 3.1.5 (2), $\mathcal{P}_X \otimes p_{\hat{X}}^*(-) : D(\hat{X}) \to D(X \times \hat{X})$ restricts to a functor $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X \times \hat{X})$. Applying Theorem 4.2.7 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "\phi\times\mathrm{Id}_{\hat{X}}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Box"] & X \times \hat{X} \arrow[d, "p_X"] \\ Y \arrow[r, "\phi"'] & X, \end{tikzcd}$$

one has an isomorphism of functors

$$L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* -) \xrightarrow{\sim} Rp_{1*} L(\phi \times \mathrm{Id}_{\hat{X}})^*(\mathcal{P}_X \otimes p_{\hat{X}}^* -) : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y). \tag{21}$$

There are isomorphisms

$$\begin{aligned} L\phi^* \circ \mathcal{F}_X &= L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* -) \\ &\overset{(a)}{\xrightarrow{\sim}} Rp_{1*} L(\phi \times \mathrm{Id}_{\hat{X}})^*(\mathcal{P}_X \otimes p_{\hat{X}}^* -) \\ &\cong Rp_{1*}\left(L(\phi \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}_X \otimes^L L(\phi \times \mathrm{Id}_{\hat{X}})^* p_{\hat{X}}^* -\right) \\ &\cong Rp_{1*}\left((\phi \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}_X \otimes p_2^* -\right) \\ &\overset{(b)}{\cong} Rp_{1*}\left((\mathrm{Id}_Y \times \hat{\phi})^*\mathcal{P}_Y \otimes p_2^* -\right) \\ &\cong Rp_{Y*} R(\mathrm{Id}_Y \times \hat{\phi})_*\left(L(\mathrm{Id}_Y \times \hat{\phi})^*\mathcal{P}_Y \otimes p_2^* -\right) \\ &\overset{(c)}{\xleftarrow{\sim}} Rp_{Y*}\left(\mathcal{P}_Y \otimes R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* -\right) \\ &\overset{(d)}{\xleftarrow{\sim}} Rp_{Y*}(\mathcal{P}_Y \otimes p_{\hat{Y}}^* R\hat{\phi}_* -) \\ &=: \mathcal{F}_Y \circ R\hat{\phi}_* \end{aligned}$$

of functors $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y)$, where (a), (b), (c) and (d) use (21), (19), Fact 3.2.4 and (20) respectively. This proves (17). $\square$

Commutativity with external tensor product

Let M, N be two complex analytic spaces. Let $p : M \times N \to M$ and $q : M \times N \to N$ be the projections. The derived external tensor product is the bifunctor

$$\boxtimes^L : D(M) \times D(N) \to D(M \times N), \quad (-, +) \mapsto (p^* -) \otimes^L (q^* +).$$

Proposition 6.1.2. *Let X, Y be two complex tori and $Z = X \times Y$. Then there is an isomorphism $\mathcal{F}_Z(- \boxtimes^L +) \cong \mathcal{F}_X(-) \boxtimes^L \mathcal{F}_Y(+)$ of bifunctors $D_{\mathrm{gd}}(\hat{X}) \times D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(Z)$.*

Lemma 6.1.3.

(1) Let X, Y and T be complex analytic spaces. Let $f : X \to Y$ be a *proper* morphism. Then there is a canonical isomorphism

$$(Rf_* -) \boxtimes^L (+) \to R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +)$$

of bifunctors $D_{\mathrm{gd}}(X) \times D(T) \to D(Y \times T)$.

(2) Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be *proper* morphism of complex analytic spaces. Then there is a canonical isomorphism of bifunctors

$$(Rf_{1*}-)\boxtimes^L(Rf_{2*}+) \to R(f_1 \times f_2)_*(-\boxtimes^L +) : D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2).$$

Proof.

(1) Consider the notation in the commutative diagram

$$\begin{tikzcd} & X \times T \arrow[r, "u"] \arrow[dl, "v"'] \arrow[d, dashed, "f \times \mathrm{Id}_T"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T & Y \times T \arrow[l, "q"] \arrow[r, "p"'] & Y, \end{tikzcd}$$

where u, v, p and q are projections. Since $v = q \circ (f \times \mathrm{Id}_T)$, there is a canonical isomorphism of functors

$$v^* \cong L(f \times \mathrm{Id}_T)^* q^* : D(T) \to D(X \times T).$$

By Fact 3.2.4 and properness of $f \times \mathrm{Id}_T : X \times T \to Y \times T$, there is an isomorphism of bifunctors

$$(R(f \times \mathrm{Id}_T)_* u^* -) \otimes^L q^*(+) \to R(f \times \mathrm{Id}_T)_*(u^* - \otimes^L v^* +) : D(X) \times D(T) \to D(Y \times T). \tag{22}$$

By Theorem 1.3.2, one has an isomorphism of functors

$$p^* Rf_* \to R(f \times \mathrm{Id}_T)_* u^* : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y \times T). \tag{23}$$

Therefore, there are canonical isomorphisms

$$\begin{aligned} (Rf_* -) \boxtimes^L (+) :=& (p^* Rf_* -) \otimes^L (q^* +) \\ \overset{(a)}{\xrightarrow{\sim}}& (R(f \times \mathrm{Id}_T)_* u^* -) \otimes^L (q^* +) \\ \overset{(b)}{\xrightarrow{\sim}}& R(f \times \mathrm{Id}_T)_*(u^* - \otimes v^* +) \\ =& R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +), \end{aligned}$$

of bifunctors $D_{\mathrm{gd}}(X) \times D(T) \to D(Y \times T)$, where (a) (resp. (b)) uses (23) (resp. (22)).

(2) As in Corollary 3.2.11, since $f_i : X_i \to Y_i$ are proper, $R(f_1 \times f_2)_*(- \boxtimes^L +) : D(X_1) \times D(X_2) \to D(Y_1 \times Y_2)$ restricts to a bifunctor $D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2)$. By Part (1), there are canonical isomorphisms of bifunctors

$$(Rf_{1*}-) \boxtimes^L (+) \xrightarrow{\sim} R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +) : D_{\mathrm{gd}}(X_1) \times D(X_2) \to D(Y_1 \times X_2),$$
$$(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \xrightarrow{\sim} R(\mathrm{Id}_{Y_1} \times f_2)_* \Big((Rf_{1*}-) \boxtimes^L +\Big) : D(X_1) \times D_{\mathrm{gd}}(X_2) \to D(Y_1 \times Y_2)$$

and hence an isomorphism

$$\begin{aligned}(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) &\xrightarrow{\sim} R(\mathrm{Id}_{Y_1} \times f_2)_* \Big((Rf_{1*}-) \boxtimes^L +\Big)\\ &\xrightarrow{\sim} R(\mathrm{Id}_{Y_1} \times f_2)_* R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +)\\ &\xrightarrow{\sim} R(f_1 \times f_2)_*(- \boxtimes^L +)\end{aligned}$$

of bifunctors $D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2)$. □

Proof of Proposition 6.1.2. By the seesaw principle, there is an isomorphism $\mathcal{P}_Z \cong \mathcal{P}_X \boxtimes^L \mathcal{P}_Y$. Then there exist isomorphisms

$$\begin{aligned}\mathcal{F}_Z(- \boxtimes^L +) :=& Rp_{Z*} \Big(\mathcal{P}_Z \otimes^L Lp_{\hat{Z}}^*(- \boxtimes^L +)\Big)\\ \cong& Rp_{Z*} \Big((\mathcal{P}_X \boxtimes^L \mathcal{P}_Y) \otimes^L (Lp_{\hat{X}}^*(-) \boxtimes^L Lp_{\hat{Y}}^*(+))\Big)\\ \xrightarrow{\sim}& R(p_X \times p_Y)_* \Big((\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L (\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+))\Big)\\ \overset{\text{(a)}}{\xleftarrow{\sim}}& Rp_{X*}(\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L Rp_{Y*}(\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+))\\ =& \mathcal{F}_X(-) \boxtimes^L \mathcal{F}_Y(+)\end{aligned}$$

of bifunctors $D_{\mathrm{gd}}(\hat{X}) \times D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(Z)$, where (a) uses Lemma 6.1.3 (2). □

Skew commutativity with duality

We summarize classical facts about the duality theory on complex manifolds.

Fact 6.1.4. Let X be a complex manifold of pure dimension n, and let $\omega_X = \bigwedge^n \Omega_X^1$ be its canonical line bundle.

(1) ([RR70, p.81, p.90]) The dualizing functor $D_X = R\mathcal{H}om_X(-, \omega_X)[n] : D(X)^{\mathrm{op}} \to D(X)$ restricts to an equivalence $D^b_{\mathrm{coh}}(X)^{\mathrm{op}} \cong D^b_{\mathrm{coh}}(X)$, and the natural transformation $\mathrm{Id} \to D_X \circ D_X : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$ is an isomorphism of functors.

(2) ([RRV71, p.264]) There is a canonical isomorphism $R\mathcal{H}om_X(-, +) \cong D_X(- \otimes^L D_X +)$ of bifunctors $D^b_{\mathrm{coh}}(X)^{\mathrm{op}} \times D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$

Proposition 6.1.5 ([Muk81, (3.8)]). *Let X be a complex torus of dimension g. Then there exist isomorphisms of functors*

$$D_X \circ \mathcal{F} \cong ([-1]_X^* \circ \mathcal{F} \circ D_{\hat{X}})[g] : D^b_{\mathrm{coh}}(\hat{X}) \to D^b_{\mathrm{coh}}(X);$$
$$D_{\hat{X}} \circ \hat{\mathcal{F}} \cong ([-1]_{\hat{X}}^* \circ \hat{\mathcal{F}} \circ D_X)[g] : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(\hat{X}).$$

Proposition 6.1.5 follows from Lemma 6.1.6 and Corollary 6.1.8.

Lemma 6.1.6. Let E be a vector bundle on a complex manifold X, and let $E^\vee$ be the dual vector bundle. Then there is a canonical isomorphism $E^\vee \otimes_{O_X} D_X(-) \cong D_X(E \otimes_{O_X} -)$ of contravariant functors $D(X) \to D(X)$.

Proof. Since E is a vector bundle, one has isomorphisms

$$E \otimes (-) \cong \mathcal{H}om_X(E^\vee, -) \cong R\mathcal{H}om_X(E^\vee, -)$$

of functors $D(X) \to D(X)$. Then

$$D_X(E \otimes -) \cong R\mathcal{H}om_X(R\mathcal{H}om_X(E^\vee, -), \omega_X)[\dim X].$$

By [Sta26, Tag 0G40], as $E^\vee$ is a perfect object of $D(X)$ in the sense of [Sta26, Tag 08CM], one has $D_X(E \otimes -) \cong R\mathcal{H}om_X(-, \omega_X)[\dim X] \otimes^L E^\vee \cong E^\vee \otimes D_X(-)$. $\square$

Lemma 6.1.7 is an adaption of [Har66, Ch.II, Prop. 5.8] and [Sta26, Tag 0C6I].

