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Integral points, monodromy, generic vanishing and Fourier-Mukai transform

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祝我的妈妈六十岁生日快乐！

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Résumé

Cette thèse est une compilation de plusieurs résultats vaguement liés. Ils concernent la non-densité des points entiers sur les variétés algébriques, la méthode de Lawrence-Venkatesh-Sawin et la géométrie analytique complexe.

Dans Chapitre 2, parallèlement au principe alternatif d'Ullmo et Yafaev sur les points rationnels des variétés de Shimura, nous montrons que la conjecture de Lang sur les points intégraux des variétés de Shimura est soit vraie, soit très fausse.

Le Chapitre 3 est un complément à la comparaison des monodromies dans les travaux respectifs de Lawrence-Sawin et Krämer-Maculan. Nous prouvons qu'il existe de nombreux caractères, tels que le groupe de monodromie correspondant est normal dans le groupe tannakien générique.

Le Chapitre 4 contient un théorème de l'annulation générique pour les variétés dans la classe Fujiki $\mathcal{C}$. En particulier, cela s'applique aux variétés algébriques complexes propres lisses ainsi qu'aux variétés kählériennes compactes.

Dans Chapitre 5, nous prouvons un analogue de la formule d'inversion de Fourier pour la transformation de Fourier-Mukai sur des tores complexes. Il corrige une inexactitude dans la littérature. En application, nous retrouvons la classification de Matsushima-Morimoto des fibrés vectoriels homogènes sur des tores complexes.

Le Chapitre 6 est une transformation de Fourier-Mukai analytique sur les D-modules, dont la version algébrique a été étudiée par Laumon et Rothstein. Nous étendons leur résultat de dualité des variétés abéliennes aux tores complexes. En application, nous réprouvons le théorème de Morimoto, selon lequel sur un tore complexe, tout fibré vectoriel admettant une connexion admet une connexion intégrable.

Mots-clés

Conjecture de Lang, groupe de monodromie, annulation générique, transformation de Fourier-Mukai, D-module.

Abstract

This dissertation is a compilation of several loosely related results. They concern the nondensity of integral points on algebraic varieties, the Lawrence-Venkatesh-Sawin's method and complex analytic geometry.

In Chapter 2, parallel to Ullmo and Yafaev's alternative principle on rational points of Shimura varieties, we show that Lang's conjecture about integral points on Shimura varieties is either true or very false.

Chapter 3 is a complement to the monodromy comparison step in Lawrence-Sawin's and Krämer-Maculan's respective work. We prove that there are many characters, such that the corresponding monodromy group is normal in the generic Tannakian group.

Chapter 4 contains a generic vanishing theorem for Fujiki class $\mathcal{C}$. In particular, it applies to smooth proper complex algebraic varieties as well as compact Kähler manifolds.

In Chapter 5, we prove an analog of the Fourier inversion formula for the Fourier-Mukai transform on complex tori. It corrects a misstatement in the literature. As an application, we recover Matsushima-Morimoto's classification of homogeneous vector bundles on complex tori.

Chapter 6 is a lift of the analytic Fourier-Mukai to D-modules, whose algebraic version is studied by Laumon and Rothstein. We extend their duality result from abelian varieties to complex tori. As an application, we reprove Morimoto's theorem that on a complex torus, every vector bundle admitting a connection admits a flat connection.

Keywords

Lang conjecture, monodromy group, generic vanishing, Fourier-Mukai transform, D-module.

Contents

Chapter 1

Introduction

1.1 Rational points

Intuitively, given an algebraic variety over a number field, the complexity of its geometry affects how many rational points (over finite extensions of the base field) it can posses. In Chapter 1, by an algebraic variety, we mean a geometrically integral, finite type, separated scheme over a field. An algebraic variety of dimension one is called a curve. Subvarieties means closed subvarieties.

1.1.1 Mordell conjecture

Let K be a number field. Let S be a finite subset of places of K containing all the infinite ones. Let $O_{K,S}$ be the ring of S-integers. For an affine algebraic variety X over K, choose an affine embedding (*i.e.*, a closed immersion to an affine space) $i : X \hookrightarrow \mathbf{A}^n_K$ over K. An *$O_{K,S}$-integral point* of X relative to i refers to an element $p \in X(K)$ such that the coordinates of $i(P) \in \mathbf{A}^n_K(K)$ lie in $O_{K,S}$.

Remark 1.1.1.1. Let $\mathcal{X} \subset \mathbf{A}^n_{O_{K,S}}$ be the scheme-theoretic image of the composition $X \xrightarrow{i} \mathbf{A}^n_K \to \mathbf{A}^n_{O_{K,S}}$. Because X is reduced, the generic fiber of $\mathcal{X} \to \operatorname{Spec} O_{K,S}$ is X. The set of $O_{K,S}$-integral points of X relative to i coincides with $\mathcal{X}(O_{K,S})$. By [Ser97, p.94], this set is quasi-integral relative to $O_{K,S}$. It may depend on the choice of the affine embedding i.

Let C be either the projective line $\mathbf{P}^1_K$ with at least three punctures, or a genus one curve over K with at least one puncture. By Siegel's theorem [Sie29, p.252] (see also [Ser97, p.95]), relative to every affine embedding of C, there are only finitely many $O_{K,S}$-integral points. For curves of higher genus, Mordell [Mor22, (5), p.192] conjectures the finiteness of rational points, which is proved by Faltings.

Fact 1.1.1.2 (Faltings, [Fal83, Satz 7])**.** *Let Y be a smooth projective curve over K of genus ≥ 2. Then $Y(K)$ is finite.*

A sample application of Faltings's theorem is a partial solution to Fermat's Last Theorem: for every integer $n \geq 4$, there are only finitely many pairwise coprime integer solutions to the equation $x^n + y^n = z^n$. Indeed, the projective plane curve (known as the n-th Fermat curve) in $\mathbf{P}^2_{\mathbb{Q}}$ cut out by this equation has genus $(n-1)(n-2)/2 (\geq 2)$. It is more than a decade earlier than Andrew Wiles's complete solution in 1994 to Fermat's conjecture.

Parshin [Par68] constructed a family of curves over Y, with which he showed that Mordell's conjecture (Fact 1.1.1.2) is a consequence of Shafarevich's conjecture for curves. This conjecture in turn follows from Shafarevich's conjecture for abelian varieties and Torelli's theorem.

We recall the statement of Shafarevich's conjecture. A smooth proper variety (resp. abelian variety) over a discrete valuation field E is said to have *good reduction* if it is isomorphic to the generic fiber of a smooth proper scheme (resp. an abelian scheme) over the integer ring O_E of E. By [Mil20, Prop. 6.4], there is at most one such abelian scheme. A smooth proper variety (resp. an abelian variety) over the number field K is said to have good reduction at a finite place v of K, if its base change to the completion K_v has good reduction.

Fact 1.1.1.3 (Shafarevich conjecture, [Fal83, Korollar 1, p.365 (resp. Satz 6)])**.** *For every integer g at least two (resp. one), up to K-isomorphism there are only finitely many smooth projective curves (resp. abelian varieties) defined over K of genus (resp. dimension) g, with good reduction outside S.*

In 1983, Faltings proved Shafarevich's conjecture for abelian varieties and hence Mordell's conjecture, "opening thereby a new chapter in number theory".[1] Faltings's proof can be decomposed into two parts, Facts 1.1.1.4 and 1.1.1.5.

Fact 1.1.1.4 ([Fal83, Satz 5])**.** *For every integer $g > 0$, up to K-isogeny there are only finitely many abelian varieties over K of dimension g, with good reduction outside S.*

Fact 1.1.1.4 is weaker than Shafarevich's conjecture for abelian varieties. Its proof is to consider the representations of the absolute Galois group Γ_K of K on the Tate modules of abelian varieties over K. For one thing, by Tate's conjecture over number fields [Fal83, Korollar 2], the Galois representation on the Tate module determines the abelian variety up to K-isogeny. For another, by Weil's conjecture proved by Deligne [Del74, Thm. 1.6], there are only finitely many such representations up to isomorphism.

Fact 1.1.1.5. *Let A be an abelian variety over K. Then up to K-isomorphism, there are only finitely many abelian varieties over K which are K-isogenous to A.*

[1] quotation from [Blo84, p.41]

Faltings introduced a differential height function, now known as Faltings's height, to measure the "complexity" of abelian varieties. Height function is a tool of global nature, as it collects the information at every place of the base number field. The core of the proof of Fact 1.1.1.5 is [Fal83, Lem. 5], which proves that Faltings's height does not change much under isogenies.

1.1.2 Lang conjectures

From [CHM97, p.2], "one natural generalization to higher dimensions of the notion of 'curve of geometric genus $g \geq 2$' is 'variety of general type'." For a smooth projective variety X over a field, let ω_X be its canonical line bundle. For an integer $d \geq 0$, let $P_d(X) = h^0(X, \omega_X^{\otimes d})$ be the d-th *plurigenus* of X. The *Kodaira dimension* $\kappa(X)$ is defined to be $-\infty$ (or -1 depending on the convention) if $P_d(X) = 0$ for every integer $d > 0$; otherwise, it is the minimum real number r such that the sequence $\{P_d(X)/d^r\}_{d>0}$ is bounded. From [Laz04, Eg. 2.1.5], the Kodaira dimension is the "most basic" integer birational invariant of X. If $\kappa(X) = \dim X$, then X is called of *general type.* For instance, a smooth projective curve is of general type if and only of its genus is at least two.

A high-dimensional analog of Mordell's conjecture (Fact 1.1.1.2) is conjectured by Lang (see, *e.g.*, [CHM97, Conjecture A]).

Conjecture 1.1.2.1 (Weak Lang conjecture)**.** *Let X be a positive dimensional smooth projective variety of general type over a number field K. Then $X(K)$ is not Zariski dense in X.*

Using techniques from Diophantine approximation, Faltings proves the weak Lang conjecture (Conjecture 1.1.2.1) for subvarieties of abelian varieties, which gives a second proof of Mordell's conjecture (Fact 1.1.1.2). From [Hin98, p.95], a subvariety of an abelian variety is of general type if and only if its stabilizer is finite.

Fact 1.1.2.2 ([Fal91, Thm. 1])**.** *Let A be an abelian variety over a number field K. Let $X \subset A$ be a subvariety of general type. Then $X(K)$ is finite.*

Based on Faltings's work [Fal94], Moriwaki proves another particular case of the weak Lang conjecture (Conjecture 1.1.2.1). By [Deb05b, p.1445], a smooth projective variety with ample cotangent bundle is of general type.

Fact 1.1.2.3 ([Mor95, p.114])**.** *Let X be a smooth projective variety over a number field K. If the cotangent bundle $\Omega^1_{X/K}$ is ample and generated by global sections, then $X(K)$ is finite.*

Conjecture 1.1.2.1 is stronger than the uniformity conjecture.

Fact 1.1.2.4 ([CHM97, Thm. 1.1])**.** *Assume Conjecture 1.1.2.1 for all number fields. Then for every number L and every integer $g \geq 2$, there is an* *integer $B(L,g)$ such that every smooth curve C over L of genus g, one has $\#C(L) \leq B(L,g)$.*

Conjecture 1.1.2.1 for algebraic surfaces was independently raised by Bombieri, so also known as the Bombieri-Lang conjecture. It gives a conditional solution to the Erdös-Ulam problem.

A rational distance set in $\mathbb{R}^2$ is a subset such that every pairwise distance between its points is rational. Erdös and Ulam conjectured in 1945 that there is no dense rational distance set in $\mathbb{R}^2$.

Fact 1.1.2.5 ([Sha18, Cor. 1.4])**.** *Assume Conjecture 1.1.2.1 for algebraic surfaces over all number fields. Let S be an infinite rational distance set. Then either all but at most 4 points of S are on a line, or all but at most 3 points of S are on a circle.*

A complex manifold M is called *Brody hyperbolic* if every morphism $\mathbb{C} \to M$ of complex manifolds is constant. By [DR16, p.417], a compact Riemann surface is Brody hyperbolic if and only if its genus is at least two.

Conjecture 1.1.2.6 ([Lan86, Conjeture 5.6], see also [BD21, Conjecture, p.2])**.** *A complex smooth projective variety is hyperbolic if and only if every subvariety is of general type.*

Conjecture 1.1.2.6 is known as the *geometric Lang conjecture*. It lies between algebraic geometry and complex analytic geometry. Both directions of it are unknown till now. For subvarieties of abelian varieties, Conjecture 1.1.2.6 is confirmed by [Yam19, Cor. 1.3] (and Brody's theorem [Bro78, p.213] that Brody hyperbolicity agrees with Kobayashi hyperbolicity for compact complex manifolds).

Conjecture 1.1.2.7 would follow from Conjectures 1.1.2.1 and 1.1.2.6.

Conjecture 1.1.2.7 ([Lan74, (1.3)])**.** *Let X be a smooth projective variety over a number field K. If a complex analytification of X is Brody hyperbolic, then $X(K)$ is finite.*

1.1.3 Lang conjecture for Shimura varieties

Shimura varieties are higher-dimensional analogs of modular curves. As Alex Youcis puts it, the reason to study Shimura varieties is multiple: They are highly symmetrical objects with rich actions of various Lie groups; They are moduli spaces of abelian varieties (with extra structures); They are moduli spaces of motives; They are objects conjectured to realize the global Langlands correspondence, *etc.* However, to define Shimura varieties requires an exceptional amount of technical sophistication. See [Mil17b] for a reference.

Let (G,X) be a Shimura datum, and let $K \leq G(\mathbb{A}_f)$ be a sufficiently small, neat, compact open subgroup. Let S be a connected component of the complex manifold $\mathrm{Sh}_K(G,X)$. From Nadel's work [Nad89, Thm. 0.2], the Baily-Borel compactification S^* of S is Brody hyperbolic. As the canonical model of $\mathrm{Sh}_K(G,X)$ exists (see, *e.g.*, [Mil17b, p.128]), S is naturally a smooth quasi-projective variety defined over a number field F. Then Lang's conjecture (Conjecture 1.1.2.7) predicts that $S(F')$ is finite for every finite extension F'/F. Similar speculation for integral points on Shimura varieties of abelian type is confirmed by Ullmo. His proof relies on Faltings's solution to Shafarevich's conjecture for abelian varieties (Fact 1.1.1.3).

Fact 1.1.3.1 ([Ull04, Thm. 3.2 (a)]). *Suppose that the Shimura datum (G,X) is of adjoint abelian type. Let $\Gamma \leq G(\mathbb{Q})$ be a net arithmetic lattice. Then for every number field F, every finite set of places Σ of F and every model* (in the sense of [Ull04, Def. 2.2]) *$\mathcal{M}$ of X^+/Γ over $O_{F,\Sigma}$, the set $\mathcal{M}(O_{F,\Sigma})$ is finite.*

Concerning the rational points on general Shimura varieties, Lang's conjecture (Conjecture 1.1.2.1) is related to an alternative principle [UY10, Thm. 1.1]. For a projective variety Z over a number field, Ullmo and Yafaev [UY10, (1)] define its *Lang locus* Z^L to be the Zariski closure of $\cup_M \overline{Z(M)}^{>0}$, where M runs through finite extensions of the definition field of Z (inside a fixed algebraic closure), and $\overline{Z(M)}^{>0}$ is the union of positive-dimensional irreducible components of the Zariski closure $\overline{Z(M)}$.

The Lang locus measures the failure of Lang's conjecture, since $Z^L = \emptyset$ if and only if Z satisfies Conjecture 1.1.2.7. For Shimura varieties, Fact 1.1.3.2 shows that Lang's conjecture is either true or very false.

Fact 1.1.3.2 (Ullmo-Yafaev's all-or-nothing principle, [UY10, Thm. 1.1]). *Let S be a (connected) Shimura variety of sufficiently high level. Then $S \cap (S^*)^L$ is either $\emptyset$ or S.*

As Shimura varieties are not proper in general, it is equally natural to consider integral points instead of rational points. For quasi-projective varieties over $\bar{\mathbb{Q}}$, we define an "integral Lang locus" (Definition 2.6.0.1) measuring the infiniteness of integral points of a chosen integral model. This locus is independent of the choice of the integral model. It is empty if and only if the variety has only finitely many integral points over each number field where the variety can be defined. We give a result parallel to Fact 1.1.3.2 for integral points. It shows that the Lang conjecture on integral points [Lan91, IX, Conjecture 5.1] is either true or very false for Shimura varieties.

Theorem (Theorem 2.6.0.13). *The integral Lang locus of a (connected) Shimura variety S is either $\emptyset$ or S.*

In fact, we form several axioms for an abstract locus formation, and prove that such an alternative principle results from the axioms. Both Lang locus and integral Lang locus satisfy the axioms.

1.2 Lawrence-Venkatesh technique

Lawrence-Venkatesh's new proof [LV20] of Faltings's theorem (Fact 1.1.1.2) sheds light on the weak Lang conjecture (Conjecture 1.1.2.1). This technique, compared with Faltings's strategy, is of local nature. We give a highly sketchy review, and refer the reader to [LV20] for more details.

1.2.1 Setting

Let K, S be as in Section 1.1.1. Let $f : X \to Y$ be a smooth proper morphism of smooth algebraic varieties over K. By enlarging S, one may choose a smooth proper morphism $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ between smooth $O_{K,S}$-schemes whose base change to K is $f : X \to Y$. Lawrence-Venkatesh's idea uses the induced variation of local Galois representations, to prove that $\mathcal{Y}(O_{K,S})$ is not Zariski dense in Y.

Remark 1.2.1.1. If Y is as in Mordell's conjecture (Fact 1.1.1.2), then by properness of Y over K and [Poo17, Thm. 3.2.13 (ii)], the natural map $\mathcal{Y}(O_{K,S}) \to Y(K)$ is bijective. By $\dim Y = 1$, every non-Zariski-dense subset of Y is finite. That is why one only needs nondensity of integral points to get finiteness of rational points in this case.

Lawrence and Venkatesh [LV20] apply the following machinery to a variant of the relative curve constructed by Parshin (and of a construction due to Kodaira).

1.2.2 Galois representations

Choose a finite place v of K, such that underlying rational prime p is unramified in K and no place dividing p is in S. Let $O_v \subset K_v$ be the integer ring of the completion. There is a natural inclusion $\mathcal{Y}(O_{K,S}) \subset \mathcal{Y}(O_v)$. For every $y \in \mathcal{Y}(O_v)$, the fiber $\mathcal{X}_y$ is a smooth proper scheme over O_v with generic fiber X_y:

$$\begin{tikzcd}
\mathcal{X}_y \arrow[r] \arrow[d] & y \arrow[d] \\
\mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[d] \\
 & \operatorname{Spec} O_v
\end{tikzcd}
\qquad
\begin{tikzcd}
X_y \arrow[r] \arrow[d] & y \arrow[d] \\
X_{K_v} \arrow[r] \arrow[dr] & Y_{K_v} \arrow[d] \\
 & \operatorname{Spec} K_v.
\end{tikzcd}$$

Let $\operatorname{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})$ the category of (continuous) $\mathbb{Q}_p$-representations of the absolute Galois group Γ_{K_v}. For every integer $d \geq 0$, there is a *local* Galois representation

$$\rho_y^d : \Gamma_{K_v} \to \mathrm{GL}(H^d_{\text{ét}}(X_{\bar{y}}/\overline{K_v}, \mathbb{Q}_p))$$

on the d-th étale cohomology group. For a locally small category $\mathcal{C}$, let $\mathcal{C}/\sim$ be the set of isomorphism classes of objects of $\mathcal{C}$. Hence, one gets a map $\rho : \mathcal{Y}(O_v) \to \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})/\sim$. Representations are more or less "linear" data.

1.2.3 p-adic Hodge theory

The functor D_{cris} in p-adic Hodge theory induces a functor from the category $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})$ to the category FVec_{K_v} of filtered vector spaces over K_v. Because the K_v-algebraic variety X_y has a smooth proper model $\mathcal{X}_y$ over O_v, the p-adic Galois representation ρ_y^d is crystalline in the sense of [BC09, p.133]. By Fontaine's conjecture proved by Faltings [Fal88, Cor., p.69], this functor sends ρ_y^d to the de Rham cohomology $H^d_{\mathrm{dR}}(X_y/K_v)$ equipped with its Hodge filtration (in the sense of [Sta24, Tag 0FM8]). That is informally depicted below.

$$\mathcal{Y}(O_S) \overset{\text{choosing a suitable place } v|p}{\subset} \mathcal{Y}(O_v) \overset{\rho}{\to} \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})/\sim \overset{D_{\mathrm{cris}}}{\to} \mathrm{FVec}_{K_v}/\sim .$$

Locally, one can interpret the map

$$\mathcal{Y}(O_v) \to \mathrm{FVec}_{K_v}/\sim \tag{1.1}$$

as a period map. There is an algebraic vector bundle $V = \mathcal{H}^d_{\mathrm{dR}}(X/Y)$ on Y, and a decreasing Hodge filtration $F^\bullet V$ by vector subbundles, whose fiber at every $y \in Y(K_v)$ is $H^d_{\mathrm{dR}}(X_y/K_v)$ with its Hodge filtration. By [KO68], there is a natural *flat* connection ∇_{GM} on V, known as the *Gauss-Manin connection.*

1.2.4 Complex period map

To define the period map, we begin reviewing the complex analytic analog. Consider a variation of Hodge structure $(V, F^\bullet V, \nabla)$ on a connected complex manifold Y, where $V \to Y$ is a (holomorphic) vector bundle, $F^\bullet V$ is a decreasing filtration of V by vector subbundles, and ∇ is a flat connection on V. (On a complex manifold, by connection we mean a holomorphic connection in the sense of [Huy05, Def. 4.2.17].) Take a base point $y_0 \in Y$ and a small open disk $\Omega \subset Y$ around y_0. As ∇ is flat, for $y \in \Omega$, the parallel transport induces a $\mathbb{C}$-linear isomorphism of fibers $V_y \to V_{y_0}$. In general, the connection ∇ does not respect the filtration $F^\bullet V$. Still, the fiberwise filtration $F^\bullet V_y$ is transported to a filtration on the fiber V_{y_0}, which has the same dimensional data as the filtration $F^\bullet V_{y_0}$. Let Flag be the complex projective variety parameterizing the filtrations of V_{y_0} of this common dimension data. In this way, one gets a holomorphic map

$$\Phi_{\mathbb{C}} : \Omega \to \mathrm{Flag}, \quad y \mapsto \text{transport of } F^\bullet V_y \text{ to } V_{y_0}.$$

It is only locally defined on Y, and called a *period map.*

1.2.5 p-adic period map

Let $k/\mathbb{Q}_p$ be a finite field extension. Let O_k be the integer ring of k. Let m_k be the maximal ideal of O_k. Let $\mathcal{Y}$ be a smooth O_k-scheme with generic fiber Y. Let $\bar{\mathcal{Y}}$ be the special fiber of $\mathcal{Y}$:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] & \bar{\mathcal{Y}} \arrow[l] \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} O_k & \operatorname{Spec} O_k/m_k. \arrow[l] \end{tikzcd}$$

Fix a base point $y_0 \in \mathcal{Y}(O_k)$. Let $r : \mathcal{Y}(O_k) \to \bar{\mathcal{Y}}(O_k/m_k)$ be the reduction map. Set $\Omega = r^{-1}(r(y_0))$, and call it the *residue disk* around y_0. Then Ω is an open neighborhood of y_0 in the analytic manifold Y^{an} over k. Consider a triple $(V, F^\bullet V, \nabla)$, where V is a vector bundle on Y, $F^\bullet V$ be a decreasing filtration on V by vector subbundles, and ∇ a flat connection on V. As in Section 1.2.4, one can define a flag variety Flag over k, and a p-adic period map $\Phi_p : \Omega \to \mathrm{Flag}$ which is k-analytic.

1.2.6 Ax-Schanuel property of period map

In the notation of Section 1.2.1, take $k = K_v$. Take the triple $(V, F^\bullet V, \nabla)$ to be $(\mathcal{H}^d_{\mathrm{dR}}(X/Y) \otimes_K K_v, \text{Hodge filtration}, \nabla_{\mathrm{GM}})$ on Y_{K_v}. When the fibers of V on a residue disk Ω are identified by the Gauss-Manin connection ∇_{GM}, the restriction of the map (1.1) to Ω coincides with the p-adic period map Φ_p. Because $\mathcal{Y}(O_v)$ is covered by finitely many residue disks, to prove that $\mathcal{Y}(O_{K,S})$ is not Zariski dense in Y, it suffices to prove the nondensity of $\Omega \cap \mathcal{Y}(O_{K,S})$. Fact 1.2.6.1 counts essentially on Bakker-Tsimerman's Ax-Schanuel type result [BT19]. Let $\mathfrak{H}_p$ be the Zariski closure of $\Phi_p(\Omega)$ in Flag.

Fact 1.2.6.1 ([KM23, Prop. 7.10 (4)]). *Let $Z \subset \mathfrak{H}_p$ be a subvariety with $\dim \mathfrak{H}_p \geq \dim Z + \dim Y$. Then $\Phi_p^{-1}(Z)$ is* not *Zariski dense in Y_{K_v}.*

The situation is summarized as follows.

$$\begin{tikzcd} & \Phi_p^{-1}(Z) \arrow[r] \arrow[dl, "\text{nondense}"'] \arrow[dd, hook] & Z \arrow[d, hook] \\ & & \mathfrak{H}_p \arrow[d, hook] \\ Y_{K_v} & \Omega \arrow[l, hook'] \arrow[r, "\Phi_p"'] \arrow[ur, dashed] & \mathrm{Flag}. \end{tikzcd}$$

Take Z to be the Zariski closure of $\Phi_p(\Omega(O_{K,S}))$ in $\mathfrak{H}_p$. If $\dim \mathfrak{H}_p \geq \dim Z + \dim Y$, then by Fact 1.2.6.1, the subset $\Omega \cap \mathcal{Y}(O_{K,S}) \subset Y$ is not Zariski dense.

1.2.7 Summary

To show nondensity of integral points in Lawrence-Venkatesh's method, one needs to show that $\dim \mathfrak{H}_p$ is "large" when compared with the dimension of the image of $O_{K,S}$-integral points. Let $\mathfrak{H}_{\mathbb{C}}$ be the image of the complex period map $\Phi_{\mathbb{C}}$ corresponding to the variation of Hodge structures induced by $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$. For one thing, as the Gauss-Manin connection ∇_{GM} is defined on K, one gets $\dim \mathfrak{H}_p \geq \dim \mathfrak{H}_{\mathbb{C}}$. Using Hodge theory, one proves that $\mathfrak{H}_{\mathbb{C}}$ contains the orbit of $\Phi_{\mathbb{C}}(y_0)$ under the monodromy action. For another thing, using Faltings's finiteness theorem (see, *e.g.*, [LV20, Lem. 2.3]), one gets an upper bound (involving the centralizer of the crystalline Frobenius operator arising from the comparison of de Rham cohomology and crystalline cohomology) on $\Phi_p(\Omega(O_{K,S}))$.

1.3 Lawrence-Sawin technique

The technique of Lawrence-Venkatesh is a promising approach to the weak Lang conjecture (Conjecture 1.1.2.1), because it is successfully applied in higher dimension. For example, based on this technique, Lawrence and Sawin establish in an innovatory way the Shafarevich conjecture for hypersurfaces in abelian varieties.

1.3.1 Statements

Let K, S be as in Section 1.1.1. Let A be an abelian variety over K of dimension g, with good reduction outside S. A subvariety $V \subset A$ is said to have *good reduction* at a place $v \notin S$ of K, if the Zariski closure of V in the *unique* abelian scheme $\mathcal{A}/O_{K_v}$ with generic fiber A_{K_v} is smooth.

Fact 1.3.1.1 (Lawrence-Sawin, [LS20, Thm. 1.1])**.** *Suppose $g \geq 4$. Fix an ample class ϕ in the Néron-Severi group of A. Then there are only finitely many hypersurfaces $H \subset A$ over K representing ϕ, with good reduction outside S, up to translation by points in $A(K)$.*

Using a similar technique, Krämer and Maculan obtained an analog for subvarieties of dimension less than half the dimension of the ambient abelian variety.

Fact 1.3.1.2 (Krämer-Maculan, [KM23, Cor., p.3])**.** *Fix a polynomial $P \in \mathbb{Q}[z]$ of degree $d < (g-1)/2$ and an ample line bundle L on A. Then up to translation by points in $A(K)$, there are only finitely many nondivisible geometric complete intersections of ample divisors $X \subset A$ over K, with good reduction outside S, that have Hilbert polynomial P with respect to L and satisfy $2\chi(X \times X, \Omega^d_{X\times X}) \leq \chi_{\mathrm{top}}(X \times X)$.*

In both cases of Facts 1.3.1.1 and 1.3.1.2, the dimension of the base algebraic variety Y in Section 1.2 is grater than one. So nondensity of the set of integral points is strictly weaker than finiteness. To get finiteness of integral points, an idea suggested in [LV20, Sec. 10.2] is to iterate the Lawrence-Venkatesh argument by replacing Y with the Zariski closure of integral points. In this manner, an estimate of topological monodromy group that is *uniform* in subvarieties of the variety Y under consideration is needed. The main novelty of [LS20] is to compare the monodromy with a Tannakian group. (The comparison involved in the proof of Fact 1.3.1.2 leans on [JKLM23].) This idea is similar to the study of monodromy groups of variation of Hodge structures *via* Mumford-Tate groups in [And92].

This Tannakian group arises from sheaf convolution developed fundamentally by Krämer-Weissauer [KW15b]. We give a cursory review of the comparison.

1.3.2 Tannakian theory of sheaf convolution

Tannakian formalism is a way to reconstruct a group from its representations. By the Tannaka–Krein duality, a compact group can be recovered from the abelian category of its complex representations *together with the tensor product operation.*

Definition 1.3.2.1 ([DM22, Def. 2.19])**.** A rigid, symmetric, monoidal abelian category $(C, \otimes)$ of unit object $\mathbb{1}$ is a *neutral Tannakian category* over a field k if it admits an exact faithful k-linear tensor functor $\omega : C \to \mathrm{Vec}_k$ (called a fiber functor) and if $\mathrm{End}(\mathbb{1}) = k$ as rings.

Fact 1.3.2.2 ([DM22, Thm. 2.11], [Del90, Sec. 9.2])**.** *Let $(C, \otimes)$ be a neutral Tannakian category over a field k with a fiber functor $\omega : C \to \mathrm{Vec}_k$. Then there is a natural affine group scheme $\mathrm{Aut}^{\otimes}(C, \omega)$ over k (called the Tannakian group of $(C, \otimes)$ at ω), such that ω factors through an equivalence $C \to \mathrm{Rep}_k(\mathrm{Aut}^{\otimes}(C, \omega))$ of symmetric monoidal categories. If k is algebraically closed, then $\mathrm{Aut}^{\otimes}(C, \omega)$ is independent of the choice of ω up to k-isomorphism.*

Let X be an algebraic variety. Let ℓ be a prime number invertible on X. Let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Let $D^b_c(X)$ be the triangulated category of complexes of $\bar{\mathbb{Q}}_\ell$-étale sheaves on X with bounded, constructible cohomologies. Perverse sheaves on X are the singular version of local systems. They form a full, abelian subcategory $\mathrm{Perv}(X)$ of $D^b_c(X)$. This abelian category is Noetherian and Artinian. For every smooth subvariety $Y \subset X$, the complex of sheaves $\bar{\mathbb{Q}}_{\ell,Y}[\dim Y]$ is a perverse sheaf on X.

We review the work of Krämer and Weissauer. Let A be an abelian variety over a field of characteristic zero. Let $a : A \times A \to A$ be the group law. Let $p_i : A \times A \to A$ $(i = 1, 2)$ be the projection to the i-th factor. The bifunctor

$$(\cdot) * (\cdot) : D^b_c(A) \times D^b_c(A) \to D^b_c(A), \quad (-, +) \mapsto Ra_*(p_1^* - \otimes^L p_2^* +)$$

is called the *convolution* on A. In general, $\mathrm{Perv}(A)$ is not stable under the convolution. Still, its quotient modulo the subcategory of "negligible objects" is stable under the convolution. Let $N(A) \subset \mathrm{Perv}(A)$ be the full subcategory comprised of (so-called negligible) objects with Euler characteristic zero.

Fact 1.3.2.3 ([Krä22, 1.b]). *The subcategory $N(A)$ is a Serre subcategory* (in the sense of [Sta24, Tag 02MO (1)]) *of* $\mathrm{Perv}(A)$*. Let $\bar{P}(A)$ be the quotient abelian category* (in the sense of [Sta24, Tag 02MS])*. Then the convolution descends to a bifunctor $* : \bar{P}(A) \times \bar{P}(A) \to \bar{P}(A)$. Moreover, $(\bar{P}(A), *)$ is a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$.*

Every object $F \in \mathrm{Perv}(A)$ generates a Tannakian subcategory $\langle F \rangle$ of $\bar{P}(A)$. Let $G(F)$ be the (unique up to isomorphism) Tannakian group of $\langle F \rangle$. The computation of the Tannakian group in [LS20] follows essentially the general approach in Krämer's work [Krä22; Krä21].

1.3.3 Monodromy group and generic Tannakian group

Let A be an abelian variety over an algebraically closed field of characteristic zero. In Lawrence-Sawin [LS20, Sec. 11], the strategy of Lawrence-Venkatesh is applied to the universal family of hypersurfaces $f : U \to \mathrm{Hilb}$ over the corresponding Hilbert scheme

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, "f"'] & \mathrm{Hilb} \times A \arrow[d] \\ & \mathrm{Hilb}. \end{tikzcd}$$

More generally, Krämer and Maculan [KM23, Sec. 1.4] consider (the complex analytic analog of) the following setting. Let Y be a smooth algebraic variety over k with generic point η. Let $X \subset A \times Y$ be a subvariety, such that the projection $f : X \to Y$ is smooth of relative dimension d:

$$\begin{tikzcd} & X \arrow[r, hook] \arrow[dl, "\pi"'] \arrow[dr, "f"] & A \times Y \arrow[d] \\ A & & Y. \end{tikzcd}$$

Then $R^d f_* \bar{\mathbb{Q}}_\ell$ is a lisse sheaf on Y. To get "big monodromy", they "twist" this lisse sheaf as follows. Let $\mathcal{C}(A)_\ell$ be the scheme (recalled in Section 3.4) parameterizing pro-ℓ characters of the étale fundamental group $\pi_1^{\text{ét}}(A, 0)$. For every character $\chi \in \mathcal{C}(A)_\ell$, let L_χ be the corresponding rank one lisse sheaf on A. Then $V_\chi := R^d f_* \pi^* L_\chi$ is a lisse sheaf on Y and $V_1 = R^d f_* \bar{\mathbb{Q}}_\ell$. Let $\mathrm{Mon}(\chi)$ be the Zariski closure of the image of the monodromy representation $\pi_1^{\text{ét}}(Y, \bar{\eta}) \to \mathrm{GL}(V_{\chi,\bar{\eta}})$ of V_χ. We need to find χ such that the monodromy group $\mathrm{Mon}(\chi)$ is big enough to carry out Lawrence-Venkatesh-Sawin's method.

Denote the perverse sheaf $\bar{\mathbb{Q}}_{\ell,X_\eta}[d]$ on A_η by P_η. Let $P_{\bar{\eta}}$ be its scalar extension along the extension $k(\bar{\eta})/k(\eta)$. By Krämer-Weissauer's theorem (Fact 1.5.1.1), for a generic $\chi \in \mathcal{C}(A)_\ell$, the functor

$$\omega_\chi : \langle P_\eta \rangle (\subset \bar{P}(A_\eta)) \to \mathrm{Vec}_{\mathbb{C}}, \quad F \mapsto H^0(A_{\bar{\eta}}, F \otimes^L L_\chi)$$

is a *fiber functor*. Using $\omega_\chi(P_\eta) = V_{\chi,\bar{\eta}}$, one can compare $\mathrm{Mon}(\chi)$ with $G(P_\eta)$: By [JKLM23, p.28], for a generic character $\chi \in \mathcal{C}(A)_\ell$, the monodromy group $\mathrm{Mon}(\chi)$ is a closed subgroup of $G(P_\eta)$.

To apply Bakker-Tsimerman's Theorem (Fact 1.2.6.1), one needs a lower bound on $\mathrm{Mon}(\chi)$. The proof of the lower bound uses the normality of the geometric generic Tannakian group $G(P_{\bar{\eta}})$ inside the generic Tannakian group $G(P_\eta)$.

1.3.4 Normality of geometric generic Tannakian group

Let k be a field of characteristic zero with an algebraic closure $\bar{k}$. Let A be an abelian variety over k.

Fact 1.3.4.1 ([LS20, Lem. 3.7])**.** *Let G_k (resp. $G_{k'}$) be the Tannakian fundamental group of the category of* geometrically semisimple *perverse sheaves on A (resp. the summands of the pullbacks to $A_{\bar{k}}$ of geometrically semisimple perverse sheaves on A), modulo the full subcategory of "negligible objects". Then $G_{k'}$ is naturally a closed* normal *subgroup of G_k, with quotient isomorphic to the Tannakian group of the neutral Tannakian category* $(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \otimes)$*.*

The assumption of geometric semisimplicity in Fact 1.3.4.1 is removed in [JKLM23]. Let K/k be a field extension. Let k' be the algebraic closure of k in K. Assume that k'/k is Galois. Let $\mathcal{C} \subset \bar{P}(A)$ be a full abelian tensor subcategory. Let $\mathcal{C}_K \subset \bar{P}(A_K)$ be the full subcategory of subquotients of the pullbacks to A_K of perverse sheaves on A. Fix a fiber functor $\omega : \mathcal{C}_K \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$.

Fact 1.3.4.2 ([JKLM23, Thm. 4.3])**.** *There is a short exact sequence of proalgebraic groups*

$$1 \to \mathrm{Aut}^{\otimes}(\mathcal{C}_K, \omega) \to \mathrm{Aut}^{\otimes}(\mathcal{C}, \omega) \to \mathrm{Aut}^{\otimes}(\mathcal{C} \cap \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{Gal}(k'/k)), \omega) \to 1.$$

1.3.5 Monodromy group and geometric generic Tannakian group

The normality result (Fact 1.3.4.1) and an analog of Larsen's alternative [LS20, Lem. 5.4] permit one to get a lower bound on $\mathrm{Mon}(\chi)$. Under certain geometric conditions on the family $f : X \to Y$, by [LS20, Thm. 5.6] and

[JKLM23, Thm. 4.10], for a generic character $\chi \in \mathcal{C}(A)_\ell$, the corresponding monodromy group $\mathrm{Mon}(\chi)$ contains the geometric generic Tannakian group $G(P_{\bar\eta})$

$$\begin{tikzcd}
\mathrm{Mon}(\chi) \arrow[r, hook] & G(P_\eta) \\
G(P_{\bar\eta}). \arrow[u, hook, dashed] \arrow[ur, hook, "\text{normal}"']
\end{tikzcd}$$

1.3.6 Summary

In brief, in the work of Lawrence-Sawin and that of Krämer-Maculan, the uniform estimation on the monodromy group $\mathrm{Mon}(\chi)$ follows from a comparison to the Tannakian group $G(P_{\bar\eta})$ (of perverse sheaves on the geometric generic fiber). Both $\mathrm{Mon}(\chi)$ and $G(P_{\bar\eta})$ are embedded as closed subgroups in the Tannakian group $G(P_\eta)$ on the generic fiber. Moreover, $G(P_{\bar\eta})$ is *normal* in $G(P_\eta)$. This normality is used to prove $\mathrm{Mon}(\chi) \supset G(P_{\bar\eta})$ for most characters χ.

1.4 Normality of monodromy group

Complementing the normality of geometric Tannakian groups (Facts 1.3.4.1 and 1.3.4.2), we prove that for many characters $\chi \in \mathcal{C}(A)_\ell$, the associated monodromy group $\mathrm{Mon}(\chi)$ is also normal in the generic Tannakian group $G(P_\eta)$. This result poses a restriction on what the monodromy group can be.

1.4.1 Relative perverse sheaves

In Section 1.3.3, one uses a perverse sheaf P_η on the generic fiber A_η of an abelian scheme $A \times Y \to Y$. Hansen and Scholze's work [HS23] on relative perverse sheaves provides a way to study a family of perverse sheaves. Let $f : X \to Y$ be a morphism of algebraic varieties. Assume that the prime ℓ is invertible in the base field. By [HS23, Thm. 1.1], the category $D^b_c(X)$ has a unique t-structure, called the *relative perverse t-structure*, which restricts to the perverse t-structure on every geometric fiber of f. The heart $\mathrm{Perv}(X/Y)$ of the relative perverse t-structure is called the category of *relative perverse sheaves* in [HS23, p.636]. For every $y \in Y$, restricting to the fiber over y induces a functor $\mathrm{Perv}(X/Y) \to \mathrm{Perv}(X_y)$. An object of $\mathrm{Perv}(X/Y)$ should be thought as a family of perverse sheaves. However, the abelian category $\mathrm{Perv}(X/Y)$ may not be Artinian.

To get an abelian category with many of the same properties familiar in the absolute setting, Hansen and Scholze add a condition, *i.e.*, universal local acyclicity recalled in Definition 3.2.2.1. Roughly, an object of $D^b_c(X)$ is universally locally acyclic if it satisfies the base change theorem. The relative perverse t-structure preserves universally locally acyclic complexes. The resulting abelian subcategory $\mathrm{Perv}^{\mathrm{ULA}}(X/Y) \subset \mathrm{Perv}(X/Y)$ is Noetherian, Artinian and compatible with Verdier duality. Moreover, if Y is smooth with generic point η, then the functor $\mathrm{Perv}^{\mathrm{ULA}}(X/Y) \to \mathrm{Perv}(X_\eta)$ exhibits a Serre subcategory. In this sense, a universally locally acyclic, relative perverse sheaf is determined by the perverse sheaf on the generic fiber.

1.4.2 Statement

Let k be an algebraically closed field of characteristic zero. Let A be an abelian variety over k. Let Y be an algebraic variety over k with generic point η. Let $p_Y : A \times Y \to Y$ be the projection, which is a constant abelian scheme. Let $K \in \mathrm{Perv}(A \times Y/Y)$ be semisimple in $D_c^b(A \times Y)$ in the sense of Definition 3.2.1.3.

Theorem 1.4.2.1 (Theorem 3.1.2.2)**.** *Assume* $\dim A > 0$. *Then there are uncountably many characters* $\chi : \pi_1^{ét}(A) \to \bar{\mathbb{Q}}_\ell^*$ *with the following properties:* $Rp_{Y*}(K \otimes L_\chi)$ *is a lisse sheaf on* Y*, and its monodromy group is a closed* normal *subgroup of the generic Tannaka group* $G(K|_{A_\eta})$ *(defined on p.20).*

In spirit, Theorem 1.4.2.1 is similar to André's normality theorem.

Fact 1.4.2.2 ([And92, Thm. 1])**.** *Let* X *be a smooth complex algebraic variety. For a polarizable good variation of mixed Hodge structure over* X *and every* x *in the complement of some meager subset of* X*, the corresponding connected monodromy group is a normal subgroup of the corresponding derived Mumford-Tate group.*

Fact 1.4.2.2 is proved *via* the theorem of the fixed part due to Griffiths-Schmidt-Steenbrink-Zucker. In our case, an analog of the fixed part theorem is Theorem 1.4.2.3.

Theorem 1.4.2.3. *Assume that* Y *is smooth and* $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times Y/Y)$*. Then there is a subobject* $K^0 \subset K$ *in* $\mathrm{Perv}^{\mathrm{ULA}}(A \times Y/Y)$ *with the following property: For a general pro-*ℓ *character* $\chi : \pi_1^{ét}(A) \to \bar{\mathbb{Q}}_\ell^*$*, one has*

$$H^0(A_{\bar\eta}, K^0|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta}) = H^0(A_{\bar\eta}, K|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta})^{\Gamma_{k(\eta)}}.$$

1.5 Generic vanishing

1.5.1 Known results

In the construction of the Tannakian category in Fact 1.3.2.3, the existence of a fiber functor is deduced from Krämer-Weissauer's generic vanishing theorem.

Fact 1.5.1.1 (Krämer-Weissauer, [KW15b, Thm. 1.1])**.** *Let P be a perverse sheaf on a complex abelian variety A. Let $\Pi(A) := \mathrm{Hom}(\pi_1^{\top}(A), \mathbb{C}^*)$. Then there is a finite union $\mathcal{S}(P)$ of translates of strict algebraic subtori of $\Pi(A)$, such that for every character $\chi \in \Pi(A) \setminus \mathcal{S}(P)$ and every integer $i \neq 0$, one has $H^i(A, P \otimes^L L_\chi) = 0$.*

The proof of Fact 1.5.1.1 relies on two ingredients. The first is Deligne's [Del02] characterization of rigid symmetric monoidal abelian categories. It shows that a construction of André-Kahn [AKO02] leads to a super Tannakian category. The other is Kashiwara's conjecture for semisimple perverse sheaves. Its solution [Dri01] in turn relies on de Jong's conjecture proved in [BK06; Gai07]. Fact 1.5.1.1 shows that the super Tannakian category is in fact a neutral Tannakian category. As [KW15b, Sec. 3] explains, Krämer-Weissauer's theorem is a (partial) generalization of Green-Lazarsfeld's generic vanishing theorem.

Fact 1.5.1.2 ([GL87, Thm. 2])**.** *Let X be a compact Kähler manifold. Let*

$$w(X) = \max\{\mathrm{codim}_X Z(\omega) : \omega \in H^0(X, \Omega_X^1) \setminus \{0\}\},$$

where $Z(\omega)$ denotes the zero-locus of a holomorphic one form ω. Then for any integers $i, j \geq 0$ with $i + j < w(X)$ and a generic line bundle $L \in \mathrm{Pic}^0(X)$, one has $H^i(X, \Omega_X^j \otimes L) = 0$.

Green-Lazarsfeld's theorem is an analog of the Kodaira-Nakano vanishing theorem and answers a problem of Beauville [Uen83, Problem 8, p.620] affirmatively. Fact 1.5.1.1 implies generic vanishing theorem for compact Kähler manifolds whose Albanese manifolds are abelian varieties (for example, projective manifolds). In this sense, it generalizes Fact 1.5.1.2 partially. The reason for such restriction is that Fact 1.5.1.1 is stated for abelian varieties. To recover generic vanishing for all compact Kähler manifolds, one needs a generalization of Fact 1.5.1.1 to all complex tori.

Fact 1.5.1.3 (Bhatt-Schnell-Scholze, [BSS18, Thm. 1.1])**.** *Let P be a perverse sheaf on a complex torus A. Then there is a strict Zariski closed subset $\mathcal{S}(P)$ of the algebraic torus $\Pi(A)$ such that for every character $\chi \in \Pi(A) \setminus \mathcal{S}(P)$ and every integer $i \neq 0$, one has $H^i(A, P \otimes^L L_\chi) = 0$.*

1.5.2 Results for Fujiki class $\mathcal{C}$

Existing vanishing results are mainly stated for Kähler manifolds. Deligne [Del68] shows that parallel to the Kähler setting, every complex smooth proper algebraic variety (not necessarily Kähler) admits a Hodge theory. We show that generic vanishing results hold for such varieties. Instead of giving a demonstration parallel to the Kähler situation, one can give a uniform proof in Fujiki class $\mathcal{C}$.

A compact complex manifold is called in *Fujiki class $\mathcal{C}$*, if it is the meromorphic image of a compact Kähler manifold. Compact Kähler manifolds and smooth *proper* complex algebraic varieties are in this class. Fujiki class $\mathcal{C}$ admits a Hodge theory. We give a generic vanishing theorem for Fujiki class $\mathcal{C}$. Let X be a complex manifold in Fujiki class $\mathcal{C}$. Let $\alpha : X \to \mathrm{Alb}(X)$ be an Albanese morphism of X. Let $r(\alpha)$ be its defect of semismallness in the sense of Definition 4.5.2.1. Let F be a flat unitary vector bundle on X in the sense of Definition 4.2.2.2.

Theorem 1.5.2.1 (Theorem 4.7.1.3)**.** *For any integers $p, q \geq 0$ with $\dim X - p - q > r(\alpha)$ and a generic line bundle $L \in \mathrm{Pic}^0(X)$, one has*

$$H^q(X, \Omega_X^p \otimes_{O_X} F \otimes_{O_X} L) = 0.$$

Corollary (Corollary 4.7.2.6)**.** *Suppose that X is the analytification of a complex smooth proper algebraic variety. Then for any integers $p, q \geq 0$ with $\dim X - p - q > r(\alpha)$, the locus*

$$\{L \in \mathrm{Pic}^0(X) : H^q(X, \Omega_X^p \otimes_{O_X} F \otimes_{O_X} L) \neq 0\} \tag{1.2}$$

is contained in a finite union of translates of strict abelian subvarieties of the Picard variety $\mathrm{Pic}^0_{X/\mathbb{C}}$.

The strategy of the proof of Theorem 1.5.2.1 is considering the unitary local system L corresponding to F (provided by the Riemann-Hilbert correspondence). Its derived pushout $R\alpha_* L$ along the Albanese morphism is a complex of constructible sheaves on the complex torus $\mathrm{Alb}(X)$. Taking the perverse sheaf cohomologies, one deduces a generic vanishing result for $R\alpha_* L$ from the Bhatt-Schnell-Scholze theorem (Fact 1.5.1.3). Theorem 1.5.2.1 follows.

Fact 1.5.1.3 generalizes Krämer-Weissauer's theorem (Fact 1.5.1.1) to complex tori, but with a coarser control of the jump locus $\mathcal{S}(P)$ of a perverse sheaf P. The finer control in Fact 1.5.1.1 results from the classification [KW15b, Prop. 10.1 (a)] of negligible (*i.e.*, of Euler characteristic zero) simple perverse sheaves.

Do Krämer-Weissauer's theorem (Fact 1.5.1.1) and the classification have generalizations to all complex tori? A positive answer would allow one to describe the failure locus of generic vanishing theorem for all compact Kähler manifolds. The proof of [KW15b, Prop. 10.1 (a)] uses Poincaré's reducibility theorem for abelian varieties, which fails for complex tori. Still, Schnell [Sch15, Thm. 7.6] gives an independent proof of Fact 1.5.1.1 as well as that classification. Schnell's proof is relatively elementary and makes use of a lift of the Fourier-Mukai transform.

1.6 Fourier-Mukai transform

Fourier-Mukai transform on abelian varieties, initiated by Mukai [Muk81], is an analog of the classical Fourier transform. For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the category of O_X-modules. Let $D(O_X)$ be the derived category of the abelian category $\mathrm{Mod}(O_X)$.

1.6.1 Construction

Let k be an algebraically closed field. Let A be an abelian variety over k with dual abelian variety B. Let $p_A : A \times B \to A$ (resp. $p_B : A \times B \to B$) be the projection to A (resp. B). Denote the normalized Poincaré bundle on $A \times B$ by $\mathcal{P}$.

Definition 1.6.1.1. The pair of functors

$$R\hat{S} = Rp_{B*}(\mathcal{P} \otimes^L p_A^* \cdot) : D(O_A) \to D(O_B),$$
$$RS = Rp_{A*}(\mathcal{P} \otimes^L p_B^* \cdot) : D(O_B) \to D(O_A)$$

is called the *Fourier-Mukai transform* between A and B.

The Fourier-Mukai transform has found many applications in algebraic geometry: the Künneth decomposition for Chow motives [DM91], the study of stable bundles on elliptic surfaces [Bri98], a new proof of Torelli's theorem [BP01], *etc.* Motivated by noncommutative geometry, [BBP07] studies the Fourier-Mukai transform on complex tori. Similar to the classical Fourier inversion, a duality result for the Fourier-Mukai transform is stated in [Muk81, Thm. 2.2] (resp. [BBP07, Thm. 2.1]) in the algebraic (resp. analytic) case. However, both statements are imprecise (Lemma 5.2.0.6). In the algebraic case, the minor problem is bypassed by adding the quasi-coherence condition. For an algebraic variety X, let $D_{\mathrm{qc}}(O_X) \subset D(O_X)$ be the full subcategory of objects with quasi-coherent cohomologies.

Fact 1.6.1.2 (Mukai)**.** *The functor $R\hat{S}$ (resp. RS) restricts to a functor $D_{\mathrm{qc}}(O_A) \to D_{\mathrm{qc}}(O_B)$ (resp. $D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_A)$). Moreover, there are canonical isomorphisms of functors*

$$RS \circ R\hat{S} \cong T^{-g}[-1]_A^* : D_{\mathrm{qc}}(O_A) \to D_{\mathrm{good}}(O_A);$$
$$R\hat{S} \circ RS \cong T^{-g}[-1]_B^* : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{good}}(O_B),$$

where T denotes the degree shift of triangulated categories. In particular, $R\hat{S} : D_{\mathrm{qc}}(O_A) \to D_{\mathrm{qc}}(O_B)$ is an equivalence with a quasi-inverse $T^g[-1]_A^* RS$.

1.6.2 Complex tori

In the analytic setting, there are several competing definitions of "quasi-coherent" sheaves. A choice is the so-called *good* sheaves proposed by Kashiwara [Kas03, Def. 4.22] (Definition A.1.4.1). With good sheaves, we give a way to correct [BBP07, Thm. 2.1] in Chapter 5. First, we show that good sheaves are analytic analogs of quasi-coherent sheaves. For a complex manifold X, let $\mathrm{Good}(O_X) \subset \mathrm{Mod}(O_X)$ (resp. $D_{\mathrm{good}}(O_X) \subset D(O_X)$) be the full subcategory of good sheaves (resp. objects with good cohomologies).

Proposition (GAGA, Proposition B.3.0.2, Theorem B.4.0.2)**.** *Let X be a complex smooth proper algebraic variety. Then analytification induces an equivalence abelian (resp. triangulated) categories* $\mathrm{Qch}(O_X) \to \mathrm{Good}(O_{X^{\mathrm{an}}})$ *(resp.* $D_{\mathrm{qc}}(O_X) \to D_{\mathrm{good}}(O_{X^{\mathrm{an}}})$*).*

Let A be a complex torus of dimension g. Let B be its dual torus. Set $RS : D(O_B) \to D(O_A)$ and $R\hat{S} : D(O_A) \to D(O_B)$ for the corresponding Fourier-Mukai transform.

Theorem 1.6.2.1 (Mukai, Ben-Bassat, Block, Pantev, Theorem 5.4.1.1)**.** *The functor $R\hat{S}$ (resp. RS) restricts to a functor $D_{\mathrm{good}}(O_A) \to D_{\mathrm{good}}(O_B)$ (resp. $D_{\mathrm{good}}(O_B) \to D_{\mathrm{good}}(O_A)$). Moreover, there are canonical isomorphisms of functors*

$$RS \circ R\hat{S} \cong T^{-g}[-1]_A^* : D_{\mathrm{good}}(O_A) \to D_{\mathrm{good}}(O_A);$$
$$R\hat{S} \circ RS \cong T^{-g}[-1]_B^* : D_{\mathrm{good}}(O_B) \to D_{\mathrm{good}}(O_B).$$

In particular, $R\hat{S} : D_{\mathrm{good}}(O_A) \to D_{\mathrm{good}}(O_B)$ is an equivalence with a quasi-inverse $T^g[-1]_A^* RS$.

Mukai's proof of Fact 1.6.1.2 uses the flat base change theorem, of which we need an analytic analogue to prove Theorem 1.6.2.1. Our analytic replacement (Theorem 5.3.2.3) concerns only *smooth* base changes, but this weak version suffices for our purpose.

1.6.3 Homogeneous vector bundles

As an application of the analytic Fourier-Mukai transform, we recover Matsushima-Morimoto's classification of homogeneous vector bundles on complex tori ([Mat59; Mor59], see also [FL14, Thm. 7.1]).

The classification of vector bundles on a complex manifold X is completely worked out by Grothendieck [Gro57a] if X is the Riemann sphere $\mathbf{P}^1_{\mathbb{C}}$, and by Atiyah [Ati57b] if X is an elliptic curve. From [Muk78, p.239], when X is an abelian variety of higher dimension, "there are 'too' many vector bundles on X". Still, there are classification results for some special classes of vector bundles.

A vector bundle on a complex torus is called *homogeneous* if it is invariant under all translations. For example, a line bundle on the complex torus A is homogeneous if and only if its isomorphism class is in $\mathrm{Pic}^0(A)$.

For an abstract group G, a finite dimensional complex representation $G \to \mathrm{GL}(V)$ is called *unipotent*, if G acts on V by unipotent endomorphisms. By [Mil17a, Cor. 14.2], there is a basis of V for which G acts through upper triangular matrices with ones on the diagonal. An extension of finitely many O_A is called a *unipotent vector bundle* on A. By [FL14, Lem. 5.1], for every unipotent vector bundle U on A of rank r, there is a unipotent representation $\rho : \pi_1(A, 0) \to \mathrm{GL}_r(\mathbb{C})$ inducing U. More precisely, let $\mathbb{C}_\rho$ be the local system of rank r on A corresponding to ρ. Then $\mathbb{C}_\rho \otimes_{\mathbb{C}_A} O_A$ is isomorphic to U. The extension of two homogeneous vector bundles is still homogeneous, so every unipotent vector bundle is homogeneous.

Theorem (Matsushima-Morimoto, Theorem 5.5.3.6). *A vector bundle F on the complex torus A is homogeneous if and only F is isomorphic to $\oplus_{i=1}^n L_i \otimes_{\mathbb{C}_X} \mathbb{C}_{\rho_i}$, where $n \geq 0$ is an integer, $L_i \in \mathrm{Pic}^0(X)$ and $\rho_i : \pi_1(X, 0) \to \mathrm{GL}_{r_i}(\mathbb{C})$ is a unipotent representation of dimension r_i.*

1.7 Laumon-Rothstein transform

Laumon and Rothstein independently lift the Fourier-Mukai transform to D-modules and establish a duality result similar to Mukai's duality (Fact 1.6.1.2). The lift is referred to as the Laumon-Rothstein transform.

1.7.1 D-modules

On a complex manifold X, an O_X-module with a flat connection is called a D_X-module. A D_X-module is a flat vector bundle if and only if it is coherent over O_X. The reason that we need D-modules is twofold. For one thing, by the Riemann-Hilbert correspondence (see, *e.g.*, [HT07, Thm. 7.2.1]), perverse sheaves on X are equivalent to regular holonomic D_X-modules. For another, Krämer-Weissauer's convolution theory relies on [KW15b, Prop. 10.1 (a)]. Its proof (resp. an independent proof of Schnell [Sch15]) uses characteristic cycles (resp. the Laumon-Rothstein transform) of D-modules.

1.7.2 Construction

Let A be an abelian variety over an algebraically closed field. Let B be the abelian variety dual to A. Set $g = \dim A$. The Laumon-Rothstein transform turns left D_A-modules to modules over a commutative ringed space: the universal vector extension (in the sense of Fact F.1.0.2). By independent work of Rosenlicht [Ros58] and Serre [Ser88, Ch. VII], there is a universal vectorial extension $\pi : B^\natural \to B$, where $B^\natural$ is a connected commutative algebraic group of dimension $2g$. By Mazur-Messing's theorem (see, *e.g.*, [Lau96, Thm. 2.1.2]), $B^\natural$ is the moduli space of flat line bundles on A. This allows Schnell [Sch15] to apply Simpson's nonabelian Hodge theory [Sim93] there.

Let $D^b(D_A)$ be the bounded derived category of the category of left D_A-modules. Let $D^b_{\mathrm{qc}}(D_A)$ (resp. $D^b_c(D_A)$) be the full subcategory of $D^b(D_A)$ of objects with O-quasi-coherent (resp. D-coherent) cohomologies. Let $D^b_c(O_{B^\natural})$ be the full subcategory of $D^b(O_{B^\natural})$ of objects with coherent cohomologies. Let $\mathcal{P}^\natural$ be the pullback of the Poincaré bundle $\mathcal{P}$ along the morphism $\pi \times \mathrm{Id}_A : B^\natural \times A \to B \times A$. There is a natural flat connection $\nabla^\natural$ relative to $B^\natural$ on $\mathcal{P}^\natural$. Then the pair $(\mathcal{P}^\natural, \nabla^\natural)$ is naturally a $D_{B^\natural \times A/B^\natural}$-module. Let $\widetilde{\mathrm{pr}} : B^\natural \times A \to A$ and $\widetilde{\mathrm{pr}}^\natural : B^\natural \times A \to B^\natural$ be the projections.

Definition 1.7.2.1 ([Lau96, p.14])**.** The Laumon-Rothstein transform between A and B is a pair of functors

$$\tilde{\mathcal{F}} = R\widetilde{\mathrm{pr}}^\natural_* \mathrm{DR}_{B^\natural \times A/B^\natural}((\mathcal{P}^\natural, \nabla^\natural) \otimes^L_{O_{B^\natural \times A}} \widetilde{\mathrm{pr}}^* \cdot) : D^b_{\mathrm{qc}}(D_A) \to D^b_{\mathrm{qc}}(O_{B^\natural}),$$
$$\tilde{\mathcal{F}}^\natural = R\widetilde{\mathrm{pr}}_*((\mathcal{P}^\natural, \nabla^\natural) \otimes^L_{O_{B^\natural \times A}} L\widetilde{\mathrm{pr}}^{\natural *} \cdot) : D^b_{\mathrm{qc}}(O_{B^\natural}) \to D^b_{\mathrm{qc}}(D_A).$$

Fact 1.7.2.2 (Laumon-Rothstein, [Lau96, Thm. 3.2.1, Cor. 3.1.3], [Rot96, Thm. 4.5, Thm. 6.2], [Rot97])**.** *There are canonical isomorphism of functors*

$$\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}} \cong T^{-g}[-1]^*_A : D^b_{\mathrm{qc}}(D_A) \to D^b_{\mathrm{qc}}(D_A), \ \tilde{\mathcal{F}} \tilde{\mathcal{F}}^\natural \cong T^{-g}[-1]^*_{B^\natural} : D^b_{\mathrm{qc}}(O_{B^\natural}) \to D^b_{\mathrm{qc}}(O_{B^\natural}).$$

In particular, the functor $\tilde{\mathcal{F}} : D^b_{\mathrm{qc}}(D_A) \to D^b_{\mathrm{qc}}(O_{B^\natural})$ is an equivalence of categories. It restricts to an equivalence $D^b_c(D_A) \to D^b_c(O_{B^\natural})$.

1.7.3 Schnell's proof of Fact 1.5.1.1

The Laumon-Rothstein transform is a geometric tool to study generic vanishing theorems for perverse sheaves. The Riemann-Hilbert correspondence induces an isomorphism $\Phi : (B^\natural)^{\mathrm{an}} \to \Pi(A)^{\mathrm{an}}$ of complex manifolds. By [Sch15, Sec. 3], for a holonomic D_A-module M, the support of its Laumon-Rothstein transform $\mathrm{Supp}\,\tilde{\mathcal{F}}(M) \subset B^\natural$ is identified (*via* Φ) with the failure locus (in $\Pi(A)$) of generic vanishing for M. Schnell "deforms" the Laumon-Rothstein transform to a transform for Higgs bundles.

On a complex manifold X, a connection on a vector bundle may not be O_X-linear. Higgs bundles can be regarded as degenerations of vector bundles with flat "linear" connection.

Definition 1.7.3.1. A Higgs bundle is a vector bundle E on X with a holomorphic one form $\phi \in \Gamma(X, \mathcal{E}nd(E))$ taking values in the bundle of endomorphisms of E with $\phi \wedge \phi = 0$.

Deligne's λ-connection is a notion interpolating between flat bundles and Higgs bundles.

Definition 1.7.3.2. For $\lambda \in \mathbb{C}$, a λ-connection of a vector bundle E on X is a $\mathbb{C}$-linear morphism of sheaves $\nabla : E \to \Omega^1_X \otimes_{O_X} E$ such that for every open subset $U \subset X$, every $f \in O_X(U)$ and every $s \in \Gamma(U, E)$, one has $\nabla(f \cdot s) = f \cdot \nabla s + \lambda df \otimes s$. A λ-connection is *flat* if the O_X-linear curvature operator $\nabla \circ \nabla : E \to \Omega^2_X \otimes_{O_X} E$ is zero.

Then 1-connection is exactly a connection, and a vector bundle with a flat 0-connection is the same as a Higgs bundle. The moduli spaces of λ-connections on projective varieties are studied in Simpson's work [Sim97].

For a complex abelian variety A, Schnell [Sch15, Sec. 10] analyzes the moduli space $E(A)$ of line bundles on A with λ-connections. Let $\lambda : E(A) \to \mathbf{A}^1_{\mathbb{C}}$ be the morphism taking the parameter of generalized connections. By [Sch15, Lem. 10.7, 10.9], one has $\lambda^{-1}(1) = B^\natural$, and $\lambda^{-1}(0)$ is the moduli space of rank one Higgs bundles on A (*i.e.*, $M_{\mathrm{Dol}}(A)$ in [Sim93, p.363]). The morphism λ is real-analytically trivial, recovering the isomorphism of real Lie groups $M_{\mathrm{Dol}}(A) \to B^\natural$ in [Sim93, p.364].

Schnell [Sch15, Sec. 11] introduces an "extended Fourier-Mukai transform" taking values in $D(O_{E(A)})$. Restricting to $\lambda^{-1}(1)$, it coincides with the Laumon-Rothstein transform. Restricting to $\lambda^{-1}(0)$, it is essentially the Fourier transform for Higgs bundles in [Bon06; Bon10].

Schnell deforms the holonomic D_A-module M to an O_{T^*A}-module M', which is a "generalized" Higgs bundle (more precisely, a holonomic Higgs module as [Sab07, Example 5.1.6 (1)] shows). By definition of holonomicity, the subspace $\operatorname{Supp} M' \subset T^*A$ has pure dimension g. Therefore, the support of the Fourier transform of M' is a strict subset of $\lambda^{-1}(0)$. It is the intersection of the support of the extended Fourier-Mukai transform of M in $E(A)$ with $\lambda^{-1}(0)$. Using the real analytic isomorphism $\lambda^{-1}(0) \to \lambda^{-1}(1)$, Schnell proves that the support of $\tilde{\mathcal{F}}(M)$ is also a strict subset of $B^\natural$. The generic vanishing theorem (Fact 1.5.1.1) follows from this strictness. Details can be found in [Sch15, Prop. 18.2].

1.7.4 Complex tori

An analytic Laumon-Rothstein transform may help to extend Schnell's method to all complex tori. By [Fav12, Thm. 3], an abelian variety A is determined by $D^b_c(D_A)$. From the Laumon-Rothstein duality (Fact 1.7.2.2), it is also determined by $D^b_c(O_{B^\natural})$. However, this fails in the analytic case. Let A, B be complex tori dual to each other, of dimension g. By Proposition F.5.4.5 1, the universal vectorial extension $\pi : B^\natural \to B$ still exists. Contrary to the algebraic case, the complex Lie group $B^\natural$ is isomorphic to $(\mathbb{C}^*)^{2g}$. Then $B^\natural$ can only tell the dimension of A. In particular, one can no longer recover the complex structure of A from $B^\natural$! That is why we need to replace $B^\natural$ by something else in an analytic Laumon-Rothstein duality.

In fact, we construct an O_B-subalgebra $\mathcal{A}_B$ of $\pi_* O_{B^\natural}$, and define a pair of functors

$$\tilde{\mathcal{F}} : D(D_A) \to D(\mathcal{A}_B), \quad \tilde{\mathcal{F}}^\natural : D(\mathcal{A}_B) \to D(D_A).$$

A coherent D_A-module is called *good* if it admits global good filtration. (In the algebraic case, every coherent D-module admits a global good filtration. The complex analytic analog is false.) Let $D^b_{\mathrm{good}}(D_A)$ (resp. $D_{O-\mathrm{good}}(D_A)$) be the full subcategory of $D^b(D_A)$ (resp. $D(D_A)$) of objects with good (resp. O_A-good) cohomology. Theorem 1.7.4.1 is a "lift" of the analytic Mukai duality (Theorem 1.6.2.1).

Theorem 1.7.4.1 (Theorem 6.1.2.2). *1. The pair $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^\natural)$ is a lift of Fourier-Mukai transform in the sense that the following squares are commutative:*

$$\begin{tikzcd} D(D_A) \arrow[r, "\tilde{\mathcal{F}}"] \arrow[d] & D(\mathcal{A}_B) \arrow[d] \\ D(O_A) \arrow[r, "R\hat{S}"] & D(O_B), \end{tikzcd} \qquad \begin{tikzcd} D(D_A) \arrow[d] & D(\mathcal{A}_B) \arrow[l, "\tilde{\mathcal{F}}^\natural"'] \arrow[d] \\ D(O_A) & D(O_B), \arrow[l, "RS"'] \end{tikzcd}$$

where the vertical functors are forgetful.

2. *One has*

$$\tilde{\mathcal{F}}\tilde{\mathcal{F}}^\natural \cong T^{-g}[-1]_B^* : D_{O-\mathrm{good}}(\mathcal{A}_B) \to D_{O-\mathrm{good}}(\mathcal{A}_B),$$
$$\tilde{\mathcal{F}}^\natural\tilde{\mathcal{F}} \cong T^{-g}[-1]_A^* : D_{O-\mathrm{good}}(D_A) \to D_{O-\mathrm{good}}(D_A).$$

Moreover, $\tilde{\mathcal{F}}^\natural\tilde{\mathcal{F}}$ restricts to a functor $D^b_{\mathrm{good}}(D_A) \to D^b_{\mathrm{good}}(D_A)$.

1.7.5 Vector bundles with connection

A smooth vector bundle on a smooth manifold always admits a smooth connection. In the complex analytic case, a (holomorphic) vector bundle many not admit any (holomorphic) connection.

Fact 1.7.5.1 (Atiyah, [Ati57a, Theorems 2, 5, 6]). *Let X be a compact Kähler manifold. Let E be a vector bundle on X admitting a connection. Then for every integer $k > 0$, the k-th Chern class $c_k(E)$ is zero in $H^{2k}(X, \mathbb{R})$.*

Fact 1.7.5.1 leads to Question 1.7.5.2, which is attributed to Atiyah in [BD24, p.1].

Question 1.7.5.2. Does every vector bundle on a compact Kähler manifold admitting a connection also admit a flat connection?

Using the analytic Laumon-Rothstein transform, we recover a result of Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], which answers Question 1.7.5.2 affirmatively for complex tori.

Theorem. *1.* (Theorem 6.3.3.1) *Let E be a coherent module on a complex torus with a connection ∇. Then E is a homogeneous vector bundle and the pair (E, ∇) is translation invariant.*

2. (Proposition 6.5.2.1) *A homogeneous vector bundle on a complex torus admits a flat connection.*

1.8 Future directions

Several possible topics for further research are as follows. Depending on the limit of the author's knowledge, they vary from a vague idea to a relatively concrete plan.

1.8.1 Analytic quasi-coherent sheaves

As the proof of Theorem 1.6.2.1 needs a bit six-functor formalism in complex analytic geometry, there are a few natural questions: Does Theorem 1.6.2.1 have an analog for analytic quasi-coherent sheaves in Scholze and Clausen's sense [Sch19; Sch22]? What is the relation between the notions of analytic quasi-coherence existing in the literature: the one of Scholze and Clausen, good sheaves proposed by Kashiwara and quasi-coherent sheaves in the sense of [RR74, p.100]? Moreover, the Gabriel-Rosenberg reconstruction theorem (see, *e.g.*, [Bra18, Thm. 1.1]) shows that a quasi-separated scheme X can be recovered from $\mathrm{Qch}(X)$. Does it have an analytic analog?

Question 1.8.1.1. Let X and Y be compact complex manifolds. If $\mathrm{Good}(O_X)$ and $\mathrm{Good}(O_Y)$ are equivalent as abelian categories, then is X isomorphic to Y?

1.8.2 Analytic Krämer-Weissauer's vanishing theorem

With the analytic Laumon-Rothstein transform and Theorem 1.7.4.1, we can study analytic holonomic D-modules (instead of perverse sheaves only) following Schnell [Sch15]. This shall lead to a convolution theory on complex tori, extending that on abelian varieties. The resulting analytic Krämer-Weissauer theorem would hopefully give a finer control of the loci (1.2) for not only projective manifolds but also compact Kähler manifolds.

1.8.3 Lawrence-Venkatesh's method

Faltings [Fal83] deduces Mordell's conjecture (Fact 1.1.1.2) from Shafarevich's conjecture [Fal83, Satz 6]. For any integers $g \geq 1$ and $n \geq 3$, the Siegel variety $A_{g,n}$ is a Shimura variety parameterizing principally polarized abelian varieties of dimension g with a level n-structure. Using Shafarevich's conjecture (Fact 1.1.1.3), Ullmo [Ull04, Prop. 3.1 (a)] shows that every integral model of $A_{g,n}$ has only finitely many integral points. Now that Lawrence-Venkatesh's method [LV20] can recover Faltings's theorem, a natural question is whether it can also prove the finiteness of integral points of $A_{g,n}$.

The situation should be compared to that in [LS20, p.7], where the authors consider the universal hypersurface inside a *constant* abelian scheme and compare its Tannakian group with the monodromy group. Similarly, to study Shafarevich's conjecture (Fact 1.1.1.3), we can consider the convolution of relative perverse sheaves on the universal abelian variety over $A_{g,n}$. Then we may relate the Tannakian group associated with the universal theta divisor computed in [KW15a] to the corresponding monodromy group.

1.9 Overview

The thesis consists of several independent chapters. Chapter 2 contains an arithmetic result related to Conjecture 1.1.2.1. The geometric foundation of the work [LS20; KM23] has inspired the study in Chapters 3, 4 and 6. Chapter 3 is related to the monodromy comparison part of [LS20; KM23]. Chapters 4, 5 and 6 are of complex analytic nature, completely independent of arithmetic. Appendix A reviews generalities of sheaves of modules over ringed spaces and supplements Chapter 5. Appendix B extends the classical GAGA theorem from coherent sheaves to quasi-coherent sheaves. Appendix C shows that quasi-coherent sheaves on complex analytic spaces form an abelian category. Appendix D complements Chapter 4 by giving more details. Appendix E concerns basics of D-modules and adds a detail on the Laumon-Rothstein theorem (Fact 1.7.2.2). Appendix F details a construction used in Chapter 6 and investigates related group-theoretic problems.

Chapter 2

Integral points of Shimura varieties: an “all or nothing” principle

2.1 Introduction

A complex analytic space X is called *Brody hyperbolic*, if every morphism $\mathbb{C} \to X$ is constant. For example, by [Cos05, p.78], a genus g connected compact Riemann surface is Brody hyperbolic if and only if $g \geq 2$. Conjecture 2.1.0.1 predicts that hyperbolicity restricts the behavior of rational points.

Conjecture 2.1.0.1 (Lang, [Lan74, (1.3)], [Lan86, p.160])**.** *Let X be an integral projective variety over a number filed $F(\subset \mathbb{C})$. If the complex analytification $X(\mathbb{C})$ is Brody hyperbolic, then the set of rational points $X(F)$ is finite.*

Ullmo and Yafaev [UY10] study Conjecture 2.1.0.1 in the case of Shimura varieties. Let (G, X) be a Shimura datum in the sense of [Mil17b, Def. 5.5]. Let K be a small compact open subgroup of $G(\mathbb{A}_f)$. For every connected component S of $\mathrm{Sh}_K(G, X)$, denote the Baily-Borel compactification of S by S^*. In [UY10, p.691], Fact 2.1.0.2 is derived from [Nad89, Thm. 0.2].

Fact 2.1.0.2 (Nadel)**.** *There is an open subgroup K' of K, such that for every induced finite étale cover $S' \to S$, the Baily-Borel compactification $S'^*(\mathbb{C})$ is Brody hyperbolic.*

For one thing, by Fact 2.1.0.2, shrinking K to a sufficiently small open subgroup, one may assume that the Shimura variety $S(\mathbb{C})$ is Brody hyperbolic. For another, S^* has a natural structure of projective variety over a number field $F(\subset \mathbb{C})$. Then Conjecture 2.1.0.1 predicts $S(F')$ to be finite for every finite extension F'/F. Ullmo and Yafaev [UY10] introduce “Lang locus” (recalled in Example 2.2.0.2) for algebraic varieties over $\bar{\mathbb{Q}}$ to measure the possible failure of Conjecture 2.1.0.1. By construction, the Lang locus of an algebraic variety X is empty if and only if it has only finitely many rational points over every number field on which X can be defined. The Lang locus of Shimura varieties satisfies an alternative principle.

Fact 2.1.0.3 ([UY10, Thm. 1.1])**.** *Let S be a Shimura variety of sufficiently high level. Then its Lang locus is either empty or full S.*

As Ullmo and Yafaev put it, Fact 2.1.0.3 means that for Shimura varieties, Conjecture 2.1.0.1 is either true or very false.

Shimura varieties may not be proper, so it is natural to consider integral points. Conjecture 2.1.0.1 predicts that S has only finitely many integral points. We define a notion of "integral Lang locus" (Definition 2.6.0.1) for algebraic varieties over $\bar{\mathbb{Q}}$ that measures the failure of finiteness of integral points. Then we derive an analogue of Fact 2.1.0.3 for this locus.

Theorem 2.1.0.4 (Theorem 2.6.0.13)**.** *The integral Lang locus of a Shimura variety is either empty or full.*

A minor difference is that by the Chevalley-Weil theorem, we don't need the level structure to be high. As Example 2.6.0.5 shows, Theorem 2.1.0.4 is not a direct corollary of Fact 2.1.0.3.

Notation and conventions

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $\mathbb{A}_f$ be the ring of finite adèles of $\mathbb{Q}$. For a topological space X, let $X^{>0}$ be the union of irreducible components of X of positive Krull dimension. Then for every subspace $Y \subset X$, one has $Y^{>0} \subset X^{>0}$.

An algebraic variety means a finite type, separated scheme over a field. The closure of a subset of an algebraic variety is taken in the Zariski topology. A subvariety is assumed to be Zariski closed. By an *étale cover* $X \to Y$, we mean a finite étale morphism between integral algebraic varieties. In particular, it is surjective. If the automorphism group $\mathrm{Aut}(X/Y)$ of the étale cover acts transitively on each fiber, then $X \to Y$ is called a *Galois cover*, of Galois group $\mathrm{Aut}(X/Y)$.

2.2 Locus formation

We shall show that an alternative principle (Corollary 2.5.0.4) for an abstract locus is a consequence of some axioms listed in Section 2.2.

Suppose that for every integral algebraic variety X over $\bar{\mathbb{Q}}$, one has a subvariety $X^L \subset X$. For a reducible algebraic variety Z over $\bar{\mathbb{Q}}$, let $Z = \cup_{i=1}^n Z_i$ be the irreducible decomposition. Set $Z^L := \cup_{i=1}^n Z_i^L$. Suppose that the formation $(\cdot)^L$ satisfies Assumption 2.2.0.1.

Assumption 2.2.0.1. For any integral algebraic varieties X, Y over $\bar{\mathbb{Q}}$:

1. Every irreducible component of X^L has positive dimension;
2. For every closed immersion $i : X \to Y$ over $\bar{\mathbb{Q}}$, one has $i(X^L) \subset Y^L$;
3. For every étale cover $f : X \to Y$ over $\bar{\mathbb{Q}}$, one has $f(X^L) \subset Y^L$;
4. One has $X^L \subset (X^L)^L$;
5. For every finite birational morphism $f : X \to Y$ over $\bar{\mathbb{Q}}$, one has $f(X^L) \subset Y^L$.

By Assumption 2.2.0.1 2, for every integral algebraic variety X over $\bar{\mathbb{Q}}$, one has $X^L \supset (X^L)^L$. From Assumption 2.2.0.1 4, one has $X^L = (X^L)^L$.

Example 2.2.0.2. We recall the Lang locus defined in [UY10, Sec. 2.2] and the reason why it satisfies Assumption 2.2.0.1. By [Sta24, Tag 01ZM (1), Tag 01ZQ], for every integral algebraic variety X over $\bar{\mathbb{Q}}$, there exists a number field F, an algebraic variety X_F over F called a model of X, and an isomorphism $\phi : X_F \otimes_F \bar{\mathbb{Q}} \to X$ over $\bar{\mathbb{Q}}$. For every finite subextension M/F inside $\bar{\mathbb{Q}}$, let $X(X_F, M)$ be the image of the injection[1] $\phi_* : X_F(M) \to X(\bar{\mathbb{Q}})$. The *Lang locus* X^{UY} of X relative to (X_F, ϕ) is defined to be the Zariski closure of

$$\cup_M \overline{X(X_F, M)}^{>0}$$

in X, where M runs through all finite subextensions of F. By Lemma 2.2.0.3, the Lang locus X^{UY} depends only on X. It measures the failure of finiteness of rational points, since $X^{\mathrm{UY}}\emptyset$ if and only if $X_F(M)$ is finite for every finite subextension M/F.

Every irreducible component of X^{UY} of dimension 0 is an isolated point p. Then p is contained in $\overline{X(X_F, M)}^{>0}$ for some finite extension M/F, so there is a positive dimensional irreducible component C of $\overline{X(X_F, M)}$ containing p. Since $C \subset X^{\mathrm{UY}}$, one has a contradiction. This contradiction proves Assumption 2.2.0.1 1. From [UY10, Lemme 2.3 (resp. 2.5)], the formation $(\cdot)^{\mathrm{UY}}$ satisfies Assumption 2.2.0.1 3 and 5 (resp. 2 and 4).

Lemma 2.2.0.3. *The Lang locus X^{UY} of X is independent of the choice of (X_F, ϕ).*

Proof. Take another model $X_{F'}$ over a number field F' and an isomorphism $\phi' : X_{F'} \otimes_{F'} \bar{\mathbb{Q}} \to X$. Then $\phi'^{-1} \circ \phi : X_F \otimes_F \bar{\mathbb{Q}} \to X_{F'} \otimes_{F'} \bar{\mathbb{Q}}$ is an isomorphism over $\bar{\mathbb{Q}}$. By [EGA IV 2, Prop. 4.8.13], because $X_{F'}$ is separated over F', the morphism is defined over a number field F'' containing both F and F'. For

[1] The natural map $X_F(M) \to X_F$ may not be injective. For instance, let $F = \mathbb{Q}$, $M = \mathbb{Q}(\sqrt{2})$ and $X_F = \mathbf{A}^1_{\mathbb{Q}}$. Then $\pm\sqrt{2} \in X_F(M)$ are mapped to the same closed point of X_F corresponding to the maximal ideal $(x^2 - 2) \subset \mathbb{Q}[x]$.

every finite extension M/F, there is a number field M' containing M and F''. Then $X(X_F, M) \subset X(X_{F'}, M')$, so $\overline{X(X_F, M)}^{>0} \subset \overline{X(X_{F'}, M')}^{>0}$ and hence the Lang locus relative to (X_F, ϕ) is contained in that relative to $(X_{F'}, \phi')$. The reverse inclusion follows by symmetry. $\square$

Remark 2.2.0.4.

1. The Lang locus X^{UY} in Example 2.2.0.2 is different from the Zariski closed subset X_F^L of X_F defined by [UY10, (1)]. Let $\varphi : X \to X_F$ be the morphism of schemes induced by $\phi : X_F \otimes_F \bar{\mathbb{Q}} \to X$. Then φ is integral. For every finite extension M/F, let $X_F[M]$ be the image of the natural map $X_F(M) \to X_F$. Then $\varphi(X(X_F, M)) = X_F[M]$. Because surjective integral morphisms preserve the dimension, one has $\varphi(\overline{X(X_F, M)}) = \overline{X_F[M]}$ and $\varphi(\overline{X(X_F, M)}^{>0}) = \overline{X_F[M]}^{>0}$. Therefore, $\varphi(X^{\mathrm{UY}}) = X_F^L$.

2. For a finite birational morphism $f : X \to Y$ of integral algebraic varieties over $\bar{\mathbb{Q}}$, it is not clear whether the induced map $X^{\mathrm{UY}} \to Y^{\mathrm{UY}}$ is surjective. That is why we require only inclusion but not equality in Assumption 2.2.0.1 5.

We gather some consequences of Assumption 2.2.0.1.

Lemma 2.2.0.5. *Let X be an algebraic variety over $\bar{\mathbb{Q}}$.*

1. *If $X = \cup_{i=1}^r Z_i$, where each Z_i is a subvariety of X, then $X^L = \cup_{i=1}^r Z_i^L$.*

2. *If Z is an irreducible component of X^L, then $Z^L = Z$.*

3. *If $f : X \to Y$ is Galois cover over $\bar{\mathbb{Q}}$, then $f^{-1}(f(X^L)) = X^L$. Let $Z \subset Y$ be an irreducible subvariety, and let Z' be an irreducible component of $f^{-1}(Z)$. Then $f(f^{-1}(Z)^L) = f(Z'^L)$.*

Proof.

1. By Assumption 2.2.0.1 2, one has $\cup_{i=1}^r Z_i^L \subset X^L$. If Y is an irreducible component of X, then there exists an index i with $Y \subset Z_i$. From Assumption 2.2.0.1 2, one has $Y^L \subset Z_i^L$ and hence $X^L \subset \cup_{i=1}^r Z_i^L$.

2. Write $X^L = \cup_{i=1}^n Z_i$ for the irreducible decomposition with $Z_1 = Z$. By Assumption 2.2.0.1 4, one has

$$Z \subset X^L = (X^L)^L = \cup_{i=1}^n Z_i^L.$$

   As Z is irreducible, there is an index i with $Z \subset Z_i^L \subset Z_i$. As $Z = Z_1$ is an irreducible component of X^L, one has $i = 1$ and $Z = Z^L$.

3. For every $x \in f^{-1}(f(X^L))$, there is $x' \in X^L$ with $f(x') = f(x)$. Let Θ be the Galois group of $f : X \to Y$. There is $\theta \in \Theta$ with $\theta(x') = x$, so $x \in X^L$ by Assumption 2.2.0.1 2. Therefore, $f^{-1}(f(X^L)) = X^L$.

Since Θ permutes transitively the irreducible components of $f^{-1}(Z)$, one has $f^{-1}(Z) = \Theta \cdot Z'$. By Part 1, one has $f^{-1}(Z)^L = \Theta \cdot Z'^L$ and hence $f(f^{-1}(Z)^L) = f(Z'^L)$.

