Lifting line bundles to schemes over p-adic base

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Abstract

Given a scheme X proper over a complete DVR of mixed characteristic, and a line bundle L on the special fiber X_s, we consider cohomological conditions under which L lifts to a line bundle on X. This result removes the smoothness assumption in the Berthelot-Ogus theorem. Our criterion involves the crystalline first Chern class $c_{1,\mathrm{cris}}(L) \in H^2_{\mathrm{cris}}(X_s)$ in the case of low ramification, and the rigid first Chern class $c_{1,\mathrm{rig}}(L) \in H^2_{\mathrm{rig}}(X_s)$ in general case.

1 Introduction

Given a formal scheme proper over a complete discrete valuation ring (DVR) of unequal characteristic (such as $\mathbb{Z}_p$), we study the obstruction of lifting line bundles.

Setup 1.0.1. Let V be a complete DVR, with fraction field K of characteristic 0 and perfect residue field k of characteristic $p > 0$. Let $S := \operatorname{Spec} V$. Let s, η be the closed point and the generic point of S respectively. Let W be the ring of Witt vectors with coefficient k.

Given an integer $i \geq 0$ and a scheme Z over K, let $H^i_{\mathrm{dR}}(Z/K) := H^i R\Gamma(Z, \Omega^\bullet_{Z/K})$ be the i-th de Rham cohomology group. Let $F^1 H^i_{\mathrm{dR}}(Z/K)$ be the first Hodge filtration, i.e., the image of $H^i(Z, \sigma_{\geq 1}\Omega^\bullet_{Z/K}) \to H^i_{\mathrm{dR}}(Z/K)$. For a scheme Y over k to which the divided power structure on W extends, let $H^i_{\mathrm{cris}}(Y/W)$ be the i-th crystalline cohomology which is defined as in [BO78, 7.26.1]. Given a line bundle $L \in \operatorname{Pic}(Y)$, let $c_{1,\mathrm{cris}}(L) \in H^2_{\mathrm{cris}}(Y/W)$ be its crystalline first Chen class, reviewed in Section 2.2. Let $\pi \in V$ be a uniformizer. Let $e \geq 0$ be the absolute ramification index of V, i.e., the integer with $pV = \pi^e V$. In the case of low ramification $e < p$, we give a cohomological criterion of when a line bundle on the special fiber lifts to a global line bundle.

Theorem 1.0.2 (Corollary 4.2.2)**.** *Let V be as in Setup 1.0.1. Assume $e < p$. Let X be a scheme flat proper over V. Let $L \in \operatorname{Pic}(X_s)$. Then L lifts to an element of* $\operatorname{Pic}(X)[1/p]$ *if and only if the image of $c_{1,\mathrm{cris}}(L) \in H^2_{\mathrm{cris}}(X_s/W)$ under the natural morphism*

$$\tau_{\mathrm{cris}} : H^2_{\mathrm{cris}}(X_s/W) \otimes_W K \to H^2_{\mathrm{dR}}(X_\eta/K)$$

lies in $F^1H^2_{\mathrm{dR}}(X_\eta/K)$.

Remark 1.0.3. Let X be a scheme *smooth* proper over V. For every $i \geq 0$, Berthelot and Ogus [BO83, Theorem 2.4] construct a canonical isomorphism

$$\sigma_{\mathrm{cris}} : H^i_{\mathrm{dR}}(X_\eta/K) \to H^i_{\mathrm{cris}}(X_s/W) \otimes_W K,$$

using which they obtain the criterion [BO83, Theorem 3.8] without restriction $e < p$. Weakening smoothness to *semistable* reduction, one has the Hyodo-Kato isomorphism between $H^i_{\mathrm{dR}}(X_\eta/K)$ and the i-th log-crystalline cohomology of X_s. In the semistable case, Yamashita [Yam11, Theorem 3.1] obtains a similar criterion involving the log-crystalline first Chern class of L. In Theorem 1.0.2, we use the assumption $e < p$ to construct the morphism τ_{cris}. Contrary to the two previous results, as X_η may be singular, τ_{cris} may not be an isomorphism.

Berthelot [Ber86, p.19] introduces rigid cohomology as a generalization of crystalline cohomology. Using rigid cohomology, we give a result similar to Theorem 1.0.2 which removes the hypothesis $e < p$. For an integer $i \geq 0$ and an algebraic variety Y over k, let $H^i_{\mathrm{rig}}(Y/K)$ be the i-th rigid cohomology. Petrequin [Pet03, Définition 4.1] defines the first Chern class in rigid cohomology

$$c_{1,\mathrm{rig}} : \mathrm{Pic}(Y) \to H^2_{\mathrm{rig}}(Y/K).$$

Theorem 1.0.4. *Let S be as in Setup 1.0.1. Let $f : X \to S$ be a flat proper morphism of schemes. Let $L \in \mathrm{Pic}(X_s)$. Then L lifts to an element of* $\mathrm{Pic}(X)[1/p]$ *if and only if the image of* $c_{1,\mathrm{rig}}(L)$ *under the natural morphism* $H^2_{\mathrm{rig}}(X_s/K) \to H^2_{\mathrm{dR}}(X_\eta/K)$ *lies in* $F^1H^2_{\mathrm{dR}}(X_\eta/K)$.

Notation

A scheme separated of finite type over a field k is called an *algebraic variety* over k. For an integer $i \geq 0$ and a morphism of schemes $X \to S$, let $H^i_{\mathrm{dR}}(X/S) := H^i(R\Gamma(X, \Omega^\bullet_{X/S}))$ be the i-th *de Rham cohomology* group. For an integer $q \geq 0$, let $F^qH^i_{\mathrm{dR}}(X/S) := \mathrm{Im}(H^i(X, \sigma_{\geq q}\Omega^\bullet_{X/S}) \to H^i_{\mathrm{dR}}(X/S))$ be the q-th Hodge filtration.

2 Review on crystalline Chern class

2.1 Nilpotent case

We recall the construction of crystalline first Chern class in [BI70].

Let (S, I, γ) be a divided power scheme in the sense of [Sta25, Tag 07I3]. Let p be a prime *nilpotent* on S. Let $f : X \to S$ be a morphism of schemes. Assume that γ extends to X. As p is nilpotent on S, the small crystalline site $\mathrm{Cris}(X/S)$ defined in [Ber74, III, Définition 1.1.1] coincides with that in [BI70]. Let

$$i_{X/S} : X_{\mathrm{Zar}} \to (X/S)_{\mathrm{cris}}, \quad f_{X/S} : (X/S)_{\mathrm{cris}} \to S_{\mathrm{Zar}}$$

be the morphisms of topoi in [Ber74, Propositions III 3.3.2 and V 3.5.2]. Let $J_{X/S} \subset O_{X/S}$ be the canonical divided power ideal in [Ber74, p.184]. Then there is a short exact sequence

$$0 \to J_{X/S} \to O_{X/S} \to i_{X/S*}O_X \to 0 \tag{1}$$

in the abelian category $\mathrm{Mod}(O_{X/S})$ of $O_{X/S}$-modules on $\mathrm{Cris}(X/S)$. By [Ber74, III, Remarques 1.1.2 i)], as p is nilpotent on S, for every object $(U,T,\delta) \in \mathrm{Cris}(X/S)$, $U \to T$ is a thickening. Hence, $J_{X/S}$ is a sheaf of nilideals, and (1) induces a short exact sequence

$$0 \to 1 + J_{X/S} \to O^*_{X/S} \to i_{X/S*}O^*_X \to 0 \tag{2}$$

of abelian sheaves in the crystalline topos $(X/S)_{\mathrm{cris}}$. This multiplicative sequence induces a coboundary morphism

$$i_{X/S*}O^*_X[-1] \xrightarrow{\partial} 1 + J_{X/S}$$

in $D(\mathrm{Ab}(\mathrm{Cris}(X/S)))$.

Let δ be the canonical divided power structure on $J_{X/S}$. Define a logarithmic morphism of abelian sheaves in $(X/S)_{\mathrm{cris}}$

$$\log : 1 + J_{X/S} \to J_{X/S}, \quad \log(1+x) := \sum_{m>0} (-1)^{m-1}(m-1)!\delta_m(x). \tag{3}$$

As p is nilpotent on S, the series is a finite sum. Then there are morphisms

$$Rf_*O^*_X[-1] = Rf_{X/S*}i_{X/S*}O^*_X[-1] \xrightarrow{\partial} Rf_{X/S*}(1+J_{X/S}) \xrightarrow{\log} Rf_{X/S*}J_{X/S} \to Rf_{X/S*}O_{X/S} \tag{4}$$

in $D(\mathrm{Ab}(S))$. Applying the functor $R\Gamma(S,-) : D(\mathrm{Ab}(S)) \to D(\mathrm{Ab})$ and taking cohomology, one gets a group morphism called the first crystalline Chern class

$$c_{1,\mathrm{cris}} : \mathrm{Pic}(X) = H^1(X, O^*_X) \to H^2_{\mathrm{cris}}(X/S).$$

2.2 P-adic base

We review the construction of the crystalline first Chern class over a P-adic base.

Setup 2.2.1. Fix a prime p. Let A be a Noetherian ring. Let (I,γ) be a divided power ideal of A. Let $I_0 \subset I$ be a sub divided power ideal, containing a power of p. Assume that A is separated and complete for the I_0-adic topology. In particular, A is an adic ring. For every integer $m \geq 0$, set $A_m = A/I_0^{m+1}$, $S_m = \mathrm{Spec}\, A_m$. Write γ for the induced PD structure on I/I_0^{m+1}. Then $(S_m, I/I_0^{m+1}, \gamma)$ is a PD scheme. Set $\hat{S} := \mathrm{Spf}\, A$.

Let X be a scheme over S_0 to which γ extends. Let $\mathrm{Cris}(X/\hat{S})$ be the site defined in [BO78, p.7-22]. Then one can define the crystalline first Chern class over $\hat{S}$

$$c_{1,\mathrm{cris}} : \mathrm{Pic}(X) \to H^2_{\mathrm{cris}}(X/\hat{S})$$

in a way parallel to Section 2.1. Let $J_{X/\hat{S}}$ be the kernel of the canonical morphism $O_{X/\hat{S}} \to i_{X/\hat{S}*}O_X$ in the abelian category of $O_{X/\hat{S}}$-modules on the site $\mathrm{Cris}(X/\hat{S})$. By definition of $\mathrm{Cris}(X/\hat{S})$, for every object (U, T, δ) of it, p is nilpotent on T. Therefore, the exact sequence

$$0 \to J_{X/\hat{S}} \to O_{X/\hat{S}} \to i_{X/\hat{S}*}O_X \to 0$$

induces an exact sequence

$$0 \to 1 + J_{X/\hat{S}} \to O^*_{X/\hat{S}} \to i_{X/\hat{S}*}O^*_X \to 0. \tag{5}$$

One can define a logarithm morphism $\log : 1 + J_{X/\hat{S}} \to J_{X/\hat{S}}$ by a formula similar to (3), which is again a finite sum as p is nilpotent on every object of $\mathrm{Cris}(X/\hat{S})$.