Lemma 6.1.7. Let $f : X \to Y$ be a *flat* morphism of complex analytic spaces. Then:

(1) There is a canonical natural transformation of bifunctors

$$f^* R\mathcal{H}om_Y(-, +) \to R\mathcal{H}om_X(f^*-, f^*+) : D(Y) \times D(Y) \to D(X). \quad (24)$$

(2) The natural transformation (24) restricts to an isomorphism of bifunctors $D^b_{\mathrm{coh}}(Y) \times D(Y) \to D(X)$.

Proof. Consider $G \in D(Y)$.

(1) By [Spa88, Thm. D], there is a *functorial* quasi-isomorphism $G \to I^\bullet$, where $I^\bullet$ is a K-injective complex over $\mathrm{Mod}(O_Y)$. There are natural transformations of functors $D(Y) \to D(X)$

$$\begin{aligned} f^* R\mathcal{H}om_Y(-, G) &\to f^* \mathcal{H}om_Y(-, I^\bullet) \to \mathcal{H}om_X(f^*-, f^* I^\bullet) \\ \to & R\mathcal{H}om_X(f^*-, f^* I^\bullet) \xleftarrow{\sim} R\mathcal{H}om_X(f^*-, f^* G). \end{aligned}$$

(2) By [Har66, I, Examples 1], the contravariant functors

$$f^* R\mathcal{H}om_Y(-, G), R\mathcal{H}om_X(f^*-, f^* G) : D(Y) \to D(X)$$

are bounded below. By [Har66, I, Prop. 7.1 (ii)], for $F \in D^b_{\mathrm{coh}}(Y)$, to show that the natural morphism

$$f^* R\mathcal{H}om_Y(F, G) \to R\mathcal{H}om_X(f^* F, f^* G) \quad (25)$$

is an isomorphism, one may assume $F \in \mathrm{Coh}(Y)$. By [Sta26, Tag 08DL], one may shrink Y to open subsets. Thus, from Lemma A.3.1, one may assume that there is a quasi-isomorphism $K^\bullet \to F$, where $K^\bullet$ is a complex of finite free O_Y-modules. As $f : X \to Y$ is flat, $f^* K^\bullet \to f^* F \to 0$ is a globally free resolution of $f^* F$. The morphism (25) is identified with $f^* \mathcal{H}om_Y(K^\bullet, G) \to \mathcal{H}om_X(f^* K^\bullet, f^* G)$, which is an isomorphism. $\square$

Corollary 6.1.8. Let $f : X \to Y$ be a flat morphism of complex manifolds of relative dimension n. Write ω_f for the relative dualizing line bundle $\omega_X \otimes_{O_X} f^* \omega_Y^\vee$. Then there is a canonical isomorphism of functors $D_X f^* D_Y \cong \omega_f \otimes_{O_X} f^*(-)[n] : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$.

Proof. One has

$$\begin{aligned}
&D_X f^* D_Y O_Y = D_X\big(f^* R\mathcal{H}om_Y(O_Y,\omega_Y)[\dim Y]\big) = D_X(f^*\omega_Y[\dim Y]) \\
=&R\mathcal{H}om_X(f^*\omega_Y,\omega_X)[\dim X - \dim Y] \overset{(a)}{=} \mathcal{H}om_X(f^*\omega_Y,\omega_X)[n] \\
=&f^*\omega_Y^\vee \otimes_{O_X} \omega_X[n] = \omega_f[n],
\end{aligned} \tag{26}$$

where (a) uses that $f^*\omega_Y$ is a line bundle on X.

By Fact 6.1.4 (1) and (2), there is an isomorphism of functors

$$D_Y \xrightarrow{\sim} R\mathcal{H}om_Y(-, D_Y O_Y) : D^b_{\mathrm{coh}}(Y)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(Y).$$

From flatness of $f : X \to Y$ and Lemma 6.1.7 (2), there are isomorphisms

$$f^* D_Y \xrightarrow{\sim} f^* R\mathcal{H}om_Y(-, D_Y O_Y) \xrightarrow{\sim} R\mathcal{H}om_X\,(f^*(-), f^* D_Y O_Y)$$

of functors $D^b_{\mathrm{coh}}(Y)^{\mathrm{op}} \to D^b_{\mathrm{coh}}(X)$. Then by Fact 6.1.4 (1) and (2) again, there are isomorphisms

$$\begin{aligned}
&D_X f^* D_Y \xrightarrow{\sim} f^*(-) \otimes^L D_X f^* D_Y O_Y \\
\overset{(a)}{=}&f^*(-) \otimes^L_{O_X} \omega_f[n] \overset{(b)}{=} f^*(-) \otimes_{O_X} \omega_f[n]
\end{aligned}$$

of functors $D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$, where (a) (resp. (b)) uses (26) (resp. local freeness of ω_f). ☐

6.2 Unipotent vector bundles

In Section 6.2, we show that extensions of O_X on the complex torus X correspond precisely to coherent sheaves supported in the origin of the dual $\hat{X}$ *via* the Fourier-Mukai transform.

Definition 6.2.1. We say that W.I.T. (weak index theorem) holds for a coherent sheaf F on the complex torus X if there is an integer $i(F)$ with $H^i\hat{\mathcal{F}}(F) = 0$ for all integers $i \neq i(F)$. In that case, the integer $i(F)$ is called the *index* of F and the coherent sheaf $\hat{F} := H^{i(F)}\hat{\mathcal{F}}(F)$ on $\hat{X}$ is called the Fourier transform of F. We say that I.T. (index theorem) holds for F if there is an integer i_0 such that for every $L \in \mathrm{Pic}^0(X)$ and every integer $i \neq i_0$, one has $H^i(X, F \otimes_{O_X} L) = 0$.

Definition 6.2.2. A vector bundle U on a complex analytic space X is *unipotent* if it has a filtration by vector subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

with $U_i/U_{i-1} \cong O_X$ for all $1 \leq i \leq n$. Denote the full subcategory of $\mathrm{Coh}(X)$ consisting of unipotent vector bundles by $\mathrm{Uni}(X)$.

A rank 2 unipotent vector bundle on a complex analytic space X is equivalent to an extension $0 \to O_X \to U \to O_X \to 0$. Such extensions are classified by $\mathrm{Ext}^1(O_X, O_X) = H^1(X, O_X)$. By [FL14, Lem. 5.1], every unipotent vector bundle on a complex torus admits a flat holomorphic connection such that the underlying local system is unipotent. Atiyah [Ati57, Thm. 5] proves that when X is an elliptic curve, for every $r > 0$, there is (up to isomorphism) a unique indecomposable unipotent vector bundle of rank r.

Proposition 6.2.3. *Let X be a complex torus of dimension g, $\hat{X}$ be its dual complex torus.*

(1) *W.I.T. with index g holds for all unipotent vector bundles on X.*

(2) *The functor $H^g\hat{\mathcal{F}} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to an equivalence $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$, with a quasi-inverse $H^0\mathcal{F} = \mathcal{F} : \mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$.*

Lemma 6.2.4. Let $F \to A$ be a ring map, with F a field and (A, m) an Artinian local ring. If $[A/m : F]$ is finite, then $\dim_F A$ is finite.

Proof. By [Ati69, Prop. 8.4], because A is an Artinian local ring ring, there is an integer $n > 0$ with $m^n = 0$. For every integer $i \geq 0$, the A-module m^i is finitely generated, so the A/m-module m^i/m^{i+1} is finitely generated. Thus, $\dim_F(m^i/m^{i+1}) = [A/m : F] \dim_{A/m}(m^i/m^{i+1})$ is finite. Then $\dim_F A = \sum_{i=0}^{n} \dim_F(m^i/m^{i+1})$ is finite. □

For a commutative ring R, let $\mathrm{Mod}_f(R) \subset \mathrm{Mod}(R)$ be the full subcategory comprised of R-modules of finite length.

Lemma 6.2.5. Let X be a complex analytic space. Let $x \in X$.

(1) The functor $i_x^{-1} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X,x})$ taking the stalk at x restricts to a functor $\mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$. In particular, if X is a singleton, then $\dim_{\mathbb{C}} O_X$ is finite.

(2) The functor $i_{x,*} : \mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_X)$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.

(3) The functor $i_x^{-1} : \mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$ is an equivalence.

Proof. (1) For every $F \in \mathrm{Coh}_x(X)$, to prove that F_x is a finite length $O_{X,x}$-module, one may assume $F_x \neq 0$. As F is a finite type O_X-module, F_x is a finite $O_{X,x}$-module. Then $\mathrm{Supp}_{O_{X,x}}(F_x)$ is a nonempty subset of $\mathrm{Spec}\, O_{X,x}$. Let m_x be the maximal ideal of $O_{X,x}$. For every $f \in m_x$, there is an open neighborhood U of x in X such that f is the stalk of some $\bar{f} \in O_X(U)$. Then $\bar{f}$ vanishes on $\mathrm{Supp}(F)$. By the Rückert Nullstellensatz (see, *e.g.*, [CAS, p.67]), there is an integer $n \geq 1$ such that $\bar{f}^n F = 0$ near x. In particular, one has $f \in \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}$, which implies $m_x \subset \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}$. By [CAS, Corollary, p.44], the ideal m_x is finitely generated, so there is an integer $N \geq 1$ with $m_x^N \subset \mathrm{Ann}_{O_{X,x}}(F_x)$. By [Sta26, Tag 00L6], $\mathrm{Supp}_{O_{X,x}}(F_x)$ is the unique closed point of $\mathrm{Spec}(O_{X,x})$. By [Sta26, Tag 00L5], the $O_{X,x}$-module F_x has finite length. The second statement follows from Lemma 6.2.4.

(2) Every simple $O_{X,x}$-module is isomorphic to the residue field $\mathbb{C}$. Every $M \in \mathrm{Mod}_f(O_{X,x})$ has a composition series with successive quotients isomorphic to $\mathbb{C}$. Thus, the O_X-module $M_{(x)}$ has a filtration with successive quotients isomorphic to $\mathbb{C}_{(x)}$. By [Sta26, Tag 01BY (4)], since $\mathbb{C}_{(x)}$ is coherent, so is $M_{(x)}$. Therefore, $i_{x,*}$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.