$\square$

2.3 Shimura varieties

We review some basic facts about Shimura varieties, the main objects of interest in this note. We use essentially results on the geometry of Hecke correspondences and special subvarieties from [UY10; UY14].

Complex structure

Let G be an affine algebraic group over $\mathbb{Q}$.

Definition 2.3.0.1 ([Pin90, Sec. 0.1, p.13])**.** For every prime number p, choose an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$.

1. For an element $g = (g_p)_p \in \mathrm{GL}_n(\mathbb{A}_f)$, let $\Gamma_p \leq \bar{\mathbb{Q}}_p^\times$ be the subgroup generated by all eigenvalues of $g_p \in \mathrm{GL}_n(\mathbb{Q}_p)$. If the intersection of torsion subgroups
$$\cap_p(\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\mathrm{tor}}$$
for p running through all primes is trivial, then g is called *neat*.

2. An element of $G(\mathbb{A}_f)$ is *neat* if its image under some faithful algebraic representation of $G \to \mathrm{GL}_{n/\mathbb{Q}}$ is neat.

3. A subgroup of $G(\mathbb{A}_f)$ is *neat* if all its elements are neat.

Fact 2.3.0.2 ([Pin90, p.13])**.**

1. *Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then there is a normal open subgroup K' of K that is neat.*

2. *Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Then $K \cap G(\mathbb{Q})$ is a neat subgroup of $G(\mathbb{Q})$* (in the sense of [Mil17b, p.34]).

Let (G, X) be a Shimura datum. The set $G(\mathbb{R})$ is naturally a (real) Lie group. For a Lie group L, let L^+ be its identity component. Let G^{ad} be the quotient of G by its center. Set $G(\mathbb{R})_+$ to be the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the natural morphism $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$ of Lie groups. By [Noo06, p.168] and [Mil17b, Prop. 5.9], X is naturally a finite disjoint union of isomorphic hermitian symmetric domains. Let X^+ be a connected component of X. By [Mil17b, Prop. 5.7 (b)], the stabilizer of X^+ in $G(\mathbb{Q})$ is $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Set $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$. Then $G(\mathbb{Q})^+ \subset G(\mathbb{Q})_+ \subset G(\mathbb{Q})$.

Let K be a compact open subgroup of $G(\mathbb{A}_f)$. From Fact 2.3.0.2 1, by passing to an open subgroup of K, we may and *always* assume that K is neat. Then by [Pin90, Prop. 3.3 (b)], $\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is naturally a complex manifold. For every $g \in G(\mathbb{A}_f)$, put $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$ and $S_g := \Gamma_g\backslash X^+$. By Fact 2.3.0.2 2 and [Mil17b, Prop. 4.1], Γ_g is a neat (hence torsion-free) arithmetic subgroup of $G(\mathbb{Q})$ in the sense of [Mil17b, p.33]. From [Mil17b, Prop. 3.1], $S_g = [X^+, g]_K$ is naturally a connected complex submanifold of $\mathrm{Sh}_K(G,X)$. Let C be a set of representatives for the double coset space $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. From [Mil17b, Lemmas 5.12 and 5.13], the set C is finite, and as complex manifold[2] $\mathrm{Sh}_K(G,X) = \sqcup_{g\in C} S_g$.

By [Mil17b, Thm. 3.12, Cor. 3.16], the complex manifold S_g has a canonical structure of a complex algebraic variety. The algebraic variety S_g is an irreducible, *smooth*, arithmetic locally symmetric variety in the sense of [Mil17b, p.58]. From [Mil17b, p.40], it is Zariski-open in its Baily-Borel compactification S_g^*, which is a projective variety. Thus, $\mathrm{Sh}_K(G,X)$ is naturally a smooth quasi-projective (possibly disconnected) complex algebraic variety.

Lemma 2.3.0.3. *Let* (H, X_H) *be s Shimura subdatum of* (G,X) in the sense of [CLZ16, p.894]. *Then there is a compact open subgroup* K_0 *of* $G(\mathbb{A}_f)$*, such that for every open subgroup* K *of* K_0*, the induced morphism* $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G,X)$ *is a closed immersion.*

Proof. By [Mil17b, Aside, p.58], there is a compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup K of K_0, there is an open subgroup K' of $K\cap H(\mathbb{A}_f)$, such that the induced morphism $\mathrm{Sh}_{K'}(H, X_H) \to \mathrm{Sh}_K(G,X)$ is a closed immersion. The morphism $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G,X)$ is separated, so the morphism $p_{K',K\cap H(\mathbb{A}_f)} : \mathrm{Sh}_{K'}(H, X_H) \to \mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H)$ is a closed immersion. By [Sta24, Tag 025G], $p_{K',K\cap H(\mathbb{A}_f)}$ is an isomorphism. Therefore, $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G,X)$ is a closed immersion. □

Remark 2.3.0.4. In [DJK20, p.6], it writes that $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G,X)$ is a closed immersion for every compact open subgroup K of $G(\mathbb{A}_f)$. This is a typo, as the first paragraph of [Mil17b, p.59] explains.

Canonical model

Let $E(G,X) \subset \bar{\mathbb{Q}}$ be the reflex field in the sense of [Mil17b, Def. 12.2] of the Shimura datum (G,X). By [Mil17b, Rk. 12.3 (a)], $E(G,X)$ is a number field. From [Mil99, Rk. 2.4 (b)] and [Mil17b, p.128], $\mathrm{Sh}_K(G,X)$ admits a unique (up to a unique isomorphism) *canonical model* over $E(G,X)$ in the sense of [Mil17b, Def. 12.8]. Hence, one obtains a *smooth* quasi-projective

[2]Compare with [EY03, Sec. 5, p.634] and [Noo06, p.169].

variety $\mathrm{Sh}_K(G,X)$ over $E(G,X)$. By [Moo98b, p.282] and [GN20, Remark (3), p.56], every connected component S of $\mathrm{Sh}_K(G,X)$ and its inclusion $S \to S^*$ to the Baily-Borel compactification are defined over a finite abelian extension of $E(G,X)$. Such an S is called a *Shimura variety* associated with (G,X,K). By [Moo98b, Prop. 2.9] and [Mil17b, Thm. 5.17], when K is sufficiently small, S is a connected Shimura variety in the sense of [Mil17b, Def. 4.10].

By [Del71b, Cor. 5.4], for every morphism of Shimura data $\phi : (G',X') \to (G,X)$ and every compact open subgroup $K \leq G(\mathbb{A}_f)$ containing $\phi(K')$, the induced morphism $\mathrm{Sh}_{K'}(G',X') \to \mathrm{Sh}_K(G,X)$ is defined over a number field. Assume that ϕ is the identity of (G,X). Then the induced morphism $p_{K',K} : \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$ is surjective finite étale and defined over $E(G,X)$. For every $g \in G(\mathbb{A}_f)$, the image of the irreducible component $[X^+,g]_{K'}$ of $\mathrm{Sh}_{K'}(G,X)$ is the irreducible component $[X^+,g]_K$ of $\mathrm{Sh}_K(G,X)$. The restriction $[X^+,g]_{K'} \to [X^+,g]_K$ is an étale cover defined over a finite extension of $E(G,X)$, called an *étale cover of Shimura varieties.* From [CK16, p.1901], when K' is normal in K, this étale cover is Galois with Galois group $(gKg^{-1} \cap G(\mathbb{Q})_+)/(gK'g^{-1} \cap G(\mathbb{Q})_+)$. Moreover, K/K' acts on the complex algebraic variety $\mathrm{Sh}_{K'}(G',X')$, and this action permutes the irreducible components of $p_{K',K}^{-1}([X^+,g]_K)$.

Remark 2.3.0.5. The last paragraph of [UY10, p.694] is a typo. The family of étale covers of Shimura varieties over S is not directed, in the sense that for such covers $S_1 \to S$ and $S_2 \to S$, there may not be any étale cover of Shimura varieties $S_{1,2} \to S$ that covers both S_1 and S_2.

Hecke correspondences

For every $g \in G(\mathbb{A}_f)$, let $T(g)$ be the morphism of complex manifolds

$$\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{g^{-1}Kg}(G,X), \quad [x,a]_K \mapsto [x,ag]_{g^{-1}Kg}.$$

By [Mil17b, Thm. 13.6], it descends to an isomorphism of algebraic varieties over $E(G,X)$. For every $h \in G(\mathbb{A}_f)$, the morphism $T(g)$ sends the connected component $[X^+,h]_K$ of $\mathrm{Sh}_K(G,X)$ isomorphically to the connected component $[X^+,hg]_{g^{-1}Kg}$ of $\mathrm{Sh}_{g^{-1}Kg}(G,X)$. The algebraic correspondence

$$\mathrm{Sh}_K(G,X) \overset{p_{K\cap gKg^{-1},K}}{\longleftarrow} \mathrm{Sh}_{K\cap gKg^{-1}}(G,X) \overset{p_{K\cap gKg^{-1},gKg^{-1}}}{\longrightarrow} \mathrm{Sh}_{gKg^{-1}}(G,X) \overset{T(g)}{\to} \mathrm{Sh}_K(G,X)$$

over $E(G,X)$ is denoted by $T_g^{\mathbb{A}}$, and called the *adelic Hecke correspondence* induced by g.

Let $S = (K \cap G(\mathbb{Q})_+)\backslash X^+$. For every $q \in G(\mathbb{Q})_+$, let $S_q := (K \cap q^{-1}Kq \cap G(\mathbb{Q})_+)\backslash X^+$. Then S_q (resp. S) is a connected component of $\mathrm{Sh}_{K\cap q^{-1}Kq}(G,X)$ (resp. $\mathrm{Sh}_K(G,X)$). The map $\mathrm{Id}_{X^+} : X^+ \to X^+$ (resp. the map $X^+ \to X^+, \quad x \mapsto q \cdot x$) induces an étale cover $\alpha_q : S_q \to S$ (resp. $\beta_q : S_q \to S$). There is a commutative diagram

$$\begin{tikzcd}
S \arrow[d, hook] & S_q \arrow[l, "\alpha_q"'] \arrow[r, "\beta_q"] \arrow[d, hook] & S \arrow[d, hook] \\
\mathrm{Sh}_K(G,X) & \mathrm{Sh}_{K\cap q^{-1}Kq}(G,X) \arrow[l, "p_{K\cap q^{-1}Kq,K}"] \arrow[r, "T(q^{-1})p_{K\cap q^{-1}Kq,q^{-1}Kq}"'] & \mathrm{Sh}_K(G,X)
\end{tikzcd}$$

of complex manifolds. Therefore, the correspondence

$$S \overset{\alpha_q}{\leftarrow} S_q \overset{\beta_q}{\rightarrow} S$$

is algebraic and defined over a number field. It is called the *(rational) Hecke correspondence* induced by q, and denoted by T_q.

Let $\{q_i\}_{i=1}^n$ be elements of $G(\mathbb{Q})_+ \cap KgK$ satisfying

$$G(\mathbb{Q})_+ \cap KgK = \sqcup_{i=1}^n \Gamma q_i^{-1}\Gamma, \quad \Gamma := K \cap G(\mathbb{Q})_+.$$

By [KY14, p.881], the correspondence on $S \subset \mathrm{Sh}_K(G,X)$ induced by $T_g^{\mathbb{A}}$ decomposes as $\sum_{i=1}^n T_{q_i}$. For instance, the correspondences $T_1^{\mathbb{A}}$ and T_1 are the identity.

Special subvarieties

Definition 2.3.0.6. [Moo98a, Def. 2.5] An irreducible subvariety Z of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is called *special*, if there exists a connected, reductive algebraic subgroup H of G defined over $\mathbb{Q}$, an element $g \in G(\mathbb{A}_f)$ and a connected component D_H^+ of

$$D_H := \{x \in X | h_x : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}} \text{ factors through } H_{\mathbb{R}}\},$$

such that $Z(\mathbb{C})$ is the image of $D_H^+ \times gK$ in $\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$.

By [Moo98a, 2.4], D_H is a finite union of $H(\mathbb{R})$-conjugacy classes. Let C be the $H(\mathbb{R})$-conjugacy class containing D_H^+. Then (H,C) is a Shimura subdatum of (G,X). From [Del71b, Cor. 5.4] and [Moo98a, Rk. 2.6], the inclusion of a special subvariety into $\mathrm{Sh}_K(G,X)$ is defined over a number field.

Example 2.3.0.7.

1. A complex point $s \in \mathrm{Sh}_K(G,X)$ is a special subvariety, if and only if there is a special point $x \in X$ (in the sense of [Mil17b, Def. 12.5]) and $g \in G(\mathbb{A}_f)$ with $s = [x,g]_K$.

2. When $H = G$, the corresponding special subvarieties of $\mathrm{Sh}_K(G,X)$ are precisely the connected components.

For every $g \in G(\mathbb{A}_f)$, an irreducible subvariety Z of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is special if and only if $T(g)(Z)$ is special in $\mathrm{Sh}_{g^{-1}Kg}(G,X)$. By [Moo98a, Sec. 2.9], an irreducible component of the intersection of a family of special subvarieties of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is again special. Therefore, for a complex irreducible subvariety Y of $\mathrm{Sh}_K(G,X)$, there is a smallest special subvariety Z_Y of $\mathrm{Sh}_K(G,X)$ containing Y. We say that Y is *Hodge generic* in Z_Y. Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ be the Deligne torus.

Definition 2.3.0.8. The *generic Mumford-Tate group* (denoted by $\mathrm{MT}(X)$) of the Shimura datum (G,X) is the smallest closed subgroup H of G over $\mathbb{Q}$, such that every $h : \mathbb{S} \to G_{\mathbb{R}}$ in X factors through $H_{\mathbb{R}}$. If $\mathrm{MT}(X) = G$, then the Shimura datum (G,X) is called *irreducible*.

The subgroup $\mathrm{MT}(X)$ is normal in G, connected and reductive. By [Che09, Def. 1.3.3], $(\mathrm{MT}(X), X)$ is a Shimura subdatum of (G,X). Fact 2.3.0.9 characterizes special subvarieties as Hecke image of irreducible components of a Shimura subvariety. Recall that $K \leq G(\mathbb{A}_f)$ is a *neat*, compact open subgroup. For $g \in G(\mathbb{A}_f)$, the quotient $S_g = \Gamma_g\backslash X^+$ is an irreducible component of $\mathrm{Sh}_K(G,X)$, and $gKg^{-1} \cap H(\mathbb{A}_f)$ is neat.

Fact 2.3.0.9 ([UY10, Lem. 2.7])**.** *Let $(H,X_H) \subset (G,X)$ be an* irreducible *Shimura subdatum. Let X_H^+ be a connected component of X_H contained in X^+. Set $\Gamma_{H,g} = gKg^{-1} \cap H(\mathbb{Q})_+$ and $\tilde{Z}_g := \Gamma_{H,g}\backslash X_H^+$ (an irreducible component of $\mathrm{Sh}_{gKg^{-1}\cap H(\mathbb{A}_f)}(H,X_H)$). Then the image Z_g of $\tilde{Z}_g$ under the morphism over $\mathbb{C}$*

$$\mathrm{Sh}_{gKg^{-1}\cap H(\mathbb{A}_f)}(H,X_H) \to \mathrm{Sh}_K(G,X), \quad [x,h] \mapsto [x,hg]$$

is a special subvariety of S_g. The induced morphism $\pi : \tilde{Z}_g \to Z_g$ is finite and birational. Conversely, every special subvariety of S_g arises in this way.

Remark 2.3.0.10. If the special subvariety Z_g in Fact 2.3.0.9 is normal, then by Zariski's main theorem (see, *e.g.*, [Liu06, Cor. 4.6]), $\pi : \tilde{Z}_g \to Z_g$ is an isomorphism.

2.4 Locus at infinite level

Given an étale cover $f : X \to Y$ over $\bar{\mathbb{Q}}$, the induced map $f : X^L \to Y^L$ may not be surjective. Ullmo and Yafaev introduce a sublocus that lifts along étale covers of Shimura varieties.

Definition 2.4.0.1. For a Shimura variety S, its *locus at infinite level* is

$$S^{L_\infty} := \cap_{f:S'\to S} f(S'^L),$$

where $f : S' \to S$ runs through all étale covers of Shimura varieties over $\bar{\mathbb{Q}}$.

Remark 2.4.0.2. By Assumption 2.2.0.1 3, one has $S^{L_\infty} \subset S^L$. By Assumption 2.2.0.1 1, every irreducible component of S^{L_∞} is *positive dimensional*. For any $g, h \in G(\mathbb{A}_f)$, the Hecke isomorphism $T(g) : [X^+, h]_K \to [X^+, hg]_{g^{-1}Kg}$ identifies the loci at infinite level.

Remark 2.4.0.3. Given an étale cover $f : T \to S$ of a Shimura variety S over $\bar{\mathbb{Q}}$, we don't know whether $f(T^L)$ contains S^{L_∞}. The following example shows that f may not be dominated by any étale cover of Shimura varieties over S.

Let $G = \mathrm{GL}_2$. For every integer $n > 1$, let $K(n)$ be the compact open subgroup of $G(\mathbb{A}_f)$ defined in [Mil17b, Proof of Prop. 4.1]. Then $K(n) \cap G(\mathbb{Q}) = \Gamma(n)$ is a principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $X = \mathcal{H}^{\pm} \subset \mathbb{C}$ be the union of the upper and the lower upper half-planes. Let (G, X) be the Siegel Shimura datum in the sense of [Mil17b, p.69]. Let X^+ be the upper half-plane. Then the modular curve $Y(n) := \Gamma(n)\backslash X^+$ coincides with the Shimura variety $[X^+, 1]_{K(n)}$. By Fact 2.3.0.2 1, there is $n_0 > 1$ such that $K(n)$ is neat for every $n \geq n_0$.

The congruence subgroup problem of $\mathrm{SL}_2(\mathbb{Z})$ has a negative solution. Take a finite-index, noncongruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Replacing Γ with $\Gamma \cap \Gamma(n_0)$, one may assume $\Gamma \subset \Gamma(n_0)$. Let $Y := \Gamma\backslash X^+$. Then the projection $f : Y \to Y(n_0)(\mathbb{C})$ is a finite-sheeted cover of complex manifolds. By [Har77, Appendix B, Thm. 3.2], Y has a unique structure of normal complex algebraic variety making $f : Y \to Y(n_0)$ a finite morphism of schemes. From [SGA 1, XII, Prop. 3.1 (iii)], f is an étale cover over $\mathbb{C}$. By [Ser07, Thm. 6.3.3], f descends to an étale cover over $\bar{\mathbb{Q}}$.

There is no étale cover of Shimura varieties $f' : Y' \to Y(n_0)$ that covers $f : Y \to Y(n_0)$. Assume the contrary. Let K' be the open subgroup of $K(n_0)$ corresponding to f'. As f' factors through a finite-sheeted cover $Y'(\mathbb{C}) \to Y(\mathbb{C})$, taking fundamental groups, one has $K' \cap G(\mathbb{Q}) \subset \Gamma$. By [Mil17b, Prop. 4.1], $K' \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$, which contradicts the choice of Γ.

Let S be a Shimura variety associated with (G, X, K). Let $\mathrm{Sh}_K(G, X)^{L_\infty}$ be the union of the loci at infinite level of its connected components.

Lemma 2.4.0.4. *1. One has $\mathrm{Sh}_K(G, X)^{L_\infty} = \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L)$, where K' runs through the normal open subgroups of K.*

2. There exists an open subgroup K_1 of K, such that every étale cover of Shimura varieties $f_1 : S_1 \to S$ induced by $p_{K_1,K}$ satisfies $f_1(S_1^L) = S^{L_\infty}$.

Proof. 1. For $x \in \mathrm{Sh}_K(G, X)$, one may assume that S is the irreducible component of $\mathrm{Sh}_K(G, X)$ containing x. Assume first $x \in \mathrm{Sh}_K(G, X)^{L_\infty}$. Then $x \in S^{L_\infty}$. For every open normal subgroup K' of K, choose an irreducible component S' of $\mathrm{Sh}_{K'}(G, X)$ with $p_{K',K}(S') = S$. As

$S' \to S$ is an étale cover of Shimura varieties, one has $x \in p_{K',K}(S'^L) \subset p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)$. One obtains $\mathrm{Sh}_K(G,X)^{L_\infty} \subset \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)$. Conversely, assume $x \in \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)$. For every étale cover of Shimura varieties $f : S_1 \to S$, there is an open subgroup K_1 of K, such that S_1 is an irreducible component of $\mathrm{Sh}_{K_1}(G,X)$. There is a normal open subgroup K_2 of K with $K_2 \subset K_1$. Choose an irreducible component S_2 of $\mathrm{Sh}_{K_2}(G,X)$ with $p_{K_2,K_1}(S_2) = S_1$. By assumption, there is an irreducible component S'_2 of $p^{-1}_{K_2,K}(S)$ with $p_{K_2,K_1}(S'^L_2) \ni x$. As $K_2 \leq K$ is normal, K/K_2 permutes the irreducible components of $p^{-1}_{K_2,K}(S)$. Thus, there is $g \in K$ such that $T(g) : \mathrm{Sh}_{K_2}(G,X) \to \mathrm{Sh}_{K_2}(G,X)$ restricts to an isomorphism $S'_2 \to S_2$ over $\bar{\mathbb{Q}}$. By Assumption 2.2.0.1 3, one has $p_{K_2,K_1}(T(g)S'^L_2) \subset S^L_1$. Then $x \in f(p_{K_2,K_1}(T(g)S'^L_2)) \subset f(S^L_1)$. Thus, one has $x \in S^{L_\infty} \subset \mathrm{Sh}_K(G,X)^{L_\infty}$, which shows

$$\mathrm{Sh}_K(G,X)^{L_\infty} \supset \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L).$$

2. The algebraic variety $\mathrm{Sh}_K(G,X)_{\bar{\mathbb{Q}}}$ is topologically Noetherian, so the family of closed subsets $\{p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)\}$ for normal open subgroups K' of K has a minimal member $p_{K_1,K}(\mathrm{Sh}_{K_1}(G,X)^L)$. Let $f_1 : S_1 \to S$ an étale cover of Shimura varieties induced by $p_{K_1,K}$. For every étale cover of Shimura varieties $f_2 : S_2 \to S$ associated with an open subgroup K_2 of K, there is a normal open subgroup $K_{1,2}$ of K with $K_{1,2} \subset K_1 \cap K_2$. Let S'_2 be an irreducible component of $\mathrm{Sh}_{K_{1,2}}(G,X)$ with $p_{K_{1,2},K_2}(S'_2) = S_2$. One has

$$\begin{aligned} f_1(S^L_1) &\subset p_{K_1,K}(\mathrm{Sh}_{K_1}(G,X)^L) \overset{\text{(a)}}{=} p_{K_{1,2},K}(\mathrm{Sh}_{K_{1,2}}(G,X)^L) \\ &\overset{\text{(b)}}{=} p_{K_{1,2},K}(K/K_{1,2} \cdot S'^L_2) = p_{K_{1,2},K}(S'^L_2) \overset{\text{(c)}}{\subset} f_2(S^L_2), \end{aligned}$$

where (a) uses the choices of K_1, (b) is because $K/K_{1,2}$ permutes the irreducible components of $\mathrm{Sh}_{K_{1,2}}(G,X)$, and (c) uses Assumption 2.2.0.1 3. □

In Lemma 2.4.0.4 2, for another étale cover of Shimura varieties $S_2 \to S_1$, the composition $S^L_2 \to S^L_1 \xrightarrow{f_1} S^{L_\infty}$ is still surjective.

Lemma 2.4.0.5. *Let $f : T \to S$ be an étale cover of Shimura varieties over $\bar{\mathbb{Q}}$. Then $f^{-1}(S^{L_\infty}) = T^{L_\infty}$. In particular, $T^{L_\infty} = T$ is equivalent to $S^{L_\infty} = S$, and $S^{L_\infty} = S^L$ implies $T^{L_\infty} = T^L$.*

Proof. There is an open subgroup K_1 of K, such that T is an irreducible component of $\mathrm{Sh}_{K_1}(G,X)$, and $p_{K_1,K} : \mathrm{Sh}_{K_1}(G,X) \to \mathrm{Sh}_K(G,X)$ restricts to $f : T \to S$.

- We show $T^{L_\infty} \subset f^{-1}(S^{L_\infty})$.

Fix $t \in T^{L_\infty}$, and set $s = f(t)$. For every open subgroup K' of K, there is $t' \in \mathrm{Sh}_{K' \cap K_1}(G,X)^L$ with $p_{K' \cap K_1, K_1}(t') = t$. By Assumption 2.2.0.1 3, one has $s' := p_{K' \cap K_1, K'}(t') \in \mathrm{Sh}_{K'}(G,X)^L$. Then $s = p_{K',K}(s') \in p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)$. By Lemma 2.4.0.4 1, one has $s \in \mathrm{Sh}_K(G,X)^{L_\infty}$ and hence $s \in S^{L_\infty}$.

- We show $T^{L_\infty} \supset f^{-1}(S^{L_\infty})$.

Take $t \in f^{-1}(S^{L_\infty})$. Then $s := f(t) \in S^{L_\infty}$. For every étale cover of Shimura varieties $u : Z \to T$ associated with an open subgroup K_2 of K_1, there is a normal open subgroup K_3 of K with $K_3 \subset K_2$. Let N be an irreducible component of $p^{-1}_{K_3,K_2}(Z)$. Then the restriction $v : N \to Z$ is an étale cover of Shimura varieties, and the composition $N \xrightarrow{v} Z \xrightarrow{u} T \xrightarrow{f} S$ is a Galois cover. One has

$$\begin{aligned}(u \circ v)^{-1}(t) &\subset (f \circ u \circ v)^{-1}(s) \subset (f \circ u \circ v)^{-1}(S^{L_\infty}) \\ &\subset (f \circ u \circ v)^{-1}((f \circ u \circ v)(N^L)) \overset{\text{(a)}}{=} N^L,\end{aligned}$$

where (a) uses Lemma 2.2.0.5 3. Thus, one has $u^{-1}(t) \subset v(N^L) \subset Z^L$ and $t \in u(Z^L)$. Hence, one has $t \in T^{L_\infty}$.

- The equality $T^{L_\infty} = T$ is equivalent to $S^{L_\infty} = S$.

If $T^{L_\infty} = T$, then $S^{L_\infty} = f(f^{-1}(S^{L_\infty})) = f(T^{L_\infty}) = f(T) = S$. Conversely, if $S^{L_\infty} = S$, then $T^{L_\infty} = f^{-1}(S^{L_\infty}) = f^{-1}(S) = T$.

- The equality $S^{L_\infty} = S^L$ implies $T^{L_\infty} = f^{-1}(S^{L_\infty}) = f^{-1}(S^L) \overset{\text{(b)}}{\supset} T^L$, where (b) uses Assumption 2.2.0.1 3. Hence, one obtains $T^{L_\infty} = T^L$.

□

Assume $S = [X^+, 1]_K$. Let Z be a special subvariety of S over $\bar{\mathbb{Q}}$. Let $(H, X_H) \subset (G, X)$ be an irreducible Shimura subdatum and $\pi : \tilde{Z} \to Z$ be as in Fact 2.3.0.9.

Lemma 2.4.0.6. *There is a open subgroup K' of K, such that*

1. *the induced étale cover of Shimura varieties $f : S'(:= [X^+, 1]_{K'}) \to S$ is Galois and satisfies $S'^L = S'^{L_\infty}$;*
2. *for every irreducible component Z' of $f^{-1}(Z)$, the restriction $f|_{Z'} : Z' \to Z$ factors through an étale cover $h : Z' \to \tilde{Z}$ with $h(Z'^L) = \tilde{Z}^{L_\infty}$.*

Proof. By Lemma 2.4.0.4 2, there is an open subgroup $K_{0,H} \leq K \cap H(\mathbb{A}_f)$, such that the induced étale cover $g_0 : \tilde{Z}_0(:= [X_H^+, 1]_{K_{0,H}}) \to \tilde{Z}$ satisfies $g_0(\tilde{Z}_0^L) = \tilde{Z}^{L_\infty}$. Similarly, there is an open subgroup $K_1 \leq K$, such that $K_1 \cap H(\mathbb{A}_f) \subset K_{0,H}$ and the étale cover $f_1 : S_1(:= [X^+, 1]_{K_1}) \to S$ satisfies $f_1(S_1^L) = S^{L_\infty}$. By Lemma 2.4.0.5, one has $S_1^L = S_1^{L_\infty}$. From Lemma 2.3.0.3, there is a normal open subgroup K' of K contained in K_1, such that the induced morphism $\mathrm{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K'}(G, X)$ is a closed immersion. Set $S' := [X^+, 1]_{K'}$. Let $f : S' \to S$ be the restriction of $p_{K',K} : \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$.

1. Since K' is normal in K, the cover f is Galois. By Lemma 2.4.0.5 and $S_1^L = S_1^{L_\infty}$, one has $S'^L = S'^{L_\infty}$.

2. The morphism of Shimura varieties $\tilde{Z}_1(:= [X_H^+, 1]_{K' \cap H(\mathbb{A}_f)}) \to S'$ is a closed immersion, so is the induced morphism $\pi' : \tilde{Z}_1 \to f^{-1}(Z)$. Let $h : \tilde{Z}_1 \to \tilde{Z}$ be the restriction of the morphism

$$p_{K' \cap H(\mathbb{A}_f), K \cap H(\mathbb{A}_f)} : \mathrm{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H).$$

   Then h is an étale cover of Shimura varieties. The situation is depicted as a commutative diagram

$$\begin{tikzcd}
\tilde{Z}_1 \arrow[r, hook, "\pi'"] \arrow[d] \arrow[dd, bend right, "h"'] & f^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \arrow[dd, bend left, "f"] \\
\tilde{Z}_0 \arrow[r] \arrow[d] & f_1^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S_1 \arrow[d, "f_1"'] \\
\tilde{Z} \arrow[r, two heads, "\pi"] & Z \arrow[r, hook] & S.
\end{tikzcd}$$

   Since h factors through $\tilde{Z}_0$, one has $h(\tilde{Z}_1^L) = \tilde{Z}^{L_\infty}$.

   Claim 2.4.0.7. The closed immersion π' identifies $\tilde{Z}_1$ with an irreducible component of $f^{-1}(Z)$.

   The Galois group of the Galois cover $f : S' \to S$ permutes the irreducible components of $f^{-1}(Z)$, so Z' has similar properties.

□

Proof of Claim 2.4.0.7. Since $\tilde{Z}_1$ is irreducible, it is contained in an irreducible component C of $f^{-1}(Z)$. By [Sta24, Tag 0BAC], as $\pi : \tilde{Z} \to Z$ is birational, there is a nonempty open subset $\tilde{U} \subset \tilde{Z}$, such that $U := \pi(\tilde{U})$ is open in Z, and $\pi|_{\tilde{U}} : \tilde{U} \to U$ is an isomorphism. Consider the commutative square of algebraic varieties

$$\begin{array}{ccc} h^{-1}(\tilde{U}) & \xrightarrow{\pi'|_{h^{-1}(\tilde{U})}} & f^{-1}(U) \\ {\scriptstyle h|_{h^{-1}(\tilde{U})}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{U} & \xrightarrow{\pi|_{\tilde{U}}} & U. \end{array}$$

The morphism $h^{-1}(\tilde{U}) \to \tilde{U}$ (resp. $f^{-1}(U) \to U$) is a base change of the étale morphism $h : \tilde{Z}_1 \to \tilde{Z}$ (resp. $f : S' \to S$), so it is étale. By [Sta24, Tag 03PC (10)], the morphism $\pi'|_{h^{-1}(\tilde{U})} : h^{-1}(\tilde{U}) \to f^{-1}(U)$ is étale. From [Sta24, Tag 03PC (9)], $h^{-1}(\tilde{U})$ is an open subset of $f^{-1}(U)$, hence a nonempty open subset of C. Since C is irreducible, $h^{-1}(\tilde{U})$ is dense in C. Therefore, $C = \tilde{Z}_1$. □

Let $S = S_g$ be a Shimura variety associated with (G, X, K). Lemma 2.4.0.8 is used in the proof of Theorem 2.5.0.1.

Lemma 2.4.0.8. *If $S^{L_\infty} \neq \emptyset$ is a finite union of special subvarieties of S, then $S^L = S$.*

Proof. The Hecke isomorphism $T(g) : \mathrm{Sh}_{gKg^{-1}}(G, X) \to \mathrm{Sh}_K(G, X)$ sends $[X^+, 1]_{gKg^{-1}}$ to S_g. It keeps the special subvarieties. Thus, one may assume $g = 1$ (by replacing K with gKg^{-1}). Write $S^{L_\infty} = \cup_{i=1}^n Z_i$ for the irreducible decomposition. By assumption, for every $1 \leq i \leq n$, the subvariety $Z_i \subset S$ is special. Let $\pi_i : \tilde{Z}_i \to Z_i$ be a finite birational morphism given by Fact 2.3.0.9.

Claim 2.4.0.9. One has $\tilde{Z}_1^{L_\infty} = \tilde{Z}_1$.

For every $q \in G(\mathbb{Q})_+$, we prove

$$T_q Z_1 \subset S^L. \tag{2.1}$$

Let $(H, X_H) \subset (G, X)$ be an irreducible Shimura subdatum, $\tilde{Z}_1 = [X_H^+, 1]_{K \cap H(\mathbb{A}_f)}$, and $\pi_1 : \tilde{Z}_1 \to Z_1$ be as in Fact 2.3.0.9. For every irreducible component $Z_q \subset \alpha_q^{-1}(Z_1)$, there is an irreducible component $\tilde{Z}_q$ of $\mathrm{Sh}_{K \cap q^{-1}Kq \cap H(\mathbb{A}_f)}(H, X_H)$ with the following properties:

- The morphism $\mathrm{Sh}_{K \cap q^{-1}Kq \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$ restricts to an étale cover of Shimura varieties $\alpha'_q : \tilde{Z}_q \to \tilde{Z}_1$.

- The image of $\tilde{Z}_q$ under the morphism

$$\mathrm{Sh}_{K \cap q^{-1}Kq \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K \cap q^{-1}Kq}(G, X)$$

  is Z_q.

Conjugating by q gives another irreducible Shimura subdatum $(qHq^{-1}, q \cdot X_H) \subset (G,X)$, and a morphism of Shimura data $(H, X_H) \to (qHq^{-1}, q \cdot X_H)$. Let $\tilde{Z}'_q$ be the image of $\tilde{Z}_q$ under the induced morphism

$$\mathrm{Sh}_{K\cap q^{-1}Kq\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1}, q \cdot X_H).$$

Then $\tilde{Z}'_q$ is an irreducible component of $\mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1}, q\cdot X_H)$, and the restriction $\beta'_q : \tilde{Z}_q \to \tilde{Z}'_q$ is an étale cover of Shimura varieties. By Fact 2.3.0.9, the morphism $\mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1}, q\cdot X_H) \to \mathrm{Sh}_K(G,X)$ restricts to a finite birational morphism $\pi'_q : \tilde{Z}'_q \to \beta_q(Z_q)$. Consider the commutative diagram

$$\begin{tikzcd}
\tilde{Z}_1 \arrow[d, two heads, "\pi_1"'] & \tilde{Z}_q \arrow[l, "\alpha'_q"'] \arrow[r, "\beta'_q"] \arrow[d, two heads] & \tilde{Z}'_q \arrow[d, "\pi'_q"] \\
Z_1 & Z_q \arrow[l, "\alpha_q"] \arrow[r, "\beta_q"'] & \beta_q(Z_q).
\end{tikzcd}$$

From Claim 2.4.0.9 and Lemma 2.4.0.5, one has $\tilde{Z}'_q = \tilde{Z}'^{L_\infty}_q = \tilde{Z}'^{L}_q$. Then $\beta_q(Z_q) = \pi'_q(\tilde{Z}'_q) = \pi'_q(\tilde{Z}'^{L}_q) \overset{(a)}{\subset} \beta_q(Z_q)^L$, where (a) uses Assumption 2.2.0.1 5. By Assumption 2.2.0.1 2, one has $\beta_q(Z_q) \subset S^L$, which proves (2.1).

From [KUY18, Lem. 2.5], the special subvariety Z_1 of S contains a special point z. By [LZ19, Rk. 2.7], $\{T_q z\}_{q\in G(\mathbb{Q})_+}$ is dense in the complex manifold $S(\mathbb{C})$. By (2.1), the algebraic Zariski closed subset $S^L \subset S$ contains $\{T_q z\}_{q\in G(\mathbb{Q})_+}$, which implies $S^L = S$. □

Proof of Claim 2.4.0.9. First, we prove

$$S^{L_\infty} \subset \cup_{i=1}^n \pi_i(\tilde{Z}_i^{L_\infty}). \tag{2.2}$$

Let K_i be a normal open subgroup of K given by Lemma 2.4.0.6 for π_i. There is a normal open subgroup K' of K with $K' \subset \cap_{i=1}^n K_i$. Set $S_i := [X^+, 1]_{K_i}$ and $S' := [X^+, 1]_{K'}$. Let $g_i : S' \to S_i$, $f_i : S_i \to S$, and $f : S' \to S$ be the induced Galois cover of Shimura varieties. Then for every $1 \le i \le n$, one has $f_i g_i = f$. By construction of $f : S' \to S$, one has $S'^L = S'^{L_\infty}$ and $f(S'^L) = S^{L_\infty}$.

For every irreducible component $C \subset S'^L$, the subset $f(C)$ of S^{L_∞} is irreducible. Then there is $1 \le i \le n$ with $f(C) \subset Z_i$. Thus, $g_i(C)$ is an irreducible subset of $f_i^{-1}(Z_i)$. There is an irreducible component Z' of $f_i^{-1}(Z_i)$ containing $g_i(C)$. By Lemma 2.4.0.6, the morphism $f_i|_{Z'} : Z' \to Z_i$ factors through an étale cover $Z' \to \tilde{Z}_i$. Therefore, Z' and $g_i^{-1}(Z')$ are smooth. One has

$$g_i^{-1}(Z') \subset f^{-1}(Z_i) \subset f^{-1}(S^{L_\infty}) \overset{(a)}{=} S'^L,$$

where (a) uses Lemma 2.4.0.5. Then C is an irreducible component of $g_i^{-1}(Z')$. As $g_i^{-1}(Z')$ is smooth, $g_i|_C : C \to Z'$ is an étale cover. The setting is summarized in the diagram

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[dr] & g_i^{-1}(Z') \arrow[rr, hook] \arrow[d] & & S' \arrow[d, "g_i"] \\
& Z' \arrow[r, hook] \arrow[d, dashed] & f_i^{-1}(Z_i) \arrow[r, hook] \arrow[d] \arrow[ur, phantom, "\square"] & S_i \arrow[d, "f_i"] \arrow[dl, phantom, "\square"] \\
& \tilde{Z}_i \arrow[r, two heads, "\pi_i"] & Z_i \arrow[r, hook] & S.
\end{tikzcd}$$

One has

$$f(C) \overset{(a)}{=} f(C^L) = f_i g_i(C^L) \overset{(b)}{\subset} f_i(Z'^L) \overset{(c)}{\subset} \pi_i(\tilde{Z}_i^{L_\infty}),$$

where (a), (b) and (c) use Lemma 2.2.0.5 2, Assumption 2.2.0.1 3 and Lemma 2.4.0.6 2 respectively. This proves (2.2).

From (2.2), one has $Z_1 \subset \cup_{i=1}^n \pi_i(\tilde{Z}_i^{L_\infty})$. Since Z_1 is irreducible, there is $1 \le j \le n$ with $Z_1 \subset \pi_j(\tilde{Z}_j^{L_\infty}) \subset Z_j$. As Z_1 is an irreducible component of S^{L_∞}, one has $j = 1$. Then $\dim \tilde{Z}_1^{L_\infty} \ge \dim Z_1 = \dim \tilde{Z}_1$. The irreducibility of $\tilde{Z}_1$ proves $\tilde{Z}_1^{L_\infty} = \tilde{Z}_1$. □

2.5 Ullmo-Yafaev alternative principle

Let (G, X) be a Shimura datum. Let $(\cdot)^L$ be a locus formation satisfying Assumption 2.2.0.1. Let $(\cdot)^{L_\infty}$ be the corresponding locus formation at infinite level for Shimura varieties. Theorem 2.5.0.1 is an alternative principle due to Ullmo and Yafaev.

Theorem 2.5.0.1. *Let S be a Shimura variety associated with (G, X, K). Then either $S^{L_\infty} = \emptyset$ or $S^{L_\infty} = S$.*

Proof. By Hecke isomorphisms, one may assume $S = [X^+, 1]_K$. By Lemma 2.4.0.5, one may replace S by an étale cover induced by an open subgroup of K. One may thereby assume $S^L = S^{L_\infty} \neq \emptyset$. By Assumption 2.2.0.1 1 (resp. Lemma 2.2.0.5 2), for every irreducible component Z of S^L, one has $\dim Z > 0$ (resp. $Z^L = Z$).

Claim 2.5.0.2. The subvariety $Z \subset S$ is special.

By Claim 2.5.0.2, the locus S^L is a finite union of special subvarieties. From Lemma 2.4.0.8, one has $S^{L_\infty} = S$. □

Proof of Claim 2.5.0.2. Let $S_M \subset S$ be the smallest special subvariety containing Z. From Fact 2.3.0.9, there is a Shimura subdatum $(H, X_H) \subset (G, X)$, such that the restriction $\pi : \tilde{S}_M(:= [X_H^+, 1]_{K\cap H(\mathbb{A}_f)}) \to S_M$ of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G, X)$ is finite birational.

Take a Galois cover $f : S' \to S$ given by Lemma 2.4.0.6 for the special subvariety $S_M \subset S$. Since f is finite surjective, there is an irreducible component $T \subset f^{-1}(Z)$ with $f(T) = Z$. Since $Z \subset S^L$ is an irreducible component, T is an irreducible component of

$$f^{-1}(S^L) = f^{-1}(S^{L_\infty}) \overset{\text{(a)}}{=} S'^{L_\infty} \overset{\text{(b)}}{=} S'^L.$$

Here (a) and (b) use Lemma 2.4.0.5. There is an irreducible component S'_M of $f^{-1}(S_M)$ containing T.

By Lemma 2.4.0.6, $f : S'_M \to S_M$ factors through an étale cover $h : S'_M \to \tilde{S}_M$. Consider the commutative diagram

$$\begin{tikzcd} T \arrow[r, hook] & S'_M \arrow[r, hook] \arrow[d, dashed, "h"] & f^{-1}(S_M) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d, "f"] \\ & \tilde{S}_M \arrow[r, "\pi"] & S_M \arrow[r, hook] & S. \end{tikzcd}$$

One has

$$h(T) \overset{\text{(a)}}{=} h(T^L) \subset h(S'^L_M) \overset{\text{(b)}}{=} \tilde{S}_M^{L_\infty}, \tag{2.3}$$

where (a) and (b) uses Lemma 2.2.0.5 2 and Lemma 2.4.0.6 2 respectively.

We show that the nonempty, irreducible, closed subset $h(T)$ is Hodge generic in $\tilde{S}_M$. Since $\pi : \tilde{S}_M \to S_M$ is finite surjective, there is an irreducible component $\tilde{Z}$ of $\pi^{-1}(Z)$ with $\pi(\tilde{Z}) = Z$. By [KY14, p.879], for every special subvariety V of $\tilde{S}_M$ containing $h(T)$, the image $\pi(V)$ is a special subvariety of S containing $\pi h(T) = f(T) = Z$. Hence, one has $\pi(V) = S_M$. Therefore, $\dim V \geq \dim S_M = \dim \tilde{S}_M$. Since $\tilde{S}_M$ is irreducible, one has $V = \tilde{S}_M$.

Then by (2.3) and Lemma 2.5.0.3, one has $\tilde{S}_M = \tilde{S}_M^{L_\infty} = \tilde{S}_M^L$. As a result, $S_M = \pi(\tilde{S}_M) = \pi(\tilde{S}_M^L) \overset{\text{(c)}}{\subset} S_M^L \subset S^L$, where (c) uses Assumption 2.2.0.1 5. Since Z is an irreducible component of S^L and S_M is irreducible, one has $Z = S_M$. $\square$

Lemma 2.5.0.3. *Let $S = [X^+, 1]_K \subset \mathrm{Sh}_K(G, X)$. If S^{L_∞} contains a nonempty irreducible closed subset that is Hodge generic in S, then $S^{L_\infty} = S$.*

Proof. By Lemma 2.4.0.5, for every $q \in G(\mathbb{Q})^+$, one has $T_q S^{L_\infty} = \beta_q(\alpha_q^{-1}(S^{L_\infty})) = \beta_q(S_q^{L_\infty}) = S^{L_\infty}$. Write $S^{L_\infty} = U_1 \cup U_2$, where U_1 is the union of irreducible components of S^{L_∞} that are Hodge generic in S, and U_2 is the union of the remaining irreducible components. By Remark 2.4.0.2 and assumption, one has $\dim U_1 > 0$.

Let C be an irreducible component of $T_q U_2$. There is an irreducible subvariety $C_q \subset S_q$ with $\beta_q(C_q) = C$ and $\alpha_q(C_q) \subset U_2$. Then $\alpha_q(C_q)$ is not Hodge generic in S. Thus, there is a strict, special subvariety V of S containing $\alpha_q(C_q)$. Then $C \subset T_q(\alpha_q(C_q)) \subset T_qV$. There is an irreducible component W of T_qV containing C. By [LZ19, Remark 2.7], W is a special subvariety of S. Since $\dim W \leq \dim V < \dim S$, the subvariety C of S is not Hodge generic. As every irreducible component of T_qU_2 is not Hodge generic in S, by $U_1 \subset T_qS^{L_\infty} = T_qU_1 \cup T_qU_2$, one has $U_1 \subset T_qU_1$. By $\dim U_1 > 0$ and [UY10, Thm. 1.2], one has $U_1 = S$ and $S^{L_\infty} = S$. □

Corollary 2.5.0.4. *There is a compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup K of K_0 and every Shimura variety S associated with (G, X, K), either $S^L = \emptyset$ or $S^L = S$.*

Proof. By Lemmas 2.4.0.4 2 and 2.4.0.5, there is a neat compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup $K \leq K_0$, one has $S^L = S^{L_\infty}$. The result follows from Theorem 2.5.0.1. □

2.6 "All or nothing" principle for integral points

We define an locus concerning integral points, analogous to the Lang locus $(\cdot)^{\mathrm{UY}}$ concerning rational points. For this locus, we verify Assumption 2.2.0.1. Then an alternative principle follows.

Let X be an integral algebraic variety over $\bar{\mathbb{Q}}$. As in Example 2.2.0.2, there is a number field $F \subset \bar{\mathbb{Q}}$, an integral algebraic variety X_F over F and an isomorphism $X_F \otimes_F \bar{\mathbb{Q}} \to X$ over $\bar{\mathbb{Q}}$. For every finite set Σ of places of F including all archimedean ones, let $O_{F,\Sigma}$ be the ring of Σ-integers. When Σ is sufficiently large, there exists an integral scheme $\mathcal{X}$ and a dominant, separated morphism $\mathcal{X} \to \operatorname{Spec} O_{F,\Sigma}$ of finite type, whose generic fiber is X_F. (From [Har77, III, Prop. 9.7], $\mathcal{X}$ is flat over $O_{F,\Sigma}$.) We call $\mathcal{X}$ an *integral model* for X relative to (F, Σ). By a finite extension $(M, \Omega)/(F, \Sigma)$, we mean a finite extension M/F inside $\bar{\mathbb{Q}}$ together with a finite set Ω of places of M containing all the places above Σ. For every $(M, \Omega)/(F, \Sigma)$, let $X(\mathcal{X}, M, \Omega)$ be the image of the injection

$$\mathcal{X}(O_{M,\Omega}) \to X(\bar{\mathbb{Q}}), \quad x \mapsto x|_{\operatorname{Spec} \bar{\mathbb{Q}}}.$$

Definition 2.6.0.1. Let $\mathcal{X}^I$ be the Zariski closure of

$$\cup_{(M,\Omega)/(F,\Sigma)} \overline{X(\mathcal{X}, M, \Omega)}^{>0}$$

inside X, where (M, Ω) runs though all finite extensions of (F, Σ). We call $\mathcal{X}^I$ the *integral Lang locus* of X relative to the integral model $(\mathcal{X}, F, \Sigma)$.

The integral Lang locus $\mathcal{X}^I$ is a subvariety of the Lang locus X^{UY} of X.

Lemma 2.6.0.2. *Given two integral models $\mathcal{X}_i$ over O_{F_i,Σ_i} $(i = 1, 2)$ for X, one has $\mathcal{X}_1^I = \mathcal{X}_2^I$.*

Proof. For $i = 1, 2$, let $\phi_i : X_{F_i} \otimes_{F_i} \bar{\mathbb{Q}} \to X$ be the isomorphism in the corresponding model. By [EGA IV 3, Cor. 8.8.2.5], there is a common finite extension (F_3, Σ_3) of (F_1, Σ_1) and (F_2, Σ_2), such that there is an O_{F_3,Σ_3}-isomorphism

$$\mathcal{X}_1 \otimes_{O_{F_1,\Sigma_1}} O_{F_3,\Sigma_3} \to \mathcal{X}_2 \otimes_{O_{F_2,\Sigma_2}} O_{F_3,\Sigma_3}$$

extending the isomorphism $\phi_2^{-1}\phi_1 : X_{F_1} \otimes_{F_1} \bar{\mathbb{Q}} \to X_{F_2} \otimes_{F_2} \bar{\mathbb{Q}}$. For every finite extension $(M_1, \Omega_1)/(F_1, \Sigma_1)$, there is a common finite extension (M_2, Ω_2) of (F_3, Σ_3) and (M_1, Ω_1). Then

$$\mathcal{X}_1(O_{M_1,\Omega_1}) \subset \mathcal{X}_1(O_{M_2,\Omega_2}) = \mathcal{X}_2(O_{M_2,\Omega_2}),$$

so $X(\mathcal{X}_1, M_1, \Omega_1) \subset X(\mathcal{X}_2, M_2, \Omega_2)$. Therefore,

$$\overline{X(\mathcal{X}_1, M_1, \Omega_1)}^{>0} \subset \overline{X(\mathcal{X}_2, M_2, \Omega_2)}^{>0} \subset \mathcal{X}_2^I.$$

Hence $\mathcal{X}_1^I \subset \mathcal{X}_2^I$. The other inclusion follows by symmetry. □

By Lemma 2.6.0.2, one may use the notation X^I for $\mathcal{X}^I$ and call it the *integral Lang locus* of X. We extend the definition to reducible algebraic varieties as in Section 2.2.

Remark 2.6.0.3. Assume that X is proper over $\bar{\mathbb{Q}}$. Then there is an integral model $(\mathcal{X}, F, \Sigma)$ for X, such that $\mathcal{X}$ is proper over $O_{F,\Sigma}$. Then by [Poo17, Thm. 3.2.13 (ii)], X^I coincides with the Lang locus X^{UY}.

Definition 2.6.0.4. [Ull04, Déf. 2.3] An integral algebraic variety X over $\bar{\mathbb{Q}}$ is *arithmetically hyperbolic* if $X^I = \emptyset$.

An integral algebraic variety X over $\bar{\mathbb{Q}}$ is arithmetically hyperbolic if and only if for one (hence for every by Lemma 2.6.0.2) integral model $(\mathcal{X}, F, \Sigma)$, the set of integral points $\mathcal{X}(O_{M,\Omega})$ is finite for every finite extension $(M, \Omega)/(F, \Sigma)$. Thus, [Ull04, Lem. 2.4] follows from Lemma 2.6.0.2.

Example 2.6.0.5. Let $X = \mathbf{P}^1 \setminus \{0, 1, \infty\} = Y(2)$ be a modular curve over $\bar{\mathbb{Q}}$. Its Baily-Borel compactification is $X^* = \mathbf{P}^1$, and $X^{\mathrm{UY}} = X$. By the Siegel-Mahler theorem (see, *e.g.*, [HS00, Thm. D.8.1]), X is arithmetically hyperbolic.

A complex analytic space is called *Kobayashi hyperbolic*, if its Kobayashi pseudo-distance (in the sense of [Kob98, p.50]) is a metric. Every Kobayashi hyperbolic, complex analytic space is Brody hyperbolic. Conversely, Brody [Bro78, p.213] proves that every compact, Brody hyperbolic complex analytic space is Kobayashi hyperbolic. In view of Remark 2.6.0.3, Conjecture 2.6.0.6 implies Conjecture 2.1.0.1.

Conjecture 2.6.0.6 ([Lan91, IX, Conjecture 5.1], [Ull04, Conjecture 2.5])**.** *Let X be a quasi-projective, integral algebraic variety over $\bar{\mathbb{Q}}$. If the complex analytic space $X(\mathbb{C})$ is Kobayashi hyperbolic, then X is arithmetically hyperbolic.*

Fact 2.6.0.7 is an evidence of Conjecture 2.6.0.6. It relies on Faltings's solution [Fal83, Satz 6] to Shafarevich's conjecture.

Fact 2.6.0.7 ([Ull04, Thm. 3.2 (a)])**.** *Let (G, X) be an adjoint Shimura datum of abelian type* (in the sense of [Ull04, p.4118])*. Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Then every irreducible component of $\mathrm{Sh}_K(G, X)_{\bar{\mathbb{Q}}}$ is arithmetically hyperbolic.*

Remark 2.6.0.8. By [Moo98b, 2.17], the model over $\bar{\mathbb{Q}}$ of $\mathrm{Sh}_K(G, X)$ defined by Faltings [Fal82] (used in [Ull04, Thm. 3.2 (a)]) is the scalar extension of the canonical model along the field embedding $E(G, X) \to \bar{\mathbb{Q}}$.

We prove that an alternative principle holds for integral points on Shimura varieties, by checking Assumption 2.2.0.1. Since an irreducible component of X^I with dimension 0 would be an isolated point, Assumption 2.2.0.1 1 holds. Lemma 2.6.0.9 verifies Assumptions 2.2.0.1 2, 3 and 5.

Lemma 2.6.0.9. *Let $f : Z_1 \to Z_2$ be a morphism of integral algebraic varieties over $\bar{\mathbb{Q}}$. If f has finite geometric fibers, then $f(Z_1^I) \subset Z_2^I$.*

Proof. One may choose a number field F, a finite set Σ of places of F containing all the archimedean ones, an integral model $\mathcal{Z}_i$ over $O_{F,\Sigma}$ for Z_i ($i = 1, 2$) and a morphism $f' : \mathcal{Z}_1 \to \mathcal{Z}_2$ over $O_{F,\Sigma}$ extending f. For every finite extension $(M, \Omega)/(F, \Sigma)$, one has $f'(\mathcal{Z}_1(O_{M,\Omega})) \subset \mathcal{Z}_2(O_{M,\Omega})$. Hence, one obtains

$$f(Z_1(\mathcal{Z}_1, M, \Omega)) \subset Z_2(\mathcal{Z}_2, M, \Omega), \quad f(\overline{Z_1(\mathcal{Z}_1, M, \Omega)}) \subset \overline{Z_2(\mathcal{Z}_2, M, \Omega)}.$$

Let $C \subset \overline{Z_1(\mathcal{Z}_1, M, \Omega)}$ be an irreducible component of positive dimension. Then $f(C)$ is irreducible but not a singleton. (For otherwise, C is a finite set by assumption, which is a contradiction). Hence

$$f(C) \subset \overline{Z_2(\mathcal{Z}_2, M, \Omega)}^{>0} \subset Z_2^I.$$

Therefore, $f(\overline{Z_1(\mathcal{Z}_1, M, \Omega)}^{>0}) \subset Z_2^I$ and $f(Z_1^I) \subset Z_2^I$. □

Corollary 2.6.0.10 ([Ull04, Prop. 2.6])**.** *A locally closed subvariety of an arithmetically hyperbolic variety is also arithmetically hyperbolic.*

Proof. It follows from Lemma 2.6.0.9. □

Lemma 2.6.0.11 verifies Assumption 2.2.0.1 4 for integral Lang locus.

Lemma 2.6.0.11. *Let X be an integral algebraic variety over $\bar{\mathbb{Q}}$. Then $X^I \subset (X^I)^I$.*

Proof. Write $X^I = \cup_{i=1}^n Y_i$ for the irreducible decomposition. Take an integral model $(\mathcal{X}, F, \Sigma)$ for X. Let $\mathcal{Y}_i$ be the scheme-theoretic image of the composition $Y_i \to X \to \mathcal{X}$. Then $\mathcal{Y}_i$ is an integral model of Y_i relative to (F, Σ). For every finite extension $(M, \Omega)/(F, \Sigma)$, the Zariski closed subset $\overline{X(\mathcal{X}, M, \Omega)}$ of X is the disjoint union of $\overline{X(\mathcal{X}, M, \Omega)}^{>0}$ with a finite subset $\{p_1, \ldots, p_t\}$ of $X(\bar{\mathbb{Q}})$.

Consider $x \in \mathcal{X}(O_{M,\Omega})$, *i.e.*, a section $x : \mathrm{Spec}(O_{M,\Omega}) \to \mathcal{X}$ to the structure morphism $\mathcal{X} \to \mathrm{Spec}(O_{M,\Omega})$. If $x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \notin \{p_1, \ldots, p_t\}$, then

$$x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \in \overline{X(\mathcal{X}, M, \Omega)}^{>0} \subset X^I.$$

Thus, there exists an index $1 \leq i \leq n$ with $x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \in Y_i$. Since $\mathcal{Y}_i$ is Zariski closed in $\mathcal{X}$, the section x factors through $\mathcal{Y}_i$, *i.e.*, $x \in \mathcal{Y}_i(O_{M,\Omega})$. Therefore,

$$X(\mathcal{X}, M, \Omega) \subset \cup_{i=1}^n Y_i(\mathcal{Y}_i, M, \Omega) \cup \{p_1, \ldots, p_t\}.$$

Then

$$\overline{X(\mathcal{X}, M, \Omega)}^{>0} \subset \cup_{i=1}^n \overline{Y_i(\mathcal{Y}_i, M, \Omega)}^{>0} \subset \cup_{i=1}^n Y_i^I = (X^I)^I,$$

so $X^I \subset (X^I)^I$. $\square$

Lemma 2.6.0.12 implies [Ull04, Prop. 2.8].

Lemma 2.6.0.12 (Chevalley-Weil)**.** *If $f : X \to Y$ is an étale cover over $\bar{\mathbb{Q}}$, then $f(X^I) = Y^I$. In particular, $X^I = X$ (resp. $X^I = \emptyset$) is equivalent to $Y^I = Y$ (resp. $Y^I = \emptyset$).*

Proof. By Lemma 2.6.0.9, one has $f(X^I) \subset Y^I$. There is a number field F, a finite set Σ of places of F containing all the archimedean ones, and a finite étale morphism $f' : \mathcal{X} \to \mathcal{Y}$ between integral models over $O_{F,\Sigma}$ extending f.

From the Chevalley-Weil theorem (see, *e.g.*, [Ser97, p.50]), for every finite extension $(M, \Omega)/(F, \Sigma)$, there is a finite extension $(M', \Omega')/(M, \Omega)$ with $Y(\mathcal{Y}, M, \Omega) \subset f(X(\mathcal{X}, M', \Omega'))$. Since zero dimensional schemes are discrete, one has

$$\overline{Y(\mathcal{Y}, M, \Omega)}^{>0} \subset f(\overline{X(\mathcal{X}, M', \Omega')}^{>0}) \subset f(X^I).$$

Hence $Y^I \subset f(X^I)$.

If $X^I = X$, then $Y = f(X) = f(X^I) = Y^I$. Conversely, if $Y^I = Y$, then $\dim X = \dim Y = \dim Y^I \leq \dim X^I$. Since X is irreducible, one has $X^I = X$.

If $X^I = \emptyset$, then $Y^I = f(X^I) = \emptyset$. Conversely, if $Y^I = \emptyset$, then by Assumption 2.2.0.1 3, one has $X^I = \emptyset$. $\square$

Theorem 2.6.0.13. *The integral Lang locus of a Shimura variety is either empty or full.*

Proof. Take the locus formation $(\cdot)^L := (\cdot)^I$ to be that of integral Lang locus. By Lemmas 2.6.0.9 and 2.6.0.11, it satisfies Assumption 2.2.0.1. By Lemma 2.6.0.12, one has $S^I = S^{L_\infty}$. The result follows from Theorem 2.5.0.1. $\square$

Chapter 3

Normality of monodromy group in generic Tannakian group

3.1 Introduction

3.1.1 Background

Constructing local systems (or ℓ-adic lisse sheaves) with a prescribed monodromy group is an important problem having a long history.

In positive characteristics, Katz and his collaborators exhibit local systems whose monodromy groups are the simple algebraic group G_2 ([Kat88, 11.8]), $2.J_2$ ([KRL19]), the finite symplectic groups ([KT19b]), the special unitary groups ([KT19a]), *etc.* In particular, the exceptional Lie groups appear unexpectedly in algebraic geometry.

In characteristic zero, such constructions help to understand Galois groups of number fields. Dettweiler and Reiter [DR10] prove the existence of a local system on $\mathbf{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ whose monodromy group is G_2. It produces a motivic Galois representations with image dense in G_2. Their proof relies on Katz's middle convolution of perverse sheaves. Yun [Yun14] constructs local systems with some other exceptional groups as monodromy groups. As applications, he answers a long standing question of Serre, and solves new cases of the inverse Galois problem. His construction uses the geometric Langlands correspondence.

A new proof of Mordell's conjecture [LV20], and its potential generalization to higher dimensional varieties over number fields, rely on the existence of local systems with big monodromy over the variety in question. Lawrence and Sawin [LS20] use this technique to prove Shafarevich's conjecture for hypersurfaces in abelian varieties. Krämer and Maculan [KM23] apply roughly the same strategy to obtain an arithmetic finiteness result for very irregular varieties of dimension less than half the dimension of their Albanese variety. In both cases, the construction of local systems uses perverse sheaves.

In [LS20], that construction rests on comparing the monodromy group with the Tannakian group from Krämer-Weissauer's convolution theory [KW15b]. As [JKLM23, p.4] comments, this comparison is similar to the study of monodromy groups *via* Mumford-Tate groups in [And92].

We briefly outline their argument. On an abelian variety A, a quotient of the abelian category $\mathrm{Perv}(A)$ (of perverse sheaves) is a Tannakian category under sheaf convolution. Let X be an irreducible algebraic variety with generic point η. Let K be a universally locally acyclic, relative perverse sheaf on the constant abelian scheme $p_X : A \times X \to X$ (intuitively, a family of perverse shaves on A parameterized by X). The Tannakian group $G(K|_{A_{\bar{\eta}}})$ of $K|_{A_{\bar{\eta}}} \in \mathrm{Perv}(A_{\bar{\eta}})$ is normal in the Tannakian group $G(K|_{A_\eta})$ of $K|_{A_\eta} \in \mathrm{Perv}(A_\eta)$ ([LS20, Lem. 3.7], [JKLM23, Thm. 4.3]). This normality is used to prove that for most character sheaves L_χ on A, the monodromy groups $\mathrm{Mon}(K \otimes p_A^* L_\chi)$ in the sense of Definition 3.4.3.4 contain $G(K|_{A_{\bar{\eta}}})$. Then Lawrence-Venkatesh's machinery works for these twists $K \otimes p_A^* L_\chi$.

3.1.2 Statements

In the main result (Theorem 3.1.2.2), we prove that the generic Tannakian group of a semisimple, relative perverse sheaf is reductive. Moreover, for many characters, the monodromy group is a normal subgroup of this reductive group. This normality puts further restriction on the monodromy group. Using Krämer's method ([Krä22, Thm. 6.2.1]), Lawrence and Sawin [LS20, Lem. 4.6] even show that the geometric generic Tannakian group is simple.

Setting 3.1.2.1. Let k be an algebraically closed field of characteristic zero. Let X be an integral algebraic variety over k with generic point η. Let A be an abelian variety over k. Denote by $p_X : A \times X \to X$ and $p_A : A \times X \to A$ the projections.

Let ℓ be a prime number. Let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Let $D_c^b(A \times X)$ be the triangulated category of bounded constructible $\bar{\mathbb{Q}}_\ell$-sheaves on $A \times X$. Let $\pi_1^{\text{ét}}(A)$ be the étale fundamental group of A based at the geometric origin point. For every character $\chi : \pi_1^{\text{ét}}(A) \to \bar{\mathbb{Q}}_\ell^\times$, let $\chi_\eta : \pi_1^{\text{ét}}(A_\eta) \to \bar{\mathbb{Q}}_\ell^\times$ be the pullback of χ along $(p_A|_{A_\eta}) : \pi_1^{\text{ét}}(A_\eta) \to \pi_1^{\text{ét}}(A)$. Fix a relative perverse sheaves (Definition 3.2.3.2) K for $p_X : A \times X \to X$ that is a semisimple object of $D_c^b(A \times X)$ (in the sense of Definition 3.2.1.3). Let $\mathrm{Mon}(K, \chi_\eta)$ be the corresponding monodromy group. Let $G_{\omega_\chi}(K|_{A_\eta})$ be the Tannakian monodromy group (Definition 3.4.2.1) of $K|_{A_\eta}$, referred to as the *generic Tannakian group.*

Theorem 3.1.2.2 (Theorems 3.5.1.1, 3.5.3.1)**.** *Assume* $\dim A > 0$*. Then there are uncountably many characters* $\chi : \pi_1^{\acute{e}t}(A) \to \bar{\mathbb{Q}}_\ell^\times$*, such that* $G_{\omega_\chi}(K|_{A_\eta})$

is a well-defined reductive group. It contains $\mathrm{Mon}(K, \chi_\eta)$ *as a closed, reductive,* normal *subgroup.*

The line of the proof of Theorem 3.1.2.2 is similar to that of André's normality theorem [And92, Thm. 1]. André proves that for a polarizable good variation of mixed Hodge structure, the connected monodromy group outside a meager locus is normal in the derived Mumford-Tate group. As [And92, p.10] explains, the normality is a consequence of the theorem of the fixed part due to Griffiths-Schmidt-Steenbrink-Zucker. In our case, an analog of the theorem of the fixed part is Theorem 3.1.2.3.

Let $\mathcal{C}(A)_\ell$ be the cotorus parameterizing pro-ℓ characters of $\pi_1^{\text{ét}}(A)$ (Definition 3.3.1.3). For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$ and every $\chi_\ell \in \mathcal{C}(A)_\ell$, set $\chi = \chi_{\ell'}\chi_\ell$. Let $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X) \subset D_c^b(A \times X)$ be the full subcategory of p_X-universally locally acyclic (ULA, Definition 3.2.2.1) relative perverse sheaves. It is an abelian category.

Theorem 3.1.2.3 (Theorem 3.5.2.1)**.** *Assume that X is smooth and $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Then there is a subobject $K^0 \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ with the following property: For every character $\chi_{\ell'} : \pi_1^{\acute{e}t}(A) \to \bar{\mathbb{Q}}_\ell^\times$ of finite order prime to ℓ, there is a nonempty Zariski open subset $U \subset \mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in U$, one has*

$$H^0(A_{\bar\eta}, K^0|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta}) = H^0(A_{\bar\eta}, K|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta})^{\Gamma_{k(\eta)}}.$$

The proof of Theorem 3.1.2.3 uses the projection $p_A : A \times X \to A$, which restricts our results to constant abelian schemes. We leave the question whether Theorem 3.1.2.2 has an analog for relative perverse sheaves on an arbitrary (non-constant) abelian scheme.

Notation and conventions

An object of an abelian category is *semisimple* if it is the direct sum of finitely many simple objects. An abelian category is *semisimple* if every object is semisimple. For a field k, its absolute Galois group is denoted by Γ_k. An algebraic variety means a scheme of finite type and separated over k. A linear algebraic group is *reductive*, if its identity component is reductive (in the sense of [Mil17a, 6.46, p.135]). For a topological group, its $\bar{\mathbb{Q}}_\ell$-characters are assumed to be continuous. For an irreducible algebraic variety X (on which ℓ is invertible) and a $\bar{\mathbb{Q}}_\ell$-character χ of its étale fundamental group $\pi_1^{\text{ét}}(X)$, let L_χ be the corresponding rank one lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X.

3.2 Recollections on constructible sheaves

No originality is claimed in Section 3.2. Let k be a field. Let ℓ be a prime number invertible in k. Fix an algebraic closure $\bar{k}$ of k. For every algebraic variety X over k, denote by $D_c^b(X) := D_c^b(X, \bar{\mathbb{Q}}_\ell)$ the triangulated category of complexes of $\bar{\mathbb{Q}}_\ell$-sheaves on X with bounded constructible cohomologies defined in [Bei+82, p.74]. Let $\mathbb{D}_X : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$ be the Verdier duality functor. The heart of the standard t-structure on $D_c^b(X)$ is denoted by $\mathrm{Cons}(X)$, which is the category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X. For $F \in \mathrm{Cons}(X)$, set $\mathrm{Supp}\, F := \{x \in X | F_x \neq 0\}$ to be its support. Then $\mathrm{Supp}\, F$ is a quasi-constructible subset of X in the sense of [EGA IV 3, 10.1.1]. Let $\mathrm{Loc}(X) \subset \mathrm{Cons}(X)$ be the full subcategory of lisse $\bar{\mathbb{Q}}_\ell$-sheaves on X. For every integer n, let $\mathcal{H}^n : D_c^b(X) \to \mathrm{Cons}(X)$ be the functor taking the n-th cohomology sheaf.

For every subset $S \subset X$, let $\bar{S}$ be its Zariski closure. Let ${}^pD^{\leq 0}(X) \subset D_c^b(X)$ be the full subcategory of objects K with $\dim \overline{\mathrm{Supp}\, \mathcal{H}^n K} \leq -n$ for every integer n. Let ${}^pD^{\geq 0}(X) \subset D_c^b(X)$ be the full subcategory of objects K with $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. Then $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ defines the (absolute) perverse t-structure on $D_c^b(X)$, whose heart is denoted by $\mathrm{Perv}(X)$. The functor $\mathbb{D}_X$ interchanges ${}^pD^{\leq 0}(X)$ and ${}^pD^{\geq 0}(X)$. For every integer n, let ${}^p\mathcal{H}^n : D_c^b(X) \to \mathrm{Perv}(X)$ be the functor taking the n-th perverse cohomology sheaf. For a morphism $f : X' \to X$ of schemes and $K \in D_c^b(X)$, set $K|_{X'} := f^*K$.

3.2.1 Basics

Fact 3.2.1.1 (Projection formula, [FK88, Rk. (2), p.100], [Sta24, Tag 0F10 (1)])**.** *Let $f : X \to Y$ be a morphism of algebraic varieties over $\bar{k}$. Let $L \in D_c^b(Y)$ be an object with $\mathcal{H}^n L \in \mathrm{Loc}(X)$ for every integer n. Then there is a natural isomorphism $(Rf_*-) \otimes^L L \to Rf_*(- \otimes^L f^*L)$ of functors $D_c^b(X) \to D_c^b(Y)$.*

Let X be an algebraic variety over k.

Fact 3.2.1.2 ([FK88, Prop. 12.10])**.** *For every $F \in \mathrm{Cons}(X)$, there is a nonempty Zariski open subset $U \subset X$ with $F|_U \in \mathrm{Loc}(U)$.*

Definition 3.2.1.3 ([BC18, Def. 78])**.** An object $K \in D_c^b(X)$ is called semisimple if it is isomorphic to a finite direct sum of degree shifts of semisimple objects of $\mathrm{Perv}(X)$.

If $K \in D_c^b(X)$ is semisimple, then it is isomorphic to $\oplus_{n \in \mathbb{Z}} {}^p\mathcal{H}^n(K)[-n]$ in $D_c^b(X)$, and each ${}^p\mathcal{H}^n(K)$ is a semisimple object of $\mathrm{Perv}(X)$. A degree shift of a semisimple object of $D_c^b(X)$ is still semisimple.

Example 3.2.1.4. Every perverse cohomology sheaf of a semisimple object of $D_c^b(X)$ is semisimple. By contrast, its cohomology sheaves may be no longer semisimple in $D_c^b(X)$.

Consider $k = \mathbb{C}$ and $X = A^1$. Let $j : U = A^1 \setminus \{0, 1\} \to X$ be the inclusion. Then the topological fundamental group $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1)$ is the free group generated by two loops a and b, surrounding 0 and 1 respectively. There is a unique morphism

$$\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1) \to \mathrm{SL}_2(\mathbb{Z}) \tag{3.1}$$

sending a, b to

$$A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$$

respectively. By Grauert-Remmert's theorem (see, *e.g.*, [SGA 1, XII, Cor. 5.2]), the étale fundamental group $\pi_1^{\text{ét}}(U, -1)$ is the profinite completion of $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1)$. Since $\mathrm{SL}_2(\mathbb{Z}_\ell)$ is a profinite group, the morphism (3.1) extends naturally to a continuous morphism

$$\pi_1^{\text{ét}}(U, -1) \to \mathrm{SL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell^2). \tag{3.2}$$

The representation (3.2) is irreducible. Otherwise, assume that $v := (x, y)^T \neq 0 \in \bar{\mathbb{Q}}_\ell^2$ generates a 1-dimensional subrepresentation. Then $Av = (-x-2y, -y)^T$ is parallel to v. Therefore, $y = 0$. Similarly, $Bv = (-x, -2x-y)^T$ is parallel to v, then $x = 0$, a contradiction.

Let L be the rank two *simple* lisse $\bar{\mathbb{Q}}_\ell$-sheaf on U corresponding to (3.2). Then L^{an} is the local system on U^{an} corresponding to (3.1). For every small open ball $B_0 \subset X^{\mathrm{an}}$ centered at 0, the $\mathbb{C}$-vector space $H^0(B_0, j_*^{\mathrm{an}} L^{\mathrm{an}})$ is the kernel of the linear operator $A - 1$ on the stalk L_{-1}^{an}. Since $A - 1$ is invertible, one has $H^0(B_0, j_*^{\mathrm{an}} L^{\mathrm{an}}) = 0$. Therefore, the stalk $(j_*^{\mathrm{an}} L^{\mathrm{an}})_0 = 0$. Similarly, the stalk $(j_*^{\mathrm{an}} L^{\mathrm{an}})_1 = 0$. In conclusion, the natural morphism $j_!^{\mathrm{an}} L^{\mathrm{an}} \to j_*^{\mathrm{an}} L^{\mathrm{an}}$ is an isomorphism in $\mathrm{Cons}(X^{\mathrm{an}})$.

We prove that $H^1(U^{\mathrm{an}}, L^{\mathrm{an}}) = H^1(\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1), L_{-1}^{\mathrm{an}})$ is nonzero. Define a map $f : \pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1) \to \bar{\mathbb{Q}}_\ell^2$ inductively. Set $f(e) = 0$, $f(a) = f(b) = (1, 0)^T$, $f(a^{-1}) = -A^{-1} f(a)$, and $f(b^{-1}) = -B^{-1} f(b)$. Once f is defined for every element of $\pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1)$ with length $n \geq 1$, we define it on elements of length $n + 1$ as follows. For every element $g \in \pi_1^{\mathrm{top}}(U^{\mathrm{an}}, -1)$ of length n, set

$$f(ag) = Af(g) + f(a), \quad f(bg) = Bf(g) + f(b),$$
$$f(a^{-1}g) = A^{-1} f(g) + f(a^{-1}), \quad f(b^{-1}g) = B^{-1} f(g) + f(b^{-1}).$$

The map f is a crossed homomorphism. It is not a boundary, because the equation $(A - 1)x = (B - 1)x = (1, 0)^T$ admits no solution in $\bar{\mathbb{Q}}_\ell^2$.

Therefore, L^{an} is in the cohomology support loci of U^{an} (in the sense of [Bud+17, p.295]). From [HT07, Example 8.1.35 (ii)], one has $j_! L[1] \in \mathrm{Perv}(X)$. By [Bud+17, p.299], $j_!^{\mathrm{an}} L^{\mathrm{an}}[1]$ is not semisimple in $\mathrm{Perv}(X^{\mathrm{an}})$.

By [Bei+82, Thm. 4.3.1 (ii)], the intermediate extension $K := j_{!*} L[1]$ is a simple object of $\mathrm{Perv}(X)$. We claim that $\mathcal{H}^{-1} K$ is not semisimple in $D_c^b(X)$. From [HT07, Prop. 8.2.11], K is isomorphic to $\tau^{\leq -1} Rj_* L[1]$, where $\tau^{\leq -1} : D_c^b(X) \to D_c^b(X)$ is the truncation functor with respect to the standard t-structure. Thus, $\mathcal{H}^{-1}K$ is isomorphic to $\mathcal{H}^{-1}(Rj_*L[1]) = j_*L$ in $\mathrm{Cons}(X)$. Then $(\mathcal{H}^{-1}K)^{\mathrm{an}}$ is isomorphic to $j_!^{\mathrm{an}}L^{\mathrm{an}}$ in $\mathrm{Cons}(X^{\mathrm{an}})$. From [Kat90, p.375], one has $(\mathcal{H}^{-1}K)[1] \in \mathrm{Perv}(X)$. Since $(\mathcal{H}^{-1}K)^{\mathrm{an}}[1]$ is not semisimple in $\mathrm{Perv}(X^{\mathrm{an}})$, by [Kat90, Lem. 12.7.1.1], $(\mathcal{H}^{-1}K)[1]$ is not semisimple in $\mathrm{Perv}(X)$. The claim is proved.

Lemma 3.2.1.5 is used in the proof of Theorem 3.5.1.1.

Lemma 3.2.1.5. *Let $U \subset X$ be an open subset of X. Then the functor $(-)|_U : \mathrm{Perv}(X) \to \mathrm{Perv}(U)$ sends every simple object of $\mathrm{Perv}(X)$ to a simple or zero object of $\mathrm{Perv}(U)$. In particular, the functor $(-)|_U : D_c^b(X) \to D_c^b(U)$ preserves semisimplicity.*

Proof. Let K be a simple object of $\mathrm{Perv}(X)$. By [Bei+82, Thm. 4.3.1 (ii)], there is an irreducible, locally closed and geometrically smooth subvariety $j : V \to X$ and a simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf on V, such that K is isomorphic to $j_{!*}L[\dim V]$. If V is disjoint from U, then $K|_U = 0$. Otherwise, take a geometric point $\bar{x}$ on $V \cap U$. From [SGA 1, V, Prop. 8.2], the morphism $\pi_1^{\text{ét}}(U \cap V, \bar{x}) \to \pi_1^{\text{ét}}(V, \bar{x})$ is surjective. Thus, the composite representation $\pi_1^{\text{ét}}(U \cap V, \bar{x}) \to \mathrm{GL}(L_{\bar{x}})$ is also simple, *i.e.*, the lisse $\bar{\mathbb{Q}}_\ell$-sheaf $L|_{U\cap V}$ is simple. Let $h : U \cap V \to U$ be the base change of $j : V \to X$ along the inclusion $U \to X$. Then $K|_U$ is isomorphic to $h_{!*}L|_{U\cap V}[\dim(U \cap V)]$, hence simple in $\mathrm{Perv}(U)$. ∎

When $k = \mathbb{C}$, Fact 3.2.1.6 1 follows from Kashiwara's conjecture for semisimple perverse sheaves and the paragraph following [Bei+82, Thm. 6.2.5]. Kashiwara's conjecture is formulated in [Kas98, Sec. 1]; see also [Dri01, Sec. 1.2, 1]. It is reduced to de Jong's conjecture by Drinfeld [Dri01], which in turn is proved in [BK06] and [Gai07]. The case of general k follows *via* Lemma 3.2.1.7.

Fact 3.2.1.6. *Let k be an algebraically closed field of characteristic 0. Let $f : X \to Y$ be a proper morphism of algebraic varieties over k. Let K be a semisimple object of $D_c^b(X)$.*

1. (Decomposition theorem) *Then Rf_*K is a semisimple object of $D_c^b(Y)$.*
2. (Global invariant cycle theorem, [Bei+82, Cor. 6.2.8]) *Let i be an integer. By Fact 3.2.1.2, there is a nonempty connected open subset $V \subset Y$ such that $\mathcal{H}^iRf_*K|_V$ is a lisse sheaf. Then for every $y \in V(k)$, the canonical map*

$$H^i(X, K) \to H^i(X_y, K|_{X_y})^{\pi_1^{\acute{e}t}(V,y)}$$

is surjective.

Lemma 3.2.1.7. *Let E/F be an extension of algebraically closed fields. Let X be an algebraic variety over F. Then:*

1. *The functor $(-)|_{X_E} : D^b_c(X) \to D^b_c(X_E)$ is fully faithful. It induces an exact functor* $\mathrm{Perv}(X) \to \mathrm{Perv}(X_E)$.
2. *An object of* $\mathrm{Perv}(X)$ *is simple (resp. semisimple) if and only if its image under $(-)|_{X_E} : \mathrm{Perv}(X) \to \mathrm{Perv}(X_E)$ is simple (resp. semisimple).*

Proof. 1. In characteristic zero, it is the first half of [JKLM23, Lem. A.1]. That proof works in positive characteristic as well.

2. Let $K \in \mathrm{Perv}(X)$. By [Bei+82, Thm. 4.3.1 (ii)] and [Esn17, Prop. 5.3], K is simple if and only if $K|_{X_E}$ is simple. Thus, if K is semisimple, so is $K|_{X_E}$. Now assume that $K|_{X_E}$ is semisimple. For every subobject $P \subset K$ in $\mathrm{Perv}(X)$, there is a morphism $r : K|_{X_E} \to P|_{X_E}$ in $\mathrm{Perv}(X_E)$ with $r|_{P|_{X_E}} = \mathrm{Id}_{P|_{X_E}}$. By Part 1, there is a morphism $r' : K \to P$ in $\mathrm{Perv}(X)$ with $r'|_{K|_{X_E}} = r$ and $r'|_P = \mathrm{Id}_P$. Thus, P admits a direct complement in K. By Lemma 3.2.1.8 2, K is semisimple.

□

Lemma 3.2.1.8. *Let $\mathcal{A}$ be an abelian category. Let $X \in \mathcal{A}$ be a Noetherian and Artinian object.*

1. *Let Y be a simple subquotient of X. Then there is a composite series of X with one graded piece isomorphic to Y. In particular, up to isomorphism X has only finitely many simple subquotients.*
2. *If every subobject of X admits a direct complement, then X is semisimple.*

Proof.

1. There is a subobject $i : X_0 \subset X$ and a quotient $q : X_0 \to Y$ in $\mathcal{A}$. Let $N = \ker(q)$. By [Sta24, Tag 0FCH, Tag 0FCI], both N and X/X_0 are Noetherian and Artinian. From [Sta24, Tag 0FCJ], they admit composite series. A composite series of X/X_0 is equivalent to a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ by subobjects such that X_i/X_{i-1} is simple for every $1 \leq i \leq n$. This filtration and every composite series of N glue to a composite series of X with a step $N \subset X_0$, whose factor is isomorphic to Y. By the Jordan-Hölder lemma [Sta24, Tag 0FCK], up to isomorphism Y has finitely many choices.
2. One may assume $X \neq 0$. Let $\mathcal{P}$ be the family of nonzero semisimple subobjects of X. As X is Artinian, it has a simple subobject, so $\mathcal{P}$ is nonempty. Since X is Noetherian, $\mathcal{P}$ has a maximal element $i : X_0 \to X$. By assumption, there is a subobject $F \subset X$ with

$X_0 \oplus F = X$. Then $F = 0$. (Otherwise, by [Sta24, Tag 0FCJ], F has a nonzero simple subobject F_0. Then $X_0 \oplus F_0 \in \mathcal{P}$ is strictly larger than X_0, which is a contradiction.) Therefore, i is an isomorphism and X is semisimple.

□

Remark 3.2.1.9. In a Noetherian and Artinian abelian category, an object may have infinitely many distinct (non semisimple) subobjects up to isomorphism.

Lemma 3.2.1.10. *Let L be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf of rank one on X. Then $-\otimes^L L : D_c^b(X) \to D_c^b(X)$ is an equivalence of categories. It is t-exact for the perverse t-structures.*

Proof. Let L^{-1} be the lisse sheaf dual to L. By associativity of the derived tensor product $\otimes^L$, the pair of functors $(-\otimes^L L, -\otimes^L L^{-1})$ is an equivalence.

1. Right t-exactness: The functor is t-exact for the standard t-structures. Thus, for every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\mathcal{H}^n(K \otimes^L L) = \mathcal{H}^n(K) \otimes^L L$. Therefore, one has $\operatorname{Supp} \mathcal{H}^n(K \otimes^L L) = \operatorname{Supp} \mathcal{H}^n(K)$. Thus, $K \otimes^L L \in {}^pD^{\leq 0}(X)$.

2. Left t-exactness: By Part 1, for every $K \in {}^pD^{\geq 0}(K)$, one has $L^{-1} \otimes^L \mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. By [KW01, II, Cor. 7.5 f)], one has isomorphisms
$$\mathbb{D}_X(K \otimes^L L) \to R\mathcal{H}om(L, \mathbb{D}_X K) \to L^{-1} \otimes^L \mathbb{D}_X K$$
in $D_c^b(X)$. Therefore, one gets $K \otimes^L L \in {}^pD^{\geq 0}(X)$.

□

3.2.2 Universal local acyclicity

In Section 3.2.2, all schemes are assumed to be quasi-compact and quasi-separated. For a scheme X and a geometric point $\bar{x}$ on X, denote by $O^{\mathrm{sh}}_{X,\bar{x}}$ the strict henselization (in the sense of [Sta24, Tag 04GQ (3)]) of $O_{X,\bar{x}}$. Set $X_{(\bar{x})} := \operatorname{Spec} O^{\mathrm{sh}}_{X,\bar{x}}$.

Let $f : X \to S$ be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes.

Definition 3.2.2.1 ([Sta24, Tag 0GJM], [Bar23, Def. 1.2])**.** Let K be an object of $D_c^b(X)$.