For every integer $n \geq 0$, let $i_n : (X/S_n)_{\mathrm{cris}} \to (X/\hat{S})_{\mathrm{cris}}$ be the canonical morphism of topoi. From [BO78, p.7-22], for a sheaf F on $\mathrm{Cris}(X/\hat{S})$, i_n^*F is the restriction of F to $\mathrm{Cris}(X/S_n)$. Thus, $i_n^* : \mathrm{Ab}(\mathrm{Cris}(X/\hat{S})) \to \mathrm{Ab}(\mathrm{Cris}(X/S_n))$ is exact, and

$$i_n^* J_{X/\hat{S}} = J_{X/S_n}. \tag{6}$$

The exact sequence (5) induces a morphism $\partial : i_{X/\hat{S}*}O^*_X[-1] \to 1 + J_{X/\hat{S}}$ in $D(\mathrm{Ab}(\mathrm{Cris}(X/\hat{S})))$, which is turned to $\partial_{(n)} : i_{X/S_n}O^*_X[-1] \to 1 + J_{X/S_n}$ constructed in Section 2.1 by i_n^*. By construction, i_n^* also turns $\log : 1 + J_{X/\hat{S}} \to J_{X/\hat{S}}$ to $\log_{(n)} : 1 + J_{X/S_n} \to J_{X/S_n}$.

3 Crystalline Chern classes in de Rham cohomology

3.1 Nilpotent case

The goal of Section 3.1 is to prove Lemma 3.1.5, an intermediate result helping to compare the de Rham first Chern class of a global line bundle L and the crystalline first Chern class of the restriction of L to the special fiber.

Let (S, I, γ) be a divided power scheme. Let p be a prime *nilpotent* on S. Let $f : X \to S$ be a morphism of schemes such that γ extends to X. By [BO78, p. 3.9], such an extension is unique. Let $\bar{\gamma}$ be the divided power structure on $I \cdot O_X$.

Let Y be another scheme over S and $g : X \to Y$ be a morphism over S. Let Y-HPD-Strat(X/S) be the site in [Ber74, III, Définition 1.2.1], which is a full subcategory of $\mathrm{Cris}(X/S)$. Let $(X/S)_{Y\text{-HPD-Strat}}$ be the associated topos. Let

$$j_{X\to Y/S} : (X/S)_{Y\text{-HPD-Strat}} \to (X/S)_{\mathrm{cris}}$$

be the morphism of PD-ringed topoi from [Ber74, III, (1.2.1)]. Let $u'_{X/S}$ and $f'_{X/S}$ be respectively the compositions

$$(X/S)_{Y\text{-HPD-Strat}} \xrightarrow{j_{X\to Y/S}} (X/S)_{\mathrm{cris}} \xrightarrow{u_{X/S}} X_{\mathrm{Zar}},$$

$$(X/S)_{Y\text{-HPD-Strat}} \xrightarrow{j_{X\to Y/S}} (X/S)_{\mathrm{cris}} \xrightarrow{f_{X/S}} S_{\mathrm{Zar}}$$

of morphisms of topoi. By adjunction, there are canonical morphisms of functors

$$Ru_{X/S*} \to Ru'_{X/S} j^*_{X\to Y/S} : D^+(\mathrm{Ab}(\mathrm{Cris}(X/S))) \to D^+(\mathrm{Ab}(X)), \tag{7}$$

$$\mathrm{Id} \to Ru'_{X/S} j^*_{X\to Y/S} i_{X/S*} : D^+(\mathrm{Ab}(X)) \to D^+(\mathrm{Ab}(X)) \tag{8}$$

respectively. (By [Ber74, III, Proposition 1.2.3], if Y is quasi-smooth over S, then $j_{X\to Y/S}$ is an equivalence of categories, and (7) is an isomorphism.)

Remark 3.1.1. From [Ber74, V, Théorème 1.2.5], by computing the Čech-Alexander cosimplicial complex and its associated cochain complex, one proves that if $g : X \to Y$ is an immersion, then (8) is an isomorphism of functors on $D^+(\mathrm{Ab}(X))$. Because of $R\Gamma(X, Ru'_{X/S*}-) = R\Gamma((X/S)_{Y\text{-HPD-Strat}}, -)$, it induces an isomorphism of functors

$$R\Gamma(X, \cdot) \xrightarrow{\sim} R\Gamma((X/S)_{Y\text{-HPD-Strat}}, j^*_{X\to Y/S} i_{X/S*} \cdot) : \quad D^+(\mathrm{Ab}(X)) \to D^+(\mathrm{Ab}).$$

Remark 3.1.2. By [Ber74, IV, Lemme 1.6.1 i)], there is a natural functor from the category of O_Y-modules with HPD stratification relative to S, with morphisms being horizontal, to the category of $O_{X/S}$-modules on Y-HPD-Strat(X/S). The image of O_Y equipped with the natural HPD stratification is $O_{X/S}$.

Let $S_0 = V(I)$ be the closed subscheme of S defined by I. Let $X_0 = X \times_S S_0$. By [Sta25, Tag 07GR], as p is nilpotent on S and I is quasi-coherent, $S_0 \to S$ is a finite order thickening. Then one has $1 + I \cdot O_X \subset O_X^*$ and $X_0 \to X$ is a thickening. Therefore, the sequence

$$0 \to 1 + I \cdot O_X \to O_X^* \to O_{X_0}^* \to 0 \tag{9}$$

of abelian sheaves on X is exact.

Let $\Omega^\times_{X/S}$ be the *multiplicative de Rham complex*

$$O_X^* \xrightarrow{d\log} \Omega^1_{X/S} \xrightarrow{d} \Omega_{X/S} \xrightarrow{d} \dots$$

where $d\log(\alpha) = d\alpha/\alpha$. Set

$$\mathcal{K}^\times_{X/S} := \ker(\Omega^\times_{X/S} \to O_{X_0}^*[0]), \quad \mathcal{I}^\bullet_{X/S} := \ker(\Omega^\bullet_{X/S} \to O_{X_0}[0]) \tag{10}$$

in the abelian category of complexes $\mathrm{Ch}(\mathrm{Ab}(X))$. Then $\mathcal{K}_{X/S}\times, 0$ (resp. $\mathcal{I}^0_{X/S}$) is the kernel of the morphism $O_X^* \to O_{X_0}^*$ (resp. $O_X \to O_{X_0}$) in $\mathrm{Ab}(X)$ (resp. $\mathrm{Mod}(O_X)$). In particular, as ideal sheaf $\mathcal{I}^0_{X/S}$ equals $I \cdot O_X$. By (9), one has $\mathcal{K}_{X/S}\times, 0 = 1 + \mathcal{I}^0_{X/S}$, $\mathcal{K}_{X/S} n = \mathcal{I}^n_{X/S}$ for every integer $n > 0$, and the sequence in $\mathrm{Ch}(\mathrm{Ab}(X))$

$$0 \to \mathcal{K}^\times_{X/S} \to \Omega^\times_{X/S} \to O_{X_0}^*[0] \to 0 \tag{11}$$

is exact. It induces a morphism $\partial : O_{X_0}^*[-1] \to \mathcal{K}^\times_{X/S}$ in $D(\mathrm{Ab}(X))$.

We define a morphism $\log : 1 + I \cdot O_X \to I \cdot O_X$ of Zariski abelian sheaves on X as follows. For every affine open subset $U = \operatorname{Spec} A$ of X and every local section $x \in \Gamma(U, I \cdot O_X)$, let

$$\log(1 + x) := \sum_{n>0} (-1)^{n-1}(n-1)! \bar{\gamma}_n(x). \tag{12}$$

As p is nilpotent on X, the sum is finite. Thus, the local maps glue to a morphism of sheaves. Then $\log : \mathcal{K}_{X/S}\times, 0 \to \mathcal{I}^0_{X/S}$ and $\mathrm{Id} : \mathcal{K}_{X/S}n \to \mathcal{I}^n_{X/S}$ for $n > 0$ fit to a morphism $\log^\bullet : \mathcal{K}^\times_{X/S} \to \mathcal{I}^\bullet_{X/S}$ in $\mathrm{Ch}(\mathrm{Ab}(X))$.

Let $\mathcal{D}_{X_0}(X)$ be the structure sheaf of the divided power envelope $D_{X_0}(X)$ of X_0 in X defined in [Ber74, I, 4.1.3]. As the divided power structure γ on I extends to X, and the ideal sheaf of $X_0 \to X$ is $I \cdot O_X$, the morphism

$$O_X \to \mathcal{D}_{X_0}(X) \tag{13}$$

is an isomorphism. Then by Remark 3.1.2 and [Ber74, V, (2.3.2)], for the *immersion* $X_0 \to X$ over S, there is a canonical morphism $Ru'_{X_0/S*}O_{X_0/S} \to \Omega^\bullet_{X/S}$ in $D^+(X_{0,\mathrm{Zar}}, O_S)$. From the composition

$$Ru_{X_0/S*}O_{X_0/S} \to Ru'_{X_0/S*}O_{X_0/S} \to \Omega^\bullet_{X/S}$$

where the first morphism is from (7), one has a morphism

$$Rf_{X_0/S*}O_{X_0/S} \to Rf_*\Omega^\bullet_{X/S} \tag{14}$$

in $D^+(X, O_S)$.

Remark 3.1.3. The morphism (14) induces a comparison morphism

$$R\Gamma_{\mathrm{cris}}(X_0/S) \to R\Gamma_{\mathrm{dR}}(X/S), \tag{15}$$

which has an equivalent construction as follows. From [BO78, Theorem 5.17], as p is nilpotent on S and I_0 is a sub divided power ideal of I, there is a natural isomorphism

$$R\Gamma_{\mathrm{cris}}(X/S) \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(X_0/S) \tag{16}$$

By [Ber74, V, Corollaire 2.3.7], as p is nilpotent on S, there is a canonical morphism $R\Gamma_{\mathrm{cris}}(X/S) \to R\Gamma_{\mathrm{dR}}(X/S)$. Its composition with the inverse of (16) is (15). For every integer $i \geq 0$, (15) induces a canonical morphism between crystalline cohomology and de Rham cohomology $H^i_{\mathrm{cris}}(X_0/S) \to H^i_{\mathrm{dR}}(X/S)$.