(3) Let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. There is a canonical isomorphism $i_x^*(i_x)_* \xrightarrow{\sim} \mathrm{Id}_{\mathrm{Mod}(O_{X,x})}$ of functors $\mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_{X,x})$. By adjunction, $(i_x)_* : \mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_X)$ is fully faithful. By Point (2), $(i_x)_* : \mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_X)$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$. By Point (1), for every object $F \in$

$\mathrm{Coh}_x(O_X)$, the stalk F_x is an object of $\mathrm{Mod}_f(O_{X,x})$. The adjunction morphism $F \to (i_x)_*(F_x)$ is an isomorphism. Thus, $(i_x)_* : \mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$ is essentially surjective and hence an equivalence. Therefore, the functor $i_x^{-1} = i_x^* : \mathrm{Coh}_x(O_X) \to \mathrm{Mod}_f(O_{X,x})$ is an equivalence. □

Proof of Proposition 6.2.3. (1) Because $\hat{\mathcal{F}} : D(X) \to D(\hat{X})$ is a triangulated functor, the full subcategory of $\mathrm{Coh}(X)$ comprised of modules satisfying W.I.T. of a fixed index is closed under extensions. One has $\hat{\mathcal{F}}(O_X) = \hat{\mathcal{F}}\mathcal{F}(\mathbb{C}_{(0)}) \cong \mathbb{C}_{(0)}[-g]$, where the equality uses Lemma 2.0.9, and the isomorphism is from Theorem 5.0.1. Then W.I.T. with index g holds for O_X, so it holds for all unipotent vector bundles on X.

(2) By Point (1), one has an isomorphism of functors $H^g\hat{\mathcal{F}} \xrightarrow{\sim} \hat{\mathcal{F}}[g] : \mathrm{Uni}(X) \to \mathrm{Mod}(O_{\hat{X}})$. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of objects F with $\mathrm{Supp}\left(H^g\hat{\mathcal{F}}(F)\right) \subset \{0\}$ is closed under extensions and contains O_X, so it contains $\mathrm{Uni}(X)$. Since $\mathrm{Uni}(X) \subset \mathrm{Coh}(X)$, $H^g\hat{\mathcal{F}} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$.

For every $F \in \mathrm{Coh}_0(\hat{X})$, the projection $p_X : X \times \hat{X} \to X$ restricts to a finite morphism $\mathrm{Supp}(p_{\hat{X}}^* F \otimes \mathcal{P}) \to X$. By [GR04, Thm. 4, p.47], one has $\mathcal{F}(F) = H^0\mathcal{F}(F)$. By Lemma 6.2.5 (3), the $O_{\hat{X}}$-module F has a filtration

$$F = F_0 \supset F_1 \supset \cdots \supset F_n = 0$$

with the F_i coherent and $F_i/F_{i+1} \cong \mathbb{C}_{(0)}$ for all $0 \leq i < n$. Then $\mathcal{F}(F)$ has a filtration

$$\mathcal{F}(F) = E_0 \supset E_1 \supset \cdots \supset E_n = 0$$

with the E_i coherent and $E_i/E_{i+1} \cong \mathcal{F}(\mathbb{C}_{(0)}) = O_X$ for all $0 \leq i < n$. By [EGA I, Ch. 0, 5.4.9], every E_i is finite locally free over X. Therefore, $\mathcal{F}(F) \in \mathrm{Uni}(X)$ and $\mathcal{F} : D(\hat{X}) \to D(X)$ restricts to a functor $\mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$. By Theorem 5.0.1, the functor $H^g\hat{\mathcal{F}} : \mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$ is an equivalence with a quasi-inverse $\mathcal{F}$. □

6.3 Homogeneous vector bundles

As an application of the Fourier-Mukai transform, we prove Matsushima and Morimoto's classification of homogeneous vector bundles on complex tori.

Definition 6.3.1. A vector bundle E on a complex torus X is *homogeneous* if for every $x \in X$, one has $T_x^* E \cong E$. Let $H(X) \subset \mathrm{Coh}(X)$ be the full subcategory comprised of homogeneous vector bundles.

On on a complex torus X, every unipotent vector bundle is homogeneous, and a line bundle L is homogeneous exactly when $[L] \in \mathrm{Pic}^0(X)$. Moreover, $H(X)$ is closed under tensor products and finite direct sums in $\mathrm{Mod}(O_X)$. The classification of homogeneous vector bundles on complex tori is due to Matsushima [Mat59, Thm. 3] and Morimoto [Mor59]. Using the algebraic Fourier-Mukai transform, Mukai [Muk81, p.159] recovers the algebraic analog for abelian varieties due to Miyanishi [Miy73]. Similarly, we recover Matsushima-Morimoto's original theorem using the analytic Fourier-Mukai transform.

Theorem 6.3.2 (Matsushima-Morimoto)**.** *A vector bundle F on a complex torus X is homogeneous if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \ldots, U_n$ on X and $L_1, \ldots, L_n \in \mathrm{Pic}^0(X)$ with $F \cong \oplus_{i=1}^n (L_i \otimes U_i)$.*

For a complex analytic space X, let $\mathrm{Coh}_f(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of objects with finite support. It is a Serre subcategory of $\mathrm{Coh}(X)$.

Proposition 6.3.3. *Let X be a complex torus of dimension g. Let $\hat{X}$ be the complex torus dual to X.*

(1) *For every integer i, $H^i\hat{\mathcal{F}} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$.*
(2) *W.I.T. with index g holds for every homogeneous vector bundle on X.*
(3) *The functor $H^g\hat{\mathcal{F}} : H(X) \to \mathrm{Coh}_f(\hat{X})$ is an equivalence of categories, with a quasi-inverse $H^0\mathcal{F} : \mathrm{Coh}_f(\hat{X}) \to H(X)$.*

Lemma 6.3.6 is a complex analytic counterpart of [Muk81, Lem. 3.3]. The proof of [Muk81, Lem. 3.3] relies on the following fact: Every positive dimensional projective variety contains a projective curve. By contrast, its direct analog fails in the complex analytic case. Indeed, by [Pil00, Lem. 4.3], every simple non-algebraic complex torus contains no 1-dimensional analytic subset. We use Lemma 6.3.5 as a replacement of this fact.

Definition 6.3.4. Let X be a *reduced* complex analytic space. Let F be an O_X-module. For every $x \in X$, let $T(F_x)$ be the set of $s_x \in F_x$ such that there is a non zero divisor $g_x \in O_{X,x}$ with $g_x s_x = 0$. Then $T(F) := \cup_{x\in X} T(F_x)$ is an O_X-submodule of F, called the *torsion* of F.

Lemma 6.3.5. Let X be a compact Kähler manifold. Let F be a coherent O_X-module. Then for every irreducible component C of $\mathrm{Supp}(F)$, there is a connected compact Kähler manifold Z and a morphism $h : Z \to X$, such that $h(Z) = C$ and $h^*F/T(h^*F)$ is a vector bundle on Z of positive rank.

Proof. By [CAS, p.76], $\mathrm{Supp}(F)$ is an analytic subset of X. Because X is a Kähler manifold, with the reduced induced complex structure, the subspace C is a Kähler space in the sense of [Var89, II, 1.3]. Let $i : C \to X$ be the inclusion. Set

$$D = \{x \in C \mid i^*F \text{ is not locally free at } x\}.$$

From [Ros68, Prop. 3.1], D is a strict analytic subset of C. By Rossi's theorem (see, *e.g.* [Rie71, Thm. 2]), there is a reduced irreducible complex analytic space W and a proper modification $f : W \to C$, such that $W \setminus f^{-1}(D) \to C \setminus D$ is biholomorphic and that $E := N/T(N)$ is a *vector bundle* on W, where $N := f^*i^*F \in \mathrm{Mod}(O_W)$. From [GD71, Cor. 5.2.4.1], one has $\mathrm{Supp}(N) = W$. From [CD94, I, Thm. 9.12], one gets $\mathrm{Supp}(T(N)) \neq W$. Therefore, the rank r of the vector bundle E is positive.

Since $f : W \to C$ is bimeromorphic, W is in the Fujiki class $\mathcal{C}$ (defined in [Fuj78, p.34]). By [Fuj78, Lem. 4.6, 1)], there exist a connected compact Kähler manifold Z and a surjective morphism $g : Z \to W$. Denote the composition

$$Z \xrightarrow{g} W \xrightarrow{f} C \xhookrightarrow{i} X$$

by h. Then $h(Z) = C$. By [Sta26, Tag 05NJ], as E is flat over O_W, applying g^* to the natural short exact sequence

$$0 \to T(N) \to N \to E \to 0$$

in $\mathrm{Mod}(O_W)$, one gets a short exact sequence in $\mathrm{Mod}(O_Z)$:

$$0 \to g^*T(N) \to h^*F \to g^*E \to 0.$$

As g^*E is torsion free, $g^*T(N)$ contains $T(h^*F)$. One has $g^*T(N) \subset T(g^*N) = T(h^*F)$. Therefore, $T(h^*F) = g^*T(N)$ and $h^*F/T(h^*F) = g^*E$ is a vector bundle on Z of rank $r > 0$. □

Lemma 6.3.6. Let M be a coherent sheaf on a complex torus X. If $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^0(X)$, then $\mathrm{Supp}(M)$ is finite.

Proof. Suppose the contrary that $\mathrm{Supp}(M)$ is infinite. With the reduced induced complex structure, the complex subspace $\mathrm{Supp}(M)$ has positive dimension. Let C be an irreducible component of $\mathrm{Supp}(M)$ of maximal dimension. Take a morphism $h : Z \to X$ provided by Lemma 6.3.5. Then $E := h^*M/T(h^*M)$ is a vector bundle of positive rank r. Choose a base point $z_0 \in Z$, and set $x_0 := h(z_0)$. As X is a complex torus, the Albanese morphism $\mathrm{alb}_{X,x_0} : X \to \mathrm{Alb}(X)$ is biholomorphic. By functoriality, as Z is a Kähler manifold, $h : Z \to X$ induces a morphism of complex tori $\mathrm{alb}(h) : \mathrm{Alb}(Z) \to \mathrm{Alb}(X)$ fitting into a commutative diagram

$$\begin{tikzcd} Z \arrow[r, two heads] \arrow[d, "\mathrm{alb}_{Z,z_0}"'] \arrow[rr, bend right, "h"] & C \arrow[r, hook] & X \arrow[d, "\mathrm{alb}_{X,x_0}"] \\ \mathrm{Alb}(Z) \arrow[rr, dashed, "\mathrm{alb}(h)"] & & \mathrm{Alb}(X). \end{tikzcd}$$

The image of $\mathrm{alb}(h)$ contains $\mathrm{alb}_{X,x_0}(C)$, so $\mathrm{alb}(h) : \mathrm{Alb}(Z) \to \mathrm{Alb}(X)$ is nonzero. Thus, the morphism of the dual complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is nonzero. In particular, there is $L \in \mathrm{Pic}^0(X)$ such that the line bundle $(h^*L)^{\otimes r}$ is nontrivial.