- If for every geometric point $\bar{x}$ on X and every geometric point $\bar{t}$ on $S_{(\bar{s})}$ with $\bar{s} = f(\bar{x})$, the canonical morphism $R\Gamma(X_{(\bar{x})}, K) \to R\Gamma(X_{(\bar{x})} \times_{S_{(\bar{s})}} t, K)$ is an isomorphism, then K is called f-locally acyclic.

- If for every morphism $S' \to S$ of schemes, in notation of the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd} \tag{3.3}$$

g'^*K is f'-locally acyclic, then K is called f-universally locally acyclic (f-ULA). Let $D^{\mathrm{ULA}}(X/S) \subset D^b_c(X)$ be the full subcategory of f-ULA objects.

By [HS23, Thm. 4.4], an object $K \in D^b_c(X)$ is f-ULA if and only if K is universally locally acyclic in the sense of [HS23, Def. 3.2]. Thus, the notation $D^{\mathrm{ULA}}(X/S)$ agrees with that in [HS23]. It is a triangulated subcategory of $D^b_c(X)$.

Fact 3.2.2.2.

1. ([Bar23, Lem. 3.7 (ii)]) *If $S = \operatorname{Spec} k$, then $D^{\mathrm{ULA}}(X/k) = D^b_c(X)$.*

2. ([Bar23, Lem. 3.7 (i)]) *If $f : X \to S$ is an isomorphism, then $D^{\mathrm{ULA}}(X/S) \subset D^b_c(X)$ is the full subcategory of objects whose cohomology sheaves are lisse.*

3. ([HS23, Prop. 3.4 (i)]) *Let $g : S' \to S$ be a morphism of $\mathbb{Z}[1/\ell]$-schemes. Then in the notation of (3.3), the functor $g'^* : D^b_c(X) \to D^b_c(X')$ restricts to a functor $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X'/S')$.*

4. ([Ric14, Lem. 3.15], [Bar23, Lem. 3.6 (i), (ii)]) *Let $f' : Y \to S$ be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes. Let $h : X \to Y$ be a morphism of S-schemes. If h is smooth (resp. proper), then the functor $h^* : D^b_c(Y) \to D^b_c(X)$ (resp. $Rh_* : D^b_c(X) \to D^b_c(Y)$) restricts to a functor $D^{\mathrm{ULA}}(Y/S) \to D^{\mathrm{ULA}}(X/S)$ (resp. $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(Y/S)$).*

5. ([HS23, p.643]) *Let $g : S \to T$ be a smooth morphism of $\mathbb{Z}[1/\ell]$-schemes. Then $D^{\mathrm{ULA}}(X/S) \subset D^{\mathrm{ULA}}(X/T)$.*

6. ([Zhu17, Thm. A.2.5 (4)]) *Let $f_i : X_i \to S$ ($i = 1, 2$) be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes. Let $K_i \in D^{\mathrm{ULA}}(X_i/S)$. Then $K_1 \boxtimes_S K_2 \in D^{\mathrm{ULA}}(X_1 \times_S X_2/S)$.*

Although the derived external tensor product preserves universal local acyclicity (Fact 3.2.2.2 6), Example 3.2.2.3 shows that the derived tensor product of two ULA complexes may not be ULA.

Example 3.2.2.3. Let $f : X \to S$ be the morphism $\mathbf{A}^2_{\mathbb{C}} \to \mathbf{A}^1_{\mathbb{C}}, \quad (a,b) \mapsto a+b$. Let $i : A \to X$ (resp. $j : B \to X$) be the inclusion of a-axis (resp. b-axis) of X. Then $fi : A \to S$ (resp. $fj : B \to S$) is an isomorphism. Let $K = i_*\bar{\mathbb{Q}}_{\ell,A}$ and $K' = j_*\bar{\mathbb{Q}}_{\ell,B}$. By Fact 3.2.2.2 2 and 4, one has $K, K' \in D^{\mathrm{ULA}}(X/S)$. As $K \otimes^L K'$ is the skyscraper supported at the origin of X with stalk $\bar{\mathbb{Q}}_\ell$, it is not f-ULA. (Otherwise, from [Bar23, Lem. 3.6 (iv)], the skyscraper viewed as a sheaf on A is fi-ULA, which contradicts Fact 3.2.2.2 2.)

Lemma 3.2.2.4. *Assume that S is irreducible with generic point η. Let $K \in D^{\mathrm{ULA}}(X/S)$. If $K|_{X_{\bar{\eta}}} = 0$ in $D^b_c(X_{\bar{\eta}})$, then $K = 0$.*

Proof. It suffices to prove that for every $s \in S$, one has $K|_{X_{\bar{s}}} = 0$ in $D^b_c(X_{\bar{s}})$. By [EGA II, Prop. 7.1.9], there is a discrete valuation ring R and a separated morphism $g : \mathrm{Spec}(R) = S' \to S$, sending the generic (resp. closed) point ξ (resp. r) of S' to η (resp. s). Let $i : R \to R^h$ be the henselization of R (in the sense of [Sta24, Tag 04GQ (1)]). By [Sta24, Tag 0AP3], R^h is a discrete valuation ring. From [Mil80, I, Exercise 4.9], the local morphism i is injective. Then $i^* : \mathrm{Spec}(R^h) \to S'$ preserves the generic (resp. closed) point. Replacing R by R^h, one may assume further that R is henselian.

Consider the following cartesian squares

$$\begin{tikzcd}
X'_{\bar{r}} \arrow[r, "\bar{i}"] \arrow[d] \arrow[dr, phantom, "\square"] & X'_{(\bar{r})} \arrow[d] \arrow[dr, phantom, "\square"] & X'_{\bar{\xi}} \arrow[l, "\bar{j}"'] \arrow[d] \\
\bar{r} \arrow[r] & S'_{(\bar{r})} & \bar{\xi}, \arrow[l]
\end{tikzcd}$$

where every vertical morphism is a base change of $f : X \to S$. In the notation of (3.3), let $R\Phi : D^+(X') \to D^+(X'_{\bar{r}})$ be the vanishing cycle functor. Let $R\Psi : D^+(X') \to D^+(X'_{\bar{r}})$ be the nearby cycle functor. Set $K' = g'^*K$. By definition, one has $R\Psi(K') = \bar{i}^* R\bar{j}_*(K'|_{X'_{\bar{\xi}}})$. As R is henselian, from [Ill06, (1.1.3)], there is a natural exact triangle $K'|_{X'_{\bar{r}}} \to R\Psi(K') \to R\Phi(K') \xrightarrow{+1}$ in $D^+(X'_{\bar{r}})$. Since $K'|_{X'_{\bar{\xi}}}$ is a pullback of $K|_{X_{\bar{\eta}}} = 0$, one has $K'|_{X'_{\bar{\xi}}} = 0$ and $R\Psi(K') = 0$. By [Ill06, Cor. 3.5], the universal local acyclicity of K implies $R\Phi(K') = 0$. Therefore, one gets $K'|_{X'_{\bar{r}}} = 0$.

Since $K'|_{X'_{\bar{r}}}$ is the pullback of $K|_{X_{\bar{s}}}$ under the field extension $k(\bar{r})/k(\bar{s})$, by Lemma 3.2.1.7 1, one gets $K|_{X_{\bar{s}}} = 0$. $\square$

3.2.3 Relative perverse sheaves

Let $f : X \to S$ be a morphism of algebraic varieties over k. Then S is *bon* in the sense of [KL85, (1.0)]. In particular, f is separated and of finite presentation. Set $K_{X/S} := Rf^!\bar{\mathbb{Q}}_\ell \in D^b_c(X)$ to be the relative dualizing complex. The functor

$$\mathbb{D}_{X/S}(-) = R\mathcal{H}om(-, K_{X/S}) : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$$

is called the *relative Verdier dual*. By [KL85, (1.1.5)], as S is bon, there is a canonical morphism of functors $\mathrm{Id}_{D_c^b(X)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$.

Fact 3.2.3.1 is stated for ∞-categories in [HS23], but holds for the underlying triangulated categories (described in [HRS23, Lem. 7.9]) by [HS23, Footnote 1].

Fact 3.2.3.1.

1. ([HS23, Thm. 1.1]) *There is a unique t-structure $({}^{p/S}D^{\leq 0}(X/S), {}^{p/S}D^{\geq 0}(X/S))$ on $D_c^b(X)$, called the relative perverse t-structure, with the following property: An object $K \in D_c^b(X)$ lies in ${}^{p/S}D^{\leq 0}(X/S)$ (resp. ${}^{p/S}D^{\geq 0}(X/S)$) if and only if for every geometric point $\bar{s} \to S$, the restriction $K|_{X_{\bar{s}}}$ lies in ${}^{p}D^{\leq 0}(X_{\bar{s}})$ (resp. ${}^{p}D^{\geq 0}(X_{\bar{s}})$). In particular, for every $s \in S$, the functor $(-)|_{X_s} : D_c^b(X) \to D_c^b(X_s)$ is t-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse t-structure.*

2. ([HS23, Thm. 1.9]) *The relative perverse t-structure on $D_c^b(X)$ restricts to a t-structure $({}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S), {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S))$ on $D^{\mathrm{ULA}}(X/S)$.*

3. ([HS23, Prop. 3.4]) *The functor $\mathbb{D}_{X/S}$ preserves $D^{\mathrm{ULA}}(X/S)$, and the morphism $\mathrm{Id}_{D^{\mathrm{ULA}}(X/S)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ of functors $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)$ is an isomorphism. The formation of $\mathbb{D}_{X/S} : D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)^{\mathrm{op}}$ commutes with any base change in S, so $\mathbb{D}_{X/S}$ exchanges ${}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S)$ with ${}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S)$.*

Definition 3.2.3.2. Let $\mathrm{Perv}(X/S)$ (resp. $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$) be the heart of the relative perverse t-structure on $D_c^b(X)$ (resp. $D^{\mathrm{ULA}}(X/S)$).

By Fact 3.2.3.1 1, an object $K \in D_c^b(X)$ lies in $\mathrm{Perv}(X/S)$ if and only if for every geometric point $\bar{s} \to S$, one has $K|_{X_{\bar{s}}} \in \mathrm{Perv}(X_{\bar{s}})$.

Example 3.2.3.3.

1. ([HS23, p.632]) If $S = \mathrm{Spec}(k)$, then $\mathrm{Perv}(X/k) = \mathrm{Perv}(X)$.

2. If f is universally injective, then $\mathrm{Perv}(X/S) = \mathrm{Cons}(X)$.

3. ([Bar23, Lem. 3.7 (ii)]) If f is smooth of relative dimension r, then the functor $(-)[r] : \mathrm{Loc}(X) \to D_c^b(X)$ factors through $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Example 3.2.3.4. Let $i : Y \to X$ be a closed immersion of S-schemes, with $Y \to S$ smooth of relative dimension d and with geometrically connected fibers. If L is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on Y, then $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Indeed, by Fact 3.2.2.2 2, one has $L \in D^{\mathrm{ULA}}(Y/Y)$. From the smoothness of $Y \to S$ and Fact 3.2.2.2 5, one has $L \in D^{\mathrm{ULA}}(Y/S)$. Using the properness of $i : Y \to X$ and Fact 3.2.2.2 4, one has $i_*L[d] \in D^{\mathrm{ULA}}(X/S)$. For every geometric point $\bar{s} \to S$, let $i_{\bar{s}} : Y_{\bar{s}} \to X_{\bar{s}}$ be the base change of i along the morphism $X_{\bar{s}} \to X$. By the proper base change theorem, $i_*L[d]|_{X_{\bar{s}}} = (i_{\bar{s}})_*(L|_{Y_{\bar{s}}})[d] \in \mathrm{Perv}(X_{\bar{s}})$. Therefore, $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Lemma 3.2.3.5. *If S is geometrically unibranch and irreducible, then* $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ *is a Serre subcategory of* $\mathrm{Perv}(X/S)$.

Proof. By definition, $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is a strictly full subcategory of $\mathrm{Perv}(X/S)$. By Fact 3.2.3.1 2 and [Bei+82, Thm. 1.3.6], $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is an abelian subcategory of $\mathrm{Perv}(X/S)$ and closed under extensions in $D^{\mathrm{ULA}}(X/S)$. As $D^{\mathrm{ULA}}(X/S) \subset D^b_c(X)$ is a triangulated subcategory, $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under extensions in $\mathrm{Perv}(X/S)$. Because S is geometrically unibranch, from the proof of [HS23, Thm. 6.8 (ii)], $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under subquotients in $\mathrm{Perv}(X/S)$. By [Sta24, Tag 02MP], it is a Serre subcategory. □

Fact 3.2.3.6 ([HS23, Thm. 1.10 (ii)])**.** *Assume that S is geometrically unibranch and irreducible with generic point η. Then the functor*

$$(-)|_{X_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X_\eta)$$

is exact and fully faithful, and its essential image is stable under subquotients.

For every integer j, let ${}^{p/S}\mathcal{H}^j : D^b_c(X) \to \mathrm{Perv}(X/S)$ be the j-th cohomology functor associated with the relative perverse t-structure.

Lemma 3.2.3.7. *Suppose that S is smooth over k and irreducible with generic point η. Assume that $K \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is semisimple in $D^b_c(X)$. Then $K|_{X_\eta}$ is semisimple in* $\mathrm{Perv}(X_\eta)$.

Proof. By Fact 3.2.3.6, for every subobject $M \subset K|_{X_\eta}$ in $\mathrm{Perv}(X_\eta)$, there is a subobject $K' \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ with $K'|_{X_\eta} = M$. By Lemma 3.2.3.8 and smoothness of S, the morphism $K'[\dim S] \to K[\dim S]$ is a monomorphism in $\mathrm{Perv}(X)$. Because K is semisimple in $D^b_c(X)$, its shift $K[\dim S]$ is semisimple in $\mathrm{Perv}(X)$. Thus, there is a subobject $N \subset K[\dim S]$ in $\mathrm{Perv}(X)$ with $K[\dim S] = (K'[\dim S]) \oplus N$. Then $K = K' \oplus (N[-\dim S])$ in $D^b_c(X)$. For every nonzero integer j, one has

$$0 = {}^{p/S}\mathcal{H}^j(K) = 0 \oplus {}^{p/S}\mathcal{H}^j(N[-\dim S])$$

in $\mathrm{Perv}(X/S)$. Hence ${}^{p/S}\mathcal{H}^j(N[-\dim S]) = 0$ and $N[-\dim S] \in \mathrm{Perv}(X/S)$. Consequently, $K|_{X_\eta} = M \oplus (N|_{X_\eta}[-\dim S])$ in $\mathrm{Perv}(X_\eta)$. By [Bei+82, Thm. 4.3.1 (i)], the abelian category $\mathrm{Perv}(X_\eta)$ is Noetherian and Artinian. As every subobject of $K|_{X_\eta}$ in $\mathrm{Perv}(X_\eta)$ admits a direct complement, the semisimplicity follows from Lemma 3.2.1.8 2. □

Lemma 3.2.3.8 is stated without proof for regular schemes S in [HS23, p.636].

Lemma 3.2.3.8. *Assume that S is smooth over k of equidimension d. Then the shifted inclusion*

$$(-)[d] : D^{\mathrm{ULA}}(X/S) \to D^b_c(X) \tag{3.4}$$

is t-exact, where $D^{\mathrm{ULA}}(X/S)$ (resp. $D^b_c(X)$) is equipped with the relative (resp. absolute) perverse t-structure. In particular, it induces an exact functor

$$(-)[d] : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X). \tag{3.5}$$

Proof.

1. The functor $(-)[d] : D^b_c(X) \to D^b_c(X)$ is right t-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse t-structure. For every geometric point $\bar{s}$ on S, the functor $(-)|_{X_{\bar{s}}} : D^b_c(X) \to D^b_c(X_{\bar{s}})$ is t-exact for the standard t-structures. Then for every integer n and every $K \in {}^{p/S}D^{\leq 0}(X/S)$, one has $\mathcal{H}^n(K[d])|_{X_{\bar{s}}} = \mathcal{H}^{n+d}(K|_{X_{\bar{s}}})$. Hence
$$X_{\bar{s}} \cap \operatorname{Supp} \mathcal{H}^n(K[d]) = \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}).$$
As $K|_{X_{\bar{s}}} \in {}^pD^{\leq 0}(X_{\bar{s}})$, one has $\dim \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}) \leq -n-d$. By Lemma 3.2.3.11 3, one has
$$\dim \operatorname{Supp} \mathcal{H}^n(K[d]) \leq -n.$$
From Lemma 3.2.3.11 1, the Zariski closure of $\operatorname{Supp} \mathcal{H}^n(K[d])$ in X has dimension at most $-n$. Hence $K[d] \in {}^pD^{\leq 0}(X)$.

2. The functor (3.4) is left t-exact. One may assume that k is algebraically closed. For every $K \in {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S)$, by smoothness of S and the proof of [Bar23, Lem. 3.12], $\mathbb{D}_X(K[d])$ is (noncanonically) isomorphic to $(\mathbb{D}_{X/S}K)[d]$ in $D^b_c(X)$. From Fact 3.2.3.1 3, $\mathbb{D}_{X/S}K \in {}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S)$. By Part 1, one has $(\mathbb{D}_{X/S}K)[d] \in {}^pD^{\leq 0}(X)$. Hence $K[d] \in {}^pD^{\geq 0}(X)$. □

Remark 3.2.3.9. In Lemma 3.2.3.8, the functor $(-)[d] : D^b_c(X) \to D^b_c(X)$ may not send $\mathrm{Perv}(X/S)$ to $\mathrm{Perv}(X)$. Indeed, let $k = \mathbb{C}$, and let $f : X = 0 \to S = A^1_{\mathbb{C}}$ be the inclusion of the origin. By Example 3.2.3.3 1, the relative perverse t-structure on $D^b_c(X)$ coincides with the standard one (which is also the absolute perverse t-structure). Then $\mathrm{Perv}(X/S) = \mathrm{Perv}(X)$.

Lemma 3.2.3.10. *If S is integral with generic point η and $\dim S = d$, then the functor $(-)|_{X_\eta}[-d] : D^b_c(X) \to D^b_c(X_\eta)$ is t-exact for the absolute perverse t-structures. In particular, it restricts to an exact functor*

$$(-)|_{X_\eta}[-d] : \mathrm{Perv}(X) \to \mathrm{Perv}(X_\eta). \tag{3.6}$$

Proof. 1. Right t-exactness: For every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) = \operatorname{Supp} \mathcal{H}^{n-d}(K|_{X_\eta}) = X_\eta \cap \operatorname{Supp} \mathcal{H}^{n-d}(K)$. By Lemma 3.2.3.11 4, one has

$$\dim \operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) \leq \dim \operatorname{Supp}(\mathcal{H}^{n-d}(K)) - d \leq -n.$$

From Lemma 3.2.3.11 1, one has $K|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$.

2. Left t-exactness: For every $K \in D_c^b(X)$ and every integer n, one has

$$\operatorname{Supp} \mathcal{H}^n(\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])) = \operatorname{Supp} \mathcal{H}^n((\mathbb{D}_X K)|_{X_\eta}[-d]). \tag{3.7}$$

Indeed, from [SGA 4 1/2, Thm. 2.13, p.242], by shrinking S to a nonempty open subset, one may assume that $K \in D^{\mathrm{ULA}}(X/S)$. By the proof of [Bar23, Lem. 3.12], one has $\mathbb{D}_X K = (\mathbb{D}_{X/S} K)(d)[2d]$. From Fact 3.2.3.1 3, $(\mathbb{D}_X K)|_{X_\eta}[-d]$ is a Tate twist of $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])$, which proves (3.7).

Now assume $K \in {}^pD^{\geq 0}(X)$. Then $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. From Part 1, one has $(\mathbb{D}_X K)|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$. By (3.7), one has $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d]) \in {}^pD^{\leq 0}(X_\eta)$, or equivalently, $K|_{X_\eta}[-d] \in {}^pD^{\geq 0}(X_\eta)$.

□

By convention, the dimension of an empty space is $-\infty$.

Lemma 3.2.3.11. *Let X be a scheme of finite type over a field F. Let C be a quasi-constructible subset of X.*

1. *Then* $\dim C = \dim \bar{C}$.
2. *Let $\{B_i\}_{i=1}^n$ be finitely many locally closed subsets of X and $B = \cup_{i=1}^n B_i$. Then* $\dim B = \max_{i=1}^n \dim B_i$.

Let $f : X \to Y$ be a morphism between schemes of finite type over F.

3. *Let $n \geq 0$ be an integer such that* $\dim(C \cap f^{-1}(y)) \leq n$ *for every $y \in Y$. Then* $\dim C \leq \dim Y + n$.
4. *Assume that Y is integral with generic point η. Then* $\dim Y + \dim(C \cap X_\eta) \leq \dim C$.

Proof.

1. As X is a Noetherian scheme, the topological subspace C is Noetherian. Therefore, C is the union of finitely many irreducible components. Thus, one may assume further that C is nonempty and irreducible. Then the reduced induced closed subscheme $\bar{C}$ of X is integral and of finite type over F. By [Bor91, AG. Prop. 1.3], C contains a nonempty open subset of $\bar{C}$. By [Har77, II, Exercise 3.20 (e)], one has $\dim C = \dim \bar{C}$.

2. For every $1 \leq i \leq n$, since $B_i \subset B$, one has $\dim B_i \leq \dim B$. Then $\max_i \dim B_i \leq \dim B$. As B_i is quasi-constructible in X, by 1, one has $\dim B_i = \dim \overline{B_i}$. As $\{\overline{B_i}\}_{i=1}^n$ is a finite closed cover of $\bar{B}$, one gets $\dim B \leq \dim \bar{B} = \max_i \dim \overline{B_i} = \max_i \dim B_i$.

3. By 2, one may assume that C is locally closed in X. Taking irreducible components, one may assume further that C is irreducible. Let Z be the Zariski closure of $f(C)$ in Y. Then Z is irreducible. With reduced induced subscheme structures, one views C and Z as integral schemes of finite type over F. Moreover, $f : X \to Y$ induces a dominant morphism $g : C \to Z$ over F. By [Har77, II, Exercise 3.22 (b)], for every $y \in f(C) = g(C)$, one has

$$n \geq \dim C \cap f^{-1}(y) = \dim g^{-1}(y) \geq \dim C - \dim Z.$$

   Hence $\dim C \leq \dim Z + n \leq \dim Y + n$.

4. As in the proof of 3, one may assume that C is an irreducible, locally closed subset of X and view C as an integral scheme of finite type over F. One may assume that $C \cap X_\eta$ is nonempty. As C_η is homeomorphic to $C \cap X_\eta$, the morphism $C \to Y$ induced by f is dominant. Thus, by [Har77, II, Exercise 3.22 (c)], one gets $\dim C \cap X_\eta = \dim C_\eta = \dim C - \dim Y$.

□

Lemma 3.2.3.12. *Assume that S is smooth over k, integral with generic point η and* $\dim S = d$. *Then:*

1. *Let $A \in$* $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$, *and let $B[d]$ be a subquotient of $A[d]$ in* $\mathrm{Perv}(X)$. *If the image $B|_{X_\eta} \in$* $\mathrm{Perv}(X_\eta)$ *of $B[d]$ under the functor (3.6) is zero, then* $B[d] = 0$ *in* $\mathrm{Perv}(X)$.

2. *The functor* (3.5) *identifies* $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ *as a Serre subcategory of* $\mathrm{Perv}(X)$.

Proof.

1. By regularity of S and [HS23, Cor. 1.12], one has $B \in D^{\mathrm{ULA}}(X/S)$. Since $B|_{X_\eta} = 0$, by Lemma 3.2.2.4, one has $B = 0$.

2. It follows from the definition that the functor (3.5) is fully faithful. Its essential image is closed under extensions in $\mathrm{Perv}(X)$, because $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under extensions in the triangulated subcategory $D^{\mathrm{ULA}}(X/S)$ of $D^b_c(X)$.

   We claim that the essential image is closed under taking subobjects. Take $K \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$ and a subobject $L[d]$ of $K[d] \in \mathrm{Perv}(X)$. As

S is integral, Lemma 3.2.3.10 shows that $L|_{X_\eta} \subset K|_{X_\eta}$ is a subobject in $\mathrm{Perv}(X_\eta)$. By smoothness of S and Fact 3.2.3.6, there is a subobject $L' \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ with $L'|_{X_\eta} = L|_{X_\eta}$. Set $M = K/L' \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$. Let $N[d]$ be the image of $L[d]$ under the morphism $K[d] \to M[d]$ in $\mathrm{Perv}(X)$. As the sequence

$$0 \to L'[d] \cap L[d] \to L[d] \to N[d] \to 0$$

is exact in $\mathrm{Perv}(X)$, by Lemma 3.2.3.10, the sequence

$$0 \to L'|_{X_\eta} \cap L|_{X_\eta} \to L|_{X_\eta} \to N|_{X_\eta} \to 0$$

is exact in $\mathrm{Perv}(X_\eta)$. Hence $N|_{X_\eta} = 0$. Since $N[d]$ is a subobject of $M[d] \in \mathrm{Perv}(X)$, by Part 1, one has $N[d] = 0$. Then $L[d] \subset L'[d]$ is a subobject in $\mathrm{Perv}(X)$. Since $(L'[d])/(L[d])$ is a quotient of $L'[d]$ in $\mathrm{Perv}(X)$ and $(L'|_{X_\eta})/(L|_{X_\eta}) = 0$ in $\mathrm{Perv}(X_\eta)$, one gets $(L'[d])/(L[d]) = 0$ in $\mathrm{Perv}(X)$. Therefore, $L[d] = L'[d]$. The claim is proved.

Similarly, the essential image is closed under taking quotients. By [Sta24, Tag 02MP], the essential image is a Serre subcategory of $\mathrm{Perv}(X)$.

□

3.3 Cotori

3.3.1 Definition and basic properties

By [Rob00, p.127], there is a canonical absolute value on $\bar{\mathbb{Q}}_\ell$ extending the discrete absolute value $|\cdot|_\ell$ on $\mathbb{Q}_\ell$. It induces a topology on $\bar{\mathbb{Q}}_\ell$ which is totally disconnected. A subset $A \subset \bar{\mathbb{Q}}_\ell$ is closed if and only if for every finite subextension $E/\mathbb{Q}_\ell$ of $\bar{\mathbb{Q}}_\ell$, the subset $A \cap E$ is closed in the discrete valuation field E.

For a profinite group G, let $\mathcal{C}(G)$ be the group of ℓ-adic characters, *i.e.*, continuous morphisms $\chi : G \to \bar{\mathbb{Q}}_\ell^\times$. Then $\chi(G)$ are compact subgroup of $\bar{\mathbb{Q}}_\ell^\times$.

Lemma 3.3.1.1.

1. *Let C be a compact subset of $\bar{\mathbb{Q}}_\ell$. Then there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with $C \subset E$.*

2. *Let $G \leq \bar{\mathbb{Q}}_\ell^\times$ be a compact subgroup. Then there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with $G \subset O_E^\times$.*

3. *In 2, let $G^{(\ell)}$ (resp. $G^{(\ell')}$) be the ℓ-Sylow subgroup (resp. maximal prime-to-ℓ quotient) of G. Then the topological group $G \xrightarrow{\sim} G^{(\ell)} \times G^{(\ell')}$, and $G^{(\ell')}$ is finite.*

Proof. 1. Otherwise, there is a sequence of elements $x_1, x_2, \dots$ in C with $[\mathbb{Q}_\ell(x_{n+1}) : \mathbb{Q}_\ell] > [\mathbb{Q}_\ell(x_n) : \mathbb{Q}_\ell]$ for every integer $n > 0$. Let $B \subset C$ be the (infinite) set of elements of this sequence. For every subset $S \subset B$, every finite subextension $F/\mathbb{Q}_\ell$, the set $S \cap F$ is finite, so closed in F. Therefore, S is closed in $\bar{\mathbb{Q}}_\ell$. In particular, the set B is closed and hence compact in C. Every subset of B is closed in B, so B is discrete. Thus, B is finite, a contradiction.

2. By 1, there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ containing G. By [Ser92, Thm. 1 2, p.122], one has $G \subset O_E^\times$.

3. By 2 and [Ser92, Cor., p.155], G is an ℓ-adic Lie group. From Lazard's theorem (see, *e.g.*, [GSK09, p.711]), there is a pro-ℓ open subgroup $U \leq G$. By [RZ10, Cor. 2.3.6 (b)], there is an ℓ-Sylow subgroup $H \leq G$ containing U. Since G is compact, $[G : U]$ is finite. Thus, the group G/H is finite of order prime to ℓ. By [RZ10, Prop. 2.3.8], G is isomorphic to $G/H \times H$. Since G is commutative, by [RZ10, Cor. 2.3.6 (c)], G has exactly one ℓ-Sylow subgroup.

□

Let $\mathcal{C}(G)_{\ell'}$ (resp. $\mathcal{C}(G)_\ell$) be the subgroup of characters of finite order prime to ℓ (resp. that are pro-ℓ). Then there is a canonical isomorphism $\mathcal{C}(G)_\ell \xrightarrow{\sim} \mathcal{C}\big((G^{(\ell)})^{\mathrm{ab}}\big)$. By Lemma 3.3.1.1 3, one has $\mathcal{C}(G) = \mathcal{C}(G)_{\ell'} \times \mathcal{C}(G)_\ell$.

We review the contents of [GL96, Sec. 3.2]. Fix an integer $n \geq 0$. Let A_n be a free $\hat{\mathbb{Z}}$-module of rank n. Let $\{\gamma_1, \dots, \gamma_n\}$ be a $\mathbb{Z}_\ell$-basis of $A_n^{(\ell)}$. Let $\mathcal{R} = \{O_E : E/\mathbb{Q}_\ell \text{ is a finite subextension of } \bar{\mathbb{Q}}_\ell\}$, which is a directed set under inclusion. For every $R \in \mathcal{R}$, let m_R be the maximal ideal of R. Let $R[[A_n^{(\ell)}]] := \varprojlim_{i,j\geq 1} (R/m_R^i)[A_n^{(\ell)}/\ell^j]$ be the completed group ring. There is a canonical injective morphism $A_n^{(\ell)} \to R[[A_n^{(\ell)}]]^\times$ of groups.

Fact 3.3.1.2 ([GL96, p.509]). *The ring $R[[A_n^{(\ell)}]]$ is a Noetherian, regular, complete, local domain of Krull dimension $1+n$. There is an isomorphism of topological rings*

$$R[[A_n^{(\ell)}]] \to R[[X_1, \dots, X_n]], \quad \gamma_i \mapsto 1 + X_i. \tag{3.8}$$

Gabber and Loeser introduce a scheme of ℓ-adic characters.

Definition 3.3.1.3. Write $R_n = \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[A_n^{(\ell)}]]$. Define the "cotorus" associated with A_n to be $\mathcal{C}_\ell := \operatorname{Spec} R_n$.

By [GL96, Prop. A.2.2.3 (ii)], the scheme $\mathcal{C}_\ell$ is integral, Noetherian and regular. Its set of closed points coincides with $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$, and it is Zariski dense in $\mathcal{C}_\ell$. If $n > 0$, then $\mathcal{C}_\ell$ is **not** locally of finite type over $\bar{\mathbb{Q}}_\ell$.

Lemma 3.3.1.4. *Every character $\chi : A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ extends canonically to a surjective morphism $R_n \to \bar{\mathbb{Q}}_\ell$ of $\bar{\mathbb{Q}}_\ell$-algebras.*

Proof. There is a finite subextension $E/\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$ containing all the $\chi(\gamma_i)$. Then for every $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{Z}_\ell[[X_1, \dots, X_n]]$, by completeness of E, the series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha \prod_{i=1}^n (\chi(\gamma_i) - 1)^{\alpha_i}$ converges in E. Denote its limit by $f(\chi(\gamma_1)-1, \dots, \chi(\gamma_n)-1)$. The composition $\mathbb{Z}_\ell[[A_n^{(\ell)}]] \to E$ of (3.8) followed by

$$\mathbb{Z}_\ell[[X_1, \dots, X_n]] \to E, \quad f \mapsto f(\chi(\gamma_1) - 1, \dots, \chi(\gamma_n) - 1)$$

extends χ. It induces the stated surjection. The construction is independent of the choice of the $\mathbb{Z}_\ell$-basis of $A_n^{(\ell)}$. $\square$

By Lemma 3.3.1.4, for every $\chi \in \mathcal{C}(A_n)_\ell$, the corresponding character $A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ induces a surjection $R_n \to \bar{\mathbb{Q}}_\ell$. Let $\Psi(\chi)$ be the corresponding element of $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$. Hence a map

$$\Psi : \mathcal{C}(A_n)_\ell \to \mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell). \tag{3.9}$$

Fact 3.3.1.5 ([GL96, p.519])**.** *The map (3.9) is bijective.*

3.3.2 Cotori are Baire

The objective of Section 3.3.2 is Lemma 3.3.2.11, used in the proof of Theorem 3.5.3.1. We show that over an uncountable algebraically closed field, a reasonable scheme has uncountably many rational points outside a countable union of strict closed subsets. Fix an uncountable, algebraically closed field k.

Baire schemes

Definition 3.3.2.1. A scheme X over k is called k-Baire, if its dimension $\dim X$ is finite and $X(k) \setminus \cup_{i \geq 1} Z_i(k)$ is uncountable for every countable sequence $\{Z_i\}_{i \geq 1}$ of closed subschemes of X with $\dim Z_i < \dim X$ for all i. A k-algebra R is called k-Baire if $\mathrm{Spec}(R)$ is k-Baire.

Remark 3.3.2.2. An algebraic curve over k is k-Baire. In Definition 3.3.2.1, The underlying reduced induced closed subscheme $X_{\mathrm{red}} \to X$ induces a bijection $X_{\mathrm{red}}(k) \to X(k)$, so X is k-Baire if and only X_{red} is k-Baire. If X is irreducible and k-Baire, then $X \setminus \cup_{i \geq 1} Z_i$ is Zariski dense in X.

Remark 3.3.2.3. Let $k = \mathbb{C}$. Let X be a complex algebraic variety with $\dim X > 0$. The analytification X^{an} of X is locally compact Hausdorff. Then by the Baire category theorem (see, *e,g.*, [Wil70, Cor. 25.4 a)]), X is $\mathbb{C}$-Baire.

Lemma 3.3.2.4. *Let $f : X \to Y$ be a finite surjective morphism of schemes over k. If Y is k-Baire, then so is X.*

Proof. Let $\{Z_i\}_i$ be a sequence of closed subschemes of X with $\dim Z_i < \dim X$. Then for every integer $i \geq 1$, since f is a closed morphism, $Y_i := f(Z_i)$ is closed in Y. Endow each Y_i with the reduced induced structure. Let $Z'_i := f^{-1}(Y_i) = Y_i \times_Y X$. Then there is a canonical closed immersion $Z_i \to Z'_i$. The restriction $Z_i \to Y_i$ of f is a finite surjective morphism. By [Sta24, Tag 0ECG], one has $\dim X = \dim Y$ and $\dim Y_i = \dim Z_i$. In particular, $\dim X$ is finite and $\dim Y_i < \dim Y$.

As k is algebraically closed, the induced map $X(k) \to Y(k)$ is surjective. Then the induced map

$$X(k) \setminus (\cup_{i\geq 1} Z'_i(k)) \to Y(k) \setminus (\cup_i Y_i(k))$$

is surjective. Because Y is k-Baire, the target is uncountable. Then $X(k) \setminus (\cup_{i\geq 1} Z_i(k))$ is also uncountable, as it contains the source. □

Lemma 3.3.2.5. *Let X be a Noetherian scheme over k.*

1. *Then X is k-Baire if and only if X has an irreducible component C with $\dim C = \dim X$, such that the underlying reduced induced closed subscheme C is k-Baire.*

2. *Assume that $n := \dim X - 1$ is finite. If X has uncountably many (pairwise set-theoretically distinct) irreducible, k-Baire, closed subschemes of dimension n, then X is k-Baire.*

Proof. 1. Assume that there is such a component C. Consider a sequence of closed subschemes $\{Z_i\}_{i\geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Then for every $i \geq 1$, one has $\dim C \cap Z_i \leq \dim Z_i < \dim X = \dim C$. Since C is k-Baire, the set $C(k) \setminus \cup_i (C \cap Z_i)(k)$ is uncountable. Therefore, $X(k) \setminus \cup_i Z_i(k)$ is also uncountable.

   Assume that every component of X of maximum dimension is not k-Baire. As X is Noetherian, one can write $X = \cup_{j=1}^n C_j$ as a finite union of the irreducible components. For every j with $\dim C_j = \dim X$, the scheme C_j is not k-Baire. Therefore, there is a sequence $\{Z_i^j\}_{i\geq 1}$ of closed subschemes of C_j such that $\dim Z_i^j < \dim C_j$ for all i and $C_j(k) \setminus \cup_i Z_i^j(k)$ is countable. The finite family of components C_k with $\dim C_k < \dim X$, joint with the sequences $\{Z_i^j\}_i$ for all j with $\dim C_j = \dim X$, gives a countable family $\{Z_s\}_s$ of closed subschemes of X with $\dim Z_s < \dim X$ for all s. Then $X(k) \setminus (\cup_s Z_s(k))$ is countable, so X is not k-Baire.

2. Consider a sequence of closed subschemes $\{Z_i\}_{i\geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Every Z_i is a Noetherian scheme, so it has only finitely many irreducible components. The set of irreducible components of the family $\{Z_i\}_i$ is countable. Thus, one may assume

that every Z_i is irreducible. By assumption, X has an n-dimensional, irreducible, k-Baire closed subscheme X' which is set-theoretically distinct from any Z_i. For every $i \geq 1$, because $\dim X' = n \geq \dim Z_i$ and Z_i is irreducible, one has $X' \not\subset Z_i$ and $X' \cap Z_i \neq X'$. Since X' is irreducible, one has $\dim(X' \cap Z_i) < \dim X'$. As X' is k-Baire, the set $X'(k) \setminus \cup_{i \geq 1}(X' \times_X Z_i)(k)$ is uncountable, which is a subset of $X(k) \setminus \cup_{i \geq 1} Z_i(k)$. Therefore, X is k-Baire.

□

Lemma 3.3.2.6 is well-known.

Lemma 3.3.2.6. *If X is a finite type scheme over k with $\dim X > 0$, then X is k-Baire.*

Proof. Since X is of finite type over k, its dimension m is finite and X has only finitely many irreducible components. Replacing X with an irreducible component of dimension m, one may that assume X is irreducible. Then by [Har77, Exercise 3.20 (e), p.94], every nonempty open subset of X has dimension m. Replacing X by an affine open, one may assume that X is affine. By Noether's normalization lemma, there is a finite surjective morphism $p : X \to \mathbf{A}_k^m$ over k. By Lemma 3.3.2.4, one may assume $X = \mathbf{A}_k^m$.

By induction on $m > 0$, we prove that $\mathbf{A}_k^m$ is k-Baire. When $m = 1$, one has $\dim \mathbf{A}_k^1 = 1$, and $\mathbf{A}_k^1(k)$ is uncountable. By Remark 3.3.2.2, $\mathbf{A}_k^1$ is k-Baire. Assume the statement for $m-1$ with $m \geq 2$. The set of hyperplanes in $\mathbf{A}_k^m$ is uncountable. By the inductive hypothesis, every hyperplane is k-Baire. From Lemma 3.3.2.5 2, so is $\mathbf{A}_k^m$. The induction is completed. □

Baireness of cotori

We show that every positive dimensional cotorus is $\bar{\mathbb{Q}}_\ell$-Baire.

Definition 3.3.2.7 ([BGR84, Def. 1, p.205])**.** Let A be a k-algebra, and let $A[X] \to B$ be an injective ring map. We say that B is *k-Rückert* over A if there is a nonempty family W of *monic* polynomials in $A[X]$ such that the following axioms are fulfilled:

1. If $f, g \in A[X]$ are monic polynomials with $fg \in W$, then $f, g \in W$.

2. For every $w \in W$, the A-algebra B/w is isomorphic to $A[X]/w$.

3. For every $b \in B \setminus \{0\}$, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ such that $u\sigma(b) \in W$.

Remark 3.3.2.8. From Axiom 1, one gets $1 \in W$. If $W = \{1\}$, then by Axiom 3, for every $b \in B \setminus \{0\}$, one has $b \in B^\times$, *i.e.*, B is a field. Conversely, if B is a field, then B is k-Rückert over A with $W = \{1\}$.

If $W \neq \{1\}$, then $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. Indeed, take $w(\neq 1) \in W$. By Axiom 2, there is an A-isomorphism $B/w \to A[X]/w$, hence an isomorphism $\operatorname{Spec}(A[X]/w) \to \operatorname{Spec}(B/w)$ of schemes over $\operatorname{Spec} A$. Because w is a monic polynomial different from 1, the ring map $A \to A[X]/w$ is injective and finite. The induced morphism $\operatorname{Spec}(A[X]/w) \to \operatorname{Spec}(A)$ is surjective, so $\operatorname{Spec}(B/w) \to \operatorname{Spec}(A)$ is surjective.

For a commutative ring R and an ideal $I \subset R$, let $V_R(I) = \operatorname{Spec} R/I (\subset \operatorname{Spec} R)$. For $r \in R$, let $V_R(r) = V_R(rR)$. Lemma 3.3.2.9 is used in the induction step of the proof of Lemma 3.3.2.11.

Lemma 3.3.2.9. *Let A be Noetherian k-algebra of dimension n. Let B be a domain, but not a field, containing $A[X]$. Assume that B is k-Rückert over A.*

1. *The ring B is Noetherian of dimension $n+1$.*
2. *Suppose that A is k-Baire. Let S be an uncountable subset of A such that for every $s \in S$, one has $\dim V_A(s) = n-1$. Suppose that the family $\{V_A(s)\}_{s\in S}$ is pairwise disjoint. Then B is k-Baire.*

Proof. For every $b \in B \setminus (B^\times \cup \{0\})$, by Axiom 3, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ such that $w := u\sigma(b)$ is in W. Since b is not a unit, one has $w \neq 1$. By Axiom 2, the A-algebra B/w is isomorphic to $A[X]/w$. Since $w(\neq 1)$ is a monic polynomial over A, the ring map $A \to A[X]/w$ is injective finite.

1. One has

$$\dim B/b = \dim B/w = \dim A[X]/w \overset{\text{(a)}}{=} \dim A = n, \tag{3.10}$$

   where (a) uses [Sta24, Tag 00OK]. The domain B is not a field, so $\dim B = n+1$. By [BGR84, Prop. 2, p.206], the ring B is Noetherian.

2. The morphism $\operatorname{Spec} A[x]/w \to \operatorname{Spec} A$ is finite surjective. Then by Lemma 3.3.2.4, the algebra $A[X]/w$ is k-Baire. As σ is over k, the k-algebra B/b is isomorphic to B/w. Then B/b is k-Baire.

   For every $s \in S$, one has $\dim V_A(s) < \dim A$, so $s \neq 0$. As B is not a field, from Remark 3.3.2.8, the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. The preimage of $V_A(s)$ under the surjection $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is $V_B(s)$, so $V_B(s)$ is nonempty. In particular, $s \notin B^\times$ and B/s is k-Baire. Moreover, the family $\{V_B(s)\}_{s\in S}$ is pairwise disjoint. By (3.10), one gets $\dim V_B(s) = n$.

   By Part 1, B is Noetherian. Then for every $s \in S$, by Lemma 3.3.2.5 1, there is a k-Baire irreducible component $C_s \subset \operatorname{Spec}(B/s)$ of dimension n. The family $\{C_s\}_{s\in S}$ is pairwise disjoint. From Lemma 3.3.2.5 2, B is k-Baire.

□

Set $S_n := \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[X_1, \ldots, X_n]])$. By [GL96, Prop. 3.2.2 (1)], the natural morphism $S_n \to \bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]$ is injective. Then the isomorphism (3.8) identifies R_n with the $\bar{\mathbb{Q}}_\ell$-subalgebra $S_n \subset \bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]$.

Fact 3.3.2.10. *For every integer $n \geq 0$,*

1. ([GL96, Thm. A.2.1, Prop, A.2.2.1]) *the ring S_n is a Noetherian, regular, Jacobson domain of Krull dimension n;*

2. ([GL96, Prop A.2.2.2, proof of A.2.2.3 (ii)]) *S_{n+1} is $\bar{\mathbb{Q}}_\ell$-Rückert over S_n.*

Lemma 3.3.2.11. *For every integer $n \geq 1$, the algebra S_n is $\bar{\mathbb{Q}}_\ell$-Baire.*

Proof. Since $\bar{\mathbb{Q}}_\ell$ is a flat $\mathbb{Z}_\ell$-module, the injection $\mathbb{Z}_\ell[X_1, \ldots, X_n] \to \mathbb{Z}_\ell[[X_1, \ldots, X_n]]$ induces an injection $\bar{\mathbb{Q}}_\ell[X_1, \ldots, X_n] \to S_n$. The natural morphism

$$\mathrm{Spec}(\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell} \tag{3.11}$$

of schemes over $\bar{\mathbb{Q}}_\ell$ factors through a morphism $p_n : \mathrm{Spec}(S_n) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell}$.

Let $\mathcal{M} = \cup_E m_E$, where E runs through all finite subextensions of $\mathbb{Q}_\ell \subset \bar{\mathbb{Q}}_\ell$, and m_E is the maximal ideal of the ring of integers of E. Then $\mathcal{M}$ is the maximal ideal of the integral closure $\overline{\mathbb{Z}_\ell}$ of $\mathbb{Z}_\ell$ inside $\bar{\mathbb{Q}}_\ell$. By [Rob00, Prop., p.128], the residue field $\overline{\mathbb{Z}_\ell}/\mathcal{M}$ is an algebraic closure of the finite field F_ℓ, so it is countable. As $\mathbb{Z}_\ell$ is uncountable, so is the set $\mathcal{M}$.

For every $(a_1, \ldots, a_n) \in \mathcal{M}^n$, there is a surjective morphism of $\bar{\mathbb{Q}}_\ell$-algebras:

$$\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]] \to \bar{\mathbb{Q}}_\ell, \quad f \mapsto f(a_1, \ldots, a_n).$$

Its kernel is a $\bar{\mathbb{Q}}_\ell$-point of $\mathrm{Spec}(\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]])$, whose image under (3.11) is $(a_1, \ldots, a_n) \in A^n_{\bar{\mathbb{Q}}_\ell}(\bar{\mathbb{Q}}_\ell)$. Hence $\mathcal{M}^n \subset p_n(\mathrm{Spec}(S_n)(\bar{\mathbb{Q}}_\ell))$. In particular, $\mathrm{Spec}(S_n)(\bar{\mathbb{Q}}_\ell)$ is uncountable.

By induction on $n > 0$, we prove that S_n is $\bar{\mathbb{Q}}_\ell$-Baire, and $\{V_{S_n}(X_1 - a)\}_{a \in \mathcal{M}}$ is a pairwise disjoint family of $(n-1)$-dimensional subsets. When $n = 1$, by Fact 3.3.2.10 1, S_1 is $\bar{\mathbb{Q}}_\ell$-Baire. Moreover, $\{V_{S_1}(X_1 - a)\}_{a \in \mathcal{M}}$ is a pairwise distinct family of closed point of $\mathrm{Spec}(S_1)$. The statement is proved for $n = 1$. Assume the statement for $n - 1$ with $n \geq 2$. By Fact 3.3.2.10, (3.10), and Lemma 3.3.2.9 2, the statement holds for n. The induction is completed. □

3.4 Krämer-Weissauer theory

Let k be a field of characteristic zero. Let Vec_k be the category of finite dimensional k-vector spaces. Choose an algebraic closure $\bar{k}$ of k. Let

$\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ be the category of continuous, finite dimensional $\bar{\mathbb{Q}}_\ell$-representations of Γ_k. Let A be an abelian variety over k. Recall that $\pi_1^{\text{ét}}(A_{\bar{k}})$ is a free $\hat{\mathbb{Z}}$-module of rank $2\dim A$. With the notation of Section 3.3, set

- $\mathcal{C}(A) = \mathcal{C}(\pi_1^{\text{ét}}(A_{\bar{k}}))$: the group of characters $\pi_1^{\text{ét}}(A_{\bar{k}}) \to \bar{\mathbb{Q}}_\ell^*$;
- $\mathcal{C}(A)_{\ell'} = \mathcal{C}(\pi_1^{\text{ét}}(A_{\bar{k}}))_{\ell'}$: the group of characters of finite order prime to ℓ;
- $\mathcal{C}(A)_\ell$: the cotorus assigned to $\pi_1^{\text{ét}}(A_{\bar{k}})$.

3.4.1 Generic vanishing theorem

For an object $K \in \mathrm{Perv}(A)$, set

$$\mathcal{S}(K) := \{\chi \in \mathcal{C}(A) | H^i(A_{\bar{k}}, K \otimes^L L_\chi) \neq 0 \text{ for some integer } i \neq 0\}.$$

Fact 3.4.1.1 ([KW15b, Thm. 1.1], [Wei16, Vanishing Theorem, p.561, Thm. 2])**.** *For every perverse sheaf $K \in \mathrm{Perv}(A)$ and every character $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, the set*

$$\{\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) | \chi_{\ell'}\chi_\ell \in \mathcal{S}(K)\}$$

is the set of $\bar{\mathbb{Q}}_\ell$-points of a strict Zariski closed subset of the scheme $\mathcal{C}(A)_\ell$.

We review [KW15a, p.725]. Because of $\mathrm{char}(k) = 0$, for every $K \in \mathrm{Perv}(A)$, its Euler characteristic satisfies

$$\chi(A, K) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\bar{\mathbb{Q}}_\ell} H^i(A_{\bar{k}}, K) \geq 0. \tag{3.12}$$

Let $N(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $\chi(A, K) = 0$. From the additivity of the function $\chi(A, -) : \mathrm{Ob}(\mathrm{Perv}(A)) \to \mathbb{N}$ and (3.12), $N(A)$ is a Serre subcategory of $\mathrm{Perv}(A)$. Let $\bar{P}(A) := \mathrm{Perv}(A)/N(A)$ be the quotient abelian category. For every $\chi \in \mathcal{C}(A)$, set

$$\mathcal{E}^\chi(A_{\bar{k}}) = \{K \in \mathrm{Perv}(A_{\bar{k}}) | H^i(A_{\bar{k}}, K \otimes^L L_\chi) = 0, \quad \forall i \in \mathbb{Z} \setminus \{0\}\}.$$

Then $\mathcal{E}^\chi(A_{\bar{k}})$ is closed under extensions in $\mathrm{Perv}(A_{\bar{k}})$. Let $P^\chi(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $Q \in \mathcal{E}^\chi(A_{\bar{k}})$ for every simple subquotient Q of $K|_{A_{\bar{k}}}$ in $\mathrm{Perv}(A_{\bar{k}})$.

By [Bei+82, Thm. 4.3.1 (i)], every object $K \in \mathrm{Perv}(A)$ is Noetherian and Artinian. For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, by Fact 3.4.1.1 and Lemma 3.2.1.8 1, the set $\{\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) | K \in P^{\chi_{\ell'}\chi_\ell}(A)\}$ is the set of $\bar{\mathbb{Q}}_\ell$-points of a strict Zariski closed subset of $\mathcal{C}(A)_\ell$.

Lemma 3.4.1.2. *Let $\mathcal{A}$ be a Noetherian and Artinian abelian category. Let $\mathcal{E}$ be a class of objects of $\mathcal{A}$ closed under isomorphisms. Let $\mathcal{S} \subset \mathcal{A}$ be the full subcategory of objects every nonzero simple subquotient of which is in $\mathcal{E}$.*

1. *Then $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$.*

2. *If further $\mathcal{E}$ is closed under extensions, then $\mathcal{S} \subset \mathcal{E}$.*

Proof.

1. (a) We prove that $\mathcal{S}$ is closed under subquotients. Let X be an object of $\mathcal{S}$ with a subquotient Y. Every simple subquotient of Y is that of X, hence in $\mathcal{E}$. Thus, $Y \in \mathcal{S}$.

   Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathcal{A}$ with $L, N \in \mathcal{S}$. Let Q be a nonzero simple subquotient of M. We prove that $Q \in \mathcal{E}$.

   (b) First, assume that Q is a quotient of M. The natural morphism $L \to Q$ is either an epimorphism or zero, in which case Q is a simple quotient of L or N respectively. Hence $Q \in \mathcal{E}$.

   (c) Now assume that Q is general. There is a subobject $M_0 \subset M$ and an epimorphism $M_0 \to Q$. Then

   $$0 \to f^{-1}(M_0) \to M_0 \to g(M_0) \to 0$$

   is a short exact sequence in $\mathcal{A}$ with $f^{-1}(M_0)$ (resp. $g(M_0)$) a subobject of L (resp. N). From Part 1a, both $f^{-1}(M_0)$ and $g(M_0)$ are in $\mathcal{S}$. From Part 1b, one has $Q \in \mathcal{E}$.

   From Part 1c, one has $M \in \mathcal{S}$, and $\mathcal{S}$ is closed under extensions. The result follows from [Sta24, Tag 02MP].

2. By [Sta24, Tag 0FCJ], every object $X \in \mathcal{S}$ admits a filtration in $\mathcal{A}$

   $$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

   by subobjects such that each X_i/X_{i-1} is a simple subquotient of X. Then $X_i/X_{i-1} \in \mathcal{E}$. As $\mathcal{E}$ is closed under extensions, one has $X \in \mathcal{E}$.

□

By Lemma 3.4.1.2 1, for every $\chi \in \mathcal{C}(A)$, the subcategory $P^\chi(A) \subset \mathrm{Perv}(A)$ is a Serre subcategory. From Lemma 3.4.1.2 2, for every $K \in P^\chi(A)$ and every integer $i \neq 0$, one has

$$H^i(A_{\bar{k}}, K \otimes^L L_\chi) = 0. \tag{3.13}$$

From the proof of [LS20, Lem. 3.4 (3)], the functor

$$\omega_\chi = H^0(A_{\bar{k}}, \cdot \otimes^L L_\chi) : P^\chi(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \tag{3.14}$$

is exact. Let $N^\chi(A)$ be the full subcategory of $P^\chi(A)$ of objects in $N(A)$. For every $K \in N^\chi(A)$, by [KW15b, Cor. 4.2], one has $\chi(A, K \otimes^L L_\chi) = 0$. From (3.13), one has $H^0(A_{\bar{k}}, K \otimes^L L_\chi) = 0$. By [Sta24, Tag 02MS], the functor ω_χ factors uniquely through an exact functor (still denoted by ω_χ)

$$P^\chi(A)/N^\chi(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}. \tag{3.15}$$

3.4.2 Tannakian groups

Let $(\mathcal{C}, \otimes)$ a neutral Tannakian category (in the sense of [DM22, Def. 2.19]) over an algebraically closed field Q of characteristic 0, with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_Q$. Let $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ be the corresponding affine group scheme over Q. By [Del90, Sec. 9.2, p.187], up to isomorphism of group schemes, $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ is independent of the choice of ω. (See [Wib22, Thm. 1.2] for an elementary proof.)

For an object $K \in \mathcal{C}$, let $\iota : \langle K \rangle \hookrightarrow \mathcal{C}$ be the full subcategory whose objects are the subquotients of $\{(K \oplus K^{\vee})^{\otimes n}\}_{n \geq 1}$. Then $(\langle K \rangle, \otimes)$ is a neutral Tannakian subcategory of $\mathcal{C}$ (in the sense of [Mil07, 1.7]), for which $\omega\iota : \langle K \rangle \to \mathrm{Vec}_Q$ is a fiber functor. The group scheme $\mathrm{Aut}^{\otimes}(\langle K \rangle, \omega\iota)$ is the image of the natural morphism $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega) \to \mathrm{GL}(\omega(K))$.

Definition 3.4.2.1. The algebraic group $\mathrm{Aut}^{\otimes}(\langle K \rangle, \omega\iota)$ is called the Tannakian monodromy group of K at ω and is denoted by $G_\omega(K)$.

By [Sim92, p.69], $G_\omega(K)$ is reductive if and only if K is semisimple in $\mathcal{C}$.

Example 3.4.2.2. With tensor product, $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ is a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. The forgetful functor $\omega : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ is a fiber functor. The Tannakian monodromy group of an object $\rho : \Gamma_k \to \mathrm{GL}(V)$ at ω is the Zariski closure of $\rho(\Gamma_k) \subset \mathrm{GL}(V)$.

3.4.3 Sheaf convolution

Let $m : A \times_k A \to A$ be the group law on A. Let $p_i : A \times_k A \to A$ be the projection to i-th factor ($i = 1, 2$). The bifunctor

$$* : D^b_c(A) \times D^b_c(A) \to D^b_c(A), \quad - * + := Rm_*(p_1^* - \otimes^L p_2^* +)$$

is called the *convolution* on A.

Example 3.4.3.1. For every closed reduced subvariety $i : X \to A$, let $\delta_X := i_* \bar{\mathbb{Q}}_{\ell,X} \in D^b_c(A)$. Then for every closed point $x \in A$, one has $\delta_x * \delta_X = \delta_{x+X}$.

By [Wei11] and [JKLM23, Sec. 3.1], the pair $(D^b_c(A), *)$ is a rigid, symmetric monoidal category, with unit δ_0. For every $K \in D^b_c(A)$, its adjoint dual is $K^\vee := [-1]_A^* \mathbb{D}_A K$.

Fact 3.4.3.2 ([KW15b, proof of Thm. 13.2], [LS20, Lem. 3.4 (4)], [JKLM23, Prop. 3.1])**.** *The convolution on A induces a bifunctor*

$$\bar{P}(A) \times \bar{P}(A) \to \bar{P}(A), \quad (-, +) \mapsto {}^p\mathcal{H}^0(- * +)$$

fitting into a commutative square

$$\begin{tikzcd}
\mathrm{Perv}(A) \times \mathrm{Perv}(A) \arrow[r, "*"] \arrow[d] & D_c^b(A) \arrow[d, "{}^p\mathcal{H}^0"] \\
\bar{P}(A) \times \bar{P}(A) \arrow[r, dashed] & \bar{P}(A).
\end{tikzcd}$$

It makes $\bar{P}(A)$ a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. For every $\chi \in \mathcal{C}(A)$, the subcategory $P^\chi(A)/N^\chi(A) \subset \bar{P}(A)$ is a Tannakian subcategory, on which (3.15) is a fiber functor.

Example 3.4.3.3. [KW15a, Example 7.1] Fix a closed point $x \in A$. Then $\delta_x \in \mathrm{Perv}(A)$. Then $\mathcal{S}(\delta_x)$ is empty and for every $\chi \in \mathcal{C}(A)$, one has $\delta_x \in P^\chi(A)$. If x is a torsion point of order n, then $G_{\omega_\chi}(\delta_x)$ is isomorphic to $\mathbb{Z}/n$. If x is not a torsion point, then $G_{\omega_\chi}(\delta_x)$ is isomorphic to $\mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$.

Let $\psi : \pi_1^{\text{ét}}(A) \to \bar{\mathbb{Q}}_\ell^\times$ be a character, and set $\psi' = \psi|_{\pi_1^{\text{ét}}(A_{\bar{k}})}$. The functor

$$\omega_\psi : \mathrm{Perv}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \quad K \mapsto H^0(A_{\bar{k}}, K \otimes^L L_\psi)$$

fits into a commutative square

$$\begin{tikzcd}
\mathrm{Perv}(A) \arrow[r, "\omega_\psi"] & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \arrow[d, "\omega"] \\
P^{\psi'}(A) \arrow[u, hook] \arrow[r, "(3.14)"] & \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}
\end{tikzcd}$$

The quotient functor $P^{\psi'}(A)/N^{\psi'}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ of $\omega_\psi|_{P^{\psi'}(A)}$ induces a morphism of affine groups schemes

$$\omega_\psi^* : \mathrm{Aut}^\otimes(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \omega) \to \mathrm{Aut}^*(P^{\psi'}(A)/N^{\psi'}(A), \omega_{\psi'}). \tag{3.16}$$

Definition 3.4.3.4. For every $K \in \mathrm{Perv}(A)$, let $\mathrm{Mon}(K, \psi)$ be the Tannakian monodromy group of $\omega_\psi(K)$ in $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$.

For every $K \in P^{\psi'}(A)$, the functor $\omega_\psi|_{\langle K \rangle} : \langle K \rangle \to \langle \omega_\psi(K) \rangle$ induces a closed immersion of linear algebraic groups $\omega_\psi^* : \mathrm{Mon}(K, \psi) \to G_{\omega_{\psi'}}(K)$, which is the projection of (3.16) in $\mathrm{GL}(\omega_{\psi'}(K))$.

3.5 Main results

Consider Setting 3.1.2.1. For every character $\chi \in \mathcal{C}(A)$, denote the pullback of χ along $(p_A|_{A_\eta})_* : \pi_1^{\text{ét}}(A_\eta) \to \pi_1^{\text{ét}}(A)$ by $\chi_\eta : \pi_1^{\text{ét}}(A_\eta) \to \bar{\mathbb{Q}}_\ell^\times$. Then the restriction $\chi_\eta|_{\pi_1^{\text{ét}}(A_{\bar{\eta}})}$ is identified with χ *via* the isomorphism $(p_A|_{A_{\bar{\eta}}})_* : \pi_1^{\text{ét}}(A_{\bar{\eta}}) \to \pi_1^{\text{ét}}(A)$. Let $K \in \mathrm{Perv}(A \times X/X)$ be an object which is semisimple in $D_c^b(A \times X)$.

We shall prove that the monodromy group of K is normal in its Tannakian group. By the normality criterion (Lemma 3.5.0.1), it suffices to show that the monodromy is reductive, and to consider the monodromy fixed part of *all* the representations of the Tannakian group. Such representations are from perverse sheaves.

Lemma 3.5.0.1. *Let G be a linear algebraic group over an algebraically closed field C. Let H be a closed, reductive subgroup of G. If for every $V \in \mathrm{Rep}_C(G)$, the subspace V^H is G-stable, then H is normal in G.*

Proof. By [Gro06, Cor. 2.4] and reductivity, H is observable in G (in the sense of [BBHM63, p.134]). From [And21, Prop. C.3], H is normal in G. $\square$

3.5.1 Reductivity

Theorem 3.5.1.1. *For every $\chi \in \mathcal{C}(A) \setminus \mathcal{S}(K|_{A_\eta})$, the monodromy group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ is reductive.*

Proof. By Lemma 3.2.1.5, when X is replaced by a nonempty open subset, the semisimplicity of K in $D_c^b(A \times X)$ is preserved. Moreover, the $\Gamma_{k(\eta)}$-representation $\omega_{\chi_\eta}(K|_{A_\eta})$ and hence the group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ remain unchanged. Thus, by [Sta24, Tag 056V], one may assume that X is smooth. As K is semisimple in $D_c^b(A \times X)$, from Lemma 3.2.1.10, so is $K \otimes^L p_A^* L_\chi$. By Fact 3.2.1.6 1, the object $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is semisimple in $D_c^b(X)$.

By the proper base change theorem (see, *e.g.*, [Sta24, Tag 095T]), for every integer n, one has

$$\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi)_{\bar{\eta}} = H^n(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi).$$

Since $\chi \notin \mathcal{S}(K|_{A_\eta})$, when $n \neq 0$, one has $H^n(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi) = 0$. By Fact 3.2.1.2, there is a nonempty open subset U_0 (resp. U_n for every integer $n \neq 0$) of X such that $[\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)]|_{U_0}$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf (resp. $[\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi)]|_{U_n} = 0$). The set

$$J := \{n \in \mathbb{Z} : \mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) \neq 0\}$$

is finite and X is irreducible, so $U := U_0 \cap \cap_{n \in J} U_n$ is a nonempty open subset of X. Shrinking X to U, one may assume further that $\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) = 0$ for every integer $n \neq 0$, and that $\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X.

Thus, the semisimple object $Rp_{X*}(K \otimes^L p_A^* L_\chi)[\dim X]$ of $D_c^b(X)$ lies in $\mathrm{Perv}(X)$, so it is semisimple in $\mathrm{Perv}(X)$. By [Ach21, Prop. 3.4.1], the object $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ of $\mathrm{Loc}(X)$ is semisimple. Therefore, the corresponding representation

$$\pi_1^{\text{ét}}(X, \bar{\eta}) \to \mathrm{GL}(H^0(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi))$$

is semisimple. Because X is smooth, the natural morphism $\eta_* : \Gamma_{k(\eta)} \to \pi_1^{\text{ét}}(X, \bar{\eta})$ is surjective. Then the composition $\Gamma_{k(\eta)} \to \mathrm{GL}(H^0(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi))$, *i.e.*, the representation $\omega_{\chi_\eta}(K|_{A_\eta})$, is semisimple. Consequently, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ is reductive. □

Example 3.5.1.2. In Theorem 3.5.1.1, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ may not be semisimple. Let X be a smooth, projective, integral algebraic curve over k of genus 1. Then $\pi_1^{\text{ét}}(X, \bar{\eta}) \cong \hat{\mathbb{Z}}^2$. There exists a character $\sigma : \pi_1^{\text{ét}}(X, \bar{\eta}) \to \bar{\mathbb{Q}}_\ell^\times$ of *infinite* order. Let $A = \mathrm{Spec}(k)$. Then $\mathcal{C}(A) = \{1\}$ and $\mathrm{Mon}(L_\sigma|_{A_\eta}, 1) = \mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$ is an algebraic torus.

Remark 3.5.1.3. In view of Example 3.2.1.4, the semisimplicity of $\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)$ in $D_c^b(X)$ is not clear *a priori*. That is why we exclude characters in the spectrum $\mathcal{S}(K|_{A_\eta})$ in Theorem 3.5.1.1.

Remark 3.5.1.4. Let $i : Y \to A \times X$ be a closed subvariety, such that the induced morphism $f : Y \to X$ is smooth with connected fibers of dimension d:

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, "f"] & A \times X \arrow[dl, "p_X"] \arrow[d, "p_A"] \\ X & A. \end{tikzcd}$$

By Example 3.2.3.4, one has $K := i_* \bar{\mathbb{Q}}_{\ell,Y}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By Fact 3.2.1.6 1, it is semisimple in $D_c^b(A \times X)$. Assume that X is smooth. Then for every $\chi \in \mathcal{C}(A) \setminus \mathcal{S}(K|_{A_\eta})$, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ coincides with the Zariski closure of the image of the monodromy representation of the lisse $\bar{\mathbb{Q}}_\ell$-sheaf $R^d f_* i^* p_A^* L_\chi$ on X, which is studied in [KM23, Sec. 1.4] (but with coefficient $\mathbb{C}$ instead of $\bar{\mathbb{Q}}_\ell$).

3.5.2 Fixed part

Theorem 3.1.2.3 follows from Theorem 3.5.2.1 and Fact 3.4.1.1, because the union in Condition 1 of Theorem 3.5.2.1 is in fact a finite union.

Theorem 3.5.2.1. *Assume that X is smooth and $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Then there exists a subobject $K^0 \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ such that for every $\chi \in \mathcal{C}(A)$ with*

1. $\chi \notin \cup_{j \in \mathbb{Z}} \mathcal{S}({}^p\mathcal{H}^j(Rp_{A*}K))$,
2. $K|_{A_\eta} \in P^\chi(A_\eta)$ *and*
3. ${}^p\mathcal{H}^0(Rp_{A*}K) \in P^\chi(A)$,

one has $\omega_{\chi_\eta}(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$.

Proof. By properness of $p_X : A \times X \to X$ and Fact 3.2.2.2 4, one has $Rp_{X*}K \in D^{\mathrm{ULA}}(X/X)$. Then from Fact 3.2.2.2 2, the sheaf $\mathcal{H}^0 Rp_{X*}K$ is lisse. Since X is smooth, by [Sta24, Tag 0BQM], the canonical morphism $\Gamma_{k(\eta)} \to \pi_1^{\text{ét}}(X, \bar{\eta})$ is surjective. Thus, from Fact 3.2.1.6 2, the natural map

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) \to \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}} \tag{3.17}$$

is surjective.

By Fact 3.2.1.1, one has

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) = H^0(A, (Rp_{A*}K) \otimes^L L_\chi). \tag{3.18}$$

By Condition 1, for any integers $i \neq 0$ and j, one has

$$H^i(A, {}^p\mathcal{H}^j(Rp_{A*}K) \otimes^L L_\chi) = 0.$$

By Lemma 3.2.1.10, the spectral sequence in [Max19, Rk. 8.1.14 (6)] becomes

$$E_2^{i,j} = H^i(A, {}^p\mathcal{H}^j(Rp_{A*}K) \otimes^L L_\chi) \Rightarrow H^{i+j}(A, (Rp_{A*}K) \otimes^L L_\chi).$$

It degenerates at page E_2. Hence

$$H^0(A, (Rp_{A*}K) \otimes^L L_\chi) = H^0(A, ({}^p\mathcal{H}^0 Rp_{A*}K) \otimes^L L_\chi). \tag{3.19}$$

Set $K^1 := p_A^* {}^p\mathcal{H}^0(Rp_{A*}K) \in D_c^b(A \times X)$. By Fact 3.2.2.2 1, one has ${}^p\mathcal{H}^0(Rp_{A*}K) \in D^{\mathrm{ULA}}(A/k)$. From Fact 3.2.2.2 3, one gets $K^1 \in D^{\mathrm{ULA}}(A \times X/X)$. For every $x \in X(k)$, the restriction $p_A|_{A_x} : A_x \to A$ is an isomorphism of abelian varieties over k, so the functor $(p_A|_{A_x})^* : \mathrm{Perv}(A) \to \mathrm{Perv}(A_x)$ is an equivalence of abelian categories. It sends ${}^p\mathcal{H}^0(Rp_{A*}K)$ to $K^1|_{A_x}$, so $K^1|_{A_x} \in \mathrm{Perv}(A_x)$ and hence $K^1 \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. From $K^1|_{A_\eta} = (p_A|_{A_\eta})^* {}^p\mathcal{H}^0(Rp_{A*}K)$ and Condition 3, one has $K^1|_{A_\eta} \in P^\chi(A_\eta)$. Then

$$\omega_\chi(K^1|_{A_\eta}) = H^0(A, {}^p\mathcal{H}^0(Rp_{A*}K) \otimes^L L_\chi). \tag{3.20}$$

Every fiber of $p_A : A \times X \to A$ has dimension $\dim X$, so by [Bei+82, 4.2.4], the functor

$$Rp_{A*}[-\dim X] : D_c^b(A \times X) \to D_c^b(A)$$

is left t-exact for the absolute perverse t-structures. From Lemma 3.2.3.8, as X is smooth and K is ULA, one has $K[\dim X] \in \mathrm{Perv}(A \times X)$ and so $Rp_{A*}K \in {}^pD^{\geq 0}(A)$. Taking the perverse truncation, one has ${}^p\tau^{\leq 0}(Rp_{A*}K) = {}^p\mathcal{H}^0(Rp_{A*}K)$. *Via* the adjunction formula (see, *e.g.*, [KW01, p.107]), the natural morphism

$${}^p\tau^{\leq 0}(Rp_{A*}K) \to Rp_{A*}K$$

in $D_c^b(A)$ (from the definition of t-structure) induces a morphism $h : K^1 \to K$ in $D_c^b(A \times X)$. Then h is a morphism in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Let K^0 be the image of h in the abelian category $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By Fact 3.2.3.1 1, the functor $\mathrm{Perv}(A \times X/X) \to \mathrm{Perv}(A_\eta)$ is exact. Then $K^0|_{A_\eta}$ is the image of $h|_{A_\eta} : K^1|_{A_\eta} \to K|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$.

Because $P^\chi(A_\eta)$ is an abelian subcategory of $\mathrm{Perv}(A_\eta)$, by Condition 2, the image of $h|_{A_\eta}$ in $P^\chi(A_\eta)$ is still $K^0|_{A_\eta}$. As the functor (3.14) is exact, the image of $\omega_\chi(h|_{A_\eta}) : \omega_\chi(K^1|_{A_\eta}) \to \omega_\chi(K|_{A_\eta})$ is $\omega_\chi(K^0|_{A_\eta})$. Combining (3.17), (3.18), (3.19) with (3.20), one gets $\omega_\chi(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$. □

3.5.3 Normality

By [JKLM23, Thm. 4.3], for every character $\chi \in \mathcal{C}(A)$, the geometric generic Tannakian group $G_{\omega_\chi}(K|_{A_{\bar\eta}})$ is a normal closed subgroup of the generic Tannakian group $G_{\omega_\chi}(K|_{A_\eta})$. Theorem 3.5.3.1 shows that for uncountably many characters, the corresponding monodromy group is also a normal closed subgroup of the generic Tannakian group.

For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$ and every $\chi_\ell \in \mathcal{C}(A)_\ell$, set $\chi = \chi_{\ell'}\chi_\ell$.

Theorem 3.5.3.1. *For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, there is a countable union $B = \cup_{i\geq 1} B_i$ of strict closed subsets of $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus B$,*

- *one has $K|_{A_\eta} \in P^\chi(A_\eta)$;*
- *the algebraic group $G_{\omega_\chi}(K|_{A_\eta})$ is reductive;*
- *and* $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ *is a* normal *closed subgroup of $G_{\omega_\chi}(K|_{A_\eta})$.*

By Lemma 3.3.2.11, when $\dim A > 0$, the set $\mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell)\setminus B$ is uncountable. Thus, Theorem 3.1.2.2 follows from Theorem 3.5.3.1. We sketch the proof of Theorem 3.5.3.1. For every representation V of the Tannakian group $G(K|_{A_\eta})$ and every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, by Theorem 3.1.2.3, there is a strict Zariski closed subset B_V of the cotorus $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_V)(\bar{\mathbb{Q}}_\ell)$, the monodromy invariant $V^{\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)}$ is a $G(K|_{A_\eta})$-subrepresentation. Choose $B = \cup_V B_V(\bar{\mathbb{Q}}_\ell)$. From Lemma 3.5.0.1, normality holds when $\chi_\ell \notin B$.

Proof. Both $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ and $G_{\omega_\chi}(K|_{A_\eta})$ depend only on the generic fiber of $p_X : A \times X \to X$. Therefore, shrinking X to a nonempty open subset does not change them. Thus, one may assume that X is smooth. By [SGA 4 1/2, Thm. 2.13, p.242], one may assume further $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By smoothness of X and Lemma 3.2.3.7, the object $K|_{A_\eta}$ of $\mathrm{Perv}(A_\eta)$ is semisimple. From Lemma 3.5.3.5 1, the object $K|_{A_\eta} \in \bar{P}(A_\eta)$ is also semisimple. Therefore, a (hence every) Tannakian group of the neutral Tannakian category $\langle K|_{A_\eta}\rangle (\subset \bar{P}(A_\eta))$ is a *reductive*, algebraic group over $\bar{\mathbb{Q}}_\ell$. Then by Lemma 3.5.3.4, there is a countable sequence of objects $\{\bar{K}_i\}_{i\geq 1}$, such that every object of $\langle K|_{A_\eta}\rangle$ is isomorphic to some $\bar{K}_i$. To apply Theorem 3.1.2.3, we need semisimple objects of $D^b_c(A \times X)$.

Claim 3.5.3.2. For every object $N \in \langle K|_{A_\eta}\rangle$, there is $L \in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ that is semisimple in $D^b_c(A\times X)$, such that $L|_{A_\eta}$ isomorphic to N in $\bar{P}(A_\eta)$.

From Claim 3.5.3.2, for every integer $i \geq 1$, there is $K_i \in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ that is semisimple in $D^b_c(A\times X)$ with $K_i|_{A_\eta}$ isomorphic to $\bar{K}_i$ in $\bar{P}(A_\eta)$. From smoothness of X and Theorem 3.1.2.3, there is a subobject $K_i^0 \subset K_i$ in $\mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ and a strict Zariski closed subset $B_i \subset \mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_i)(\bar{\mathbb{Q}}_\ell)$, one has $K_i|_{A_\eta} \in P^\chi(A_\eta)$ and

$$\omega_{\chi_\eta}(K_i|_{A_\eta})^{\Gamma_{k(\eta)}} = \omega_{\chi_\eta}(K_i^0|_{A_\eta}). \tag{3.21}$$

Set $B := \cup_{i\geq 1} B_i$. For every $\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell)\setminus B$, one has $K|_{A_\eta} \in P^\chi(A_\eta)$. For every $i \geq 1$, by $\chi_\ell \notin B_i(\bar{\mathbb{Q}}_\ell)$ and (3.21), the subspace $\omega_{\chi_\eta}(K_i|_{A_\eta})^{\mathrm{Mon}(K|_{A_\eta},\chi_\eta)}$ is $G_{\omega_\chi}(K|_{A_\eta})$-stable. By Theorem 3.5.1.1 and Lemma 3.5.0.1, the subgroup $\mathrm{Mon}(K|_{A_\eta},\chi_\eta)$ of $G_{\omega_\chi}(K|_{A_\eta})$ is *normal*. □

Proof of Claim 3.5.3.2. From Lemma 3.5.3.4, the object $N \in \bar{P}(A_\eta)$ is semisimple. There is an integer $n \geq 0$ such that N is a subquotient of $(K|_{A_\eta} \oplus K|_{A_\eta}^\vee)^{*n}$ in $\bar{P}(A_\eta)$.

We "globalize" the fiberwise convolution functors as follows. Define a bifunctor

$$\begin{aligned} *_X : & D^b_c(A\times X)\times D^b_c(A\times X) \to D^b_c(A\times X), \\ & (-,+) \mapsto R(m\times \mathrm{Id}_X)_*(p_{13}^* - \otimes^L p_{23}^* +), \end{aligned} \tag{3.22}$$

where p_{ij} are the projections on $A\times A\times X$. By the proper base change theorem, for every $x \in X(k)$, one has $(- *_X +)|_{A_x} \xrightarrow{\sim} (-|_{A_x}) * (+|_{A_x})$ as bifunctors $D^b_c(A\times X)\times D^b_c(A\times X) \to D^b_c(A_x)$. Therefore, one has $(- *_X +)|_{A_\eta} \xrightarrow{\sim} (-|_{A_\eta}) * (+|_{A_\eta})$ as bifunctors $D^b_c(A\times X)\times D^b_c(A\times X) \to D^b_c(A_\eta)$.

The bifunctor (3.22) restricts to a bifunctor $D^{\mathrm{ULA}}(A\times X/X)\times D^{\mathrm{ULA}}(A\times X/X) \to D^{\mathrm{ULA}}(A\times X/X)$. Indeed, for any $K', K'' \in D^{\mathrm{ULA}}(A\times X/X)$, by Fact 3.2.2.2 6, one has

$$p_{13}^*K' \otimes^L p_{23}^*K'' \in D^{\mathrm{ULA}}(A\times A\times X/X).$$

By Fact 3.2.2.2 4, one gets $K' *_X K'' \in D^{\mathrm{ULA}}(A\times X/X)$.

Set $K^\vee := ([-1]_A \times \mathrm{Id}_X)^*\mathbb{D}_{A\times X/X}K$. By Fact 3.2.3.1 3, one has $K^\vee \in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ and $(K^\vee)|_{A_\eta} = (K|_{A_\eta})^\vee$. Then

$$(K\oplus K^\vee)^{*_X n}) \in D^{\mathrm{ULA}}(A\times X/X).$$

Set $M := {}^{p/X}\mathcal{H}^0((K\oplus K^\vee)^{*_X n}) \in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$. Then $M|_{A_\eta} = {}^p\mathcal{H}^0([K|_{A_\eta} \oplus (K|_{A_\eta})^\vee]^{*n})$ in $\mathrm{Perv}(A_\eta)$. By Lemma 3.5.3.5 3, there is a semisimple subquotient L' of $M|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$, whose image in $\bar{P}(A_\eta)$ is N. By smoothness of X and Fact 3.2.3.6, there is a semisimple subquotient L of M in $\mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ with $L|_{A_\eta} = L'$. By smoothness of X and Lemma 3.2.3.12 2, the object $L[\dim X]$ is semisimple in $\mathrm{Perv}(A\times X)$. Then L is semisimple in $D^b_c(A\times X)$. □

Remark 3.5.3.3. When $A = \mathrm{Spec}(k)$, the bifunctor (3.22) becomes $\otimes^L : D_c^b(X) \times D_c^b(X) \to D_c^b(X)$. The derived tensor product may not preserve semisimplicity in the category $D_c^b(X)$. That is why we need semisimplicity in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ in the last paragraph of the proof of Claim 3.5.3.2.

In fact, consider $k = \mathbb{C}$, $X = A^1$ and $U = X \setminus \{0\}$. Let $j : U \to X$ be the inclusion. Then $\pi_1^{\text{ét}}(U, 1) = \hat{\mathbb{Z}}$. The unique surjective morphism $\pi_1^{\text{ét}}(U, 1) \to \mathbb{Z}/2$ corresponds to a rank one lisse $\bar{\mathbb{Q}}_\ell$-sheaf L on U. Then $L \otimes^L L \cong \bar{\mathbb{Q}}_{\ell,U}$, and L^{an} is a $\bar{\mathbb{Q}}_\ell$-local system on $U^{\mathrm{an}} = \mathbb{C} \setminus \{0\}$.

Let U_0 be a punctured ball in $X^{\mathrm{an}} = \mathbb{C}$ centered at 0 containing 1. One has $H^0(U_0, L^{\mathrm{an}}) = (L^{\mathrm{an}})^{\pi_1^{\mathrm{top}}(U_0,1)} = 0$, and $H^1(U_0, L^{\mathrm{an}})$ coincides with the group cohomology $H^1(\pi_1^{\mathrm{top}}(U_0, 1), L_1^{\mathrm{an}})$, where the $\pi_1^{\mathrm{top}}(U_0, 1) = \mathbb{Z}$-action on the stalk L_1^{an} is the monodromy. For every crossed homomorphism $f : \mathbb{Z} \to L_1^{\mathrm{an}}$, every integer j, one has $f(1 + j) = f(1) - f(j)$. Therefore, when j is even (resp. odd), $f(j)$ is 0 (resp. $f(1)$). In particular, f is a boundary and hence $H^1(\pi_1^{\mathrm{top}}(U_0, 1), L_1^{\mathrm{an}}) = 0$. Thus, L^{an} is *not* in the cohomology support loci of U_0. From [Bud+17, p.299], $j_!^{\mathrm{an}} L^{\mathrm{an}}[1]$ is a simple object of $\mathrm{Perv}(X^{\mathrm{an}})$. Set $M := j_! L[1]$. By [Bei+82, p.150], the natural morphism $j_{!*} L[1] \to M$ is an isomorphism in $D_c^b(X)$. In particular, M is a *simple* object of $\mathrm{Perv}(X)$.

From [KW01, II, Cor. 7.5 g)], one has

$$N := M \otimes^L M = j_!(L \otimes^L j^* j_! L)[2] = j_! \bar{\mathbb{Q}}_{\ell,U}[2].$$

By [HT07, Example 8.1.35 (ii)], one has $N[-1] \in \mathrm{Perv}(X)$. Let $i : 0 \to A^1$ be the inclusion. From the short exact sequence

$$0 \to j_! \bar{\mathbb{Q}}_{\ell,U} \to \bar{\mathbb{Q}}_{\ell,X} \to i_*(\bar{\mathbb{Q}}_{\ell,0}) \to 0$$

in $\mathrm{Cons}(X)$, one gets an exact sequence

$${}^p\mathcal{H}^0(\bar{\mathbb{Q}}_{\ell,X}) \to {}^p\mathcal{H}^0(i_*(\bar{\mathbb{Q}}_{\ell,0})) \to {}^p\mathcal{H}^1(j_! \bar{\mathbb{Q}}_{\ell,U}) \to {}^p\mathcal{H}^1(\bar{\mathbb{Q}}_{\ell,X}) \to {}^p\mathcal{H}^1(i_*(\bar{\mathbb{Q}}_{\ell,0}))$$

in $\mathrm{Perv}(X)$. Since $i_*(\bar{\mathbb{Q}}_{\ell,0}), \bar{\mathbb{Q}}_{\ell,X}[1] \in \mathrm{Perv}(X)$, it gives a short exact sequence

$$0 \to i_*(\bar{\mathbb{Q}}_{\ell,0}) \to N[-1] \to \bar{\mathbb{Q}}_{\ell,X}[1] \to 0$$

in $\mathrm{Perv}(X)$. This sequence does not split as $N[-1]$ is supported on U. Therefore, $N[-1]$ is not a semisimple object of $\mathrm{Perv}(X)$. It follows that $M \otimes^L M$ is not semisimple in $D_c^b(X)$. (This sequence also shows that the support of a perverse sheaf may be smaller than that of a subquotient.)

For a category $\mathcal{C}$, let $\mathcal{C}/\sim$ be the class of isomorphism classes of objects in $\mathcal{C}$.

Lemma 3.5.3.4. *Let $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over k with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_k$. Assume that $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ is a reductive, algebraic group over k. Then the underlying abelian category is semisimple, and $\mathcal{C}/\sim$ is countable.*

Proof. Set $G = \mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$. Let $\mathrm{Rep}(G)$ be the category of k-rational representations of G. Then $\mathcal{C}$ is equivalent to $\mathrm{Rep}(G)$. Because k has characteristic zero, by [Mil17a, Cor. 22.43], the abelian category $\mathrm{Rep}(G)$ is semisimple. As k is algebraically closed, by [AHR20, Thm. 2.16], there is an at most countable set X^+ and for every $\lambda \in X^+$, a unital k-algebra $\mathscr{A}^\lambda$ with the following property: The set $\mathrm{Irr}(G)$ of isomorphism classes of simple objects of $\mathrm{Rep}(G)$ is in bijection with the set of pairs (λ, E), where $\lambda \in X^+$ and E is an isomorphism class of simple left $\mathscr{A}^\lambda$-modules. For every $\lambda \in X^+$, from [AHR20, Lem. 2.19], the algebra $\mathscr{A}^\lambda$ is semisimple. Then by [Lan02, XVII, Thm. 4.3, Cor. 4.5], the set of isomorphism classes of simple left $\mathscr{A}^\lambda$-modules is finite. Therefore, $\mathrm{Irr}(G)$ is at most countable. Consequently, $\mathrm{Rep}(G)/\sim$ is countable. $\square$

Lemma 3.5.3.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Consider the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$.*

1. *Let $X \in \mathcal{A}$. Let $i : Y \to F(X)$ be a monomorphism in $\mathcal{A}/\mathcal{B}$. Then there is a monomorphism $j : Z \to X$ in $\mathcal{A}$ and an isomorphism $u : Y \to F(Z)$ in $\mathcal{A}/\mathcal{B}$ fitting into a commutative diagram in $\mathcal{A}/\mathcal{B}$*

$$\begin{tikzcd} & F(Z) \arrow[dr, "F(j)"] & \\ Y \arrow[ur, "u"] \arrow[rr, hook, "i"] & & F(X). \end{tikzcd}$$

   Dually, up to isomorphism every quotient in $\mathcal{A}/\mathcal{B}$ lifts to a quotient in $\mathcal{A}$. In particular, if $X \in \mathcal{A}$ is a simple object, then $F(X)$ is either simple or zero in $\mathcal{A}/\mathcal{B}$.