Let $\mathcal{C}$ be the category of O_X-modules, where morphisms are HPD differential operators (in the sense of [Ber74, II, Exemple 2.1.3 c)]) relative to S. Then $\mathcal{C}$ is an additive category. Let $\mathcal{D}$ be the category of crystals in $O_{X_0/S}$-module on the site X-HPD-Strat(X_0/S). Let $L : \mathcal{C} \to \mathcal{D}$ be the linearization functor, constructed in [Ber74, IV, Sec. 3.1.3]. Then $L(O_X)$ is a sheaf of rings on the site X-HPD-Strat(X_0/S).

Remark 3.1.4. We recall that the linearization of the de Rham complex is a complex. For every $n \geq 0$, let $P^n_{X/S}$ be the sheaf of principal parts of order n of the S-scheme X defined in [EGA IV 4, Définition 16.3.1], and let $D_{X/S}(n)$ be the sheaf introduced in [Ber74, p.75]. From [Ber74, II, 2.1.4 b) and Proposition 2.1.5], the connecting maps of the de Rham complex can be viewed as HPD differential operators. More explicitly, by [EGA IV 4, 16.8.1], as for every integer $i \geq 0$, the differential $d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ is a differential operator of order ≤ 1, it corresponds to an O_X-linear morphism $P^1_{X/S} \otimes_{O_X} \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$. The composition

$$D : D_{X/S}(1) \otimes_{O_X} \Omega^i_{X/S} \to P^1_{X/S} \otimes_{O_X} \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$$

is a HPD differential operator from $\Omega^i_{X/S}$ to $\Omega^{i+1}_{X/S}$. By [Ber74, IV, Proposition 3.2.7], the linearization of the de Rham complex

$$L(O_X) \overset{L(d)}{\to} L(\Omega^1_{X/S}) \overset{L(d)}{\to} L(\Omega^2_{X/S}) \to \dots$$

is a *complex* of $O_{X_0/S}$-modules on X-HPD-Strat(X_0/S), denoted by $L(\Omega^\bullet_{X/S})$.

Lemma 3.1.5. *The diagram*

$$\begin{tikzcd}
Rf_{X_0/S*} i_{X_0/S*} O^*_{X_0}[-1] \arrow[r, "\partial"] \arrow[d, equal] & Rf_{X_0/S*}(1 + J_{X_0/S}) \arrow[r, "\log"] & Rf_{X_0/S*} J_{X_0/S} \arrow[r] & Rf_{X_0/S*} O_{X_0/S} \arrow[d, "(14)"] \\
Rf_* O^*_{X_0}[-1] \arrow[r, "\partial"] & Rf_* \mathcal{K}^\times_{X/S} \arrow[r, "\log^\bullet"] & Rf_* \mathcal{I}^\bullet_{X/S} \arrow[r] & Rf_* \Omega^\bullet_{X/S}
\end{tikzcd}$$

in $D^+(\mathrm{Ab}(S))$ *is commutative, where the first row is from* (4).

Proof. we shall unfold the definition of (14) via the linearization functor.

Step 1 We recall some properties of an ideal sheaf $\mathcal{K} \subset L(O_X)$.

From [Ber74, V, (2.1.0)], there exists a natural morphism $O_{X_0/S} \to L(O_X)$ in the category $\mathcal{D}$ and a co-augmentation morphism

$$O_{X_0/S} \to L(\Omega^\bullet_{X/S})$$

in $D^+((X_0/S)_{X-\mathrm{HPD-strat}}, O_{X_0/S})$. Write j for the morphism of topoi

$$j_{X_0 \to X/S} : (X_0/S)_{X\text{-HPD-Strat}} \to (X_0/S)_{\mathrm{cris}}.$$

By [Ber74, V, (2.1.5)], there is a natural surjective morphism

$$L(O_X) \twoheadrightarrow j^* i_{X_0/S*} O_{X_0}$$

of $L(O_X)$-algebras on X-HPD-Strat(X_0/S). Let $\mathcal{K}$ be its kernel in the abelian category of $L(O_X)$-modules on X-HPD-Strat(X_0/S). There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & j^* J_{X_0/S} \arrow[r] \arrow[d, dashed] & O_{X_0/S} \arrow[r] \arrow[d] & j^* i_{X_0/S*} O_{X_0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & L(O_X) \arrow[r] & j^* i_{X_0/S*} O_{X_0} \arrow[r] & 0
\end{tikzcd} \tag{17}$$

with exact rows in the abelian category of $O_{X_0/S}$-modules on X -HPD-Strat(X_0/S). By [Ber74, pp.298–299], $\mathcal{K}$ has a natural structure of divided power ideal such that

$$(O_{X_0/S}, j^*J_{X_0/S}) \to (L(O_X), \mathcal{K}) \tag{18}$$

is a PD morphism. As p is nilpotent on S, $\mathcal{K}$ is a sheaf of nilideals. Thus, the bottom of (17) induces an exact sequence

$$0 \to 1+\mathcal{K} \to L(O_X)^* \to j^* i_{X_0/S*} O^*_{X_0} \to 0 \tag{19}$$

of abelian sheaves on X -HPD-Strat(X_0/S).

Step 2 We construct a natural subcomplex $\mathcal{K}^\bullet \subset L(\Omega^\bullet_{X/S})$ and a multiplicative version $\mathcal{K}^\times$.

Let $\mathcal{K}^\bullet$ be the subcomplex

$$\mathcal{K}^\bullet : \quad \mathcal{K} \stackrel{L(d)}{\to} L(\Omega^1_{X/S}) \stackrel{L(d)}{\to} L(\Omega^2_{X/S}) \to \dots$$

of the complex $L(\Omega^\bullet_{X/S})$ from Remark 3.1.4. The morphism

$$L(O_X)^* \to L(\Omega^1_{X/S}), \quad \alpha \mapsto L(d)(\alpha)/\alpha$$

fits into a complex of abelian sheaves on X -HPD-Strat(X_0/S)

$$L^\times : \quad L(O_X)^* \to L(\Omega^1_{X/S}) \stackrel{L(d)}{\to} L(\Omega^2_{X/S}) \to \dots$$

Let $\mathcal{K}^\times$ be the subcomplex

$$\mathcal{K}^\times : \quad 1+\mathcal{K} \to L(\Omega^1_{X/S}) \stackrel{L(d)}{\to} L(\Omega^2_{X/S}) \to \dots$$

of $L^\times$. Then (2) and (19) induce a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & j^*(1+J_{X_0/S}) \arrow[r] \arrow[d] & O^*_{X_0/S} \arrow[r] \arrow[d] & j^* i_{X_0/S*} O^*_{X_0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}^\times \arrow[r] & L^\times \arrow[r] & j^* i_{X_0/S*} O^*_{X_0} \arrow[r] & 0
\end{tikzcd} \tag{20}$$

of complexes of abelian sheaves on X -HPD-Strat(X_0/S).

Similar to (3), as p is nilpotent on S, one can define a morphism $\log : 1+\mathcal{K} \to \mathcal{K}$ of abelian sheaves on X -HPD-Strat(X_0/S). Together with $\mathrm{Id} : L(\Omega^n_{X/S}) \to L(\Omega^n_{X/S})$ for $n > 0$, it gives rise to a morphism

$$\log^\bullet : \mathcal{K}^\times \to \mathcal{K}^\bullet$$

of complexes. By (20), the left square in the diagram

$$\begin{array}{ccccccc} j^*i_{X_0/S*}O^*_{X_0}[-1] & \xrightarrow{j^*\partial} & j^*(1+J_{X_0/S}) & \xrightarrow{\log} & j^*J_{X_0/S} & \longrightarrow & O_{X_0/S} \\ \| & & \downarrow & & \downarrow & & \downarrow \\ j^*i_{X_0/S*}O^*_{X_0}[-1] & \xrightarrow{\partial} & \mathcal{K}^\times & \xrightarrow{\log^\bullet} & \mathcal{K}^\bullet & \longrightarrow & L(\Omega^\bullet_{X/S}) \end{array} \tag{21}$$

of complexes on X-HPD-Strat(X_0/S) is commutative. From the PD morphism (18), the square in the middle is also commutative.

Step 3 For the complex $\mathcal{K}^\times_{X/S}$ defined in (10), we construct a comparison morphism $\mathcal{K}^\times_{X/S} \to Ru'_{X_0/S}\mathcal{K}^\times$.

By [Ber74, V, Corollaire 2.2.4] and the isomorphism (13), for every integer $i \geq 0$ there exist canonical isomorphisms

$$Ru'_{X_0/S*}L(\Omega^i_{X/S}) \xrightarrow{\sim} \Omega^i_{X/S}, \quad Ru'_{X_0/S*}L(\Omega^\bullet_{X/S}) \xrightarrow{\sim} \Omega^\bullet_{X/S} \tag{22}$$

in $D^+(X_{0,\mathrm{Zar}}, O_S)$. When $i = 0$, the isomorphism $u'_{X_0/S*}L(O_X) \xrightarrow{\sim} O_X$ is compatible with ring structures, so it induces an isomorphism

$$O^*_X \xrightarrow{\sim} u'_{X_0/S*}L(O_X)^*$$

of abelian sheaves on X. By (22), it fits into an isomorphism of complexes $\Omega^\times_{X/S} \xrightarrow{\sim} u'_{X_0/S*}L^\times$.

As

$$u'_{X_0/S*} : \mathrm{Ab}(X\text{-HPD-Strat}(X_0/S)) \to \mathrm{Ab}(X_0)$$

is the direct image of a morphism of topoi, it is left exact. Then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}^\times_{X/S} & \longrightarrow & \Omega^\times_{X/S} & \longrightarrow & O^*_{X_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & u'_{X_0/S*}\mathcal{K}^\times & \longrightarrow & u'_{X_0/S*}L^\times & \longrightarrow & u'_{X_0/S*}j^*i_{X_0/S*}O^*_{X_0} & & \end{array} \tag{23}$$

with exact rows, where the firs row is (11), the second row is induced by the bottom of (20), and the vertical isomorphism on the right is from Remark 3.1.1. The diagram induces an isomorphism $\mathcal{K}^\times_{X/S} \xrightarrow{\sim} u'_{X_0/S*}\mathcal{K}^\times$ of complexes and hence a morphism

$$\mathcal{K}^\times_{X/S} \to Ru'_{X_0/S*}\mathcal{K}^\times \tag{24}$$

in $D^+(\mathrm{Ab}(X\text{-HPD-Strat}(X_0/S)))$.

Step 4 Using complexes $\mathcal{K}^\bullet$ and $\mathcal{K}^\times$ we decompose the stated diagram to show its commutativity.