On the other hand, we claim that the line bundle $(h^*L)^{\otimes r}$ is trivial. Indeed, by assumption $M \otimes L \cong M$, one has $h^*M \otimes h^*L \cong h^*M$. Since $T(h^*M \otimes h^*L) = T(h^*M) \otimes h^*L$, one gets $E \otimes h^*L \cong E$. Taking the determinant of both sides, one has $\det(E) \otimes (h^*L)^{\otimes r} \cong \det(E)$. As $\det(E)$ is an invertible sheaf, the line bundle $(h^*L)^{\otimes r}$ on Z is trivial. The claim is proved, which gives a contradiction. □

Proof of Proposition 6.3.3. (1) Let E be a homogeneous vector bundle on X. By Corollary 3.2.11, $H^i\hat{\mathcal{F}}(E)$ is coherent over $O_{\hat{X}}$. By the analog of [Muk81, (3.1)] for complex tori, for every $x \in X$, there exist isomorphisms

$$\hat{\mathcal{F}}(E) \cong \hat{\mathcal{F}}(T_{-x}^*E) \cong P_x^* \otimes \hat{\mathcal{F}}(E)$$

and hence $H^i\hat{\mathcal{F}}(E) \cong P_x^* \otimes H^i\hat{\mathcal{F}}(E)$. From Lemma 6.3.6, the support of $H^i\hat{\mathcal{F}}(E)$ is finite.

(2) By Point (1), for every integer $i \neq g$, one has $H^i\hat{\mathcal{F}}(E) \in \mathrm{Coh}_f(\hat{X})$. We shall prove $H^i\hat{\mathcal{F}}(E) = 0$. One has

$$\begin{aligned} 0 =& H^{i-g}([-1]_X^*E) = H^i([-1]_X^*E[-g]) \\ &\overset{\text{(a)}}{\cong} H^i\mathcal{F} \circ \hat{\mathcal{F}}(E) = R^i p_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^*\hat{\mathcal{F}}(E)) \\ &\overset{\text{(b)}}{\cong} p_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^* H^i\hat{\mathcal{F}}(E)) = H^0\mathcal{F}(H^i\hat{\mathcal{F}}(E)), \end{aligned}$$

where (a) (resp. (b)) uses Theorem 5.0.1 (resp. [GR04, Thm. 4, p.47]).

It remains to prove that for every $F \in \mathrm{Coh}_f(\hat{X})$ with $H^0\mathcal{F}(F) = 0$, one has $F = 0$. Since F is the direct sum of finitely many coherent submodules whose supports are singletons, one may assume that $\mathrm{Supp}(F)$ is a singleton. By [Muk81, (3.1)], one may assume that $F \in \mathrm{Coh}_0(\hat{X})$. From Proposition 6.2.3 (2), one has $F = 0$.

(3) By Point (1), $H^g\hat{\mathcal{F}} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$. From Point (2), one has an isomorphism $H^g\hat{\mathcal{F}} \cong \hat{\mathcal{F}}[g]$ of functors $H(X) \to \mathrm{Coh}_f(\hat{X})$. By Proposition 6.2.3 (2) and [Muk81, (3.1)], $H^0\mathcal{F} : \mathrm{Mod}(O_{\hat{X}}) \to \mathrm{Mod}(O_X)$ restricts to a functor $H^0\mathcal{F} : \mathrm{Coh}_f(\hat{X}) \to H(X)$, and the restriction is isomorphic to $\mathcal{F} : \mathrm{Coh}_f(\hat{X}) \to H(X)$. By Theorem 5.0.1, $H^g\hat{\mathcal{F}} : H(X) \to \mathrm{Coh}_f(\hat{X})$ is an equivalence with a quasi-inverse $H^0\mathcal{F}$. □

Proof of Theorem 6.3.2. It follows from the complex analytic analog of [Muk81, (3.1)], Propositions 6.2.3 (2) and 6.3.3 (3). □

A Sheaves of modules

We recall some facts about sheaves of modules. Let (X, O_X) be a ringed space.

A.1 Generalities

Definition A.1.1. An O_X-module F is called

(1) ([Sta26, Tag 01B5]) of *finite type* if every $x \in X$ admits an open neighborhood U such that $F|_U$ is generated by finitely many sections;

(2) ([Sta26, Tag 01BN]) of *finite presentation* if for every $x \in X$, there is an open neighborhood $U \subset X$, integers $n, m \geq 0$ and an exact sequence of O_U-modules
$$O_U^m \to O_U^n \to F|_U \to 0;$$

(3) ([EGA I, 5.1.3]) *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$, two sets I, J and an exact sequence of O_U-modules
$$O_U^{\oplus J} \to O_U^{\oplus I} \to F|_U \to 0;$$

(4) ([Kas03, Def. A.5 (1)]) *(K)-pseudocoherent* if for every open subset U of X, every finite type O_U-submodule of $F|_U$ is of finite presentation. Let $\mathrm{PCoh}(X) \subset \mathrm{Mod}(O_X)$ be full subcategory of (K)-pseudocoherent modules;

(5) ([Kas03, Def. A.5 (2)]) *(K)-coherent* if F is (K)-pseudocoherent and of finite type;

(6) ([Sta26, Tag 01BV]) *coherent* if F is of finite type and for every open subset U of X and every finite collection $\{s_i\}_{1\leq i\leq n}$ in $F(U)$, the kernel of the associated morphism $O_U^n \to F|_U$ is of finite type over O_U.

Every property in Definition A.1.1 is local, in the sense that it restricts to every open subset, and if it holds on each member of an open covering of X, then it holds on X. Lemma A.1.2 is well known, but we include a proof for lack of reference.

Lemma A.1.2. Let X be a ringed space. Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in $\mathrm{Mod}(O_X)$. If F and H are of finite presentation, then so is G.

Proof. By [Sta26, Tag 01B8], every $x \in X$ has an open neighborhood U such that the sequence $G(U) \xrightarrow{r_U} H(U) \to 0$ is exact. Up to shrinking U, there exist integers $m, n, p, q \geq 0$ and two exact sequences

$$O_U^m \to O_U^n \xrightarrow{f} F|_U \to 0, \quad O_U^p \to O_U^q \xrightarrow{h} H|_U \to 0.$$

The morphism h is defined by q elements $s_1, \ldots, s_q \in H(U)$. For each $1 \leq i \leq q$, choose a preimage $t_i \in G(U)$ of s_i. For $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in O(U)^n$ be the vector with a 1 in the i-th coordinate and 0's elsewhere. The morphism $\phi : O_U^{n+q} \to G|_U$ determined by $if(e_1), \ldots, if(e_n), t_1, \ldots, t_q \in G(U)$ fits into a commutative diagram with two exact middle rows

$$\begin{tikzcd}
0 \arrow[r] & O_U^m \arrow[r] \arrow[d] & \ker(\phi) \arrow[r] \arrow[d] & O_U^p \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & O_U^n \arrow[r] \arrow[d, "f"] & O_U^{n+q} \arrow[r] \arrow[d, "\phi"] & O_U^q \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & F|_U \arrow[r] \arrow[d] & G|_U \arrow[r] \arrow[d] & H|_U \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{coker}(\phi) \arrow[r] & 0. &
\end{tikzcd}$$

By the snake lemma, $\phi : O_U^{n+q} \to G|_U$ is surjective and $\ker(\phi)$ is of finite type over O_U. Shrinking U again, one may find an integer $k \geq 0$ and a surjection $O_U^k \to \ker(\phi)$. The induced sequence $O_U^k \to O_U^{n+q} \to G|_U \to 0$ is exact. Therefore, G is of finite presentation. $\square$

A.2 Pseudo-coherent modules

Lemma A.2.1. Let X be a ringed space.

(1) Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in $\mathrm{Mod}(O_X)$. If F, H are (K)-pseudocoherent, then so is G.

(2) Let I be a directed set. Let (F_i, f_{ij}) be a direct system over I consisting of (K)-pseudocoherent O_X-modules such that for any $i \leq j$ in I, $f_{ij} : F_i \to F_j$ is injective. Then $F := \mathrm{colim}_{i \in I} F_i$ in $\mathrm{Mod}(O_X)$ is (K)-pseudocoherent.

(3) Let $\{M_\alpha\}_{\alpha \in A}$ be a family of (K)-pseudocoherent O_X-modules. Then $S := \oplus_{\alpha \in A} M_\alpha$ is also (K)-pseudocoherent.

Proof. Let U be an open subset of X.

(1) Let M be a finite type submodule of $G|_U$. Then the kernel of $r|_M : M \to H|_U$ is $(F|_U) \cap M$. Thus, $r|_M$ induces an injection $M/(F|_U \cap M) \hookrightarrow H|_U$. As H is (K)-pseudocoherent, its finite type O_U-submodule $M/(F|_U \cap M)$ is of finite presentation. By [Sta26, Tag 01BP (2)], $F|_U \cap M$ is of finite type. As F is (K)-pseudocoherent, $F|_U \cap M$ is of finite presentation. By Lemma A.1.2 applied to the exact sequence $0 \to F|_U \cap M \to M \to M/(F|_U \cap M) \to 0$, the O_U-module M is of finite presentation. Thus, G is (K)-pseudocoherent.

(2) Let N be a finite type submodule of $F|_U$. From the proof of [Sta26, Tag 01BB], for every $x \in U$, there is an open neighborhood V of x in U and an index $i \in I$ with $N|_V \subset F_i|_V$. Since F_i is (K)-pseudocoherent, $N|_V$ is of finite presentation. As finite presentation is a local property, N is of finite presentation. Thus, F is (K)-pseudocoherent.

(3) Let I be the set of all finite subsets of A with the inclusion order. Then I is a directed set. For $B \in I$, set $F_B := \oplus_{\alpha \in B} M_\alpha$. By Point (1), F_B is (K)-pseudocoherent. For $B \leq B'$ in I, set $f_{B,B'} : F_B \to F_{B'}$ to be the inclusion. Thus, one gets a direct system $(F_B, f_{B,B'})$ over I. By Point (2), the O_X-module $S = \mathrm{colim}_{B \in I} F_B$ is (K)-pseudocoherent. □

Lemma A.2.2 allows one to apply the results on (K)-coherent sheaves in [Kas03] to coherent sheaves.

Lemma A.2.2. An O_X-module is (K)-coherent if and only if it is coherent.

Proof. Let U be an open subset of X. Assume that F is a (K)-coherent module. Let $\{s_i\}_{1 \leq i \leq n}$ be a finite collection in $F(U)$, and let $f : O_U^n \to F|_U$ be the associated morphism. Then $\mathrm{im}\, f$ is a finite type submodule of $F|_U$. Because F is (K)-pseudocoherent, $\mathrm{im}\, f$ is of finite presentation over O_U. From [Sta26, Tag 01BP (2)], $\ker f$ is of finite type over O_U. Therefore, F is coherent.