2. *Let $V \in \mathcal{A}$ be a Noetherian and Artinian object. If $F(V)$ is simple in $\mathcal{A}/\mathcal{B}$, then there is a simple subquotient W of V in $\mathcal{A}$ such that $F(W)$ is isomorphic to $F(V)$ in $\mathcal{A}/\mathcal{B}$.*

3. *Assume that $\mathcal{A}$ is Noetherian and Artinian. Let $X \in \mathcal{A}$. If Y is a simple subquotient of $F(X)$ in $\mathcal{A}/\mathcal{B}$, then there is a simple subquotient W of X, with $F(W)$ isomorphic to Y in $\mathcal{A}/\mathcal{B}$.*

Proof.

1. By the construction in the proof of [Sta24, Tag 02MS] and the right calculus of fractions in [Sta24, Tag 04VB], there is a diagram

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & X \end{tikzcd}$$

in $\mathcal{A}$, such that $F(f)$ is an isomorphism and $F(g) = i \circ F(f)$ in $\mathcal{A}/\mathcal{B}$. Therefore, $F(g)$ is a monomorphism. Since F is exact, one has $F(\ker(g)) = \ker(F(g)) = 0$, so $\ker(g) \in \mathcal{B}$. Let $q : M \to M/\ker(g)$ be the epimorphism in $\mathcal{A}$, and let $j : M/\ker(g) \to X$ be the monomorphism in $\mathcal{A}$ induced by g. Then $F(q)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Set $u : Y \to F(M/\ker(g))$ to be the morphism $F(q) \circ F(f)^{-1}$ in $\mathcal{A}/\mathcal{B}$. Then u is an isomorphism with the stated property.

2. Let $\mathcal{P}$ be the family of subobjects V' of V in $\mathcal{A}$ with $V/V' \in \mathcal{B}$. Then $\mathcal{P}$ is nonempty since $V \in P$. As V is Artinian in $\mathcal{A}$, there is a minimal object $U \in \mathcal{P}$. Moreover, the morphism $F(U) \to F(V)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Let $\mathcal{Q}$ be the family of subobjects of $U \in \mathcal{A}$ lying in $\mathcal{B}$. Then $\mathcal{Q}$ is nonempty since $0 \in \mathcal{Q}$. As V is Noetherian in $\mathcal{A}$, so is U. Thus, $\mathcal{Q}$ has a maximal object U_0. Then $W := U/U_0$ is a subquotient of $V \in \mathcal{A}$ and the morphism $F(U) \to F(W)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. In particular, $W \neq 0$ in $\mathcal{A}$.

   We claim that W is simple in $\mathcal{A}$. Indeed, let $U' \to W$ be a subobject in $\mathcal{A}$. Then there is a subobject U'' of U in $\mathcal{A}$ containing U_0 with $U''/U_0 = U'$. As $F(U'')$ is a subobject of a simple object $F(U)$ in $\mathcal{A}/\mathcal{B}$, either the morphism $F(U'') \to F(U)$ is an isomorphism or $F(U'') = 0$. If $F(U'') = 0$, then $U'' \in \mathcal{B}$ and $U'' \in \mathcal{Q}$. Since U_0 is maximal in $\mathcal{Q}$, one has $U_0 = U''$, so $U' = 0$. If $F(U'') \to F(U)$ is an isomorphism, then $U/U'' \in \mathcal{B}$. Since the sequence

$$0 \to U/U'' \to V/U'' \to V/U \to 0$$

   is exact in $\mathcal{A}$, and $\mathcal{B}$ is closed under extensions, one gets $V/U'' \in \mathcal{B}$ and $U'' \in \mathcal{P}$. Since U is minimal in $\mathcal{P}$, one has $U'' = U$. The morphism $U' \to W$ is thus an isomorphism in $\mathcal{A}$. The claim is proved.

3. By 1, there is a subquotient Z of X in $\mathcal{A}$ with $F(Z)$ isomorphic to Y. Then $F(Z)$ is simple in $\mathcal{A}/\mathcal{B}$. By assumption, Z is Noetherian and Artinian in $\mathcal{A}$. Thus from 2, there is a simple subquotient W of Z in $\mathcal{A}$ with $F(W)$ isomorphic to $F(Z)$ and to Y in $\mathcal{A}/\mathcal{B}$.

□

Example 3.5.3.6. Let $s : X \to A \times X$ be a section to $p_X : A \times X \to X$. Let F be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, and let $\sigma : \pi_1^{\text{ét}}(X, \bar{\eta}) \to \mathrm{GL}(F_{\bar{\eta}})$ be the corresponding monodromy representation. By Fact 3.2.2.2 2, one has $F \in D^{\mathrm{ULA}}(X/X)$. Then from Fact 3.2.2.2 4, one has $K := Rs_*F \in D^{\mathrm{ULA}}(A \times X/X)$. For every $x \in X(k)$, by the proper base change theorem, $K|_{A_x} \in D_c^b(A_x)$ is the skyscraper supported at the closed point $s(x) \in A_x$ with stalk F_x. Thus, $K|_{A_x} \in \mathrm{Perv}(A_x)$ and $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Moreover, $K|_{A_{\bar{\eta}}}$ is the skyscraper supported at $s(\bar{\eta}) \in A_{\bar{\eta}}$ with stalk $F_{\bar{\eta}}$. Therefore, the generic and the geometric generic Tannakian groups agree and are computed in Example 3.4.3.3.

For every $\chi \in \mathcal{C}(A)$, by Fact 3.2.1.1, one has $K \otimes^L p_A^* L_\chi = Rs_*(F \otimes^L s^* p_A^* L_\chi)$. Thus, $Rp_{X*}(K \otimes^L p_A^* L_\chi) = F \otimes^L s^* p_A^* L_\chi$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X. The corresponding $\pi_1^{\text{ét}}(X, \bar{\eta})$-representation is the tensor product of σ with the character

$$\pi_1^{\text{ét}}(X, \bar{\eta}) \stackrel{(p_A \circ s)_*}{\to} \pi_1^{\text{ét}}(A, p_A s(\bar{\eta})) \xrightarrow{\sim} \pi_1^{\text{ét}}(A, 0) \xrightarrow{\chi} \bar{\mathbb{Q}}_\ell^\times .$$

The $\Gamma_{k(\eta)}$-representation induced by pulling back along $\eta_* : \Gamma_{k(\eta)} \to \pi_1^{\text{ét}}(X, \bar{\eta})$ is $\omega_{\chi_\eta}(K|_{A_\eta})$.

Assume that F is semisimple in $\mathrm{Loc}(X)$. Then $F[\dim X]$ is a semisimple object of $\mathrm{Perv}(X)$. By Fact 3.2.1.6 1, the object K is semisimple in $D_c^b(A \times X)$.

Chapter 4

Generic vanishing theorem for Fujiki class $\mathcal{C}$

4.1 Introduction

Recall the historical origin of generic vanishing results. In the last paragraph of [Enr39], Enriques gave an upper bound on the dimension of the paracanonical system of curves on some class of algebraic surfaces. However, in [Enr49, p.354] he pointed out a mistake in the proof of his result as well as a similar theorem by Severi [Sev42]. Catanese [Cat83, p.103] posed Conjecture 4.1.0.1.

Conjecture 4.1.0.1. *For a smooth projective surface $S/\mathbb{C}$ without irrational pencils, the dimension of the paracanonical system $\{K_S\}$ is at most the geometric genus $p_g(S)$.*

In 1987, Green and Lazarsfeld [GL87, Theorem 4.2] provided a positive answer to Conjecture 4.1.0.1. Its proof uses a result of generic vanishing type [GL87, Prop. 4.1].

As is explained in [Uen83, pp.619–620], the dimension of $\{K_S\}$ in Conjecture 4.1.0.1 is related to Conjecture 4.1.0.2, which is also of generic vanishing type.

Conjecture 4.1.0.2 ([Uen83, Problem 8, p.620])**.** *Let X be a projective manifold and $\alpha : X \to \mathrm{Alb}(X)$ be an Albanese morphism. If $\dim \alpha(X) > 1$, then $H^1(X, L) = 0$ for generic $L \in \mathrm{Pic}^0(X)$.*

Green and Lazarsfeld [GL87] proved a strengthening of Conjecture 4.1.0.2. Since then, the theory of generic vanishing results has been very much investigated and numerous authors have contributed to its development, so the overview in Section 4.1.1 is by no means complete.

For a finitely generated $\mathbb{Z}$-module H, let H_{tor} be the submodule of H comprised of torsion elements and $H_{\mathrm{free}} := H/H_{\mathrm{tor}}$. We use the words “locally free sheaf” and “vector bundle” interchangeably. Let $F \to X$ be a (holomorphic) vector bundle on a complex manifold. The dimension of a complex space always means the complex dimension. For any three integers $p, q, m \geq 0$, the corresponding jumping locus is defined as

$$S_m^{p,q}(X, F) := \{L \in \mathrm{Pic}^0(X) : h^q(X, \Omega_X^p \otimes_{O_X} L \otimes_{O_X} F) \geq m\}.$$

For simplicity, p (resp. m, resp. F) is omitted when $p = 0$ (resp. $m = 1$, resp. $F = O_X$). Roughly speaking, generic vanishing results show that these loci are small (in some sense) and study their structure when F is flat unitary (in the sense of Definition 4.2.2.2).

4.1.1 Known results

Let X be a connected compact Kähler manifold, $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map associated with some base point and $F \to X$ be a flat unitary vector bundle. Each locus $S_m^{p,q}(X, F)$ is an analytic subset of the complex torus $\mathrm{Pic}^0(X)$ (see the proof of Theorem 4.7.1.3 1) and "generic" means outside a strict analytic subset. In the literature, generic vanishing results concerning $S^q(X, F)$ (resp. $S^{p,q}(X, F)$) are usually called of Kodaira type (resp. Nakano type). Such results typically involve the following invariants:

- $\dim \alpha(X)$;
- $w(X) := \max\{\mathrm{codim}(Z(\eta), X) : 0 \neq \eta \in H^0(X, \Omega_X^1)\}$, where $Z(\eta)$ denotes the zero-locus of the 1-form η;
- the defect of semismallness $r(\alpha)$ of α (in the sense of Definition 4.5.2.1).

Using deformation theory of cohomology groups, Green and Lazarsfeld [GL87, Remarks (1), p.401] proves Fact 4.1.1.1, which is of Kodaira type and implies Conjecture 4.1.0.2.

Fact 4.1.1.1. *For every integer $k \geq 0$, one has*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)}(S^k(X, F)) \geq \dim \alpha(X) - k.$$

In particular, if $k < \dim \alpha(X)$, then $H^k(X, F \otimes_{O_X} L) = 0$ for a generic line bundle $L \in \mathrm{Pic}^0(X)$.

Green and Lazarsfeld also give a Nakano-type generic vanishing theorem.

Fact 4.1.1.2 ([GL87, Remarks (1), p.404])**.** *For any integers $i, j \geq 0$ with $i + j < w(X)$, one has $S^{i,j}(X, F) \neq \mathrm{Pic}^0(X)$.*

In another direction, there are known results concerning the structure of the jumping loci.

Fact 4.1.1.3 ([GL91, Thm. 0.1 (1)])**.** *For any two integers $k, m \geq 0$, the subset $S_m^k(X)$ is a finite union of translates of subtori of* $\mathrm{Pic}^0(X)$.

Beauville and Catanese conjectured in [Cat91, Problem 1.25] and [Bea92, p.1] that for every integer $q \geq 0$, the locus $S^q(X)$ is a finite union of *torsion* translates of subtori of $\mathrm{Pic}^0(X)$. When X is a projective manifold, this conjecture is proved by Simpson [Sim93, Sec. 5].

Fact 4.1.1.4 (Simpson)**.** *If X is furthermore projective, then for any two integers $k, m \geq 0$, the locus $S^k_m(X)$ is a finite union of* torsion *translates of subtori of* $\mathrm{Pic}^0(X)$*.*

Some arguments of [Sim93] are of arithmetic nature, so they do not apply to the Kähler case. Campana [Cam01, Sec. 1.5.2] provided a partial answer for not only Kähler manifolds but also for Fujiki class $\mathcal{C}$ (Definition 4.7.1.1).

Later on, Wang [Wan16, Cor. 1.4] answered affirmatively Beauville and Catanese's conjecture in full generality.

Fact 4.1.1.5 (Wang)**.** *For any three integers $p, q, m \geq 0$, the subset $S^{p,q}_m(X)$ of* $\mathrm{Pic}^0(X)$ *is a finite union of* torsion *translates of subtori.*

Hacon [Hac04, Cor. 4.2] uses Fourier-Mukai transforms of coherent modules on complex abelian varieties to recover Fact 4.1.1.1 when X is a projective manifold. This algebraic viewpoint sheds new insight on this topic. Similarly, as a byproduct of the theory on convolution of perverse sheaves on abelian varieties, Krämer and Weissauer obtain a Nakano-type generic vanishing theorem. The proof of [KW15b, Thm. 3.1] gives Fact 4.1.1.6.

Fact 4.1.1.6. *If furthermore the Albanese torus* $\mathrm{Alb}(X)$ *is algebraic, then for any two integers $p, q \geq 0$ with $p+q < \dim X - r(\alpha)$, the locus $S^{p,q}(X, F)$ is contained in a finite union of translates of strict subtori of* $\mathrm{Pic}^0(X)$*.*

Around the same time, by different methods Popa and Schnell [PS13, Thm. 1.2] obtained precise codimension bounds.

Fact 4.1.1.7. *If furthermore X is a projective manifold, then*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)}(S^{p,q}(X)) \geq |p + q - \dim X| - r(\alpha)$$

for any two integers $p, q \geq 0$. Moreover, for every X there exist p and q for which the inequality becomes an equality.

4.1.2 The main result and a sketch

An algebraic variety means an integral scheme of finite type and separated over a field. By [Del68, Prop. 5.3] and [Del71a, Thm. 3.2.5], even though not necessarily Kähler, a complex smooth proper algebraic variety also admits Hodge theory. It is natural to ask if generic vanishing results also hold for such varieties. The aim of this note is to show that generic vanishing result is not only true for Kähler manifolds, but also for complex manifolds in Fujiki class $\mathcal{C}$. This class contains compact Kähler manifolds as well as smooth proper algebraic varieties.

Here is the main result that is of Nakano type.

Theorem (Theorem 4.7.1.3). *Let X be an n-dimensional complex manifold in Fujiki class $\mathcal{C}$ with an Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$, and let F be a flat unitary vector bundle on X. Then for any two integers $p, q \geq 0$ with $p+q < n - r(\alpha)$, the locus $S^{p,q}(X, F)$ is a strict analytic subset of the complex torus* $\mathrm{Pic}^0(X)$.

For smooth proper algebraic varieties, the following finer result follows from Corollary 4.7.2.6 and Lemma 4.6.1.2. It is not immediate from previously known generic vanishing results.

Corollary 4.1.2.1. *Let X be an n-dimensional smooth proper, complex algebraic variety with an algebraic Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$. Let $\mathcal{L}$ be a unitary local system on the analytification X^{an}, and let $F = \mathcal{L} \otimes_{\mathbb{C}} O_{X^{\mathrm{an}}}$ be the corresponding holomorphic vector bundle. Then, for any two integers $p, q \geq 0$ with $p+q < n - r(\alpha)$, the subset $S^{p,q}(X, F)$ is contained in a finite union of translates (*torsion *translates if $\mathcal{L}$ is semisimple of geometric origin* in the sense of [Bei+82, p.163]*) of strict abelian subvarieties of the Picard variety* $\mathrm{Pic}^0_{X/\mathbb{C}}$.

Here is the outline of the proof of Theorem 4.7.1.3. By the Riemann-Hilbert correspondence restricted to unitary objects, we pass from flat unitary vector bundles to unitary local systems. The corresponding cohomology groups are related by the Hodge decomposition (Fact 4.7.1.2). In this way, the initial generic vanishing problem for a flat unitary vector bundle twisted by line bundles is reduced to a generic vanishing problem for a unitary local system twisted by rank 1 local systems.

By pushing forward along the Albanese map, the problem about the local system on a manifold in Fujiki class $\mathcal{C}$ is converted to a problem about a complex of sheaves on a complex torus. The last problem is solved by Krämer and Weissauer [KW15b] for perverse sheaves (on complex abelian varieties) and by the subsequent generalization (to all complex tori) due to Bhatt, Schnell and Scholze [BSS18].

This text is organized as follows. Sections 4.2 reviews the unitary Riemann-Hilbert correspondence. Section 4.3 and 4.4 construct the Jacobian and the Albanese map for regular manifolds, relaxing the usual Kähler condition. Several definitions of defect of semismallness are proved to be equivalent in Section 4.5. The work of Krämer and Weissauer on generic vanishing for perverse sheaves is recalled in Section 4.6. Finally in Section 4.7, the previous results are applied to prove the main result, Theorem 4.7.1.3, for Fujiki class $\mathcal{C}$.

4.2 Riemann-Hilbert correspondence

In Section 4.2, we review how the classical Riemann-Hilbert correspondence restricts to an equivalence between unitary local systems and flat unitary vector bundles on complex manifolds. The reason to introduce this restricted equivalence is that unitary local systems on manifolds in Fujiki class $\mathcal{C}$ admit Hodge decomposition (Fact 4.7.1.2).

4.2.1 Unitary local systems

Let X be a path-connected, locally path-connected and locally simply connected topological space with a base point $x_0 \in X$. Let $\mathrm{Loc}(X)$ be the category of local systems (of finite dimensional $\mathbb{C}$-vector spaces) on X. Let $\pi_1(X, x_0)$ be the fundamental group of X at x_0 and $\mathrm{Rep}_{\mathbb{C}}(\pi_1(X, x_0))$ be the category of its finite dimensional complex representations. By [Del70, Cor. 1.4, p.4], the functor taking the stalk at x_0 gives rise to an equivalence

$$\mathrm{Loc}(X) \to \mathrm{Rep}_{\mathbb{C}}(\pi_1(X, x_0)) \tag{4.1}$$

compatible with tensor products. The image under (4.1) of a local system on X is called the corresponding monodromy representation.

Every compact subgroup of $\mathrm{GL}_r(\mathbb{C})$ can be conjugated into the unitary subgroup $U_r(\mathbb{C})$. Therefore, for every representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(V)$, the following conditions are equivalent:

1. The closure of $\rho(\pi_1(X, x_0))$ inside $\mathrm{GL}(V)$ is compact;
2. There is a hermitian inner product $h : V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C}$ such that $\rho(\pi_1(X, x_0))$ is contained in the corresponding unitary group $U(V, h)$.

When the conditions hold, ρ is called *unitary*. Let $\mathrm{Rep}^u_{\mathbb{C}}(\pi_1(X, x_0)) \subset \mathrm{Rep}_{\mathbb{C}}(\pi_1(X, x_0))$ be the full subcategory of unitary representations. Let $\mathrm{Loc}^u(X)$ be the full subcategory of $\mathrm{Loc}(X)$ corresponding to $\mathrm{Rep}^u_{\mathbb{C}}(\pi_1(X, x_0))$ *via* the equivalence (4.1). Its objects are called unitary local systems on X. Every unitary local system is semisimple, since every unitary representation is so.

4.2.2 Flat unitary bundles

Let $E \to X$ be a holomorphic vector bundle on a complex manifold with a hermitian metric h. By [Huy05, Prop. 4.2.14], there exists a unique hermitian connection ∇_h that is compatible with the holomorphic structure (in the sense of [Huy05, Def. 4.2.12], *i.e.*, $\nabla^{0,1} = \bar{\partial}^E$), which is called the Chern connection of (E, h). The corresponding curvature form, called the Chern curvature, is an $\mathrm{End}(E, h)$-valued $(1, 1)$-form, (see, *e.g.*, [Huy05, Prop. 4.3.8 iii)]).

For every integer $k \geq 0$ (resp. any two integers $i, j \geq 0$), let A_X^k (resp. $A_X^{i,j}$) be the sheaf of *smooth* k (resp. (i,j)) forms on X. Then there is a direct sum decomposition $A_X^k = \oplus_{i+j=k} A_X^{i,j}$. In general, a (smooth) *flat* connection ∇ on E that is compatible with the holomorphic structure needs not to be a holomorphic connection (in the sense of [Huy05, Def. 4.2.17]).

Lemma 4.2.2.1. *Let $E \to X$ be a holomorphic vector bundle with a* flat *connection $\nabla : E \to E \otimes_{A_X^0} A_X^1$. If ∇ is compatible with the holomorphic structure, then ∇ is a holomorphic connection.*

Proof. Take a local holomorphic frame $\{e_1, \ldots, e_r\}$ of E, and denote the corresponding local smooth connection matrix 1-form by Ω. As $\nabla^{0,1} = \bar{\partial}^E$, one has $\Omega^{0,1} = 0$. By flatness, $d\Omega + \Omega \wedge \Omega = 0$. Taking the $(1,1)$ part of it, one gets $\bar{\partial}\Omega = 0$, *i.e.*, Ω is a holomorphic form. This shows that ∇ is holomorphic. □

Let $\mathrm{Mod}(O_X)$ be the category of O_X-modules, and let $\mathrm{VB}(X) \subset \mathrm{Mod}(O_X)$ be the full subcategory of finite locally free O_X-modules. Let $\mathrm{DE}(X)$ be the category of holomorphic vector bundles with a flat *holomorphic* connection. Forgetting the connection gives a functor $\mathrm{DE}(X) \to \mathrm{VB}(X)$. Let $\mathrm{DE}^u(X) \subset \mathrm{DE}(X)$ be the full subcategory comprised of objects (F, ∇) such that there exists a hermitian metric on F whose Chern connection is ∇.

Definition 4.2.2.2. An object in the essential image of $\mathrm{DE}^u(X)$ under the forgetful functor $\mathrm{DE}(X) \to \mathrm{VB}(X)$ is called a *flat unitary* vector bundle on X.

From [Huy05, Eg. 4.2.15], the trivial line bundle O_X is flat unitary. By Lemma 4.2.2.1, a holomorphic vector bundle is flat unitary if and only if it admits a hermitian metric whose Chern connection is flat.

4.2.3 An equivalence

Let X be a connected complex manifold. By the Riemann-Hilbert correspondence [Del70, Thm. 2.17, p.12], the pair of functors

$$\mathrm{Loc}(X) \to \mathrm{DE}(X), \quad \mathcal{L} \mapsto (\mathcal{L} \otimes_{\mathbb{C}} O_X, \mathrm{Id}_{\mathcal{L}} \otimes d); \tag{4.2}$$

$$\mathrm{DE}(X) \to \mathrm{Loc}(X), \quad (E, \nabla) \mapsto \ker(\nabla) \tag{4.3}$$

forms an equivalence of categories. It is compatible with tensor products and preserves the rank.

Theorem 4.2.3.1 (Unitary Riemann-Hilbert correspondence)**.** *The equivalence (4.2), (4.3) restricts to an equivalence between* $\mathrm{Loc}^u(X)$ *and* $\mathrm{DE}^u(X)$.

Proof. First, we prove that the functor (4.2) sends $\mathrm{Loc}^u(X)$ to $\mathrm{DE}^u(X)$. Consider a unitary local system $\mathcal{L}$ on X. Since the corresponding monodromy representation is unitary, we may choose a hermitian inner product h_{x_0} on the stalk $\mathcal{L}_{x_0}$ such that the representation factors through $U(\mathcal{L}_{x_0}, h_{x_0})$. For any $x \in X$, choose a path γ from x_0 to x and propagate h_{x_0} along this curve, *i.e.*, using the linear isomorphism $\gamma_* : \mathcal{L}_{x_0} \to \mathcal{L}_x$ induced by γ, we translate h_{x_0} to a hermitian inner product h_x of $\mathcal{L}_x$. This h_x is independent of the choice of γ by assumption. Hence a positive definite hermitian form h on $\mathcal{L}$ that is invariant under the monodromy action. Then h extends naturally to a (smooth) hermitian metric h' on the associated holomorphic vector bundle $\mathcal{L} \otimes_{\mathbb{C}} O_X$ on X and the corresponding flat holomorphic connection $\mathrm{Id}_{\mathcal{L}} \otimes d$ is a hermitian connection. Therefore, $\mathrm{Id}_{\mathcal{L}} \otimes d$ is the Chern connection of $(\mathcal{L} \otimes_{\mathbb{C}} O_X, h')$ and $(\mathcal{L} \otimes_{\mathbb{C}} O_X, \mathrm{Id}_{\mathcal{L}} \otimes d) \in \mathrm{DE}^u(X)$.

Conversely, we prove that the functor (4.3) sends $\mathrm{DE}^u(X)$ to $\mathrm{Loc}^u(X)$. Consider a holomorphic hermitian vector bundle (E, h) on X whose Chern connection ∇_h is flat. Around every point we can find a local ∇_h-horizontal holomorphic frame $\{e_1, \ldots, e_r\}$ of E. For any $1 \leq i, j \leq r$, since the connection ∇_h is compatible with h, we have

$$d[h(e_i, e_j)] = h(\nabla_h e_i, e_j) + h(e_i, \nabla_h e_j) = 0.$$

Therefore, the local function $h(e_i, e_j)$ is locally constant and the parallel transport along every closed path on X preserves the hermitian inner products on the fibers of E. The sheaf $\ker(\nabla_h)$ of horizontal sections of E forms a local system on X, whose stalks are exactly the fibers of E. Thus, it admits a monodromy-invariant positive definite hermitian form and is consequently unitary.[1] □

4.3 Hodge theory and Jacobian

In Section 4.3, we review the definition of Jacobian and show that for every complex manifold admitting Hodge theory (Definition 4.3.1.1), its Jacobian has nice expected properties.

4.3.1 Regular manifolds

Let X be a complex manifold. Let $d : A_X^\bullet \to A_X^{\bullet+1}$ be the exterior derivative. Then $d = \partial + \bar{\partial}$, where $\partial : A_X^{\bullet,\bullet} \to A_X^{\bullet+1,\bullet}$ and $\bar{\partial} : A_X^{\bullet,\bullet} \to A_X^{\bullet,\bullet+1}$ are the $(1,0)$ and $(0,1)$ part of d respectively. For every $\mathcal{E} \in \mathrm{Loc}^u(X)$, every integer $k \geq 0$, define a decreasing filtration of $A_X^k \otimes_{\mathbb{C}} \mathcal{E}$ by

$$F^p = F^p(A_X^k \otimes_{\mathbb{C}} \mathcal{E}) := \oplus_{i \geq p} A_X^{i,k-i} \otimes_{\mathbb{C}} \mathcal{E}. \tag{4.4}$$

[1] The definition of unitary local system in [Tim87, p.152] seems to forget this invariance.

Then $(d \otimes \mathrm{Id}_{\mathcal{E}})(F^p) \subset F^p$. Therefore, this filtration induces a spectral sequence, called the Frölicher spectral sequence:

$$E_1^{p,q} = H^q(X, \Omega_X^p \otimes_{\mathbb{C}} \mathcal{E}) \Rightarrow H^{p+q}(X, \mathcal{E}), \tag{4.5}$$

where the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by the operator $\partial : A_X^{p,q} \to A_X^{p+1,q}$ on X. It is the classical notion in [Voi02, Sec. 8.3.3] when $\mathcal{E}$ is the constant sheaf $\mathbb{C}_X$.

Although the Hodge theory for the first cohomology groups H^1 suffices for most properties of the Jacobian and the Albanese, in the sequel we mainly work with manifolds admitting Hodge theory in all degrees. Such manifolds are called "regular" for convenience.

Definition 4.3.1.1 (Regular manifold, [Del+75, 5.21 (2)])**.** Assume that X is compact. Let $\mathcal{E} \in \mathrm{Loc}^u(X)$. If the following conditions are satisfied:

1. The corresponding spectral sequence (4.5) degenerates at page E_1;
2. For every integer $k \geq 0$, the filtration induced by $F^\bullet(A_X^\bullet \otimes_{\mathbb{C}} \mathcal{E})$ on $H^k(X, \mathcal{E})$ gives a complex Hodge structure of weight k, in particular a Hodge decomposition
$$H^k(X, \mathcal{E}) = \oplus_{p+q=k} H^q(X, \Omega_X^p \otimes_{\mathbb{C}} \mathcal{E}); \tag{4.6}$$
3. For any integers $p, q \geq 0$, the conjugation map induces a $\mathbb{C}$-anti-linear isomorphism
$$H^q(X, \Omega_X^p \otimes_{\mathbb{C}} \mathcal{E}) \to H^p(X, \Omega_X^q \otimes_{\mathbb{C}} \mathcal{E}^\vee),$$
where $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathbb{C}_X)$ is the dual local system.

Then X is called $\mathcal{E}$-regular (and simply *regular* when $\mathcal{E} = \mathbb{C}_X$).

For instance, classical Hodge theory asserts that compact Kähler manifolds are regular (see *e.g.*, [Voi02, Sec. 6.1.3]). Because of Fact 4.3.1.2, regular manifolds are also called $\partial\bar{\partial}$-manifolds.

Fact 4.3.1.2 ($\partial\bar{\partial}$-lemma, [Del+75, 5.14, 5.21], [Var86, Prop. 3.4], [Huy05, Cor. 3.2.10])**.** *Assume that X is compact. Then X is regular if and only if for every d-closed smooth (p, q)-form η on X, the following conditions are equivalent:*

1. *η is d-exact;*
2. *η is ∂-exact;*
3. *η is $\bar{\partial}$-exact;*

4. η is $\partial\bar{\partial}$-exact.

If the above conditions hold and η is real, then there is a real smooth $(p-1, q-1)$-form ρ on X with $\eta = i\partial\bar{\partial}\rho$.

Remark 4.3.1.3. For Fact 4.3.1.2, it is important that the decomposition (4.6) is induced by the filtration (4.4). In fact, [Ceb+16, Prop. 4.3] constructs a non-regular, compact complex manifold X of dimension 3, such that the spectral sequence (4.5) for $\mathcal{E} = \mathbb{C}_X$ degenerates at page E_1, with numerical Hodge symmetry $h^{p,q}(X) = h^{q,p}(X)$ for any two integers $p, q \geq 0$. In this case, there is a non canonical decomposition of the form (4.6).

For the rest of Section 4.3.1, **we assume that X is a regular manifold**. For every integer $k \geq 0$ (resp. any two integers $p, q \geq 0$), the space of global ∂-closed, $\bar{\partial}$-closed smooth k (resp. (p,q)) forms on X is denoted by $Z^k(X)$ (resp. $Z^{p,q}(X)$). For any two integers $p, q \geq 0$, the Dolbeault cohomology group $H^q(X, \Omega_X^p)$ is denoted by $H^{p,q}(X)$.

Corollary 4.3.1.4. *For any integers $p, q \geq 0$ and $k := p + q$, there is a canonical commutative diagram*

$$\begin{tikzcd} Z^{p,q}(X) \arrow[r, hook] \arrow[d] & Z^k(X) \arrow[d] \\ H^{p,q}(X) \arrow[r, "\iota^{p,q}"] & H^k(X), \end{tikzcd}$$

where the first row is the natural inclusion and each vertical map is surjective. Moreover,

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} \mathrm{im}(\iota^{p,q}), \tag{4.7}$$

where each $\mathrm{im}(\iota^{p,q})$ can be identified with $H^{p,q}(X)$. The complex conjugation map $Z^{p,q}(X) \to Z^{q,p}(X)$ descends to a $\mathbb{C}$-antilinear isomorphism $H^{p,q}(X) \to H^{q,p}(X)$ (Hodge symmetry).

Proof. For each $\bar{\partial}$-closed (p,q)-form η on X, $\partial\eta$ is a d-closed, ∂-exact $(p+1, q)$-form. By Fact 4.3.1.2, there is a $(p, q-1)$-form ρ on X with $\partial\eta + \partial\bar{\partial}\rho = 0$, then the (p,q)-form $\eta + \bar{\partial}\rho$ is in $Z^{p,q}(X)$. Therefore, the map taking Dolbeault cohomology class $Z^{p,q}(X) \to H^{p,q}(X)$ is surjective.

Note that $\eta + \bar{\partial}\rho$ is d-closed. Its de Rham cohomology class is independent of the choice of ρ. Indeed, if ρ' is another $(p, q-1)$-form with $\eta + \bar{\partial}\rho'$ also d-closed, then $\bar{\partial}(\rho - \rho')$ is d-closed and $\bar{\partial}$-exact. By Fact 4.3.1.2, it is d-exact.

Thus the map

$$\iota^{p,q} : H^q(X, \Omega_X^p) \to H_{\mathrm{dR}}^{p+q}(X, \mathbb{C}), \quad [\eta] \to [\eta + \bar{\partial}\rho]$$

is a well-defined $\mathbb{C}$-linear map. By a third application of Fact 4.3.1.2, the map $\iota^{p,q}$ is injective. Thus, $H^{p,q}$ is identified with $\mathrm{im}(\iota^{p,q})$.

We claim that the sum $\sum_{p+q=k} \operatorname{im}(\iota^{p,q})$ is direct. In fact, if $\alpha^{p,q} \in Z^{p,q}(X)$ for each pair (p,q) with $p+q=k$ and the de Rham class of $\sum_{p+q=k} \alpha^{p,q}$ is 0 in $H^k_{\mathrm{dR}}(X,\mathbb{C})$, then there is a $(k-1)$-form β on X with $d\beta = \sum_{p+q=k} \alpha^{p,q}$. Thus,

$$\alpha^{p,q} = \partial(\beta^{p-1,q}) + \bar{\partial}(\beta^{p,q-1}).$$

The ∂-exact form $\partial(\beta^{p-1,q})$ is thereby $\bar{\partial}$-closed, so d-closed. By Fact 4.3.1.2 again, $\partial(\beta^{p-1,q})$ is $\bar{\partial}$-exact, hence $[\alpha^{p,q}] = 0$ in $H^{p,q}(X)$ for every (p,q). The claim is proved.

By assumption,

$$\dim_{\mathbb{C}} H^k(X,\mathbb{C}) = \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}(X),$$

hence the decomposition (4.7). In particular, the map taking de Rham cohomology class $Z^k(X) \to H^k(X,\mathbb{C})$ is surjective. The complex conjugate of $Z^{p,q}(X)$ is exactly $Z^{q,p}(X)$, the Hodge symmetry follows. □

Lemma 4.3.1.5 is used in the proof of Corollary 4.3.2.2.

Lemma 4.3.1.5. *For every integer $k \geq 0$, the map $H^k(X,\mathbb{C}) \to H^k(X,O_X)$ induced by the inclusion $\mathbb{C} \to O_X$ coincides with the projection $H^k(X,\mathbb{C}) \to H^{0,k}(X)$ given by the Hodge decomposition (4.7).*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}_X \arrow[r, hook] \arrow[d, hook] & A^0_X \arrow[r, "d"] \arrow[d, "p^{0,0}"] & A^1_X \arrow[r, "d"] \arrow[d, "p^{0,1}"] & \dots \\
O_X \arrow[r, hook] & A^{0,0}_X \arrow[r, "\bar{\partial}"] & A^{0,1}_X \arrow[r, "\bar{\partial}"] & \dots
\end{tikzcd}$$

The first row is an acyclic resolution of $\mathbb{C}_X$ by (smooth) Poincaré lemma, and the second row is the Dolbeault resolution. The first vertical map is the inclusion and each $p^{0,j} : A^j_X \to A^{0,j}_X$ is taking the $(0,j)$-part of a j-form. It is a morphism of complexes. Taking global sections, the induced map on k-th cohomology groups is the first map in the statement.

For a class $[\alpha] \in H^k(X,\mathbb{C})$, we may assume that the representative k-form α is ∂-closed and $\bar{\partial}$-closed by Corollary 4.3.1.4. Then its image under the first map $H^k(X,\mathbb{C}) \to H^k(X,O_X)$ is represented by the $(0,k)$-part of α, which is still ∂-closed and $\bar{\partial}$-closed. This describes exactly the projection induced by the Hodge decomposition (4.7). □

4.3.2 Jacobian

For a connected compact complex manifold X, let $b_1(X) := \dim_{\mathbb{C}} H^1(X,\mathbb{C})$ be its first Betti number. The exponential short exact sequence

$$0 \to \mathbb{Z} \to O_X \overset{f\mapsto\exp(2\pi i f)}{\longrightarrow} O_X^* \to 1$$

induces a long exact sequence

$$\begin{aligned} &H^0(X,O_X) \overset{f\mapsto\exp(2\pi i f)}{\longrightarrow} H^0(X,O_X^*) \to H^1(X,\mathbb{Z}) \\ \to &H^1(X,O_X) \to H^1(X,O_X^*) \overset{\delta}{\to} H^2(X,\mathbb{Z}). \end{aligned} \tag{4.8}$$

Set $\mathrm{Pic}(X) := H^1(X,O_X^*)$ for the Picard group, $\mathrm{NS}(X) := \mathrm{im}(\delta)$ for the Néron-Severi group, $\mathrm{Pic}^0(X) = \ker(\delta)$ and $\mathrm{Pic}^\tau(X) := \delta^{-1}(H^2(X,\mathbb{Z})_{\mathrm{tor}})$. As X is compact connected, one has $H^0(X,O_X) = \mathbb{C}$, $H^0(X,O_X^*) = \mathbb{C}^*$ and the first map in (4.8) is surjective. Accordingly, the third map $H^1(X,\mathbb{Z}) \to H^1(X,O_X)$ is injective and

$$\mathrm{Pic}^0(X) = \frac{H^1(X,O_X)}{H^1(X,\mathbb{Z})}. \tag{4.9}$$

If X is a complex torus, then $H^2(X,\mathbb{Z})$ is torsion free and

$$\mathrm{Pic}^0(X) = \mathrm{Pic}^\tau(X). \tag{4.10}$$

For general X, let $\mathrm{Loc}^1(X)$ (resp. $\mathrm{Loc}^{u,1}(X)$) be the set of isomorphism classes of rank-1 (resp. and unitary) local systems on X. Then $\mathrm{Loc}^1(X)$ is a group under tensor product and $\mathrm{Loc}^{u,1}(X)$ is a subgroup. For each $\mathcal{L} \in \mathrm{Loc}^1(X)$, $L := \mathcal{L} \otimes_{\mathbb{C}} O_X$ is a flat line bundle on X. By [Dem12, Ch. V, § 9], $L \in \mathrm{Pic}^\tau(X)$, whence a group morphism

$$\mathrm{Loc}^1(X) \to \mathrm{Pic}^\tau(X), \quad \mathcal{L} \mapsto \mathcal{L} \otimes_{\mathbb{C}} O_X. \tag{4.11}$$

Remark 4.3.2.1. Theorem 4.2.3.1 implies that a line bundle on X is flat unitary if and only if its class in $\mathrm{Pic}(X)$ lies in the image of the restriction of (4.11):

$$\mathrm{Loc}^{u,1}(X) \to \mathrm{Pic}^\tau(X). \tag{4.12}$$

The image of (4.12) may not to be contained in $\mathrm{Pic}^0(X)$. For instance, let X be an Enriques surface, then $\pi_1(X,x_0) = \mathbb{Z}/2$, $\#\,\mathrm{Loc}^{u,1}(X) = \mathbb{Z}/2$. By Corollary 4.4.2.2 2 below, the map (4.12) is an isomorphism, while $\mathrm{Pic}^0(X)$ is trivial.

Corollary 4.3.2.2. *Assume that X is regular. Then $\mathrm{Pic}^\tau(X)$ has a natural structure of compact complex Lie group with identity component $\mathrm{Pic}^0(X)$ that is a complex torus of dimension $b_1(X)/2$. Moreover, $\pi_0(\mathrm{Pic}^\tau(X)) = \mathrm{NS}(X)_{\mathrm{tor}}$.*

Proof. The inclusion $\mathbb{R} \subset O_X$ induces an $\mathbb{R}$-linear map

$$\phi : H^1(X, \mathbb{R}) \to H^1(X, O_X). \tag{4.13}$$

Because of Lemma 4.3.1.5 and the Hodge symmetry in Corollary 4.3.1.4, taking complex conjugate inside $H^1(X, \mathbb{C})$ induces an $\mathbb{R}$-linear map

$$\bar{\phi} : H^1(X, \mathbb{R}) \to H^0(X, \Omega^1_X). \tag{4.14}$$

If $\xi \in \ker(\phi)$, then the image of ξ under the injection $H^1(X, \mathbb{R}) \to H^1(X, \mathbb{C})$ is $\phi(\xi)+\bar{\phi}(\xi) = 0$, so $\xi = 0$. This shows that ϕ is injective. But $\dim_{\mathbb{R}} H^1(X, \mathbb{R}) = \dim_{\mathbb{R}} H^1(X, O_X) = b_1(X)$, so ϕ is a linear isomorphism.

The map $H^1(X, \mathbb{Z}) \to H^1(X, O_X)$ in (4.8) factors through ϕ. Since $H^1(X, \mathbb{Z})$ is a full lattice of $H^1(X, \mathbb{R})$, it remains a full lattice in $H^1(X, O_X)$. Therefore, the quotient $\mathrm{Pic}^0(X)$ is a complex torus of dimension $b_1(X)/2$. The $\mathbb{Z}$-module $\mathrm{Pic}^0(X)$ is divisible, so the short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^\tau(X) \to \mathrm{NS}(X)_{\mathrm{tor}} \to 0$$

spits. Therefore, there is a natural structure of compact complex Lie group on $\mathrm{Pic}^\tau(X)$ satisfying the stated properties. $\square$

The complex torus $\mathrm{Pic}^0(X)$ in Corollary 4.3.2.2 is called the *Jacobian* of the regular manifold X.

Example 4.3.2.3. Here are two examples showing how Corollary 4.3.2.2 fails for non-regular compact complex manifolds.

1. Let X be a Hopf surface. The first Betti number $b_1(X)$ is one, $H^1(X, \mathbb{Z}) = \mathbb{Z}$ and $H^1(X, O_X) = \mathbb{C}$, so the complex manifold $\mathrm{Pic}^0(X) = \mathbb{C}/\mathbb{Z}$ is not compact. However, by [Kod64], the Frölicher spectral sequence of $\mathbb{C}_X$ degenerates.

2. Let Y be a Calabi-Eckmann manifold in the sense of [BS17, Sec 1.2]. Then $H^1(Y, O_Y) = \mathbb{C}$ and Y is simply connected, so $H^1(Y, \mathbb{Z}) = 0$ and $b_1(Y) = 0$, but $\mathrm{Pic}^0(Y) = \mathbb{C}$ is not compact and $b_1(Y)/2 < \dim \mathrm{Pic}^0(Y)$.

4.4 Albanese torus

We turn to the conception of Albanese torus and Albanese map. They help to reduce some problems about general complex manifolds to those about complex tori. They are also tools to study the Jacobian. Again, Section 4.4 conveys the fact that Hodge theory guarantees the usual properties of the Albanese torus and Albanese map.

Fix a connected regular manifold X and a base point $x_0 \in X$.

4.4.1 Basics of Albanese torus

From [Uen06, Cor. 9.5, p.101], every element of $H^0(X, \Omega_X^1)$ is d-closed, so there is a well-defined natural map

$$\iota : H_1(X, \mathbb{Z}) \to H^0(X, \Omega_X^1)^\vee, \quad [\gamma] \mapsto (\beta \mapsto \int_\gamma \beta), \tag{4.15}$$

where γ runs through closed paths on X. Set

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^\vee / \mathrm{im}(\iota). \tag{4.16}$$

Lemma 4.4.1.1. *On* $\mathrm{Alb}(X)$*, there is a natural structure of* $h^{1,0}(X)$*-dimensional complex torus with* $H_1(\mathrm{Alb}(X), \mathbb{Z}) = \mathrm{im}(\iota)$*.*

Proof. Using the $\mathbb{R}$-linear isomorphism (4.14) and de Rham isomorphism

$$H^1_{\mathrm{dR}}(X, \mathbb{R}) \to H^1(X, \mathbb{R}),$$

the map (4.15) is identified with the natural map $H_1(X, \mathbb{Z}) \to H^1_{\mathrm{dR}}(X, \mathbb{R})^\vee$. The latter extends to an $\mathbb{R}$-linear isomorphism $H_1(X, \mathbb{R}) \to H^1_{\mathrm{dR}}(X, \mathbb{R})^\vee$ by Poincaré duality. Therefore,

$$\ker(\iota) = H_1(X, \mathbb{Z})_{\mathrm{tor}} \tag{4.17}$$

and $\mathrm{im}(\iota)$ is a full lattice in $H^0(X, \Omega_X^1)^\vee$ isomorphic to $H_1(X, \mathbb{Z})_{\mathrm{free}}$. Thus, the quotient $\mathrm{Alb}(X)$ is a complex torus with the stated properties. $\square$

The complex torus $\mathrm{Alb}(X)$ in Lemma 4.4.1.1 is called the *Albanese torus* of X. For each $x \in X$, choose two paths γ_x, γ'_x connecting x_0 to x. Then the composition γ of γ_x followed by the reverse of γ'_x is a closed path on X and

$$\int_{\gamma_x} \bullet - \int_{\gamma'_x} \bullet = \int_\gamma \bullet = \iota([\gamma])$$

belongs to $\mathrm{im}(\iota)$. Therefore, $[\int_{\gamma_x} \bullet] = [\int_{\gamma'_x} \bullet]$ in $\mathrm{Alb}(X)$. As $[\int_{\gamma_x} \bullet]$ is independent of the choice of γ_x, we write it as $\int_{x_0}^x \bullet$. For the fixed base point $x_0 \in X$, the associated Albanese morphism is

$$\alpha_{X,x_0} : X \to \mathrm{Alb}(X), \quad x \mapsto \int_{x_0}^x \bullet.$$

The subscripts X and x_0 are omitted when they are clear from the context.

Proposition 4.4.1.2.

1. *The Albanese map* $\alpha_{X,x_0} : X \to \mathrm{Alb}(X)$ *is a morphism of complex manifolds and the formation of Albanese map is functorial for the pair* (X, x_0)*.*

2. *The induced morphism* $\alpha_{x_0,*} : H_1(X, \mathbb{Z}) \to H_1(\mathrm{Alb}(X), \mathbb{Z})$ *is surjective with kernel* $H_1(X, \mathbb{Z})_{\mathrm{tor}}$.

3. *The morphism* α_{x_0} *satisfies the following universal property: every morphism of pointed complex manifolds* $(X, x_0) \to (A, 0)$ *with* A *a complex torus factors uniquely through a morphism of complex tori* $\mathrm{Alb}(X) \to A$. *In particular, the complex subtorus of* $\mathrm{Alb}(X)$ *generated by* $\alpha_{x_0}(X)$ *is* $\mathrm{Alb}(X)$.

4. *The pullback morphism* $\alpha_{x_0}^* : H^1(\mathrm{Alb}(X), \mathbb{Z}) \to H^1(X, \mathbb{Z})$ *is an isomorphism of weight* 1 $\mathbb{Z}$*-Hodge structures independent of the choice of* x_0.

5. *The pullback* $\alpha_{x_0}^* : \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ *is an isomorphism of complex tori independent of the choice of* x_0. *In particular, the complex tori* $\mathrm{Alb}(X)$ *and* $\mathrm{Pic}^0(X)$ *are dual to each other (*in the sense of [BL04, p.34]*).*

Proof.

1. When X is Kähler, it is proved in [Huy05, Prop. 3.3.8]. The general case is similar.

2. By Lemma 4.4.1.1, $H_1(\mathrm{Alb}(X), \mathbb{Z}) = \mathrm{im}(\iota)$. Let $\gamma : [0, 1] \to X$ be a closed path on X based at x_0. It defines a path

$$\zeta : [0, 1] \to H^0(X, \Omega_X^1)^\vee, \quad \zeta(t) = \int_{\gamma(0)}^{\gamma(t)} \bullet,$$

where the integral is along a part of γ. Then

$$\zeta \pmod{\mathrm{im}(\iota)} = \alpha_{x_0} \circ \gamma : [0, 1] \to \mathrm{Alb}(X).$$

Therefore, $\alpha_{x_0,*}[\gamma] = \zeta(1) - \zeta(0) = \int_\gamma \bullet = \iota([\gamma])$. Hence a commutative triangle

$$\begin{tikzcd} & \mathrm{im}(\iota) = H_1(\mathrm{Alb}(X), \mathbb{Z}) \arrow[dr, hook] & \\ H_1(X, \mathbb{Z}) \arrow[ur, "\alpha_{x_0,*}"] \arrow[rr, "\iota"] & & H^0(X, \Omega_X^1)^\vee \end{tikzcd}$$

Therefore, $\alpha_{x_0,*}$ is surjective and $\ker(\alpha_{x_0,*}) = \ker(\iota) = H_1(X, \mathbb{Z})_{\mathrm{tor}}$, where the last equality uses (4.17).

3. The universal property follows from Point 1. Let T be the complex subtorus of $\mathrm{Alb}(X)$ generated by $\alpha_{x_0}(X)$. Then the pointed morphism $\alpha_{x_0} : (X, x_0) \to (T, 0)$ factors through $\alpha_{x_0} : (X, x_0) \to (\mathrm{Alb}(X), 0)$, so $T = \mathrm{Alb}(X)$.

4. From [BL04, Thm. 1.4.1 b)], the map $\alpha_{x_0}^* : H^{1,0}(\mathrm{Alb}(X)) \to H^{1,0}(X)$ is a $\mathbb{C}$-linear isomorphism. By [BL04, Sec 1.3, p.13], $H^1(\mathrm{Alb}(X), \mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}(\mathrm{im}(\iota), \mathbb{Z})$. By Poincaré duality, the latter is identified with $H^1(X, \mathbb{Z})$, so

$$\alpha_{x_0}^* : H^1(\mathrm{Alb}(X), \mathbb{Z}) \to H^1(X, \mathbb{Z})$$

is an isomorphism of weight 1 $\mathbb{Z}$-Hodge structures. Up to translation, different base points give rise to the same Albanese map. More precisely, for $x \in X$, $T_{\alpha_x(x_0)} \circ \alpha_{x_0} = \alpha_x$, where

$$T_a : \mathrm{Alb}(X) \to \mathrm{Alb}(X), \quad u \mapsto u + a$$

is the translation by a on $\mathrm{Alb}(X)$. The independence stated in Point 4 follows.

5. As the isomorphism (4.9) is functorial in X, there is a commutative diagram with exact rows

$$\begin{tikzcd}
H^1(\mathrm{Alb}(X), \mathbb{Z}) \arrow[r] \arrow[d, "\alpha_{x_0}^*"] & H^1(\mathrm{Alb}(X), O_{\mathrm{Alb}(X)}) \arrow[r] \arrow[d, "\alpha_{x_0}^*"] & \mathrm{Pic}^0(\mathrm{Alb}(X)) \arrow[r] \arrow[d, "\alpha_{x_0}^*"] & 0 \\
H^1(X, \mathbb{Z}) \arrow[r] & H^1(X, O_X) \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & 0.
\end{tikzcd}$$

By Point 4, the left two vertical maps are isomorphisms independent of x_0. Therefore, the right vertical map is an isomorphism independent of x_0. As $\mathrm{Alb}(X)$ is a complex torus, by [BL04, Proposition 2.4.1], $\mathrm{Pic}^0(\mathrm{Alb}(X))$ is the dual torus of $\mathrm{Alb}(X)$. As $\alpha_{x_0}^* : \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ is an isomorphism, $\mathrm{Pic}^0(X)$ is dual to $\mathrm{Alb}(X)$.

□

Remark 4.4.1.3. By [Uen06, Cor. 9.5, p.101], for every connected regular manifold X the formation of $\mathrm{Alb}(X)$ and α_{x_0} agrees with the construction in [Bla56, §2]. Then [Bla56, p.163] gives another proof of the universal property stated in Proposition 4.4.1.2 3.

Example 4.3.2.3 1 (continued). If X were a Hopf surface, then $H_1(X, \mathbb{Z}) = \mathbb{Z}$ and $H^0(X, \Omega_X^1) = 0$. Equation (4.16) would define a point and Proposition 4.4.1.2 2 would fail.

4.4.2 Back to Jacobian

Albanese torus helps to understand the Jacobian. Corollary 4.4.2.1 is used to show the jumping loci are analytic subsets.

Corollary 4.4.2.1 (Universal line bundle)**.** *There exists a unique (up to isomorphism) line bundle L on $X \times \mathrm{Pic}^0(X)$ such that its pullback module to $\{x_0\} \times \mathrm{Pic}^0(X)$ is trivial and for every point $y \in \mathrm{Pic}^0(X)$, the isomorphism class of the pullback line bundle $L|_{X \times \{y\}}$ in $\mathrm{Pic}(X)$ is y.*

Proof. Consider the map

$$f = \alpha_{x_0} \times \mathrm{Id}_{\mathrm{Pic}^0(X)} : X \times \mathrm{Pic}^0(X) \to \mathrm{Alb}(X) \times \mathrm{Pic}^0(X).$$

By Proposition 4.4.1.2 5 and [GH78, Lemma, p.328], there is a holomorphic line bundle $\mathcal{P}$ on $\mathrm{Alb}(X) \times \mathrm{Pic}^0(X)$ that is trivial on $\{0\} \times \mathrm{Pic}^0(X)$ such that for every $y \in \mathrm{Pic}^0(X)$, the line bundle $\mathcal{P}|_{\mathrm{Alb}(X) \times \{y\}}$ is of class y in $\mathrm{Pic}^0(\mathrm{Alb}(X))$. Let $L = f^*\mathcal{P}$, then $L|_{\{x_0\} \times \mathrm{Pic}^0(X)} = f^*(\mathcal{P}|_{\{0\} \times \mathrm{Pic}^0(X)})$ is trivial. For every $y \in \mathrm{Pic}^0(X)$, the line bundle

$$L|_{X \times \{y\}} = f^*(\mathcal{P}|_{\mathrm{Alb}(X) \times \{y\}}) = \alpha_{x_0}^*(\mathcal{P}|_{\mathrm{Alb}(X) \times \{y\}})$$

is of class y in $\mathrm{Pic}^0(X)$. The existence is proved. The uniqueness follows from [BL04, Cor. A.9]. □

Let

$$\mathrm{Char}(X) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$$

be the group of characters of the first homology of X. Every $\chi \in \mathrm{Char}(X)$ induces a line bundle $L_\chi := \mathcal{L}_\chi \otimes_{\mathbb{C}} O_X$. By [Hat05, Cor. A.8, A.9], the abelian group $H_1(X, \mathbb{Z})$ is finitely generated. From [Mil17a, Ch. 12 b.], $\mathrm{Char}(X)$ has a natural structure of diagonalizable algebraic group over $\mathbb{C}$, with identity component $\mathrm{Char}^\circ(X)$ isomorphic to $\mathbb{G}_m^{b_1(X)}$. Moreover, $\mathrm{Char}^u(X) := \mathrm{Hom}(H_1(X, \mathbb{Z}), S^1)$ is a real Lie subgroup of $\mathrm{Char}(X)$ of dimension $b_1(X)$. There is a canonical group isomorphism by taking character sheaves

$$\mathrm{Char}^u(X) \to \mathrm{Loc}^{u,1}(X), \quad \chi \mapsto \mathcal{L}_\chi. \tag{4.18}$$

Set $T(X) := \mathrm{Hom}(H_1(X, \mathbb{Z})_{\mathrm{free}}, S^1)$. Then $T(X)$ is the identity component of $\mathrm{Char}^u(X)$. From Corollary 4.3.2.2, composing the isomorphism (4.18) and the map (4.12) gives a morphism of real Lie groups

$$T(X) \to \mathrm{Pic}^0(X), \quad \chi \mapsto L_\chi. \tag{4.19}$$

In Corollary 4.4.2.2 1, the isomorphism allows one to identify certain characters with topologically trivial line bundles. This identification is used in the proof of Theorem 4.7.1.3. When X is in Fujiki Class $\mathcal{C}$ (resp. Kähler), Corollary 4.4.2.2 2 is also in [Ara90, Lem. 2] (resp. the proof of [Wan16, Cor. 1.4]).

Corollary 4.4.2.2.

1. *The morphism (4.19) is an isomorphism of real Lie groups.*

2. The map (4.12) is a group isomorphism and $\mathrm{NS}(X)_{\mathrm{tor}} = H^2(X,\mathbb{Z})_{\mathrm{tor}}$*. In particular, every element of* $\mathrm{Pic}^\tau(X)$ *is a flat unitary line bundle.*

Proof.

1. Lemma 4.4.1.1 gives an identification $H_1(X,\mathbb{Z})_{\mathrm{free}} = \mathrm{im}(\iota)$. By [BL04, Prop. 2.2.2], the natural group morphism
$$\mathrm{Hom}(\mathrm{im}(\iota), S^1) \to \mathrm{Pic}^0(\mathrm{Alb}(X)) \tag{4.20}$$
defined *via* factors of automorphy [BL04, p.30] is an isomorphism. The map (4.19) is the composition of (4.20) with the isomorphism $\alpha_{x_0}^* : \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ in Proposition 4.4.1.2 5.
To sum it up:
$$\begin{tikzcd}
\mathrm{Char}^u(\mathrm{Alb}(X)) = \mathrm{Hom}(\mathrm{im}(\iota), S^1) \arrow[r, "\sim"] \arrow[d, "(4.20)"] & T(X) \arrow[r, hook] \arrow[d, "(4.19)"] & \mathrm{Char}^u(X) = \mathrm{Loc}^{u,1}(X) \arrow[d, "(4.12)"] \\
\mathrm{Pic}^0(\mathrm{Alb}(X)) \arrow[r, "\alpha_{x_0}^*", "\sim"'] & \mathrm{Pic}^0(X) \arrow[r, hook] & \mathrm{Pic}^\tau(X).
\end{tikzcd}$$
2. The commutative diagram of abelian sheaves on X
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbb{R} \arrow[r] \arrow[d, hook] & S^1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & O_X \arrow[r, "f \mapsto \exp(2\pi i f)"] & O_X^* \arrow[r] & 0
\end{tikzcd}$$
has exact rows. Moreover, the $\mathbb{Z}$-module $\mathbb{R}$ is injective. Therefore, there is a commutative diagram with exact columns
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
H^1(X,\mathbb{Z}) \arrow[d] & H^1(X,\mathbb{Z}) \arrow[l, "="'] \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \arrow[d] & \\
H^1(X,O_X) \arrow[d] & H^1(X,\mathbb{R}) \arrow[l, "(4.13)"'] \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{R}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^1(X,O_X^*) \arrow[d, "\delta"] & H^1(X,S^1) \arrow[l] \arrow[r, "\simeq"] \arrow[d] & \mathrm{Char}^u(X) \arrow[r, "r"] \arrow[d] & \mathrm{Hom}(H_1(X,\mathbb{Z})_{\mathrm{tor}}, S^1) \arrow[d, "\simeq"] \arrow[dll, dashed, "\psi"] \\
H^2(X,\mathbb{Z}) & H^2(X,\mathbb{Z}) \arrow[l, "="'] & \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{Z}) \arrow[l, hook', "\xi"] \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X,\mathbb{Z})_{\mathrm{tor}},\mathbb{Z}) \arrow[d] \\
 & & 0 & 0,
\end{tikzcd}$$

where r is the restriction, and the horizontal morphisms in the middle column are from [Hat05, Thm. 3.2]. By [Hat05, p.196], the image of the injection $\xi : \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X,\mathbb{Z}),\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is $H^2(X,\mathbb{Z})_{\mathrm{tor}}$. Thus, there is a natural isomorphism $\psi : \mathrm{Hom}(H_1(X,\mathbb{Z})_{\mathrm{tor}}, S^1) \to H^2(X,\mathbb{Z})_{\mathrm{tor}}$ fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] \arrow[d, "(4.19)"] & \mathrm{Loc}^{u,1}(X) \arrow[r, "r"] \arrow[d, "(4.12)"] & \mathrm{Hom}(H_1(X,\mathbb{Z})_{\mathrm{tor}}, S^1) \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}^\tau(X) \arrow[r, "\delta"] & H^2(X,\mathbb{Z})_{\mathrm{tor}} \arrow[r] & 0,
\end{tikzcd}$$

where the first row is exact. Thus, δ is surjective and the second row is also exact. By the five lemma, the middle vertical map (4.12) is an isomorphism.

□

4.5 Defect of semismallness

We review the defect of semismallness of a morphism, an invariant introduced by de Cataldo and Migliorini that plays a crucial role in the decomposition theorem and Lefschetz's theorem. It appears in Fact 4.1.1.6 and Theorem 4.7.1.3. Its main property that we need is Proposition 4.5.3.2. Complex analytic spaces are in the sense of [CAS, p.7].

4.5.1 Stratifications and constructible sheaves

We refer to [BF84, Sec. 2.1] for the definitions of **constructible stratifications** and **Whitney stratifications** of a complex analytic space.

Theorem 4.5.1.1 is about the semicontinuity of fiber dimension. Although it is well-known, a short proof is included due to the lack of reference. Its analogue in algebraic geometry is a celebrated theorem of Chevalley [EGA IV 3, Cor. 13.1.5].

Theorem 4.5.1.1 (Analytic Chevalley theorem)**.** *Let $f : X \to Y$ be a proper morphism of reduced complex analytic spaces. For every integer $n \geq 0$, let $Y_n = \{y \in Y : \dim f^{-1}(y) = n\}$ and $Y_{\geq n} = \cup_{m \geq n} Y_m$. Then $Y_{\geq n}$ is an analytic subset of Y. In particular, $\{Y_n\}_{n\in\mathbb{N}}$ is a constructible stratification of Y.*

Proof. Let $F_n := \{x \in X : \dim_x f^{-1}(f(x)) \geq n\}$. By [Fis76, Thm. 3.6, p.137], F_n is an analytic subset of X. By the definition of global dimension [CAS, p.94], one has $Y_{\geq n} = f(F_n)$. By Remmert theorem (see, *e.g.*, [Whi72, Thm. 4A, p.150]), the subset $Y_{\geq n}$ is analytic in Y. □

Definition 4.5.1.2. ([BF84, p.125]) Let $f : X \to Y$ be a morphism of complex analytic spaces. Assume that two Whitney stratifications $\mathfrak{X} : X = \sqcup_\alpha X_\alpha$ and $\mathfrak{Y} : Y = \sqcup_\lambda Y_\lambda$ satisfy that:

1. For each α, there is λ with $f(X_\alpha) \subset Y_\lambda$;

2. For each pair (α, λ) with $f(X_\alpha) \subset Y_\lambda$, the restricted morphism $f : X_\alpha \to Y_\lambda$ is smooth.

Then such a pair $(\mathfrak{X}, \mathfrak{Y})$ is called a *Whitney stratification* of f.

Fact 4.5.1.3 ([Hir77, Thm. 1], [BF84, Lem. 2.4], [GM88, Thm, p.43])**.** *Let $f : X \to Y$ be a* proper *morphism of complex analytic spaces. Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are constructible stratifications of X and Y respectively. Then there exists a Whitney stratification $(\mathfrak{X}', \mathfrak{Y}')$ of f such that $\mathfrak{X}'$ and $\mathfrak{Y}'$ refine $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.*

Corollary 4.5.1.4 is useful but implicit in the literature.

Corollary 4.5.1.4. *Let X be a complex analytic space. For finitely many constructible stratifications of X, there exists a Whitney stratification of X refining all of them.*

Proof. It suffices to consider the case of two constructible stratifications $\mathfrak{X}_1$ and $\mathfrak{X}_2$ of X. By Fact 4.5.1.3, there is a Whitney stratification $(\mathfrak{X}, \mathfrak{X}')$ of $\mathrm{Id}_X : X \to X$ such that $\mathfrak{X}$ (resp. $\mathfrak{X}'$) refines $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$). Moreover, $\mathfrak{X}$ refines $\mathfrak{X}'$ by Definition 4.5.1.2. Then $\mathfrak{X}$ is a Whitney stratification refining both $\mathfrak{X}_1$ and $\mathfrak{X}_2$. $\square$

For a complex analytic space X, using analytic constructible stratifications, one can define constructible sheaves. Let $D^b_c(X)$ be the triangulated category of complexes of sheaves of $\mathbb{C}$-vector spaces whose cohomology is bounded and constructible (see, *e.g.*, [Dim04, p.82]).

Fact 4.5.1.5 ([KS90, Prop. 8.5.7 (b)], [Dim04, Thm. 4.1.5 (b)])**.** *Let $f : X \to Y$ be a morphism of complex analytic spaces and $\mathcal{K} \in D^b_c(X)$. If f is proper on* $\mathrm{Supp}(\mathcal{K})$*, then $Rf_*\mathcal{K} \in D^b_c(Y)$.*

Corollary 4.5.1.6. *Let $f : X \to Y$ be a* proper *morphism of complex analytic spaces and $\mathcal{K} \in D^b_c(X)$. Then there exists a Whitney stratification $(\mathfrak{X}, \mathfrak{Y})$ of f such that for every integer i and every stratum S of $\mathfrak{Y}$, the restriction $\mathcal{H}^i(Rf_*\mathcal{K})|_S$ is a local system on S.*

Proof. By Fact 4.5.1.5, $Rf_*\mathcal{K} \in D^b_c(Y)$. In particular, there are only finitely many $j \in \mathbb{Z}$ with $\mathcal{H}^j(Rf_*\mathcal{K}) \neq 0$. For each such j, there is an admissible partition (in the sense of [Dim04, p.81]) $\mathcal{P}_j$ on Y such that the restriction of $\mathcal{H}^j(Rf_*\mathcal{K})$ to each stratum of $\mathcal{P}_j$ is a local system. By Corollary 4.5.1.4, there exists a Whitney stratification $\mathfrak{Y}^0$ of Y refining the finitely many $\mathcal{P}_j$. By Fact 4.5.1.3, there is a Whitney stratification $(\mathfrak{X}, \mathfrak{Y})$ of f satisfying the properties. $\square$

4.5.2 Equivalent definitions

The defect of semismallness measures how far a morphism of complex manifolds is from being semismall (see, *e.g.*, [KW01, Def. 7.3, p.156]). However, in the literature there exist multiple seemingly different definitions. We review some of them and show that they are equivalent.

Definition 4.5.2.1. Let $f : X \to Y$ be a proper morphism of complex manifolds with $\dim X = n$.

- ([EV89, Def. 1.1]) Define
$$r_1(f) = \max_Z(\dim Z - \dim f(Z) - \operatorname{codim}_X(Z)), \tag{4.21}$$
where Z runs through all irreducible analytic subsets of X.
- ([Max19, Def. 9.3.7]) For a Whitney stratification $(X = \sqcup S_\alpha, Y = \sqcup T_\lambda)$ of f, we choose a point $y_\lambda \in T_\lambda$ in each stratum, and define
$$r_2(f) = \max_\lambda\{2\dim f^{-1}(y_\lambda) + \dim T_\lambda - n\}. \tag{4.22}$$
(By convention, the empty space has dimension $-\infty$.)
- ([CM05, Def. 4.7.2]) For each integer $i \geq 0$, let $Y_i = \{y \in Y : \dim f^{-1}(y) = i\}$. Define
$$r_3(f) = \max_{i\geq 0}(2i + \dim Y_i - n).$$
- ([PS13, Def. 2.8]) For each integer $i \geq 0$, let $Y_{\geq i} = \{y \in Y : \dim f^{-1}(y) \geq i\}$ for each $i \geq 0$. Define
$$r_4(f) = \max_{i\geq 0}(2i + \dim Y_{\geq i} - n).$$
- ([CM09, Sec. 3.3.2, part 2]) Define $r_5(f) = \dim X \times_Y X - n$.
- ([Wil16, Sec 3.2]) Define
$$r_6(f) = \max\{i \in \mathbb{Z} : {}^p\mathcal{H}^i(Rf_*\mathbb{C}_X[n]) \neq 0\}.$$

Proposition 4.5.2.2. *The first five numbers in Definition 4.5.2.1 are all equal.*

This common integer is called the *defect of semismallness* of f and denoted by $r(f)$. We shall show $r(f) = r_6(f)$ in Proposition 4.5.3.2 2.

Proof.

- $r_3(f) = r_4(f)$: As each Y_i is a subset of $Y_{\geq i}$, one has $r_3(f) \leq r_4(f)$. There are only finitely many integers $i \geq 0$ with $Y_{\geq i}$ nonempty, so the maximum defining $r_4(f)$ is attained at some $i_0 (\geq 0)$. Then

$$2(i_0 + 1) + \dim Y_{\geq i_0+1} \leq 2i_0 + \dim Y_{\geq i_0}.$$

Since $Y_{\geq i_0} = Y_{\geq i_0+1} \cup Y_{i_0}$, one has $\dim Y_{\geq i_0} = \dim Y_{i_0}$. Then

$$r_4(f) = 2i_0 + \dim Y_{\geq i_0} - n \leq r_3(f).$$

Therefore, $r_3(f) = r_4(f)$.

- $r_2(f) = r_5(f)$: By Thom's first isotopy lemma (see, *e.g.*, [Mat12, Prop. 11.1]), for every λ, the restriction $f|_{f^{-1}(T_\lambda)} : f^{-1}(T_\lambda) \to T_\lambda$ is a topologically locally trivial fibration. Therefore, $\dim f^{-1}(y_\lambda)$ is independent of $y_\lambda \in T_\lambda$ and

$$\dim f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda) = \dim T_\lambda + 2 \dim f^{-1}(y_\lambda). \tag{4.23}$$

As $\{f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda)\}_\lambda$ is a locally finite partition of $X \times_Y X$ into locally closed subsets (in the analytic Zariski topology), one has

$$\dim X \times_Y X = \max_\lambda [\dim f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda)]. \tag{4.24}$$

Plugging (4.23) into (4.24) we get $r_5(f) = r_2(f)$. In particular, $r_2(f)$ is independent of the choice of the stratifications.

- $r_1(f) \leq r_2(f)$: For every irreducible analytic subset $Z \subset X$, $f(Z)$ is an irreducible analytic subset of Y. Then $\{Y \setminus f(Z), f(Z)\}$ is a constructible stratification of Y. Fact 4.5.1.3 yields a Whitney stratification $(X = \sqcup S_\alpha, Y = \sqcup T_\lambda)$ of f with $Y = \sqcup T_\lambda$ refining $\{Y \setminus f(Z), f(Z)\}$. There exists λ_0 such that T_{λ_0} is an open subset of $f(Z)$, hence $\dim T_{\lambda_0} \leq \dim f(Z)$. Then $f^{-1}(T_{\lambda_0}) \cap Z$ is a nonempty open subset of Z. Therefore,

$$\dim Z = \dim(f^{-1}(T_{\lambda_0}) \cap Z) \leq \dim f^{-1}(T_{\lambda_0}).$$

Then

$$2 \dim Z - \dim f(Z) \leq 2 \dim f^{-1}(T_{\lambda_0}) - \dim T_{\lambda_0} = 2 \dim f^{-1}(y_{\lambda_0}) + \dim T_{\lambda_0}.$$

This shows $r_1(f) \leq r_2(f)$. In particular, the maximum in (4.21) is indeed attained.

- $r_2(f) \leq r_1(f)$: Fix a Whitney stratification $Y = \sqcup_\lambda T_\lambda$ defining $r_2(f)$. For every λ with $f^{-1}(y_\lambda)$ nonempty, $\overline{T_\lambda}$ is an analytic subset of Y of dimension $\dim T_\lambda$. Then $f^{-1}(\overline{T_\lambda})$ is a nonempty analytic subset of

X. Let Z_0 be an irreducible component of $f^{-1}(\overline{T_\lambda})$ with $\dim Z_0 = \dim f^{-1}(\overline{T_\lambda})$. Then $f(Z_0) \subset \overline{T_\lambda}$ and $\dim f(Z_0) \leq \dim T_\lambda$. Therefore,

$$2\dim f^{-1}(y_\lambda)+\dim T_\lambda = 2\dim f^{-1}(T_\lambda)-\dim T_\lambda \leq 2\dim Z_0-\dim f(Z_0).$$

This shows $r_2(f) \leq r_1(f)$.

- $r_2(f) \leq r_3(f)$: By Theorem 4.5.1.1, $\{Y_i\}$ is a constructible stratification of Y. By Fact 4.5.1.3, there is a Whitney stratification ($X = \sqcup S_\alpha, Y = \sqcup T_\lambda$) of f such that the stratification $Y = \sqcup T_\lambda$ refines $Y = \sqcup_i Y_i$. For every λ, there is i_0 with $T_\lambda \subset Y_{i_0}$. In particular, for every $y_\lambda \in T_\lambda$, one has $\dim f^{-1}(y_\lambda) = i_0$, so

$$2\dim f^{-1}(y_\lambda) + \dim T_\lambda \leq 2i_0 + \dim Y_{i_0}.$$

  This shows $r_2(f) \leq r_3(f)$.

- $r_3(f) \leq r_2(f)$: For every integer $i \geq 0$ with Y_i nonempty, $Y_i = \sqcup_\lambda(Y_i \cap T_\lambda)$ is a constructible stratification, so there is an index λ_0 with $\dim(Y_i \cap T_{\lambda_0}) = \dim Y_i$. Then $\dim Y_i \leq \dim T_{\lambda_0}$. One may take $y_{\lambda_0} \in Y_i \cap T_{\lambda_0}$. Then

$$2i + \dim Y_i \leq 2\dim f^{-1}(y_{\lambda_0}) + \dim T_{\lambda_0},$$

  which shows $r_3(f) \leq r_2(f)$.

□

From the diagonal inclusion $X \to X \times_Y X$, one gets $\dim X \leq \dim X \times_Y X$, so $r(f) = r_5(f) \geq 0$. If $r(f) = 0$, then f is said to be semismall.

Example 4.5.2.3.

1. If $f : X \to Y$ is a proper morphism of complex manifolds that is flat of relative dimension r, then $r(f) = r$.

2. Let X be projective manifold such that $-K_X$ is nef and $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map associated with some base point. Then $r(\alpha) = \dim X - \dim \alpha(X)$ by [Lu+10, Theorem].

4.5.3 Direct image of local systems

Defect of semismallness is an important invariant appearing in the decomposition of direct image of perverse sheaves. Proposition 4.5.3.2 is an elementary instance. We begin with a well-known estimation of cohomological dimension of a complex analytic space, used in the proof of Proposition 4.5.3.2. An analogue for topological manifolds is [KS90, Prop. 3.2.2 (iv)].

Lemma 4.5.3.1. *Let X be a paracompact* (in the sense of [Mun00, Def., p.253]) *complex analytic space of complex dimension n. Then $H^q(X, F) = 0$ for every abelian sheaf F on X and every integer $q > 2n$.*

Proof. By [CAS, Prop., p.94], there is an open covering $\{U_\alpha\}_\alpha$ of X such that for each α, there is a finite morphism $f_\alpha : U_\alpha \to B_\alpha$ of complex analytic spaces to an open ball $B_\alpha \subset \mathbb{C}^n$. As X is Hausdorff paracompact, by [Mun00, Lemma 41.6], there exists a locally finite open covering $\{V_\alpha\}$ on X such that $\overline{V_\alpha} \subset U_\alpha$ for each α.

From [Mun00, p.314], for every α, the topological dimension $\operatorname{covdim}(B_\alpha)$ in the sense of [Mun00, Def., p.305] is $2n$. By [KK83, Prop. 51 A.2], the topological space X is metrizable. From [Mun00, Thm. 32.2], each U_α is normal. Therefore, by [Eng95, Thm. 3.3.10, p.200], $\operatorname{covdim}(U_\alpha) \leq 2n$. By [Eng95, Theorem 3.1.3, p.169], $\operatorname{covdim}(\overline{V_\alpha}) \leq 2n$. Similarly, X is normal, so $\operatorname{covdim}(X) \leq 2n$ by [Eng95, Thm. 3.1.10, p.172]. By Alexandroff theorem (see, *e.g.*, [Bre12, p.122]), the cohomological dimension $\dim_{\mathbb{Z}} X$ in the sense of [Eng95, p.75] is at most $2n$. □

The category $D^b_c(X)$ has a natural perverse t-structure (p being the middle perversity)

$$({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X)),$$

whose heart $\operatorname{Perv}(X)$ is a $\mathbb{C}$-linear abelian category ([Bei+82], see also [HT07, Thm. 8.1.27]). An object of $\operatorname{Perv}(X)$ is called a perverse sheaf on X. For every integer i, the functor taking the i-th perverse cohomology sheaf is denoted by ${}^p\mathcal{H}^i : D^b_c(X) \to \operatorname{Perv}(X)$. For any two integers $a \leq b$, set

$$\begin{aligned} {}^pD^{[a,b]}(X) &:= \{\mathcal{K} \in D^b_c(X) : {}^p\mathcal{H}^i(\mathcal{K}) = 0, \forall i \notin [a, b]\}; \\ D^{[a,b]}(X) &:= \{\mathcal{K} \in D^b_c(X) : \mathcal{H}^i(\mathcal{K}) = 0, \forall i \notin [a, b]\}. \end{aligned}$$

Verdier duality $\mathcal{D}_X : D^b_c(X) \to D^b_c(X)$ is a contravariant auto-equivalence that interchanges ${}^pD^{\leq 0}(X)$ and ${}^pD^{\geq 0}(X)$ (see, *e.g.*, [HT07, p.192]).

Proposition 4.5.3.2 is an analytic analogue of [CM03, Prop. 10.0.7]. It allows local coefficients and in our case permits to descend some problems about local systems on X to problems about complexes of sheaves on Y. The proof is different from that in [CM03], in particular it does not use the decomposition theorem [CM03, Thm. 10.0.6].

Proposition 4.5.3.2. *Let $f : X \to Y$ be a* proper *morphism of complex manifolds, where X is of pure dimension n. Let $\mathcal{L}$ a local system on X. Then:*

1. *$Rf_*(\mathcal{L}[n]) \in {}^pD^{[-r(f),r(f)]}(Y)$. In particular, $Rf_*\mathcal{L}[n] \in \operatorname{Perv}(Y)$ when f is moreover semismall.*

2. *When $\mathcal{L} = \mathbb{C}_X$, for $j = \pm r(f)$, one has ${}^p\mathcal{H}^j(Rf_*\mathbb{C}_X[n]) \neq 0$. Hence $r(f) = r_6(f)$.*

Proof. From Corollary 4.5.1.6, there exists a Whitney stratifications ($X = \sqcup_\alpha X_\alpha, Y = \sqcup_\lambda Y_\lambda$) of f such that for every λ, every integer j, the restriction $\mathcal{H}^j(Rf_*\mathcal{L}[n])|_{Y_\lambda}$ is a local system. For each λ, choose a point $y_\lambda \in Y_\lambda$.

1. First, we show that $Rf_*\mathcal{L}[n] \in {}^pD^{\leq r(f)}(Y)$. Fix an integer i. If $\dim Y_\lambda > r(f) - i$, then by (4.22), one has $i + n > 2\dim f^{-1}(y_\lambda)$. Since the fiber $f^{-1}(y_\lambda)$ is a compact complex analytic space, by Lemma 4.5.3.1,
$$H^{i+n}(f^{-1}(y_\lambda), \mathcal{L}|_{f^{-1}(y_\lambda)}) = 0.$$
By proper base change theorem (see, *e.g.*, [Mil13, Thm. 17.2]),
$$\mathcal{H}^i(Rf_*\mathcal{L}[n])_{y_\lambda} = H^{i+n}(f^{-1}(y_\lambda), \mathcal{L}|_{f^{-1}(y_\lambda)}).$$
So $\mathcal{H}^i(Rf_*\mathcal{L}[n]) = 0$ on every stratum Y_λ with $\dim Y_\lambda > r(f) - i$. Therefore, $\dim \operatorname{Supp} \mathcal{H}^i(Rf_*\mathcal{L}[n]) \leq r(f) - i$ and hence $Rf_*\mathcal{L}[n] \in {}^pD^{\leq r(f)}(Y)$.
It remains to show $Rf_*\mathcal{L}[n] \in {}^pD^{\geq -r(f)}(Y)$. By what we have proved, $Rf_*\mathcal{L}^\vee[n] \in {}^pD^{\leq r(f)}(Y)$. Since $\mathcal{D}_X(\mathcal{L}[n]) = \mathcal{L}^\vee[n]$, one has
$$Rf_*\mathcal{L}^\vee[n] = Rf_*\mathcal{D}_X(\mathcal{L}[n]) = \mathcal{D}_Y(Rf_*\mathcal{L}[n]).$$
The last equality uses Verdier's duality (see, *e.g.*, [Max19, Prop. 5.3.9]). This shows $Rf_*\mathcal{L}[n] \in {}^pD^{\geq -r(f)}(Y)$.

2. By (4.22), there exists λ_0 with $r(f) = 2\dim f^{-1}(y_{\lambda_0}) + \dim Y_{\lambda_0} - n$. In particular, $f^{-1}(y_{\lambda_0})$ is nonempty. Let $i_0 = r(f) - \dim Y_{\lambda_0}$, then $i_0 + n = 2\dim f^{-1}(y_{\lambda_0})$. By proper base change theorem again,
$$\mathcal{H}^{i_0}(Rf_*\mathbb{C}[n])_{y_\lambda} = H^{i_0+n}(f^{-1}(y_\lambda), \mathbb{C}) \neq 0.$$
Therefore, $Y_{\lambda_0} \subset \operatorname{Supp} \mathcal{H}^{i_0}(Rf_*\mathbb{C}[n])$ and hence
$$\dim \operatorname{Supp} \mathcal{H}^{i_0}(Rf_*\mathbb{C}[n]) \geq \dim Y_{\lambda_0} = r(f) - i_0.$$
Then $Rf_*\mathbb{C}[n] \notin {}^pD^{\leq r(f)-1}(Y)$. Together with Point 1, this shows
$${}^p\mathcal{H}^{r(f)}(Rf_*\mathbb{C}_X[n]) \neq 0.$$
The other part follows from Verdier's duality.

□

4.6 Generic vanishing for constructible sheaves

In Section 4.6, we review the generic vanishing theorem for (complexes of) constructible sheaves on a complex torus. The case of abelian varieties is treated in [KW15b] and the general case in [BSS18]. We shall reduce the generic vanishing problem on a manifold in Fujiki class $\mathcal{C}$ to results on its Albanese torus.

4.6.1 Thin subsets

To state Krämer-Weissauer's theorem, we recall the terminology "thin subset" introduced in [KW15b, p.532 and p.536].

Fix a complex torus A. Then $\mathrm{Char}(A)$ has a natural structure of algebraic torus over $\mathbb{C}$ of dimension $2\dim A$ and $T(A) = \mathrm{Char}^u(A)$ is a real Lie subgroup of $\mathrm{Char}(A)$. For each complex subtorus $B \subset A$, let $K(B)$ be the kernel of the morphism of algebraic tori $\mathrm{Char}(A) \to \mathrm{Char}(B)$ induced by functoriality. The induced morphism $\pi_1(B,0) \to \pi_1(A,0)$ is injective with torsion-free cokernel of rank $2\dim A - 2\dim B$, so $K(B)$ is an algebraic subtorus of $\mathrm{Char}(A)$.

Definition 4.6.1.1. A thin subset of $\mathrm{Char}(A)$ is a finite union of translates $\chi_i \cdot K(A_i)$ for certain characters $\chi_i \in \mathrm{Char}(A)$ and certain *nonzero* complex subtori $A_i \subset A$. If every χ_i can be chosen to be a torsion point of $\mathrm{Char}(A)$, then such a thin subset is called *arithmetic*.

A thin subset of $\mathrm{Char}(A)$ is strict and Zariski closed. If the complex torus A is nonzero and *simple*, then a subset of $\mathrm{Char}(A)$ is thin if and only if it is finite.

For each complex subtorus $B \subset A$, we have a functorial commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Char}^u(A) & \xrightarrow{(4.18)} & \mathrm{Loc}^{u,1}(A) & \xrightarrow{(4.12)} & \mathrm{Pic}^\tau(A) & \xleftarrow[(4.10)]{} & \mathrm{Pic}^0(A) \\
\big\downarrow \phi & & \big\downarrow & & \big\downarrow & & \big\downarrow \psi \\
\mathrm{Char}^u(B) & \longrightarrow & \mathrm{Loc}^{u,1}(B) & \longrightarrow & \mathrm{Pic}^\tau(A) & \longleftarrow & \mathrm{Pic}^0(B)
\end{array} \tag{4.25}$$

where all the horizontal maps are isomorphisms by Corollary 4.4.2.2 2.

A subset of $\mathrm{Pic}^0(A)$ is called (arithmetic and) thin, if it is the intersection of $\mathrm{Char}^u(A)$ with a (arithmetic and) thin subset of $\mathrm{Char}(A)$ when $\mathrm{Pic}^0(A)$ is identified with $\mathrm{Char}^u(A)$ *via* the diagram (4.25).