From [Ber74, V, Théorème 1.2.5, Proposition 2.2.2], as $X_0 \to X$ is an immersion over S, there are isomorphisms

$$\mathcal{I}^0_{X/S} \xrightarrow{\sim} Ru'_{X_0/S*}\mathcal{K}, \tag{25}$$

$$\mathcal{I}^\bullet_{X/S} \xrightarrow{\sim} Ru'_{X_0/S*}\mathcal{K}^\bullet \tag{26}$$

in the abelian category of $u'_{X_0/S*}O_{X_0/S}$-modules on X_0, and in $D^+(X_{0,\mathrm{Zar}}, u'_{X_0/S*}O_{X_0/S})$ respectively. The morphism (25) is compatible with the divided power structures, so the diagram

$$\begin{tikzcd}
\mathcal{K}^\times_{X/S} \arrow[r, "\log^\bullet"] \arrow[d, "(24)"] & \mathcal{I}^\bullet_{X/S} \arrow[d, "(26)"] \\
Ru'_{X_0/S*}\mathcal{K}^\times \arrow[r, "\log^\bullet"] & Ru'_{X_0/S*}\mathcal{K}^\bullet
\end{tikzcd} \tag{27}$$

is commutative.

One has a diagram

$$\begin{tikzcd}
Rf_{X_0/S*}i_{X_0/S*}O^*_{X_0}[-1] \arrow[r, "\partial"] \arrow[d] & Rf_{X_0/S*}(1+J_{X_0/S}) \arrow[r, "\log"] \arrow[d] & Rf_{X_0/S*}J_{X_0/S} \arrow[r] \arrow[d] & Rf_{X_0/S*}O_{X_0/S} \arrow[d] \arrow[dd, bend left, "(14)"] \\
Rf'_{X_0/S*}j^*i_{X_0/S*}O^*_{X_0}[-1] \arrow[r, "\partial"] & Rf'_{X_0/S*}\mathcal{K}^\times \arrow[r, "\log^\bullet"] & Rf'_{X_0/S*}\mathcal{K}^\bullet \arrow[r] & Rf'_{X_0/S*}L(\Omega^\bullet_{X/S}) \\
Rf_*O^*_{X_0}[-1] \arrow[r, "\partial"] \arrow[u, "\cong"] & Rf_*\mathcal{K}^\times_{X/S} \arrow[r, "\log^\bullet"] \arrow[u] & Rf_*\mathcal{I}^\bullet_{X/S} \arrow[r] \arrow[u] & Rf_*\Omega^\bullet_{X/S} \arrow[u, "\cong"]
\end{tikzcd}$$

in $D^+(\mathrm{Ab}(S))$. The isomorphism $Rf_*O^*_{X_0}[-1] \xrightarrow{\sim} Rf'_{X_0/S*}j^*i_{X_0/S*}O^*_{X_0}[-1]$ on the left is from Remark 3.1.1. The isomorphism $Rf_*\Omega^\bullet_{X/S} \to Rf'_{X_0/S*}L(\Omega^\bullet_{X/S})$ on the right is induced by (22). By (21), the squares on the top are commutative. From (23), the square in the lower left corner is commutative. By (27), the square in the middle of the bottom is commutative. Therefore, the outer rectangular is commutative.

□

3.2 P-adic base

We shall compare the de Rham first Chern class and the crystalline first Chern class for a scheme over a P-adic base. For this, we apply the result obtained in the nilpotent case and pass to limit.

We recall the de Rham cohomology of formal schemes. For a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of locally Noetherian formal schemes, let $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ be the sheaf of differential, and $d : O_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ be the canonical derivation defined in [TLR07, p.1354]. From this, one constructs the de Rham complex $\Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}}$, which is an $O_{\mathfrak{Y}}$-linear complex of $O_{\mathfrak{X}}$-modules. Let $R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{Y}) := R\Gamma(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}})$. By [TLR07, Proposition 3.3], when $f : \mathfrak{X} \to \mathfrak{Y}$ is of finite type, $\Omega^i_{\mathfrak{X}/\mathfrak{Y}}$ is coherent over $O_{\mathfrak{X}}$ for every $i \geq 0$. Lemma 3.2.1 can be deduced from [EGA III 1, Corollaires 3.4.4, 4.1.7] in a way similar to [MP12, Sec. 4.5.3].

Lemma 3.2.1. (a) *Let R be a Noetherian adic ring, $\hat{S} := \mathrm{Spf}(R)$. Let I be an ideal of definition of R. Let $\mathfrak{X}$ be a Noetherian formal scheme proper over R. For every $n \geq 0$, let $S_n = \mathrm{Spec}(R/I^{n+1})$, $X_n := \mathfrak{X} \times_{\hat{S}} S_n$. Then there is an isomorphism*

$$R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \xrightarrow{\sim} R\lim_n R\Gamma_{\mathrm{dR}}(X_n/S_n)$$

in $D(R)$.

(b) *Let $f : X \to Y$ be a* proper *morphism of locally Noetherian schemes. Let Y' be a closed subset of Y. Let $X' = f^{-1}(Y')$. Let $\hat{f} : \hat{X} \to \hat{Y}$ be the morphism of completions in the sense of [GD71, 10.9.1].*

i) *Let $F^\bullet$ be a bounded below $f^{-1}O_Y$-linear complex of* coherent *O_X-modules. Let $\hat{F}^\bullet$ be the completion of $F^\bullet$, which is an $\hat{f}^{-1}O_{\hat{Y}}$-linear complex of coherent $O_{\hat{X}}$-modules. Then for every integer n, $R^n\hat{f}\hat{F}^\bullet$ is a coherent $O_{\hat{Y}}$-module. The natural morphism $\widehat{R^n f_* F^\bullet} \xrightarrow{\sim} R^n\hat{f}\hat{F}^\bullet$ of $O_{\hat{Y}}$-modules is an isomorphism.*

ii) *Assume that $Y = \mathrm{Spec}\, A$ is affine. Take $F^\bullet = \Omega^\bullet_{X/Y}$ and an ideal $I \subset A$ defining Y'. Let $\hat{A} := \lim_{n\geq 0} A/I^{n+1}$. Then for every $q \geq 0$, the isomorphism $H^n_{\mathrm{dR}}(X/Y) \otimes_A \hat{A} \xrightarrow{\sim} H^n_{\mathrm{dR}}(\hat{X}/\hat{Y})$ from Part* i) *restricts to an isomorphism of Hodge filtrations*

$$(F^q H^n_{\mathrm{dR}}(X/Y)) \otimes_A \hat{A} \xrightarrow{\sim} F^q H^n_{\mathrm{dR}}(\hat{X}/\hat{Y}).$$

Moreover, one has $F^q H^n_{\mathrm{dR}}(\hat{X}/\hat{Y}) = \lim_{n\geq 0} F^q H^n_{\mathrm{dR}}(X_n/Y_n)$.

Consider Setup 2.2.1. Let $f : \mathfrak{X} \to \hat{S}$ be a flat adic morphism of Noetherian formal schemes. For every integer $m \geq 0$, let $X_m = \mathfrak{X} \times_{\hat{S}} S_m$. Then $X_m \to S_m$ is a flat morphism of schemes.

Lemma 3.2.2. *Assume further that $f : \mathfrak{X} \to \hat{S}$ is proper. Then for every integer $i \geq 0$, there a natural morphism $H^i_{\mathrm{cris}}(X_0/\hat{S}) \to H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$.*

Proof. From [Ber74, I, Corollaire 2.7.4], for every $n \geq 0$, since $X_n \to S_n$ is flat, the divided power structure on S_n extends to X_n. Then by Remark 3.1.3, as p is nilpotent on S_n, there is a canonical morphism

$$R\Gamma_{\mathrm{cris}}(X_0/S_n) \to R\Gamma_{\mathrm{dR}}(X_n/S_n). \tag{28}$$

From Lemma 3.2.1 (a), as A is a Noetherian adic ring, and $\mathfrak{X} \to \hat{S}$ is proper, there is an isomorphism

$$R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \xrightarrow{\sim} R\lim_n R\Gamma_{\mathrm{dR}}(X_n/S_n) \tag{29}$$

in $D^+(A)$. By [BO78, 7.22.2], as X_0 is a scheme over S_0, there is an isomorphism

$$R\Gamma_{\mathrm{cris}}(X_0/\hat{S}) \cong R\lim_n R\Gamma_{\mathrm{cris}}(X_0/S_n). \tag{30}$$

Taking $R\lim_n$ of the inverse system (28), one gets a comparison morphism

$$\tau_{\mathrm{cris}} : R\Gamma_{\mathrm{cris}}(X_0/\hat{S}) \to R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$$

in $D^+(A)$. Taking cohomology groups one gets the stated morphism. □

Remark 3.2.3. The comparison morphism in Lemma 3.2.2 is functorial in $f : \mathfrak{X} \to \hat{S}$ in the following sense. Let (A', I', γ') be another divided power ring, and $I'_0 \subset I'$ be a sub divided power ideal, which also satisfies the assumption in Setup 2.2.1. Let $(A, I, \gamma) \to (A', I', \gamma')$ be a morphism of divided power rings inducing a map $I_0 \to I'_0$. Let $g : \hat{S}' \to \hat{S}$ be the induced morphism of formal spectra. Consider a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \hat{S}' & \xrightarrow{g} & \hat{S} \end{array}$$

of Noetherian formal schemes, with both $f : \mathfrak{X} \to \hat{S}$ and $f' : \mathfrak{X}' \to \hat{S}'$ flat and proper. As I_0 is mapped to I'_0, $g' : \mathfrak{X}' \to \mathfrak{X}$ restricts to a morphism $X'_0 \to X_0$ of schemes. Then for every integer $i \geq 0$, there is a commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{cris}}(X_0/\hat{S}) & \xrightarrow{g'^*} & H^i_{\mathrm{cris}}(X'_0/\hat{S}') \\ \downarrow & & \downarrow \\ H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) & \xrightarrow{g'^*} & H^i_{\mathrm{dR}}(\mathfrak{X}'/\hat{S}'). \end{array}$$

Corollary 3.2.4. *In the notation of Setup 2.2.1, let $f : X \to S$ be a flat proper morphism of schemes. Let $X_0 := X \otimes_A A/I_0$. Then for every $i \geq 0$, there is a natural morphism $\tau_{\mathrm{cris}} : H^i_{\mathrm{cris}}(X_0/\hat{S}) \to H^i_{\mathrm{dR}}(X/A)$.*

Proof. Let $\mathfrak{X}$ be the formal completion of X along X_0. Let $\hat{f} : \mathfrak{X} \to \hat{S}$ be the completion of $f : X \to S$, which is proper by [EGA III 1, (3.4.1)]. From [FK18, I, Corollary 4.8.2], $\hat{f} : \mathfrak{X} \to \hat{S}$ is flat. By Lemma 3.2.1 i), as A is complete, there is a canonical isomorphism $H^i_{\mathrm{dR}}(X/A) \cong H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$. Then the result follows from Lemma 3.2.2. □

Similar to Section 3.1, we define complexes of abelian sheaves

$$\mathcal{K}^{\times}_{\mathfrak{X}/\hat{S}} := \ker(\Omega^{\times}_{\mathfrak{X}/\hat{S}} \to O^{*}_{X_0}), \quad \mathcal{I}^{\bullet}_{\mathfrak{X}/\hat{S}} := \ker(\Omega^{\bullet}_{\mathfrak{X}/\hat{S}} \to O_{X_0}[0])$$

on $\mathfrak{X}$. Then $\mathcal{I}^{0}_{\mathfrak{X}/\hat{S}} = I_0 \cdot O_{\mathfrak{X}}$. For every integer $n \geq 0$, as p is nilpotent on S_n, and γ extends to X_n, there is an exact sequence

$$0 \to \mathcal{I}^{\bullet}_{X_n/S_n} \to \Omega^{\bullet}_{X_n/S_n} \to O_{X_0} \to 0$$

in $\mathrm{Ch}(\mathrm{Ab}(X_n))$. As inverse limit is left exact, one has $\lim_n \mathcal{I}^{\bullet}_{X_n/S_n} = \mathcal{I}^{\bullet}_{\mathfrak{X}/\hat{S}}$.