Conversely, assume that F is a coherent O_X-module. Let M be a finite type submodule of $F|_U$. By [Sta26, Tag 01BY (1)], M is coherent over O_U. From [Sta26, Tag 01BW], M is of finite presentation. Thus, F is (K)-pseudocoherent and hence (K)-coherent. □

We discuss the comparison between (K)-pseudocoherence and quasi-coherence. For a ringed space (X, O_X), O_X is quasi-coherent over itself, but may not be (K)-pseudocoherent.

Lemma A.2.3. Let X be a locally Noetherian scheme. Then every quasi-coherent O_X-module is (K)-pseudocoherent.

Proof. By [EGA I, Cor. 9.4.9], a quasi-coherent module is a direct limit of coherent submodules, hence (K)-pseudocoherent by Lemma A.2.1 (2). □

Example A.2.4. The converse of Lemma A.2.3 fails. We give a (K)-pseudocoherent sheaf on an algebraic variety that is not quasi-coherent. Let $X = \mathbf{A}_k^1$ be the affine line over a field k. Let $U = X \setminus \{0\}$, and let $j : U \to X$ be the inclusion. By [Har77, II, Example 5.2.3], the O_X-module $j_! O_U$ is not quasi-coherent. From [Har77, II, Exercise 1.19 (c)], it is a submodule of the coherent module O_X. Hence, $j_! O_U$ is (K)-pseudocoherent.

Lemma A.2.5 shows that if O_X is coherent, then a coherent sheaf is "small" in certain sense.

Lemma A.2.5. Let (X, O_X) be a ringed space. If O_X is coherent over itself, then for every coherent O_X-module F and every integer $q \geq 0$, the functor $\mathcal{E}xt^q_X(F, -) : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_X)$ commutes with filtered colimits.

Proof. By [Sta26, Tags 01BW and 0GMV], the statement with $q = 0$ holds. Now assume $q > 0$. From the proof of [EGA III 1, Prop. 12.3.3], as O_X is coherent, for every $x \in X$, there is an open neighborhood U and an exact sequence of O_U-modules

$$0 \to M \to L_{q-1} \to \cdots \to L_0 \to F|_U \to 0$$

such that every $L_i \cong O_U^{n_i}$ $(0 \leq i < q)$ is free of finite rank, and M is coherent. Let $G = \mathrm{colim}_{i \in I} G_i$ be a filtered colimit of O_X-modules. For every $i \in I$, the sequence

$$\mathcal{H}om_U(L_{q-1}, G_i|_U) \to \mathcal{H}om_U(M, G_i|_U) \to \mathcal{E}xt^q_U(F|_U, G_i|_U) \to 0$$

is exact. As $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, one gets a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{colim}_{i\in I}\mathcal{H}om_U(L_{q-1}, G_i|_U) \arrow[r] \arrow[d, "\cong"] & \mathrm{colim}_{i\in I}\mathcal{H}om_U(M, G_i|_U) \arrow[r] \arrow[d, "\cong"] & \mathrm{colim}_{i\in I}\mathcal{E}xt^q_U(F|_U, G_i|_U) \arrow[r] \arrow[d] & 0 \\
\mathcal{H}om_U(L_{q-1}, G|_U) \arrow[r] & \mathcal{H}om_U(M, G|_U) \arrow[r] & \mathcal{E}xt^q_U(F|_U, G|_U) \arrow[r] & 0.
\end{tikzcd}$$

The first two vertical morphisms are isomorphisms, so is the one on the right. Therefore, the natural morphism $\mathrm{colim}_{i\in I}\mathcal{E}xt^q_X(F, G_i) \to \mathcal{E}xt^q_X(F, G)$ is an isomorphism. □

Definition A.2.6 defines a local property. It is weaker than [Bjö93, A:III, 2.24] and [Kas03, Def. A.7].

Definition A.2.6. Let (X, O_X) be a ringed space. If O_X is coherent over itself, and if for every open subset U of X and every family of coherent ideal sheaves $\{I_i\}_{i\in A}$ in O_U, the ideal sheaf $\sum_{i\in A} I_i$ is coherent over O_U, then O_X is called a *quasi-Noetherian sheaf of rings.*

Example A.2.7. (1) If (X, O_X) is a locally Noetherian scheme, then O_X is quasi-Noetherian.

(2) If (X, O_X) is a complex analytic space, then by the Oka-Cartan theorem (see, *e.g.*, [Kas03, Thm. A.12]), O_X is quasi-Noetherian.

A.3 Analytic coherent modules

Let X be a complex analytic space. We show that a coherent O_X-module admits a local free resolution, from which we deduce that coherence is preserved by derived pullbacks and derived tensor products. An analog of Lemma A.3.1 for algebraic varieties is [Har77, III, Example 6.5.1].

Lemma A.3.1. Let X be a complex analytic space. Then every $x \in X$ admits an open neighborhood U, such that for every coherent O_X-module F, there is a (possibly infinite-length) resolution

$$\cdots \to O_U^{n_1} \to O_U^{n_0} \to F|_U \to 0,$$

where $n_i \geq 0$ are integers.

Proof. Shrinking X to an open neighborhood of x, one may assume that X is Stein. By [GR04, Thm. 8, p.108], there is a compact neighborhood $K \subset X$ of x, such that Theorem B is valid on K in the sense of [GR04, Def. 1, p.100]. Let $U = K^\circ$.

For every coherent O_X-module F, we construct inductively a sequence of morphisms. From [GR04, Cor. p.101], there is an integer $n_0 \geq 0$, an open neighborhood U_0 of K in X and a morphism $f_0 : O_{U_0}^{n_0} \to F|_{U_0}$ in $\mathrm{Mod}(O_{U_0})$ such that $f_0|_U : O_U^{n_0} \twoheadrightarrow F|_U$ is an epimorphism in $\mathrm{Mod}(O_U)$. By convention, set $\ker(f_{-1})|_{U_0} := F|_{U_0}$. By [Sta26, Tag 01BY (3)], for an integer $j \geq 0$, given an open neighborhood U_j of K in X and such a morphism $f_j : O_{U_j}^{n_j} \to \ker(f_{j-1})|_{U_j}$, the O_{U_j}-module $\ker(f_j)$ is coherent. By [GR04, Cor. p.101], there is an open neighborhood U_{j+1} of K in U_j, an integer $n_{j+1} \geq 0$ and a morphism $f_{j+1} : O_{U_{j+1}}^{n_{j+1}} \to \ker(f_j)|_{U_{j+1}}$ in $\mathrm{Mod}(O_{U_{j+1}})$ such that $f_{j+1}|_U : O_U^{n_{j+1}} \to \ker(f_j)|_U$ is an epimorphism. Thus, one gets a sequence

$$\cdots \to O_U^{n_2} \xrightarrow{f_2|_U} O_U^{n_1} \xrightarrow{f_1|_U} O_U^{n_0} \xrightarrow{f_0|_U} F|_U \to 0$$

in $\mathrm{Mod}(O_U)$. By construction, it is exact, hence a resolution of $F|_U$. □

Example A.3.2. By local syzygies [GH78, p.696], on a complex manifold X, every coherent module locally admits a resolution by finite free modules such that the length of the resolution is at most $\dim X$. On the other hand, when a complex analytic space X has a singular point $x \in X$, Lemma A.3.1 fails if we consider only finite-length resolutions. For instance, for the coherent skyscraper sheaf $M := \mathbb{C}_{(x)}$ and every open neighborhood U of x in X, we prove that there is no resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M|_U \to 0$$

such that the F_i $(0 \leq i \leq n)$ are flat O_U-modules.

Assume to the contrary that such a flat resolution exists. Then by [Sta26, Tag 05NE], as the F_i are flat,

$$0 \to F_{n,x} \to \cdots \to F_{0,x} \to M_x \to 0$$

is a flat resolution of the $O_{X,x}$-module $M_x = O_{X,x}/m_x$. Hence, the flat dimension of $O_{X,x}/m_x$ over $O_{X,x}$ is at most n. From [Wei95, Prop. 4.1.5 1], as $O_{X,x}$ is a Noetherian ring, this flat dimension equals the projective dimension $\mathrm{pd}_{O_{X,x}}(O_{X,x}/m_x)$. By Serre's theorem (see, *e.g.*, [Sta26, Tag 00OC]), $O_{X,x}$ is regular. From [Ser56, p.6], x is a smooth point of X, which is a contradiction.

Lemmas A.3.3 and A.3.4 are well known, of which we include proofs for lack of reference.

Lemma A.3.3. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then the derived pullback $Lf^* : D(Y) \to D(X)$ restricts to a functor $D^-_{\mathrm{coh}}(Y) \to D^-_{\mathrm{coh}}(X)$.

Proof. Let $F \in D^-_{\mathrm{coh}}(Y)$. By [Har66, I, Prop. 7.3 (ii)], as $Lf^* : D^-(Y) \to D(X)$ is bounded above, to prove $Lf^*F \in D^-_{\mathrm{coh}}(X)$, one may assume $F \in \mathrm{Coh}(O_Y)$. By Lemma A.3.1, for every $x \in X$, there is an open neighborhood V of $f(x) \in Y$, such that there is a resolution

$$\cdots \to E_1 \to E_0 \to F|_V \to 0$$

in $\mathrm{Mod}(O_V)$, where the E_i are finite free O_V-modules. Let $g : f^{-1}(V) \to V$ be the restriction of $f : X \to Y$. Then the morphism $g^*E_\bullet \to (Lf^*F)|_{f^{-1}(V)}$ in $D(f^{-1}(V))$

is an isomorphism. For every integer $j \geq 0$, the $O_{f^{-1}(V)}$-module g^*E_j is finite free. Thus, $(L^{-j}f^*F)|_{f^{-1}(V)}$ is coherent over $O_{f^{-1}(V)}$. Since coherence is a local property, $L^{-j}f^*F$ is coherent over O_X. $\square$

Lemma A.3.4. Let X be a complex analytic space. Then the derived tensor product $(+) \otimes^L (-) : D(X) \times D(X) \to D(X)$ restricts to a bifunctor $D^-_{\mathrm{coh}}(X) \times D^-_{\mathrm{coh}}(X) \to D^-_{\mathrm{coh}}(X)$.

Proof. Let $F, G \in D^-_{\mathrm{coh}}(X)$. From [Har66, I, Prop. 7.3 (ii)], to prove that $F \otimes^L G \in D^-_{\mathrm{coh}}(X)$, one may assume $F, G \in \mathrm{Coh}(O_X)$. By Lemma A.3.1, for every $x \in X$, there is an open neighborhood U of x in X and a resolution $E_\bullet \to F|_U \to 0$ by finite free O_U-modules. The natural morphism

$$E_\bullet \otimes_{O_U} G|_U \xrightarrow{\sim} F|_U \otimes^L_{O_U} G|_U$$

in $D(U)$ is an isomorphism. For every integer n, the O_U-module

$$H^n(F \otimes^L_{O_X} G)|_U = H^n(F|_U \otimes^L_{O_U} G|_U) = H^n(E_\bullet \otimes_{O_U} G|_U)$$

is coherent. Since coherence is a local property, the O_X-module $H^n(F \otimes^L_{O_X} G)$ is coherent. $\square$

A.4 Good modules

Let (X, O_X) be a ringed space whose underlying topological space is locally compact Hausdorff. Let $D_{\mathrm{gd}}(X)$ be the full subcategory of $D(X)$ comprised of complexes whose cohomologies are good sheaves on X.