Lemma 4.6.1.2. *Every thin subset of* $\mathrm{Pic}^0(A)$ *is a finite union of translates of strict complex subtori.*

Proof. Let B be a subtorus of A. As the induced morphism $\pi_1(B,0) \to \pi_1(A,0)$ is injective with torsion-free cokernel of rank $2(\dim A - \dim B)$, the restriction morphism $\phi : \mathrm{Char}^u(A) \to \mathrm{Char}^u(B)$ in (4.25) is surjective, and its kernel $K(B) \cap \mathrm{Char}^u(A)$ is the group of unitary characters of $\pi_1(A,0)/\pi_1(B,0)$. Therefore, the kernel of the morphism $\psi : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(B)$ in (4.25) is a complex subtorus of dimension $\dim A - \dim B$. ∎

For a connected regular manifold X, let $\alpha : X \to \mathrm{Alb}(X)$ be its Albanese morphism corresponding to some base point. Then α induces a morphism $\alpha^* : \mathrm{Char}(\mathrm{Alb}(X)) \to \mathrm{Char}(X)$ of algebraic groups. By Proposition 4.4.1.2

2, this map identifies $\mathrm{Char}(\mathrm{Alb}(X))$ with the identity component $\mathrm{Char}^\circ(X)$ of $\mathrm{Char}(X)$. Thus we can define thin subsets of $\mathrm{Char}^\circ(X)$. By Proposition 4.4.1.2 5, $\mathrm{Pic}^0(X)$ is naturally identified with $\mathrm{Pic}^0(\mathrm{Alb}(X))$, thus we can define (arithmetic and) thin subsets of $\mathrm{Pic}^0(X)$.

4.6.2 Generic vanishing result on regular manifolds

Roughly speaking, Krämer-Weissauer's theorem controls the failure of vanishing for perverse sheaves on complex tori, measured by the following loci.

Let X be a compact complex manifold of dimension d. For any integers $k \geq 0$, i and for every $\mathcal{K} \in D_c^b(X)$, consider the cohomology support locus

$$\Sigma^i(X, \mathcal{K}) := \{\chi \in \mathrm{Char}(X) : H^i(X, \mathcal{L}_\chi \otimes \mathcal{K}) \neq 0\}.$$

Let $\Sigma^{\neq 0}(X, \mathcal{K}) := \cup_{i \neq 0, i \in \mathbb{Z}} \Sigma^i(X, \mathcal{K})$. Similarly, let $\Sigma^{>j}(X, \mathcal{K}) := \cup_{i>j} \Sigma^i(X, \mathcal{K})$ for every integer j. By Verdier's duality, $H^{2d-i}(X, \mathcal{K}^\vee \otimes \mathcal{L}_{\chi^{-1}})$ is the $\mathbb{C}$-linear dual of $H^i(X, \mathcal{K} \otimes \mathcal{L}_\chi)$. Therefore,

$$\Sigma^{2d-i}(X, \mathcal{K}^\vee) = \{\chi^{-1} : \chi \in \Sigma^i(X, \mathcal{K})\}. \tag{4.26}$$

Fact 4.6.2.1. *Let X be a compact Kähler manifold, and let $\mathcal{K} \in D_c^b(X)$. Then:*

1. ([Wan16, p.547]) *For every integer i, the subset $\Sigma^i(X, \mathcal{K})$ of* $\mathrm{Char}(X)$ *is Zariski closed.*

2. ([BSS18, Thm. 1.1]) *If X is a complex torus, and if $\mathcal{K} \in \mathrm{Perv}(X)$, then $\Sigma^{\neq 0}(X, \mathcal{K})$ is a strict subset of* $\mathrm{Char}(X)$.

3. ([KW15b, Thm. 1.1 and Lem. 11.2 (c)]) *If further X is a complex abelian variety, then $\Sigma^{\neq 0}(X, \mathcal{K})$ is contained in a thin (and arithmetic when $\mathcal{K}$ is semisimple of geometric origin) subset of* $\mathrm{Char}(X)$.

Corollary 4.6.2.2. *Let X be a compact Kähler manifold, and let $\mathcal{K} \in D_c^b(X)$. Then:*

1. *There are only finitely many integers i such that $\Sigma^i(X, \mathcal{K}) \neq \emptyset$. In particular, $\Sigma^{\neq 0}(X, \mathcal{K})$ and for every integer j, $\Sigma^{>j}(X, \mathcal{K})$ are Zariski closed in* $\mathrm{Char}(X)$.

2. *If X is a complex torus, and if $\mathcal{K} \in {}^pD^{\leq m}(X)$ for some integer m, then $\Sigma^{>m}(X, \mathcal{K}) \neq \mathrm{Char}(X)$.*

3. *If X is a complex abelian variety, and $\mathcal{K} \in {}^pD^{\leq m}(X)$ for some integer m, then $\Sigma^{>m}(X, \mathcal{K})$ is contained in a thin (and arithmetic when $\mathcal{K}$ is semisimple of geometric origin) subset of* $\mathrm{Char}(X)$.

Proof. The proof is sketched in [KW15b, p.533].

1. There exist two integers $c < d$ such that $\mathcal{K} \in D^{[c,d]}(X)$. Applying [KS90, Proposition 10.2.12] to the proper morphism $X \to p$, where p is a point, one gets two integers $a < b$ such that $Rf_*(D^{[c,d]}(X)) \subset D^{[a,b]}(p)$. For every character sheaf $\mathcal{L}$ on X, the functor $* \otimes^L \mathcal{L} : D^b_c(X) \to D^b_c(X)$ is t-exact with respect to the standard t-structure. Consequently, $\mathcal{K} \otimes^L \mathcal{L} \in D^{[c,d]}(X)$ and hence $Rf_*(\mathcal{K} \otimes^L \mathcal{L}) \in D^{[a,b]}(p)$. For all integers $i \notin [a,b]$, $\Sigma^i(X, \mathcal{K}) = \emptyset$. This shows the first part of the assertion. The second part of the assertion follows from Fact 4.6.2.1 1.

2. By shifting degree, one may assume $m = 0$. From [KW15b, Prop. 4.1], for every character sheaf $\mathcal{L}$ on X, the functor $\cdot \otimes^L \mathcal{L} : D^b_c(X) \to D^b_c(X)$ is t-exact for the perverse t-structure. Then for every integer j, one has ${}^p\mathcal{H}^j(\mathcal{K} \otimes^L \mathcal{L}) = {}^p\mathcal{H}^j(\mathcal{K}) \otimes^L \mathcal{L}$. Consider the subset
$$W = \cup_{j \in \mathbb{Z}} \Sigma^{\neq 0}(X, {}^p\mathcal{H}^j(\mathcal{K})) \tag{4.27}$$
of $\mathrm{Char}(X)$. It is in fact a finite union, because by [Dim04, Remark 5.1.19], ${}^p\mathcal{H}^j(K) \neq 0$ for only finitely many integers j. By Fact 4.6.2.1 2, $W \neq \mathrm{Char}(X)$.

   For every $\chi \in \mathrm{Char}(X) \backslash W$, consider the Grothendieck spectral sequence from [CM09, p.545]
$$E_2^{i,j} = H^i(X, {}^p\mathcal{H}^j(\mathcal{K}) \otimes^L \mathcal{L}_\chi) \Rightarrow H^{i+j}(X, \mathcal{K} \otimes^L \mathcal{L}_\chi). \tag{4.28}$$
   For any integers $i \neq 0$ and j, one has $H^i(X, {}^p\mathcal{H}^j(\mathcal{K}) \otimes^L \mathcal{L}_\chi) = 0$, so the spectral sequence (4.28) degenerates (in the sense of [Sta24, Tag 011O (2)]) at page E_2 and hence
$$H^j(X, \mathcal{K} \otimes^L \mathcal{L}_\chi) = H^0(X, {}^p\mathcal{H}^j(\mathcal{K}) \otimes^L \mathcal{L}_\chi)$$
   for every integer j. Now that $\mathcal{K} \in {}^pD^{\leq 0}(X)$, for every $i > 0$ one has ${}^p\mathcal{H}^i(\mathcal{K}) = 0$ and hence $H^i(X, \mathcal{K} \otimes^L \mathcal{L}_\chi) = 0$. This shows $\chi \notin \Sigma^{>0}(X, \mathcal{K})$. One concludes that $\Sigma^{>0}(X, \mathcal{K}) \subset W$.

3. As ${}^p\mathcal{H}^j(\mathcal{K}) \neq 0$ for only finitely many integers j, by Fact 4.6.2.1 3, the subset W defined by (4.27) is contained in a thin (and arithmetic when $\mathcal{K}$ is semisimple of geometric origin) subset of $\mathrm{Char}(X)$.

□

Theorem 4.6.2.3 is a generic vanishing result for local systems on a manifold admitting Hodge theory. When X is a projective manifold, [PS13, Theorem 1.5] gives a dimension estimate of $\Sigma^k(X, \mathbb{C}_X)$.

Theorem 4.6.2.3. *Let X be a connected regular manifold of dimension n. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map associated with some base point and $\mathcal{E}$ be a local system on X. Let k be an integer either $< n - r(\alpha)$ or $> n + r(\alpha)$. Then:*

1. *$\Sigma^k(X, \mathcal{E}) \cap \mathrm{Char}^\circ(X)$ is a strict Zariski closed subset of $\mathrm{Char}^\circ(X)$.*
2. *If furthermore $\mathrm{Alb}(X)$ is algebraic, then $\Sigma^k(X, \mathcal{E}) \cap \mathrm{Char}^\circ(X)$ is contained in a thin subset of $\mathrm{Char}^\circ(X)$.*

Proof. In view of (4.26), one may assume $k > d + r(\alpha)$. Set $\mathcal{K} := R\alpha_*\mathcal{E}[d + r(\alpha)]$. We first prove

$$\Sigma^k(X, \mathcal{E}) \cap \mathrm{Char}^\circ(X) = \Sigma^{k-d-r(\alpha)}(\mathrm{Alb}(X), \mathcal{K}) \subset \Sigma^{>0}(\mathrm{Alb}(X), \mathcal{K}). \quad (4.29)$$

This is used in the proof of both 1 and 2.

Indeed, by Proposition 4.5.3.2, the complex of sheaves $\mathcal{K}$ lies in ${}^pD^{\leq 0}(\mathrm{Alb}(X))$. For every $\chi \in \mathrm{Char}^\circ(X)$, let $\mathcal{D}_\chi$ (resp. $\mathcal{L}_\chi$) be the corresponding character sheaf on $\mathrm{Alb}(X)$ (resp. on X). Then $\alpha^*\mathcal{D}_\chi = \mathcal{L}_\chi$. By [KW01, Cor. 7.5 (g), p.109], $R\alpha_*(\mathcal{E} \otimes^L \mathcal{L}_\chi) = (R\alpha_*\mathcal{E}) \otimes^L \mathcal{D}_\chi$ in $D^b_c(\mathrm{Alb}(X))$. It follows that

$$H^k(X, \mathcal{E} \otimes \mathcal{L}_\chi) = H^k(\mathrm{Alb}(X), (R\alpha_*\mathcal{E}) \otimes^L \mathcal{D}_\chi) = H^{k-d-r(\alpha)}(\mathrm{Alb}(X), \mathcal{K} \otimes^L \mathcal{D}_\chi),$$

whence (4.29). Now Point 1 follows from Fact 4.6.2.1 1 and Corollary 4.6.2.2 2, and Point 2 follows from Corollary 4.6.2.2 3. □

4.7 Generic vanishing result for manifolds in Fujiki class $\mathcal{C}$

In Section 4.7.1, we recall the definition of Fujiki class $\mathcal{C}$, the object of central interest in this note. Then we restrict mainly to algebraic varieties in Section 4.7.2.

4.7.1 Fujiki class $\mathcal{C}$

Definition 4.7.1.1 (Fujiki class $\mathcal{C}$, [Uen80, Def. 1])**.** A compact complex manifold is called in Fujiki class $\mathcal{C}$ if it is the meromorphic image of a compact Kähler manifold.

Every compact Kähler manifold is in Fujiki class $\mathcal{C}$. The reason why Fujiki class $\mathcal{C}$ is interesting is two-fold. For one thing, this class is large enough in practice. For another, in this class there is a Hodge theory with unitary local systems as coefficients.

Fact 4.7.1.2 ([Tim87, Cor. 5.3], [Ara90, Thm. 1, Thm. 2, Cor. 2])**.** *Let X be a complex manifold in Fujiki class $\mathcal{C}$. Then for every unitary local system $\mathcal{E}$ on it, X is $\mathcal{E}$-regular.*

In particular, from Fact 4.7.1.2, every manifold in Fujiki class $\mathcal{C}$ is regular. As is explained in Section 4.3 and Section 4.4, the Jacobian and Albanese of a complex manifold in Fujiki class $\mathcal{C}$ behave well.

Theorem 4.7.1.3. *Let X be a connected n-dimensional complex manifold in Fujiki class $\mathcal{C}$, and let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map associated with some base point. Let $E \to X$ be a flat unitary vector bundle. Then for any integers $p, q \geq 0$:*

1. *The locus $S^{p,q}(X, E)$ is an analytic subset of $\mathrm{Pic}^0(X)$.*
2. *One has $S^{n-p,n-q}(X, E^\vee) = \{L \in \mathrm{Pic}^0(X) | L^\vee \in S^{p,q}(X, E)\}$.*
3. *If $p + q < n - r(\alpha)$ or $p + q > n + r(\alpha)$, then $S^{p,q}(X, E)$ is contained in a strict (and thin when $\mathrm{Alb}(X)$ is algebraic) subset of $\mathrm{Pic}^0(X)$.*

Proof.

1. The projection $p_2 : X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ is a regular family in the sense of [CAS, p.207]. Let $p_1 : X \times \mathrm{Pic}^0(X) \to X$ be the other projection. Let $\mathcal{P}$ be the universal line bundle on $X \times \mathrm{Pic}^0(X)$ given by Corollary 4.4.2.1. Applying the upper semi-continuity theorem [CAS, p.210] to the vector bundle $\mathcal{P} \otimes p_1^* \Omega_X^p$ and the regular family p_2, one gets that $S^{p,q}(E)$ is an analytic subset of $\mathrm{Pic}^0(X)$.

2. By the Serre duality (see, *e.g.*, [Huy05, Prop. 4.1.15]), for every $L \in \mathrm{Pic}(X)$, there is a perfect pairing
$$H^q(X, \Omega_X^p \otimes_{O_X} L \otimes_{O_X} E) \times H^{n-q}(X, \Omega_X^{n-p} \otimes_{O_X} L^\vee \otimes_{O_X} E^\vee) \to \mathbb{C},$$
so $L \in S^{p,q}(X, E)$ if and only if $L^\vee \in S^{n-p,n-q}(X, E^\vee)$.

3. By Theorem 4.2.3.1, there is a unitary local system $\mathcal{E}$ on X such that $\mathcal{E} \otimes_{\mathbb{C}} O_X$ is isomorphic to E. For every $\chi \in \mathrm{Char}^u(X)$, the Hodge decomposition (4.6) of $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{L}_\chi$ provided by Fact 4.7.1.2 is
$$\begin{aligned} H^k(X, \mathcal{E} \otimes_{\mathbb{C}} \mathcal{L}_\chi) &= \oplus_{p+q=k} H^q(X, \Omega_X^p \otimes_{\mathbb{C}} \mathcal{E} \otimes_{\mathbb{C}} \mathcal{L}_\chi) \\ &= \oplus_{p+q=k} H^q(X, \Omega_X^p \otimes_{O_X} E \otimes_{O_X} L_\chi). \end{aligned}$$
Therefore, under the isomorphism (4.19), one has
$$\Sigma^k(X, \mathcal{E}) \cap T(X) = \cup_{p+q=k} S^{p,q}(X, E). \tag{4.30}$$
The result follows from Theorem 4.6.2.3.

□

Remark 4.7.1.4. Theorem 4.7.1.3 3 extends Fact 4.1.1.6 from Kähler manifolds to Fujiki class $\mathcal{C}$. As $\dim X - r(\alpha) \leq \dim \alpha(X)$, the numerical hypothesis in Theorem 4.7.1.3 is more restrictive than that in Fact 4.1.1.1. An example from [GL87, Remark, p.401] is reconsidered in the last paragraph of [KW15b, Sec. 3], to show that the bound $p + q < \dim X - r(\alpha)$ is optimal for Fact 4.1.1.6.

4.7.2 Moishezon manifolds

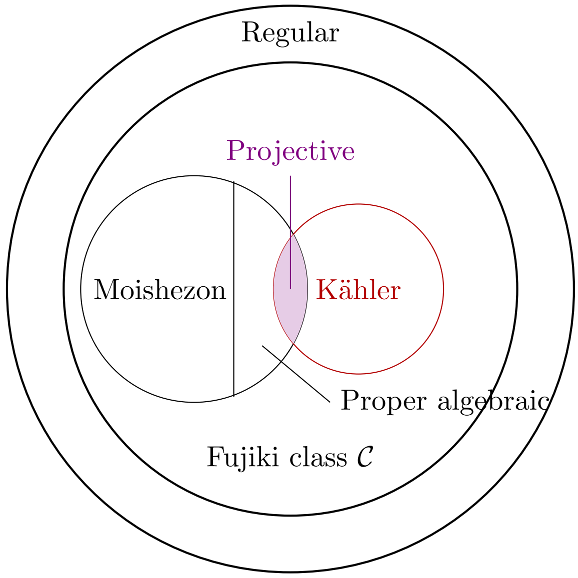
Moishezon manifolds are examples of manifolds in Fujiki class $\mathcal{C}$.

Definition 4.7.2.1 (Moishezon manifold, [MM07, Def. 2.2.12])**.** A connected compact complex manifold X is called *Moishezon*, if it has $\dim X$ algebraically independent meromorphic functions.

By [MM07, Thm. 2.2.16], for every Moishezon manifold X, there is a proper modification $\pi : X' \to X$ with X' a projective manifold. In particular, X is the meromorphic image of a projective manifold, hence in Fujiki class $\mathcal{C}$. Conversely, by the proof of [Voi02, Cor. 12.12], a connected compact complex manifold that is the meromorphic image of a projective manifold must be Moishezon. For more references, see [JM22, Sec. 1].

The intersection of the two subclasses, Kähler and Moishezon, is exactly the class of projective manifolds. More precisely, Moishezon's theorem (see, *e.g.*, [Voi02, Thm. 12.13]) asserts that a Moishezon manifold is Kähler if and only if it is projective. By [Ogu94, Thm. 1], a Moishezon manifold may not be homotopy equivalent to any Kähler manifold. The Kodaira-Spencer stability theorem (see, *e.g.*, [Voi02, Thm. 9.1]) shows that small deformations of a Kähler manifold are Kähler. Similarly, by [AT13, Cor. 3.7], small deformations of a regular manifold are regular. By contrast, from [Cam91, Sec. 0], there is a small deformation of a Moishezon manifold that is not in Fujiki class $\mathcal{C}$. In particular, there exists a regular manifold that is not in Fujiki class $\mathcal{C}$.

Moishezon manifolds are abundant. By [Har77, p.442], for every smooth proper, complex algebraic variety X, its analytification X^{an} is a Moishezon manifold. Hironaka [Hir60] (see also [Har77, p.443]) gives examples of Moishezon manifolds that are not algebraic, and smooth proper algebraic varieties that are not projective. The situation is depicted below. Every inclusion in this graph is strict.



We need Proposition 4.7.2.2 on the algebraicity of Picard torus and Albanese torus to compare them with the Picard variety and Jacobian variety of an algebraic variety.

Proposition 4.7.2.2. *If X is a Moishezon manifold, then* $\mathrm{Alb}(X)$ *and* $\mathrm{Pic}^0(X)$ *are complex abelian varieties dual to each other.*

Proof. By [MM07, Thm. 2.2.16], X admits a proper modification $\pi : X' \to X$ with X' a projective manifold. By [Voi02, Prop. 7.16], the Jacobian $\mathrm{Pic}^0(X')$ is projective. From Proposition 4.4.1.2 5, the torus $\mathrm{Alb}(X')$ is dual to $\mathrm{Pic}^0(X')$, so $\mathrm{Alb}(X')$ is algebraic. By [Uen06, Prop. 9.12, p.107], the morphism $\pi_* : \mathrm{Alb}(X') \to \mathrm{Alb}(X)$ given by Proposition 4.4.1.2 1 is an isomorphism. □

Remark 4.7.2.3. By [BL04, p.70], the analytic dual torus of a complex abelian variety is an abelian variety. Moreover, by [MRM74, p.86], the (algebraic) dual abelian variety (defined in [MRM74, p.78]) of a complex abelian variety coincides with its analytic dual torus, so we do not distinguish the two duals in this case.

Remark 4.7.2.4. Another proof of Proposition 4.7.2.2 is as follows. From Lemma D.3.0.1, there is an integer $n \geq 1$ such that $\mathrm{Alb}(X)$ is the image of X^n under certain morphism. As the product of finitely many Moishezon manifolds, X^n is a Moishezon manifold. Then the complex torus $\mathrm{Alb}(X)$ is Moishezon, so projective by Moishezon Theorem.

Let X be a smooth proper complex algebraic variety of dimension n with a base point $x_0 \in X(\mathbb{C})$. Let $\mathrm{Sch}/\mathbb{C}$ (resp. Set) be the category of $\mathbb{C}$-schemes (resp. sets). The fppf-sheaf associated to the functor

$$P_{X/\mathbb{C}} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}, \quad T \mapsto \mathrm{Pic}(X \times_{\mathbb{C}} T)$$

is called the *relative Picard functor* of X. From [BLR90, p.211, p.231 and p.233], the relative Picard functor of X is represented by a smooth group scheme $\mathrm{Pic}_{X/\mathbb{C}}$ over $\mathbb{C}$. In particular, the group $\mathrm{Pic}_{X/\mathbb{C}}(\mathbb{C}) = \mathrm{Pic}(X)$. By [BLR90, Thm. 3, p.232], the identity component $\mathrm{Pic}^0_{X/\mathbb{C}}$ of $\mathrm{Pic}_{X/\mathbb{C}}$ is proper over $\mathbb{C}$, hence a complex abelian variety called the *Picard variety* of X.

From [Ser58, Thm. 5], there is a complex abelian variety $\mathrm{Alb}(X)$ with a $\mathbb{C}$-morphism $\alpha_{X,x_0} : (X, x_0) \to (\mathrm{Alb}(X), 0)$ of pointed varieties satisfying the following universal property (similar to that stated in Proposition 4.4.1.2 3): every $\mathbb{C}$-morphism of pointed varieties $(X, x_0) \to (A, 0)$ with A a complex abelian variety factors uniquely through a morphism of abelian varieties $\mathrm{Alb}(X) \to A$. Such morphism α_{x_0} is unique up to a unique isomorphism. We call $\mathrm{Alb}(X)$ the *algebraic Albanese variety* of X and $\alpha_{X,x_0} : (X, x_0) \to (\mathrm{Alb}(X), 0)$ the *algebraic Albanese morphism* corresponding to x_0.

For every O_X-module F, let F^{an} be the corresponding $O_{X^{\mathrm{an}}}$-module defined in [SGA 1, Exp. XII, 1.3]. Hence a functor

$$\mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X^{\mathrm{an}}}), \quad F \mapsto F^{\mathrm{an}}.$$

By Serre's GAGA [SGA 1, Exp. XII, Thm. 4.4], the natural group morphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{an}}), \quad L \mapsto L^{\mathrm{an}}$$

is an isomorphism. Corollary 4.7.2.5 2 of GAGA type compares the algebraic Picard variety and the analytic Jacobian. Once again, it is well-known, but a proof is given for the lack of reference.

Corollary 4.7.2.5.

1. *The analytification of* $\mathrm{Alb}(X)$ *(resp.* $\alpha_{X,x_0} : X \to \mathrm{Alb}(X)$ *) is* $\mathrm{Alb}(X^{\mathrm{an}})$ *(resp.* $\alpha_{X^{\mathrm{an}},x_0} : X^{\mathrm{an}} \to \mathrm{Alb}(X^{\mathrm{an}})$*).*

2. *The analytification of* $\mathrm{Pic}^0_{X/\mathbb{C}}$ *is* $\mathrm{Pic}^0(X^{\mathrm{an}})$*.*

Proof.

1. Since X^{an} is a Moishezon manifold, by Proposition 4.7.2.2, its Albanese torus $\mathrm{Alb}(X^{\mathrm{an}})$ is projective. By Chow's theorem [BL04, Cor. A.4], the map $\alpha_{X^{\mathrm{an}},x_0}$ is algebraic. By Proposition 4.4.1.2 3, every algebraic morphism $(X, x_0) \to (A, 0)$ to a complex abelian variety A factors uniquely through an analytic (hence algebraic by Chow's theorem again) morphism of complex tori $\mathrm{Alb}(X^{\mathrm{an}}) \to A^{\mathrm{an}}$. The result follows.

2. By [Moc12, Prop. A.6], the (algebraic) dual abelian variety of $\mathrm{Pic}^0_{X/\mathbb{C}}$ is $\mathrm{Alb}(X)$. By Proposition 4.4.1.2 5, $\mathrm{Pic}^0(X^{\mathrm{an}})$ is the (analytic) dual torus of $\mathrm{Alb}(X^{\mathrm{an}}) = \mathrm{Alb}(X)^{\mathrm{an}}$, so $\mathrm{Pic}^0(X^{\mathrm{an}})$ is the analytification of $\mathrm{Pic}^0_{X/\mathbb{C}}$.

□

Identifying $\mathrm{Pic}^0_{X/\mathbb{C}}$ with $\mathrm{Pic}^0(X^{\mathrm{an}})$ *via* Corollary 4.7.2.5 2, one can define thin subsets of $\mathrm{Pic}^0_{X/\mathbb{C}}$. Define the defect of semismallness of a proper morphism $f : M \to N$ between complex algebraic varieties by $r(f) = r(f^{\mathrm{an}})$. With this terminology, we get the following generic vanishing result for smooth proper algebraic varieties.

Corollary 4.7.2.6. *Let $\mathcal{E}$ be a* unitary *local system on X^{an}, and let $E = \mathcal{E} \otimes_{\mathbb{C}} O_{X^{\mathrm{an}}}$ be the corresponding holomorphic vector bundle. Then for any integers $p, q \geq 0$ with $p + q > n + r(\alpha)$ or $p + q < n - r(\alpha)$, the locus $S^{p,q}(X^{\mathrm{an}}, E)$ is contained in a thin (and arithmetic when $\mathcal{E}$ is semisimple of geometric origin in $D^b_c(X^{\mathrm{an}})$) subset of* $\mathrm{Pic}^0_{X/\mathbb{C}}$.

Proof. By Corollary 4.7.2.5 1, the analytification $\alpha^{\mathrm{an}}_{X,x_0} : X^{\mathrm{an}} \to \mathrm{Alb}(X)^{\mathrm{an}}$ coincides with $\alpha_{X^{\mathrm{an}},x_0} : X^{\mathrm{an}} \to \mathrm{Alb}(X^{\mathrm{an}})$, and by definition, $r(\alpha) = r(\alpha^{\mathrm{an}})$. From Theorem 4.7.1.3 3, the locus $S^{p,q}(X^{\mathrm{an}}, E)$ is contained in a thin subset of $\mathrm{Pic}^0(X)$.

What remains to show is the assertion in the parentheses. Assume that $\mathcal{E}$ is semisimple of geometric origin. By the decomposition theorem [Bei+82, Thm. 6.2.5], $\mathcal{K} := R\alpha_*\mathcal{E}[n + r(\alpha)]$ is semisimple of geometric origin in $D^b_c(\mathrm{Alb}(X^{\mathrm{an}}))$. By Theorem 4.7.1.3 2, one may assume that $p+q > n+r(\alpha)$, so that

$$S^{p,q}(X^{\mathrm{an}}, E) \subset \Sigma^{p+q}(X^{\mathrm{an}}, \mathcal{E}) \cap T(X) \subset \Sigma^{>0}(\mathrm{Alb}(X), \mathcal{K}),$$

where the first inclusion follows from (4.30) and the second from (4.29). From Corollary 4.6.2.2 3, $\Sigma^{>0}(\mathrm{Alb}(X), \mathcal{K})$ is contained in an arithmetic thin subset of $\mathrm{Pic}^0_{X/\mathbb{C}}$. □

Remark 4.7.2.7. By Chow's theorem, every analytic subset of X^{an} is algebraic. Therefore, $D^b_c(X^{\mathrm{an}})$ coincides with $D^b_c(X(\mathbb{C}), \mathbb{C})$ defined in [Bei+82, p.66] using *algebraic* Whitney stratifications.

Chapter 5

Fourier-Mukai transform on complex tori, revisited

5.1 Introduction

For a ringed space (Z, O_Z), let $D(Z)$ be the derived category of the abelian category of O_Z-modules. A scheme of finite type and separated over a field is called an algebraic variety. For two algebraic varieties (resp. complex analytic spaces) M, N, let $p_M : M \times N \to M$ and $p_N : M \times N \to N$ be the projections. For an object $K \in D(M \times N)$, the *integral transform* $\phi_K^{[M \to N]} : D(M) \to D(N)$ with integral kernel K is defined as

$$\phi_K^{[M \to N]}(\cdot) = Rp_{N,*}(K \otimes^L p_M^* \cdot). \tag{5.1}$$

When Z is a complex analytic space, let $D_{\mathrm{good}}(Z) \subset D(Z)$ be the full subcategory consisting of complexes whose cohomology sheaves are good (Definition A.1.4.1). Roughly speaking, an analytic sheaf of modules is good if it can be approximated by coherent submodules. For a complex torus X of dimension g, let $\hat{X}$ be the dual complex torus. Let $\mathcal{P}$ be the normalized[1] Poincaré line bundle on $X \times \hat{X}$. Define functors $RS : D(\hat{X}) \to D(X)$ and $R\hat{S} : D(X) \to D(\hat{X})$ by $RS = \phi_{\mathcal{P}}^{[\hat{X} \to X]}$, $\quad R\hat{S} = \phi_{\mathcal{P}}^{[X \to \hat{X}]}$. The pair $(RS, R\hat{S})$ is called the *Fourier-Mukai transform* of X. Theorem 5.1.0.1 establishes an analog of the Fourier inversion formula for this pair.

Theorem 5.1.0.1 (Theorem 5.4.1.1)**.** *The functor $R\hat{S}$ (resp. RS) restricts to a functor $D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(\hat{X})$ (resp. $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(X)$). Moreover, there are natural isomorphisms of functors*

$$RS \circ R\hat{S} \cong [-1]_X^*[-g] : D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(X),$$
$$R\hat{S} \circ RS \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(\hat{X}),$$

[1] *i.e.*, both pullback *modules* $\mathcal{P}|_{X \times 0}$ and $\mathcal{P}|_{0 \times \hat{X}}$ are trivial

where $[-g]$ denotes degree shift.

Theorem 5.1.0.1 is a complex analytic variant of [Muk81, Thm. 2.2] (Statement 5.2.0.4, which has a minor problem for lack of quasi-coherence condition). For complex tori, a parallel false assertion is made as [BBP07, Thm. 2.1] (Statement 5.2.0.5). Theorem 5.1.0.1 shows that "good sheaves" on complex manifolds serve as substitutes for "quasi-coherent sheaves" on algebraic varieties in this case. As an application, we recover Matsushima-Morimoto's classification of homogeneous vector bundles on complex tori.

Theorem (Theorem 5.5.3.6)**.** *A vector bundle F on the complex torus X is translation invariant if and only if there is an integer $n \geq 0$, unipotent vector bundles[2] $U_1, \ldots, U_n$ on X and $P_1, \ldots, P_n \in \mathrm{Pic}^0(X)$, such that F is isomorphic to $\oplus_{i=1}^n (P_i \otimes U_i)$.*

Notation and conventions

For a topological space M, the category of abelian sheaves on M is denoted by $\mathrm{Ab}(M)$. The category of ringed spaces is denoted by RingS. For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the category of O_X-modules. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of quasi-coherent (resp. coherent) O_X-modules in the sense of Definition A.1.1.1 3 (resp. 6) is denoted by $\mathrm{Qch}(X)$ (resp. $\mathrm{Coh}(X)$). For a closed subset $Z \subset X$, let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of modules with support contained in Z.

Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*(X)$ refers to the unbounded/bounded below/bounded above/bounded derived category of $\mathrm{Mod}(O_X)$ in order. The full subcategory of $D^*(X)$ consisting of the complexes whose cohomologies are coherent (resp. quasi-coherent) is denoted by $D^*_c(X)$ (resp. $D^*_{\mathrm{qc}}(X)$). Denote by $R\mathcal{H}om_X : D(X)^{\mathrm{op}} \times D(X) \to D(X)$ the internal hom bifunctor constructed in [Sta24, Tag 08DH].

For a locally ringed space X and $x \in X$, let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. For an $O_{X,x}$-module M, the O_X-module $(i_x)_* M$ is denoted by M_x.

All complex analytic spaces (in the sense of [KK83, Def. 43.2]) are assumed to be paracompact. Let An be the category of complex analytic spaces. The dimension of a complex manifold always refers to the complex dimension, which is assumed to be finite.

When X is an abelian variety (resp. complex torus), its dual abelian variety (resp. complex torus) is denoted by $\hat{X}$. The normalized Poincaré bundle on $X \times \hat{X}$ is denoted by $\mathcal{P}$. For $y \in \hat{X}$ (resp. $x \in X$), let P_y (resp. P_x) denote the line bundle $\mathcal{P}|_{X \times y}$ (resp. $\mathcal{P}|_{x \times \hat{X}}$).

[2] Definition 5.5.2.6

5.2 Fourier-Mukai transform

Complex tori are generalizations of complex abelian varieties. Every complex torus of dimension 1 is an abelian variety. By contrast, for every integer $g \geq 2$, a very general complex torus of dimension g is not[3] an abelian variety (see, *e.g.*, [BZ23b, p.21]).

The Fourier-Mukai transform is an analog of the classical Fourier transform. It is proposed by Mukai [Muk81] on abelian varieties and complex tori. Let k be an algebraically closed field. Let X be an abelian variety over k (resp. a complex torus) of dimension g. Write RS and $R\hat{S}$ for $\phi_{\mathcal{P}}^{[\hat{X}\to X]}$ and $\phi_{\mathcal{P}}^{[X\to\hat{X}]}$ respectively. The pair $(RS, R\hat{S})$ is called the Fourier-Mukai transform of X. The functor RS (resp. $R\hat{S}$) restricts to a functor $D^b(\hat{X}) \to D^b(X)$ (resp. $D^b(X) \to D^b(\hat{X})$).

Let X be an abelian variety. The usual exchange of translation and time shifting (resp. multiplication and convolution) of Fourier transform finds analog for Fourier-Mukai transform, namely the exchange of translation and line bundle twisting (resp. tensor product and Pontrjagin product) in [Muk81, (3.1) (resp. (3.7))]. Moreover, Mukai proves a duality theorem similar to the classical Fourier inversion formula.

Fact 5.2.0.1. *[Algebraic Mukai duality] There are canonical isomorphisms of functors*

$$RS \circ R\hat{S} \cong [-1]_X^*[-g] : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X);$$
$$R\hat{S} \circ RS \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(\hat{X}).$$

In particular, the functor $RS : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(X)$ *is an equivalence of categories, with a quasi-inverse* $[-1]_{\hat{X}}^* \circ R\hat{S}[g]$.

Example 5.2.0.2 ([Muk81, Eg. 2.6])**.** For every $y \in \hat{X}(k)$, one has $RS(k_y) = P_y$ and $R\hat{S}(P_y) = k_{-y}[-g]$.

Remark 5.2.0.3. Combining Fact 5.2.0.1, the natural equivalence $D(\mathrm{Qch}(X)) \to D_{\mathrm{qc}}(X)$ from [BN93, Cor. 5.5] with the compatibility of derived direct images [TT90, Cor. B.9], one gets [Rot96, Mukai's Theorem, p.569] stated for $D^b(\mathrm{Qch}(*))$ instead of $D_{\mathrm{qc}}(*)$. The quasi-coherence restriction is essential for Čech resolution with respect to affine covers in [Rot96, p.571].

The proof of Fact 5.2.0.1 uses the projection formula and the flat base change theorem (see, *e.g.*, [Lip60, Prop. 3.9.4, Prop. 3.9.5]). Compared

[3] To the contrary, it is incorrectly implied in [BBR94, p.151] that every complex torus of dimension 2 admits a compatible structure of algebraic complex surface. In fact, it fails for each 2-dimensional complex torus X that is not a projective manifold. For otherwise, assume there is a complex algebraic surface V with $V^{\mathrm{an}} \cong X$. Then V is proper by [SGA 1, XII, Prop. 3.2 (v)]. In consequence, the algebraic variety V is projective by [Har77, p.357]. Thus, X is a projective manifold, a contradiction.

with Fact 5.2.0.1, the original statement (Statement 5.2.0.4) has no quasi-coherence restriction.

Statement 5.2.0.4 ([Muk81, Thm. 2.2]). The functor RS gives an equivalence of categories between $D(\hat{X})$ and $D(X)$, and its quasi-inverse is $[-1]^*_{\hat{X}} \circ R\hat{S}[g]$.

In [BBP07, Thm. 2.1], an assertion similar to Statement 5.2.0.4 is made for complex tori.

Statement 5.2.0.5. Let X be a complex torus. Then the integral transform $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence of triangulated categories.

However, Lemma 5.2.0.6 shows that Statement 5.2.0.4 (resp. Statement 5.2.0.5) holds if and only if $g = 0$.

Lemma 5.2.0.6 ([(ht16]). *Let X be an abelian variety or a complex torus. If the functor $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories, then* $\dim X = 0$.

Proof. When X is a complex torus, let $k = \mathbb{C}$. In both cases, let $F = k_0^{\mathbb{N}}$ be the product of a countable infinite family of k_0 in $\mathrm{Mod}(O_{\hat{X}})$. Since $k^{\mathbb{N}} = k^{\oplus I}$ as k-module for some index set I, the direct sum sheaf $k_0^{\oplus I}$ is isomorphic to F. Therefore, by [Sta24, Tag 07D9 (2)], F is the direct sum of I copies of k_0 in $D^b(\hat{X})$. We claim that F is the product of $\mathbb{N}$ copies of k_0 in $D^b(\hat{X})$.

By [Gro57b, p.129], the abelian category $\mathrm{Mod}(O_{\hat{X},0})$ satisfies the AB 4*) axiom. From [Sta24, Tag 07KC (2)], the inclusion $\mathrm{Mod}(O_{\hat{X},0}) \to D^b(\mathrm{Mod}(O_{\hat{X},0}))$ commutes with countable products. Let $i : 0 \to \hat{X}$ be the closed immersion. Since $i_* : \mathrm{Mod}(O_{\hat{X},0}) \to \mathrm{Mod}(O_{\hat{X}})$ is exact, there is a commutative square

$$\begin{tikzcd}
\mathrm{Mod}(O_{\hat{X},0}) \arrow[r, "i_*"] \arrow[d, hook] & \mathrm{Mod}(O_{\hat{X}}) \arrow[d, hook] \\
D^b(\mathrm{Mod}(O_{\hat{X},0})) \arrow[r, "Ri_*"] & D^b(\hat{X}).
\end{tikzcd}$$

Since $Ri_* : D^b(\mathrm{Mod}(O_{\hat{X},0})) \to D^b(\hat{X})$ has a left adjoint, it commutes with products. As $F = i_*(k^{\mathbb{N}})$, the claim is proved.

As $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence, inside $D^b(X)$, the object $RS(F)$ is the direct sum of I copies of $RS(k_0)$, as well as the product of $\mathbb{N}$ copies of $RS(k_0)$. By Example 5.2.0.2 (when X is an abelian variety) and Lemma 5.2.0.8 (when X is a complex torus), one has $RS(k_0) = O_X$. Therefore, $RS(F)$ is isomorphic to $O_X^{\oplus I}$ and to $O_X^{\mathbb{N}}$ in $\mathrm{Mod}(O_X)$.

Assume the contrary $\dim X > 0$. Then there is a nonempty *connected* open subset $V \subset X$, such that $O_X(V)$ is an integral domain but not a field. In particular, the ring $O_X(V)$ is not Artinian. By [Har77, II, Exercise 1.11] (when X is an abelian variety) and Corollary A.1.5.4 (when X is a complex torus), the $O_X(V)$-module $\Gamma(V, RS(F))$ is isomorphic to $O_X(V)^{\oplus I}$ and to $O_X(V)^{\mathbb{N}}$. However, this contradicts Fact 5.2.0.7. $\square$

Fact 5.2.0.7 ([Len68, Thm, p.211])**.** *If A is a commutative ring such that $A^{\mathbb{N}}$ is a free A-module, then A is Artinian.*

For algebraic varieties, the analog of Lemma 5.2.0.8 follows from the flat base change theorem and the projection formula.

Lemma 5.2.0.8. *Let X, Y be two complex analytic spaces, let $K \in D(X \times Y)$, and let $x \in X$. Consider the closed embedding $h_x : Y \to X \times Y, \quad y \mapsto (x, y)$. Then $\phi_K^{[X\to Y]}(\mathbb{C}_x) = Lh_x^* K$.*

Proof. Let $p : X \times Y \to X$, $q : X \times Y \to Y$ be the two projections. Denote the closed embedding of complex analytic spaces $x \to X$ by j_x. The cartesian square

$$\begin{tikzcd} Y \arrow[r, "p_0"] \arrow[d, "h_x"] \arrow[dr, phantom, "\square"] & x \arrow[d, "j_x"] \\ X \times Y \arrow[r, "p"] & X \end{tikzcd}$$

in the category An induces a natural morphism $\phi : p^*\mathbb{C}_x \to Rh_{x,*}O_Y$ in $\mathrm{Mod}(O_{X\times Y})$. Both sheaves are supported on $\{x\} \times Y$.

For two (Hausdorff) locally convex topological vector spaces E, F over $\mathbb{C}$, the completed projective topological tensor product $E \hat{\otimes}_{\mathbb{C}} F$ is defined in [Gro55, Ch. I, Déf. 2, p.32]. For every $y \in Y$, by [CAS, p.27], the stalk $O_{X\times Y,(x,y)} = O_{X,x} \hat{\otimes}_{\mathbb{C}} O_{Y,y}$. Then

$$(p^*\mathbb{C}_x)_{(x,y)} = \mathbb{C} \otimes_{O_{X,x}} O_{X\times Y,(x,y)} = O_{Y,y}.$$

Therefore, $\phi_{(x,y)} : (p^*\mathbb{C}_x)_{(x,y)} \to (h_{x,*}O_Y)_{(x,y)}$ is an isomorphism. Thus, ϕ is an isomorphism.

By [Sta24, Tag 0B55], the natural morphism $(Rh_{x,*}O_Y) \otimes^L K \to Rh_{x,*}(Lh_x^* K)$ is an isomorphism. Then

$$\begin{aligned} \phi_K^{[X\to Y]}(\mathbb{C}_x) &= Rq_*(p^*\mathbb{C}_x \otimes^L K) \cong Rq_*(Rh_{x,*}O_Y \otimes^L K) \\ \cong & Rq_* Rh_{x,*}(Lh_x^* K) \cong R(qh_x)_*(Lh_x^* K) = Lh_x^* K. \end{aligned}$$

$\square$

The minor problem with Statement 5.2.0.4 occurs in the proof of [Muk81, Prop. 1.3], when the flat base change theorem [Har66, Prop. 5.12] stated for objects of $D_{\mathrm{qc}}(*)$ is applied to objects in $D^-(*)$. Similarly, the minor problem with Statement 5.2.0.5 originates from a lack of certain analytic quasi-coherence in the wrong Statement 5.2.0.9 (a counterpart of [Muk81, Prop. 1.3]). A modification of Statement 5.2.0.9 is Proposition 5.4.2.3.

Statement 5.2.0.9 ([BBP07, p.427]). If M, N, and P are compact complex manifolds and $K \in D^b(M \times N)$ and $L \in D^b(N \times P)$, then one has a natural isomorphism of functors from $D^b(M)$ to $D^b(P)$:

$$\phi_L^{[N \to P]} \circ \phi_K^{[M \to N]} \cong \phi_{K*L}^{[M \to P]},$$

where

$$K * L = Rp_{M \times P *}(p_{M \times N}^* K \otimes^L p_{N \times P}^* L) \in D^b(M \times P),$$

and $p_{M \times N}$, $p_{M \times P}$, $p_{N \times P}$ are the natural projections $M \times N \times P \to M \times N$, *etc.*

When X is an abelian variety of positive dimension, by Fact 5.2.0.1, $RS(F)$ is the product of $\mathbb{N}$ copies of O_X in $\mathrm{Qch}(X)$. It is not isomorphic to $O_X^{\mathbb{N}}$ by Lemma 5.2.0.10.

Lemma 5.2.0.10. *Let X be an integral scheme with generic point η. If the O_X-module $O_X^{\mathbb{N}}$ is quasi-coherent, then the natural morphism $\eta \to X$ is an isomorphism.*

Proof. Consider an arbitrary affine open $U = \mathrm{Spec}(A) \subset X$. Then A is an integral domain of fraction field $\kappa(\eta)$. We show that the natural inclusion $A \to \kappa(\eta)$ is an isomorphism.

For otherwise, there exists $f \in A \setminus (A^* \cup \{0\})$. Let $D_f \subset U$ be the corresponding standard open subset. Note $\Gamma(U, O_X^{\mathbb{N}}) = A^{\mathbb{N}}$ and $\Gamma(D_f, O_X^{\mathbb{N}}) = (A_f)^{\mathbb{N}}$. As $O_X^{\mathbb{N}} \in \mathrm{Qch}(X)$, the natural A_f-module morphism $\Gamma(U, O_X^{\mathbb{N}})_f \to \Gamma(D_f, O_X^{\mathbb{N}})$ is an isomorphism. Or equivalently, the natural map $\phi : (A^{\mathbb{N}})_f \to (A_f)^{\mathbb{N}}$ is an isomorphism.

In particular, there exists $a = (a_0, a_1, \dots) \in A^{\mathbb{N}}$ and an integer $m \geq 0$ such that $\phi(a/f^m) = (1/f^i)_{i \geq 0}$. Then $a_{m+1} = f^{-1}$ in A_f. There exists an integer $n \geq 0$ such that $(a_{m+1}f - 1)f^n = 0$ in A. Since A is a domain, $a_{m+1}f - 1 = 0$ in A. This contradicts the fact that $f \notin A^*$.

Therefore, the natural morphism $\eta \to U$ is an isomorphism. The proof is completed as U is taken arbitrarily. $\square$

Lemma 5.2.0.11 computes the derived restriction of a relatively flat module, which is a partial converse to [Huy06, Lemma 3.31] in the analytic setting.

Lemma 5.2.0.11. *Let $f : S \to X$ be a flat morphism of complex analytic spaces, and let K be an O_S-module flat over X. For $x \in f(S)$, let $i_x : S_x \to S$ be the inclusion of the fiber over x. Then $Li_x^* K = i_x^* K$.*

Proof. To simplify the notation, we denote i_x by i. By [Sta24, Tag 0B55], the natural morphism

$$Ri_* O_{S_x} \otimes_{O_S}^L K \to Ri_*(Li^* K) \tag{5.2}$$

is an isomorphism. They are supported on S_x, since for every integer n one has

$$H^n(Ri_*(Li^*K)) = Ri_*H^n(Li^*K) = i_*H^n(Li^*K).$$

For every $s \in S_x$, the morphism $j : (s, O_{S,s}) \to S$ of ringed spaces is flat and $j^* : \mathrm{Mod}(O_S) \to \mathrm{Mod}(O_{S,s})$ is taking the stalk at s. Let m_x be the maximal ideal of $O_{X,x}$. As the ring map $f_s^\# : O_{X,x} \to O_{S,s}$ is flat, one has

$$O_{S_x,s} = (O_{X,x}/m_x) \otimes_{O_{X,x}} O_{S,s} = (O_{X,x}/m_x) \otimes^L_{O_{X,x}} O_{S,s}. \tag{5.3}$$

By [Sta24, Tag 079U], one has

$$\begin{aligned} &Lj^*(Ri_*O_{S_x} \otimes^L_{O_S} K) = Lj^*Ri_*O_{S_x} \otimes^L_{O_{S,s}} Lj^*K \\ =&O_{S_x,s} \otimes^L_{O_{S,s}} K_s = [(O_{X,x}/m_x) \otimes^L_{O_{X,x}} O_{S,s}] \otimes^L_{O_{S,s}} K_s \\ =&(O_{X,x}/m_x) \otimes^L_{O_{X,x}} (O_{S,s} \otimes^L_{O_{S,s}} K_s) \\ =&(O_{X,x}/m_x) \otimes^L_{O_{X,x}} K_s = (O_{X,x}/m_x) \otimes_{O_{X,x}} K_s, \end{aligned} \tag{5.4}$$

where the third (resp. fourth, resp. last) equality uses (5.3) (resp. Lemma 5.4.2.1, resp. the flatness of the $O_{X,x}$-module K_s).

Then for every integer $n \neq 0$, every $s \in S_x$, the stalk

$$[H^n(Ri_*O_{S_x} \otimes^L_{O_S} K)]_s = H^n[Lj^*(Ri_*O_{S_x} \otimes^L_{O_S} K)] = H^n((O_{X,x}/m_x) \otimes_{O_{X,x}} K_s) = 0,$$

where the second equality uses (5.4). Hence

$$i_*H^n(Li^*K) = H^n[Ri_*(Li^*K)] \cong H^n(Ri_*O_{S_x} \otimes^L_{O_S} K) = 0,$$

where the second equality uses (5.2). Thus, for every integer $n \neq 0$, $H^n(Li^*K) = 0$ in $\mathrm{Mod}(O_{S_x})$. □

Remark 5.2.0.12. Lemmas 5.2.0.8 and 5.2.0.11 yield an analytic version of [Huy06, Eg. 5.4 vi)]: Let X, Y be two complex analytic spaces. Let $x \in X$ and K be an $O_{X \times Y}$-module flat over X. Then $\phi_K^{[X \to Y]}(\mathbb{C}_x) = K|_{\{x\} \times Y}$.

Remark 5.2.0.13. Here is an example showing the necessity of the flatness of f in Lemma 5.2.0.11.

Let $A = \mathbb{C}[t]$ and $B = \mathbb{C}[x, y]/xy$. Then the B-module xB (resp. yB) is isomorphic to B/y (resp. B/x). Let $S = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A) = A^1_{\mathbb{C}}$. The morphism $A \to B$ of k-algebras defined by $t \mapsto x$ induces a morphism $f : S \to X$ of schemes. Let K be the coherent O_S-module corresponding to the B-module B/y. Then K is flat over X, because the ring map composition $A \to B \to B/y$ is an isomorphism. Let $i : S_0 \to S$ be the inclusion of the fiber over $0 \in X(\mathbb{C})$. Then i is a closed immersion defined by ideal $xB \subset B$, so Li^*K is induced by $K \otimes^L_B (B/x)$. By [Osb12, Exercise 9, b), p.72],

$$\mathrm{Tor}_2^B(B/y, B/x) = (yB) \otimes_B (xB) \cong (B/x) \otimes_B (B/y) = B/(x, y) = \mathbb{C}.$$

In particular, $Li^*K \neq i^*K$. Taking analytification one gets $L(i^{\mathrm{an}})^*K^{\mathrm{an}} \neq (i^{\mathrm{an}})^*K^{\mathrm{an}}$.

Corollary 5.2.0.14 follows from Lemma 5.2.0.11, and it is an analytic counterpart of [Huy06, Example 5.4 vi)].

Corollary 5.2.0.14. *In Lemma 5.2.0.8, if $K \in \mathrm{Mod}(O_{X\times Y})$ is flat over X, then $\phi_K^{[X\to Y]}(\mathbb{C}_x) = i^*K$.*

By Corollary 5.2.0.14 and Theorem 5.4.1.1, Example 5.2.0.2 remains true when X is a complex torus.

5.3 Good modules

As Section 5.2 explains, to obtain an analytic analogue of Fact 5.2.0.1, it is necessary to find a substitute for quasi-coherence on complex manifolds. We show that goodness introduced by Kashiwara (Definition A.1.4.1) can be used as such.

5.3.1 Functoriality

In Corollary 5.3.1.16, we prove that goodness is preserved by integral transforms. To prove this, we show that goodness is preserved by the operations involved in (5.1).

Example 5.3.1.1. [Har66, Example 1., p.68] Let $f : X \to Y$ be a morphism of ringed spaces. Then the derived pullback $Lf^* : D(Y) \to D(X)$ (constructed in [Spa88, Prop. 6.7 (a)]) is bounded above (in the sense of [Lip60, 1.11.1]), and the derived pushout $Rf_* : D(X) \to D(Y)$ is bounded below.

Proposition 5.3.1.2 (Pullback)**.** *Let $f : X \to Y$ be a morphism of complex analytic spaces. Then $Lf^* : D(Y) \to D(X)$ restricts to a functor*

1. *$D^b_c(Y) \to D^b_c(X)$ when Y is a complex manifold or f is flat;*
2. *$D_{\mathrm{good}}(Y) \to D_{\mathrm{good}}(X)$.*

Proof.

1. Because Y is smooth or f is flat, by Lemma 5.3.1.3, the morphism f has finite tor-dimension. Thus, Lf^* restricts to a functor $D^b(Y) \to D^b(X)$.

   Consider $F \in D^b_c(Y)$. To prove that $Lf^*F \in D^b_c(X)$, by [Har66, I, Prop. 7.3 (i)], one may assume $F \in \mathrm{Coh}(Y)$. This case is proved by Lemma A.1.3.3.

2. (a) Let $G \in D^-_{\mathrm{good}}(Y)$. By Example 5.3.1.1, Lemma A.1.4.3 3 and a dual of [Har66, Prop. 7.3 (ii)], to prove $Lf^*G \in D_{\mathrm{good}}(X)$, one may assume $G \in \mathrm{Good}(Y)$. Let U be a relatively compact open subset of X. Then $f(\bar{U})$ is a compact subset of Y, so contained

in a relatively compact open subset V of Y. Since G is good, its restriction $G|_V = \sum_{i\in I} G_i$ is the sum of a directed family of coherent O_V-submodules of $G|_V$. Let $g : f^{-1}(V) \to V$ be the base change of f along the inclusion $V \to Y$. As Lf^* commutes with colimits, one has

$$(Lf^*G)|_{f^{-1}(V)} = \mathrm{colim}_{i\in I} Lg^*G_i.$$

For every integer n, in $\mathrm{Mod}(O_{f^{-1}(V)})$ one has

$$\begin{aligned} &H^n(Lf^*G)|_{f^{-1}(V)} = H^n\big((Lf^*G)|_{f^{-1}(V)}\big) \\ =&H^n(\mathrm{colim}_{i\in I} Lg^*G_i) = \mathrm{colim}_{i\in I} H^n(Lg^*G_i). \end{aligned}$$

Since G_i is coherent, by Lemma A.1.3.3, the $O_{f^{-1}(V)}$-module $H^n(Lg^*G_i)$ is coherent. By Lemma A.1.4.3 3, the $O_{f^{-1}(V)}$-module $H^n(Lf^*G)|_{f^{-1}(V)}$ is good. Since $\bar{U}$ is a compact subset of $f^{-1}(V)$, the subset U is relatively compact in $f^{-1}(V)$. Hence, $H^n(Lf^*G)|_U$ is the sum of a directed family of coherent submodules. Hence $Lf^*G \in D_{\mathrm{good}}(X)$.

(b) Then consider the general case $C \in D_{\mathrm{good}}(Y)$. For every integer $m \geq 0$, the m-th canonical truncation $C_m := \tau^{\leq m}C$ (in the sense of [Sta24, Tag 0118 (4)]) is in $D^-_{\mathrm{good}}(Y)$. From the proof of [Lip60, Prop. 2.5.5], there is a bounded above complex of flat O_Y-modules Q_m with a quasi-isomorphism $Q_m \to C_m$ that is functorial in C_m. Moreover, the complex $Q := \mathrm{colim}_m Q_m$ is K-flat (in the sense of [Spa88, Def. 5.1]), and the canonical morphism $Q \to C$ is a quasi-isomorphism. Because $Lf^* : D(Y) \to D(X)$ admits a right adjoint, it commutes with colimits. Thus, the resulting morphisms

$$\mathrm{colim}_m Lf^*Q_m \to Lf^*Q \to Lf^*C$$

are isomorphisms in $D(X)$.

Let $\mathrm{Ch}(\mathrm{Mod}(O_X))$ be the category of chain complexes over $\mathrm{Mod}(O_X)$. The directed set $\mathbb{N}$ can be seen naturally as a category. Define a functor $\mathbb{N} \to \mathrm{Ch}(\mathrm{Mod}(O_X)), \quad m \mapsto f^*Q_m$. Because $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, for every integer n, by [Hov99, Lem. 1.5], the natural morphism

$$\mathrm{colim}_m H^n(f^*Q_m) \to H^n(\mathrm{colim}_m f^*Q_m)$$

in $\mathrm{Mod}(O_X)$ is an isomorphism. Hence an isomorphism $H^n(Lf^*C) \cong \mathrm{colim}_m H^n(Lf^*Q_m)$ in $\mathrm{Mod}(O_X)$. Since $Q_m \in D^-_{\mathrm{good}}(Y)$, by Case 2a, the O_X-module $H^n(Lf^*Q_m)$ is good. By Lemma A.1.4.3 3, so is the O_X-module $H^n(Lf^*C)$.

□

The tor-dimension tor-dim f of a morphism $f : X \to Y$ of ringed spaces is defined to be the lower dimension (in the sense of [Lip60, 1.11.1]) of the functor $Lf^* : D^-(Y) \to D(X)$. If f is flat, then tor-dim $f = 0$. If f has finite tor-dimension, then $Lf^* : D^-(Y) \to D(X)$ restricts to a functor $D^b(Y) \to D^b(X)$. The weak dimension wgld(R) of a commutative ring R is defined to be the supremum of flat dimension of all R-modules.

Lemma 5.3.1.3. *Let $f : X \to Y$ be a morphism of complex analytic spaces, with Y a complex manifold. Then f has finite tor-dimension.*

Proof. From [Lip60, (2.7.6.4)], one only needs to show that for every $x \in X$, the flat dimension of the $O_{Y,f(x)}$-module $O_{X,x}$ is uniformly bounded. By definition, the flat dimension of every $O_{Y,f(x)}$-module is bounded by the weak dimension of the ring $O_{Y,f(x)}$. Because Y is a complex manifold, the local ring $O_{Y,f(x)}$ is Noetherian regular. By Lemma 5.3.1.4, wgld $O_{Y,f(x)}$ is the Krull dimension of $O_{Y,f(x)}$, which coincides with the dimension of the complex manifold Y near $f(x)$. □

Lemma 5.3.1.4 (Serre)**.** *Let R be a commutative, Noetherian, regular local ring. Then* wgld(R) *coincides with the Krull dimension of R, hence finite.*

Proof. From [Osb12, Cor. 4.21], the weak dimension coincides with the global dimension of R. By Serre's theorem (see, *e.g.*, [Osb12, p.332]), the global dimension equals the Krull dimension, which is finite. □

Proposition 5.3.1.5 (Tensor product)**.** *Let X be a complex analytic space. Then the bifunctor* (constructed in [Spa88, Thm. A. (ii)]) $\otimes^L : D(X) \times D(X) \to D(X)$ *restricts to a bifunctor*

1. *$D^b(X) \times D^b(X) \to D^b(X)$ (resp. $D^b_c(X) \times D^b_c(X) \to D^b_c(X)$) when X is a complex manifold;*
2. *$D_{\mathrm{good}}(X) \times D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(X)$.*

Proof.

1. The weak dimension of a ringed space (M, O_M) is defined to be $\sup_{x\in M}$ wgld$(O_{M,x})$. By [HT07, (C.2.20)], to prove the statement for $D^b(X)$, it suffices to bound the weak dimension of X. As X is smooth, for every $x \in X$, the stalk $O_{X,x}$ is a Noetherian, regular local ring. Thus, by Lemma 5.3.1.4, its weak dimension wgld$(O_{X,x})$ is equal to the dimension of the complex manifold X near x. Therefore, the weak dimension of X is at most $\dim X$

   Consider any $F, G \in D^b_c(X)$. To prove that $F \otimes^L G \in D^b_c(X)$, by [Har66, I, Prop. 7.3 (i)], one may assume $F, G \in \mathrm{Coh}(X)$. Then the conclusion follows from [GH78, 4., p.700].

2. Take $F, G \in D_{\mathrm{good}}(X)$. To prove that $F \otimes^L G \in D_{\mathrm{good}}(X)$, as in the proof of Proposition 5.3.1.2 2, one may assume that $F, G \in D^-_{\mathrm{good}}(X)$. By a dual of [Har66, I, Prop. 7.3 (ii)], one may assume that $F, G \in \mathrm{Good}(X)$. Let U be a relatively compact open subset of X.

   For every integer n, we claim that the O_U-module $H^n(F \otimes^L_{O_X} G)|_U$ is good. By assumption, the restrictions $F|_U = \sum_{i\in I} F_i$ and $G|_U = \sum_{j\in J} G_j$ can be written as sums of directed families of coherent submodules. By [Sta24, Tag 08DJ], the functor $\otimes^L_{O_U}(G|_U) : D(U) \to D(U)$ has a right adjoint, so

$$(F \otimes^L G)|_U = \mathrm{colim}_{i\in I}[F_i \otimes^L (G|_U)]. \tag{5.5}$$

   By [Sta24, Tag 05NI (2)], there exists a complex $C^\bullet$ of flat O_U-modules and a quasi-isomorphism $C^\bullet \to G|_U$. Then for every $i \in I$, in $D(U)$

$$F_i \otimes_{O_U} C^\bullet \xrightarrow{\sim} F_i \otimes^L_{O_U} G|_U. \tag{5.6}$$

   Define a functor $I \to \mathrm{Ch}(\mathrm{Mod}(O_X))$ by $i \mapsto F_i \otimes C^\bullet$. By [Hov99, Lem. 1.5], the natural morphism

$$\mathrm{colim}_{i\in I} H^n(F_i \otimes C^\bullet) \to H^n(\mathrm{colim}_{i\in I}(F_i \otimes C^\bullet))$$

   in $\mathrm{Mod}(O_U)$ is an isomorphism. Combining it with (5.5) and (5.6), one gets an isomorphism in $\mathrm{Mod}(O_U)$

$$\mathrm{colim}_{i\in I} H^n(F_i \otimes^L_{O_U} G|_U) \to H^n(F \otimes^L_{O_X} G)|_U.$$

   Because $\mathrm{Good}(U)$ is closed under colimits in $\mathrm{Mod}(O_U)$ by Lemma A.1.4.3 3, one may assume that $F|_U$ is coherent. Similarly, one may assume further that $G|_U$ is coherent. Then the claim follows from Lemma A.1.3.4.

□

Remark 5.3.1.6. Proposition 5.3.1.5 2 can also be derived from Proposition 5.3.1.2 2 as in the proof of [Bjö93, Thm. 3.2.13 (3)].

As the proof of Theorem 5.3.1.7 is lengthy, we split it into a series of lemmas.

Theorem 5.3.1.7 (Pushout). *Let $f : X \to Y$ be a* proper *morphism of complex analytic spaces. If $\dim X$ is finite, then $Rf_* : D(X) \to D(Y)$ restricts to a functor $D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(Y)$ (resp. $D^b_{\mathrm{good}}(X) \to D^b_{\mathrm{good}}(Y)$).*

Proof. By Lemma 5.3.1.11, the functor Rf_* restricts to a functor $D^b(X) \to D^b(Y)$. We show that $Rf_* F \in D_{\mathrm{good}}(Y)$ for every $F \in D_{\mathrm{good}}(X)$. By [Har66, I, Prop. 7.3 (iii)], Lemmas 5.3.1.11 and A.1.4.3 3, one may assume that $F \in \mathrm{Good}(X)$. For every relatively compact open subset $V \subset Y$, its closure $\bar{V}$ is compact in Y. As f is proper, the preimage $f^{-1}(\bar{V})$ is compact. Thus, $U := f^{-1}(V)$ is a relatively compact open subset of X. Since F is good, $F|_U = \mathrm{colim}_{i\in I} F_i$, where $\{F_i\}_{i\in I}$ is a directed family of coherent O_U-submodules of $F|_U$. Let $g : U \to V$ be the base change of f. Fix an integer n. By Lemma 5.3.1.9, in $\mathrm{Mod}(O_V)$

$$(R^n f_* F)|_V = R^n g_*(F|_U) = \mathrm{colim}_{i\in I} R^n g_* F_i.$$

As a base change of f, the morphism g is proper. Then by Fact 5.3.1.8, for every $i \in I$, the O_V-module $R^n g_* F_i$ is coherent. By $\mathrm{Coh}(V) \subset \mathrm{Good}(V)$ and Lemma A.1.4.3 3, the O_V-module $(R^n f_* F)|_V$ is good. Therefore, $Rf_* F \in D_{\mathrm{good}}(Y)$.

□

Fact 5.3.1.8 (Grauert direct image theorem, see *e.g.*, [CAS, p.207])**.** *Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then $Rf_* : D(X) \to D(Y)$ restricts to a functor* $\mathrm{Coh}(X) \to D_c(Y)$.

Lemma 5.3.1.9. *Let $f : X \to Y$ be a* proper *map between locally compact, Hausdorff spaces. Then for every integer $n \geq 0$, the functor $R^n f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ commutes with filtrant colimits.*

Proof. Let $(F_i, f_{ij})_{i\in I}$ be a filtrant inductive system with colimit F in $\mathrm{Ab}(X)$. Since the abelian category $\mathrm{Ab}(Y)$ is Grothendieck, the filtrant colimit $G = \mathrm{colim}_{i\in I} R^n f_* F_i$ exists and there is a canonical morphism $\phi : G \to R^n f_* F$ in $\mathrm{Ab}(Y)$. For every $y \in Y$, the functor $\mathrm{Ab}(Y) \to \mathrm{Ab}$ taking the stalk at y commutes with colimits, so $G_y = \mathrm{colim}_{i\in I}(R^n f_* F_i)_y$. By [Mil13, Thm. 17.2], for every i the stalk $(R^n f_* F_i)_y = H^n(X_y, F_i|_{X_y})$. Then by [God58, Thm. 4.12.1], the morphism $\phi_y : G_y \to (R^n f_* F)_y$ is an isomorphism. Therefore, ϕ is an isomorphism. □

The proof of Fact 5.3.1.10 is similar to that of [KS90, Prop. 3.2.2].

Fact 5.3.1.10. *Let X be a locally compact, Hausdorff topological space which is countable at infinity. Suppose that there is an integer $n \geq 0$ such that every point of X has an open neighborhood homeomorphic to a locally closed subset of $\mathbb{R}^n$. Then for every abelian sheaf F on X and every integer $j > n$, one has $H^j(X, F) = 0$.*

Lemma 5.3.1.11. *Let X be a complex analytic space of finite dimension n. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then for an object $E \in D(X)$ with $H^m(E) = 0$ for every integer $m > 0$, one has $H^i(Rf_* E) = 0$ for every integer $i > 2n$. In particular, the functor $Rf_* : D(X) \to D(Y)$ is bounded.*

Proof. For every open subset $V \subset Y$ and every O_X-module M, from $i > 2n$ and Fact 5.3.1.10, one has $H^i(f^{-1}(V), M) = 0$. Applying Lemma 5.3.1.14 to the functor $\Gamma(f^{-1}(V), \cdot) : \mathrm{Mod}(O_X) \to \mathrm{Ab}$, one gets

$$H^i(R\Gamma(f^{-1}(V), E)) = H^i(R\Gamma(f^{-1}(V), \tau^{\geq 1}E)) = 0.$$

By Lemma 5.3.1.13, the O_Y-module $H^i(Rf_*E) = 0$. □

Remark 5.3.1.12. The finite dimension condition in Lemma 5.3.1.11 is necessary. For every integer $m \geq 1$, let T_m be a complex torus of dimension m, and let $f_m : T_m \to \operatorname{Specan} \mathbb{C}$ be the canonical morphism. Let $f : X \to Y$ be $\sqcup_{m\geq 1} f_m : \sqcup_{m\geq 1} T_m \to \sqcup_{m\geq 1} \operatorname{Specan} \mathbb{C}$. Then f is proper. For every integer $q \geq 1$, the sheaf $R^q f_* O_X \neq 0$.

Lemma 5.3.1.13 is a derived version of [Har77, III, Prop. 8.1].

Lemma 5.3.1.13. *Let $f : X \to Y$ be a continuous map of topological spaces. Then for every integer i and every $F \in D(\mathrm{Ab}(X))$, the sheaf $H^i(Rf_*F)$ on Y is the sheaf associated to the abelian presheaf $V \mapsto H^i R\Gamma(f^{-1}(V), F)$.*

Proof. By [Spa88, Thm. D], there is a quasi-isomorphism $F \to I$, where I is a K-injective complex of abelian sheaves on X. Then the canonical morphism $Rf_*F \to f_*I$ is an isomorphism in $D(\mathrm{Ab}(Y))$. By [Mur06, Lem. 3], $H^i(Rf_*F)$ is the sheaf associated the presheaf

$$V \mapsto H^i\big(\Gamma(V, f_*I)\big) = H^i\big(\Gamma(f^{-1}(V), I)\big) = H^i\big(R\Gamma(f^{-1}(V), F)\big).$$

□

Lemma 5.3.1.14. *Let X be a ringed space as in Fact 5.3.1.10. Let $F : \mathrm{Mod}(O_X) \to \mathrm{Ab}$ be an additive functor. Assume that F commutes with countable products, and there is an integer $N \geq 0$ with $R^pF(M) = 0$ for every integer $p \geq N$ and every $M \in \mathrm{Mod}(O_X)$. Then the right derived functor $RF : D(X) \to D(\mathrm{Ab})$ exists. Moreover, for any integers $i \geq j$, the natural transformation*

$$H^i(RF\cdot) \to H^i(RF(\tau^{\geq j-N+1}\cdot)) : D(X) \to \mathrm{Ab}$$

is an isomorphism.

Proof. The existence of N and [Wei95, Cor. 10.5.11] show that $RF : D^+(X) \to D^+(\mathrm{Ab})$ extends to a right derived functor $RF : D(X) \to D(\mathrm{Ab})$ of F.

For every integer m and every $E \in D(X)$, set $E_m := \tau^{\geq -m}E$. Then $\{E_m\}_{m\in\mathbb{Z}}$ forms an inverse system in $D(X)$. Let n be as in Fact 5.3.1.10. Then for every open subset $U \subset X$, any integers $p (> n)$ and q, one has $H^p(U, H^q(E)) = 0$. Then by [Sta24, Tag 0D64], the canonical morphism $E \to R\lim_m E_m$ is an isomorphism in $D(X)$. Since F commutes with countable products, from [Sta24, Tag 08U1], in $D(\mathrm{Ab})$ one has $RF(E) \xrightarrow{\sim}$

$R\lim_m RF(E_m)$. For every integer i, by [Sta24, Tag 08U5], there is a short exact sequence in the category Ab

$$0 \to R^1 \lim_m H^{i-1}(RF(E_m)) \to H^i(RF(E)) \to \lim_m H^i(RF(E_m)) \to 0. \quad (5.7)$$

We claim that $R^1 \lim_m H^{i-1}(RF(E_m)) = 0$.

For every integer $m \geq N - i$, by [Sta24, Tag 08J5], there is an exact triangle

$$H^{-m}(E)[m] \to E_m \to E_{m-1} \xrightarrow{+1} H^{-m}(E)[m+1] \quad (5.8)$$

in $D(X)$. By assumption, one has

$$H^i(RF(H^{-m}(E)[m])) = R^{i+m}F((H^{-m}(E)) = 0;$$
$$H^i(RF(H^{-m}(E)[m+1])) = R^{i+m+1}F((H^{-m}(E)) = 0.$$

Taking the long exact sequence associated with (5.8), one concludes that the canonical morphism $H^i(RF(E_m)) \to H^i(RF(E_{m-1}))$ in Ab is an isomorphism. Since the inverse system $\{H^i RF(E_m)\}_{m\geq 1}$ is constant starting with $m = N - i - 1$, it satisfies the Mittag-Leffler condition in the sense of [Sta24, Tag 02N0]. From [Sta24, Tag 07KW (3)], one obtains

$$R^1 \lim_m H^i(RF(E_m)) = 0,$$

which proves the claim.

When $i \geq j$, as the inverse system is constant from $m = N - j - 1$, one has $\lim_m H^i(RF(E_m)) = H^i[RF(E_{N-j-1})]$. Then the sequence (5.7) induces an isomorphism $H^i(RF(E)) \to H^i(RF(\tau^{\geq j-N+1}E))$. □

Remark 5.3.1.15. In the statement of Lemma 5.3.1.14, because $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, it has enough injectives. By [Ver66, p.338], the total right derived functor $RF : D^+(X) \to D^+(\mathrm{Ab})$ exists (even if F may not be left exact).

Corollary 5.3.1.16. *Let X, Y be complex manifolds (resp. complex analytic spaces), with X compact and Y finite dimensional. If F is an object of $D^b_c(X\times Y)$ (resp. $D_{\mathrm{good}}(X\times Y)$), then $\phi_F^{[X\to Y]}$ restricts to a functor $D^b_c(X) \to D^b_c(Y)$ (resp. $D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(Y)$).*

Proof. Because X is compact, its dimension is finite and the projection $X \times Y \to Y$ is proper. Thus, $X \times Y$ is finite dimensional. The result is a combination of Proposition 5.3.1.2 1 (resp. 2), Proposition 5.3.1.5 1 (resp. 2), Fact 5.3.1.8 and Lemma 5.3.1.11 (resp. Theorem 5.3.1.7). □

Remark 5.3.1.17. Although we don't need the functors $R\mathcal{H}om$, $f_!$ and $f^!$, it is interesting to know whether they preserve goodness or not.

5.3.2 Base change theorems

As a replacement for the (algebraic) flat base change theorem (used in Mukai's proof of Fact 5.2.0.1), we give an analytic smooth base change theorem. It is a consequence of Theorem 5.3.2.3 and Fact 5.3.2.2.

Consider a cartesian square in the category An:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S. \end{tikzcd} \tag{5.9}$$

Then [Sta24, Tag 08HY] gives a natural transformation of functors $D(X) \to D(S')$

$$Lg^* Rf_* \to Rf'_* Lg'^*, \tag{5.10}$$

coming from the adjunction in [Sta24, Tag 079W].

Smooth base change

Definition 5.3.2.1. A morphism $g : S' \to S$ of complex analytic spaces is called locally product, if for every $s' \in S'$, there is an open neighborhood U of $s' \in S'$ and a complex analytic space Z, such that $g(U)$ is open in S and there is a $g(U)$-isomorphism $U \to g(U) \times Z$.

By [CD94, II, Cor. 2.7], a locally product morphism is flat.

Fact 5.3.2.2 ([Gro60b, Thm. 3.1])**.** *A morphism of complex analytic spaces is smooth (in the sense of in the sense of [Gro60b, Déf. 3.2]) if and only if it is a submersion (in the sense of [Fis76, p.100]). In particular, a smooth morphism is locally product.*

Theorem 5.3.2.3. *Consider the square (5.9) with both* $\dim X$ *and* $\dim X'$ *finite,* $f : X \to S$ *proper and* $g : S' \to S$ *locally product. Then (5.10) restricts to an isomorphism of functors* $D_{\text{good}}(X) \to D_{\text{good}}(S')$*.*

We begin the proof with several lemmas.

Definition 5.3.2.4. A morphism of complex analytic spaces $g : S' \to S$ is said to satisfy property $\mathcal{Q}_S$ if for every *proper* morphism $f : X \to S$ of complex analytic spaces, every coherent O_X-module F and every integer $i \geq 0$, the base change morphism $g^* R^i f_* F \to R^i f'_*(g'^* F)$ induced by (5.9) is an isomorphism in $\mathrm{Mod}(O_{S'})$.

Lemma 5.3.2.5 shows that the property $\mathcal{Q}$ is local on the source and the target.

Lemma 5.3.2.5. *Let* $g : S' \to S$ *and be a morphism of complex analytic spaces.*

1. *Let $h : S'' \to S'$ be another morphism of complex analytic spaces. If g and h satisfy $\mathcal{Q}_S$ and $\mathcal{Q}_{S'}$ respectively, then gh satisfies $\mathcal{Q}_S$.*

2. *Assume that $\{S'_i\}_{i\in I}$ (resp. $\{S_j\}_{j\in J}$) is an open covering of S' (resp. S) such that for every $i \in I$ (resp. $j \in J$), the morphism $g|_{S'_i} : S'_i \to S$ (resp. $g^{-1}(S_j) \to S_j$) satisfies $\mathcal{Q}_S$ (resp. $\mathcal{Q}_{S_j}$). Then g satisfies $\mathcal{Q}_S$.*

3. *If g is an open embedding of complex analytic spaces, then g satisfies $\mathcal{Q}_S$.*

Proof. 1. The proof is similar to that of [Day23, Lem. 2.13 (2)].

2. It follows from the local nature of sheaves.

3. The proof is similar to that of [Har77, III, Cor. 8.2].

□

Lemma 5.3.2.6. *Let $f : X \to S$ be a proper morphism of complex analytic spaces, with S Stein. Then for every coherent O_X-module F and every integer $n \geq 0$, one has $H^n(X, F) = H^0(S, R^n f_* F)$.*

Proof. By properness of f and Fact 5.3.1.8, the O_S-module $R^n f_* F$ is coherent. As S is Stein, from Cartan's Theorem B (see, *e.g.*, [KK83, Sec. 52, Thm. B]), for every integer $m > 0$ one has $H^m(S, R^n f_* F) = 0$. The conclusion follows from [Sta24, Tag 01F4 (2)]. □

Remark 5.3.2.7. As an application of Lemma 5.3.2.6, we give an enhancement of Lemma 5.3.1.11 for good modules. Let $f : X \to Y$ be a proper morphism of complex analytic spaces with $\dim X$ finite. Then for every good O_X-module G and every integer $n > \dim X$, one has $R^n f_* G = 0$.

Assume first that G is coherent. For every Stein open subset $V \subset Y$, from Cartan's Theorem A (see *e.g.*, [GR04, Theorem A, p.XVIII]), the restriction $R^n f_* G|_V$ is generated by sections $H^0(V, R^n f_* G|_V)$. By Lemma 5.3.2.6, one has
$$H^0(V, R^n f_* G|_V) = H^n(f^{-1}(V), G|_{f^{-1}(V)}),$$
which vanishes by [Rei64, Cor., p.2333]. Thus, $R^n f_* G|_V = 0$. Hence $R^n f_* G = 0$.

Assume now that $G \in \mathrm{Good}(X)$ is arbitrary. For every relatively compact open subset $W \subset Y$, the open subset $f^{-1}(W)$ of X is relatively compact. Then there is a directed family of coherent submodules $\{G_i\}_{i\in I}$ of $G|_{f^{-1}(W)}$ such that $G|_{f^{-1}(W)} = \mathrm{colim}_{i\in I} G_i$. By Lemma 5.3.1.9, one gets $(R^n f_* G)|_W = \mathrm{colim}_{i\in I} R^n (f|_{f^{-1}(W)})_* G_i = 0$. Hence $R^n f_* G = 0$.

Lemma 5.3.2.8. *Let X, Y be complex analytic spaces, with Y Stein. Let $p : X \times Y \to X$ be the projection. Then for every coherent O_X-module F and every integer $i \geq 0$, the natural morphism $H^i(X, F) \hat{\otimes}_{\mathbb{C}} O_Y(Y) \to H^i(X \times Y, p^* F)$ of locally convex topological vector spaces is an isomorphism.*

Proof. Choose a Stein covering $\mathcal{U}$ of X. Let $C^\bullet$ be the Čech complex of F relative to $\mathcal{U}$. Then $H^i(C^\bullet) = H^i(X, F)$. By [EP+96, Prop. 4.1.5], for every integer q, the q-th term C^q of the complex $C^\bullet$ is a Fréchet space. Moreover, $\{U \times Y : U \in \mathcal{U}\}$ forms a Stein covering of $X \times Y$. By [EP+96, Prop. 4.2.3, Thm. 4.2.4], the Čech complex of p^*F relative to this Stein covering is $C^\bullet \hat{\otimes}_{\mathbb{C}} O(Y)$. Therefore, $H^i(C^\bullet \hat{\otimes}_{\mathbb{C}} O(Y)) = H^i(X \times Y, p^*F)$. By [EP+96, Prop. 4.1.5], $O(Y)$ is a unital Fréchet nuclear algebra, so from [EP+96, Thm. A1.6 (d)], the functor $* \hat{\otimes}_{\mathbb{C}} O(Y)$ preserves exact sequences, hence commutes with taking cohomology groups of the Čech complexes. $\square$

We consider the special case of products.

Corollary 5.3.2.9. *Let S, Z be two complex analytic spaces. Then the projection $S \times Z \to S$ satisfies $\mathcal{Q}_S$.*

Proof. Fix a proper morphism $X \to S$ of complex analytic spaces and a coherent O_X-module F. By Lemma 5.3.2.5, we may assume that S, Z are Stein spaces. Then the result follows from Lemma 5.3.2.6, Lemma 5.3.2.8 and [EP+96, Prop. 4.2.3, Thm. 4.2.4]. $\square$

Corollary 5.3.2.10. *Every locally product morphism $g : S' \to S$ of complex analytic spaces satisfies $\mathcal{Q}_S$.*

Proof. Fix $s' \in S'$, and let $s = g(s')$. Since g is locally product, there is an open neighborhood U (resp. V) of $s' \in S'$ (resp. $s \in S$), a complex analytic space Z and an isomorphism $\psi : U \to Z \times V$ of complex analytic spaces such that the diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "g|_U"'] & Z \times V \arrow[dl, "p_2"] \\ V & \end{tikzcd}$$

commutes, where p_2 is the projection to the second factor. By Corollary 5.3.2.9, $g|_U : U \to V$ satisfies $\mathcal{Q}_V$. By Lemma 5.3.2.5, the morphism $g : S' \to S$ satisfies $\mathcal{Q}_S$. $\square$

Proof of Theorem 5.3.2.3. The morphism f' is a base change of f, hence a proper morphism. Because $\dim X, \dim X'$ are finite, by Theorem 5.3.1.7 and Proposition 5.3.1.2 2, the functors $Lg^* Rf_*$ and $Rf'_* Lg'^*$ restrict to functors $D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(S')$.

For every $K \in D_{\mathrm{good}}(X)$, we prove that the base change morphism $Lg^* Rf_* K \to Rf'_* Lg'^* K$ in $D(S')$ is an isomorphism. By Lemma 5.3.1.11, the functors $Rf_* : D(X) \to D(S)$ and $Rf'_* : D(X') \to D(S')$ are bounded. From [Har66, I, Prop. 7.1 (iii)] and Lemma A.1.4.3 3, one may assume that $K \in \mathrm{Good}(X)$. For every $s' \in S'$, there is a relatively compact open neighborhood $V \subset S$ of $g(s')$. The preimage $f^{-1}(V)$ is a relatively compact open subset of X. Consider the base change of the square (5.9) along the open embedding $V \to S$:

$$\begin{tikzcd} f^{-1}(V) \times_V g^{-1}V \arrow[r, "v'"] \arrow[d, "u'"] \arrow[dr, phantom, "\square"] & f^{-1}(V) \arrow[d, "u"] \\ g^{-1}(V) \arrow[r, "v"] & V. \end{tikzcd}$$

Because g is locally product, so is v. One can write $K|_{f^{-1}(V)} = \mathrm{colim}_{i \in I} G_i$, where $\{G_i\}_{i \in I}$ is a directed family of coherent submodules of $K|_{f^{-1}(V)}$. By Lemma 5.3.1.9, the natural morphism

$$(g^* R^i f_* K)|_{g^{-1}(V)} \to R^i f'_*(g'^* K)|_{g^{-1}(V)} \tag{5.11}$$

in $\mathrm{Mod}(O_{g^{-1}(V)})$ is the colimit of the morphisms

$$v^* R^i u_* G_i \to R^i u'_* v'^* G_i.$$

By Corollary 5.3.2.10, for all $i \in I$, they are isomorphisms. Then (5.11) is an isomorphism. $\square$

Remark 5.3.2.11. In the proof of [BBR94, Lem. 5], an analytic flat base change result is applied without further justification. In [MS08, p.153], a flat base change theorem for cartesian squares in the category of complex manifolds is stated, referring to [Spa88] for the proof. However, the cited result [Spa88, Prop. 6.20] is for cartesian squares in the category RingS. In general, a cartesian square in the category of complex manifolds is not cartesian in RingS. For example, the complex vector space $\mathbb{C}^2$ is the product of two copies of $\mathbb{C}$ in the category of complex manifolds, but is not the product even in the subcategory LRS $\subset$ RingS of locally ringed space. (By contrast, from [Sta24, Tag 01JN], every cartesian square in the category of schemes remains cartesian in LRS.)

In fact, by [Gil11, Cor. 5], the product E of two copies of $\mathbb{C}$ in LRS exists. By the universal property of E, there is a unique morphism $f : \mathbb{C}^2 \to E$ in LRS induced by the two projections $p_i : \mathbb{C}^2 \to \mathbb{C}$. Let $o = f(0) \in E$. We claim that the local ring $O_{E,o}$ is not Noetherian.

The local ring $A := O_{\mathbb{C},0} = \mathbb{C}\{z\}$ is the ring of convergent power series. Let $B = A \otimes_{\mathbb{C}} A$. Let $\epsilon : B \to A$ be the surjective (diagonal) morphism defined by $\epsilon(f \otimes g) = fg$. Set $I = \ker(\epsilon)$. Let $c : A \to \mathbb{C}$ be the ring map taking the constant term. Then $c\epsilon : B \to \mathbb{C}$ is surjective, so $m = \ker(c\epsilon)$ is a maximal ideal of B containing I. Set $S = B \setminus m$. Then $O_{E,o} = S^{-1}B$. From [Tu97, p.367], I/I^2 is a free B/I-module of *infinite* rank. Thus, $S^{-1}(I/I^2) = (S^{-1}I)/(S^{-1}I^2)$ is a free $S^{-1}(B/I) = (S^{-1}B)/(S^{-1}I)$-module of infinite rank. In particular, the ideal $S^{-1}I$ of the ring $S^{-1}B$ is not finitely generated. The claim is proved.

By [GH78, p.679], the ring $\mathbb{C}\{x, y\}$ is Noetherian. Thus, the local morphism $f_0^{\#} : O_{E,o} \to O_{\mathbb{C}^2,0} = \mathbb{C}\{x, y\}$ is not an isomorphism. Hence, f is not an isomorphism in LRS.