Lemma 3.2.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of locally Noetherian formal schemes. Let $\mathfrak{I} \subset O_{\mathfrak{Y}}$ be an ideal of definition. Let $\mathfrak{J} = f^{*}(\mathfrak{I}) \cdot O_{\mathfrak{X}}$ and $X_0 = (\mathfrak{X}, O_{\mathfrak{X}}/\mathfrak{J})$. Then X_0 is a locally Noetherian scheme. The short exact sequence*

$$0 \to \mathfrak{J} \to O_{\mathfrak{X}} \to O_{X_0} \to 0$$

induces a short exact sequence

$$0 \to 1 + \mathfrak{J} \to O^{*}_{\mathfrak{X}} \to O^{*}_{X_0} \to 0.$$

Proof. As $f : \mathfrak{X} \to \mathfrak{Y}$ is an adic morphism, $\mathfrak{J}$ is an ideal of definition of $\mathfrak{X}$. Then from [GD71, 10.5.2], X_0 is a locally Noetherian scheme. Exactness is a local property, so one may assume that $\mathfrak{X}$, $\mathfrak{Y}$ are affine, $f : \mathfrak{X} \to \mathfrak{Y}$ is induced by a morphism $A \to B$ of Noetherian adic rings, and that $\mathfrak{I}$ is induced by an ideal of definition I of A. By [GD71, Corollaire 7.1.12], as $I_0 B$ is an ideal of definition of the adic ring B, one has $1 + I_0 B \subset B^{*}$, and the sequence $0 \to 1 + I_0 B \to B^{*} \to (B/IB)^{*} \to 0$ is exact. □

From Lemma 3.2.5, one has $\mathcal{K}^{\times,0}_{\mathfrak{X}/\hat{S}} = 1 + \mathcal{I}^{0}_{\mathfrak{X}/\hat{S}}$ and hence $\lim_n \mathcal{K}^{\times}_{X_n/S_n} = \mathcal{K}^{\times}_{\mathfrak{X}/\hat{S}}$. Whence, the termwise inverse limit of the inverse system of exact sequences

$$0 \to \mathcal{K}^{\times}_{X_n/S_n} \to \Omega^{\times}_{X_n/S_n} \to O^{*}_{X_0} \to 0$$

is

$$0 \to \mathcal{K}^{\times}_{\mathfrak{X}/\hat{S}} \to \Omega^{\times}_{\mathfrak{X}/\hat{S}} \to O^{*}_{X_0} \to 0,$$

which is also exact. For every $n \geq 0$, the induced morphism $\partial : O^{*}_{X_0}[-1] \to \mathcal{K}^{\times}_{\mathfrak{X}/\hat{S}}$ restricts to $\partial_{(n)} : O^{*}_{X_0}[-1] \to \mathcal{K}^{\times}_{X_n/S_n}$ on X_n constructed in Section 3.1.

Let $\mathrm{Spf}\, B$ be an affine open subset of $\mathfrak{X}$. Since a power of p lies in I_0, and since B is complete, the series analogous to (12) converges in this case. Every ideal of a Noetherian adic ring is closed, so for every $x \in I_0 B$, the corresponding limit of the series is in $I_0 B$. Thus, using the divided power structure γ on the ideal $I_0 \subset A$, one similarly defines a logarithm morphism

$$\log : 1 + I_0 \cdot O_{\mathfrak{X}} \to I_0 \cdot O_{\mathfrak{X}} \tag{31}$$

of abelian sheaves on $\mathfrak{X}$. Then one gets a morphism $\log^{\bullet} : \mathcal{K}^{\times}_{\mathfrak{X}/\hat{S}} \to \mathcal{I}^{\bullet}_{\mathfrak{X}/\hat{S}}$ of complexes of sheaves, which by construction is the inverse limit of the logarithm morphisms $\log^{\bullet}_{(n)} : \mathcal{K}^{\times}_{X_n/S_n} \to \mathcal{I}^{\bullet}_{X_n/S_n}$ of complexes.

Remark 3.2.6. We show that if the divided power structure γ is I_0-adically nilpotent, i.e., for every integer $n > 0$, there is an integer $m > 0$ with $\gamma_j(I_0) \subset I_0^n$ for all integers $j \geq m$, then (31) is an isomorphism. Let $\bar{\gamma}$ be the divided power structure on $I_0 \cdot O_{\mathfrak{X}}$ induced by γ. Then $\bar{\gamma}_j(I_0 \cdot O_{\mathfrak{X}}) \subset I_0^n \cdot O_{\mathfrak{X}}$ for all integers $j \geq m$. One can define a morphism

$$\exp : I_0 \cdot O_{\mathfrak{X}} \to 1 + I_0 \cdot O_{\mathfrak{X}}, \quad x \mapsto \sum_{n \geq 0} \bar{\gamma}_n(x)$$

as the series converges. This exponential morphism is inverse to (31).

Lemma 3.2.7. *Let $\mathfrak{X}$ be a Noetherian formal scheme, and let $\mathfrak{X} \to \hat{S}$ be a flat proper morphism. Then the comparison morphism from Lemma 3.2.2 fits into a commutative diagram*

$$\begin{tikzcd}
\operatorname{Pic}(X_0) \arrow[rrr, "c_{1,\mathrm{cris}}"] \arrow[d, "\cong"] & & & H^2_{\mathrm{cris}}(X_0/\hat{S}) \arrow[d, dashed] \\
H^1(\mathfrak{X}, O^*_{X_0}) \arrow[r, "\partial"] \arrow[dr, "\partial"'] & H^2(\mathfrak{X}, \mathcal{K}^\times_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^\bullet"] \arrow[d] & H^2(\mathfrak{X}, \mathcal{I}^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r] \arrow[d] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[d] \\
& H^2(\mathfrak{X}, \mathcal{K}^{\times,0}_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log"] & H^2(\mathfrak{X}, \mathcal{I}^0_{\mathfrak{X}/\hat{S}}) \arrow[r] & H^2(\mathfrak{X}, O_{\mathfrak{X}}).
\end{tikzcd}$$

Proof. From Lemma 3.2.5, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}^\times_{\mathfrak{X}/\hat{S}} \arrow[r] \arrow[d] & \Omega^\times_{\mathfrak{X}/\hat{S}} \arrow[r] \arrow[d] & O^*_{X_0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}^0_{\mathfrak{X}/\hat{S}} \arrow[r] & O^*_{\mathfrak{X}} \arrow[r] & O^*_{X_0} \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{Ch}(\mathrm{Ab}(\mathfrak{X}))$ is commutative and has exact rows. Thus, the triangle in the lower left corner of the stated diagram is commutative. By construction, the squares at the bottom are commutative.

By Lemma 3.1.5, for every integer $n \geq 0$, as p is nilpotent on S_n, there is a canonical commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(X_0/S_n, i_{X_0/S_n*}O_{X_0^*})[-1] \arrow[r, "\partial_{(n)}"] \arrow[d, "\cong"] & R\Gamma_{\mathrm{cris}}(X_0/S_n, 1 + J_{X_0/S_n}) \arrow[r, "\log_{(n)}"] & R\Gamma_{\mathrm{cris}}(X_0/S_n, J_{X_0/S_n}) \arrow[r] & R\Gamma_{\mathrm{cris}}(X_0/S_n) \arrow[d] \\
R\Gamma(X_n, O^*_{X_0})[-1] \arrow[r, "\partial_{(n)}"] & R\Gamma(X_n, \mathcal{K}^\times_{X_n/S_n}) \arrow[r, "\log^\bullet_{(n)}"] & R\Gamma(X_n, \mathcal{I}^\bullet_{X_n/S_n}) \arrow[r] & R\Gamma_{\mathrm{dR}}(X_n/S_n)
\end{tikzcd} \tag{32}$$

in $D^+(\mathrm{Ab})$. The vertical morphism on the left is an isomorphism, and we identify both its source and target with $R\Gamma(X_0, O^*_{X_0})[-1]$.

By (6), there exist morphisms

$$R\Gamma(X_0/\hat{S}, J_{X_0/\hat{S}}) \to R\lim_n R\Gamma(X_0/S_n, J_{X_0/S_n}), \quad R\Gamma(X_0/\hat{S}, 1+J_{X_0/\hat{S}}) \to R\lim_n R\Gamma(X_0/S_n, 1+J_{X_0/S_n}).$$