Lemma A.4.1 (Goodness *vs.* (K)-pseudocoherence)**.**

(1) ([Kas03, p.77]) One has $\mathrm{Coh}(X) \subset \mathrm{Good}(X) \subset \mathrm{PCoh}(X)$.

(2) Let E be a (K)-pseudocoherent O_X-module. If on every relatively compact open subset U of X, the O_U-module $E|_U$ is the sum of its finite type submodules, then E is good.

Proof.

(1) By definition, every coherent O_X-module is good. Let E be a good O_X-module. Let W be an open subset of X, and let $F \subset E|_W$ be a finite type O_W-submodule. We show that F is of finite presentation over O_W. Replacing (X, E) with $(W, E|_W)$, one may assume $W = X$. Because X is locally compact, for every $x \in X$, there exists a *relatively compact* open neighborhood U of x in X and finitely many sections $s_1, \ldots, s_n \in F(U)$ generating $F|_U$. As E is good, $E|_U = \sum_{i \in I} E_i$ is the sum of a directed family of coherent submodules. There exists an index $i_0 \in I$ and an open neighborhood V of x in U with $s_r|_V \in E_{i_0}(V)$ for all $1 \leq r \leq n$. Then $F|_V$ is a finite type submodule of $E_{i_0}|_V$. By [Sta26, Tag 01BY (1)], $F|_V$ is O_V-coherent. As coherence is a local property, F is coherent. From [Sta26, Tag 01BW], F is of finite presentation.

(2) Since the sum of two finite type submodules is of finite type, the family of finite type submodules of $E|_U$ is directed. As E is (K)-pseudocoherent, for every relatively compact open subset U of X, every finite type submodule of $E|_U$ is (K)-pseudocoherent and hence coherent. Thus, E is good. $\square$

Basic properties of good modules (similar to those of quasi-coherent modules on algebraic varieties) are recapped in Lemma A.4.2.

Lemma A.4.2.

(1) For every family of good sheaves $\{F_i\}_{i\in I}$ on X, the direct sum $\oplus_{i\in I} F_i$ in $\mathrm{Mod}(O_X)$ is good.

(2) The subcategory $D_{\mathrm{gd}}(X)$ is closed under direct sums in $D(X)$. Moreover, the inclusion functor $\mathrm{Good}(X) \to D_{\mathrm{gd}}(X)$ commutes with direct sums.

Suppose further that O_X is quasi-Noetherian.

(3) Then $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$ and closed under filtered colimits in $\mathrm{Mod}(O_X)$. In particular, $\mathrm{Good}(X)$ is a locally Noetherian category in the sense of [Gab62, p.356].

(4) The inclusion functor $D_{\mathrm{gd}}(X) \to D(X)$ is a triangulated subcategory.

Proof.

(1) Over every relatively compact open subset U of X, the direct sum $(\oplus_{i\in I} F_i)|_U$ is the sum of its coherent O_U-submodules. By Lemmas A.2.1 (3) and A.4.1 (1), $\oplus_{i\in I} F_i$ is (K)-pseudocoherent. By Lemma A.4.1 (2), it is good.

(2) By [Sta26, Tag 07D9], since $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, the category $D(X)$ has arbitrary direct sums and they are computed by taking termwise direct sums of any representative complexes. Then by [Wei95, Exercise 1.2.1], for every integer q, the functor $H^q : D(X) \to \mathrm{Mod}(O_X)$ commutes with direct sums. The result follows from Point (1).

(3) By [Sta26, Tag 0754] and the proof of [Kas03, Prop. 4.23], as O_X is quasi-Noetherian, $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$. From [KS06, Thm. 18.1.6 (v)], $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category. By Point (1) and [Sta26, Tag 002P], the filtered colimits in $\mathrm{Good}(X)$ exist and agree with the filtered colimits in $\mathrm{Mod}(O_X)$. Thus, filtered colimits in $\mathrm{Good}(X)$ are exact. From [Sta26, Tag 01BC], there is a set of coherent O_X-modules $\{F_i\}_{i\in I}$ such that each coherent O_X-module is isomorphic to exactly one of the F_i. Then $\{F_i\}_i$ is a family of Noetherian generators of $\mathrm{Good}(X)$. Therefore, the category $\mathrm{Good}(X)$ is locally Noetherian.

(4) It follows from [Yek19, Prop. 7.4.5] and Point (3). ☐

For the sake of completeness, we mention that $R\mathcal{H}om$ preserves goodness. Lemma A.4.3 is not used elsewhere.

Lemma A.4.3. If O_X is quasi-Noetherian, then $R\mathcal{H}om_X(-,+) : D(X)^{\mathrm{op}} \times D(X) \to D(X)$ restricts to a bifunctor $D^b_{\mathrm{coh}}(X)^{\mathrm{op}} \times D^+_{\mathrm{gd}}(X) \to D^+_{\mathrm{gd}}(X)$.

Proof. Take $F \in D^b_{\mathrm{coh}}(X)$ and $G \in D^+_{\mathrm{gd}}(X)$. By [Har66, I, Prop. 7.3 (ii)], because $R\mathcal{H}om_X(-,G) : D(X)^{\mathrm{op}} \to D(X)$ is bounded below, to prove $R\mathcal{H}om_X(F,G) \in D^+_{\mathrm{gd}}(X)$, one may assume $F \in \mathrm{Coh}(X)$. Similarly, because $R\mathcal{H}om_X(F,\cdot) : D^+(X) \to D(X)$ is bounded below, one may assume $G \in \mathrm{Good}(X)$. It remains to prove that for every $q \geq 0$, $\mathcal{E}xt^q_X(F,G)$ is a good sheaf on X. Let U be a relatively compact open subset of X. Then one can write $G|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ is a directed family of coherent submodules of $G|_U$. By Lemma A.2.5, as F is coherent, one has $\mathcal{E}xt^q_U(F|_U, G|_U) = \mathrm{colim}_{i\in I}\mathcal{E}xt^q_U(F|_U, G_i)$. From [EGA III 1, Prop. 12.3.3], as O_X is coherent, $\mathcal{E}xt^q_U(F|_U, G_i)$ is coherent over O_U for every $i \in I$. From Lemma A.4.2 (3), as O_X is quasi-Noetherian, $\mathcal{E}xt^q_U(F|_U, G|_U)$ is a good sheaf on U. Therefore, $\mathcal{E}xt^q_X(F,G)$ is a good O_X-module. ☐

A.5 Sections of direct sum of sheaves

By [Har77, II, Exercise 1.11], on a Noetherian topological space, taking section commutes with (possibly infinite) direct sum of sheaves. This fails on complex manifolds, as Example A.5.1 shows.

Example A.5.1. Let $X = \mathbb{C}$. Let F be the O_X-module $\oplus_{n\geq 0}\mathbb{C}_{(n)}$. There is a section $s \in \Gamma(X, F^{\oplus\mathbb{N}})$, such that for every integer $n \geq 0$, the stalk $s_n \in (F^{\oplus\mathbb{N}})_n = (F_n)^{\oplus\mathbb{N}} = \mathbb{C}^{\oplus\mathbb{N}}$ is $(1, 1, \dots, 1, 0, 0, \dots)$, where the first $n+1$ entries are 1 and all the other entries are 0. Then s has no preimage under the canonical map $\Gamma(X, F)^{\oplus\mathbb{N}} \to \Gamma(X, F^{\oplus\mathbb{N}})$. For otherwise, let $(t^n)_{n\geq 0} \in \Gamma(X, F)^{\oplus\mathbb{N}}$ be a preimage of s. Then there are only finitely many integers $n \geq 0$ with $t^n \neq 0$. Every t^n has only finitely many nonzero stalks. However, s has infinitely many nonzero stalks, which is a contradiction.

Let X be a complex analytic space. An O_X-module F *satisfies the property (id)* if for every irreducible open subset U of X and every $x \in U$, $\Gamma(U, F) \to F_x$ is injective. By the identity theorem (see, *e.g.*, [CAS, Ch. 9, Sec. 3, Identitätssatz]), if X is reduced, then O_X satisfies the property (id).

Lemma A.5.2. Let X be an irreducible complex analytic space. Let $\{F_i\}_{i\in I}$ be a family of O_X-modules satisfying the property (id). Then $\oplus_{i\in I}\Gamma(X, F_i) \xrightarrow{\sim} \Gamma(X, \oplus_{i\in I}F_i)$ is a group isomorphism.

Proof. Let P be the presheaf direct sum of $\{F_i\}_{i\in I}$. Let $\theta : P \to \oplus_{i\in I}F_i$ be the sheafification morphism. Then $P(X) = \oplus_{i\in I}\Gamma(X, F_i)$ and $\theta_X : \oplus_{i\in I}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i)$ is the colimit of

$$\theta_X^{(J)} : \oplus_{i\in J}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i),$$

where J runs through the finite subsets of I. By [Sta26, Tag 01AH (4)], for every such J, the presheaf direct sum of $\{F_i\}_{i\in J}$ is a subsheaf of $\oplus_{i\in I}F_i$, so $\theta_X^{(J)}$ is injective. Therefore, θ_X is also injective. We prove that it is surjective.

By construction of sheafification (see, *e.g.*, [Har77, p.64]), for every $s \in \Gamma(X, \oplus_{i\in I}F_i)$, there is a covering $\{U_\alpha\}_{\alpha\in A}$ of X by nonempty open subsets and an element $t_\alpha \in \Gamma(U_\alpha, P)$ for each $\alpha \in A$, such that $s_x = t_{\alpha,x}$ in $(\oplus_{i\in I}F_i)_x = \oplus_{i\in I}F_{i,x}$ for all $x \in U_\alpha$. As X is irreducible, so are the U_α.

Fix $x_0 \in X$ and $\alpha_0 \in A$ with $x_0 \in U_{\alpha_0}$. Then there is a finite subset I_0 of I such that $t_{\alpha_0} \in \Gamma(X, \oplus_{i\in I_0}F_i) \subset \Gamma(X, P)$. Let $B := \{\alpha \in A \mid t_\alpha \notin \Gamma(U_\alpha, \oplus_{i\in I_0}F_i)\}$. Set $V = \cup_{\alpha\in B}U_\alpha$. Then V is open in X. One has

$$X \setminus V \subset \cup_{\alpha\in A\setminus B}U_\alpha. \tag{27}$$

For every $\alpha \in A\setminus B$, we claim $U_\alpha \subset X \setminus V$. In fact, for every $y \in U_\alpha$, every $\beta \in A$ with $y \in U_\beta$ and every $i \in I \setminus I_0$, one has $t^i_{\beta,y} = s^i_y = t^i_{\alpha,y} = 0$ in $F_{i,y}$. Since F_i satisfies the property (id) and U_β is irreducible, one obtains $t^i_\beta = 0$ in $\Gamma(U_\beta, F_i)$. Therefore, $t_\beta \in \Gamma(X, \oplus_{i\in I_0}F_i)$, *i.e.*, $\beta \notin B$. Hence $y \notin V$.