Non-smooth base change

Remark 5.3.2.12. A base change theorem for algebraic varieties may not have a direct generalization to complex analytic spaces. For example, the affine base change theorem [Sta24, Tag 02KG] fails for morphism of Stein manifolds. In the cartesian square (5.9), assume that $S = \operatorname{Specan}\mathbb{C}$ is a point, $X = \mathbb{C}$ and S' is a positive-dimensional complex manifold. Then there is an open subset $U \subset S'$ isomorphic to an open ball in $\mathbb{C}^n$ with $n > 0$. On the one hand, by Cartan's Theorem B, $Rf_*O_X = f_*O_X = O_{\mathbb{C}}(\mathbb{C})$. Thus, $g^*Rf_*O_X$ is a free $O_{S'}$-module of infinite rank $\dim_{\mathbb{C}} O_{\mathbb{C}}(\mathbb{C})$. From Corollary A.1.5.4, $\Gamma(U, g^*Rf_*O_X) = O_U(U) \otimes_{\mathbb{C}} O_{\mathbb{C}}(\mathbb{C})$. On the other hand, one has $f'^{-1}(U) = U \times \mathbb{C}$ and $g'^*O_X = O_{X'}$, so

$$\begin{aligned}\Gamma(U, f'_*g'^*O_X) &= \Gamma(f'^{-1}(U), O_{X'})\\ =&\Gamma(U \times \mathbb{C}, O_{U\times\mathbb{C}}) \overset{\text{(a)}}{=} O_U(U)\hat{\otimes}_{\mathbb{C}}O_{\mathbb{C}}(\mathbb{C}),\end{aligned}$$

where (a) uses [EP+96, p.75]. The natural morphism $\Gamma(U, g^*Rf_*O_X) \to \Gamma(U, f'_*g'^*O_X)$ is not an isomorphism, so the base change morphism $g^*f_*O_X \to f'_*g'^*O_X$ is not an isomorphism.

Lemma 5.3.2.13 is used in the proof of Proposition 5.5.1.2.

Lemma 5.3.2.13 (Base change)**.** *Consider the cartesian square* (5.9) *with* $\dim X, \dim S'$ *finite and f flat proper. Then (5.10) induces an isomorphism* $Lg^*Rf_* \to Rf'_*Lg'^*$ *of functors* $D_{\text{good}}(X) \to D_{\text{good}}(S')$*.*

Proof. Because $\dim X$ is finite, by Theorem 5.3.1.7 and Proposition 5.3.1.2 2, the functor $Lg^*Rf_* : D(X) \to D(S')$ restricts to a functor $D_{\text{good}}(X) \to D_{\text{good}}(S')$. Consider the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "i'"'] \arrow[d, "f'"] \arrow[rr, bend left, "g'"] & S' \times X \arrow[r, "p'"'] \arrow[d, "\mathrm{Id}_{S'}\times f"] & X \arrow[d, "f"] \\
S' \arrow[r, "i"] \arrow[rr, bend right, "g"'] & S' \times S \arrow[r, "p"] & S,
\end{tikzcd}$$

where the morphism $i : S' \to S' \times S$ is defined by $i(s') = (s', g(s'))$, and $p : S' \times S \to S$ is the projection. Then i is a closed embedding of complex analytic spaces.

Because p is locally product, by Theorem 5.3.2.3, the natural transformation $Lp^*Rf_* \to R(\mathrm{Id}_{S'} \times f)_*Lp'^* : D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(S' \times S)$ is an isomorphism. Because f is flat proper, so is $\mathrm{Id}_{S'} \times f$. Moreover, $\dim(S' \times X) = \dim S' + \dim X$ is finite. Thus, there are isomorphism of functors $D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(S')$

$$\begin{aligned} Lg^*Rf_* &\cong Li^*Lp^*Rf_* \xrightarrow{\sim} Li^*R(\mathrm{Id}_{S'} \times f)_*Lp'^* \\ &\overset{(a)}{\xrightarrow{\sim}} Rf'_*Li'^*Lp'^* \cong Rf'_*Lg'^*, \end{aligned} \tag{5.12}$$

where the isomorphism (a) uses Lemma 5.3.2.14 2. By [Sta24, Tag 0E47], the isomorphism (5.12) is induced by (5.10). □

Lemma 5.3.2.14. *In the cartesian square (5.9), assume that g is a closed embedding of complex analytic spaces. Then:*

1. *The base change morphism $f^*g_*O_{S'} \to g'_*O_{X'}$ in $\mathrm{Mod}(O_X)$ is an isomorphism.*

2. *If f is flat proper and X has finite dimension, then (5.10) is an isomorphism.*

Proof. 1. Let I be the kernel of the canonical surjection $O_S \to g_*O_{S'}$ in $\mathrm{Mod}(O_S)$. Since $f^* : \mathrm{Mod}(O_S) \to \mathrm{Mod}(O_X)$ is right exact, the sequence

$$f^*I \to O_X \to f^*g_*O_{S'} \to 0$$

is exact in $\mathrm{Mod}(O_X)$. Because g is a closed embedding, by [Gro60a, Remarque 2.10], the square (5.9) is cartesian in the category RingS. Then from [Gro60a, 9–05], the cokernel of the morphism $f^*I \to O_X$ in $\mathrm{Mod}(O_X)$ is $g'_*O_{X'}$. Therefore, the morphism $f^*g_*O_{S'} \to g'_*O_{X'}$ is an isomorphism.

2. As g is a closed embedding, the functor $g_* : \mathrm{Ab}(S') \to \mathrm{Ab}(S)$ is exact and $g^{-1}g_* = \mathrm{Id}_{\mathrm{Ab}(S')}$. Therefore, the functor $Rg_* = g_* : D(S') \to D(S)$ is conservative in the sense of [Rie17, p.180]. Thus, it suffices to show that the natural transformation

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lg'^*E \xrightarrow{\sim} Rf_*Rg'_*Lg'^*E \tag{5.13}$$

of functors $D(X) \to D(S)$ is an isomorphism. By [Sta24, Tag 0B55], the natural morphisms

$$(Rg_*O_{S'}) \otimes^L_{O_S} Rf_*E \to Rg_*Lg^*Rf_*E,$$
$$(Rg'_*O_{X'}) \otimes^L_{O_X} E \to Rg'_*Lg'^*E$$

are isomorphisms. One has

$$Rg'_*O_{X'} = g'_*O_{X'} \overset{\text{(a)}}{\xleftarrow{\sim}} f^*g_*O_{S'} \overset{\text{(b)}}{=} Lf^*Rg_*O_{S'},$$

where (a) uses Point 1, and (b) uses the flatness of f. Thus, the natural transformation (5.13) becomes

$$(Rg_*O_{S'}) \otimes^L_{O_S} Rf_*E \to Rf_*(Lf^*Rg_*O_{S'} \otimes^L_{O_X} E).$$

It is an isomorphism by the finiteness of $\dim X$, the properness of f and Fact 5.3.2.15.

□

From Fact 5.3.1.10, one gets Fact 5.3.2.15 as a special case of [Spa88, Prop. 6.18]. A slight variant can also be derived from [KS90, Prop. 2.6.6] and Lemma 5.4.2.1.

Fact 5.3.2.15 (Projection formula)**.** *Let $f : X \to Y$ be a morphism of complex analytic spaces. If $\dim X$ is finite, then there is a canonical isomorphism $(Rf_!-) \otimes^L_{O_Y} (+) \to Rf_!(- \otimes^L_{O_X} Lf^*+)$ of bifunctors $D(X) \times D(Y) \to D(Y)$.*

5.3.3 Compatibility

For a complex algebraic variety X, let $\psi_X : X^{\mathrm{an}} \to X$ be its complex analytification. With quasi-coherence condition, the algebraic and analytic integral transforms are compatible.

Corollary 5.3.3.1. *Let X, Y be two complex algebraic varieties, with X* proper*. Then for every $K \in D_{\mathrm{qc}}(X \times Y)$, the natural square*

$$\begin{CD} D(X) @>{\phi_K^{[X\to Y]}}>> D(Y) \\ @V{\psi_X^*}VV @VV{\psi_Y^*}V \\ D(X^{\mathrm{an}}) @>>{\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}}\to Y^{\mathrm{an}}]}}> D(Y^{\mathrm{an}}), \end{CD}$$

restricts to a commutative *square*

$$\begin{CD} D_{\mathrm{qc}}(X) @>{\phi_K^{[X\to Y]}}>> D_{\mathrm{qc}}(Y) \\ @V{\psi_X^*}VV @VV{\psi_Y^*}V \\ D_{\mathrm{good}}(X^{\mathrm{an}}) @>>{\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}}\to Y^{\mathrm{an}}]}}> D_{\mathrm{good}}(Y^{\mathrm{an}}). \end{CD} \tag{5.14}$$

Proof. From [Sta24, Tag 08DW (1)], [Sta24, Tag 08DX (1)] and [Sta24, Tag 08D5 (1)], the functor $\phi_K^{[X\to Y]}$ restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$. By Corollary 5.3.1.16 and compactness of X^{an}, the functor $\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}}\to Y^{\mathrm{an}}]}$ restricts to a functor $D_{\mathrm{good}}(X^{\mathrm{an}}) \to D_{\mathrm{good}}(Y^{\mathrm{an}})$. By Lemma B.2.0.3, the functor ψ_X^* (resp. ψ_Y^*) restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{good}}(X^{\mathrm{an}})$ (resp. $D_{\mathrm{qc}}(Y) \to D_{\mathrm{good}}(Y^{\mathrm{an}})$).

By [Sta24, Tag 0D5S] (resp. [Sta24, Tag 079U]), analytification commutes with derived pullback (resp. tensor product). As X is proper over $\mathbb{C}$, the projection $p_Y : X\times Y \to Y$ is proper. By Proposition B.3.0.1, analytification commutes with derived direct image. Thus, the square (5.14) is commutative. □

Remark 5.3.3.2. Fact B.2.0.2 (Theorem B.4.0.2) proves Corollary 5.4.1.2 (resp. Theorem 5.4.1.1) for complex tori that are *algebraic*. Because if X is a complex abelian variety, then every functor in the square

$$\begin{tikzcd} D_c^b(X) \arrow[r, "R\hat{S}"] \arrow[d, "\psi_X^*"] & D_c^b(\hat{X}) \arrow[d, "\psi_{\hat{X}}^*"] \\ D_c^b(X^{\mathrm{an}}) \arrow[r, "R\hat{S}"] & D_c^b(\hat{X}^{\mathrm{an}}) \end{tikzcd} \qquad \begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, "R\hat{S}"] \arrow[d, "\psi_X^*"] & D_{\mathrm{qc}}(\hat{X}) \arrow[d, "\psi_{\hat{X}}^*"] \\ D_{\mathrm{qc}}(X^{\mathrm{an}}) \arrow[r, "R\hat{S}"] & D_{\mathrm{qc}}(\hat{X}^{\mathrm{an}}) \end{tikzcd}$$

is an equivalence. In fact, by [Huy06, Def. 5.1] and the natural equivalence $D^b(\mathrm{Coh}(X)) \to D_c^b(X)$ in [SGA 6, Exp. II, Cor. 2.2.2.1], the functor $R\hat{S} : D(X) \to D(\hat{X})$ restricts to a functor $D_c^b(X) \to D_c^b(\hat{X})$. The functor on the top of the square is an equivalence by Fact 5.2.0.1. From Fact B.2.0.2, the vertical functors are also equivalences. From Corollary 5.3.1.16, the functor $R\hat{S}$ restricts to a functor $D_c^b(X^{\mathrm{an}}) \to D_c^b(\hat{X}^{\mathrm{an}})$. The commutativity of the square follows from Corollary 5.3.3.1.

5.4 Analytic Mukai duality

5.4.1 Statement

Let X be a complex torus of dimension g.

Theorem 5.4.1.1 (Mukai, Ben-Bassat, Block, Pantev)**.** *There are natural isomorphisms of functors*

$$RS \circ R\hat{S} \xrightarrow{\sim} [-1]_X^*[-g] : D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(X);$$
$$R\hat{S} \circ RS \xrightarrow{\sim} [-1]_{\hat{X}}^*[-g] : D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(\hat{X}).$$

In particular, $RS : D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(X)$ *is an equivalence of categories, with a quasi-inverse* $[-1]_{\hat{X}}^* R\hat{S}[g]$.

Corollary 5.4.1.2. [4] *The functors $RS : D^b_c(\hat{X}) \to D^b_c(X)$ and $R\hat{S} : D^b_c(X) \to D^b_c(\hat{X})$ are equivalences of triangulated categories.*

Proof. It follows from Corollary 5.3.1.16 and Theorem 5.4.1.1. □

Remark 5.4.1.3. A Mukai duality for complex tori similar to Corollary 5.4.1.2 is stated in [Blo10, p.314], with $D^b(\mathrm{Coh}(*))$ at the place of $D^b_c(*)$. However, Prof. Jonathan Block told the author that here we should stick to $D^b_c(*)$. In fact, in general the abelian category $\mathrm{Coh}(X)$ does not have enough injectives, so it is unclear how to define the derived direct image involved in [Blo10, p.314]. Moreover, recently Prof. Alexey Bondal announced[5] that for a generic complex torus X of dimension > 2, the natural functor $D^b(\mathrm{Coh}(X)) \to D^b_c(X)$ is *not* an equivalence.

5.4.2 Proof

We follow the strategy of [BBP07, Thm. 2.1] to prove Theorem 5.4.1.1.

Preliminaries

Lemma 5.4.2.1 (Associativity)**.** *Let A, B be two sheaves of rings on a topological space X. For $M \in D(\mathrm{Mod}(A))$, $N \in D(\mathrm{BiMod}(A, B))$,*[6] *and $K \in D(\mathrm{Mod}(B))$, there is a canonical isomorphism $M \otimes^L_A (N \otimes^L_B K) = (M \otimes^L_A N) \otimes^L_B K$ in $D(\mathrm{BiMod}(A, B))$.*

Proof. By [Sta24, Tag 06YF], there exists a quasi-isomorphism $M' \to M$ (resp. $K' \to K$) in $D(\mathrm{Mod}(A))$ (resp. $D(\mathrm{Mod}(B))$), where M' (resp. K') is a K-flat complex of A (resp. B) modules. From [Sta24, Tag 06YH], one has

$$\begin{aligned} &M \otimes^L_A (N \otimes^L_B K) = M' \otimes_A (N \otimes^L_B K) \\ =&M' \otimes_A (N \otimes_B K') = (M' \otimes_A N) \otimes_B K' \\ =&(M \otimes^L N) \otimes_B K' = (M \otimes^L_A N) \otimes^L_B K. \end{aligned}$$

□

Lemma 5.4.2.2, an analytic analog of [Muk81, Example 1.2], exhibits the derived pullback and direct image as particular examples of integral transforms.

Lemma 5.4.2.2. *Let $f : X \to Y$ be a morphism of complex analytic spaces. Let $i : \Gamma_f \to X \times Y$ be the inclusion of the graph of f. Set $F = i_* O_{\Gamma_f} \in \mathrm{Mod}(O_{X\times Y})$. Then there are canonical isomorphism of functors*

$$\phi_F^{[X\to Y]} \xrightarrow{\sim} Rf_* : D(X) \to D(Y); \tag{5.15}$$

$$\phi_F^{[Y\to X]} \xrightarrow{\sim} Lf^* : D(Y) \to D(X). \tag{5.16}$$

[4] [PPS17, Thm. 13.1] relies on Statement 5.2.0.5.

[5] `https://www.mathnet.ru/eng/present35371`

[6] Here, $\mathrm{BiMod}(A, B)$ denotes the category of sheaves of (A, B)-bimodules.

Proof. Let $g : \Gamma_f \to X$ be the projection. Since g is an isomorphism of complex analytic spaces, one has a canonical isomorphism

$$Lg^* \xrightarrow{\sim} R(g^{-1})^* \tag{5.17}$$

of functors $D(X) \to D(\Gamma_f)$. Consider the following diagram

$$\begin{tikzcd}
\Gamma_f \arrow[r, hook, "i"] \arrow[d, "g"] & X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"] & \\
X \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

As i is a closed embedding of complex analytic spaces, by [Sta24, Tag 0B55], the natural transformation

$$Ri_*O_{\Gamma_f} \otimes^L Lp_X^*(\cdot) \to Ri_*Li^*Lp_X^*(\cdot) \tag{5.18}$$

is an isomorphism of functors $D(X) \to D(X \times Y)$. One has

$$\begin{aligned}
\phi_F^{[X\to Y]} :=& Rp_{Y*}(F \otimes^L p_X^* \cdot) = Rp_{Y*}(Ri_*O_{\Gamma_f} \otimes^L Lp_X^* \cdot) \\
\overset{(a)}{\xrightarrow{\sim}}& Rp_{Y*}Ri_*Li^*Lp_X^* \overset{(b)}{\xrightarrow{\sim}} Rp_{Y*}Ri_*Lg^* \\
\overset{(c)}{\xrightarrow{\sim}}& Rp_{Y*}Ri_*R(g^{-1})^* \overset{(d)}{\xrightarrow{\sim}} Rf_*,
\end{aligned}$$

where (a) (resp. (c)) uses (5.18) (resp. (5.17)), and (b), (d) are from [Spa88, Thm. A (iii)].

Thus, (5.15) is proved. The proof of (5.16) is similar. $\square$

Proposition 5.4.2.3 is the first ingredient of the proof of Theorem 5.4.1.1, which expresses the composition of two integral transforms as another integral transform.

Proposition 5.4.2.3. *Let M, N, P be complex analytic spaces, with M, N* compact *and* $\dim P$ *finite. Let p_{ij} be the projections of the product $M \times N \times P$. For $K \in D_{\text{good}}(M \times N)$ and $L \in D(N \times P)$, set*

$$H = Rp_{13*}(p_{12}^*K \otimes^L p_{23}^*L)(\in D(M \times P)).$$

Then there is a natural isomorphism $\phi_L^{[N\to P]}\phi_K^{[M\to N]} \xrightarrow{\sim} \phi_H^{[M\to P]}$ of functors $D_{\text{good}}(M) \to D(P)$.

Proof. Let

$$\begin{aligned}
a &: M \times N \to M, \quad & b &: N \times P \to P, \\
p &: M \times N \to N, \quad & q &: N \times P \to N, \\
u &: M \times P \to M, \quad & v &: M \times P \to P
\end{aligned}$$

be projections.

The morphism q is locally product. Properness of p follows from the compactness of M. By Propositions 5.3.1.2 2 and 5.3.1.5 2, the functor $K\otimes^L a^*\cdot : D(M) \to D(M\times N)$ restricts to a functor $D_{\text{good}}(M) \to D_{\text{good}}(M\times N)$. Then one can apply Theorem 5.3.2.3 to the cartesian square

$$\begin{tikzcd} M\times N\times P \arrow[r, "p_{12}"] \arrow[d, "p_{23}"'] \arrow[dr, phantom, "\square"] & M\times N \arrow[d, "p"] \\ N\times P \arrow[r, "q"'] & N, \end{tikzcd}$$

so the base change natural transformation induces an isomorphism

$$q^*Rp_*(K\otimes^L a^*\cdot) \to Rp_{23*}p_{12}^*(K\otimes^L a^*\cdot) \tag{5.19}$$

of functors $D_{\text{good}}(M) \to D_{\text{good}}(N\times P)$. Thus, one has isomorphisms

$$\begin{aligned}\phi_L^{[N\to P]}\phi_K^{[M\to N]} &= Rb_*[L\otimes^L q^*Rp_*(K\otimes^L a^*\cdot)] \\ &\overset{(a)}{\xrightarrow{\sim}} Rb_*[L\otimes^L Rp_{23*}p_{12}^*(K\otimes^L a^*\cdot)] \\ &\overset{(b)}{\xrightarrow{\sim}} Rb_*Rp_{23*}[p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*\cdot)] \\ &\cong Rp_{3*}[p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*\cdot)] \\ &\cong Rv_*Rp_{13*}(p_{12}^*K\otimes^L p_{23}^*L\otimes^L p_1^*\cdot) \\ &\overset{(c)}{\xleftarrow{\sim}} Rv_*[H\otimes^L u^*\cdot] = \phi_H^{[M\to P]},\end{aligned}$$

of functors $D_{\text{good}}(M) \to D(P)$ where (a) uses (5.19), and (b) (resp. (c)) is from the compactness of M (resp. N) and Fact 5.3.2.15. $\square$

Fact 5.4.2.4, the other ingredient of the proof of Theorem 5.4.1.1, calculates the cohomology of the Poincaré bundle.

Fact 5.4.2.4 ([Kem91, Thm. 3.15])**.** *Let X be a complex torus of dimension g. Let $p_X : X\times\hat{X} \to X$, $p_{\hat{X}} : X\times\hat{X}\to\hat{X}$ be the two projections. Then for the normalized Poincaré bundle $\mathcal{P}$, one has $Rp_{X*}\mathcal{P} = \mathbb{C}_0[-g]$ in $D^b(X)$ and $Rp_{\hat{X}*}\mathcal{P} = \mathbb{C}_0[-g]$ in $D^b(\hat{X})$.*

Proof of Theorem 5.4.1.1

By Corollary 5.3.1.16, the functor RS (resp. $R\hat{S}$) restricts to a functor $D_{\text{good}}(\hat{X}) \to D_{\text{good}}(X)$ (resp. $D_{\text{good}}(X) \to D_{\text{good}}(\hat{X})$). Let p_{ij} be the projections of $X\times X\times\hat{X}$. Set

$$H = Rp_{12,*}(p_{13}^*\mathcal{P}\otimes^L p_{23}^*\mathcal{P}).$$

By Propositions 5.3.1.2 1 and 5.3.1.5 1, Fact 5.3.1.8 and Lemma 5.3.1.11, one has $H \in D_c^b(X \times X)$. By Proposition 5.4.2.3, one has an isomorphism of $RS \circ R\hat{S} \xrightarrow{\sim} \phi_H^{[X \to X]}$ of functors $D_{\text{good}}(X) \to D_{\text{good}}(X)$. Let $m : X \times X \to X$ be the group law. Since the $O_{X \times X \times \hat{X}}$-module $p_{13}^* \mathcal{P}$ is flat, one has $p_{13}^* \mathcal{P} \otimes^L p_{23}^* \mathcal{P} = p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}$. By [BL04, Lem. 14.1.7],[7] the $O_{X \times X \times \hat{X}}$-module $p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}$ is isomorphic to $(m \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}$. Then $H \xrightarrow{\sim} Rp_{12,*}(m \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}$.

Because the morphism m is smooth, applying Theorem 5.3.2.3 to the cartesian square

$$\begin{tikzcd} X \times X \times \hat{X} \arrow[r, "m \times \mathrm{Id}_{\hat{X}}"] \arrow[d, "p_{12}"'] \arrow[dr, phantom, "\square"] & X \times \hat{X} \arrow[d, "p_X"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

in the category An, one has an isomorphism $m^* Rp_{X,*} \mathcal{P} \to H$ in $D_c^b(X \times X)$. Let $i : \Gamma_{[-1]} \to X \times X$ be the inclusion of the graph of $[-1]_X : X \to X$. From Fact 5.4.2.4, one has $H \xrightarrow{\sim} m^* \mathbb{C}_0[-g] = i_* O_{\Gamma_{[-1]}}[-g]$. By Lemma 5.4.2.2, there is an isomorphism $\phi_H^{[X \to X]} \xrightarrow{\sim} [-1]_X^*[-g]$ of functors $D(X) \to D(X)$, which shows the isomorphism $RS \circ R\hat{S} \xrightarrow{\sim} [-1]_X^*[-g]$ of functors $D_{\text{good}}(X) \to D_{\text{good}}(X)$. The proof of the second isomorphism is similar.

5.5 Properties of Fourier-Mukai transform

For later reference purposes, we check that each result starting from Theorem 2.2 to (3.12') in [Muk81] has an analytic version. We only indicate the necessary modifications in statements and proofs.

For a complex torus X, let g_X be its dimension. Let $(RS_X, R\hat{S}_X)$ be the Fourier-Mukai transform of X. The subscripts are omitted when there is only one complex torus in context. Let $p_X : X \times \hat{X} \to X$, $p_{\hat{X}} : X \times \hat{X} \to \hat{X}$ be the projections. For a morphism $\phi : X \to Y$ of complex tori, let $\hat{\phi} : \hat{Y} \to \hat{X}$ be the dual morphism.

5.5.1 Functoriality

Exchange of translations and twists

For every point x of the complex torus X, let $T_x : X \to X, \quad x' \mapsto x' + x$ be the translation by x.

Proposition 5.5.1.1. *For every $x \in X$ and every $\hat{x} \in \hat{X}$, there are canonical isomorphisms*

$$RS \circ T_{\hat{x}}^* \cong (\cdot \otimes_{O_X} P_{-\hat{x}}) \circ RS, \tag{5.20}$$

$$RS \circ (\cdot \otimes_{O_{\hat{X}}} P_x) \cong T_x^* \circ RS \tag{5.21}$$

[7] It is stated for abelian varieties, but its proof works for complex tori.

of funtors $D(\hat{X}) \to D(X)$.

Proof. We prove (5.20). From [BL04, Cor. A.9], one gets

$$T^*_{(0,-\hat{x})}\mathcal{P} \xrightarrow{\sim} \mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_X P_{-\hat{x}}; \tag{5.22}$$
$$T^*_{(x,0)}\mathcal{P} \xrightarrow{\sim} \mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}} P_x. \tag{5.23}$$

Then there are isomorphisms

$$\begin{aligned}
RS(T^*_{\hat{x}}\cdot) =& Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}} T^*_{\hat{x}}\cdot) \\
=& Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(0,\hat{x})} p^*_{\hat{X}}\cdot) \\
=& Rp_{X*}T^*_{(0,\hat{x})}(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
\xrightarrow{\sim}& Rp_{X*}R(T_{(0,-\hat{x})})_*(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
\cong& Rp_{X*}(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
\overset{\text{(a)}}{\xrightarrow{\sim}}& Rp_{X*}(p^*_X P_{-\hat{x}} \otimes \mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
\overset{\text{(b)}}{\xleftarrow{\sim}}& P_{-\hat{x}} \otimes Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
=& P_{-\hat{x}} \otimes RS(\cdot)
\end{aligned}$$

of functors $D(\hat{X}) \to D(X)$, where (a) (resp. (b)) uses (5.22) (resp. Fact 5.3.2.15).

We prove (5.21) as follows:

$$\begin{aligned}
RS(P_x \otimes \cdot) =& Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}(P_x \otimes \cdot)) \\
=& Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}} P_x \otimes p^*_{\hat{X}}\cdot)) \\
\overset{\text{(a)}}{\xrightarrow{\sim}}& Rp_{X*}(T^*_{(x,0)}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
=& Rp_{X*}T^*_{(x,0)}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(-x,0)} p^*_{\hat{X}}\cdot) \\
\xrightarrow{\sim}& Rp_{X*}R(T_{(-x,0)})_*(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(-x,0)} p^*_{\hat{X}}\cdot) \\
\cong& R(T_{-x})_* Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot) \\
\cong& T^*_x RS(\cdot),
\end{aligned}$$

where (a) uses (5.23). □

Exchange of the direct image and the inverse image

A result similar to Proposition 5.5.1.2 is stated as [Lau96, Prop. 1.3.1]. As Laumon omits its proof, we give one. The Fourier-Mukai transform is functorial.

Proposition 5.5.1.2. *For a morphism $\phi : Y \to X$ of complex tori, there are canonical isomorphisms of functors*

$$L\phi^* \circ RS_X \cong RS_Y \circ R\hat{\phi}_* : D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(Y), \tag{5.24}$$
$$R\phi_* \circ RS_Y \cong RS_X \circ L\hat{\phi}^*(\cdot)[g_X - g_Y] : D_{\mathrm{good}}(\hat{Y}) \to D_{\mathrm{good}}(X). \tag{5.25}$$

Proof. The isomorphism (5.25) follows from (5.24) as follows. There are isomorphisms

$$\begin{aligned} R\phi_* RS_Y &\overset{\text{(a)}}{\xrightarrow{\sim}} [-1]_X^* RS_X R\hat{S}_X R\phi_* RS_Y(\cdot)[g_X] \\ &\overset{\text{(b)}}{\xrightarrow{\sim}} [-1]_X^* RS_X L\hat{\phi}^* R\hat{S}_Y RS_Y(\cdot)[g_X] \\ &\overset{\text{(c)}}{\xrightarrow{\sim}} [-1]_X^* RS_X L\hat{\phi}^* [-1]_Y^*(\cdot)[g_X - g_Y] \\ &= RS_X L\hat{\phi}^*(\cdot)[g_X - g_Y] \end{aligned}$$

of functors $D_{\mathrm{good}}(\hat{Y}) \to D_{\mathrm{good}}(X)$, where (a) and (c) use Theorem 5.4.1.1, and (b) uses (5.24).

To prove (5.24), we show

$$(\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \cong (\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y. \tag{5.26}$$

Set $L := (\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \otimes_{O_{Y\times\hat{X}}} (\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y^{-1}$. By definition, on the one hand for every $\hat{x} \in \hat{X}$, one has $L|_{Y\times\hat{x}} \xrightarrow{\sim} \phi^* P_{\hat{x}} \otimes P^{-1}_{\hat{\phi}(\hat{x})} \xrightarrow{\sim} O_Y$; on the other hand, one has $L|_{0\times\hat{X}} \xrightarrow{\sim} \hat{\phi}^* O_{\hat{Y}} \xrightarrow{\sim} O_{\hat{X}}$. By the seesaw principle [BL04, Cor. A.9], these imply $L \xrightarrow{\sim} O_{Y\times\hat{X}}$.

By applying Theorem 5.3.2.3 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "p_2"] \arrow[d, "\mathrm{Id}_Y \times \hat{\phi}"'] \arrow[dr, phantom, "\square"] & \hat{X} \arrow[d, "\hat{\phi}"] \\ Y \times \hat{Y} \arrow[r, "p_{\hat{Y}}"'] & \hat{Y} \end{tikzcd}$$

in the category An, the base change natural transformation

$$p_{\hat{Y}}^* R\hat{\phi}_* \to R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* \tag{5.27}$$

induces an isomorphism of functors $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(Y\times\hat{Y})$. By Propositions 5.3.1.2 2 and 5.3.1.5 2, the functor $\mathcal{P}_X \otimes p_{\hat{X}}^*(\cdot) : D(\hat{X}) \to D(X \times \hat{X})$ restricts to a functor $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(X \times \hat{X})$. Because p_X is smooth proper, by applying Lemma 5.3.2.13 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "\phi\times\mathrm{Id}_{\hat{X}}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X \times \hat{X} \arrow[d, "p_X"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

in the category An, the base change natural transformation induces an isomorphism

$$L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \to Rp_{1*}L(\phi \times \mathrm{Id}_{\hat{X}})^*(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \tag{5.28}$$

of functors $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(Y)$.

There are isomorphisms

$$\begin{aligned} L\phi^* \circ RS_X =& L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \\ &\overset{(a)}{\xrightarrow{\sim}} Rp_{1*}L(\phi \times \mathrm{Id}_{\hat{X}})^*(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \\ &\cong Rp_{1*}[L(\phi \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}_X \otimes^L L(\phi \times \mathrm{Id}_{\hat{X}})^* p_{\hat{X}}^* \cdot] \\ &\cong Rp_{1*}[(\phi \times \mathrm{Id}_{\hat{X}})^*\mathcal{P}_X \otimes p_2^* \cdot] \\ &\overset{(b)}{\xrightarrow{\sim}} Rp_{1*}[(\mathrm{Id}_Y \times \hat{\phi})^*\mathcal{P}_Y \otimes p_2^* \cdot] \\ &\cong Rp_{Y*}R(\mathrm{Id}_Y \times \hat{\phi})_*[L(\mathrm{Id}_Y \times \hat{\phi})^*\mathcal{P}_Y \otimes p_2^* \cdot] \\ &\overset{(c)}{\xleftarrow{\sim}} Rp_{Y*}[\mathcal{P}_Y \otimes R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* \cdot] \\ &\overset{(d)}{\xleftarrow{\sim}} Rp_{Y*}[\mathcal{P}_Y \otimes p_{\hat{Y}}^* R\hat{\phi}_* \cdot] \\ &= RS_Y R\hat{\phi}_* \end{aligned}$$

of functors $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(Y)$, where (a) (resp. (b), resp. (c), resp. (d)) uses (5.28) (resp. (5.26), resp. Fact 5.3.2.15, resp. (5.27)). This proves (5.24). ☐

Remark 5.5.1.3. In Proposition 5.5.1.2, if ϕ is an isogeny, then

$$\begin{aligned} \phi^* \circ RS_X \cong RS_Y \circ \hat{\phi}_* : D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(Y); \\ \phi_* \circ RS_Y \cong RS_X \circ \hat{\phi}^* : D_{\mathrm{good}}(\hat{Y}) \to D_{\mathrm{good}}(X). \end{aligned}$$

In fact, ϕ is finite flat and $g_Y = g_X$. By [GR04, Thm. 4, p.47], the functor $\phi_* : \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ is exact, so $R\phi_* = \phi_*$ as a functor $D(Y) \to D(X)$. By the flatness, the inverse image $\phi^* : \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ is exact and $L\phi^* = \phi^*$ as a functor $D(X) \to D(Y)$.

In [Muk81, (3.4)]), for an isogeny $\phi : Y \to X$ of abelian varieties, the derived functor $R\phi_* : D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X))$ is also written as ϕ_*, but for a different reason [Sta24, Tag 08D7].

For the first half of [Muk81, Prop. 3.11 (4)], the result [MRM74, Sec. 23, Lem. 3] cited in its proof still holds for complex tori, with a similar (and simpler) proof.

Exchange of the Pontrjagin product and the tensor product

Let p_i be the two projections $X \times X \to X$. Define a bifunctor $*^R : D(X) \times D(X) \to D(X)$ by $- *^R + = Rm_*(p_1^* - \otimes^L p_2^* +)$. As in Corollary 5.3.1.16, the bifunctor $*^R$ restricts to a bifunctor $D_{\mathrm{good}}(X) \times D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(X)$ (resp. $D_c^b(X) \times D_c^b(X) \to D_c^b(X)$).

Fact 5.5.1.4 ([Muk81, (3.7)]). *For every $F \in D_{\mathrm{good}}(\hat{X})$, there are canonical isomorphisms*

$$RS(F *^R \cdot) \cong RS(F) \otimes^L RS(\cdot),$$
$$RS(F \otimes^L \cdot) \cong RS(F) *^R RS(\cdot)[g]$$

of functors $D_{\mathrm{good}}(\hat{X}) \to D_{\mathrm{good}}(X)$.

Commutativity with external tensor product

Let M, N be two complex analytic spaces. Let $p : M \times N \to M$ and $q : M \times N \to N$ be the projections. The bifunctor $D(M) \times D(N) \to D(M \times N), \quad (-, +) \mapsto (p^*-) \otimes^L (q^*+)$ is denoted by $(\cdot) \boxtimes^L (\cdot)$.

Proposition 5.5.1.5. *Let X, Y be two complex tori and $Z = X \times Y$. Then there is a canonical isomorphism $RS_Z(- \boxtimes^L +) = RS_X(-) \boxtimes^L RS_Y(+)$ of bifunctors $D_{\mathrm{good}}(\hat{X}) \times D_{\mathrm{good}}(\hat{Y}) \to D_{\mathrm{good}}(Z)$.*

Proof. By the seesaw principle, one has $\mathcal{P}_Z \xrightarrow{\sim} \mathcal{P}_X \boxtimes^L \mathcal{P}_Y$. Then there are canonical isomorphisms

$$\begin{aligned} RS_Z(- \boxtimes^L +) =& Rp_{Z*}[\mathcal{P}_Z \otimes^L Lp_{\hat{Z}}^*(- \boxtimes^L +)] \\ \xrightarrow{\sim}& Rp_{Z*}[(\mathcal{P}_X \boxtimes^L \mathcal{P}_Y) \otimes^L (Lp_{\hat{X}}^*(-) \boxtimes^L Lp_{\hat{Y}}^*(+))] \\ \xrightarrow{\sim}& R(p_X \times p_Y)_*[(\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L (\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+))] \\ \overset{(a)}{\xleftarrow{\sim}}& Rp_{X*}(\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L Rp_{Y*}(\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+)) \\ =& RS_X(-) \boxtimes^L RS_Y(+) \end{aligned}$$

of bifunctors $D_{\mathrm{good}}(\hat{X}) \times D_{\mathrm{good}}(\hat{Y}) \to D_{\mathrm{good}}(Z)$, where (a) uses Lemma 5.5.1.6 2. $\square$

Lemma 5.5.1.6.

1. *Let X, Y, T be complex analytic spaces, with X, T finite dimensional. Let $f : X \to Y$ be a* proper *morphism. Then there is a canonical isomorphism*

$$Rf_*(-) \boxtimes^L (+) \to R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +)$$

of bifunctors $D_{\mathrm{good}}(X) \times D(T) \to D(Y \times T)$.

2. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be* proper *morphism of complex analytic spaces. If X_1, X_2 and Y_1 are finite dimensional, then there is a canonical isomorphism*

$$(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(f_1 \times f_2)_*(- \boxtimes^L +)$$

of bifunctors $D_{\mathrm{good}}(X_1) \times D_{\mathrm{good}}(X_2) \to D_{\mathrm{good}}(Y_1 \times Y_2)$.

Proof.

1. Consider the notation in the commutative diagram

$$\begin{tikzcd} & X \times T \arrow[r, "u"] \arrow[dl, "v"'] \arrow[d, dashed, "f\times \mathrm{Id}_T"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T & Y \times T \arrow[l, "q"] \arrow[r, "p"'] & Y, \end{tikzcd}$$

where u, v, p and q are projections. Since $v = q \circ (f \times \mathrm{Id}_T)$, there is a canonical isomorphism $v^* \xrightarrow{\sim} L(f \times \mathrm{Id}_T)^* q^*$ of functors $D(T) \to D(X \times T)$. As $f \times \mathrm{Id}_T$ is a base change of f, it is also proper. As $\dim(X \times T)$ is finite, by Fact 5.3.2.15, the canonical morphism

$$[R(f \times \mathrm{Id}_T)_* u^* -] \otimes^L q^* + \to R(f \times \mathrm{Id}_T)_*[u^* - \otimes^L v^* +] \tag{5.29}$$

of bifunctors $D(X) \times D(T) \to D(Y \times T)$ is an isomorphism.

By Theorem 5.3.2.3, one has a canonical isomorphism

$$p^* Rf_* \to R(f \times \mathrm{Id}_T)_* u^* : D_{\mathrm{good}}(X) \to D_{\mathrm{good}}(Y \times T). \tag{5.30}$$

Therefore, there are canonical isomorphisms

$$\begin{aligned} (Rf_*-) \boxtimes^L + &= (p^* Rf_* -) \otimes^L q^* + \\ &\overset{(a)}{\xrightarrow{\sim}} [R(f \times \mathrm{Id}_T)_* u^* -] \otimes^L q^* + \\ &\overset{(b)}{\xrightarrow{\sim}} R(f \times \mathrm{Id}_T)_*[u^* - \otimes v^* +] \\ &= R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +), \end{aligned}$$

of bifunctors $D_{\mathrm{good}}(X) \times D(T) \to D(Y \times T)$, where (a) (resp. (b)) uses (5.30) (resp. (5.29)).

2. Since $\dim(X_1 \times X_2)$ is finite, as in Corollary 5.3.1.16, the bifunctor $R(f_1 \times f_2)_*(- \boxtimes^L +)$ restricts to a bifunctor $D_{\mathrm{good}}(X_1) \times D_{\mathrm{good}}(X_2) \to D_{\mathrm{good}}(Y_1 \times Y_2)$.

   As $\dim Y_1, \dim X_2$ are finite, by Point 1, there are canonical isomorphisms of bifunctors

$$(Rf_{1*}-) \boxtimes^L + \to R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +) : D_{\mathrm{good}}(X_1) \times D(X_2) \to D(Y_1 \times X_2),$$
$$(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(\mathrm{Id}_{Y_1} \times f_2)_*[(Rf_{1*}-) \boxtimes^L +] : D(X_1) \times D_{\mathrm{good}}(X_2) \to D(Y_1 \times Y_2).$$

   Then there is a canonical isomorphism of bifunctors

$$\begin{aligned}
&(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(\mathrm{Id}_{Y_1} \times f_2)_*[(Rf_{1*}-) \boxtimes^L +] \\
\to &R(\mathrm{Id}_{Y_1} \times f_2)_* R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +) \\
\to &R(f_1 \times f_2)_*(- \boxtimes^L +) : D_{\mathrm{good}}(X_1) \times D_{\mathrm{good}}(X_2) \to D_{\mathrm{good}}(Y_1 \times Y_2).
\end{aligned}$$

□

Skew commutativity with duality

We summarize classical facts about the duality theory on complex manifolds.

Fact 5.5.1.7. *Let X be a complex manifold of pure dimension n, and let $\omega_X = \bigwedge^n \Omega_X$ be the canonical line bundle.*

1. *(*[RR70, p.81, p.90]*) The dualizing functor $D_X = R\mathcal{H}om_X(\cdot, \omega_X)[n] : D(X) \to D(X)$ restricts to a functor $D_c(X) \to D_c(X)$ and the natural transformation $\mathrm{Id} \to D_X \circ D_X : D_c(X) \to D_c(X)$ is an isomorphism. If X is compact, then D_X exchanges*[8] *$D_c^+(X)$ with $D_c^-(X)$, and induces an equivalence $D_c^b(X) \to D_c^b(X)$.*

2. *(*[RRV71, p.264]*) There is a canonical isomorphism $R\mathcal{H}om_X(-,+) \to D_X(- \otimes^L D_X+)$ of bifunctors $D_c(X) \times D_c^+(X) \to D(X)$.*

3. *(*[RRV71, p.264], [Bjö93, p.122]*) Let $f : X \to Y$ be a proper morphism of complex manifolds. Then there is a canonical isomorphism of functors $Rf_* D_X \to D_Y Rf_* : D_c(X) \to D(Y)$.*

Proposition 5.5.1.8 ([Muk81, (3.8)])**.** *There are canonical isomorphisms of functors*

$$D_X \circ RS \xrightarrow{\sim} ([-1]_X^* \circ RS \circ D_{\hat{X}})[g] : D_c^+(\hat{X}) \to D_c^-(X);$$
$$D_{\hat{X}} \circ R\hat{S} \xrightarrow{\sim} ([-1]_{\hat{X}}^* \circ R\hat{S} \circ D_X)[g] : D_c^+(X) \to D_c^-(\hat{X}).$$

[8]By [FS13, p.4971], in general the functor $R\mathcal{H}om_X(\cdot, \omega_X) : D(X) \to D(X)$ does not exchange $D_c^{b,\leq 0}(X)$ and $D_c^{b,\geq 0}(X)$.

We make some preparation for the proof of Proposition 5.5.1.8. Lemma 5.5.1.9 is an adaption of [Har66, Ch.II, Prop. 5.8] and [Sta24, Tag 0C6I].

Lemma 5.5.1.9. *Let $f : X \to Y$ be a* flat *morphism of complex analytic spaces. Then:*

1. *There is a canonical natural transformation of bifunctors*

$$f^* R\mathcal{H}om_Y(-,+) \to R\mathcal{H}om_X(f^*-, f^*+) : D(Y) \times D(Y) \to D(X). \tag{5.31}$$

2. *The natural transformation (5.31) restricts to an isomorphism of bifunctors $D^-_c(Y) \times D(Y) \to D(X)$.*

Proof. Set $G \in D(Y)$.

1. By [Spa88, Thm. D], there is a *functorial* quasi-isomorphism $G \to G'$, where G' is a K-injective complex over $\mathrm{Mod}(O_Y)$. There are natural transformations of functors $D(Y) \to D(X)$

$$\begin{aligned} &f^* R\mathcal{H}om_Y(\cdot, G) \to f^*\mathcal{H}om_Y(\cdot, G') \to \mathcal{H}om_X(f^*\cdot, f^*G') \\ \to &R\mathcal{H}om_X(f^*\cdot, f^*G') \xleftarrow{\sim} R\mathcal{H}om_X(f^*\cdot, f^*G). \end{aligned}$$

2. By [Har66, I, Examples 1], the (contravariant) functors

$$f^* R\mathcal{H}om_Y(\cdot, G), R\mathcal{H}om_X(f^*\cdot, f^*G) : D(Y) \to D(X)$$

are bounded below. Consider $F \in D^-_c(Y)$. To show the natural morphism $f^*R\mathcal{H}om_Y(F,G) \to R\mathcal{H}om_X(f^*F, f^*G) : D^-_c(Y) \to D(X)$ is an isomorphism, by [Har66, I, Prop. 7.1 (ii)], one may assume $F \in \mathrm{Coh}(Y)$. By [Sta24, Tag 08DL], one may shrink Y to open subsets. Thus, from Lemma A.1.3.1, one may assume that there is a quasi-isomorphism $K \to F$, where K is a complex of finite free O_Y-modules. The morphism f is flat, so $f^*K \to f^*F \to 0$ is a globally free resolution of f^*F. The morphism (5.31) is identified with $f^*\mathcal{H}om_Y(K,G) \to \mathcal{H}om_X(f^*K, f^*G)$, which is an isomorphism.

□

Lemma 5.5.1.10. *Let $E \to X$ be a holomorphic vector bundle on a complex manifold, and let $E^\vee$ be the dual vector bundle. Then there is an isomorphism of functors $E^\vee \otimes D_X \cdot \to D_X(E \otimes \cdot) : D(X) \to D(X)$.*

Proof. Since E is a vector bundle, one has isomorphisms

$$E \otimes \cdot \xrightarrow{\sim} \mathcal{H}om_X(E^\vee, \cdot) \xrightarrow{\sim} R\mathcal{H}om_X(E^\vee, \cdot)$$

of functors $D(X) \to D(X)$. Then

$$D_X(E \otimes \cdot) = R\mathcal{H}om_X(R\mathcal{H}om_X(E^\vee, \cdot), \omega_X)[\dim X].$$

As $E^\vee$ is a perfect object of $D(X)$ (in the sense of [Sta24, Tag 08CM]), by [Sta24, Tag 0G40], one has $D_X(E \otimes \cdot) = R\mathcal{H}om_X(\cdot, \omega_X)[\dim X] \otimes^L E^\vee = E^\vee \otimes D_X \cdot$. □

Corollary 5.5.1.11. *Let $f : X \to Y$ be a flat morphism of complex manifolds of relative dimension n. Write $\omega_f = \omega_X \otimes_{O_X} f^*\omega_Y^\vee$ for the relative dualizing line bundle. Then there is a canonical isomorphism of functors $D_X f^* D_Y \to \omega_f \otimes_{O_X} f^*(\cdot)[n] : D_c^-(Y) \to D_c^-(X)$.*

Proof. One has

$$\begin{aligned}
&D_X f^* D_Y O_Y = D_X\big(f^* R\mathcal{H}om_Y(O_Y, \omega_Y)[\dim Y]\big) = D_X(f^*\omega_Y[\dim Y]) \\
=&R\mathcal{H}om_X(f^*\omega_Y, \omega_X)[\dim X - \dim Y] \overset{\text{(a)}}{=} \mathcal{H}om_X(f^*\omega_Y, \omega_X)[n] \\
=&f^*\omega_Y^\vee \otimes_{O_X} \omega_X[n] = \omega_f[n],
\end{aligned} \tag{5.32}$$

where (a) uses that $f^*\omega_Y$ is a line bundle on X.

By Fact 5.5.1.7 1 and 2, there is an isomorphism $D_Y \xrightarrow{\sim} R\mathcal{H}om_Y(\cdot, D_Y O_Y)$ of functors $D_c^-(Y) \to D_c^+(Y)$. From Lemma 5.5.1.9 2, there are isomorphisms

$$f^* D_Y \xrightarrow{\sim} f^* R\mathcal{H}om_Y(\cdot, D_Y O_Y) \xrightarrow{\sim} R\mathcal{H}om_X(f^*\cdot, f^* D_Y O_Y)$$

of functors $D_c^-(Y) \to D_c^+(X)$. Then by Fact 5.5.1.7 1 and 2 again, there are isomorphisms

$$\begin{aligned}
&D_X f^* D_Y \xrightarrow{\sim} f^*(\cdot) \otimes^L D_X f^* D_Y O_Y \\
\overset{\text{(a)}}{=}& f^*(\cdot) \otimes^L_{O_X} \omega_f[n] \overset{\text{(b)}}{=} f^*(\cdot) \otimes_{O_X} \omega_f[n]
\end{aligned}$$

of functors $D_c^-(Y) \to D_c^-(X)$, where (a) (resp. (b)) equality uses (5.32) (resp. local freeness of ω_f). □

Lemma 5.5.1.12. *There is an isomorphism $Rp_{X*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* \cdot) = [-1]_X^* RS$ of functors $D(\hat{X}) \to D(X)$.*

Proof. By [BL04, Cor. A.9], one has $\mathcal{P}^{-1} \xrightarrow{\sim} ([-1]_X \times [1]_{\hat{X}})^*\mathcal{P}$. Since $p_{\hat{X}} \circ ([-1]_X \times [1]_{\hat{X}}) = p_{\hat{X}}$, there are isomorphisms

$$\begin{aligned}
&Rp_{X*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} Rp_{X*}([-1]_X \times [1]_{\hat{X}})^*(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \\
\xleftarrow{\sim}&[-1]_X^* Rp_{X,*}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) = [-1]_X^* RS
\end{aligned}$$

of functors $D(\hat{X}) \to D(X)$. □

Proof of Proposition 5.5.1.8. By Fact 5.5.1.7 1 and 3, There are isomorphisms

$$D_X \circ RS = D_X Rp_{X,*}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} Rp_{X,*} D_{X \times \hat{X}}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot)$$

of functors $D_c^+(\hat{X}) \to D_c^-(X)$. From Lemma 5.5.1.10, there is an isomorphism $D_{X \times \hat{X}}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} \mathcal{P}^{-1} \otimes^L D_{X \times \hat{X}} p_{\hat{X}}^* \cdot$ of functors $D(\hat{X}) \to D(X \times \hat{X})$. By Fact 5.5.1.7 1, the functor $D_{\hat{X}}$ restricts to a functor $D_c^+(\hat{X}) \to D_c^-(\hat{X})$, whence Corollary 5.5.1.11 yields an isomorphism $D_{X \times \hat{X}} p_{\hat{X}}^* = (p_{\hat{X}}^* D_{\hat{X}} \cdot)[g]$ of functors $D_c^+(\hat{X}) \to D_c^-(X \times \hat{X})$. Therefore, there are isomorphisms

$$D_X \circ RS \xrightarrow{\sim} Rp_{X,*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* D_{\hat{X}} \cdot)[g] \overset{\text{(a)}}{\xrightarrow{\sim}} [-1]_X^* RS(D_{\hat{X}} \cdot)[g]$$

of functors $D_c^+(\hat{X}) \to D_c^-(X)$, where (a) uses Lemma 5.5.1.12.

The second isomorphism follows from the first by swapping X and $\hat{X}$. □

5.5.2 Unipotent vector bundles

Definition 5.5.2.1 ([Muk81, Def. 2.3])**.** We say that W.I.T. (weak index theorem) holds for a coherent module F on the complex torus X if there is an integer $i(F)$ such that $H^i R\hat{S}(F) = 0$ for every integer $i \neq i(F)$. In that case, the integer $i(F)$ is called the *index* of F and the coherent module $\hat{F} := H^{i(F)} R\hat{S}(F)$ on $\hat{X}$ is called the Fourier transform of F. We say that I.T. (index theorem) holds for F if there is an integer i_0 such that for every $L \in \mathrm{Pic}^0(X)$ and every integer $i \neq i_0$, one has $H^i(X, F \otimes_{O_X} L) = 0$.

Fact 5.5.2.2 ([Nak94, p.80])**.** *Let F be a coherent O_X-module, then I.T. holds for F if and only if W.I.T holds for F and $\hat{F}$ is locally free on $\hat{X}$.*

Example 5.5.2.4 show that that the word "Artinian" in Statement 5.5.2.3 is a typo. It should be "finite length" as in [Muk78, Thm. 4.12 (1)].

Statement 5.5.2.3 ([Muk81, Eg. 2.9]). Let X be an abelian variety. Let $\mathrm{Mod}^{\mathrm{Ar}}(O_{\hat{X},0}) \subset \mathrm{Mod}(O_{\hat{X},0})$ be the full subcategory comprised of Artinian $O_{\hat{X},0}$-modules. Then the functor $\mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X},0})$ taking the stalk at 0 restricts to an equivalence $\mathrm{Coh}_0(\hat{X}) \to \mathrm{Mod}^{\mathrm{Ar}}(O_{\hat{X},0})$ of categories.

Example 5.5.2.4. When $\dim X = 1$, the ring $O_{\hat{X},0}$ is a discrete valuation ring (DVR). Let $\mathbb{C}(\hat{X})$ be the fraction field of $O_{\hat{X},0}$ (or equivalently, the field of rational functions on $\hat{X}$). By Lemma 5.5.2.5 2, the $O_{\hat{X},0}$-module $\mathbb{C}(\hat{X})/O_{\hat{X},0}$ is Artinian but not finitely generated, so cannot be the stalk at $0 \in \hat{X}$ of any coherent $O_{\hat{X}}$-module.

Lemma 5.5.2.5. *Let R be a DVR with a uniformizer π and fraction field K, then:*

1. *For every nonzero proper R-submodule $M \subsetneq K$, there is an integer n such that $M = \pi^n R$.*

2. *The R-module K/R is Artinian but not finitely generated.*

Definition 5.5.2.6. A vector bundle U on a complex analytic space M is called unipotent if it has a filtration by vector subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong O_M$ for all $1 \leq i \leq n$. Denote the full subcategory of $\mathrm{Coh}(M)$ consisting of unipotent vector bundles by $\mathrm{Uni}(M)$.

By [FL14, Lem. 5.1], every unipotent vector bundle on a complex torus admits a flat holomorphic connection whose underlying local system is unipotent.

Proposition 5.5.2.7. *1. W.I.T. with index g holds for every unipotent vector bundle on X.*

2. *The functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to an equivalence $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$, with a quasi-inverse $H^0 RS = RS : \mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$.*

3. *For every unipotent vector bundle $U \to X$ and every integer $i \geq 0$, one has $H^i(X, U) = \mathrm{Ext}^i_{O_{\hat{X},0}}(\mathbb{C}, \hat{U})$.*

Proof. 1. Because $R\hat{S}$ is a triangulated functor, the full subcategory of $\mathrm{Coh}(X)$ comprised of modules satisfying W.I.T. of a fixed index is closed under extensions. By Lemma 5.2.0.8 and Theorem 5.4.1.1, one has $R\hat{S}(O_X) = R\hat{S}RS(\mathbb{C}_0) \xrightarrow{\sim} \mathbb{C}_0[-g]$. Then W.I.T. with index g holds for O_X, so it holds for every unipotent vector bundle on X.

2. By Point 1, one has an isomorphism of functors $H^g R\hat{S} \xrightarrow{\sim} R\hat{S}[g] : \mathrm{Uni}(X) \to \mathrm{Mod}(O_{\hat{X}})$. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of modules F with $\mathrm{Supp}\left(H^g R\hat{S}(F)\right) \subset \{0\}$ is closed under extensions and contains O_X, so it contains Uni_X. Since $\mathrm{Uni}(X) \subset \mathrm{Coh}(X)$, the functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$.

   For every $F \in \mathrm{Coh}_0(\hat{X})$, the restriction $\mathrm{Supp}(p^*_{\hat{X}} F \otimes \mathcal{P}) \to X$ of p_X is finite. By [GR04, Thm. 4, p.47], one has $RS(F) = H^0 RS(F)$. By Lemma 5.5.2.8 3, the $O_{\hat{X}}$-module F has a filtration with successive quotients isomorphic to $\mathbb{C}_0$. Then $RS(F)$ has a filtration with successive quotients isomorphic to $RS(\mathbb{C}_0) = O_X$. By [EGA I, Ch. 0, 5.4.9], every term of this filtration is finite locally free. Therefore, $RS(F) \in \mathrm{Uni}(X)$ and RS restricts to a functor $\mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$. By Theorem 5.4.1.1, the functor $H^g R\hat{S} : \mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$ is an equivalence with a quasi-inverse RS.

3. It follows from [Muk81, Prop. 2.7] and Point 1.

□

For a commutative ring R, let $\mathrm{Mod}_f(R) \subset \mathrm{Mod}(R)$ be the full subcategory comprised of R-modules of finite length. Lemma 5.5.2.8 1 confirms a guess in [Gro60a, 9–12] for complex field.

Lemma 5.5.2.8. *Let X be a complex analytic space. Let $x \in X$.*

1. *The functor $i_x^{-1} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X,x})$ taking the stalk at x restricts to a functor $\mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$. In particular, if X is a singleton, then $\dim_{\mathbb{C}} O_X$ is finite.*

2. *The functor $i_{x,*} : D(O_{X,x}) \to D(O_X)$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.*

3. *The functor $i_x^{-1} : \mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$ is an equivalence.*

Proof.

1. For every $F \in \mathrm{Coh}_x(X)$, to prove that F_x is a finite length $O_{X,x}$-module, one may assume that $F_x \neq 0$. As F is a finite type O_X-module, F_x is a finite $O_{X,x}$-module. Then $\mathrm{Supp}_{O_{X,x}}(F_x)$ is nonempty. Let m_x be the maximal ideal of $O_{X,x}$. For every $f \in m_x$, there is an open neighborhood U of $x \in X$ such that f is the stalk of some $\bar{f} \in O_X(U)$. Then $\bar{f}$ vanishes on $\mathrm{Supp}(F)$. By the Rückert Nullstellensatz (see, *e.g.*, [CAS, p.67]), there is an integer $n \geq 1$ such that $\bar{f}^n F = 0$ near x. In particular, $f \in \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}$. Therefore,
$$m_x \subset \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}.$$
By [CAS, Corollary, p.44], the ideal m_x is finitely generated, so there is an integer $N \geq 1$ with $m_x^N \subset \mathrm{Ann}_{O_{X,x}}(F_x)$. By [Sta24, Tag 00L6], $\mathrm{Supp}_{O_{X,x}}(F_x)$ is the unique closed point of $\mathrm{Spec}(O_{X,x})$. By [Sta24, Tag 00L5], the $O_{X,x}$-module F_x has finite length. The second statement follows from Lemma 5.5.2.9.

2. Up to isomorphism, the only simple $O_{X,x}$-module is the residue field $\mathbb{C}$. Every $M \in \mathrm{Mod}_f(O_{X,x})$ has a composite series with successive quotients isomorphic to $\mathbb{C}$. Thus, M_x has a filtration with successive quotients isomorphic to $\mathbb{C}_x$. Since $\mathbb{C}_x$ is coherent, by [Sta24, Tag 01BY (4)], M_x is coherent. Therefore, $i_{x,*}$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.

3. Let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. There is a canonical isomorphism $i_x^*(i_x)_* \xrightarrow{\sim} \mathrm{Id}_{\mathrm{Mod}(O_{X,x})}$ of functors $\mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_{X,x})$. By adjunction, $(i_x)_* : \mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_X)$ is fully faithful. By Point 2, pushout $(i_x)_*$ restricts to a

functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$. For every object F of $\mathrm{Coh}_x(O_X)$, by Point 1, F_x is an object of $\mathrm{Mod}_f(O_{X,x})$. The adjunction morphism $F \to (i_x)_*(F_x)$ is an isomorphism. Thus, $(i_x)_* : \mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$ is essentially surjective and hence an equivalence. Therefore, the functor $i_x^* : \mathrm{Coh}_x(O_X) \to \mathrm{Mod}_f(O_{X,x})$ (taking the stalk at x) is an equivalence.

□

Lemma 5.5.2.9. *Let $F \to A$ be a ring map, with F a field and (A, m) an Artinian local ring. If $\dim_F A/m$ is finite, then $\dim_F A$ is finite.*

Proof. Because A is an Artinian local ring ring, by [Ati69, Prop. 8.4], there is an integer $n > 0$ with $m^n = 0$. For every integer $i \geq 0$, the A-module m^i is finitely generated, so the A/m-module m^i/m^{i+1} is finitely generated. Thus, $\dim_F m^i/m^{i+1} = \dim_F A/m \cdot \dim_{A/m} m^i/m^{i+1}$ is finite. Then $\dim_F A = \sum_{i=0}^n \dim_F m^i/m^{i+1}$ is finite. □

5.5.3 Homogeneous vector bundles

Definition 5.5.3.1. A vector bundle E on the complex torus X is *homogeneous* if for every $x \in X$, one has $T_x^* E \cong E$. Let $H(X) \subset \mathrm{Coh}(X)$ be the full subcategory comprised of homogeneous vector bundles.

For a complex analytic space M, let $\mathrm{Coh}_f(M) \subset \mathrm{Coh}(M)$ be the full subcategory consisting of objects with finite support.

Proposition 5.5.3.2. *1. For every integer i, the functor $H^i R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$.*

2. W.I.T. holds for every homogeneous vector bundle on X with index g.

3. The functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to an equivalence of categories $H(X) \to \mathrm{Coh}_f(\hat{X})$, with a quasi-inverse $H^0 RS$.

Proof. 1. Let E be a homogeneous vector bundle on X. By Corollary 5.3.1.16, the $O_{\hat{X}}$-module $H^i R\hat{S}(E)$ is coherent. For every $x \in X$, by Proposition 5.5.1.1, one has $R\hat{S}(E) \xrightarrow{\sim} R\hat{S}(T_{-x}^* E) \xrightarrow{\sim} P_x^* \otimes R\hat{S}(E)$, so $H^i R\hat{S}(E) \xrightarrow{\sim} P_x^* \otimes H^i R\hat{S}(E)$. From Lemma 5.5.3.4, the support of $H^i R\hat{S}(E)$ is finite.

2. For every integer $i \neq g$, by Point 1, one has $H^i R\hat{S}(E) \in \mathrm{Coh}_f(\hat{X})$ and

$$\begin{aligned}
0 =& H^{i-g}([-1]_X^* E) \\
=& H^i([-1]_X^* E[-g]) \\
\overset{(a)}{\xrightarrow{\sim}}& H^i RS \circ R\hat{S}(E) \\
=& H^i Rp_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^* R\hat{S}(E)) \\
\overset{(b)}{\xrightarrow{\sim}}& H^0 Rp_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^* H^i R\hat{S}(E)) \\
=& H^0 RS(H^i R\hat{S}(E)),
\end{aligned}$$

where (a) (resp. (b)) uses Theorem 5.4.1.1 (resp. [GR04, Thm. 4, p.47]).

It remains to prove that for every $F \in \mathrm{Coh}_f(\hat{X})$ with $H^0 RS(F) = 0$, one has $F = 0$. Since F is the direct sum of finitely many coherent submodules whose supports are singletons, one may assume that $\mathrm{Supp}(F)$ is a singleton. By Proposition 5.5.1.1, one may assume that $F \in \mathrm{Coh}_0(\hat{X})$. From Proposition 5.5.2.7 2, one has $F = 0$.

3. By Point 1, the functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$. From Point 2, one has an isomorphism of functors $H^g R\hat{S} \cong R\hat{S}[g] : H(X) \to \mathrm{Coh}_f(\hat{X})$.

By Propositions 5.5.1.1 and 5.5.2.7, the functor $H^0 RS : \mathrm{Mod}(O_{\hat{X}}) \to \mathrm{Mod}(O_X)$ restricts to a functor $H^0 RS = RS : \mathrm{Coh}_f(\hat{X}) \to H(X)$. By Theorem 5.4.1.1, the functor $H^g R\hat{S} : H(X) \to \mathrm{Coh}_f(\hat{X})$ is an equivalence with a quasi-inverse $H^0 RS$.

□

For a sheaf of module F on a complex analytic space, denote the torsion part of F (in the sense of [CD94, p.60]) by $T(F)$.

Lemma 5.5.3.3. *Let X be a compact Kähler manifold. Let F be a coherent O_X-module. Then for every irreducible component $C \subset \mathrm{Supp}(F)$, there is a connected compact Kähler manifold Z and a morphism $h : Z \to X$, such that $h(Z) = C$ and $h^*F/T(h^*F)$ is a vector bundle on Z of positive rank.*

Proof. By [CAS, p.76], $\mathrm{Supp}(F)$ is an analytic subset of X. Because X is a Kähler manifold, with the induced reduced complex structure, the subspace C is a Kähler space in the sense of [Var89, II, 1.3]. Let $i : C \to X$ be the inclusion. Set

$$D = \{x \in C : i^*F \text{ is not locally free at } x\}.$$

From [Ros68, Prop. 3.1], D is a strict analytic subset of C. By Rossi's theorem (see, *e.g.* [Rie71, Thm. 2]), there is a reduced irreducible complex analytic space W and a proper modification $f : W \to C$, such that $W \setminus f^{-1}(D) \to C \setminus D$ is biholomorphic and $E := N/T(N)$ is a *vector bundle* on W, where $N = f^*i^*F$. From [GD71, Cor. 5.2.4.1], one has $\mathrm{Supp}(N) = W$. From [CD94, I, Thm. 9.12], one gets $\mathrm{Supp}(T(N)) \neq W$. Therefore, the rank r of the vector bundle E is positive.

Since $f : W \to C$ is bimeromorphic, the space W is in the Fujiki class $\mathcal{C}$ (defined in [Fuj78, p.34]). By [Fuj78, Lem. 4.6, 1)], there is a connected compact Kähler manifold Z with a surjective morphism $g : Z \to W$. Denote the composition $Z \xrightarrow{g} W \xrightarrow{f} C \xrightarrow{i} X$ by h. Then $h(Z) = C$. As E is flat over O_W, by [Sta24, Tag 05NJ], applying g^* to the natural short exact sequence

$$0 \to T(N) \to N \to E \to 0$$

in $\mathrm{Mod}(O_W)$, one gets a short exact sequence in $\mathrm{Mod}(O_Z)$:

$$0 \to g^*T(N) \to h^*F \to g^*E \to 0.$$

As g^*E is torsion free, $g^*T(N) \supset T(h^*F)$. One has $g^*T(N) \subset T(g^*N) = T(h^*F)$. Therefore, $T(h^*F) = g^*T(N)$ and $h^*F/T(h^*F) = g^*E$ is a vector bundle on Z of rank $r > 0$. □

Lemma 5.5.3.4. *Let M be a coherent sheaf on the complex torus X. If $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^0(X)$, then* $\mathrm{Supp}(M)$ *is finite.*

Proof. Suppose the contrary that $\mathrm{Supp}(M)$ is infinite. With the reduced induced complex structure, the complex subspace $\mathrm{Supp}(M)$ has positive dimension. Let C be an irreducible component of $\mathrm{Supp}(M)$ of maximal dimension. Take a morphism $h : Z \to X$ provided by Lemma 5.5.3.3. Then the rank r of the vector bundle $E := h^*M/T(h^*M)$ is positive. As $h(Z) = C$, the morphism of complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is nonzero. In particular, there is $L \in \mathrm{Pic}^0(X)$ such that the line bundle $(h^*L)^{\otimes r}$ is nontrivial.

On the other hand, we claim that the line bundle $(h^*L)^{\otimes r}$ is trivial. Indeed, by assumption $M \otimes L \cong M$, so $h^*M \otimes h^*L \cong h^*M$. Since $T(h^*M \otimes h^*L) = T(h^*M) \otimes h^*L$, one gets $E \otimes h^*L \cong E$. Taking the determinant of both sides, one has $\det(E) \otimes (h^*L)^{\otimes r} \cong \det(E)$. As $\det(E)$ is an invertible sheaf, the line bundle $(h^*L)^{\otimes r}$ on Z is trivial. The claim is proved, which gives a contradiction. □

Remark 5.5.3.5. The proof of [Muk81, Lem. 3.3] (the algebraic counterpart of Lemma 5.5.3.4) relies on the following fact: Every positive dimensional projective variety contains a projective curve. By contrast, from [Pil00, Lem. 4.3], every simple non-algebraic complex torus contains no one-dimensional analytic subset.

The classification of homogeneous vector bundles on complex tori is due to Matsushima [Mat59] and Morimoto [Mor59]. Using the Fourier-Mukai transform, Mukai [Muk81, p.159] proves an analog for abelian varieties. We can similarly recover Matsushima-Morimoto's theorem.

Theorem 5.5.3.6. *A vector bundle F on the complex torus X is homogeneous if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \ldots, U_n$ on X and $P_1, \ldots, P_n \in \mathrm{Pic}^0(X)$, such that F is isomorphic to $\oplus_{i=1}^n P_i \otimes U_i$.*

Proof. It follows from Propositions 5.5.1.1, 5.5.2.7 2 and 5.5.3.2 3. $\square$

Chapter 6

Sheaves with connection on complex tori

6.1 Introduction

6.1.1 Background

Mukai [Muk81, Sec. 2] introduces an analog of the Fourier transform for sheaves of modules on abelian varieties, known as the *Fourier-Mukai transform.* Laumon [Lau96] and Rothstein [Rot96] study independently its lift to sheaves with connection (integrable or not). They both prove the Fourier inversion formula for the lift. Laumon [Lau96, Thm. 6.3.3] applies it to investigate generalized 1-motives. Meanwhile, as an application, Rothstein [Rot96, Thm. 3.2] recovers Matsushima's theorem [Mat59]: every vector bundle on an abelian variety admitting a connection is translation invariant. Schnell's work [Sch15] about holonomic D-modules on abelian varieties relies upon the lift of the Fourier-Mukai transform.

Let k be an algebraically closed field. Let A, B be abelian varieties over k dual to each other. Set $g = \dim A$. Let p_A (resp. p_B) denote the projection from $A \times B$ to A (resp. B). Let $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times B$. We adopt the following sign convention for the Fourier-Mukai transform:

$$
\begin{aligned}
R\mathcal{S}_1 &= Rp_{A*}(\mathcal{P} \otimes^L p_B^* \cdot) : D(O_B) \to D(O_A); \\
R\mathcal{S}_2 &= Rp_{B*}(\mathcal{P}^{-1} \otimes^L p_A^* \cdot) : D(O_A) \to D(O_B),
\end{aligned}
\tag{6.1}
$$

For a triangulated category, let T denote the degree shift automorphism. For an algebraic variety V over k, denote by $D_{\mathrm{qc}}(O_V) \subset D(O_V)$ (resp. $D^b_c(O_V) \subset D^b(O_V)$) the full subcategory of objects whose cohomologies are quasi-coherent (resp. coherent) O_V-modules. Mukai establishes an analog of the Fourier inversion formula for this triangulated subcategory.

Fact 6.1.1.1 (Mukai, [Muk81, Thm. 2.2], [Rot96, p.569])**.** *1. There are natural isomorphisms of functors*

$$RS_1 \circ RS_2 \cong T^{-g} : D_{\mathrm{qc}}(O_A) \to D_{\mathrm{qc}}(O_A),$$
$$RS_2 \circ RS_1 \cong T^{-g} : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_B).$$

In particular, $RS_1 : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_A)$ is an equivalence of triangulated categories, with a quasi-inverse $T^g RS_2$.

2. The functor $RS_1 : D(O_B) \to D(O_A)$ restricts to an equivalence $D^b_c(O_B) \to D^b_c(O_A)$.

Let $0 \to H^0(A, \Omega^1_A) \to B^\natural \xrightarrow{p} B \to 0$ be the universal vectorial extension of B (constructed in [Ros58, Prop. 11]). For an algebraic variety V, denote the forgetful functor $D(D_V) \to D(O_V)$ by for_V. Let $D_{\mathrm{qc}}(D_A) \subset D(D_A)$ (resp. $D^b_c(D_A) \subset D^b(D_A)$) be the full subcategory of objects whose cohomologies are quasi-coherent O_A-modules (resp. coherent D_A-modules). Laumon and Rothstein lift the Fourier-transform to D-modules and establish a duality result similar to Fact 6.1.1.1.

Fact 6.1.1.2 (Laumon, Rothstein)**.**

1. *There are functors $RS_1 : D(O_{B^\natural}) \to D(D_A)$ and $RS_2 : D(D_A) \to D(O_{B^\natural})$ fitting into commutative squares*

$$\begin{tikzcd} D_{\mathrm{qc}}(O_{B^\natural}) \arrow[r, "RS_1"] \arrow[d, "Rp_*"] & D_{\mathrm{qc}}(D_A) \arrow[d, "\mathrm{for}_A"] \\ D_{\mathrm{qc}}(O_B) \arrow[r, "RS_1"] & D_{\mathrm{qc}}(O_A), \end{tikzcd} \qquad \begin{tikzcd} D_{\mathrm{qc}}(O_{B^\natural}) \arrow[d, "Rp_*"] & D_{\mathrm{qc}}(D_A) \arrow[l, "RS_2"'] \arrow[d, "\mathrm{for}_A"] \\ D_{\mathrm{qc}}(O_B) & D_{\mathrm{qc}}(O_A). \arrow[l, "RS_2"'] \end{tikzcd}$$

2. (Remark 6.1.1.4) *There are natural isomorphisms of functors*

$$RS_1 RS_2 \cong T^{-g} : D_{\mathrm{qc}}(D_A) \to D_{\mathrm{qc}}(D_A),$$
$$RS_2 RS_1 \cong T^{-g} : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(O_{B^\natural}).$$

Thus, $RS_1 : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(D_A)$ is an equivalence of triangulated categories.

3. ([Lau96, Cor. 3.1.3], [Rot96, Thm. 6.2]) *The functor $RS_1 : D(O_{B^\natural}) \to D(D_A)$ restricts to an equivalence $RS_1 : D^b_c(O_{B^\natural}) \to D^b_c(D_A)$.*

Remark 6.1.1.3. Laumon and Rothstein use apparently different definitions for the functors on D-modules. We sketch why the two definitions agree.

In the notation of [Lau96, p.14] and [Vig21, Sec. 2.1.1], one has functors $\tilde{\mathcal{F}} : D^b_{\mathrm{qc}}(D_A) \to D^b_{\mathrm{qc}}(O_{B^\natural})$ and $\tilde{\mathcal{F}}^\natural : D^b_{\mathrm{qc}}(O_{B^\natural}) \to D^b_{\mathrm{qc}}(D_A)$ defined by composition

$$\begin{tikzcd} D^b_{\mathrm{qc}}(D_A) \arrow[r, "\tilde{\mathcal{F}}"] \arrow[d, "\tilde{\mathrm{pr}}^{!(B^\natural)}"'] & D^b_{\mathrm{qc}}(O_{B^\natural}) \\ D^b_{\mathrm{qc}}(D_{B^\natural\times A/B^\natural}) \arrow[r, "(\tilde{\mathcal{P}},\tilde{\nabla})\otimes^L_{O_{B^\natural\times A}}\cdot"'] & D^b_{\mathrm{qc}}(D_{B^\natural\times A/B^\natural}) \arrow[u, "\tilde{\mathrm{pr}}^\natural_{+/B^\natural}"'] \end{tikzcd}; \qquad \begin{tikzcd} D^b_{\mathrm{qc}}(O_{B^\natural}) \arrow[r, "\tilde{\mathcal{F}}^\natural"] \arrow[d, "\tilde{\mathrm{pr}}^{\natural,\flat}_{/B^\natural}"'] & D^b_{\mathrm{qc}}(D_A) \\ D^b_{\mathrm{qc}}(D_{A\times B^\natural/B^\natural}) \arrow[r, "(\tilde{\mathcal{P}},\tilde{\nabla})\otimes^L_{O_{A\times B^\natural}}\cdot"'] & D^b_{\mathrm{qc}}(D_{A\times B^\natural/B^\natural}) \arrow[u, "\tilde{\mathrm{pr}}^{(B^\natural)}_{+}"'] \end{tikzcd}.$$

Applying the projection formula (to $\mathrm{Id}_A\times p : A\times B^\natural \to A\times B$) and the flat base change to the cartesian square

$$\begin{tikzcd} A\times B^\natural \arrow[r, "\mathrm{Id}_A\times p"] \arrow[d, "p_{B^\natural}"] & A\times B \arrow[d, "p_B"] \\ B^\natural \arrow[r, "p"] & B, \end{tikzcd}$$

one gets an isomorphism $\mathrm{for}_A\tilde{\mathcal{F}}^\natural \cong \mathcal{F}'Rp_*$ of functors $D^b_{\mathrm{qc}}(O_{B^\natural}) \to D^b_{\mathrm{qc}}(O_A)$. This shows the compatibility with the Fourier-Mukai transform, as well as that $[-1]^*_A\tilde{\mathcal{F}}^\natural$ is the restriction of [Rot96, (4.16)] to $D^b_{\mathrm{qc}}(O_{B^\natural})$. (The sign is due to different conventions.) Fact 6.1.1.2 1 is not mentioned in [Lau96], and is implicitly used in the derivation of [Rot96, (2.25)]. For Rothstein's definition, the compatibility can be proved as in Proposition 6.3.1.2.

Remark 6.1.1.4. The direct image functor $p_{A*} : \mathrm{Mod}(O_{A\times B}) \to \mathrm{Mod}(O_A)$ restricts to a left exact functor $p_{A*} : \mathrm{Qch}(O_{A\times B}) \to \mathrm{Qch}(O_A)$. Let $R(\mathrm{qc})p_{A*} : D(\mathrm{Qch}(O_{A\times B})) \to D(\mathrm{Qch}(O_A))$ be the right derived functor of the restriction. Denote the functor

$$R(\mathrm{qc})p_{A*}(\mathcal{P}\otimes_{O_{A\times B}} p_B^*\pi_*\cdot) : D(\mathrm{Qch}(O_{B^\natural})) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_A))$$

by $R(\mathrm{qc})S_1$. Strictly speaking, [Rot96, Thm. 4.5] and [Rot97] demonstrate that the functor $R(\mathrm{qc})S_1$ is an equivalence. In comparison, Laumon's result [Lau96, Thm. 3.2.1] is stated for bounded derived categories D^b_{qc} and needs the characteristic of k to be 0.

We sketch how to get Fact 6.1.1.2 2 from Rothstein's original statement. For every algebraic variety V, by [Sta24, Tag 077P (1)], the abelian category $\mathrm{Qch}(O_V)$ has enough injectives. Furthermore, from [Con00, Lem. 2.1.3], the inclusion $\iota_V : \mathrm{Qch}(O_V) \to \mathrm{Mod}(O_V)$ preserves injectives. Let $\mathrm{Mod}(O_B)_{\mathrm{sp}}$ be as in Example 6.2.1.5 (resp. $\mathrm{Mod}(O_{A\times B})_{-1-\mathrm{cxn}}$ denote $\mathrm{Mod}(O_{A\times B})_{\pi_B,-\pi_B^*1-\mathrm{cxn}}$). Let $\mathrm{Qch}(O_B)_{\mathrm{sp}}$ (resp. $\mathrm{Qch}(O_{A\times B})_{-1-\mathrm{cxn}}$) be the full subcategory of quasi-coherent objects.

Then the exact functor $\pi_* : \mathrm{Qch}(O_{B^\natural}) \to \mathrm{Qch}(O_B)_{\mathrm{sp}}$ is the restriction of $R\pi_* : D(O_{B^\natural}) \to D(\mathrm{Mod}(O_B)_{\mathrm{sp}})$. Using [Lip60, Prop. 3.9.2] and [Har66, I, Prop. 7.1 (iii)], one proves that the canonical square

$$\begin{tikzcd}
D(\mathrm{Qch}(O_{B^\natural})) \arrow[r, "L\iota_{B^\natural}"] \arrow[d, "\pi_*"] & D_{\mathrm{qc}}(O_{B^\natural}) \arrow[d, "R\pi_*"] \\
D(\mathrm{Qch}(O_B)_{\mathrm{sp}}) \arrow[r] & D_{\mathrm{qc}}(\mathrm{Mod}(O_B)_{\mathrm{sp}})
\end{tikzcd}$$

is commutative. Similarly, using [Kas04, Remark 3.2], one proves that the canonical square

$$\begin{tikzcd}
D(\mathrm{Qch}(O_{A\times B})_{-1-\mathrm{cxn}}) \arrow[r] \arrow[d, "p_{A*}"] & D_{\mathrm{qc}}(\mathrm{Mod}(O_{A\times B})_{-1-\mathrm{cxn}}) \arrow[d, "Rp_{A*}"] \\
D(\mathrm{Mod}_{\mathrm{qc}}(D_A)) \arrow[r, "L\iota'_A"] & D_{\mathrm{qc}}(D_A)
\end{tikzcd}$$

is commutative. Therefore, the following square is commutative

$$\begin{tikzcd}
D(\mathrm{Qch}(O_{B^\natural})) \arrow[r, "R(\mathrm{qc})S_1"] \arrow[d, "L\iota_{B^\natural}"] & D(\mathrm{Mod}_{\mathrm{qc}}(D_A)) \arrow[d, "L\iota'_A"] \\
D_{\mathrm{qc}}(O_{B^\natural}) \arrow[r, "RS_1"] & D_{\mathrm{qc}}(D_A).
\end{tikzcd} \tag{6.2}$$

By [Sta24, Tag 09T4] and Theorem E.1.0.4, the two vertical functors in (6.2) are equivalences. As $R(\mathrm{qc})S_1$ is an equivalence, so is the bottom row.

Remark 6.1.1.5. From [Sch14, p.97] and the square in [HT07, p.38], the bifunctor $\otimes_O$ on relative D-modules is compatible with that on the underlying O-modules. However, the following triangles

$$\begin{tikzcd}
D^b_{\mathrm{qc}}(D_{A\times B^\natural/B^\natural}) \arrow[rr, "(\tilde{\mathcal{P}},\tilde{\nabla})\otimes^L_{O_{A\times B^\natural}}\cdot"] \arrow[dr, "\mathrm{for}"'] & & D^b_{\mathrm{qc}}(D_{A\times B^\natural/B^\natural}) \\
& D^b_{\mathrm{qc}}(O_{A\times B^\natural}), \arrow[ur, "(\tilde{\mathcal{P}},\tilde{\nabla})\otimes^L_{O_{A\times B^\natural}}\cdot"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
D^b_{\mathrm{qc}}(O_{A\times B^\natural}) \arrow[rr, "(\tilde{\mathcal{P}},\tilde{\nabla})\otimes^L_{O_{A\times B^\natural}}\cdot"] \arrow[dr, "\tilde{\mathcal{P}}\otimes^L_{O_{A\times B^\natural}}\cdot"'] & & D^b_{\mathrm{qc}}(D_{A\times B^\natural/B^\natural}) \arrow[dl, "\mathrm{for}"] \\
& D^b_{\mathrm{qc}}(O_{A\times B^\natural}) &
\end{tikzcd}$$

are not commutative in general. Thus, the first remark in [Vig21, p.58] is not true. In particular, the last but one equations in the proofs of [Vig21, Propositions 2.2.12 and 2.2.13] are wrong. Similarly, in [Lau96, p.14], the relative integrable connection on $(\tilde{\mathcal{P}},\tilde{\nabla})\otimes_{O_{B^\natural\times A}}\tilde{\mathrm{pr}}^{\natural *}M^\natural$ is induced by not only $\tilde{\nabla}$, *but also* the canonical relative connection on $\tilde{\mathrm{pr}}^{\natural *}M^\natural$.

6.1.2 Extension to complex tori

Let X, Y be complex tori dual to each other and of dimension g. Define the analytic Fourier-Mukai transform $R\mathcal{S}_1 : D(O_X) \to D(O_Y)$ and $R\mathcal{S}_2 : D(O_Y) \to D(O_X)$ by formulae similar to (6.1). For a complex manifold Z, let $D_{\mathrm{good}}(O_Z) \subset D(O_Z)$ be the full subcategory of objects whose cohomologies are good O_Z-modules (in the sense of [Kas03, Def. 4.22]). In [BBP07, Thm. 2.1], a result similar to Fact 6.1.1.1 is established for complex tori.

Fact 6.1.2.1 (Mukai, Ben-Bassat, Block, Pantev)**.**

1. (Theorem 5.4.1.1) *There are natural isomorphisms of functors*

$$R\mathcal{S}_1 R\mathcal{S}_2 \cong T^{-g} : D_{\mathrm{good}}(O_Y) \to D_{\mathrm{good}}(O_Y),$$
$$R\mathcal{S}_2 R\mathcal{S}_1 \cong T^{-g} : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_X).$$

   In particular, $R\mathcal{S}_1 : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_Y)$ is an equivalence of categories with a quasi-inverse $T^g R\mathcal{S}_2$.

2. ([PPS17, Thm. 13.1]) *The functor $R\mathcal{S}_1 : D(O_X) \to D(O_Y)$ restricts to an equivalence $D^b_c(O_X) \to D^b_c(O_Y)$.*

We lift the analytic Fourier-Mukai transform to D-modules, and give an analog of Fact 6.1.1.2. Good D-modules are reviewed in Section 6.6.1. For a complex manifold Z and an O_Z-algebra $\mathcal{R}$, let $D_{O-\mathrm{good}}(\mathcal{R}) \subset D(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b(\mathcal{R})$) be the full subcategory of objects whose cohomologies are good over O_Z (resp. $\mathcal{R}$).

Theorem 6.1.2.2.

- (Prop. 6.5.1.3) *There is a canonical commutative O_X-algebra $\mathcal{A}_X$, such that the functors $R\mathcal{S}_1$ and $R\mathcal{S}_2$ lift naturally to triangulated functors $RS_1 : D(\mathcal{A}_X) \to D(D_Y)$ and $RS_2 : D(D_Y) \to D(\mathcal{A}_X)$ respectively.*
- (Thm. 6.5.1.4) *The functors RS_1 and RS_2 restrict to equivalences $RS_1 : D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y)$ and $RS_2 : D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X)$ respectively.*
- (Thm. 6.6.3.1) *The functors RS_1 and RS_2 restrict to equivalences $RS_1 : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$ and $RS_2 : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$ respectively.*

Remark 6.1.2.3. Arinkin [Fav12, Thm. 3] uses Fact 6.1.1.2 3 to show that an abelian variety A can be recovered from the triangulated category $D^b_c(D_A)$. By Proposition F.5.4.7, however, for a complex abelian variety A, the complex Lie group $(A^\natural)^{\mathrm{an}}$ (associated with $A^\natural$) is isomorphic to $(\mathbb{C}^*)^{2g}$. So an analytic version of Fact 6.1.1.2 3 needs a modification.