From Section 2.2, each square in the diagram

$$\begin{tikzcd}
R\Gamma(X_0/\hat{S}, i_{X_0/\hat{S}}O^*_{X_0})[-1] \arrow[r, "\partial"] \arrow[d, "\cong"] & R\Gamma(X_0/\hat{S}, 1+J_{X_0/\hat{S}}) \arrow[r, "\log"] \arrow[d] & R\Gamma(X_0/\hat{S}, J_{X_0/\hat{S}}) \arrow[r] \arrow[d] & R\Gamma_{\mathrm{cris}}(X_0/\hat{S}) \arrow[d, "\cong"] \\
R\lim_n R\Gamma(X_0/S_n, i_{X_0/S_n*}O_{X_0^*})[-1] \arrow[r, "R\lim_n \partial_{(n)}"'] & R\lim_n R\Gamma(X_0/S_n, 1+J_{X_0/S_n}) \arrow[r, "R\lim_n \log_{(n)}"'] & R\lim_n R\Gamma(X_0/S_n, J_{X_0/S_n}) \arrow[r] & R\lim_n R\Gamma_{\mathrm{cris}}(X_0/S_n)
\end{tikzcd} \tag{33}$$

is commutative, where the vertical isomorphism on the right is from (30). Similarly, each square in the diagram

$$\begin{tikzcd}
R\Gamma(\mathfrak{X}, O^*_{X_0})[-1] \arrow[r, "\partial"] \arrow[d, "\cong"] & R\Gamma(\mathfrak{X}, \mathcal{K}^\times_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^\bullet"] \arrow[d] & R\Gamma(\mathfrak{X}, I^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r] \arrow[d] & R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[d, "\cong"] \\
R\lim_n R\Gamma(X_n, O_{X_0^*})[-1] \arrow[r, "R\lim_n \partial_{(n)}"'] & R\lim_n R\Gamma(X_n, \mathcal{K}^\times_{X_n/S_n}) \arrow[r, "R\lim_n \log^\bullet_{(n)}"'] & R\lim_n R\Gamma(X_n, I^\bullet_{X_n/S_n}) \arrow[r] & R\lim_n R\Gamma_{\mathrm{dR}}(X_n/S_n)
\end{tikzcd} \tag{34}$$

is also commutative, where the isomorphism on the right is from (29). Taking $R\lim_n$ of (32), and combining it with (33) and (34), one has a commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(X_0/\hat{S}, i_{X_0/\hat{S}}O^*_{X_0})[-1] \arrow[r, "\partial"] \arrow[d, "\cong"] & R\Gamma_{\mathrm{cris}}(X_0/\hat{S}, 1+J_{X_0/\hat{S}}) \arrow[r, "\log"] & R\Gamma_{\mathrm{cris}}(X_0/\hat{S}, J_{X_0/\hat{S}}) \arrow[r] & R\Gamma_{\mathrm{cris}}(X_0/\hat{S}) \arrow[d] \\
R\Gamma(\mathfrak{X}, O^*_{X_0})[-1] \arrow[r, "\partial"] & R\Gamma(\mathfrak{X}, \mathcal{K}^\times_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^\bullet"] & R\Gamma(\mathfrak{X}, I^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\hat{S})
\end{tikzcd}$$

Taking cohomology groups, one gets the upper half of the stated diagram. □

Remark 3.2.8. We recall the construction of the de Rham first Chern class on formal schemes. For every abelian category $\mathcal{A}$ and every complex $\mathcal{K}^\bullet \in \mathrm{Ch}^{\geq 0}(\mathcal{A})$, let $\sigma_{\geq 1}\mathcal{K}^\bullet$ be a naive truncation of complex. The differential $d : K^0 \to K^1$ induces a morphism $\partial : K^0 \to \sigma_{\geq 1}\mathcal{K}^\bullet[1]$ in $\mathrm{Ch}^{\geq 0}(\mathcal{A})$. It represents the connecting morphism in $D(\mathcal{A})$ induced by the short exact sequence

$$0 \to \sigma_{\geq 1}\mathcal{K}^\bullet \to \mathcal{K}^\bullet \to K^0 \to 0.$$

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of finite type of locally Noetherian formal schemes. One has $\sigma_{\geq 1}\Omega^\times_{\mathfrak{X}/\mathfrak{Y}} = \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}}$. The morphism $d\log : O^*_{\mathfrak{X}}[-1] \to \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}}$ in $\mathrm{Ch}(\mathrm{Ab}(\mathfrak{X}))$ represents the connecting morphism induced by the short exact sequence

$$0 \to \sigma_{\geq 1}\Omega^\times_{\mathfrak{X}/\mathfrak{Y}} \to \Omega^\times_{\mathfrak{X}/\mathfrak{Y}} \to O^*_{\mathfrak{X}} \to 0.$$

It induces a boundary map $\gamma_1 : H^1(\mathfrak{X}, O^*_{\mathfrak{X}}) \to H^2(\mathfrak{X}, \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}})$. The composition

$$c_{1,\mathrm{dR}} : \mathrm{Pic}(\mathfrak{X}) \xrightarrow{\gamma_1} H^2(\mathfrak{X}, \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\mathfrak{Y}}) \to H^2_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{Y})$$

is called the *first Chern class in de Rham cohomology.*

As a consequence of Lemma 3.2.7, the crystalline first Chern class on X_0 is compatible with the de Rham first Chen class on $\mathfrak{X}$.

Corollary 3.2.9. *In the setting of Lemma 3.2.7, the diagram*

$$\begin{tikzcd}
\mathrm{Pic}(X_0) \arrow[rrr] & & & H^2_{\mathrm{cris}}(X_0/\hat{S}) \arrow[d] \\
\mathrm{Pic}(\mathfrak{X}) \arrow[u] \arrow[r, "\gamma_1"] \arrow[rrr, bend left, "c_{1,\mathrm{dR}}"] & H^2(\mathfrak{X}, \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r, two heads] & F^1 H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[r, hook] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
H^1(\mathfrak{X}, O^*_{\mathfrak{X}}) \arrow[r, "\gamma_1"] \arrow[d] & H^2(\mathfrak{X}, \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[d] \arrow[rrd] & & \\
H^1(\mathfrak{X}, O^*_{X_0}) \arrow[r, "\partial"] & H^2(\mathfrak{X}, \mathcal{K}^\times_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^\bullet"] & H^2(\mathfrak{X}, I^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \\
\mathrm{Pic}(X_0) \arrow[u, "\cong"] \arrow[rrr, "c_{1,\mathrm{cris}}"] & & & H^2_{\mathrm{cris}}(X_0/\hat{S}). \arrow[u]
\end{tikzcd}$$

The commutativity of the square in the upper left corner follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\hat{S}} \arrow[r] \arrow[d, dashed] & \Omega^\times_{\mathfrak{X}/\hat{S}} \arrow[r] \arrow[d, equal] & O^*_{\mathfrak{X}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}^\times_{\mathfrak{X}/\hat{S}} \arrow[r] & \Omega^\times_{\mathfrak{X}/\hat{S}} \arrow[r] & O^*_{X_0} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The composition $\sigma_{\geq 1}\Omega^\bullet_{\mathfrak{X}/\hat{S}} \to \mathcal{K}^\times_{\mathfrak{X}/\hat{S}} \xrightarrow{\log^\bullet} \mathcal{I}^\bullet_{\mathfrak{X}/\hat{S}} \hookrightarrow \Omega^\bullet_{\mathfrak{X}/\hat{S}}$ is the inclusion, so the triangle is commutative. By Lemma 3.2.7, as $\mathfrak{X} \to \hat{S}$ is proper, the rectangular on the bottom is commutative. $\square$

4 Singular analog of Berthelot-Ogus lifting theorem

We remove the smoothness assumption of [BO83, Theorem 3.8] in Theorem 4.2.1, at the cost of adding the low ramification restriction. The notation is as in Setup 1.0.1. Let $\hat{S} := \mathrm{Spf}\, V$ and $\Sigma = \mathrm{Spf}\, W$. For an integer $n \geq 0$, let $S_n = \mathrm{Spec}\, V/p^{n+1}$ and $\Sigma_n = \mathrm{Spec}\, W/p^{n+1}$.

4.1 Comparison morphism for changing base

We recall the morphism comparing crystalline cohomology groups over different base rings.

By [BO83, Lemma 3.9], for every integer $n \geq e/(p-1)$, $\pi^n V$ admits a unique divided power structure γ, which is π-adically nilpotent if and only if $n > e/(p-1)$. By [Ber74, I, Corollaire 1.2.4], as k is perfect, the maximal ideal $pW \subset W$ has a unique divided power structure γ_W. By [Ser79, II, Theorem 4], there is a unique morphism of rings $W \to V$ fitting to a commutative diagram

$$\begin{tikzcd} W \arrow[rr, dashed] \arrow[dr] & & V \arrow[dl] \\ & k. & \end{tikzcd}$$

It makes V a finite free W-module and gives rise to a morphism of divided power rings $(W, pW, \gamma_W) \to (V, pV, \gamma)$.

Let $\mathfrak{X} \to \hat{S}$ be a proper morphism of formal schemes, with special fiber X_s. For integer $n \geq 0$, let $X_n := \mathfrak{X} \times_{\hat{S}} S_n$, which is a scheme proper over S_n. From [Ber74, I, Proposition 2.1.1], as the divided power ideals we use are principal, the DP structure always extends.

By [BO78, p. 7.11], for every integer $n \geq 0$, there is a base change morphism

$$R\Gamma_{\mathrm{cris}}(X_s/\Sigma_n) \otimes^L_{W/p^{n+1}} \frac{V}{p^{n+1}} \to R\Gamma_{\mathrm{cris}}(X_s/S_n, p, \gamma). \tag{35}$$

From the flat base change theorem [Har66, Proposition 5.12, p.111], since V is a flat W-module, there is a natural isomorphism of functors

$$(-) \otimes^L_W V \xrightarrow{\sim} (-) \otimes_{W/p^{n+1}} V/p^{n+1} : D(W/p^{n+1}) \to D(V).$$

Thus, (35) is identified with a natural morphism

$$R\Gamma_{\mathrm{cris}}(X_s/\Sigma_n) \otimes^L_W V \to R\Gamma_{\mathrm{cris}}(X_s/S_n, p, \gamma)$$

in $D^+(V)$. Taking derived limits, one gets a morphism

$$R\lim_n (R\Gamma_{\mathrm{cris}}(X_s/\Sigma_n) \otimes^L_W V) \to R\lim_n R\Gamma_{\mathrm{cris}}(X_s/S_n, p, \gamma) \tag{36}$$

in $D(V)$. As $V \in D(W)$ is a perfect complex, the functor $(-) \otimes^L_W V : D(W) \to D(V)$ commutes with $R\lim$. Combined with [BO78, 7.22.2], (36) identifies with a morphism

$$R\Gamma_{\mathrm{cris}}(X_s/\Sigma) \otimes^L_W V \to R\Gamma_{\mathrm{cris}}(X_s/\hat{S}, p, \gamma). \tag{37}$$

As V is flat over W, for every integer $i \geq 0$, (37) induces a natural morphism

$$H^i_{\mathrm{cris}}(X_s/\Sigma) \otimes_W V \to H^i_{\mathrm{cris}}(X_s/\hat{S}, p, \gamma) \tag{38}$$

of V-modules. Its composition with the inclusion $\mathrm{Id} \otimes 1 : H^i_{\mathrm{cris}}(X_s/\Sigma) \to H^i_{\mathrm{cris}}(X_s/\Sigma) \otimes_W V$ is the pullback $H^i_{\mathrm{cris}}(X_s/\Sigma) \to H^i_{\mathrm{cris}}(X_s/\hat{S}, p, \gamma)$.

4.2 A singular p-adic Lefschetz $(1,1)$ theorem

We first give a version of the Berthelot-Ogus theorem for *singular* formal schemes. As a consequence, we derive a version for singular schemes, both in the case of low ramification.