From (27) and the claim, $X \setminus V = \cup_{\alpha\in A\setminus B}U_\alpha$ is also open in X and contains U_{α_0}. Since X is connected, one has $V = B = \emptyset$. Consequently, one has $t_\alpha \in \Gamma(U_\alpha, \oplus_{i\in I_0}F_i)$ for every $\alpha \in A$. Then the family $\{t_\alpha\}_{\alpha\in A}$ glues to a preimage of s in $\Gamma(X, \oplus_{i\in I_0}F_i) \subset \Gamma(X, P)$. Thus, θ_X is surjective and hence a group isomorphism. $\square$

Lemma A.5.3. Let X be a reduced complex analytic space. If F is a locally free (possibly of infinite rank) O_X-module, then F satisfies the property (id).

Proof. Let U be an irreducible open subset of X. For every $x_0 \in U$, we prove that $\Gamma(U, F) \to F_{x_0}$ is injective. Take $s \in \Gamma(U, F)$ with $s_{x_0} = 0$. By [Har77, II, Exercise 1.14], $Z := \{x \in U \mid s_x = 0\}$ is open in U.

We claim that Z is closed in U. Let $\{x_n\}_{n \geq 1}$ be a sequence of points in Z converging to y in U. Because F is locally free, there is an open neighborhood V of y in U, a set I and an isomorphism $\psi : F|_V \xrightarrow{\sim} O_V^{\oplus I}$ of O_V-modules. There is an integer $N > 0$ with $x_N \in V$. As U is irreducible, so is V. Since X is reduced, O_V satisfies the property (id). Then from Lemma A.5.2, the map on the bottom of the commutative square

$$\begin{tikzcd}
\Gamma(V, F) \arrow[r] \arrow[d, "\Gamma(V,\psi)"] & F_{x_N} \arrow[d, "\psi_{x_N}"] \\
\Gamma(V, O_V^{\oplus I}) \arrow[r, hook] & O_{V,x_N}^{\oplus I}
\end{tikzcd}$$

is injective. Then so is the map on the top. Since $s_{X_N} = 0$, one has $s|_V = 0$ in $\Gamma(V, F)$ and hence $s_y = 0$ in F_y. It implies $y \in Z$. The claim is proved.

Because U is connected and $x_0 \in Z$, by the claim one has $Z = U$, which implies $s = 0$ in $\Gamma(U, F)$. ☐

Corollary A.5.4. Let X be a reduced irreducible complex analytic space. Let $\{F_i\}_{i \in I}$ be a family of locally free O_X-modules. Then $\oplus_{i \in I} \Gamma(X, F_i) \to \Gamma(X, \oplus_{i \in I} F_i)$ is a group isomorphism.

Proof. It follows from Lemmas A.5.2 and A.5.3. ☐

A.6 Comparison of notions of complex analytic quasi-coherent sheaves

We review several notions of "complex analytic quasi-coherent sheaves" in the literature. A compact subset K of a complex analytic space X is a *Stein compact* if K admits a fundamental system of Stein open neighborhoods in X.

Lemma A.6.1. A good sheaf F on a complex analytic space X is a quasi-coherent O_X-module.

Proof. From [Fri67, Thm. I, 9, Rem. I, 10], every $x \in X$ admits a neighborhood K that is a Stein compact with $O(K)$ a Noetherian ring. There is a relative compact open subset U of X containing K. As F is good, $F|_U = \sum_{i \in I} F_i$ is the sum of a directed family of coherent subsheaves. By [Tay02, Prop. 11.9.2], as $O(K)$ is Noetherian, applying the functor $\Gamma(K, -)$ to the directed family $\{F_i\}_{i \in I}$ in $\mathrm{Coh}(U)$, one gets a directed family of finitely generated $O(K)$-submodule $\{M_i\}_{i \in I}$ of $\Gamma(K, F)$, whose associated family in $\mathrm{Mod}(O_K)$ is $\{F_i|_K\}_{i \in I}$. Set $M := \mathrm{colim}_{i \in I} M_i$ in $\mathrm{Mod}(O(K))$. Since the localization functor $\mathrm{Mod}(O(K)) \to \mathrm{Mod}(O_K)$ is left adjoint to $\Gamma(K, -) : \mathrm{Mod}(O_K) \to \mathrm{Mod}(O(K))$, the localization preserves colimits. Then $F|_K$ is associated to M. By [Liu25b, Lem. 2.5], F is quasi-coherent. ☐

Remark A.6.2. In Lemma A.6.1, the restriction of a good O_X-module to an open subset U is a good O_U-module. Quasi-coherence in the sense of [EGA I, 5.1.3] is a local property, while goodness is not so. In fact, by Lemma A.4.2 (3), every free module on a complex manifold is good. However, Gabber [Con06, Eg. 2.1.6] gives a locally free (hence quasi-coherent and (K)-pseudocoherent) module of *infinite rank* on the unit open disk in $\mathbb{C}$ which is not good. (In particular, the converse of Lemma A.6.1 is wrong for noncompact complex manifolds.) Contrary to Lemma A.4.2 (1), the direct sum of countably many quasi-coherent sheaves on X may not be quasi-coherent.

On a complex analytic space X, a *Fréchet sheaf* (resp. a sheaf of *FN type*) is an O_X-module F such that for every open subset $U \subset X$, $F(U)$ is equipped a structure of Fréchet (resp. FN) $O(U)$-module, and for any open subsets $U' \subset U$, the restriction $F(U) \to F(U')$ is continuous. A sheaf of *DFN type* is an O_X-module G such that for every compact subset $Z \subset X$, $G(Z)$ is equipped with a structure of DFN $O(Z)$-module, such that for any compact subsets $Z' \subset Z$ of X, the restriction $G(Z) \to G(Z')$ is continuous. Ramis and Ruget [RR74, p.100] introduce the concepts of FN-quasi-coherent sheaves and DFN-quasi-coherent sheaves. (EP)-quasi-coherent sheaves [EP96, Def. 4.3.1] are generalizations of FN-quasi-coherent sheaves.

Definition A.6.3. Let X be a complex analytic space.

(1) A Fréchet sheaf F is *(EP)-quasi-coherent* if for any Stein open subsets $V \subset U$ of X, $F(U)$ and $O(V)$ are transversal over $O(U)$, and the natural morphism $F(U)\hat{\otimes}_{O(U)}O(V) \to F(V)$ is an isomorphism of topological vector spaces.

(2) A sheaf of FN type F is *FN-quasi-coherent* if it is (EP)-quasi-coherent.

(3) A sheaf of DFN type F is *DFN-quasi-coherent* if for any Stein compact subsets $L \subset K$ of X, $F(K)$ and $O(L)$ are transversal over $O(K)$, and the natural morphism $F(K)\hat{\otimes}_{O(K)}O(L) \to F(L)$ is an isomorphism of topological vector spaces.

From Example 4.2.1 and [EP96, Cor. 4.2.5], a coherent sheaf on a complex analytic space is FN-quasi-coherent. As an analytic analog of Serre's vanishing theorem and a generalization of Cartan's Theorem B, (EP)-quasi-coherent sheaves are acyclic on Stein spaces ([EP96, Prop. 4.3.3 (b)]).

Fact A.6.4. Let $f : X \to S$ be a morphism of Stein spaces.

(1) ([RR74, p.101]) Then for every coherent sheaf F on X, f_*F is FN-quasi-coherent over O_Y.

(2) ([EP96, Prop. 4.3.10]) For every (EP)-quasi-coherent sheaf G on X, f_*G is (EP)-quasi-coherent over O_Y.

In contrast to Fact A.6.4, Example 3.2.9 implies that for the open inclusion $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ of Stein spaces, $j_*O_{\mathbb{C}^\times}$ is not good.

Compared with Lemma A.4.2 (3), given an exact sequence $0 \to F \to G \to H \to 0$ of Fréchet sheaves on a Stein space X such that G and H are (EP)-quasi-coherent, [EP96, Open problem 4.5.10] asks whether F is (EP)-quasi-coherent. It is unclear whether any notion in Definition A.6.3 defines an abelian category of "quasi-coherent" sheaves on X. Lemma A.6.5 is folklore.

Lemma A.6.5. Let X be a complex analytic space. Let F be a coherent O_X-module.

(1) Then F is a DFN-quasi-coherent O_X-module.

(2) For a compact subset Z of X and a morphism of coherent sheaves $\varphi : F \to G$, $F(Z) \to G(Z)$ is continuous. If $\varphi : F \to G$ is injective, then $F(Z) \to G(Z)$ is a topological closed embedding.

Proof. (1) Let Z be a compact subset of X. One has $F(Z) = \operatorname{colim}_U F(U)$, where U runs through open neighborhood of Z in X. With the inductive limit topology, $F(Z)$ is a locally convex space. From the proof of [BS76, I, Prop. 2.10], $F(Z) \to \prod_{x\in Z}\prod_{n>0} F_x/m_x^n F_x$ is injective continuous, where every finite dimensional vector space $F_x/m_x^n F_x$ is equipped with the unique structure of Hausdorff topological vector space. In particular, $F(Z)$ is Hausdorff.

Choose a fundamental system of open neighborhoods $(W_n)_{n>0}$ of Z in X such that W_{n+1} is relatively compact in W_n for all $n > 0$. Then $F(W_n) \to F(W_{n+1})$ is a compact map. A *DFS space* is the strong dual of some Fréchet-Schwartz space. By [BS76, p.16], as every $F(W_n)$ is a Fréchet space, $F(Z)$ is a DFS space. From [Gro55, II, p.48, Cor. 1], as every $F(W_n)$ is nuclear, $F(Z)$ is a DFN space. In particular, $O(Z)$ is a DFN algebra. As the structure morphism $O(U)\times F(U) \to F(U)$ is continuous, so is $O(Z)\times F(Z) \to F(Z)$. Thus, $F(Z)$ is a DFN $O(Z)$-module. For a compact subset $Z' \subset Z$, the restriction $F(U) \to F(Z')$ is continuous, so is $F(Z) \to F(Z')$. Therefore, F is of DFN type.