The proof of Fact 6.1.1.2 due to Laumon [Lau96] and that of Rothstein [Rot97] are different. Let $\pi^\natural : A^\natural \to \mathrm{Spec}(k)$ be the structural morphism. As an immediate step, Laumon [Lau96, Thm. 2.4.1] proves that the adjunction morphism $O_{\mathrm{Spec}(k)} \to R\pi^\natural_* O_{A^\natural}$ is an isomorphism in $D^b_{\mathrm{qc}}(O_{\mathrm{Spec}(k)})$. By contrast, when $k = \mathbb{C}$, the adjunction morphism $O_{\mathrm{Specan}(\mathbb{C})} \to R(\pi^\natural)^{\mathrm{an}}_* O_{(A^\natural)^{\mathrm{an}}}$ is not an isomorphism. Still, the proof of [Rot97] works for complex tori. We follow it closely, except that the underived Fourier-Mukai transforms [Rot97, (2.14), (2.15)] are ignored. Instead, we define the corresponding functors on the derived categories directly. We should notice four misprints therein.

- [Rot97, (2.11)] should be
$$\widetilde{\pi_{12}}^*\mathcal{P} = [(1_X \times \tilde{m})^*\mathcal{P}] \otimes_{O_{X\times Y\times\tilde{Y}}} [\widetilde{\pi_{13}}^*\tilde{\mathcal{P}}^{-1}],$$
where $\widetilde{\pi_{ij}}$ denotes the projections on $X\times Y\times\tilde{Y}$ and $\tilde{Y}$ is the first-order neighborhood of 0 in Y.

- [Rot97, (2.23)] should be
$$\pi_{13}^*\mathcal{P}^{-1} \otimes \pi_{23}^*\mathcal{P} = (\epsilon_X \times 1_Y)^*\mathcal{P},$$
where π_{ij} denotes the projections on $X \times X \times Y$.

- In [Rot97, (2.24)], the starting equation should be
$$\tilde{\pi}_{12}^* O_\Delta \otimes \tilde{\pi}_{13}^*\tilde{\mathcal{P}}^{-1} \otimes \tilde{\pi}_{23}^*\tilde{\mathcal{P}}.$$

- In [Rot97, Prop. 2.4], the notation $\mathrm{Mod}(X \times X)_{(-1,1)-\mathrm{sp}}$ should be $\mathrm{Mod}(X \times X)_{(1,-1)-\mathrm{sp}}$.

Notation and conventions

For a sheaf F on a topological space, let $\mathrm{Supp}\,F$ be its support. For a (not necessarily commutative) ringed space $(X, \mathcal{R})$, let $\mathrm{Mod}(\mathcal{R})$ be the category of left $\mathcal{R}$-modules. Let $\mathrm{Coh}(\mathcal{R}) \subset \mathrm{Mod}(\mathcal{R})$ be the full subcategory of coherent $\mathcal{R}$-modules. Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*(\mathcal{R})$ refers to the unbounded/bounded below/bounded above/bounded derived category of the abelian category $\mathrm{Mod}(\mathcal{R})$ in order. Let $D^*_c(\mathcal{R}) \subset D^*(\mathcal{R})$ be the full subcategory of objects whose cohomologies are coherent $\mathcal{R}$-modules (in the sense of [Sta24, Tag 01BV]).

Let k be an algebraically closed field. An algebraic variety refers to an integral scheme of finite type and separated over k. For a complex manifold Z and $z \in Z$, let $i_z : (z, \mathbb{C}) \to (Z, O_Z)$ be the closed embedding of complex manifolds. Set $\mathbb{C}_z := (i_z)_*\mathbb{C}$, which is a coherent O_Z-module. Let X, Y be complex tori dual to each other and of dimension g.

6.2 Preliminaries

For the convenience of the reader, we recall the notation of [Rot97, Sec. 2.1].

6.2.1 Categories of splittings

For a complex manifold Z and a (holomorphic) vector bundle $M \to Z$, by [Har77, III, Prop. 6.3 (c)], one has $H^1(Z, M) = \mathrm{Ext}^1(O_Z, M)$. Thus, every $\alpha \in H^1(Z, M)$ determines a short exact sequence in $\mathrm{Mod}(O_Z)$

$$0 \to M \to \mathcal{E}_\alpha \overset{\mu_\alpha}{\to} O_Z \to 0. \tag{6.3}$$

Since O_Z is a flat O_Z-module, by [Sta24, Tag 05NJ], for every $F \in \mathrm{Mod}(O_Z)$, the sequence (6.3) remains exact after tensored with F:

$$0 \to M \otimes_{O_Z} F \to \mathcal{E}_\alpha \otimes_{O_Z} F \overset{\mu_\alpha \otimes \mathrm{Id}_F}{\to} F \to 0.$$

Definition 6.2.1.1. Define a category $\mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$ as follows: the objects are pairs (F, ψ), where $F \in \mathrm{Mod}(O_Z)$ and $\psi : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ is an α-splitting on F, *i.e.*, an O_Z-linear splitting of $\mu_\alpha \otimes \mathrm{Id}_F$. The morphisms in $\mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$ are required to be compatible with the splittings.

Example 6.2.1.2. If $\alpha = 0$ and $M = \Omega^1_Z$, then an α-splitting ϕ on a vector bundle $E \to Z$ is exactly a holomorphic 1-form on Z with values in $\mathcal{E}nd(E)$. The pair (E, ϕ) is a Higgs bundle (in the sense of [Sim92, p.6]) if and only if $[\phi, \phi] = 0$.

Lemma 6.2.1.3. *For an O_Z-module F, there is an α-splitting on F if and only if the map $i_* : H^1(Z, M) \to H^1(Z, M \otimes_{O_Z} \mathcal{E}nd(F))$ (induced by the natural morphism $O_Z \to \mathcal{E}nd(F)$) sends α to 0. In that case, the set of α-splittings on F has a natural simple transitive action of the abelian group $\mathrm{Hom}_{O_Z}(F, M \otimes_{O_Z} F)$.*

Proof. The natural morphism $O_Z \to \mathcal{E}nd(F)$ induces a morphism

$$i : M \to \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F), \quad i(m)(f) = m \otimes f.$$

There is a canonical evaluation morphism $\mathrm{ev} : \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F) \otimes F \to M \otimes_{O_Z} F, \quad \mathrm{ev}(\phi \otimes f) = \phi(f)$. The five-term exact sequence of the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i(O_Z, \mathcal{E}xt^j(F, M \otimes_{O_Z} F)) \Rightarrow \mathrm{Ext}^{i+j}(F, M \otimes_{O_Z} F)$$

gives an injection $\iota : \mathrm{Ext}^1(O_Z, \mathcal{H}om(F, M \otimes_{O_Z} F)) \to \mathrm{Ext}^1(F, M \otimes_{O_Z} F)$, which is $\mathrm{Ext}^1(F, \mathrm{ev}) \circ (\cdot \otimes F)$:

$$\begin{tikzcd}[column sep=small]
& \mathrm{Ext}^1(F, M\otimes_{O_Z} F) \arrow[d, "="] \arrow[dr, "(i\otimes \mathrm{Id}_F)_*"] & \\
\mathrm{Ext}^1(O_Z, M) \arrow[ur, "\cdot\otimes F"] \arrow[dr, "i_*"'] & \mathrm{Ext}^1(F, M\otimes_{O_Z} F) & \mathrm{Ext}^1(F, \mathcal{H}om(F, M\otimes_{O_Z} F)\otimes F) \arrow[l, "\mathrm{Ext}^1(F,\mathrm{ev})"] \\
& \mathrm{Ext}^1(O_Z, \mathcal{H}om(F, M\otimes_{O_Z} F)). \arrow[u, hook, "\iota"] \arrow[ur, "\cdot\otimes F"'] &
\end{tikzcd}$$

One has

$$\mathrm{ev}\circ(i\otimes \mathrm{Id}_F)(m\otimes f) = \mathrm{ev}(i(m)\otimes f) = i(m)(f) = m\otimes f,$$

so $\mathrm{ev}\circ(i\otimes\mathrm{Id}_F) = \mathrm{Id}_{M\otimes_{O_Z}F}$ as morphisms $M\otimes_{O_Z}F \to M\otimes_{O_Z}F$. Therefore, the diagram is commutative. Then F admits an α-splitting if and only if $\alpha\otimes F = 0$ if and only if $i_*(\alpha) = 0$. Any two α-splittings on F differ by a unique element of $\mathrm{Hom}(F, M\otimes_{O_Z} F)$. □

To each object $(F,\psi)\in \mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$, we assign an element

$$[\psi,\psi]\in\Gamma(Z,(\wedge^2 M)\otimes_{O_Z}\mathcal{E}nd(F)) \tag{6.4}$$

as follows. The sequence (6.3) induces a short exact sequence

$$0\to\wedge^2 M\to\wedge^2\mathcal{E}_\alpha\overset{\omega_\alpha}{\to} M\to 0,$$

where

$$\omega_\alpha(\rho_1\wedge\rho_2) = \mu_\alpha(\rho_1)\rho_2 - \mu_\alpha(\rho_2)\rho_1.$$

The flatness of M ensures the exactness when tensoring with F:

$$0\to(\wedge^2 M)\otimes F\to(\wedge^2\mathcal{E}_\alpha)\otimes F\overset{\omega_\alpha\otimes\mathrm{Id}_F}{\to} M\otimes_{O_Z} F\to 0. \tag{6.5}$$

Let $a:\mathcal{E}_\alpha\otimes\mathcal{E}_\alpha\to\wedge^2\mathcal{E}_\alpha$ be the morphism defined by $e\otimes e'\mapsto e\wedge e'$. Let ψ^1 be the composition

$$\mathcal{E}_\alpha\otimes F\overset{\mathrm{Id}_{\mathcal{E}_\alpha}\otimes\psi}{\to}\mathcal{E}_\alpha\otimes(\mathcal{E}_\alpha\otimes F)\overset{\sim}{\to}(\mathcal{E}_\alpha\otimes\mathcal{E}_\alpha)\otimes F\overset{a\otimes\mathrm{Id}_F}{\to}(\wedge^2\mathcal{E}_\alpha)\otimes F,$$

where the isomorphism in the middle is from the associativity of tensor product.

Lemma 6.2.1.4. *One has* $(\omega_\alpha\otimes\mathrm{Id}_F)\psi^1\psi = 0$.

Proof. Locally, the vector bundle $\mathcal{E}_\alpha$ has a (holomorphic) frame $\{e_1,\dots,e_r\}$. For a local section $f\in F$, write $\psi(f)=\sum_{i=1}^r e_i\otimes f_i$, where f_i are local sections of F. For every $1 \leq i \leq r$, write $\psi(f_i) = \sum_{j=1}^r e_j \otimes f_j^{(i)}$, where $f_j^{(i)}$ are local sections of F. As ψ is a section to $\mu_\alpha \otimes \mathrm{Id}_F$, one has

$$f = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f) = \sum_{i=1}^r \mu_\alpha(e_i) f_i; \tag{6.6}$$

$$f_i = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f_i) = \sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)}. \tag{6.7}$$

From (6.6), one has

$$\psi(f) = \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i). \tag{6.8}$$

By construction, one has $\psi^1\psi(f) = \sum_{i,j=1}^r (e_i \wedge e_j) \otimes f_j^{(i)}$. Then

$$\begin{aligned}
(\omega_\alpha \otimes \mathrm{Id}_F)\psi^1\psi(f) &= \sum_{i,j=1}^r [\mu_\alpha(e_i)e_j - \mu_\alpha(e_j)e_i] \otimes f_j^{(i)} \\
&= \sum_{i=1}^r \mu_\alpha(e_i) \sum_{j=1}^r e_j \otimes f_j^{(i)} - \sum_{i=1}^r e_i \otimes [\sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)}] \\
&\overset{(a)}{=} \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i) - \sum_{i=1}^r e_i \otimes f_i \\
&\overset{(b)}{=} \psi(f) - \psi(f) = 0,
\end{aligned}$$

where (a) and (b) use (6.7) and (6.8) respectively. □

From Lemma 6.2.1.4 and (6.5), one has $\psi^1\psi(F) \subset (\wedge^2 M) \otimes F$. The morphism $\psi^1\psi : F \to (\wedge^2 M) \otimes F$ gives an element $[\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes_{O_Z} \mathcal{E}nd(F))$.

Example 6.2.1.5. For the complex torus X, set $\mathfrak{g} = H^1(X, O_X)$. Then

$$H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = \mathfrak{g}^* \otimes_{\mathbb{C}} \mathfrak{g} = \mathrm{End}(\mathfrak{g}).$$

Hence a category $\mathrm{Mod}(O_X)_{T-\mathrm{sp}}$ for each $T \in \mathrm{End}(\mathfrak{g})$. The identity element $1 \in \mathrm{End}(\mathfrak{g})$ corresponds to the tautological exact sequence [Rot96, (1.3)]:

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X \to \mathcal{E} \to O_X \to 0. \tag{6.9}$$

We also write $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ for $\mathrm{Mod}(O_X)_{1-\mathrm{sp}}$. For $(F, \psi) \in \mathrm{Mod}(O_X)_{\mathrm{sp}}$, the element $[\psi, \psi]$ lies in

$$\Gamma(X, \wedge^2\mathfrak{g}^* \otimes_{\mathbb{C}} O_X \otimes_{O_X} \mathcal{E}nd(F)) = \wedge^2\mathfrak{g}^* \otimes_{\mathbb{C}} \mathrm{End}(F),$$

and we recover [Rot96, (4.8)]. Similarly, $H^1(X \times X, \mathfrak{g}^* \otimes O_{X\times X}) = \mathrm{End}(g) \oplus \mathrm{End}(g)$, so for every pair $T_1, T_2 \in \mathrm{End}(g)$, the category $\mathrm{Mod}(O_{X\times X})_{(T_1,T_2)-\mathrm{sp}}$ is defined.

6.2.2 Categories of twisted connection

We continue to review the twisted (relative) connection introduced in [Rot97, p.206]. Consider a smooth morphism of complex manifolds $f : Z \to S$, with relative cotangent sheaf Ω_f^1. As f is smooth, Ω_f^1 is a vector bundle on Z. Let $d_f : O_Z \to \Omega_f^1$ denote the differential relative to f. An element $\alpha \in H^1(Z, \Omega_f^1)$ determines an extension

$$0 \to \Omega_f^1 \to \mathcal{E}_\alpha \stackrel{\mu_\alpha}{\to} O_Z \to 0. \tag{6.10}$$

Definition 6.2.2.1. On an O_Z-module F, an α-connection is an $f^{-1}(O_S)$-linear splitting $\nabla : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ to $\mu_\alpha \otimes \mathrm{Id}_F$, satisfying the Leibniz rule

$$\nabla(h\phi) = h\nabla(\phi) + d_f(h) \otimes \phi, \tag{6.11}$$

where h and ϕ are local sections of O_Z and F respectively. Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ be the category of pairs (F, ∇), where $F \in \mathrm{Mod}(O_Z)$ and ∇ is an α-connection on F.

Example 6.2.2.2. If $\alpha = 0$, then α-connection are exactly f-relative connection. Define a sheaf $\tilde{D}_{Z/S}$ of noncommutative O_Z-algebras by gluing the following local data. If $\{\xi_1, \dots, \xi_n\}$ is a local frame of $(\Omega_f^1)^\vee$ (the vector bundle dual to Ω_f^1) on an open subset $U \subset Z$, then a multiplication law on $O_U\{\xi_1, \dots, \xi_n\}$ is introduced by imposing the commutation relation $[\xi_i, h] = \xi_i(h)$ for local sections h of O_Z. Let it be $\tilde{D}_{Z/S}|_U$. Then $\mathrm{Mod}(Z)_{f,0-\mathrm{cxn}} = \mathrm{Mod}(\tilde{D}_{Z/S})$. The category $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$ is denoted by $\mathrm{Mod}(O_Z)_{\mathrm{cxn}}$ when f is the structure morphism $Z \to \mathrm{Specan}(\mathbb{C})$.

Remark 6.2.2.3. In fact, a twisted connection is a particular splitting. Define another extension

$$0 \to \Omega_f^1 \to \mathcal{E}_{\alpha'} \to O_Z \to 0 \tag{6.12}$$

in $\mathrm{Mod}(O_Z)$ as follows. As an extension of abelian sheaves, (6.12) is same as (6.10). Let h (resp. s') be a local section of O_Z (resp. $\mathcal{E}_{\alpha'}$) and s denote the local section of $\mathcal{E}_\alpha$ induced by s'. The O_Z-module structure on $\mathcal{E}_{\alpha'}$ is defined such that the local section $hs + \mu_\alpha(s)d_f h$ of $\mathcal{E}_\alpha$ induces the local section hs' of $\mathcal{E}_{\alpha'}$.

We claim this indeed defines an O_Z-module structure on $\mathcal{E}_{\alpha'}$. For local sections h_1, h_2 of O_Z, let t be the local section of $\mathcal{E}_\alpha$ induced by $h_2 s'$. Then $t = h_2 s + \mu_\alpha(s) d_f h_2$, so $\mu_\alpha(t) = h_2\mu_\alpha(s)$. Thus, the local section of $\mathcal{E}_\alpha$ corresponding to $h_1(h_2 s')$ is

$$h_1 t + \mu_\alpha(t) d_f h_1 = h_1 h_2 s + h_1 \mu_\alpha(s) d_f h_2 + h_2 \mu_\alpha(s) d_f h_1 = (h_1 h_2)s + \mu_\alpha(s) d_f(h_1 h_2).$$

Therefore, $h_1(h_2 s') = (h_1 h_2)s'$. The claim is proved.

By construction, the morphisms in (6.12) are O_Z-linear. Then (6.12) is indeed an extension in $\mathrm{Mod}(O_Z)$, hence a new extension class $\alpha' \in$

$\mathrm{Ext}(O_Z, \Omega^1_f)$. An α-connection on $F \in \mathrm{Mod}(O_Z)$ is equivalent to an α'-splitting on F. Hence an equivalence of categories

$$\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{\alpha'-\mathrm{sp}}.$$

There is a notion of integrable α-connection in [Rot97, Remark, p.206]. Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn,fl}}$ be the full subcategory of $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ of objects whose connection are integrable. Then $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn,fl}}$ coincides with $\mathrm{MIC}(f)$ defined in [ABC20, 4.3.7], which is further equivalent to $\mathrm{Mod}(D_{Z/S})$. Here $D_{Z/S}$ is the sheaf of ring of relative differential operators on Z/S defined in [SS94, p.9].

Example 6.2.2.4. For the dual complex tori X, Y, consider the projection $p_X : X \times Y \to X$. Since $\Omega^1_{p_X} = p_X^*(\mathfrak{g}^* \otimes_{\mathbb{C}} O_X)$, there is a natural morphism

$$p_X^* : \mathrm{End}(\mathfrak{g}) = H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \to H^1(X \times Y, \Omega^1_{p_X}).$$

For every $T \in \mathrm{End}(g)$, the category $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^* T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^* T-\mathrm{cxn,fl}}$) is also written as $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn,fl}}$).

Fact 6.2.2.5 is taken from the two remarks in [Rot97, pp.206–207].

Fact 6.2.2.5. *The Poincaré bundle $\mathcal{P}$ is naturally an object of* $\mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn,fl}}$.

In local coordinates, the $p_X^*(-1)$-connection on $\mathcal{P}$ is explained in [Rot96, (1.10) and p.575*ff.*], except that we use a Stein open cover of X instead of Rothestein's affine open cover.

6.2.3 Functors between them

Recall that the Fourier-Mukai transform (6.1) is the composition of the pullback, the tensor product with $\mathcal{P}$ as well as the derived direct image. Rothstein's lift to modules with connection keeps an extra track of the splittings and connection.

Remark 6.2.3.1. Rothstein [Rot97, Sec. 2.2] uses Čech resolutions for quasi-coherent sheaves, while we are dealing with *all* O-modules. Combining [Rot97, (2.21)] with the fact that twisted relative connection are kinds of splittings (Remark 6.2.2.3), the categories under consideration ($\mathrm{Mod}(O_X)_{\mathrm{sp}}$, $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$, *etc.*) are equivalent to categories of modules over sheaves of certain noncommutative *flat* O-algebras. In particular, each of them is a Grothendieck abelian category. By [Sta24, Tag 079P], each category has enough K-injectives. From [HT07, Lem. 1.5.2 (ii)], each category has enough objects flat over O. By [Sta24, Tag 070K, Tag 079P], all the (left exact) direct image functors involved below admit right derived functors on the unbounded derived categories.

From splittings to connection

Given $T \in \mathrm{End}(\mathfrak{g})$ and $(F, \psi) \in \mathrm{Mod}(O_X)_{T-\mathrm{sp}}$, the induced morphism

$$p_X^{-1}\psi : p_X^{-1}F \to p_X^{-1}\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F$$

is $p_X^{-1}O_X$-linear. By Example 6.2.2.4, the sequence (6.9) induces a short exact sequence in $\mathrm{Mod}(O_{X\times Y})$

$$0 \to \Omega^1_{p_X} \to p_X^*\mathcal{E}_T \to O_{X\times Y} \to 0.$$

Its extension class is $p_X^*T \in H^1(X \times Y, \Omega^1_{p_X})$. Define another $p_X^{-1}O_X$-linear morphism

$$\nabla_\psi : p_X^*F(= O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}F) \to p_X^*\mathcal{E}_T \otimes_{O_{X\times Y}} p_X^*F(=$$
$$p_X^*\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F = O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}\mathcal{E}_T \otimes_{p_X^{-1}O_X} p_X^{-1}F)$$

by

$$\nabla_\psi(h \otimes s) = d_{p_X}(h) \otimes s + h \otimes ((p_X^{-1}\psi)(s)),$$

where h (resp. s) is a local section of $O_{X\times Y}$ (resp. $p_X^{-1}F$). By construction, ∇_ψ satisfies the Leibniz rule (6.11). So it is a p_X^*T-connection on p_X^*F. Thus, we get the *exact* functor in [Rot97, (2.5)]:

$$p_X^* : \mathrm{Mod}(O_X)_{T-\mathrm{sp}} \to \mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}. \tag{6.13}$$

Tensoring with Poincaré bundle

By Fact 6.2.2.5 and [Rot97, (2.10)], the functor

$$\cdot \otimes_{O_{X\times Y}} \mathcal{P} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \tag{6.14}$$

restricts to $\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn,fl}}(\cong \mathrm{Mod}(D_{X\times Y/X}))$. The functor (6.14) is an equivalence of abelian categories, with a quasi-inverse $\cdot \otimes_{O_{X\times Y}} \mathcal{P}^{-1}$.

From connection to splittings

For every $(F, \nabla) \in \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}}$, the morphism

$$\nabla : F \to p_X^*\mathcal{E} \otimes_{O_{X\times Y}} F(= p_X^{-1}\mathcal{E} \otimes_{p_X^{-1}O_X} F)$$

is a $p_X^{-1}O_X$-linear splitting to $(p_X^{-1}\mu_1) \otimes \mathrm{Id}_F$. By the projection formula (see *e.g*, [KS90, Prop. 2.6.6]), the induced morphism

$$p_{X*}\nabla : p_{X*}F \to \mathcal{E} \otimes_{O_X} p_{X*}F$$

is an O_X-linear splitting to $\mu_1 \otimes_{O_X} \mathrm{Id}_{p_{X*}F}$. Hence $(p_{X*}F, p_{X*}\nabla) \in \mathrm{Mod}(O_X)_{\mathrm{sp}}$. Thus, one has a left exact functor (a special case of [Rot97, (2.13)]):

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_X)_{\mathrm{sp}}. \tag{6.15}$$

If (F, ∇) is integrable, then $[p_{X*}\nabla, p_{X*}\nabla]$ defined in (6.4) is zero.

Between connection

We define the inverse image and the direct image of relative connection on changing bases. Consider a cartesian square of complex manifolds

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "f"] \\ T \arrow[r, "g"] & S, \end{tikzcd} \tag{6.16}$$

where f is smooth. For every $(F, \nabla) \in \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$, by [ABC20, Sec. 4.2], the relative connection ∇ is equivalent to an O_Z-linear splitting to the natural projection $P^1_f \otimes_{O_Z} F \to F$, where $P^1_\bullet$ denotes the sheaf of first order jets (defined in [ABC20, Sec. 4.1.2]). Applying g'^* to the induced splitting, we get an O_W-linear splitting to the natural projection $P^1_{f'} \otimes_{O_W} g'^* F \to g'^* F$. This is equivalent to an f'-connection on $g'^* F$. Hence an inverse image functor

$$g'^* : \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}} \to \mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}}. \tag{6.17}$$

It is right exact. By [ABC20, Sec. 5.1], the connection induced by ∇ is integrable if ∇ is so.

Now for direct image. Fix $\alpha \in F^1(Z, \Omega^1_f)$. For every

$$(F, \nabla) \in \mathrm{Mod}(O_W)_{f',g'^*\alpha-\mathrm{cxn}},$$

by projection formula (see *e.g*, [Har77, II, Ex. 5.1 (d)]), one has

$$g'_*(F \otimes_{O_W} g'^* \mathcal{E}_\alpha) = (g'_* F) \otimes_{O_Z} \mathcal{E}_\alpha.$$

Then the induced morphism

$$g'_* \nabla : g'_* F \to (g'_* F) \otimes_{O_Z} \mathcal{E}_\alpha$$

is $f^{-1}(O_S)$-linear. Since $d_{f'} : O_W \to \Omega^1_{f'}$ and $d_f : O_Z \to \Omega^1_f$ are related by $g'^* d_f = d_{f'}$, the induced map $g'_* \nabla$ satisfies the Leibniz rule (6.11). Hence, the pair $(g'_* F, g'_* \nabla) \in \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$. In this manner, we get a left exact functor

$$g'_* : \mathrm{Mod}(O_W)_{f',g'^*\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}. \tag{6.18}$$

When $\alpha = 0$, the functor (6.18) sends $\mathrm{MIC}(f')$ to $\mathrm{MIC}(f)$.

Example 6.2.3.2. Take (6.16) to be

$$\begin{tikzcd} X \times Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Specan}(\mathbb{C}), \end{tikzcd}$$

then $p_Y^* : \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}$ sits on the left of the diagram [Rot97, (2.15)] and

$$p_{Y*} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \tag{6.19}$$

is [Rot97, (2.12)]. They restrict respectively to functors

$$p_{Y*} : \mathrm{Mod}(D_{X\times Y/X}) \to \mathrm{Mod}(D_Y); \tag{6.20}$$

$$p_Y^* : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_{X\times Y/X}). \tag{6.21}$$

Remark 6.2.3.3. Take $\alpha = 0 \in H^1(Z, \Omega_f^1)$. From another point of view, the morphism $O_Z \to g'_* O_W$ between sheaves of rings extends to a morphism $\tilde{D}_{Z/S} \to g'_* \tilde{D}_{W/T}$. Then (6.17) and (6.18) are respectively the pullback and the pushout along the induced morphism $(W, \tilde{D}_{W/T}) \to (Z, \tilde{D}_{Z/S})$ of ringed spaces. By [Sta24, Tag 0096], the functor (6.17) is the left adjoint to (6.18). Then from [Sta24, Tag 09T5], the derived functor

$$Lg'^* : D(\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}})$$

is the left adjoint to

$$Rg'_* : D(\mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}}) \to D(\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}).$$

6.3 Rothstein transform on modules with connection

6.3.1 Construction

Definition 6.3.1.1. Define functors $R\mathfrak{S}_1 : D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ and $R\mathfrak{S}_2 : D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D(\mathrm{Mod}(O_X)_{\mathrm{sp}})$ by

$$R\mathfrak{S}_1 = Rp_{Y*}(\mathcal{P} \otimes_{O_{X\times Y}} p_X^* \cdot), \quad R\mathfrak{S}_2 = Rp_{X*}(\mathcal{P}^{-1} \otimes_{O_{X\times Y}} p_Y^* \cdot).$$

Here Rp_{Y*} (resp. Rp_{X*}) is the right derived functor of (6.19) (resp. (6.15)). The pair $(R\mathfrak{S}_1, R\mathfrak{S}_2)$ is called the *Rothstein transform*.

Let $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \subset D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ (resp. $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \subset D(\mathrm{Mod}(O_X)_{\mathrm{sp}})$) be the full subcategory of objects whose cohomologies are good O-modules (in the sense of [Kas03, Def. 4.22]). In view of Proposition 6.3.1.2, the Rothstein transform is compatible with the Fourier-Mukai transform.

Proposition 6.3.1.2. *There are commutative squares*

$$\begin{array}{ccc} D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) & \xrightarrow{R\mathfrak{S}_1} & D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \\ \downarrow & & \downarrow \\ D(O_X) & \xrightarrow{R\mathcal{S}_1} & D(O_Y), \end{array} \qquad \begin{array}{ccc} D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) & \xrightarrow{R\mathfrak{S}_2} & D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \\ \downarrow & & \downarrow \\ D(O_Y) & \xrightarrow{R\mathcal{S}_2} & D(O_X), \end{array}$$

where the vertical functors are forgetful. In particular, $R\mathfrak{S}_1$ *and* $R\mathfrak{S}_2$ *restrict to functors* $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ *and* $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})$*) respectively.*

Proof. All the functors $p_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X\times Y})$, (6.13), (6.14) and

$$\mathcal{P} \otimes_{O_{X\times Y}} \cdot : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_{X\times Y})$$

are exact. To prove the commutativity of the first square, it remains to do so for the square

$$\begin{tikzcd} D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \arrow[r, "Rp_{Y*}"] \arrow[d, "\mathrm{for}_{X\times Y}"] & D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \arrow[d, "\mathrm{for}_Y"] \\ D(O_{X\times Y}) \arrow[r, "Rp_{Y*}"] & D(O_Y). \end{tikzcd}$$

Since the forgetful functor $\mathrm{for}_Y : \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \to \mathrm{Mod}(O_Y)$ is exact, the composition $\mathrm{for}_Y Rp_{Y*} : D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \to D(O_Y)$ is the right derived functor of

$$\mathrm{for}_Y \circ p_{Y*} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y).$$

From Remark 6.2.3.1, [Sta24, Tag 0096] and [Sta24, Tag 08BJ], the functor $\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})$ preserves K-injective complexes. By Lemma E.1.0.11, the composition $Rp_{Y*}\mathrm{for}_{X\times Y} : D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \to D(O_Y)$ is the right derived functor of

$$p_{Y*}\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y).$$

Since $\mathrm{for}_Y \circ p_{Y*} = p_{Y*} \circ \mathrm{for}_{X\times Y}$, the first square is indeed commutative.

By the commutativity of the first square and Corollary 5.3.1.16, the transform $R\mathfrak{S}_1$ preserves O-goodness. The other half about $R\mathfrak{S}_2$ is similar. □

6.3.2 Rothstein's theorem

Theorem 6.3.2.1 (Rothstein)**.** *There are natural isomorphisms $R\mathfrak{S}_1 R\mathfrak{S}_2 \cong T^{-g}$ on $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ and $R\mathfrak{S}_2 R\mathfrak{S}_1 \cong T^{-g}$ on $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})$.*

We begin the proof of Theorem 6.3.2.1 with Lemma 6.3.2.2, a direct adaption of [Rot97, Prop. 2.4] for complex tori.

Lemma 6.3.2.2. *Let $\Delta \subset X \times X$ be the diagonal. Define a morphism of complex tori $\epsilon_X : X \times X \to X, \quad (x_1, x_2) \mapsto x_2 - x_1$. Then*

$$Rp_{12*}(\epsilon_X \times 1_Y)^*\mathcal{P} \cong O_\Delta[-g]$$

in $D^b(\mathrm{Mod}(O_{X\times X})_{(1,-1)-\mathrm{sp}})$, where $p_{12} : X \times X \times Y \to X \times X$ is the projection.

Proof. The identification $Rp_{X*}\mathcal{P} \cong \mathbb{C}_0[-g]$ in $D^b(O_X)$ from [Kem91, Thm. 3.15] can be lifted to an isomorphism in $D^b(\mathrm{Mod}(O_X)_{-1-\mathrm{sp}})$. As stated in the last sentence of the proof of [Vig21, Prop. 2.1.21], a morphism of modules with splittings (or connection) is an isomorphism whenever the underlying morphism of O-modules is so. Then apply Theorem 5.3.2.3 to the cartesian square

$$\begin{array}{ccc} X \times X \times Y & \xrightarrow{\epsilon_X \times 1_Y} & X \times Y \\ \Big\downarrow{\scriptstyle p_{12}} & & \Big\downarrow{\scriptstyle p_X} \\ X \times X & \xrightarrow{\quad \epsilon_X \quad} & X. \end{array}$$

□

Arguing as in Lemma 6.3.2.2, we can prove the analytic version of [Rot97, Prop. 2.5, Prop. 3.1]. These three results are used in the proof of Theorem 6.3.2.1 below.

Proof of Theorem 6.3.2.1. Repeat the proof of [Rot97, Thm. 3.2], which requires the projection formula and the smooth base change theorem for modules with *connection*. For this, we first construct the corresponding comparison morphism that is compatible with the underlying O-module comparison morphism. The construction reduces to the adjunction between derived inverse image and derived direct image of relative connection in Remark 6.2.3.3.

The compatibility with O-module comparison morphism can be proved in a way similar to Proposition 6.3.1.2. On the level of O-modules, the comparison morphism is an isomorphism by Fact 5.3.2.15 and Theorem 5.3.2.3. (This type of arguments can also be found in the proof of [Vig21, Prop. 2.1.21, Thm. 2.1.33].) □

Remark 6.3.2.3. Rothstein's first proof of [Rot96, Thm. 2.2] is based on a problematic lemma [Rot96, Lem. 2.3]. The problem is explained in [Rot97, Sec. 1]. To save his first proof, one may attempt to replace this "lemma" by its close variant, the bounded way-out lemma (see *e.g.*, [Lip60, Lem. 1.11.3 (i)]). The difficulty is that, *a priori*, there is no canonical choice of a natural transformation between the two functors to be compared, which is required by way-out argument. For instance, in the end of the proof of Proposition 5.4.2.3, there are isomorphism arrows of opposite directions.

6.3.3 Matsushima's theorem

A holomorphic vector bundle $H \to Y$ is called homogeneous if $T_y^* H$ is isomorphic to H for all $y \in Y$, where $T_y : Y \to Y$ is the translation by y. The first half of Theorem 6.3.3.1 is a special case of [Mat59, Thm. 1].

Theorem 6.3.3.1 (Matsushima)**.** *Let E be a coherent O_Y-module with a connection ∇. Then E is a homogeneous vector bundle, and the pair (E, ∇) is translation invariant.*

Proof. By Proposition 6.3.1.2, for every integer i, the coherent O_X-module $H^i R\mathcal{S}_2(E)$ admits a 1-splitting. By Lemma 6.3.3.2, the support of $H^i R\mathcal{S}_2(E)$ is finite. Consequently, there is an isomorphism $R\mathcal{S}_2(E) \xrightarrow{\sim} \oplus_{i \in \mathbb{Z}} T^{-i} H^i R\mathcal{S}_2(E)$

in $D^b_c(O_X)$. From Proposition 5.5.3.2 3 and Fact 6.1.2.1 2, it induces an isomorphism in $D^b_c(O_Y)$

$$T^{-g}E \to \oplus_{i\in\mathbb{Z}} T^{-i} H^0 RS_1(H^i RS_2(E)),$$

and each $H^0RS_1(H^iRS_2(E))$ is a homogeneous vector bundle on Y. Then E is isomorphic to $H^0RS_1(H^gRS_2(E))$, hence a homogeneous vector bundle.

Then we adopt the argument in [BK09, Footnote (6), p.388]. For every $y \in Y$, $T^*_y\nabla$ is a connection on $T^*_yE \xrightarrow{\sim} E$ and $T^*_0\nabla = \nabla$. The map

$$Y \to H^0(Y, \Omega^1_Y \otimes \mathcal{E}nd(E)), \quad y \mapsto T^*_y\nabla - \nabla$$

is holomorphic. By Cartan-Serre's theorem, $H^0(Y, \Omega^1_Y \otimes \mathcal{E}nd(E))$ is a finite-dimensional vector space. From compactness of Y, the map is constantly zero. Therefore, $T^*_y(E,\nabla) \cong (E,\nabla)$ for all $y \in Y$. $\square$

Lemma 6.3.3.2. *Let F be a* coherent *O_X-module admitting a 1-splitting on the complex torus X. Then F is finitely supported.*

Proof. Suppose to the contrary that $\mathrm{Supp}(F)$ is infinite. By [CAS, p.76], $\mathrm{Supp}(F)$ is an analytic set in X. Then $\dim \mathrm{Supp}(F) \geq 1$. Let C be an irreducible component of $\mathrm{Supp}(F)$ of maximal dimension. Write $i : C \to X$ for the inclusion. Take a morphism $h : Z \to X$ provided by Lemma 5.5.3.3. Then $h(Z) = C$ and $F'' := F'/T(F')$ is a *vector bundle* on Z of positive rank r, where $F' = h^*F$ and $T(*)$ denotes the torsion part of a sheaf of modules. In consequence, the morphism of complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is nonzero. However, we claim that its tangent map at origin $h^* : \mathfrak{g} \to H^1(Z, O_Z)$ is zero.

Let $\mathcal{E}' = h^*\mathcal{E}$. Because O_X is flat over itself, pulling back (6.9) to Y and tensoring with F'', by [Sta24, Tag 05NJ] one has a short exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} F'' \to \mathcal{E}' \otimes_{O_Z} F'' \to F'' \to 0. \tag{6.22}$$

Since $\mathcal{E}'$ is a vector bundle on Z, one has

$$\frac{\mathcal{E}' \otimes F'}{T(\mathcal{E}' \otimes F')} = \mathcal{E}' \otimes F''.$$

Then the splitting on F induces a splitting $F'' \xrightarrow{\psi'} \mathcal{E}' \otimes F''$ of (6.22).

Let β be the natural morphism $\beta : O_Z \to \mathcal{E}nd(F'')$. By Lemma 6.2.1.3, the composition

$$\mathrm{End}(g) \overset{\mathrm{Id}_{\mathfrak{g}^*}\otimes h^*}{\to} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, O_Z) \overset{\mathrm{Id}_{\mathfrak{g}^*}\otimes H^1(Z,\beta)}{\to} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, \mathcal{E}nd(F''))$$

sends $1 \in \mathrm{End}(\mathfrak{g})$ to 0. Therefore, the map $H^1(Z,\beta)h^* : \mathfrak{g} \to H^1(Z, \mathcal{E}nd(F''))$ is zero. The morphism $\tau : \mathcal{E}nd(F'') \to O_Z$ taking trace satisfies $\tau\beta = r \cdot \mathrm{Id}_{O_Z}$. Then $h^* = \frac{1}{r}\tau_* H^1(Z,\beta)h^* = 0$ as a map $\mathfrak{g} \to H^1(Z, O_Z)$. The claim follows. The claim gives a contradiction. $\square$

Corollary 6.3.3.3. *Every local system (of finite dimensional $\mathbb{C}$-vector spaces) on a complex torus is translation invariant.*

Proof. Let L be a local system on Y. By Theorem 6.3.3.1, the pair $(L \otimes_{\mathbb{C}} O_Y, \mathrm{Id}_L \otimes d)$ is translation invariant. The result follows from the Riemann-Hilbert correspondence [Del70, I, Thm. 2.17]. $\square$

6.4 Laumon-Rothstein sheaf of algebras

6.4.1 Construction

To lift the Fourier-Mukai transform to D-modules, we recall (in Definition 6.4.1.1) the sheaf $\mathcal{A}_X$ from [Rot96, p.576]. In the notation of (6.9), fix a $\mathbb{C}$-basis $\{\omega^1, \dots, \omega^g\}$ of the $\mathbb{C}$-vector space

$$H^0(Y, \Omega^1_Y) = \mathfrak{g}^* = \Gamma(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \subset \Gamma(X, \mathcal{E}).$$

For each Stein open subset $U \subset X$, by Cartan's Theorem B (see, *e.g.*, [KK83, Sec. 52, Thm. B]) one has $H^1(U, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = 0$. Thence (6.9) induces a short exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U) \to \mathcal{E}(U) \xrightarrow{\mu} O_X(U) \to 0.$$

Whence, there is $\rho \in \mathcal{E}(U)$ with $\mu(\rho) = 1 \in O_X(U)$. For two such pairs (U, ρ) and $(\tilde{U}, \tilde{\rho})$ with $U \cap \tilde{U} \neq \emptyset$, one has $\mu(\tilde{\rho} - \rho) = 0 \in O_X(U \cap \tilde{U})$, so $\tilde{\rho} - \rho \in \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U \cap \tilde{U})$. There exists a unique tuple $f_1, \dots, f_g \in O_X(U \cap \tilde{U})$ such that

$$\tilde{\rho} - \rho = \sum_{i=1}^{g} \omega^i \otimes f_i$$

in $\mathcal{E}(U \cap \tilde{U})$.

Definition 6.4.1.1. For each chosen pair (U, ρ) as above, introduce independent variables $x_1^\rho, \dots, x_\rho^g$ and put

$$\mathcal{A}_X|_U = O_U[x_1^\rho, \dots, x_g^\rho].$$

For another choice $(\tilde{U}, \tilde{\rho})$ with the tuple $(f_1, \dots, f_g)$ as above, we glue $\mathcal{A}_X|_U$ and $\mathcal{A}_X|_{\tilde{U}}$ by the rule

$$x_i^\rho - x_i^{\tilde{\rho}}|_{U \cap \tilde{U}} = f_i. \tag{6.23}$$

The resulting sheaf $\mathcal{A}_X$ is a sheaf of commutative O_X-algebra.

Let

$$0 \to \mathfrak{g}^* \to X^\natural \xrightarrow{\pi} X \to 0 \tag{6.24}$$

be the universal vectorial extension[1] of X constructed in (F.22). Then $X^\natural$ is the moduli space of flat line bundles on Y. In coordinate-free terms, $\mathcal{A}_X$ is the O_X-subalgebra of $\pi_* O_{X^\natural}$ of sections whose restriction to each fiber of π is a polynomial on $\mathfrak{g}^*$. For every integer $m \geq 0$, let $O_{X^\natural}(m) \subset O_{X^\natural}$ denote the subsheaf of sections whose restriction to the fibers of π are homogeneous polynomials of degree m. Similar to [Bjö93, Def 1.6.1], there exists a sheaf of graded rings $O_{[X^\natural]} := \oplus_{m\geq 0} O_{X^\natural}(m) (\subset O_{X^\natural})$ on $X^\natural$. Then $\mathcal{A}_X = \pi_* O_{[X^\natural]}$ and $\Gamma(X, \mathcal{A}_X) = \mathbb{C}$.

Remark 6.4.1.2. Unlike the analytic case, if X is an abelian variety, then the notation $\mathcal{A}_X$ in [Rot96, p.576] is the *algebraic* direct image $\pi_* O_{X^\natural}$. Morally, such difference also lies between algebraic and analytic D-modules. For a complex manifold or a smooth algebraic variety V, let $p : T^*V \to V$ be the natural projection of the cotangent bundle. Denote by GD_V the associated graded ring of the degree filtration on D_V. From [HT07, p.57], in the algebraic case GD_V is $p_* O_{T^*V}$. By contrast, in the analytic case, GD_V is the O_V-submodule of $p_* O_{T^*V}$ of sections whose restriction to each fiber of p is a polynomial.

Remark 6.4.1.3. The sheaf of rings $\mathcal{A}_X$ is functorial in X in the following sense. Let $\phi : X' \to X$ be a morphism of complex tori. Let $\hat{\phi} : Y \to Y'$ be the morphism dual to ϕ. By Proposition F.5.4.7, it induces a morphism $\phi^\natural : X'^\natural \to X^\natural$ of complex Lie groups fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(Y', \Omega^1_{Y'}) \arrow[r] \arrow[d, "\hat{\phi}^*"] & X'^\natural \arrow[r, "\pi'"] \arrow[d, "\phi^\natural"] & X' \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & H^0(Y, \Omega^1_Y) \arrow[r] & X^\natural \arrow[r, "\pi"] & X \arrow[r] & 0.
\end{tikzcd}$$

For each local section of $O_{[X^\natural]}$, its $\phi^\natural$-pullback (a local section of $O_{X'^\natural}$) restricts to a polynomial on each fiber of π'. Indeed, this restriction is the $\hat{\phi}^*$-pullback of a restriction to a fiber of π. Therefore, the natural morphism $O_{X^\natural} \to \phi^\natural_* O_{X'^\natural}$ restricts to a morphism $O_{[X^\natural]} \to \phi^\natural_* O_{[X'^\natural]}$. The resulting morphism of ringed spaces $(X'^\natural, O_{[X'^\natural]}) \to (X^\natural, O_{[X^\natural]})$ descends to another morphism of ringed spaces

$$\tilde{\phi} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X), \tag{6.25}$$

which is compatible with ϕ. In particular, the following square

$$\begin{tikzcd}
D(\mathcal{A}_{X'}) \arrow[r, "R\tilde{\phi}_*"] \arrow[d] & D(\mathcal{A}_X) \arrow[d] \\
D(O_{X'}) \arrow[r, "R\phi_*"] & D(O_X)
\end{tikzcd} \tag{6.26}$$

[1]By [Rot96, p.567], it is also the $\mathfrak{g}^*$-principal bundle associated to the tautological extension (6.9).

is commutative, where the vertical functors are forgetful. If M is an O_X-module, then

$$L\tilde{\phi}^*(\mathcal{A}_X \otimes_{O_X} M) = \mathcal{A}_{X'} \otimes_{O_{X'}} L\phi^* M. \tag{6.27}$$

6.4.2 Basic properties

Notice that $\mathcal{A}_X$ has a natural degree filtration $\{\mathcal{A}_X(m)\}_{m\in\mathbb{Z}}$, where

$$\mathcal{A}_X(m) = \pi_*(\oplus_{j=0}^m O_{X^\natural}(j))$$

is the O_X-submodule of $\mathcal{A}_X$ of polynomials of degree at most m. See also [Rot96, Sec. 5.3] and the end of [Lau96, p.10]. Then $\mathcal{A}_X(0) = O_X$, $\mathcal{A}_X(1) = \mathcal{E}^\vee$ (*cf.* the start of [Lau96, p.10]), and every $\mathcal{A}_X(m)$ is a locally free O_X-module of finite rank. Moreover, for any integers $m, n \geq 0$, one has

$$\mathcal{A}_X(n)\mathcal{A}_X(m) = \mathcal{A}_X(n+m). \tag{6.28}$$

Thus, $\mathcal{A}_X$ is a sheaf of positively filtered rings (in the sense of [Bjö93, p.459, p.464]) on the complex torus X.

We review some terminology from [Bjö93, A:III]. A coherent sheaf of rings on a locally compact Hausdorff space is called *noetherian* if every increasing sequence of ideal sheaves is stationary over relatively compact subsets ([Bjö93, 2.24, p.470]). Let R be a commutative filtered ring. If the subring $\oplus_{v\in\mathbb{Z}} R_v T^v$ of $R[T, T^{-1}]$ is a noetherian ring, then R is called a *noetherian filtered ring.*

Definition 6.4.2.1 ([Bjö93, A.III, 1.7, Def. 1.11, 1.19])**.** A filtration on an R-module M is a family of additive subgroups $\{M_v\}_{v\in\mathbb{Z}}$ such that

$$M_v \subset M_{v+1}; \quad R_k M_v \subset M_{k+v}; \quad \cup_v M_v = M.$$

This filtration is called *separated* if $\cap_{v\in\mathbb{Z}} M_v = 0$, and called *good* if $\oplus_{v\in\mathbb{Z}} M_v T^v$ is a finitely generated $\oplus_{v\in\mathbb{Z}} R_v T^v$-module.

A *zariskian filtered ring* is a noetherian filtered ring such that all the good filtrations on every finitely generated module are separated. A filtered sheaf of rings is called *stalkwise zariskian* if every stalk is a zariskian filtered ring ([Bjö93, Def. 2.6, p.465]).

Lemma 6.4.2.2. *The sheaf of rings $\mathcal{A}_X$ is coherent and noetherian. The sheaf of filtered rings $\mathcal{A}_X$ is stalkwise zariskian.*

Proof. By (6.23), the graded ring associated to the degree filtration of $\mathcal{A}_X$ is

$$G\mathcal{A}_X := \oplus_{m\geq 0}\mathcal{A}_X(m)/\mathcal{A}_X(m-1) = \mathrm{Sym}(\mathfrak{g}) \otimes_{\mathbb{C}} O_X = O_X[x_1, \ldots, x_g]. \tag{6.29}$$

Here for each chosen pair (U,ρ) as in Section 6.4.1, $x_i|_U \in \Gamma(U, \mathcal{A}_X(1)/\mathcal{A}_X(0)) \subset \Gamma(U, G\mathcal{A}_X)$ is the image of $x_i^\rho \in \Gamma(U, \mathcal{A}_X(1))$. From [Bjö79, Thm. 1.26, p.460], $\mathcal{A}_X$ is stalkwise zariskian. The other part follows from [Bjö79, Prop. 1.27, p.460, Thm. 2.7, p.465]. (See also the proof of [Bjö93, Thm. 1.2.5].) $\square$

In view of the difference mentioned in Remark 6.4.1.2, the statement of [Rot96, Prop. 4.4] is slightly modified as Fact 6.4.2.3. For every $\mathcal{A}_X$-module F and every chosen pair (U,ρ) as in Section 6.4.1, define $\psi_U^\rho : F(U) \to \mathcal{E}(U) \otimes_{O_X(U)} F(U)$ by

$$\psi_U^\rho(s) = \rho \otimes s + \sum_{i=1}^{g} \omega^i|_U \otimes (x_i^\rho s).$$

Then $(\mu_1 \otimes \mathrm{Id}_F)(\psi_U^\rho(s)) = s$. In light of (6.23), the family $\{\psi_U^\rho\}_{(U,\rho)}$ glues to a 1-splitting $\psi_F : F \to \mathcal{E} \otimes_{O_X} F$. One has $\psi_F = \psi_{\mathcal{A}_X} \otimes_{\mathcal{A}_X} \mathrm{Id}_F$. By commutativity of $\mathcal{A}_X$ and [Rot96, (4.9)], one has $[\psi_F, \psi_F] = 0$.

Fact 6.4.2.3. *The resulting functor* $\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)_{\mathrm{sp}}$ *induces an equivalence from* $\mathrm{Mod}(\mathcal{A}_X)$ *to the full subcategory of* $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ *of objects* (F,ψ) *with* $[\psi,\psi] = 0$.

From Fact 6.4.2.3 and the proof of [Rot96, Prop. 4.1], the functor (6.13) restricts to an exact functor $p_X^* : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}}$. Similarly by [Rot96, Prop. 4.2], the functor (6.15) restricts to a functor

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}} \to \mathrm{Mod}(\mathcal{A}_X). \tag{6.30}$$

6.5 Laumon-Rothstein transform

6.5.1 Construction and properties

Definition 6.5.1.1. Define functors

$$RS_1 = Rp_{Y*}(\mathcal{P} \otimes^L_{O_{X\times Y}} p_X^* \cdot) : D(\mathcal{A}_X) \to D(D_Y); \tag{6.31}$$

$$RS_2 = Rp_{X*}(\mathcal{P}^{-1} \otimes^L_{O_{X\times Y}} p_Y^* \cdot) : D(D_Y) \to D(\mathcal{A}_X), \tag{6.32}$$

where $Rp_{Y*} : D(\mathrm{MIC}(p_X)) \to D(D_Y)$ (resp. $Rp_{X*} : D(\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}}) \to D(\mathcal{A}_X)$) is the right derived functor of (6.20) (resp. (6.30)), and $p_Y^* : D(D_Y) \to D(D_{X\times Y/X})$ is from the exact functor (6.21). The pair is called the *Laumon-Rothstein transform*.

Example 6.5.1.2. The $\pi_* O_{X^\natural}$-module $\pi_*\mathbb{C}_z$ is naturally an $\mathcal{A}_X$-module. Its underlying O_X-module is $\mathbb{C}_{\pi(z)}$. One has $RS_1(\pi_*\mathbb{C}_z) = (L, \nabla)$.

Proposition 6.5.1.3. *There are commutative squares*

$$\begin{array}{ccccccc} D(\mathcal{A}_X) & \xrightarrow{RS_1} & D(D_Y) & \qquad & D(D_Y) & \xrightarrow{RS_2} & D(\mathcal{A}_X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D(O_X) & \xrightarrow{RS_1} & D(O_Y); & & D(O_Y) & \xrightarrow{RS_2} & D(O_X), \end{array}$$

where the vertical functors are forgetful. In particular, RS_1 *(resp.* RS_2*) sends* $D_{O-\mathrm{good}}(\mathcal{A}_X)$ *(resp.* $D_{O-\mathrm{good}}(D_Y)$*) to* $D_{O-\mathrm{good}}(D_Y)$ *(resp.* $D_{O-\mathrm{good}}(\mathcal{A}_X)$*).*

Proof. The proof is similar to that of Proposition 6.3.1.2, as $\mathcal{A}_X$ (resp. D_Y) is flat over O_X (resp. O_Y). □

With Proposition 6.5.1.3, the proof of Theorem 6.5.1.4 is similar to that of Theorem 6.3.2.1.

Theorem 6.5.1.4 (Laumon, Rothstein)**.** *There are natural isomorphisms of functors* $RS_1RS_2 \cong T^{-g}$ *on* $D_{O-\mathrm{good}}(D_Y)$ *and* $RS_2RS_1 \cong T^{-g}$ *on* $D_{O-\mathrm{good}}(\mathcal{A}_X)$*.*

Proposition 6.5.1.5 follows from Proposition 6.5.1.3, Theorem 6.5.1.4 and Fact 6.1.1.1 1 as in the proof of [Rot96, Thm. 6.1], *cf.* [Lau96, Prop. 3.1.2, Cor. 3.2.4].

Proposition 6.5.1.5. *There are natural isomorphisms of functors*

$$RS_2(D_Y \otimes^L_{O_Y} \cdot) \cong \mathcal{A}_X \otimes^L_{O_X} RS_2(\cdot) : D_{\mathrm{good}}(O_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X);$$
$$RS_1(\mathcal{A}_X \otimes^L_{O_X} \cdot) \cong D_Y \otimes^L_{O_Y} RS_1(\cdot) : D_{\mathrm{good}}(O_X) \to D_{O-\mathrm{good}}(D_Y).$$

For $x \in X$ (resp. $y \in Y$), let $P_x = \mathcal{P}|_{x\times Y}$ (resp. $P_y = \mathcal{P}|_{X\times y}$) be the pullback line bundle on Y (resp. X). For a closed analytic subset S of a complex manifold Z, [Kas03, (3.30), p.51] defines a D_Z-module $\mathcal{B}_{S|Z}$.

Corollary 6.5.1.6. *For any* $x \in X$ *and* $y \in Y$*, one has*

$$RS_2(D_Y \otimes_{O_Y} \mathbb{C}_y) = \mathcal{A}_X \otimes_{O_X} P_{-y};$$
$$T^g RS_1(\mathcal{A}_X \otimes_{O_X} P_{-y}) = D_Y \otimes_{O_Y} \mathbb{C}_y = i_{y+}\mathbb{C} = \mathcal{B}_{\{y\}|Y};$$
$$RS_1(\mathcal{A}_X \otimes_{O_X} \mathbb{C}_x) = D_Y \otimes_{O_Y} P_x;$$
$$T^g RS_2(D_Y \otimes_{O_Y} P_x) = \mathcal{A}_X \otimes_{O_X} \mathbb{C}_x.$$

Proof. By [HT07, Example 1.6.4], one has $D_Y \otimes_{O_Y} \mathbb{C}_y = \mathcal{B}_{\{y\}|Y}$. The result follows from Theorem 6.5.1.4, Proposition 6.5.1.5, Fact 6.1.2.1 and Lemma 5.2.0.8. □

6.5.2 Matsushima-Morimoto theorem

Proposition 6.5.2.1, due to Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], is a converse to Theorem 6.3.3.1. For abelian varieties, Nakayashiki [Nak94, Prop. 5.9] gives a proof using the Fourier-Mukai transform.

Proposition 6.5.2.1. *A homogeneous vector bundle on a complex torus admits an integrable connection.*

Proof. Let $E \to Y$ be a homogeneous vector bundle. Set $\hat{E} = H^g R\mathcal{S}_2(E)$. According to Proposition 5.5.3.2 and Fact 6.1.1.1, one has $E = H^0 R\mathcal{S}_1(\hat{E})$ and $\mathrm{Supp}(\hat{E})$ is finite. By Lemma 6.5.2.2, $\hat{E}$ has an $\mathcal{A}_X$-module structure. By Proposition 6.5.1.3, the O_Y-module underlying $H^0 RS_1(\hat{E})$ is E. The D_Y-module $H^0 RS_1(\hat{E})$ carries naturally an integrable connection. □

The proof of Proposition 6.5.2.1 needs Lemma 6.5.2.2, a converse to Lemma 6.3.3.2.

Lemma 6.5.2.2. *If F is an O_X-module with finite support on the complex torus X, then F admits a 1-splitting ψ with $[\psi, \psi] = 0$.*

Proof. There is a decomposition $F = \oplus_{i=1}^m F_i$, where $\mathrm{Supp}(F_i)$ is a singleton for each i. Thus, one may assume that $\mathrm{Supp}(F)$ is a singleton. Then there exists an open neighborhood $U \subset X$ of $\mathrm{Supp}(F)$ and a morphism of complex manifolds $s : U \to X^\natural$ that is a local section to (6.24). Let $\iota : U \to X$ be the inclusion. Applying π_* to the morphism of sheaves of rings $O_{X^\natural} \to s_* O_U$, one gets a morphism $\pi_* O_{X^\natural} \to \iota_* O_U$. As $\mathcal{A}_X$ is an O_X-subalgebra of $\pi_* O_{X^\natural}$, this endows $\iota_* O_U$ an $\mathcal{A}_X$-module structure.[2] Since the canonical O_X-morphism $\mathrm{Id}_F \otimes \iota^\# : F \to F \otimes_{O_X} \iota_* U$ is an isomorphism, F also obtains an $\mathcal{A}_X$-module structure. This induces such a splitting by Fact 6.4.2.3. □

Proposition 6.5.2.1, together with Theorem 6.3.3.1, yields (a slight generalization of) Morimoto's theorem [Mor59, Thm. 2, p.91].

Corollary 6.5.2.3 (Morimoto)**.** *A coherent module admitting a connection on a complex torus is a vector bundle admitting an integrable connection.*

6.6 Good modules

6.6.1 Definition

We define good $\mathcal{A}_X$-modules. We also review several definitions of good D-modules in the literature, and show that they are equivalent.

Let Z be a complex manifold.

[2] This example shows that Lemma 6.3.3.2 fails without coherent condition.

Definition 6.6.1.1 ([Bjö93, 2.5, p.465])**.** Let $\mathcal{R}$ be a positively filtered sheaf of rings on Z such that the associated graded ring $G\mathcal{R}$ is coherent. Let M be a coherent left $\mathcal{R}$-module. A filtration on M is an increasing sequence of subsheaves $\{M_v\}_{v\in\mathbb{Z}}$ satisfying $\cup_{v\in\mathbb{Z}}M_v = M$ and $\mathcal{R}_k M_v \subset M_{k+v}$ for all integers $k \geq 0$ and v. This filtration is called

- *B-good* ([Bjö93, Remark 2.16, p.467]) if for every $x \in Z$, there exists an open neighborhood U, a finite set $\{m_1, \dots, m_s\} \subset \Gamma(U, M)$ and integers $k_1, \dots, k_s$ such that $M_v|_U = \sum_{i=1}^{s} \mathcal{R}_{v-k_i} m_i$ for all integers v.
- *locally good* ([Meb89, Prop. 2.1.12 (i)]) if every M_v is coherent over O_Z, and if for every $x \in Z$, there is an open neighborhood U of x and an integer $k_0 \geq 0$ such that $\mathcal{R}_m M_{k_0} = M_{m+k_0}$ on U for all integers $m \geq 0$.

The proof of Lemma 6.6.1.2 is similar to that of [HT07, Prop. 2.1.1, Def. 2.1.2].

Lemma 6.6.1.2. *Let $M_\bullet = (M_v)_{v\in\mathbb{Z}}$ be a filtration on a coherent $\mathcal{A}_X$-module M. Then $M_\bullet$ is B-good if and only if $M_\bullet$ is locally good. (In that case, we call $M_\bullet$ a good filtration on M.)*

Proof.

- Assume that $M_\bullet$ is B-good. By Lemma 6.4.2.2 and [Bjö93, Thm. 2.17, p.467], the $G\mathcal{A}_X$-module $\oplus_{v\in\mathbb{Z}} M_v/M_{v-1}$ is coherent. Because of (6.29) and the proof of [Bjö93, Prop. 1.4.5], for every integer v, the O_X-module M_v/M_{v-1} is coherent. From [Bjö93, Prop. 2.23, p.470], the filtration $M_\bullet$ is locally bounded blow. Then by induction on $v \in \mathbb{Z}$, one proves that the O_X-module M_v is coherent.

  For every $x \in X$, by definition, there is an open neighborhood $U \subset X$ of x, sections $m_1, \dots, m_s \in \Gamma(U, M)$ and integers $k_1, \dots, k_s$ such that $M_v|_U = \sum_{i=1}^{s} \mathcal{A}_X(v - k_i)m_i$ for all integers v. Put $k_0 = \max_{j=1}^{s} k_j$. For every integer $k \geq 0$, one has $\mathcal{A}_X(k)M_{k_0} \subset M_{k+k_0}$. Moreover,

  $$M_{k+k_0}|_U = \sum_{i=1}^{s} \mathcal{A}_X(k+k_0-k_i)m_i \overset{\text{(a)}}{\subset} \sum_{i=1}^{s} \mathcal{A}_X(k)\mathcal{A}_X(k_0-k_i)m_i \subset \mathcal{A}_X(k)M_{k_0},$$

  where (a) uses (6.28). Hence $\mathcal{A}_X(k)M_{k_0} = M_{k+k_0}$ on U.

- Conversely, assume that $M_\bullet$ is locally good. For a fixed $x \in X$, take U and k_0 provided by the definition of local goodness. Since M_{k_0} is coherent over O_X, by shrinking U, one may assume that the O_U-module $M_{k_0}|_U$ is generated by sections $s_1, \dots, s_m \in \Gamma(U, M_{k_0})$. Define a morphism of $\mathcal{A}_X$-modules $\phi : \mathcal{A}_X^m|_U \to M|_U, \quad (f_1, \dots, f_m) \mapsto \sum_{j=1}^{m} f_j s_j$. Since $M_\bullet$ is a filtration, for every integer v, one has $\mathcal{A}_X(v-k_0)M_{k_0} \subset M_v$. Hence $\phi(\mathcal{A}_X(v-k_0)^m) \subset M_v$. By construction, one has $\phi(\mathcal{A}_X(0)^m) = M_{k_0}|_U$. For every integer $k \geq k_0$, on U one has

  $$M_k = \mathcal{A}_X(k - k_0)M_{k_0} = \mathcal{A}_X(k - k_0)\phi(\mathcal{A}_X(0)^m) \subset \phi(\mathcal{A}_X(k - k_0)^m).$$

Therefore, the filtration $M_\bullet$ is B-good.

$\square$

From [HT07, Thm. 2.1.3 (i)], a coherent D_V-module on a smooth algebraic variety V admits a globally defined good filtration. By contrast, Malgrange [Mal04, p.405] gives a coherent D-module on the complex manifold $\mathbb{C}^* \times \mathbb{CP}^1$ that does not admit any global good filtration.

Definition 6.6.1.3. An O_Z-module F is called

- *countably quasi-good* ([KS97, p.942]) if every compact subset of Z has an open neighborhood U such that $F|_U$ is the union of an increasing sequence of coherent O_U-submodules.
- *quasi-good* ([KS16, p.12]) if for every relatively compact open subset $U \subset Z$, the restriction $F|_U$ is a sum of coherent O_U-submodules.

A D_Z-module M is called

- *good coherent* if for every relatively compact open subset U of Z, there is a finite filtration $\{M_k\}_{k\in\mathbb{Z}}$ of $M|_U$ such that each quotient M_k/M_{k-1} is a coherent D_U-modules admitting a good filtration. ([Sai89, p.369], [SS94, p.10] and [KS96, p.43])
- *S-quasi-good* ([KS96, p.43]) if for every relatively compact open subset $U \subset Z$, the restriction $M|_U$ admits a filtration $\{M_v\}_{v\in\mathbb{Z}}$ by coherent D_U-submodule such that each quotient M_v/M_{v-1} admits a good filtration and $M_v = 0$ for $v \ll 0$.

Proposition 6.6.1.4. *Let M be a coherent D_Z-module. Then the following are equivalent.*

1. *For every relatively compact open subset U of Z, there is a coherent O_U-submodule $F \subset M|_U$ with $D_U \cdot F = M|_U$.*
2. *For every relatively compact open subset U of Z, the D_U-module $M|_U$ admits a good filtration.*
3. *The D_Z-module M is good coherent.*
4. *The D_Z-module M is S-quasi-good.*
5. *The O_Z-module M is countably quasi-good.*
6. *The O_Z-module M is good.*
7. *The O_Z-module M is quasi-good.*

Proof. We follow the circular chain.

1 implies 2 See [Bjö93, 1.4.10].

2 implies 3 For every relatively compact open subset U of Z, define a finite filtration of $M|_U$ by $M_0 = 0$ and $M_1 = M|_U$. Then the graded piece M_1/M_0 admits a good filtration over U.

3 implies 4 For every relatively compact open subset U of Z, consider the filtration $\{M_k\}$ in the definition. By induction on k, one proves that each M_k is D_U-coherent.

4 implies 5 Every quotient M_v/M_{v-1} admits a good filtration, then by [Bjö93, Cor. 1.4.6], it is countably quasi-good. By induction on v and using [KS97, Lem. 2.1.1], one proves that every M_v is countably quasi-good. Therefore, for every integer v, there is an increasing sequence $\{M_v^k\}_{k\geq 1}$ of coherent O_U-submodules of M_v with $M_v = \cup_{k\geq 1} M_v^k$. For every integer $k \geq 1$, let $M^k := \sum_{i\leq k, v\leq k} M_v^i$. By [Sta24, Tag 01BY], M^k is a coherent O_U-submodule of M_k. Then

$$M = \cup_{v\in\mathbb{Z}} M_v = \cup_{v\in\mathbb{Z}} \cup_{i\geq 1} M_v^i = \cup_{k\geq 1} M^k,$$

so M is countably quasi-good.

5 implies 6 An increasing sequence forms a directed family.

6 implies 7 By definition.

7 implies 1 Let U be a relatively compact open subset of Z. Because M is a finite type D_Z-module, for every $x \in \bar{U}$, there is a relatively compact open neighborhood $U(x) \subset Z$ of x, an integer $n(x) \geq 1$ and sections

$$\{s_i^x\}_{1\leq i\leq n(x)} \subset \Gamma(U(x), M)$$

generating the $D_{U(x)}$-module $M|_{U(x)}$. By compactness of $\bar{U}$, the open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{1\leq j\leq r}$. Then $V = \cup_{j=1}^r U(x_j)$ is a relatively compact open subset of Z containing U. By Condition 7, one may write $M|_V = \sum_{\alpha\in I} G_\alpha$, where I is an index set, and each G_α is a coherent O_V-submodule of $M|_V$.

For every $x \in \bar{U}$, there is an open neighborhood $V(x) \subset U(x)$ of x, such that for each $1 \leq i \leq n(x)$, the restriction $s_i^x|_{V(x)} \in \Gamma(V(x), G_{\alpha(x,i)})$ for some index $\alpha(x,i) \in I$. By compactness of $\bar{U}$ again, the open cover $\{V(x)\}_{x\in\bar{U}}$ has a finite subcover $\{V(x'_k)\}_{1\leq k\leq m}$. Then

$$F := \sum_{1\leq k\leq m, 1\leq i\leq n(x'_k)} G_{\alpha(x'_k, i)}$$

is a finite type O_V-submodule of $M|_V$. By Lemma 6.6.2.7, it is coherent over O_V. Moreover, $D_U \cdot F|_U = M|_U$.

□

The proof of Proposition 6.6.1.5 is similar to that of Proposition 6.6.1.4.

Proposition 6.6.1.5. *Let M be a coherent $\mathcal{A}_X$-module on the complex torus X. Then the O_X-module M is good if and only if there is a coherent O_X-submodule $F \subset M$ with $\mathcal{A}_X \cdot F = M$.*

Let the sheaf of rings $\mathcal{R}$ be either D_Z or $\mathcal{A}_X$ on the fixed complex torus X.

Definition 6.6.1.6. [Kas03, Def. 4.24] A coherent $\mathcal{R}$-module is *good* if the underlying O-module is good.

For example, by Lemma 6.4.2.2 and [Bjö93, Thm. 1.2.5], the left $\mathcal{R}$-module $\mathcal{R}$ is good. Let $\mathrm{Good}(\mathcal{R}) \subset \mathrm{Coh}(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b_{O-\mathrm{good}}(\mathcal{R})$) be the full subcategory of good $\mathcal{R}$-modules (resp. objects whose cohomologies are good $\mathcal{R}$-modules). By Proposition 6.6.1.4, the category $D^b_{\mathrm{good}}(D_Z)$ is what Björk denotes by $D^b_{\mathrm{coh}}(D_Z)_f$ in [Bjö93, p.119].

Fact 6.6.1.7 (GAGA)**.**

- ([HK84, Thm. 1.1 (2)]) *Let V be a smooth proper complex algebraic variety. Then the analytification functor induces an equivalence* $\mathrm{Coh}(D_V) \to \mathrm{Good}(D_{V^{\mathrm{an}}})$.
- *Let A be a complex abelian variety. Then the analytification functor induces an equivalence* $\mathrm{Coh}(\mathcal{A}_A) \to \mathrm{Good}(\mathcal{A}_{A^{\mathrm{an}}})$

A coherent D_Z-module is called *holonomic* if its characteristic variety is of (minimal) dimension $\dim Z$. Malgrange ([Mal94, p.35], [Mal96, p.367], see also [Sab11, Thm. 4.3.4 (2)]) proves that every holonomic D_Z-module is generated by a coherent O_Z-submodule, so it is a good D_Z-module. Let $D^b_h(D_Z) \subset D^b(D_Z)$ be the full subcategory of objects with holonomic cohomologies.

6.6.2 Basic properties

Let $\mathcal{R}$ be either D_Z on a complex manifold Z or $\mathcal{A}_X$ on the fixed complex torus X.

Lemma 6.6.2.1 (Induced modules)**.** *The functor $\mathcal{R} \otimes_{O_Z} \cdot : \mathrm{Mod}(O_Z) \to \mathrm{Mod}(\mathcal{R})$ is exact. It restricts to a functor $\mathcal{R} \otimes_{O_Z} \cdot : \mathrm{Coh}(Z) \to \mathrm{Good}(\mathcal{R})$, and induces a t-exact functor $\mathcal{R} \otimes^L_{O_Z} \cdot : D^b_c(O_Z) \to D^b_{\mathrm{good}}(\mathcal{R})$.*

Proof. As $\mathcal{R}$ is flat over O_Z, the functor is exact. Consider the degree filtration $\{\mathcal{R}(m)\}_{m \geq 0}$ of $\mathcal{R}$, where $\mathcal{R}(m) \subset \mathcal{R}$ is the O_Z-submodule of polynomials of degree at most m. Each $\mathcal{R}(m)$ is vector bundle on Z and

$\mathcal{R} = \mathrm{colim}_m \mathcal{R}(m)$. Therefore, the O-module $\mathcal{R}$ is good. By Proposition 5.3.1.5 2, for every coherent O_Z-module F, the O-module $\mathcal{R} \otimes_{O_Z} F$ is good. Because F is an O_Z-module of finite presentation, $\mathcal{R} \otimes_{O_Z} F$ is an $\mathcal{R}$-module of finite presentation. By [Bjö93, Thm. 1.2.5] and Lemma 6.4.2.2, this $\mathcal{R}$-module is coherent. The other part follows. □

Lemma 6.6.2.2. *The category* $\mathrm{Good}(\mathcal{R})$ *is a weak Serre subcategory of* $\mathrm{Mod}(\mathcal{R})$. *In particular,* $D^b_{\mathrm{good}}(\mathcal{R})$ *is a triangulated subcategory of* $D^b(\mathcal{R})$.

Proof. The first half is a combination of [Kas03, Prop. 4.23], [Sta24, Tag 01BY] and [Sta24, Tag 0754]. The second half follows from [Yek19, Prop. 7.4.5]. □

For a morphism of complex manifolds $f : M \to N$, the direct image of D-modules $f_+ : D(D_M) \to D(D_N)$ is constructed in [Bjö93, 2.3.12].

Fact 6.6.2.3 ([Bjö93, Thm. 2.8.1, 2.8.7])**.** *Let* $f : W \to Z$ *be a morphism of complex manifolds. For every* $M \in D^b_{\mathrm{good}}(D_W)$, *if* $f|_{\mathrm{Supp}(M)} : \mathrm{Supp}(M) \to Z$ *is proper, then* $f_+ M \in D^b_{\mathrm{good}}(D_Z)$.

Lemma 6.6.2.4. *Let* $f : W \to Z$ *be a proper morphism of complex manifolds. Then the direct image functor* $f_+ : D(D_W) \to D(D_Z)$ *restricts to a functor* $D_{O-\mathrm{good}}(D_W) \to D_{O-\mathrm{good}}(D_Z)$.

Proof. Take $M \in D_{O-\mathrm{good}}(D_W)$. By [Sab11, Remark 3.3.4 (4)], the functor f_+ has finite cohomological dimension. So to prove $f_+ M \in D_{O-\mathrm{good}}(D_Z)$, by [Har66, I, Prop. 7.3 (iii)], one may assume that $M \in \mathrm{Mod}(D_W)$. Define a morphism $i : W \to W \times Z, \quad w \mapsto (w, f(w))$, which is a closed embedding. Let $q : W \times Z \to Z$ be the projection. By [Sab11, Thm. 3.3.6 (1)], one has $f_+ = q_+ i_+$. The restriction $q|_W : W \to Z$ is proper. By [Bjö93, Prop. 2.4.8], one has $f_+ M = Rq_* DR_{W\times Z/Z}(i_+ M)[\dim Z]$. As each term of the (relative) de Rham complex $DR_{W\times Z/Z}(i_+ M)$ is $O_{W\times Z}$-good and supported on W, by Theorem 5.3.1.7, $Rq_*[DR_{W\times Z/Z}(i_+ M)] \in D_{\mathrm{good}}(O_Z)$. □

From [HT07, Rk. 1.5.10], for a closed embedding $i : M \to N$ of complex manifolds, the inverse image $i^* : \mathrm{Mod}(D_N) \to \mathrm{Mod}(D_M)$ may not preserve D-coherence. For smooth morphisms, Fact 6.6.2.5 can be proved by applying [Kas03, Thm. 4.7] or repeating the proof of [HT07, Prop. 1.5.13 (ii)].

Fact 6.6.2.5. *Let* $f : M \to N$ *be a smooth morphism of complex manifolds. Then* $Lf^* : D^b(D_N) \to D^b(D_M)$ *restricts to functors* $D^b_c(D_N) \to D^b_c(D_M)$ *and* $D^b_{\mathrm{good}}(D_N) \to D^b_{\mathrm{good}}(D_M)$.

Lemma 6.6.2.6 concerns the local existence of good filtrations on coherent $\mathcal{A}_X$-modules.

Lemma 6.6.2.6. *Let M be a coherent $\mathcal{A}_X$-module on the complex torus X. For every $x \in X$, there is an open neighborhood U of x and a positive good filtration on $M|_U$.*

Proof. Let $\mathcal{A}_X^q|_U \xrightarrow{\phi} \mathcal{A}_X^p|_U \xrightarrow{\epsilon} M|_U \to 0$ be a local presentation of M on a relatively compact open neighborhood U of x. For every integer v, set $M_v = \epsilon(\mathcal{A}_X(v)^p)$, which is an O_U-submodule of $M|_U$. Then $M_v = 0$ when $v < 0$. Moreover, $\cup_{v\in\mathbb{Z}} M_v = M|_U$ and for any integers $m, k \geq 0$, one has $\mathcal{A}_X(m)M_k \subset M_{k+m}$. Thus, $\{M_v\}_{v\in\mathbb{Z}}$ is a positive filtration of $M|_U$. For every integer $k \geq 0$, one has $\mathcal{A}_X(k)M_0 = M_k$. It remains to prove that M_k is coherent over O_U.

We claim that $\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p$ is coherent over O_U. In fact, for every $y \in U$, there is an integer $s \geq \max(0, k-m)$ such that $\phi(\mathcal{A}_X(m)^q) \subset \mathcal{A}_X(m+s)^p$ near y. In side the coherent O_X-module $\mathcal{A}_X(m+s)^p$, the two O_X-submodules $\phi(\mathcal{A}_X(m)^q)$ and $\mathcal{A}_X(k)^p$ are finite type. By [Sta24, Tag 01BY], their intersection $\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p$ is coherent near y. The claim is proved.

Because $\mathcal{A}_X(k)^p$ is a noetherian O_X-module, the increasing sequence of submodules $\{\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p\}_{m\geq 0}$ is stationary on U. Therefore, the union $\phi(\mathcal{A}_X^q) \cap \mathcal{A}_X(k)^p = \ker(\epsilon) \cap \mathcal{A}_X(k)^p$ is coherent over O_U. Since the sequence

$$0 \to \ker(\epsilon) \cap \mathcal{A}_X(k)^p \to \mathcal{A}_X(k)^p \to M_k|_U \to 0$$

is exact in $\mathrm{Mod}(O_U)$, the restriction $M_k|_U$ is O_U-coherent. The constructed filtration is therefore good. □

When $\mathcal{R} = D_Z$, Lemma 6.6.2.7 is [Sab11, Exercise E.2.4 (4)]. On a complex manifold Z, an O_Z-module F is *pseudo-coherent* if for every open subset U of X, every finite type O_U-submodule of $F|_U$ is of finite presentation ([Kas03, Def. A.5]).

Lemma 6.6.2.7. *If M is a coherent $\mathcal{R}$-module, then M is pseudo-coherent over O_Z.*

Proof. Let $F \subset M$ be a finite type O-submodule. For every point x, by [Meb89, Prop. 2.1.9] (in the case $\mathcal{R} = D_Z$) and Lemma 6.6.2.6 (in the case $\mathcal{R} = \mathcal{A}_X$), there exists an open neighborhood U of x and a good filtration on $M|_U$. By [Bjö93, Cor. 1.4.6] (in the case $\mathcal{R} = D_Z$) and Lemma 6.6.1.2 (in the case $\mathcal{R} = \mathcal{A}_X$), $M|_U$ is the sum of an increasing sequence of coherent O_U-submodules. Hence $M|_U$ is good over O_U. By Lemma A.1.4.2 1, the O_U-module $M|_U$ is pseudo-coherent. As pseudo-coherence is a local property, M is pseudo-coherent over O_Z. □

Lemma 6.6.2.8. *Let M be a good $\mathcal{R}$-module. Let N be a finite type $\mathcal{R}$-submodule of M. Then N is good over $\mathcal{R}$.*

Proof. By [Sta24, Tag 01BY (1)], N is coherent over $\mathcal{R}$. For every relatively compact open subset U of X and every $x \in \bar{U}$, there is an open neighborhood $U(x) \subset X$ of x, an integer $n(x) > 0$ and sections $\{s_i(x)\}_{i=1}^{n(x)} \subset \Gamma(U(x), N)$ generating the $\mathcal{R}|_{U(x)}$-module $N|_{U(x)}$. The open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{j=1}^m$. Let N_0 be the O_U-submodule of $N|_U$ generated by the finitely many local sections

$$\{s_i(x_j)\}_{1\le j\le m, 1\le i\le n(x_j)}.$$

Then N_0 is a finite type O_U-module. Because $M|_U$ is good over $\mathcal{R}|_U$, by Lemma 6.6.2.7, the O_U-module N_0 is coherent. By construction, one has $\mathcal{R}|_U \cdot N_0 = N|_U$. Therefore, the $\mathcal{R}$-module N is good by Propositions 6.6.1.4 (in the case $\mathcal{R} = D_Z$) and 6.6.1.5 (in the case $\mathcal{R} = \mathcal{A}_X$). □

6.6.3 Preservation of goodness

Theorem 6.6.3.1. *The functor $RS_1 : D(\mathcal{A}_X) \to D(D_Y)$ restricts to an equivalence $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$, with a quasi-inverse $T^g RS_2 : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$.*

Proof. 1. For every coherent O_Y-module F, one has $RS_2(D_Y \otimes^L_{O_Y} F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

By Proposition 6.5.1.5, one has $RS_2(D_Y \otimes^L_{O_Y} F) = \mathcal{A}_X \otimes^L_{O_X} R\mathcal{S}_2(F)$. By Fact 6.1.2.1 2, one has $R\mathcal{S}_2(F) \in D^b_c(O_X)$. From Lemma 6.6.2.1, one gets $\mathcal{A}_X \otimes^L_{O_X} R\mathcal{S}_2(F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

2. For every $M \in \mathrm{Good}(D_Y)$ and every integer i, the $\mathcal{A}_X$-module $H^i RS_2(M)$ is good.

Descending induction on $i \in \mathbb{Z}$. The O_X-module underlying $H^i RS_2(M)$ is $H^i R\mathcal{S}_2(M)$. By Lemma 6.6.3.2, one has $H^i R\mathcal{S}_2(M) = 0$ when $i > 2g$. In particular, $H^i RS_2(M)$ is good over $\mathcal{A}_X$.

Assume the statement for $i+1$. By Proposition 6.6.1.4, there is a coherent O_Y-submodule $F \subset M$ with $D_Y \cdot F = M$. Let M' be the kernel of the natural epimorphism $D_Y \otimes_{O_Y} F \to M$. Then

$$0 \to M' \to D_Y \otimes_{O_Y} F \to M \to 0 \tag{6.33}$$

is a short exact sequence in $\mathrm{Mod}(D_Y)$. By Lemma 6.6.2.1, the D_Y-module $D_Y \otimes_{O_Y} F$ is good. By Lemma 6.6.2.2, so is M'. From (6.33), one has an exact sequence in $\mathrm{Mod}(\mathcal{A}_X)$

$$\begin{aligned} &H^i RS_2(M') \to H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \\ \to &H^{i+1} RS_2(M') \to H^{i+1} RS_2(D_Y \otimes_{O_Y} F). \end{aligned} \tag{6.34}$$

By 1, the $\mathcal{A}_X$-module $H^j RS_2(D_Y \otimes_{O_Y} F)$ is good for $j \in \{i, i+1\}$. By the inductive hypothesis, so is $H^{i+1}RS_2(M')$.

Let $G = \ker[H^{i+1}RS_2(M') \to H^{i+1}RS_2(D_Y \otimes_{O_Y} F)]$. By Lemma 6.6.2.2, the $\mathcal{A}_X$-module G is good (hence of finite type). The sequence (6.34) yields an exact sequence

$$H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \to G \to 0,$$

so $H^i RS_2(M)$ is a finite type $\mathcal{A}_X$-module for every coherent D_Y-module M. In particular, $H^i RS_2(M')$ is a finite type $\mathcal{A}_X$-module.

Let $N = \operatorname{im}(H^i RS_2(M') \to H^i RS_2(D_Y \otimes_{O_Y} F))$. It is a finite type $\mathcal{A}_X$-submodule of the good $\mathcal{A}_X$-module $H^i RS_2(D_Y \otimes_{O_Y} F)$. By Lemma 6.6.2.8, the $\mathcal{A}_X$-module N is a good. The sequence (6.34) yields an exact sequence

$$0 \to N \to H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \to H^{i+1}RS_2(M') \to H^{i+1}RS_2(D_Y \otimes_{O_Y} F).$$

By Lemma 6.6.2.2, the $\mathcal{A}_X$-module $H^i RS_2(M)$ is good. The induction is completed.

From 2, Lemma 6.6.2.2 and [Har66, I, Prop. 7.3 (i)], the functor $R\mathcal{S}_2$ restricts to a functor $D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$. Similarly, using Proposition 6.6.1.5, one can prove that RS_1 restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$. By Theorem 6.5.1.4, the restrictions are equivalences. $\square$

The proof of Theorem 6.6.3.1 needs a cohomological dimension estimation.

Lemma 6.6.3.2. *For an O_X-module F, one has $R\mathcal{S}_1(F) \in D^{[0,2g]}(O_Y)$. Similarly, for an O_Y-module G, one has $R\mathcal{S}_2(G) \in D^{[0,2g]}(O_X)$.*

Proof. By left exactness of the functor $p_{Y*} : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_Y)$, one has $R^i\mathcal{S}_1(F) = 0$ for every integer $i < 0$. For every $y \in Y$, let M be the restriction (as sheaves) of $\mathcal{P} \otimes_{O_{X\times Y}} p_X^* F$ to $X \times y$. For every integer j, by the proper base change theorem (see *e.g.*, [Mil13, Thm. 17.2]), one has $R^j\mathcal{S}_1(F)_y = H^j(X \times y, M)$. When $j > 2g$, by [KS90, Prop. 3.2.2 (iv)], one has $H^j(X \times y, M) = 0$. Therefore, $R^j\mathcal{S}_1(F) = 0$. The other part is similar. $\square$

6.7 Relations with other functors

The properties [Muk81, (3.1), (3.4), (3.8)] of the Fourier-Mukai transform have analogs for the Laumon-Rothstein transform.

6.7.1 Duality

Let Z be a complex manifold. Denote by Δ^{O_Z} the duality (contravariant) functor $R\mathcal{H}om_{O_Z}(\cdot, \omega_Z^{-1})[\dim Z] : D^b_c(O_Z) \to D^b_c(O_Z)$. The duality functor on D_Z-modules $\Delta^{D_Z} : D(D_Z) \to D(D_Z)$ is defined by $\Delta^{D_Z} F = G[\dim Z]$, where G is the complex of left D_Z-modules associated with the complex $R\mathcal{H}om_{D_Z}(F, D_Z)$ of *right* D_Z-modules. By [Bjö93, Def. 2.11.1], Δ^{D_Z} restricts to a functor $D^b_c(D_Z) \to D^b_c(D_Z)$, and the natural transformation $\mathrm{Id} \to \Delta^{D_Z} \circ \Delta^{D_Z}$ is an isomorphism of functors $D^b_c(D_Z) \to D^b_c(D_Z)$.