Assume $e < p$, so that πV has a unique divided power structure γ. Let $f : \mathfrak{X} \to \hat{S}$ be a flat proper morphism of formal schemes. Then by Lemma 3.2.2, for every $i \geq 0$, there is a natural morphism morphism

$$H^i_{\mathrm{cris}}(X_s/\hat{S}, \pi, \gamma) \to H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S}). \tag{39}$$

Let

$$H^i_{\mathrm{cris}}(X_s/\hat{S}, p, \gamma) \to H^i_{\mathrm{cris}}(X_s/\hat{S}, \pi, \gamma) \tag{40}$$

be the pullback morphism induced by the inclusion $pV \hookrightarrow \pi V$. Composing (38), (40) with (39), one has a natural morphism

$$\tau_{\mathrm{cris}} : H^i_{\mathrm{cris}}(X_s/\Sigma) \otimes_W V \to H^i_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \tag{41}$$

of V-modules.

Theorem 4.2.1 is a variant of [BO83, Theorem 3.8] for singular morphisms. By properness of $f : \mathfrak{X} \to \hat{S}$ and [EGA III 1, Théorème 3.4.2], $H^2(\mathfrak{X}, O_{\mathfrak{X}})$ is a finite V-module. Since $p \in \pi V$, the torsion submodule is annihilated by a power of p. Let

$$c^W_{1,\mathrm{cris}} : \mathrm{Pic}(X_s) \to H^2_{\mathrm{cris}}(X_s/\Sigma)$$

be the crystalline first Chern class map constructed in Section 2.2. Define an integer l as follows. If $p > 2$, let l be the smallest integer with $l > \log_p(e/(p-1))$. If $p = 2$, let l be the smallest integer with $l \geq 1 + \log_2 e$.

Theorem 4.2.1. *In Setup 1.0.1, assume $e < p$. Let $f : \mathfrak{X} \to \hat{S}$ be a flat proper morphism of formal schemes. Let X_s be its special fiber. Let $t \geq 0$ be the smallest integer such that $p^t\alpha = 0$ for all torsion elements $\alpha \in H^2(\mathfrak{X}, O_{\mathfrak{X}})$. Let $L_s \in \mathrm{Pic}(X_s)$. Then $L_s^{p^{t+l}}$ lifts to a line bundle on $\mathfrak{X}$ if and only if the image of $\tau_{\mathrm{cris}}(c^W_{1,\mathrm{cris}}(L_s) \otimes 1) \in H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$ in $H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \otimes_V K$ lies in $\left(F^1 H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})\right) \otimes_V K$.*

Proof. To prove the "only if" part, assume that for some integer $m > 0$, L_s^m lifts to a line bundle $\mathfrak{L}$ on $\mathfrak{X}$. By Corollary 3.2.9, as V is complete and $f : \mathfrak{X} \to \hat{S}$ is flat proper, the composition $H^2_{\mathrm{cris}}(X_s/W) \otimes_W V \to H^2_{\mathrm{cris}}(X_s/\hat{S}, \pi, \gamma)$ of (38) and (40) fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathfrak{X}) \arrow[r, "c_{1,\mathrm{dR}}"] \arrow[d] & F^1 H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[r, hook] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \\
\mathrm{Pic}(X_s) \arrow[r, "c^W_{1,\mathrm{cris}}"] \arrow[rr, bend right, "c^{(V,\pi,\gamma)}_{1,\mathrm{cris}}"'] & H^2_{\mathrm{cris}}(X_s/W) \otimes_W V \arrow[ur, "\tau_{\mathrm{cris}}"] \arrow[r, dashed] & H^2_{\mathrm{cris}}(X_s/\hat{S}, \pi, \gamma). \arrow[u, "(39)"']
\end{tikzcd}$$

Thus, one has

$$m\tau_{\mathrm{cris}}(c^{W}_{1,\mathrm{cris}}(L_s)\otimes 1) = \tau_{\mathrm{cris}}(c^{W}_{1,\mathrm{cris}}(L_s^m)\otimes 1) = c_{1,\mathrm{dR}}(\mathfrak{L}) \in F^1H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}).$$

Therefore, the image of $\tau_{\mathrm{cris}}(c^{W}_{1,\mathrm{cris}}(L_s)\otimes 1)$ lies in $\left(F^1H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})\right)\otimes_V K$.

To prove the "if" part, suppose that the image of $\tau_{\mathrm{cris}}(c^{W}_{1,\mathrm{cris}}(L_s)\otimes 1)$ lies in $\left(F^1H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})\right)\otimes_V K$. Let n be the smallest integer with $n > e/(p-1)$. Let $X' := \mathfrak{X}\times_{\hat{S}} \operatorname{Spec} V/\pi^n$. The proof of [BO83, Lemma 3.10] does not use any smoothness assumption. Thus by [BO83, Lemma 3.10], there is a line bundle L' on the scheme X' whose restriction to X_s is $L_s^{p^l}$.

Consider a diagram

$$\begin{tikzcd}
 & & H^2_{\mathrm{cris}}(X_s/W) \arrow[d, hook] \arrow[dd, bend left, "\mathrm{pb}"] \\
\mathrm{Pic}(X') \arrow[r] \arrow[d, "c_1^{X'/(V,\pi^n,\gamma)}"'] & \mathrm{Pic}(X_s) \arrow[ur, "c_1^{X_s/W}"] \arrow[d, "c_1^{X/(V,\pi,\gamma)}"'] \arrow[dr, "c_1^{X_s/(V,p,\gamma)}"'] & H^2_{\mathrm{cris}}(X_s/W)\otimes_W V \arrow[d] \\
H^2_{\mathrm{cris}}(X'/\hat{S},\pi^n,\gamma) \arrow[r, "\mathrm{pb}"] \arrow[dr] & H^2_{\mathrm{cris}}(X_s/\hat{S},\pi,\gamma) \arrow[d] & H^2_{\mathrm{cris}}(X_s/\hat{S},p,\gamma) \arrow[l, "\mathrm{pb}"'] \\
 & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}), &
\end{tikzcd}$$

where every arrow labeled with "pb" is a pullback morphism of crystalline cohomology groups. From Remark 3.2.3, the triangle at the bottom is commutative. By functoriality of Chern classes [BI70, Théorème 2.4 (i)], the remaining blocks of the diagram are commutative. Therefore, the image of $c_1^{X'/(V,\pi^n,\gamma)}(L')$ in $H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$ is the same as the image of $p^l c_1^{X_s/W}(L_s)$. By assumption, their common image in $H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})\otimes_V K$ is in $F^1H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})\otimes_V K$.

By definition of the Hodge filtration, the kernel of the map $H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \to H^2(\mathfrak{X}, O_{\mathfrak{X}})$ contains $F^1H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$. From Lemma 3.2.7, there is a commutative diagram

$$\begin{tikzcd}
H^2_{\mathrm{cris}}(X'/V,\pi^n,\gamma) \arrow[rrr] & & & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[d] \\
\mathrm{Pic}(X') \arrow[u, "c_1^{X'/(V,\pi^n,\gamma)}"] \arrow[r, "\partial"] & H^2(\mathfrak{X}, K^{\times,0}_{\mathfrak{X}/\hat{S},\pi^n,\gamma}) \arrow[r, "\log"] & H^2(\mathfrak{X}, \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma}) \arrow[r, "i"] & H^2(\mathfrak{X}, O_{\mathfrak{X}})
\end{tikzcd}$$

of abelian groups. Therefore, $i\circ\log\circ\partial(L')$ is a torsion element of $H^2(\mathfrak{X}, O_{\mathfrak{X}})$.

Since $\mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma} = \pi^n O_{\mathfrak{X}}$, the morphism $\pi^n : O_{\mathfrak{X}} \to \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma}$ of sheaves on $\mathfrak{X}$ is surjective. By [FK18, I, Proposition 4.8.1], as $\mathfrak{X}\to\hat{S}$ is flat, for every affine open subset $\operatorname{Spf} A$ of $\mathfrak{X}$, the induced morphism $V \to A$ is flat. Thus, the V-module A is torsion free, and hence the morphism $\pi^n : A \to A$ is injective.

Then $\pi^n : O_{\mathfrak{X}} \to \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma}$ is an isomorphism of $O_{\mathfrak{X}}$-modules. It induces an isomorphism

$$\wp : H^2(\mathfrak{X}, O_{\mathfrak{X}}) \xrightarrow{\sim} H^2(\mathfrak{X}, \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma})$$

such that $i \circ \wp = \pi^n$ as endomorphism of $H^2(\mathfrak{X}, O_{\mathfrak{X}})$. Thus, $\wp^{-1} \circ \log \circ \partial(L')$ is a torsion element of $H^2(\mathfrak{X}, O_{\mathfrak{X}})$. By choice of t, one has $p^t \wp^{-1} \circ \log \circ \partial(L') = 0$. Let $L'' = L'^{p^t} \in \mathrm{Pic}(X')$. Then $\wp^{-1} \circ \log \circ \partial(L'') = 0$. As $\wp$ is an isomorphism, one has $\log \circ \partial(L'') = 0$ in $H^2(\mathfrak{X}, \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma})$. By $n > e/(p-1)$, the divided power structure γ on $\pi^n V$ is π^n-adically nilpotent. So from Remark 3.2.6,

$$\log : H^2(\mathfrak{X}, \mathcal{K}^{\times,0}_{\mathfrak{X}/\hat{S},\pi^n,\gamma}) \to H^2(\mathfrak{X}, \mathcal{I}^0_{\mathfrak{X}/\hat{S},\pi^n,\gamma})$$

is an isomorphism. Thus, one has $\partial(L'') = 0$ in $H^2(\mathfrak{X}, \mathcal{K}^{\times,0}_{\mathfrak{X}/\hat{S},\pi^n,\gamma})$. By Lemma 3.2.5, there is an exact sequence

$$\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X') \xrightarrow{\partial} H^2(\mathfrak{X}, \mathcal{K}^{\times,0}_{\mathfrak{X}/\hat{S},\pi^n,\gamma}).$$

Therefore, L'' extends to a line bundle on $\mathfrak{X}$, whose restriction to X_s is $L_s^{p^{t+l}}$. $\square$

Theorem 4.2.1 stated for formal schemes implies a version for schemes as follows. Let $S := \operatorname{Spec} V$. Let $f : X \to S$ be a proper flat morphism of schemes. By Corollary 3.2.4, for every integer $i \geq 0$, there is a canonical map

$$H^i_{\mathrm{cris}}(X_s/\hat{S}, \pi, \gamma) \to H^i_{\mathrm{dR}}(X/S).$$

Then similar to (41), one obtains a natural morphism

$$\tau_{\mathrm{cris}} : H^i_{\mathrm{cris}}(X_s/\Sigma) \otimes_W V \to H^i_{\mathrm{dR}}(X/S).$$

Let $t \geq 0$ be an integer such that $p^t \alpha = 0$ for all torsion elements $\alpha \in H^2(X, O_X)$.