Let $L \subset K$ be Stein compact subsets of X. Then there exist fundamental systems of open neighborhoods $(U_n)_{n>0}$ and $(V_n)_{n>0}$ of L and K respectively, such that for every $n > 0$, U_n is contained in V_n and both are Stein open subsets of X. By [RR74, Prop. 8], as F is coherent, $F(U_n)$ and $F(V_n)$ are transversal over $O(V_n)$, and the canonical morphism $F(V_n)\hat{\otimes}_{O(V_n)}O(U_n) \xrightarrow{\sim} F(U_n)$ is an isomorphism. From Lemma 4.2.5, $F(K)$ and $O(L)$ are transversal over $O(K)$, and one has $F(K)\hat{\otimes}_{O(K)}O(L) \xrightarrow{\sim} F(L)$. (The isomorphism also follows from [JT83, Lemma 1.7]). Therefore, F is DFN-quasi-coherent.

(2) For every open neighborhood U of Z in X, $F(U) \to G(U) \to G(Z)$ is continuous, so is $F(Z) \to G(Z)$. Assume that $\varphi : F \to G$ is injective. Let $H := G/F$, which is coherent over O_X. Then $F(Z)$ is the kernel of the continuous linear map $G(Z) \to H(Z)$, so it is closed in $(G(Z), \tau_G)$. By [Kaw93, Lemma 4.2 (4)], $(F(Z), \tau_G)$ is also a DFN space. By the open mapping theorem for DFS spaces (see, *e.g.*, [BS76, p.15]), $\mathrm{Id}_{F(Z)} : (F(Z), \tau_F) \to (F(Z), \tau_G)$ is an isomorphism of topological vector spaces. □

Lemma A.6.6. Let X be a complex analytic space. Let F be a good O_X-module. Let Z be a compact subset of X. Then $F(Z)$ has a canonical topology making it a locally convex space. Moreover, $O(Z) \times F(Z) \to F(Z)$ is jointly continuous. For another compact subset $Z' \subset Z$, $F(Z) \to F(Z')$ is continuous.

Proof. Choose a relatively compact open neighborhood U of Z in X. One can choose a directed family $(F_i)_{i\in I}$ of coherent submodules of $F|_U$ with $F|_U = \sum_{i\in I} F_i$. As Z is compact, one has $F(Z) = \operatorname{colim}_{i\in I} F_i(Z)$. By Lemma A.6.5, as every F_i is coherent, $F_i(Z)$ is a DFN space. The inductive limit topology of the $F_i(Z)$ makes $F(Z)$ a locally convex space. We prove that this topology is independent of the choices. In fact, given another open neighborhood V and expression $F|_V = \sum_{j\in J} G_j$, for every $i \in I$, there is an open neighborhood W of Z in $U \cap V$ and an index $j \in J$ such that $F_i|_W \hookrightarrow F|_W$ factors through $G_j|_W$. By Lemma A.6.5 (2), $F_i(Z) \to G_j(Z)$ is a closed vector subspace. As $O(Z) \times F_i(Z) \to F_i(Z) \hookrightarrow F(Z)$ is continuous, so is $O(Z) \times F(Z) \to F(Z)$. As $F_i(Z) \to F_i(Z') \hookrightarrow F(Z')$ is continuous, so is $F(Z) \to F(Z')$. □

On a ringed space X, an O_X-module F is of *countable type* if every $x \in X$ admits an open neighborhood U such that $F|_U$ is generated by a countable subset of $F(U)$.

Proposition A.6.7. *Let X be a complex analytic space. Let F be a good O_X-module of countable type. Then F is naturally a DFN-quasi-coherent O_X-module.*

Proof. Every compact subset $K \subset X$ has a relatively compact open neighborhood U. As F is good, one can write $F|_U = \sum_{i \in I} F_i$, where I is a directed set, and for every $i \in I$, F_i is a coherent submodules of $F|_U$ such that $F_i \subset F_j$ for any $i \leq j$ in I. We shall prove that there is an open neighborhood W of K in U, and a sequence $(a_n)_{n \geq 0}$ in I with $a_0 \leq a_1 \leq \dots$ such that $F|_W = \sum_{n \geq 0} F_{a_n}|_W$.

As F is of countable type, for every $x \in K$, there is an open neighborhood V_x of x in U and countably many sections $(s_r^x)_{r \geq 0}$ in $F(V_x)$ generating $F|_{V_x}$. Choose an open neighborhood W_x of x in V_x with $\bar{W}_x \subset V_x$. As U is relatively compact in X, $\bar{W}_x$ is compact. Then by [God58, Thm 3.10.1, p.162], one has $F(\bar{W}_x) = \mathrm{colim}_{i \in I} F_i(\bar{W}_x)$. Thus, for every $n \geq 0$, there is $i_{x,n} \in I$ such that the restrictions of $s_0^x, \dots, s_n^x$ to W_x are in the image of $F_{i_{x,n}}(W_x) \hookrightarrow F(W_x)$.

Since $\{W_x\}_{x \in K}$ is an open cover of the compact space K, there is a finite subcover $\{W_{x_j}\}_{j=1}^n$. We construct the sequence $(a_n)_{n \geq 0}$ in stages. Choose $a_0 \in I$ such that $a_0 \geq i_{x_j,0}$ for all $1 \leq j \leq n$. Having defined a_n, we choose $a_{n+1} \in I$ such that $a_{n+1} \geq a_n$ and $a_{n+1} \geq i_{x_j,n+1}$ for all $1 \leq j \leq n$. Let $W := \cup_{j=1}^n W_{x_j}$, which is an open neighborhood of K in U. For every $w \in W$, there is $1 \leq j_0 \leq n$ with $w \in W_{x_{j_0}}$. As $F|_{V_{x_{j_0}}}$ is generated by the sections $(s_r^{x_{j_0}})_{r \geq 0}$, for every $\sigma \in F_w$, there is $n_0 \geq 0$ such that σ can be generated by $(s_r^{x_{j_0}})_{0 \leq r \leq n_0}$. Since $a_{n_0} \geq i_{x_{j_0},n_0}$, the stalks of the local sections $(s_r^{x_{j_0}})_{0 \leq r \leq n_0}$ at w lift to $F_{a_{n_0},w}$. Therefore, σ lifts to $F_{a_{n_0},w}$. This shows $F|_W = \sum_{n \geq 0} F_{a_n}|_W$.

Thus, one can find an increasing sequence of coherent submodules $(G_n)_{n \geq 0}$ of $F|_W$ with $F|_W = \sum_{n \geq 0} G_n$. By [God58, Thm 3.10.1, p.162], as K is Hausdorff compact, one has $F(K) = \mathrm{colim}_{n \geq 0} G_n(K)$. By Lemma A.6.5 (2), every $G_n(K) \hookrightarrow G_{n+1}(K)$ is a closed topological vector subspace. Then from [SW99, II, 6.4], the inductive limit topology on $F(K)$ makes it a Hausdorff locally convex space. By [Gro55, II, Corollaire 1, p.48], $F(K)$ is nuclear. From [SW99, IV, Exercise 24 (e)], $F(K)$ is a DF space. By Lemma A.6.6, F is of DFN type.

Assume further that K is a Stein compact subset of X. Let $L \subset K$ be another Stein compact subset. One can choose fundamental systems of neighborhoods $(V_n)_{n \geq 0}$ and $(U_n)_{n \geq 0}$ in X of L and K respectively, such that for every $n \geq 0$, V_n is contained in U_n and both are Stein open subsets of X. One has

$$F(K) = \mathrm{colim}_{n \geq 0} G_n(U_n), \quad F(L) = \mathrm{colim}_{n \geq 0} G_n(V_n),$$
$$O(K) = \mathrm{colim}_{n \geq 0} O(U_n), \quad O(L) = \mathrm{colim}_{n \geq 0} O(V_n),$$

and the four are DFN spaces.

By [RR74, Prop. 8], as G_n is coherent, $G_n(U_n)$ and $O(V_n)$ are transversal over $O(U_n)$, and the natural morphism $G_n(U_n) \hat{\otimes}_{O(U_n)} O(V_n) \xrightarrow{\sim} G_n(V_n)$ is an isomorphism. By Lemma 4.2.5, $F(K)$ and $O(L)$ are transversal over $O(K)$, and $F(K) \hat{\otimes}_{O(K)} O(L) \to F(L)$ is an isomorphism. (The isomorphism can also be proved by [JT83, Lemma 1.7].) Therefore, F is DFN-quasi-coherent. $\square$

Remark A.6.8. In Proposition A.6.7, the countable type condition is necessary. For example, if $X = \mathrm{Specan}\, \mathbb{C}$, then $\mathrm{Good}(X)$ is the category of $\mathbb{C}$-vector spaces. For a $\mathbb{C}$-vector space E, the canonical topology given by Lemma A.6.6 is the *topological direct sum* of $\dim E$ copies of $\mathbb{C}$ in the sense of [Gro73, Ch. 4, Def. 3]. If E is of uncountable dimension, then E is not metrizable, not nuclear nor a DF-space. Thus, E is neither a Fréchet sheaf nor of DFN type.

Mao [m)] shows that a DFN-quasi-coherent sheaf on a complex analytic space X is naturally a nuclear object of the ∞-category $\mathcal{D}_{\mathrm{Liq}}(X)$ of liquid quasi-coherent sheaves. According to [CS22, Warning 12.4], $\mathcal{D}_{\mathrm{Liq}}(X)$ may not have a natural t-structure, making it unclear whether there is an abelian category of "liquid quasi-coherent sheaves" analogous to $\mathrm{Qch}(M)$ for algebraic varieties M.

Acknowledgment

I thank my supervisor Anna Cadoret for her much guidance, constant support and valuable helps. Her careful reading and constructive suggestions have greatly improved the exposition. I thank the referees for their helpful suggestions. Theorem 1.3.2 was stated with $g : S' \to S$ *smooth*, and the current version is based on the idea of a referee, who also kindly corrected a mistake in the proof of Proposition 3.1.2 (2), and suggested Remark 5.2.1. I benefited a lot from enlightening communication with Oren Ben-Bassat, Jonathan Block, Arthur Douay, Julien Grivaux, Daniel Huybrechts, Joseph Lipman, Pierre Schapira, Joseph Leclere, Long Liu, Zhouhang Mao, Mihai Putinar, Junhui Qin, Xinyu Shao, Mingchen Xia, Hui Zhang and Yicheng Zhou. Lemma 3.2.6 is told by Hui Zhang. In a draft version, Theorem 5.0.1 was stated for the categories $D^b_{\mathrm{gd}}(-)$. Gabriel Ribeiro suggested the present extension to $D_{\mathrm{gd}}(-)$, to whom I am very grateful for his suggestions and much help. I express my deep gratitude to Thomas Krämer for his hospitality during my visit to the Humboldt-Universität zu Berlin. I thank Emiliano Ambrosi for his helpful comments. This work is funded by Contrat doctoral spécifique normalien de l'École normale supérieure de Paris.

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