Corollary 4.2.2. *In Setup 1.0.1, assume $e < p$. Let $X \to S$ be a flat proper morphism of schemes. Let L_s be a line bundle on X_s. Then $L_s^{p^{t+l}}$ lifts to a line bundle on X if and only if the image of $c_{1,\mathrm{cris}}(L_s) \otimes 1$ under*

$$\tau_{\mathrm{cris}} : H^2_{\mathrm{cris}}(X_s/W) \otimes_W K \to H^2_{\mathrm{dR}}(X/V) \otimes_V K$$

lies in $F^1 H^2_{\mathrm{dR}}(X_\eta/K)$.

Proof. The formal completion of S along s is $\hat{S}$. Let $\mathfrak{X}$ be the formal completion of X along the special fiber X_s. By [EGA III 1, (3.4.1)], as $f : X \to S$ is proper, the induced morphism of formal schemes $\hat{f} : \mathfrak{X} \to \hat{S}$ is proper. From [FK18, I, Corollary 4.8.2], as $f : X \to S$ is flat, so is $\hat{f} : \mathfrak{X} \to \hat{S}$. As $f : X \to S$ is proper, $H^2(X, O_X)$ is a finite V-module. As V is a complete Noetherian local ring, the V-module $H^2(X, O_X)$ is π-adically complete. Then by the theorem on formal functions [EGA III 1, Corollaire 4.1.7], there is a canonical isomorphism $H^2(X, O_X) \xrightarrow{\sim} H^2(\mathfrak{X}, O_{\mathfrak{X}})$. From Lemma 3.2.1 *ii*), as A is complete, the canonical isomorphism $H^2_{\mathrm{dR}}(X/S) \xrightarrow{\sim} H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S})$ identifies the Hodge filtrations. Moreover, by the Grothendieck algebraization theorem [EGA III 1, Corollaire 5.1.6], the natural group morphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(\mathfrak{X})$ is an isomorphism.

There is a canonical isomorphism $H^2_{\mathrm{dR}}(X/S) \otimes_V K \xrightarrow{\sim} H^2_{\mathrm{dR}}(X_\eta/K)$ fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^1H^2_{\mathrm{dR}}(X/V) \otimes_V K \arrow[r] \arrow[d] & H^2_{\mathrm{dR}}(X/V) \otimes_V K \arrow[r] \arrow[d, "\cong"] & H^2(X, O_X) \otimes_V K \arrow[d, "\cong"] \\
0 \arrow[r] & F^1H^2_{\mathrm{dR}}(X_\eta/K) \arrow[r] & H^2_{\mathrm{dR}}(X_\eta/K) \arrow[r] & H^2(X_\eta, O_{X_\eta})
\end{tikzcd}$$

with exact rows. By the flat base change theorem, the right vertical morphism is an isomorphism, so is the one on the left. Therefore, the result follows from Theorem 4.2.1. □

Remark 4.2.3. Assume that k is algebraically closed. Bhatt [Bha14, Example 3.5] shows that for a projective nodal curve Y over k, $H^2_{\mathrm{cris}}(Y/W)/p$ is infinitely generated. Roughly, for a very mildly singular projective variety Y the torsion part of $H^i_{\mathrm{cris}}(Y/W)$ can be large. What we rely on is the rational crystalline cohomology $H^i_{\mathrm{cris}}(Y/W)[1/p]$.

5 Lifting criterion by rigid cohomology

As the proof of Theorem 1.0.4 is similar to that of Corollary 4.2.2, we only give a sketch and emphasize new ingredients.

5.1 Comparison morphism of rigid cohomology and de Rham cohomology

Let $S := \mathrm{Spec}\, V$ be as in Setup 1.0.1. Let $f : X \to S$ be a flat proper morphism of schemes with special fiber X_s and generic fiber X_η. Let K_0 be the fraction field of W. Given $L \in \mathrm{Pic}(X_s)$, Petrequin [Pet03, Définition 4.1] defines the first Chern class of L in $H^2_{\mathrm{rig}}(X_s/K_0)$. Let $c_{1,\mathrm{rig}}(L)$ be its image in $H^i_{\mathrm{rig}}(X_s/K)$.

For $i \geq 0$, let $H^i_{\mathrm{DR}}(X_\eta/K)$ be the i-th algebraic de Rham cohomology in the sense of [Har75, p.24]. By definition, there is a natural K-linear morphism $H^i_{\mathrm{DR}}(X_\eta/K) \to H^i_{\mathrm{dR}}(X_\eta/K)$. Let

$$\mathrm{cosp} : H^i_{\mathrm{rig}}(X_s/K) \to H^i_{\mathrm{DR}}(X_\eta/K)$$

be the cospecialization map from [BCF04, Theorem 6.6]. By composition, one gets a natural morphism $H^i_{\mathrm{rig}}(X_s/K) \to H^i_{\mathrm{dR}}(X_\eta/K)$, which for $i = 2$ appears in Theorem 1.0.4.

5.2 Review on crystalline cohomology of higher level

We recall how the rigid cohomology of a proper variety can be written in terms of crystalline cohomology of higher level.

For an integer $m \geq 0$, the definition of m-PD structure is in [Ber90, Définition 1.1.1]. Fix a prime p. Given a scheme S over $\mathbb{Z}_{(p)}$ with a quasi-coherent m-PD-ideal (I, J, γ) and a scheme X over S on which p is locally nilpotent, let $\mathrm{Cris}(X/S)^{(m)}$ be the corresponding crystalline site of level m (see, e.g., [Cre04, p.454]). Let $(X/S)^{(m)}_{\mathrm{cris}}$ be the associated crystalline topos of level m. Let $O^{(m)}_{X/S}$ be the structure sheaf, which satisfies $O^{(m)}_{X/S,(U,T)} = O_T$ for every object $(U,T) \in \mathrm{Cris}(X/S)^{(m)}$. Let $J^{(m)}_{X/S} \subset O^{(m)}_{X/S}$ be the sheaf such that $J^{(m)}_{X/S,(U,T)}$ is the ideal sheaf defining the closed immersion $U \hookrightarrow T$. Let

$$f^{(m)}_{X/S} : (X/S)^{(m)}_{\mathrm{cris}} \to S_{\mathrm{Zar}}, \quad i^{(m)}_{X/S} : X_{\mathrm{Zar}} \to (X/S)^{(m)}_{\mathrm{cris}}$$

be the natural morphisms of topoi. Let

$$\Gamma_{\mathrm{cris},m}(X/S, -) : (X/S)^{(m)}_{\mathrm{cris}} \to \mathrm{Set}$$

be the global section functor. Let

$$R\Gamma_{\mathrm{cris},m}(X/S) := R\Gamma_{\mathrm{cris},m}(X/S, O^{(m)}_{X/S}).$$

For integer $i \geq 0$, let $H^i_{\mathrm{cris},m}(X/S) := H^i R\Gamma_{\mathrm{cris},m}(X/S)$ be the i-th crystalline cohomology group of level m.

Fact 5.2.1 due to Berthelot can be found in [Pet03, Théorème 5.22] and [Cre04, (1.2) and (1.4)].

Fact 5.2.1. *Let V, K, k be as in Setup 1.0.1. Let X be a scheme proper over k. Then there is a canonical isomorphism*

$$R\Gamma_{\mathrm{rig}}(X/K) \xrightarrow{\sim} R\lim_m \left(R\Gamma_{\mathrm{cris},m}(X/V) \otimes_V K\right).$$

For every integer $i \geq 0$, it induces an isomorphism

$$H^i_{\mathrm{rig}}(X/K) \xrightarrow{\sim} \lim_m \left(H^i_{\mathrm{cris},m}(X/V) \otimes_V K\right).$$

5.3 Sketch of proof of Theorem 1.0.4

Lemma 5.3.1 can be proved similar to Lemma 3.1.5.

Lemma 5.3.1. *Let p be a prime. Let S be a scheme on which p is nilpotent. Let (I, J, γ) be a quasi-coherent m-PD-ideal on S. Let $X \to S$ be a flat separated morphism of finite presentation of schemes. Let $S_0 := V(I)$ and $X_0 := X \times_S S_0$. Then the diagram*

$$\begin{tikzcd}
Rf^{(m)}_{X_0/S*}i^{(m)}_{X_0/S*}O^*_{X_0}[-1] \arrow[r, "\partial"] \arrow[d] & Rf^{(m)}_{X_0/S*}(1+J^{(m)}_{X_0/S}) \arrow[r, "\log"] & Rf^{(m)}_{X_0/S*}J^{(m)}_{X_0/S} \arrow[r] & Rf^{(m)}_{X_0/S*}O^{(m)}_{X_0/S} \otimes \mathbb{Q} \arrow[d] \\
Rf_*O^*_{X_0}[-1] \arrow[r, "\partial"] & Rf_*\mathcal{K}^\times_{X/S} \arrow[r, "\log^\bullet"] & Rf_*\mathcal{I}^\bullet_{X/S} \arrow[r] & Rf_*\Omega^\bullet_{X/S} \otimes \mathbb{Q}
\end{tikzcd}$$

is commutative.

Lemma 5.3.2 follows from Fact 5.2.1 and Lemma 5.3.1, and can be proved as Lemma 3.2.7 and Corollary 3.2.9.

Lemma 5.3.2. *Let V and $\hat{S}$ be as in Setup 1.0.1. Let $f : \mathfrak{X} \to \hat{S}$ be a flat proper morphism of Noetherian formal schemes. Then there is a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{Pic}(X_0) \arrow[rrrr, "c_{1,\mathrm{rig}}"] \arrow[d, "\cong"] & & & & H^2_{\mathrm{rig}}(X_0/K) \arrow[d] \\
H^1(\mathfrak{X}, O^*_{X_0}) \arrow[r, "\partial"] & H^2(\mathfrak{X}, \mathcal{K}^\times_{\mathfrak{X}/\hat{S}}) \arrow[r, "\log^\bullet"] & H^2(\mathfrak{X}, \mathcal{I}^\bullet_{\mathfrak{X}/\hat{S}}) \arrow[r] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[r] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \otimes_V K.
\end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X_0) \arrow[rr, "c_{1,\mathrm{rig}}"] & & H^2_{\mathrm{rig}}(X_k/K) \arrow[d] \\
\mathrm{Pic}(\mathfrak{X}) \arrow[u] \arrow[r, "c_{1,\mathrm{dR}}"] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \arrow[r] & H^2_{\mathrm{dR}}(\mathfrak{X}/\hat{S}) \otimes_V K.
\end{tikzcd}$$

Using Lemma 5.3.2, one can prove Theorem 1.0.4 as Corollary 4.2.2.

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