Lemma 6.7.1.1 ([KS16, p.16])**.** *The functor $\Delta^{D_Z} : D(D_Z) \to D(D_Z)$ restricts to a functor $D^b_{\mathrm{good}}(D_Z) \to D^b_{\mathrm{good}}(D_Z)$.*

Proof. Suppose F is a coherent O_Z-module and $N = D_Z \otimes_{O_Z} F$, then by [Bjö93, (ii), p.122], there is $G \in D^b_c(O_Z)$ with $\Delta^{D_Z} N = D_Z \otimes_{O_Z} G$. By Lemma 6.6.2.1, $\Delta^{D_Z} N \in D^b_{\mathrm{good}}(D_Z)$.

Take $M \in D^b_{\mathrm{good}}(D_Z)$. To prove $\Delta^{D_Z} M \in D^b_{\mathrm{good}}(D_Z)$, by [Har66, I, Prop. 7.3 (i)], one may assume $M \in \mathrm{Good}(D_Z)$. By Proposition 6.6.1.4 and [Bjö93, Thm. 1.5.8], for every relatively compact open subset $U \subset Z$, there is a finite length exact sequence in $\mathrm{Mod}(D_U)$:

$$0 \to D_U \otimes_{O_U} F^{-n} \to \cdots \to D_U \otimes_{O_U} F^0 \to M|_U \to 0,$$

where each F^i is a coherent O_U-module. For every i, one has $\Delta^{D_U}(D_U \otimes_{O_U} F^i) \in D^b_{\mathrm{good}}(D_U)$. By Lemma 6.6.2.2, one has $(\Delta^{D_Z} M)|_U = \Delta^{D_U}(M|_U) \in D^b_{\mathrm{good}}(D_U)$. Hence $\Delta^{D_Z} M \in D^b_{\mathrm{good}}(D_Z)$. $\square$

For algebraic varieties, an analogue of Fact 6.7.1.2 is stated as [HT07, Cor. 2.6.8 (iii), Prop. 3.2.1]. From [HT07, p.101], all the arguments in [HT07, Sec. 2.6] are valid for analytic D-modules.

Fact 6.7.1.2.

1. *The contravariant functor $\Delta^{D_Z} : D^b_h(D_Z) \to D^b_h(D_Z)$ an equivalence.*
2. *Let M be a coherent D_Z-module. Then M is holonomic if and only if $H^i(\Delta^{D_Z} M) = 0$ for all integers $i \neq 0$.*
3. [Bjö93, Thm. 3.2.13 (3)] *The bifunctor $- \otimes^L_{O_Z} + : D^b(D_Z) \times D^b(D_Z) \to D^b(D_Z)$ restricts to a bifunctor $D^b_h(D_Z) \times D^b_h(D_Z) \to D^b_h(D_Z)$.*

Fact 6.7.1.3. *Let $f : W \to Z$ be a morphism of complex manifolds. Then:*

1. [Bjö93, Thm. 3.2.13 (1)] *The inverse image $Lf^* : D^b(D_Z) \to D^b(D_W)$ restricts to a functor $D^b_h(D_Z) \to D^b_h(D_W)$.*
2. [Sab11, Thm. 4.4.1] *If $F \in D^b_h(D_W)$ is such that $f|_{\mathrm{Supp}(F)}$ is proper, then $f_+ F \in D^b_h(D_Z)$.*

Restricted to the complex torus Y, [Bjö93, (ii), p.122] becomes [Rot96, (6.12)]:

$$\Delta^{D_Y}(D_Y \otimes^L_{O_Y} \cdot) \cong D_Y \otimes^L_{O_Y} \Delta^{O_Y} \cdot : D^b_c(O_Y) \to D^b_c(D_Y).$$

Define the duality (contravariant) functor $\Delta^{\mathcal{A}_X} : D^b(\mathcal{A}_X) \to D^b(\mathcal{A}_X)$ as

$$\Delta^{\mathcal{A}_X} = T^g R\mathcal{H}om_{\mathcal{A}_X}(\cdot, \mathcal{A}_X).$$

It restricts to a functor $D^b_c(\mathcal{A}_X) \to D^b_c(\mathcal{A}_X)$. Similar to Lemma 6.7.1.1, it restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$. Theorem 6.7.1.4 follows from Proposition 6.7.1.5 and Fact 6.7.1.2 2, in the same way how Theorem 6.5 follows from Propositions 6.3 and 6.4 in [Rot96].

Theorem 6.7.1.4 (Rothstein)**.** *Let $F \in D^b_{\mathrm{good}}(\mathcal{A}_X)$ be an object such that $RS_1(F)$ is concentrated in a single degree $i \in \mathbb{Z}$. Then $H^i RS_1(F)$ is holonomic if and only if $RS_1 \Delta^{\mathcal{A}_X} F$ is concentrated in degree $g - i$.*

Proposition 6.7.1.5 can be deduced from Corollary 6.7.1.8, Proposition 6.5.1.5 and Proposition 5.5.1.8, in the same way that [Rot96, Prop. 6.3] is proved.

Proposition 6.7.1.5. *One has*

$$RS_2 \Delta^{D_Y} = [-1]^*_X T^{-g} \Delta^{\mathcal{A}_X} RS_2 : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X),$$
$$\Delta^{D_Y} RS_1 = [-1]^*_Y T^g RS_1 \Delta^{\mathcal{A}_X} : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y).$$

Remark 6.7.1.6. Both [Rot96, (6.13), (6.14)] miss a factor $[-1]^*$, due to a missing $[-1]^*_X$ in [Rot96, (6.15)]. Still, this sign does not affect the statement of [Rot96, Thm. 6.5].

Lemma 6.7.1.7 ([Huy06, (3.13)])**.** *For any objects $K, L \in D(O_Z)$ and $M \in D^-_c(O_Z)$, the natural morphism* (provided by [Sta24, Tag 0BYS])

$$K \otimes^L_{O_Z} R\mathcal{H}om_{O_Z}(M, L) \to R\mathcal{H}om_{O_Z}(M, K \otimes^L_{O_Z} L) \tag{6.35}$$

is an isomorphism in $D(O_Z)$.

Proof. By [Har66, I, Prop. 7.1 (ii)], one may assume that $M \in \mathrm{Coh}(O_Z)$. By [Sta24, Tag 08DL] and [GH78, p.696], one may shrink Z such that M admits a globally free resolution $F \to M$, where the complex F is

$$0 \to O_Z^{k_n} \to \cdots \to O_Z^{k_1} \to O_Z^{k_0} \to 0$$

with $O_Z^{k_i}$ placed in degree $-i$. The morphism (6.35) becomes

$$K \otimes^L_{O_Z} \mathcal{H}om_{O_Z}(F, L) \to \mathcal{H}om_{O_Z}(F, K \otimes^L_{O_Z} L),$$

which is an isomorphism. $\square$

Corollary 6.7.1.8 proves the analytic counterpart of [Rot96, (6.12)].

Corollary 6.7.1.8. *There is a canonical isomorphism $\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot) \cong \mathcal{A}_X \otimes^L_{O_X} \Delta^{O_X} \cdot$ of functors $D^b_c(O_X) \to D^b_c(\mathcal{A}_X)$.*

Proof. By [Rot96, (6.2)], one has

$$\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot) = T^g R\mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot, \mathcal{A}_X) = T^g R\mathcal{H}om_{O_X}(\cdot, \mathcal{A}_X).$$

By Lemma 6.7.1.7, it equals $T^g R\mathcal{H}om_{O_X}(\cdot, O_X) \otimes^L_{O_X} \mathcal{A}_X = \mathcal{A}_X \otimes^L_{O_X} \Delta^{O_X} \cdot$. $\square$

Example 6.7.1.9. Let $F = T^g\mathcal{A}_X \in D^b_{\text{good}}(\mathcal{A}_X)$. By Corollary 6.5.1.6, one has $RS_1(F) = D_Y \otimes_{O_Y} \mathbb{C}_0$. One has $\Delta^{\mathcal{A}_X} F = \mathcal{A}_X$, and $RS_1\Delta^{\mathcal{A}_X} F$ is concentrated in degree g. Then by Theorem 6.7.1.4, the D_Y-module $D_Y \otimes_{O_Y} \mathbb{C}_0$ is holonomic.

Example 6.7.1.9 leads to a question: When is an induced D-module holonomic? A full answer is in Proposition E.1.0.12.

Remark 6.7.1.10. There is seemingly a paradox. Suppose $g = 1$ and let $i : 0 \to Y$ be the inclusion. Then the O_Y-modules pullback $i^*\mathbb{C}_0 = \mathbb{C}$ and $i^*(D_Y \otimes_{O_Y} \mathbb{C}_0) = (i^*D_Y) \otimes_{\mathbb{C}} (i^*\mathbb{C}_0) = i^*D_Y$ is the fiber of D_Y at 0, which is nonzero. On the other hand, by [Bjö93, p.87] the derived inverse image $i^+(D_Y \otimes_{O_Y} \mathbb{C}_0)$ is a complex of $D_0 = \mathbb{C}$-modules concentrated in degree -1. From [Bjö93, 2.3.7], its underlying complex of $O_0 = \mathbb{C}$-modules is $Li^*(D_Y \otimes_{O_Y} \mathbb{C}_0)$. Its 0-th cohomology $i^*(D_Y \otimes_{O_Y} \mathbb{C}_0)$ is zero, a contradiction! We suggest catching the mistake.

In fact, D_Z has two different structures of O_Z-modules. Consider local sections h (resp. δ) of O_Z (resp. D_Z). One module structure defines $h \cdot \delta$ as $h\delta$, the product in D_Z. This O_Z-module is denoted by D^l_Z. Then $\text{for}_Z(D_Z) = D^l_Z$.

The other O_Z-module structure defines $h \cdot \delta$ as the reversed product δh in D_Z. Denote this O_Z-module by D^r_Z. Given an O_Z-module F, which one is used in the tensor product defining the induced left D_Z-module $D_Z \otimes_{O_Z} F$? In fact, it relies on the (D_Z, O_Z)-bimodule structure on D^r_Z. But $M := \text{for}_Z(D_Z \otimes_{O_Z} F)$ is NOT the O_Z-module tensor product of F with neither D^r_Z nor D^l_Z.

Return to the special case $F = \mathbb{C}_0 = O_Y/I$ on Y, where $I \subset O_Y$ is the coherent ideal sheaf corresponding to the closed embedding $i : 0 \to Y$. Take a local coordinate z around $0 \in Y$ such that $O_{Y,0} = \mathbb{C}\{z\}$ and the maximal ideal $m_0 = (z) \subset O_{Y,0}$. Let ∂ be the corresponding local vector field near $0 \in Y$. Let $M = \text{for}_Y(D_Y/D_Y I)$. The $\mathbb{C}$-vector space $M_0 = \mathbb{C}[\partial]$, and its $O_{Y,0}$-action is defined by $z \cdot v = zv$ for all $v \in M_0$. The O_0-module $\text{for}_0(H^0 i^+(D_Y \otimes_{O_Y} \mathbb{C}_0)) = i^*M = M_0/m_0M_0$.

By $[\partial, z] = 1$, for every integer $k \geq 0$, one has $\partial^k = \frac{1}{k+1}(\partial^{k+1}z - z\partial^{k+1}) \in D_{Y,0}m_0 + m_0 D_{Y,0}$. So, $M_0/m_0M_0 = 0$, even though the O-module pullback to 0 of both $D^l_Y \otimes_{O_Y} \mathbb{C}_0$ and $D^r_Y \otimes_{O_Y} \mathbb{C}_0$ are nonzero.

6.7.2 Pullback and pushout

Proposition 6.7.2.1 ([Lau96, Prop. 3.3.2]). *Let $f : X' \to X$ be a morphism of complex tori, with $\dim X' = g'$. Let $\hat{f} : Y \to Y'$ be the morphism dual to f. Let $\tilde{f} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X)$ be the induced morphism (6.25). Then there are canonical isomorphisms of functors*

1.

$$L\hat{f}^* RS'_1 \cong RS_1 R\tilde{f}_* : D_{O-\text{good}}(\mathcal{A}_{X'}) \to D_{O-\text{good}}(D_Y); \tag{6.36}$$

$$R\tilde{f}_* RS'_2 \cong T^{g-g'} RS_2 L\hat{f}^* : D_{O-\text{good}}(D_{Y'}) \to D_{O-\text{good}}(\mathcal{A}_X). \tag{6.37}$$

2.

$$RS'_2 \hat{f}_+ \cong L\tilde{f}^* RS_2 : D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(\mathcal{A}_{X'}); \tag{6.38}$$

$$\hat{f}_+ RS_1 \cong T^{g'-g} RS'_1 L\tilde{f}^* : D^b_{\text{good}}(\mathcal{A}_X) \to D^b_{\text{good}}(D_{Y'}). \tag{6.39}$$

Proof. 1. The isomorphism (6.37) follows from (6.36) as follows:

$$\begin{aligned} R\tilde{f}_* RS'_2 &\overset{(a)}{\cong} T^g RS_2 RS_1 R\tilde{f}_* RS'_2 \\ &\overset{(b)}{\cong} T^g RS_2 L\hat{f}^* RS'_1 RS'_2 \\ &\overset{(c)}{\cong} T^{g-g'} RS_2 L\hat{f}^*, \end{aligned}$$

where (6.36) (resp. Theorem 6.5.1.4) is used in (b) (resp. (a) and (c)). Then we prove (6.36).

By (6.26) (resp. the proof of [HT07, Prop. 1.5.8]), the derived direct image (resp. inverse image) functor of $\mathcal{A}$-modules (resp. D-modules) regards that of the underlying O-modules. From Proposition 5.3.1.2 2, the functor $\mathcal{P}' \otimes^L_{O_{X'\times Y'}} p^*_{X'} \cdot : D(\mathcal{A}_{X'}) \to D(O_{X'\times Y'})$ restricts to a functor $D_{O-\text{good}}(\mathcal{A}_{X'}) \to D_{\text{good}}(O_{X'\times Y'})$. An application of Lemma 5.3.2.13 to the cartesian square

$$\begin{tikzcd} X' \times Y \arrow[r, "1_{X'}\times\hat{f}"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X' \times Y' \arrow[d, "p_{Y'}"] \\ Y \arrow[r, "\hat{f}"'] & Y' \end{tikzcd}$$

yields a canonical isomorphism of functors

$$L\hat{f}^* Rp_{Y'} \to Rp_{2*} L(1_{X'} \times \hat{f})^* : D_{\text{good}}(O_{X'\times Y'}) \to D_{\text{good}}(O_Y). \tag{6.40}$$

Applying Theorem 5.3.2.3 to the cartesian square

$$\begin{tikzcd} X' \times Y \arrow[r, "p_1"] \arrow[d, "f\times 1_Y"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f"] \\ X \times Y \arrow[r, "p_X"'] & X, \end{tikzcd}$$

of complex manifolds, one gets a natural isomorphism

$$p_X^* R\tilde{f}_* \to R(f \times 1_Y)_* p_1^* \tag{6.41}$$

of functors $D_{O-\text{good}}(\mathcal{A}_{X'}) \to D(\mathrm{Mod}(O_{X\times Y})_{1-\text{cxn,fl}})$.

Then

$$\begin{aligned}
L\hat{f}^* RS_1' =& L\hat{f}^* Rp_{Y'}(\mathcal{P}' \otimes^L_{O_{X'\times Y'}} p_{X'}^* \cdot) \\
&\overset{(a)}{\cong} Rp_{2*} L(1_{X'} \times \hat{f})^*(\mathcal{P}' \otimes^L_{O_{X'\times Y'}} p_{X'}^* \cdot) \\
&\cong Rp_{2*}[L(1_{X'} \times \hat{f})^* \mathcal{P}' \otimes^L_{O_{X'\times Y}} L(1_{X'} \times \hat{f})^* p_{X'}^* \cdot] \\
&\cong Rp_{2*}[(1_{X'} \times \hat{f})^* \mathcal{P}' \otimes^L_{O_{X'\times Y}} p_1^* \cdot] \\
&\overset{(b)}{\cong} Rp_{2*}[(f \times 1_Y)^* \mathcal{P} \otimes^L_{O_{X'\times Y}} p_1^* \cdot] \\
&\cong Rp_{Y*} R(f \times 1_Y)_*[(f \times 1_Y)^* \mathcal{P} \otimes^L_{O_{X'\times Y}} p_1^* \cdot] \\
&\overset{(c)}{\cong} Rp_{Y*}[\mathcal{P} \otimes^L_{O_{X\times Y}} R(f \times 1_Y)_* p_1^* \cdot] \\
&\overset{(d)}{\cong} Rp_{Y*}[\mathcal{P} \otimes^L_{O_{X\times Y}} p_X^* R\tilde{f}_* \cdot] \\
&= RS_1 R\tilde{f}_*,
\end{aligned}$$

where (a), (b), (c) and (d)) use (6.40), (5.26), Fact 5.3.2.15 and (6.41) respectively. This proves (6.36).

2. The isomorphism (6.39) follows from (6.38) as follows:

$$\begin{aligned}
\hat{f}_+ RS_1 &\overset{(a)}{\cong} T^{g'} RS_1' RS_2' \hat{f}_+ RS_1 \\
&\overset{(b)}{\cong} T^{g'} RS_1' L\tilde{f}^* RS_2 RS_1 \\
&\overset{(c)}{\cong} T^{g'-g} RS_1' L\tilde{f}^*,
\end{aligned}$$

where (a) and (c) use Theorem 6.6.3.1, and (b) uses (6.38). Then we prove (6.38).

Using (6.27), one can prove that $L\tilde{f}^* : D(\mathcal{A}_X) \to D(\mathcal{A}_{X'})$ restricts to a functor $D^b_{\text{good}}(\mathcal{A}_X) \to D^b_{\text{good}}(\mathcal{A}_{X'})$. From Fact 6.6.2.3, the direct image functor $\hat{f}_+ : D^b(D_Y) \to D^b(D_{Y'})$ restricts to a functor $D^b_{\text{good}}(D_Y) \to$

$D^b_{\text{good}}(D_{Y'})$. There are canonical isomorphism of bifunctors $D^b_{\text{good}}(D_Y)^{\text{op}} \times D^b_{\text{good}}(\mathcal{A}_{X'}) \to \text{Ab}$:

$$\begin{aligned}
\operatorname{Hom}_{D^b_{\text{good}}(\mathcal{A}_{X'})}(RS_2'\hat{f}_+-,+) &\overset{\text{(a)}}{\cong} \operatorname{Hom}_{D^b_{\text{good}}(D_{Y'})}(\hat{f}_+-, T^{g'}RS_1'+)\\
&\overset{\text{(b)}}{\cong} \operatorname{Hom}_{D^b_{\text{good}}(D_Y)}(-, T^g L\hat{f}^* RS_1'+)\\
&\overset{\text{(c)}}{\cong} \operatorname{Hom}_{D^b_{\text{good}}(D_Y)}(-, T^g RS_1 R\tilde{f}_*+)\\
&\overset{\text{(d)}}{\cong} \operatorname{Hom}_{D^b_{\text{good}}(\mathcal{A}_X)}(RS_2-, R\tilde{f}_*+)\\
&\cong \operatorname{Hom}_{D^b_{\text{good}}(\mathcal{A}_{X'})}(L\tilde{f}^* RS_2-,+),
\end{aligned}$$

where (a) and (d) use Theorem 6.6.3.1, (b) uses [Bjö93, Thm. 2.11.8], and (c) uses (6.36). From Yoneda's lemma, there is a canonical isomorphism $RS_2'\hat{f}_+ \cong L\tilde{f}^* RS_2$ of functors $D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(\mathcal{A}_{X'})$.

□

6.7.3 External tensor product

For two complex manifolds U, V, recall the (exact) external tensor product bifunctor

$$(\cdot) \boxtimes_O (\cdot) : \operatorname{Mod}(D_U) \times \operatorname{Mod}(D_V) \to \operatorname{Mod}(D_{U\times V}) \tag{6.42}$$

defined in [Bjö93, 2.4.4]. By exactness, it descends to

$$D(D_U) \times D(D_V) \to D(D_{U\times V}). \tag{6.43}$$

Remark 6.7.3.1. By [Bjö93, 2.4.13], the bifunctor (6.42) restricts to bifunctors $\operatorname{Coh}(D_U)\times\operatorname{Coh}(D_V) \to \operatorname{Coh}(D_{U\times V})$ and $\operatorname{Good}(D_U)\times\operatorname{Good}(D_V) \to \operatorname{Good}(D_{U\times V})$. Then by [Har66, I, Prop. 7.3 (i)], the bifunctor (6.43) restricts to bifunctors $D^b_c(D_U)\times D^b_c(D_V) \to D^b_c(D_{U\times V})$ and $D^b_{\text{good}}(D_U)\times D^b_{\text{good}}(D_V) \to D^b_{\text{good}}(D_{U\times V})$. By [Bjö93, p.139], it also restricts to a bifunctor $D^b_h(D_U) \times D^b_h(D_V) \to D^b_h(D_{U\times V})$.

Using Lemma 5.5.1.6 (at the place of [HT07, Lem. 1.5.31]), Lemma 6.6.2.4 and [Sab11, Thm. 3.3.6 (1)], one can argue as in [HT07, Prop. 1.5.30] to get Fact 6.7.3.2.

Fact 6.7.3.2.

1. *Let U, V, Z be complex manifolds. Let $f : U \to V$ be a* proper *morphism. Then the natural transformation*

   $$f_+(-)\boxtimes_O(+) \to (f\times\operatorname{Id}_Z)_+(-\boxtimes_O+) : D_{O-\text{good}}(D_U)\times D(D_Z) \to D(D_{V\times Z})$$

   is an isomorphism.

2. *Let $f_i : U_i \to V_i$ ($i = 1, 2$) be two* proper *morphisms of complex manifolds. Then the natural transformation $(f_{1+}-) \boxtimes_O (f_{2+}+) \to (f_1 \times f_2)_+(- \boxtimes_O +)$ of bifunctors*

$$D_{O-\mathrm{good}}(D_{U_1}) \times D_{O-\mathrm{good}}(D_{U_2}) \to D_{O-\mathrm{good}}(D_{V_1 \times V_2})$$

is an isomorphism.

For a complex torus X, let $\mathrm{for}_X : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)$ be the forgetful functor. Let X' be another complex torus. Set $X'' = X \times X'$. Write $u : X'' \to X$ and $u' : X'' \to X'$ for the projections. Let Y', Y'' be the dual of X' and X'' respectively. For an $\mathcal{A}_X$-module F and an $\mathcal{A}_{X'}$-module G, denote $\tilde{u}^* F \otimes_{\mathcal{A}_{X''}} \tilde{u'}^* G$ by $F \boxtimes_{\mathcal{A}} G$. By construction, one has

$$\begin{aligned} \mathcal{A}_{X''} =& \mathcal{A}_X \boxtimes_O \mathcal{A}_{X'} = u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^* \mathcal{A}_{X'} \\ =& O_{X \times X'} \otimes_{u^{-1}O_X \otimes_{\mathbb{C}} u'^{-1}O_{X'}} (u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}). \end{aligned} \tag{6.44}$$

Since the $u^{-1}O_X \otimes_{\mathbb{C}} u'^{-1}O_{X'}$-module $O_{X \times X'}$ is flat, so is the $u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}$-module $\mathcal{A}_{X''}$. As

$$F \boxtimes_{\mathcal{A}} G = \mathcal{A}_{X''} \otimes_{u^{-1}\mathcal{A}_X \otimes_{\mathbb{C}} u'^{-1}\mathcal{A}_{X'}} (u^{-1}F \otimes_{\mathbb{C}} u'^{-1}G),$$

the bifunctor

$$- \boxtimes_{\mathcal{A}} + : \mathrm{Mod}(\mathcal{A}_X) \times \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(\mathcal{A}_{X''})$$

is exact in both arguments. Consider the diagonal morphism $\delta : X \to X^2$. There is a canonical isomorphism of bifunctors

$$L\tilde{\delta}^*[- \boxtimes_{\mathcal{A}} +] \cong (-) \otimes^L_{\mathcal{A}_X} (+) : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X). \tag{6.45}$$

Although the tensor product of two $\mathcal{A}_X$-modules is different from the tensor product of the underlying O_X-module, Lemma 6.7.3.3 shows that external products do agree. It is used in the proof of Lemma 6.7.3.5.

Lemma 6.7.3.3. *There is a natural isomorphism of bifunctors*

$$\mathrm{for}_{X''}(-\boxtimes_{\mathcal{A}}+) \to (\mathrm{for}_X-)\boxtimes_O(\mathrm{for}_{X'}+) : \mathrm{Mod}(\mathcal{A}_X)\times\mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''}).$$

Proof. There are natural isomorphisms of functors $\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_{X''})$:

$$\begin{aligned} \mathrm{for}_{X''}\tilde{u}^* :=& u^{-1} \cdot \otimes_{u^{-1}\mathcal{A}_X} \mathcal{A}_{X''} \\ \overset{(a)}{=}& u^{-1} \cdot \otimes_{u^{-1}\mathcal{A}_X} (u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^* \mathcal{A}_{X'}) \\ \cong& u^{-1} \cdot \otimes_{u^{-1}O_X} u'^* \mathcal{A}_{X'} \\ \cong& (u^{-1} \cdot \otimes_{u^{-1}O_X} O_{X''}) \otimes_{O_{X''}} u'^* \mathcal{A}_{X'} \\ \cong& u^* \mathrm{for}_X \cdot \otimes_{O_{X''}} u'^* \mathcal{A}_{X'}, \end{aligned}$$

where (a) uses (6.44). Similarly, there is a natural isomorphism of functors $\mathrm{for}_{X''}\tilde{u'}^* \cong u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}\cdot : \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''})$. One has natural isomorphisms of bifunctors

$$\begin{aligned}
&\mathrm{for}_{X''}(-\boxtimes_{\mathcal{A}}+) := \tilde{u}^* - \otimes_{\mathcal{A}_{X''}} \tilde{u'}^* +\\
\cong&(u^*\mathrm{for}_X - \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}) \otimes_{u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}} (u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}+)\\
\cong&(u^*\mathrm{for}_X-) \otimes_{O_{X''}} (u'^*\mathrm{for}_{X'}+)\\
:=&(\mathrm{for}_X-) \boxtimes_O (\mathrm{for}_{X'}+).
\end{aligned}$$

□

Remark 6.7.3.4. Using the Laumon-Rothstein transform, we can reprove (6.44) as follows:

$$\begin{aligned}
\mathcal{A}_X \boxtimes_O \mathcal{A}_{X'} &\overset{\text{(a)}}{=} \mathrm{for}(RS_2(D_Y \otimes_{O_Y} \mathbb{C}_0)) \boxtimes_O \mathrm{for}(RS'_2(D_{Y'} \otimes_{O_{Y'}} \mathbb{C}_0))\\
&\overset{\text{(b)}}{=} R\mathcal{S}_2(D_Y \otimes_{O_Y} \mathbb{C}_0) \boxtimes_O R\mathcal{S}'_2(D_{Y'} \otimes_{O_{Y'}} \mathbb{C}_0)\\
&\overset{\text{(c)}}{=} R\mathcal{S}''_2((D_Y \otimes_{O_Y} \mathbb{C}_0) \boxtimes_O (D_{Y'} \otimes \mathbb{C}_0))\\
&= R\mathcal{S}''_2((D_Y \boxtimes_O D_{Y'}) \otimes_{O_{Y''}} (\mathbb{C}_0 \boxtimes_O \mathbb{C}_0))\\
&\overset{\text{(d)}}{=} R\mathcal{S}''_2(D_{Y''} \otimes \mathbb{C}_0)\\
&\overset{\text{(e)}}{=} \mathrm{for}(RS''_2(D_{Y''} \otimes \mathbb{C}_0))\\
&\overset{\text{(f)}}{=} \mathcal{A}_{X''},
\end{aligned}$$

where (a) and (f) use Corollary 6.5.1.6, (b) and (e) use Proposition 6.5.1.3, and (c) (resp. (d)) uses Proposition 5.5.1.5 (resp. [Bjö93, 2.4.4, (i)]).

Lemma 6.7.3.5. *There are canonical isomorphisms of bifunctors*

$$\begin{aligned}
&RS''_2[-\boxtimes_O +] \cong RS_2 - \boxtimes_{\mathcal{A}} RS'_2 + :\\
&D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_{Y'}) \to D_{O-\mathrm{good}}(\mathcal{A}_{X''}),
\end{aligned} \tag{6.46}$$

$$\begin{aligned}
&RS''_1[-\boxtimes_{\mathcal{A}} +] \cong RS_1 - \boxtimes_O RS'_1 + :\\
&D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_{X'}) \to D_{O-\mathrm{good}}(D_{Y''}).
\end{aligned} \tag{6.47}$$

Proof. It follows from Proposition 5.5.1.5, Lemma 6.7.3.3 and Proposition 6.5.1.3. □

6.7.4 Convolution and tensor product

For the dual complex tori X and Y, let $m : X^2 \to X$ and $\mu : Y^2 \to Y$ be their respective group law. Let $\tilde{m} : (X^2, \mathcal{A}_{X^2}) \to (X, \mathcal{A}_X)$ be the morphism of ringed spaces induced by m *via* Remark 6.4.1.3.

Definition 6.7.4.1 (Convolution, [Lau96, p.22])**.** Define bifunctors

$$*_D : D(D_Y) \times D(D_Y) \to D(D_Y), \quad - *_D + = \mu_+[- \boxtimes^L_O +],$$
$$*_\mathcal{A} : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X), \quad - *_\mathcal{A} + = R\tilde{m}_*[- \boxtimes^L_\mathcal{A} +].$$

As μ is proper, by Fact 6.6.2.3, Lemma 6.6.2.4 and Facts 6.7.1.2 and 6.7.1.3 2, the direct image μ_+ restricts to functors $D^b_{\mathrm{good}}(D_{Y^2}) \to D^b_{\mathrm{good}}(D_Y)$, $D_{O-\mathrm{good}}(D_{Y^2}) \to D_{O-\mathrm{good}}(D_Y)$ and $D^b_h(D_{Y^2}) \to D^b_h(D_Y)$. Together with Remark 6.7.3.1, this implies that the bifunctor $*_D$ restricts to bifunctors $D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_Y)$, $D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(D_Y)$ and $D^b_h(D_Y) \times D^b_h(D_Y) \to D^b_h(D_Y)$. As an example, there is a canonical isomorphism $\mathcal{B}_{\{y\}|Y} *_D (-) \to T^*_{-y}$ of functors $D(D_Y) \to D(D_Y)$.

Lemma 6.7.4.2. *The pair $(D(D_Y), *_D)$ is a symmetric tensor triangulated category* (in the sense of [Bal10, Def. 3]) *with unit $D_Y \otimes_{O_Y} \mathbb{C}_0$.*

Proof. Let $i : \mathrm{Specan}(\mathbb{C}) \to Y$ be the inclusion of $0 \in Y$. Then $D_Y \otimes_{O_Y} \mathbb{C}_0 = i_+\mathbb{C}$. There are canonical isomorphisms

$$\begin{aligned}
(i_+\mathbb{C}) *_D \cdot :=& \mu_+[(i_+\mathbb{C}) \boxtimes_O \cdot] \\
=& \mu_+[(i_+\mathbb{C}) \boxtimes_O (\mathrm{Id}_{Y+} \cdot)] \\
\overset{(a)}{\cong}& \mu_+(i \times \mathrm{Id}_Y)_+(\mathbb{C} \boxtimes_O \cdot) \\
\overset{(b)}{\cong}& \mathrm{Id}_{Y+} = \mathrm{Id}_{D(D_Y)}
\end{aligned}$$

of functors $D(D_Y) \to D(D_Y)$, where (a) and (b) use Fact 6.7.3.2 1 and [Sab11, Thm. 3.3.6 (1)] respectively, Therefore, $D_Y \otimes_{O_Y} \mathbb{C}_0$ is the unit. The other axioms can be verified as in [Wei07, pp. 10-11]. ☐

Proposition 6.7.4.3 ([Wei11])**.** *For every $M \in D^b_{\mathrm{good}}(D_Y)$, the functor $\cdot *_D M : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_Y)$ admits a right adjoint $([-1]^*_Y \Delta^{D_Y} M) *_D \cdot$.*

Proof. Define an automorphism $f : Y^2 \to Y^2$ of the complex torus Y^2 by $f(a, b) = (a + b, -a)$. Then $p_1 f = \mu$, $p_2 f = [-1]_Y p_1$ and $\mu f = p_2$. One has $Lf^* O_{Y^2} = O_{Y^2}$ in $D^b(D_{Y^2})$.

For any objects $F, G \in D^b_{\mathrm{good}}(D_Y)$, there are canonical bijections

$$\begin{aligned}
&\mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(F *_D M, G) := \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(\mu_+(F \boxtimes_O M), G) \\
&\overset{(a)}{=} \mathrm{Hom}_{D(D_{Y^2})}(F \boxtimes_O M, T^g \mu^* G) \\
&\overset{(b)}{=} \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, \Delta^{D_{Y^2}}(F \boxtimes_O M) \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
&\overset{(c)}{=} \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, (\Delta^{D_Y} F) \boxtimes_O (\Delta^{D_Y} M) \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
&:= \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, p_1^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^* \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
&= \mathrm{Hom}_{D(D_{Y^2})}(f^* O_{Y^2}, f^*[p_1^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^* \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G]) \\
&= \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, \mu^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_1^* [-1]^*_Y \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g p_2^* G) \\
&:= \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, T^g \mu^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} ([-1]^*_Y \Delta^{D_Y} M \boxtimes_O G)) \\
&\overset{(d)}{=} \mathrm{Hom}_{D(D_{Y^2})}(O_{Y^2}, T^g \Delta^{D_Y}(\mu^* F) \otimes^L_{O_{Y^2}} ([-1]^*_Y \Delta^{D_Y} M \boxtimes_O G)) \\
&\overset{(e)}{=} \mathrm{Hom}_{D(D_{Y^2})}(\mu^* F, T^g([-1]^*_Y \Delta^{D_Y} M \boxtimes_O G)) \\
&\overset{(f)}{=} \mathrm{Hom}_{D(D_Y)}(F, \mu_+([-1]^*_Y \Delta^{D_Y} M \boxtimes_O G)) \\
&\overset{(g)}{=} \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(F, ([-1]^*_Y \Delta^{D_Y} M) *_D G),
\end{aligned}$$

where (a), (c), (d), (f) and (g) use [Bjö93, Thm. 2.11.8], Proposition 6.7.4.4, [Kas03, Thm. 4.12], [Kas03, Thm. 4.40] and Lemma 6.7.1.1 in order, and both (b), (e) use [Kas03, (3.13)]. As the bijections are functorial in F and G, the adjunction follows. $\square$

The proof of Proposition 6.7.4.3 needs the commutativity of duality with external tensor product for D-modules.

Proposition 6.7.4.4. *Let Z_i ($i = 1, 2$) be two complex manifolds. Then there is a canonical isomorphism*

$$(\Delta^{D_{Z_1}} -) \boxtimes_O (\Delta^{D_{Z_2}} +) \to \Delta^{D_{Z_1 \times Z_2}}(- \boxtimes_O +) : D^b_c(D_{Z_1}) \times D^b_c(D_{Z_2}) \to D^b_c(D_{Z_1 \times Z_2})^{\mathrm{op}}.$$

Proof. For a complex manifold Z, the sheaf $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$ is naturally a $\mathbb{C}_Z$-algebra, and D_Z is naturally a left $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$-module. For $N_i \in D(D_{Z_i^{\mathrm{op}}})$, by [HT07, p.39], there is a natural isomorphism in $D(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$N_1 \boxtimes_O N_2 = (N_1 \boxtimes_{\mathbb{C}} N_2) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2}. \tag{6.48}$$

First, we construct the natural transformation. Take $M_i \in D^b_c(D_{Z_i})$.

Claim 6.7.4.5. Then there is a natural morphism in $D^b((D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes_{\mathbb{C}} R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \\ \to &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}). \end{aligned} \tag{6.49}$$

Claim 6.7.4.6. There is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \\ \to &R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1 \times Z_2}). \end{aligned} \tag{6.50}$$

Again, there is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$R\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1 \times Z_2}) \to R\mathcal{H}om_{D_{Z_1 \times Z_2}}(M_1 \boxtimes_O M_2, D_{Z_1 \times Z_2}), \tag{6.51}$$

which can be defined by taking a $D_{Z_1 \times Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1 \times Z_2}$-injective resolution of $D_{Z_1 \times Z_2}$.

Composing the morphisms (6.48), (6.49), (6.50) and (6.51) in order, one gets a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1 \times Z_2})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes_O R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \\ \to &R\mathcal{H}om_{D_{Z_1 \times Z_2}}(M_1 \boxtimes_O M_2, D_{Z_1 \times Z_2}). \end{aligned} \tag{6.52}$$

We prove that the constructed natural transformation is an isomorphism. To show (6.52) is an isomorphism, by [Har66, I, Prop. 7.1 (i)], one may assume $M_i \in \mathrm{Coh}(D_{Z_i})$ for $i = 1, 2$. By shrinking Z_i and using [KS90, Prop. 11.2.6], one may find a bounded resolution of M_i by free D_{Z_i}-modules of finite rank. Thus, one may further assume that $M_i = D_{Z_i}$. Since $\omega_{Z_1 \times Z_2} = \omega_{Z_1} \boxtimes_O \omega_{Z_2}$ in $\mathrm{Mod}(D^{\mathrm{op}}_{Z_1 \times Z_2})$, by [HT07, Eg. 2.6.3], in this case (6.52) is an isomorphism. □

Proof of Claim 6.7.4.5. Take a $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-injective resolution $D_{Z_i} \to I_i^*$. Then $I_1^* \boxtimes_{\mathbb{C}} I_2^*$ is a complex of modules over

$$(D_{Z_1} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1}) \boxtimes_{\mathbb{C}} (D_{Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_2}) = (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}}. \tag{6.53}$$

By [Sta24, Tag 013K (2)], there exists an injective resolution $I_1^* \boxtimes_{\mathbb{C}} I_2^* \to I^*$ (hence an induced injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to I^*$) over (6.53). The natural morphism $D_{Z_i} \to D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$ is flat, so every injective $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-module is injective over D_{Z_i}. Similarly, every term of the complex I^* is injective over $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}$. Then (6.49) is defined to be the composition of the natural morphisms

$$\mathcal{H}om_{D_{Z_1}}(M_1, I_1^*) \boxtimes_{\mathbb{C}} \mathcal{H}om_{D_{Z_2}}(M_2, I_2^*) \to \mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I_1^* \boxtimes_{\mathbb{C}} I_2^*)$$
$$\to \mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I^*).$$

□

Proof of Claim 6.7.4.6. Take an injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to J^*$ over (6.53). By [Sta24, Tag 013K (2)], over $(D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1 \times Z_2}$ there exists an injective resolution $J^* \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \to K^*$. Then (6.50) is defined to be the composition of the natural morphisms

$$\begin{aligned}
&\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^*) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \\
\to &\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^* \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2}) \\
\to &\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, K^*).
\end{aligned}$$

$\square$

Corollary 6.7.4.7 ([Lau96, Cor. 3.3.3]). *The equivalence $RS_2 : (D^b_{\mathrm{good}}(D_Y), *_D) \to (D^b_{\mathrm{good}}(\mathcal{A}_X), \otimes^L_{\mathcal{A}_X})$ is a strong monoidal functor* (in the sense of [SS03, Def. 3.3]). *In fact, there are canonical isomorphisms of bifunctors*

$$RS_2(- *_D +) \cong (RS_2-) \otimes^L_{\mathcal{A}_X} (RS_2+) : D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X); \tag{6.54}$$

$$(RS_1-) *_D (RS_1+) \cong T^{-g} RS_1(- \otimes^L_{\mathcal{A}_X} +) : D^b_{\mathrm{good}}(\mathcal{A}_X) \times D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y); \tag{6.55}$$

$$RS_1(- *_{\mathcal{A}} +) \cong (RS_1-) \otimes^L_{O_Y} (RS_1+) : D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y); \tag{6.56}$$

$$(RS_2-) *_{\mathcal{A}} (RS_2+) \cong T^{-g} RS_2(- \otimes^L_{O_Y} +) : D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X). \tag{6.57}$$

Proof. Let $\delta_X : X \to X^2 =: X'$ be the diagonal morphism. Its dual morphism is $\mu : Y^2 \to Y$. There are canonical isomorphisms of bifunctors

$$\begin{aligned}
RS_2(- *_D +) :=& RS_2 \mu_+(- \boxtimes_O +) \\
\overset{(a)}{\cong}& L\tilde{\delta}^*_X RS'_2(- \boxtimes_O +) \\
\overset{(b)}{\cong}& L\tilde{\delta}^*_X (RS_2 - \boxtimes_{\mathcal{A}} RS_2 +) \\
\overset{(c)}{\cong}& (RS_2-) \otimes^L_{\mathcal{A}_X} (RS_2+),
\end{aligned}$$

where (a), (b) and (c) use (6.38), (6.46) and (6.45) respectively. This shows (6.54).

By Corollary 6.5.1.6, the functor RS_2 preserves units, so it is strong monoidal. In addition, (6.55) follows:

$$\begin{aligned}(RS_1-) *_D (RS_1+) &\overset{\text{(a)}}{\cong} T^g RS_1 RS_2(RS_1 - *_D RS_1+)\\ &\overset{\text{(b)}}{\cong} T^g RS_1(RS_2RS_1 - \otimes^L_{\mathcal{A}_X} RS_2RS_1+)\\ &\overset{\text{(c)}}{\cong} T^g RS_1(T^{-g} - \otimes^L_{\mathcal{A}_X} T^{-g}+)\\ &= T^{-g}RS_1(- \otimes^L_{\mathcal{A}_X} +),\end{aligned}$$

where (a) and (c) (resp. (b)) use Theorem 6.6.3.1, (resp. (6.54)).

Because the diagonal morphism $\delta_Y : Y \to Y^2$ is dual to $m : X' = X^2 \to X$, there are canonical isomorphisms of bifunctors

$$\begin{aligned}RS_1(- *_{\mathcal{A}} +) :=& RS_1 R\tilde{m}_*(- \boxtimes_{\mathcal{A}} +)\\ &\overset{\text{(a)}}{\cong} L\delta_Y^* RS_1'(- \boxtimes_{\mathcal{A}} +)\\ &\overset{\text{(b)}}{\cong} L\delta_Y^*(RS_1 - \boxtimes_O RS_1+)\\ &\overset{\text{(c)}}{\cong} (RS_1-) \otimes^L_{O_Y} (RS_1+),\end{aligned}$$

where (a), (b) and (c) use (6.36), (6.47) and [HT07, p.39] respectively. This demonstrates (6.56). Then (6.57) follows:

$$\begin{aligned}(RS_2-) *_{\mathcal{A}} (RS_2+) &\overset{\text{(a)}}{\cong} T^g RS_2 RS_1(RS_2 - *_{\mathcal{A}} RS_2+)\\ &\overset{\text{(b)}}{\cong} T^g RS_2(RS_1RS_2 - \otimes^L_{O_Y} RS_1RS_2+)\\ &\overset{\text{(c)}}{\cong} T^g RS_2(T^{-g} - \otimes^L_{O_Y} T^{-g}+)\\ &= T^{-g}RS_2(- \otimes^L_{O_Y} +),\end{aligned}$$

where (a) and (c) (resp. (b)) use Theorem 6.5.1.4 (resp. (6.56)). The other axioms can be verified as in [Krä22, Prop. 3.1]. □

Remark 6.7.4.8. We reprove Proposition 6.7.4.3 using the Laumon-Rothstein transform as follows. By [Sta24, Tag 08DJ], for every object $M \in D^b_{\mathrm{good}}(\mathcal{A}_X)$, the functor $\cdot \otimes^L_{\mathcal{A}_X} M : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$ admits a right adjoint $R\mathcal{H}om_{\mathcal{A}_X}(M, \cdot)$. By [Huy06, p.84], the right adjoint is naturally isomorphic to $T^{-g}\Delta^{\mathcal{A}_X}(M) \otimes^L_{\mathcal{A}_X} \cdot$. Then combining Proposition 6.7.1.5 with Corollary 6.7.4.7, one gets Proposition 6.7.4.3.

6.7.5 Translation and multiplication

For every $y \in Y$, there is a canonical isomorphism $T^*_{(0,y)}\mathcal{P} \cong \mathcal{P} \otimes_{O_{X\times Y}} p_X^* P_y$ in $\mathrm{Mod}(X \times Y)_{-1-\mathrm{cxn,fl}}$, where $p_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(D_{X\times Y/X})$ is the pullback of relative D-modules. The functor

$$\mathcal{P} \otimes_{O_{X\times Y}} (\cdot) : \mathrm{Mod}(D_{X\times Y/X}) \to \mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn,fl}}$$

is from [Rot97, (2.10)]. Arguing as in [Muk81, (3.1)], we deduce Proposition 6.7.5.1 from the projection formula.

Proposition 6.7.5.1. *There are canonical isomorphisms of functors*

$$RS_2 \circ T_y^* \cong (\cdot \otimes_{O_X} P_y) \circ RS_2 : D(D_Y) \to D(\mathcal{A}_X),$$
$$RS_1 \circ (\cdot \otimes_{O_X} P_y) \cong T_y^* \circ RS_1 : D(\mathcal{A}_X) \to D(D_Y).$$

Similar results hold for the Rothstein transform.

Remark 6.7.5.2. As the proof does not use the smooth base change, goodness over O is not necessary in Proposition 6.7.5.1.

Proposition 6.7.5.3 is a variant of [Sch15, Thm. 9.5 (a)].

Proposition 6.7.5.3. *For a flat line bundle (L, ∇) on Y, let $z \in X^\natural$ be the corresponding point. There are canonical isomorphisms of functors*

$$RS_2(\cdot \otimes_{O_Y} (L, \nabla)) \cong RS_2(\cdot) *_{\mathcal{A}} \pi_*\mathbb{C}_z : D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X), \tag{6.58}$$

$$RS_1(\cdot) \otimes_{O_Y} (L, \nabla) \cong RS_1(\cdot *_{\mathcal{A}} \pi_*\mathbb{C}_z) : D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y). \tag{6.59}$$

Proof. By Example 6.5.1.2 and Theorem 6.5.1.4, one has $RS_2(L, \nabla) = T^{-g}\pi_*\mathbb{C}_z$. Then (6.58) follows from (6.57). By Theorem 6.5.1.4, (6.58) implies (6.59). $\square$

Appendix A

Sheaves of modules

A.1 Sheaves of modules

We recall some facts about sheaves of modules. Let (X, O_X) be a ringed space.

A.1.1 Generalities

Definition A.1.1.1. An O_X-module F is called

1. ([Sta24, Tag 01B5]) of *finite type* if every $x \in X$ admits an open neighborhood U such that $F|_U$ is generated by finitely many sections;
2. ([Sta24, Tag 01BN]) of *finite presentation* if for every $x \in X$, there is an open neighborhood $U \subset X$, integers $n, m \geq 0$ and an exact sequence of O_U-modules
$$O_U^m \to O_U^n \to F|_U \to 0;$$
3. ([EGA I, 5.1.3]) *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$, two sets I, J and a morphism $O_U^{\oplus J} \to O_U^{\oplus I}$ whose cokernel is isomorphic to $F|_U$;
4. ([Kas03, Def. A.5 (1)]) *pseudo-coherent* if for every open subset $U \subset X$, every finite type O_U-submodule of $F|_U$ is of finite presentation. Let $\mathrm{PCoh}(X) \subset \mathrm{Mod}(O_X)$ be full subcategory of pseudo-coherent modules;
5. ([Kas03, Def. A.5 (2)]) *K-coherent* if F is pseudo-coherent and of finite type;
6. ([Sta24, Tag 01BV]) *coherent* if F is of finite type and for every open subset $U \subset X$ and every finite collection $\{s_i\}_{1 \leq i \leq n}$ in $F(U)$, the kernel of the associated morphism $O_U^n \to F|_U$ is of finite type over O_U.

Every property in Definition A.1.1.1 is local, in the sense that it restricts to every open subset, and if it holds on each member of an open covering of X, then it holds on X.

Lemma A.1.1.2. *Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in* $\mathrm{Mod}(O_X)$*. If F, H are of finite presentation, then so is G.*

Proof. For every $x \in X$, by [Sta24, Tag 01B8], there is an open neighborhood U of x such that the sequence $G(U) \xrightarrow{r_U} H(U) \to 0$ is exact. Up to shrinking U, there exist integers $m, n, p, q \geq 0$ and two exact sequences

$$O_U^m \to O_U^n \xrightarrow{f} F|_U \to 0, \quad O_U^p \to O_U^q \xrightarrow{h} H|_U \to 0.$$

The morphism h is defined by q elements $s_1, \dots, s_q$ of $H(U)$. For each $1 \leq i \leq q$, choose a preimage $t_i \in G(U)$ of s_i. Consider the morphism $\phi : O_U^{n+q} \to G|_U$ determined by $if(e_1), \dots, if(e_n), t_1, \dots, t_q \in G(U)$. Hence a commutative diagram with two exact middle rows

$$\begin{tikzcd}
0 \arrow[r] & O_U^m \arrow[r] \arrow[d] & \ker(\phi) \arrow[r] \arrow[d] & O_U^p \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & O_U^n \arrow[r] \arrow[d, "f"] & O_U^{n+q} \arrow[r] \arrow[d, "\phi"] & O_U^q \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & F|_U \arrow[r] \arrow[d] & G|_U \arrow[r] \arrow[d] & H|_U \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{coker}(\phi) \arrow[r] & 0. &
\end{tikzcd}$$

By the snake lemma, ϕ is surjective and $\ker(\phi)$ is finite type. Shrinking U again, one may find an integer $k \geq 0$ and a surjection $O_U^k \to \ker(\phi)$. The induced sequence $O_U^k \to O_U^{n+q} \to G|_U \to 0$ is exact. Therefore, G is of finite presentation. $\square$

A.1.2 Pseudo-coherent modules

Lemma A.1.2.1.

1. *Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in* $\mathrm{Mod}(O_X)$*. If F, H are pseudo-coherent, then so is G.*

2. *Let I be a directed set. Let (M_i, f_{ij}) be a direct system over I consisting of pseudo-coherent O_X-modules. Then $M := \mathrm{colim}_{i \in I} M_i$ in* $\mathrm{Mod}(O_X)$ *is pseudo-coherent.*

3. *If $\{M_\alpha\}_{\alpha \in A}$ is a family of pseudo-coherent O_X-modules, then $S := \oplus_{\alpha \in A} M_\alpha$ is also pseudo-coherent.*

Proof. Let U be an open subset of X.

1. Let M be a finite type submodule of $G|_U$. Then the kernel of $r|_M : M \to H|_U$ is $(F|_U) \cap M$. Thus, $r|_M$ induces an injection $M/(F|_U \cap M) \to H|_U$. As H is pseudo-coherent, the finite type O_U-submodule $M/(F|_U \cap M)$ is of finite presentation. By [Sta24, Tag 01BP (2)], $F|_U \cap M$ is of finite type. As F is pseudo-coherent, $F|_U \cap M$ is of finite presentation. By Lemma A.1.1.2 applied to the exact sequence $0 \to F|_U \cap M \to M \to M/(F|_U \cap M) \to 0$, the O_U-module M is of finite presentation. Thus, G is pseudo-coherent.

2. Let N be a finite type submodule of $M|_U$. For every $x \in U$, from the first three lines of the proof of [Sta24, Tag 01BB], there is an open neighborhood $V \subset U$ of x and $i \in I$ such that $N|_V \subset F_i|_V$. Since F_i is pseudo-coherent, $N|_V$ is of finite presentation. As finite presentation is a local property, N is of finite presentation. Thus, M is pseudo-coherent.

3. Let I be the set of all finite subsets of A with the inclusion order. Then I is a directed set. For $B \in I$, set $F_B = \oplus_{\alpha \in B} M_\alpha$. By Point 1, F_B is pseudo-coherent. For $B \leq B'$ in I, set $f_{B,B'} : F_B \to F_{B'}$ to be the inclusion. Hence a direct system $(F_B, f_{B,B'})$ over I. By Point 2, the O_X-module $S = \mathrm{colim}_{B \in I} F_B$ is pseudo-coherent.

□

Lemma A.1.2.2 allows one to apply results on K-coherent sheaves in [Kas03] to coherent sheaves.

Lemma A.1.2.2. *An O_X-module is K-coherent if and only if it is coherent.*

Proof. Let $U \subset X$ be an open subset. Assume that F is a K-coherent module. Let $\{s_i\}_{1 \leq i \leq n}$ be a finite collection in $F(U)$, and let $f : O_U^n \to F|_U$ be the associated morphism. Then $\mathrm{im} f$ is a finite type submodule of $F|_U$. Because F is pseudo-coherent, $\mathrm{im} f$ is of finite presentation over O_U. From [Sta24, Tag 01BP (2)], $\ker f$ is of finite type over O_U. Therefore, F is coherent.

Conversely, assume that F is a coherent O_X-module. Let M be a finite type submodule of $F|_U$. By [Sta24, Tag 01BY (1)], M is coherent over O_U. From [Sta24, Tag 01BW], M is of finite presentation. Thus, F is pseudo-coherent and hence K-coherent. □

The module O_X is quasi-coherent, but in general not pseudo-coherent. If it is pseudo-coherent, then O_X is called a *coherent sheaf of rings*.

Lemma A.1.2.3. *If X is a locally Noetherian scheme, then every quasi-coherent module is pseudo-coherent.*

Proof. By [EGA I, Cor. 9.4.9], a quasi-coherent module is a directed limit of coherent modules, hence pseudo-coherent by Lemma A.1.2.1 2. □

Example A.1.2.4. Let $X = \mathbf{A}^1$ be the affine line over a field. Let $U = X \setminus \{0\}$, and let $j : U \to X$ be the inclusion. By [Har77, II, Example 5.2.3], the O_X-module $j_! O_U$ is not quasi-coherent. From [Har77, II, Exercise 1.19 (c)], it is a submodule of the coherent module O_X. Hence, $j_! O_U$ is pseudo-coherent.

Definition A.1.2.5 defines a local property. It is weaker than [Bjö93, A:III, 2.24] and [Kas03, Def. A.7].

Definition A.1.2.5. Assume that O_X is a coherent sheaf of rings. If for every open subset $U \subset X$, every family of coherent ideal sheaves $\{I_i\}_i$ in O_U, the ideal sheaf $\sum_i I_i$ is O_U-coherent, then O_X is called a quasi-Noetherian sheaf of rings.

Example A.1.2.6.

1. If (X, O_X) is a locally Noetherian scheme, then O_X is quasi-Noetherian.
2. If (X, O_X) is a complex analytic space, then by the Oka-Cartan theorem (see, *e.g.*, [Kas03, Thm. A.12]), O_X is quasi-Noetherian.

A.1.3 Analytic coherent modules

Let X be a complex analytic space. We show that a coherent O_X-module admits a local free resolution, from which we deduce that coherence is preserved by derived pullbacks and tensor products. An analog of Lemma A.1.3.1 for algebraic varieties is [Har77, III, Example 6.5.1]. By local syzygies [GH78, p.696], on complex manifolds, every coherent module local admits a finite-length, finite free resolution.

Lemma A.1.3.1. *Every $x \in X$ admits an open neighborhood U, such that for every coherent O_X-module F, there is a (possibly infinite-length) resolution*

$$\cdots \to O_U^{n_1} \to O_U^{n_0} \to F|_U \to 0,$$

where $n_i \geq 0$ are integers.

Proof. Shrinking X to an open neighborhood of x, one may assume that X is Stein. By [GR04, Thm. 8, p.108], there is a compact neighborhood $K \subset X$ of x, such that Theorem B is valid on K in the sense of [GR04, Def. 1, p.100]. Let $U = K^\circ$.

For a coherent O_X-module F, we construct inductively a sequence of morphisms. From [GR04, Cor. p.101], there is an integer $n_0 \geq 0$, an open neighborhood U_0 of $K \subset X$ and a morphism $f_0 : O_{U_0}^{n_0} \to F|_{U_0}$ in $\mathrm{Mod}(O_{U_0})$ such that $f_0|_U$ is an epimorphism in $\mathrm{Mod}(O_U)$. Set $\ker(f_{-1})|_{U_0} = F|_{U_0}$.

Given such a morphism $f_j : O_{U_j}^{n_j} \to \ker(f_{j-1})|_{U_j}$ for an integer $j \geq 0$ and an open neighborhood $U_j \subset X$ of K, by [Sta24, Tag 01BY (3)], the O_{U_j}-module $\ker(f_j)$ is coherent. By [GR04, Cor. p.101], there is an open neighborhood $U_{j+1} \subset U_j$ of K, an integer $n_{j+1} \geq 0$ and a morphism $f_{j+1} : O_{U_{j+1}}^{n_{j+1}} \to \ker(f_j)|_{U_{j+1}}$ in $\mathrm{Mod}(O_{U_{j+1}})$ such that $f_{j+1}|_U$ is an epimorphism. Thus, one gets a sequence

$$\cdots \to O_U^{n_2} \stackrel{f_2|_U}{\to} O_U^{n_1} \stackrel{f_1|_U}{\to} O_U^{n_0} \stackrel{f_0|_U}{\to} F|_U \to 0$$

in $\mathrm{Mod}(O_U)$. By construction, it is exact, hence a resolution of $F|_U$. □

Example A.1.3.2. Assume that $x \in X$ is a singular point. Then $F := \mathbb{C}_x$ is a coherent O_X-module, but for every open neighborhood $U \subset X$ of x, there is no finite-length resolution of $F|_U$ by finite locally free O_U-modules. (Otherwise, such a resolution induces a finite-length free resolution of the $O_{X,x}$-module $F_x = \mathbb{C} = O_{X,x}/m_x$. From [Osb12, Ch. 4, Prop. 4.4], the projective dimension $\mathrm{pd}_{O_{X,x}} O_{X,x}/m_x$ is finite. By [Mat87, Lem. 1, p.154] and [Osb12, Prop. 4.9], the global dimension of the ring $O_{X,x}$ is finite. By Serre's theorem (see, *e.g.*, [Osb12, p.332]), the local ring $O_{X,x}$ is regular. From [Ser56, p.6], x is a smooth point of X, a contradiction.)

Therefore, Lemma A.1.3.1 fails if one consider only finite-length resolutions. See also [EP+96, Thm. 4.1.2].

Lemma A.1.3.3. *Let $f : X \to Y$ be a morphism of complex analytic spaces. Then derived pullback $Lf^* : D(Y) \to D(X)$ restricts to a functor* $\mathrm{Coh}(Y) \to D_c(X)$.

Proof. Let F be a coherent O_Y-module. For every $x \in X$, by Lemma A.1.3.1, there is an open neighborhood V of $f(x) \in Y$, such that there is a resolution $E_\bullet \to F|_V \to 0$ by finite free O_V-modules. Let $g : f^{-1}(V) \to V$ be the base change of f along the inclusion $V \to Y$. Then the morphism $g^*E_\bullet \to (Lf^*F)|_{f^{-1}(V)}$ in $D(f^{-1}(V))$ is an isomorphism. For every integer $j \geq 0$, the $O_{f^{-1}(V)}$-module g^*E_j is finite free. Thus, the $O_{f^{-1}(V)}$-module $(H^{-j}Lf^*F)|_{f^{-1}(V)}$ is coherent. Since coherence is a local property, the O_X-module $H^{-j}(Lf^*F)$ is coherent. □

Lemma A.1.3.4. *For any coherent O_X-modules F and G, one has $F \otimes^L_{O_X} G \in D_c(X)$.*

Proof. For every $x \in X$, by Lemma A.1.3.1, there is an open neighborhood $U \subset X$ of x and a resolution $E_\bullet \to F|_U \to 0$ by finite free O_U-modules. The natural morphism $E_\bullet \otimes_{O_U} G|_U \to F|_U \otimes^L_{O_U} G|_U$ in $D(U)$ is an isomorphism. For every integer n, the O_U-module $H^n(E_\bullet \otimes^L_{O_U} G|_U) = H^n(E_\bullet \otimes_{O_U} G|_U)$ is coherent. Therefore, the O_U-module $H^n(F \otimes^L_{O_X} G)|_U = H^n(F|_U \otimes^L_{O_U} G|_U)$ is coherent. Since coherence is a local property, the O_X-module $H^n(F \otimes^L_{O_X} G)$ is coherent. □

A.1.4 Good modules

Assume that the ringed space X is locally compact Hausdorff.

Definition A.1.4.1. [Kas03, Def. 4.22] An O_X-module F is called *good* if for every relatively compact open subset $U \subset X$, there exists a directed family $\{G_i\}_{i\in I}$ of coherent O_U-submodules of $F|_U$ such that $F|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ being a directed family means that for any $i, i' \in I$, there is $i'' \in I$ with $G_i + G_{i'} \subset G_{i''}$ (and hence $F|_U = \mathrm{colim}_{i\in I} G_i$). The full subcategory of $\mathrm{Mod}(O_X)$ consisting of good O_X-modules is denoted by $\mathrm{Good}(X)$.

Lemma A.1.4.2 (Goodness *vs.* pseudo-coherence)**.**

1. ([Kas03, p.77]) *One has* $\mathrm{Coh}(X) \subset \mathrm{Good}(X) \subset \mathrm{PCoh}(X)$.
2. *Let E be a pseudo-coherent O_X-module. If on every relatively compact open subset $U \subset X$, the O_U-module $E|_U$ is the sum of its finite type submodules, then E is good.*

Proof.

1. By definition, every coherent O_X-module is good. Let E be a good O_X-module. Let W be an open subset of X, and let $F \subset E|_W$ be a finite type O_W-submodule. We show that F is of finite presentation over O_W. Replacing (X, E) with $(W, E|_W)$, one may assume that $W = X$. Because X is locally compact, for every $x \in X$, there exists a *relatively compact* open neighborhood $U \subset X$ of x and finitely many sections $s_1, \ldots, s_n \in F(U)$ generating $F|_U$. As E is good, $E|_U = \sum_{i\in I} G_i$ is the sum of a directed family of coherent submodules. There exists $i_0 \in I$ and an open neighborhood V of $x \in U$ with $s_i|_V \in G_{i_0}(V)$ for all $1 \le i \le n$. Then $F|_V$ is a finite type submodule of $G_{i_0}|_V$. By [Sta24, Tag 01BY (1)], $F|_V$ is O_V-coherent. As coherence is a local property, F is coherent. From [Sta24, Tag 01BW], F is of finite presentation.
2. The family of finite type submodules of $E|_U$ is directed, since the sum of two finite type submodules is of finite type. For every relatively compact open subset $U \subset X$, as E is pseudo-coherent, every finite type submodule of $E|_U$ is pseudo-coherent and hence coherent. Thus, E is good.

□

Basic properties of good modules (similar to those of quasi-coherent modules on algebraic varieties) are recapped in Lemma A.1.4.3. Point 3 should be compared to [Con06, Lemma 2.1.8 (1)].

Lemma A.1.4.3.

1. *For every family of objects $\{F_i\}_{i\in I}$ in $\mathrm{Good}(X)$, the direct sum $\oplus_{i\in I}F_i$ in $\mathrm{Mod}(O_X)$ is good.*

2. *The subcategory $D_{\mathrm{good}}(X)$ is closed under direct sums in $D(X)$. Moreover, the inclusion functor $\mathrm{Good}(X) \to D_{\mathrm{good}}(X)$ commutes with direct sums.*

Suppose that O_X is quasi-Noetherian. Then:

3. *The subcategory $\mathrm{Good}(X) \subset \mathrm{Mod}(O_X)$ is weak Serre and closed under filtered colimits in $\mathrm{Mod}(O_X)$. In particular, $\mathrm{Good}(X)$ is a locally Noetherian category (in the sense of [Gab62, p.356]).*

4. *The inclusion functor $D_{\mathrm{good}}(X) \to D(X)$ is a triangulated subcategory.*

Proof.

1. Over every relatively compact open subset U of X, the direct sum $(\oplus_{i\in I}F_i)|_U$ is the sum of its coherent O_U-submodules. By Lemma A.1.2.1 3, the O_X-module $\oplus_{i\in I}F_i$ is pseudo-coherent. By Lemma A.1.4.2 2, it is good.

2. Since $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, by [Sta24, Tag 07D9], the category $D(X)$ has arbitrary direct sums and they are computed by taking termwise direct sums of any representative complexes. Then by [Wei95, Exercise 1.2.1], for every integer q, the functor $H^q : D(X) \to \mathrm{Mod}(O_X)$ commutes with direct sums. The result follows from Point 1.

3. As O_X is quasi-Noetherian, by [Sta24, Tag 0754] and the proof of [Kas03, Prop. 4.23], $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$. From [KS06, Thm. 18.1.6 (v)], the category $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category. By Point 1 and [Sta24, Tag 002P], the filtered colimits in $\mathrm{Good}(X)$ exist and agree with the filtered colimits in $\mathrm{Mod}(O_X)$. Thus, filtered colimits in $\mathrm{Good}(X)$ are exact.

   Because of [Sta24, Tag 01BC], there is a set of coherent O_X-modules $\{F_i\}_{i\in I}$ such that each coherent O_X-module is isomorphic to exactly one of the F_i. Then $\{F_i\}$ is a family of Noetherian generators of $\mathrm{Good}(X)$. Therefore, the category $\mathrm{Good}(X)$ is locally Noetherian.

4. It follows from [Yek19, Prop. 7.4.5] and Point 3.

□

Lemma A.1.4.4. *A good module on a complex analytic space is quasi-coherent.*

Proof. Let F be a good module on a complex analytic space X. From [Fri67, Thm. I, 9, Rem. I, 10], every $x \in X$ admits a neighborhood K that is a Noetherian Stein compactum. There is a relative compact open subset U of X containing K. As F is good, the O_U-module $F|_U = \sum_{i \in I} F_i$ is the sum of a directed family of coherent subsheaves. Applying the functor $\Gamma(K, \cdot)$ to the directed family $\{F_i\}_{i \in I}$ in $\mathrm{Coh}(U)$, by [Tay02, Prop. 11.9.2], one gets a directed family of finitely generated $\Gamma(K, O_K)$-submodule $\{M_i\}_{i \in I}$ of $\Gamma(K, F)$, whose associated family in $\mathrm{Mod}(O_K)$ is $\{F_i|_K\}_{i \in I}$. Let M be $\mathrm{colim}_{i \in I} M_i$ in $\mathrm{Mod}(\Gamma(K, O_K))$. Since the localization functor $\mathrm{Mod}(\Gamma(K, O_K)) \to \mathrm{Mod}(O_K)$ is left adjoint to $\Gamma(K, \cdot) : \mathrm{Mod}(O_K) \to \mathrm{Mod}(\Gamma(K, O_K))$, the localization preserves colimits. Then $F|_K$ is associated to M. By Lemma C.2.0.5, F is quasi-coherent. □

Remark A.1.4.5. The restriction of a good O_X-module to an open subset U is a good O_U-module. Unlike quasi-coherence on schemes, goodness is not a local property. In fact, by Lemma A.1.4.3 3, every free module on a complex manifold is good, while Gabber [Con06, Eg. 2.1.6] gives a locally free (hence quasi-coherent and pseudo-coherent), but not good module on the unit open disk in $\mathbb{C}$. (In particular, the converse of Lemma A.1.4.4 is wrong for noncompact complex manifolds.) Still, given an O_X-module F, if for *every* relatively compact open subset $U \subset X$, the O_U-module $F|_U$ is good, then F is good.

Definition A.1.4.6 ([KS06, Def. 6.3.3])**.** In a category $\mathcal{C}$ with small filtered colimits, an object X is of finite presentation, if $\mathrm{Hom}_{\mathcal{C}}(X, \cdot) : \mathcal{C} \to \mathrm{Set}$ commutes with small filtered colimits.

Remark A.1.4.7. In an additive category with arbitrary direct sums, an object of finite presentation is necessarily compact, but the converse is false. Let M be a finite module but not of finite presentation over a commutative ring R. By [Ren69, no. 2], M is a compact object of the abelian category $\mathrm{Mod}(R)$. From [Sta24, Tag 0G8P], M is not an object of finite presentation.

Lemma A.1.4.8. *Let (X, O_X) be a ringed space. If the topology is Hausdorff compact, then every O_X-module of finite presentation is an object of finite presentation of* $\mathrm{Mod}(O_X)$*.*

Proof. Let $G = \mathrm{colim}_{i \in I} G_i$ be a filtered colimit in $\mathrm{Mod}(O_X)$. Let F be an O_X-module of finite presentation. By [Sta24, Tag 0GMV], the canonical morphism $\mathrm{colim}_{i \in I} \mathcal{H}om_{O_X}(F, G_i) \to \mathcal{H}om_{O_X}(F, G)$ is an isomorphism. By compactness of X and [God58, Thm. 4.12.1], the canonical map $\mathrm{colim}_{i \in I} H^0(X, \mathcal{H}om_{O_X}(F, G_i)) \to H^0(X, \mathrm{colim}_{i \in I} \mathcal{H}om_{O_X}(F, G_i))$ is bijective. Then the canonical map $\mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathrm{Mod}(O_X)}(F, G_i) \to \mathrm{Hom}_{\mathrm{Mod}(O_X)}(F, G)$ is bijective. □

Lemma A.1.4.9. *Let X be a compact complex analytic space. Then the objects of finite presentation in* $\mathrm{Good}(X)$ *are precisely objects of* $\mathrm{Coh}(X)$*.*

Proof. Let $F \in \mathrm{Good}(X)$ be an object of finite presentation. By compactness of X, there is a directed family of coherent submodules $\{F_i\}_{i\in I}$ with $F = \sum_{i\in I} F_i$. Then the canonical morphism $\mathrm{colim}_{i\in I} \mathrm{Hom}(F, F_i) \to \mathrm{Hom}(F, F)$ of abelian groups is an isomorphism. Thus, there is $i_0 \in I$ and an element $\mathrm{Hom}(F, F_{i_0})$ lying over Id_F. As Id_F factors through F_{i_0}, one has $F = F_{i_0} \in \mathrm{Coh}(X)$.

Conversely, by [Sta24, Tag 01BW], every object of $\mathrm{Coh}(X)$ is an O_X-module of finite presentation. From Lemma A.1.4.8 and compactness of X, it is an object of finite presentation in $\mathrm{Mod}(O_X)$. By Lemma A.1.4.3 3, it is also an object of finite presentation in $\mathrm{Good}(X)$. $\square$

A.1.5 Sections of direct sum of sheaves

By [Har77, II, Exercise 1.11], on a Noetherian topological space, taking section commutes with (possibly infinite) direct sum of sheaves. This fails on complex manifolds, as Example A.1.5.1 shows.

Example A.1.5.1. Let $X = \mathbb{C}$. Let F be the O_X-module $\oplus_{n\geq 0}\mathbb{C}_n$. There is a section $s \in \Gamma(X, F^{\oplus\mathbb{N}})$, such that for every integer $n \geq 0$, the stalk $s_n \in (F^{\oplus\mathbb{N}})_n = (F_n)^{\oplus\mathbb{N}} = \mathbb{C}^{\oplus\mathbb{N}}$ is $(1, 1, \dots, 1, 0, 0, \dots)$, where the first $n+1$ entries are 1 and all the other entries are 0. Then s has no preimage under the canonical map $\Gamma(X, F)^{\oplus\mathbb{N}} \to \Gamma(X, F^{\oplus\mathbb{N}})$. For otherwise, let $(t^n)_{n\geq 0} \in \Gamma(X, F)^{\oplus\mathbb{N}}$ be a preimage of s. Then there are only finitely many integers $n \geq 0$ with $t^n \neq 0$. Every t^n has only finitely many nonzero stalks. However, s has infinitely many nonzero stalks, which is a contradiction.

Let X be a complex manifold. An O_X-module is called *privileged* if for every connected open subset $U \subset X$ and every $x \in U$, the map $\Gamma(U, F) \to F_x$ taking the stalk at x is injective. By the identity theorem (see, *e.g.*, [GH78, p.7]), O_X is privileged.

Lemma A.1.5.2. *Assume that X is connected. Let $\{F_i\}_{i\in I}$ be a family of privileged O_X-modules. Then the canonical map $\oplus_{i\in I}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i)$ is bijective.*

Proof. Let P be the presheaf direct sum of $\{F_i\}_{i\in I}$. Let $\theta : P \to \oplus_{i\in I}F_i$ be the sheafification morphism. Then $P(X) = \oplus_{i\in I}\Gamma(X, F_i)$ and $\theta_X : \oplus_{i\in I}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i)$ is the colimit of

$$\theta_X^{(J)} : \oplus_{i\in J}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i),$$

where J runs through the finite subsets of I. For every such J, by [Sta24, Tag 01AH (4)], the presheaf direct sum of $\{F_i\}_{i\in J}$ is a subsheaf of $\oplus_{i\in I}F_i$, so the map $\theta_X^{(J)}$ is injective. Therefore, their limit map θ_X is also injective. We prove that θ_X is surjective.

By construction of sheafification in [Har77, p.64], for every $s \in \Gamma(X, \oplus_{i\in I} F_i)$, there is a covering $\{U_\alpha\}_{\alpha\in A}$ of X by nonempty *connected* open subsets and an element $t_\alpha \in \Gamma(U_\alpha, P)$ for each $\alpha \in A$ such that $s_x = t_{\alpha,x}$ in $(\oplus_{i\in I} F_i)_x = \oplus_{i\in I} F_{i,x}$ for every $x \in U_\alpha$.

Fix $x_0 \in X$ and $\alpha_0 \in A$ with $x_0 \in U_{\alpha_0}$. Then there is a finite subset $I_0 \subset I$ such that $t_{\alpha_0} \in \Gamma(X, \oplus_{i\in I_0} F_i) \subset \Gamma(X, P)$. Let $B \subset A$ be the subset of indices α with $t_\alpha \notin \Gamma(U_\alpha, \oplus_{i\in I_0} F_i)$. Set $V = \cup_{\alpha\in B} U_\alpha$. Then V is open in X and its complement

$$X \setminus V \subset \cup_{\alpha\in A\setminus B} U_\alpha. \tag{A.1}$$

For every $\alpha \in A \setminus B$, we claim that $U_\alpha \subset X \setminus V$.

In fact, for every $y \in U_\alpha$, every $\beta \in A$ with $y \in U_\beta$ and every $i \in I \setminus I_0$, the stalk $t^i_{\beta,y} = s^i_y = t^i_{\alpha,y} = 0$ in $F_{i,y}$. Since F_i is privileged and U_β is connected, the map $\Gamma(U_\beta, F_i) \to F_{i,y}$ is injective. Thus, $t^i_\beta = 0$ in $\Gamma(U_\beta, F_i)$. Therefore, $t_\beta \in \Gamma(X, \oplus_{i\in I_0} F_i)$, *i.e.*, $\beta \notin B$. Hence $y \notin V$.

From the claim and (A.1), the subset $X \setminus V = \cup_{\alpha\in A\setminus B} U_\alpha$ is also open in X and contains U_{α_0}. Since X is connected, one has $V = B = \emptyset$. Consequently, $t_\alpha \in \Gamma(X, \oplus_{i\in I_0} F_i)$ for every $\alpha \in A$. Then the family $\{t_\alpha\}_{\alpha\in A}$ glues to a preimage of s in $\Gamma(X, \oplus_{i\in I_0} F_i) \subset \Gamma(X, P)$. Thus, θ_X is surjective and hence a group isomorphism. □

Corollary A.1.5.3. *If F is a locally free (possibly of infinite rank) O_X-module, then F is privileged.*

Proof. Let U be a connected open subset of X. Fix $x_0 \in U$. We prove that the map $\Gamma(U, F) \to F_{x_0}$ is injective. Take $s \in \Gamma(U, F)$ with $s_{x_0} = 0$. By [Har77, II, Exercise 1.14], the set $Z := \{x \in U : s_x = 0\}$ is open in U.

We claim that Z is closed in U. Let $\{x_n\}_{n\geq 1}$ be a sequence of points in Z converging to $y \in U$. Because F is locally free, there is a connected open neighborhood $V \subset U$ of y, a set I and an isomorphism $\phi : F|_V \xrightarrow{\sim} O_V^{\oplus I}$ of O_V-modules. There is an integer $N > 0$ with $x_N \in V$. Because O_V is privileged, from Lemma A.1.5.2, the map on the bottom of the commutative square

$$\begin{array}{ccc}
\Gamma(V, F) & \longrightarrow & F_{x_N} \\
\downarrow{\scriptstyle \phi_V} & & \downarrow{\scriptstyle \phi_{x_N}} \\
\Gamma(V, O_V^{\oplus I}) & \longrightarrow & O_{V,x_N}^{\oplus I}
\end{array}$$

is injective. Then so is the map on the top. Since $s_{X_N} = 0$, one has $s|_V = 0$ and $s_y = 0$. Hence $y \in Z$. The claim is proved.

Because U is connected and $x_0 \in Z$, by claim one has $Z = U$. Therefore, $s = 0$ in $\Gamma(U, F)$. □

Corollary A.1.5.4. *Let X be a connected complex manifold. Let $\{F_i\}_{i\in I}$ be a family of locally free O_X-modules. Then the canonical map $\oplus_{i\in I}\Gamma(X, F_i) \to \Gamma(X, \oplus_{i\in I}F_i)$ is bijective.*

Proof. It follows from Lemma A.1.5.2 and Corollary A.1.5.3. □

A.2 Gabber's example

We present an example of a locally free module on the open unit disc that is not good. It illustrates that goodness on complex manifolds, unlike quasi-coherence on algebraic varieties, is not a local property. This construction is already exhibited in the context of rigid geometry by [Con06, Example 2.1.6], which attributes the originality to Gabber. Furthermore, in [Con06, p.1058] it is mentioned that Gabber's example makes sense in complex-analytic geometry as well. We reproduce this construction with a few extra details.

Lemma A.2.0.1. *Let X be a complex manifold, U be a dense open subset of X. If F is a locally free O_X-module, then the restriction map $r : \Gamma(X, F) \to \Gamma(U, F)$ is injective.*

Proof. Every $x \in X$ admits a *connected* open neighborhood V such that $F|_V$ is free O_V-module. Then $\Gamma(V, F)$ is a free $\Gamma(V, O_X)$-module by Corollary A.1.5.4. By density of U, $V \cap U$ is nonempty. For every $s \in \ker(r)$, $s|_{V\cap U} = 0$. As $F|_V$ is free and the map $\Gamma(V, O_X) \to \Gamma(V \cap U, O_X)$ is injective, the restriction $s|_V = 0$. By local nature of sheaves, $s = 0$. □

Lemma A.2.0.2. *Let X be a Hausdorff locally compact space, K be a compact subspace and $j : K \to X$ be the inclusion. Then for every $q \in \mathbb{Z}$ and every $F \in \mathrm{Ab}(X)$, the canonical morphism $\psi^q : \mathrm{colim}_U H^q(U, F) \to H^q(K, j^{-1}F)$ is an isomorphism, where U ranges through the family of open neighborhoods of K in X. The two groups are written as $H^q(K, F)$.*

Proof. We prove that both sides are the q-th right derived functor applied to F of a same functor.

Define a category I as follows. The objects are the open subsets of X containing K. For every $U, V \in I$, if $U \supset V$, then $\mathrm{Hom}_I(U, V)$ is a singleton; else $\mathrm{Hom}_I(U, V) = \emptyset$. Thus, I is a small category. Let Ab^I be the category of functors from I to Ab. By [Wei95, Exercise 2.3.7], Ab^I is an abelian category with enough injectives. Recall that Ab is a Grothendieck abelian category, so $\mathrm{colim}_I : \mathrm{Ab}^I \to \mathrm{Ab}$ is exact. By [KS90, Prop 2.5.1], the composition of the functor $\Phi : \mathrm{Ab}(X) \to \mathrm{Ab}^I$ defined by $\Phi(F)(U) = \Gamma(U, F)$ with $\mathrm{colim}_I : \mathrm{Ab}^I \to \mathrm{Ab}$ is $\Gamma(K, j^{-1}\cdot) : \mathrm{Ab}(X) \to \mathrm{Ab}$. Therefore, the q-th right derived functor of $\Gamma(K, j^{-1}\cdot)$ is $\mathrm{colim}_I \circ R^q\Phi = \mathrm{colim}_U H^q(U, \cdot)$.

The functor $\Gamma(K, j^{-1}\cdot) : \mathrm{Ab}(X) \to \mathrm{Ab}$ is the composition of $j^{-1} : \mathrm{Ab}(X) \to \mathrm{Ab}(K)$ with $\Gamma(K, \cdot) : \mathrm{Ab}(K) \to \mathrm{Ab}(X)$. Every injective object

G of $\mathrm{Ab}(X)$ is c-soft, so $j^{-1}G$ is c-soft by [KS90, Propositon 2.5.7 (i)]. By [KS90, Proposition 2.5.10], $j^{-1}G$ is right acyclic for $\Gamma(K, \cdot)$. By [Sta24, Tag 015M], $H^q(K, j^{-1}\cdot)$ is also the q-th right derived functor of $\Gamma(K, j^{-1}\cdot)$. We conclude that ψ^q is an isomorphism. $\square$

Definition A.2.0.3 (Compact Stein set)**.** [Con06, p.1053]Let K be a compact subset of a complex manifold X. If $H^q(K, F) = 0$ for every open neighborhood U of $K \subset X$, every coherent O_U-module and every $q \in \mathbb{N}^*$, then K is called a compact Stein set in X.

Lemma A.2.0.4 ([Con06, p.1058])**.** *Let K be a compact compact Stein set in a complex manifold X, F be a good O_X-module, then $H^q(K, F) = 0$ for all $q \in \mathbb{N}^*$.*

Proof. There is a relative compact open subset $U \subset X$ containing K. By definition, $F|_U = \mathrm{colim}_i F_i$, where $\{F_i\}$ is a direct family of coherent O_U-submodules of $F|_U$. By [God58, II, Thm. 4.12.1], $H^q(K, F) = \mathrm{colim}_i H^q(K, F_i) = 0$. $\square$

Example A.2.0.5 (Gabber)**.** Let Δ be the open unit disc in $\mathbb{C}$ and let $K = \{z \in \mathbb{C} : |z| \le 1/2\}$. Then $B(0, 2/3)$ is a relatively compact open subset of Δ containing K. By [Dou66, Thm. 3 (B), p.51, (a) p, 53], K is a compact Stein set in Δ.

Let x', x'' be two distinct points of the interior of K. Let $U' = \Delta \setminus \{x'\}$, $U'' = \Delta \setminus \{x''\}$ and define $U = U' \cap U''$. Let

$$F' = \oplus_{n \in \mathbb{Z}} O_{U'} e'_n, \quad F'' = \oplus_{n \in \mathbb{Z}} O_{U''} e''_n$$

be two free sheaves with countably infinite rank on U' and U'' respectively.

We glue F' and F'' to define a locally free O_Δ-module F as follows. Define $h \in O_\Delta(U)$ by

$$h(z) = e^{\frac{1}{z-x'} + \frac{1}{z-x''}}, \quad \forall z \in U.$$

Then h has essential singularities at x' and x''. Define F by identifying $F'|_U$ and $F''|_U$ with the free sheaf $\oplus_{n \in \mathbb{Z}} O_U e_n$ *via* the conditions

$$e_{2m} = e'_{2m}|_U = e''_{2m}|_U + h e''_{2m+1}|_U,$$
$$e_{2m+1} = e''_{2m+1}|_U = e'_{2m+1}|_U + h e'_{2m+2}|_U$$

for every $m \in \mathbb{Z}$ respectively.

We prove that $\Gamma(K, F) = 0$. For every $s \in \Gamma(K, F)$, by Lemma A.2.0.2, there is an open subset W of Δ containing K such that s lifts to an element of $\Gamma(W, F)$. By Corollary A.1.5.4, $\Gamma(U, F) = \oplus_{n \in \mathbb{Z}} \Gamma(U, O_\Delta) e_n$. So, $s|_{U \cap W} = \sum_{n \in \mathbb{Z}} f_n e_n$ with $f_n \in O_\Delta(U \cap W)$ that vanish for all but finitely many n. Note that

$$s|_{U''} = \sum_{n \in \mathbb{Z}} (f_{2n} e''_{2n} + (f_{2n} h + f_{2n+1}) e''_{2n+1})|_{U''}.$$

Therefore, f_{2n} and $f_{2n}h + f_{2n+1}$ are holomorphic near x' for all $n \in \mathbb{Z}$. Similarly, f_{2n+1} and $f_{2n+1}h + f_{2n+2}$ are holomorphic near x'' for all $n \in \mathbb{Z}$.

We claim that $s|_{U\cap W} = 0$. Otherwise, let n_0 be the maximum with $f_{n_0} \neq 0$. If n_0 is odd (resp. even), f_{n_0} and hf_{n_0} are holomorphic near x'' (resp. x'). The ratio $h = hf_{n_0}/f_{n_0}$ is meromorphic near x'' (resp. x'). It contradicts the choice of h. The claim is proved.

By Lemma A.2.0.1, the restriction map $\Gamma(W, F) \to \Gamma(W \cap U, F)$ is injective, so $s = 0$.

We prove that F is not good. Let t be the standard coordinate on Δ, then $0 \to F \xrightarrow{t} F \to F/tF \to 0$ is a short exact sequence in $\mathrm{Mod}(O_\Delta)$. The associated cohomology sequence induces an injection $H^0(K, F/tF) \to H^1(K, F)$ by Lemma A.2.0.2. As F/tF is the skyscraper supported at the origin, we have $H^0(K, F/tF) \neq 0$ and hence $H^1(K, F) \neq 0$. By Lemma A.2.0.4, the O_Δ-module F is not good. and $F|_K$ is not induced by a $\Gamma(K, O_K)$-module. In particular, F is not quasi-coherent in the sense of last paragraph of [BBP07, p.443]. Nevertheless, F is quasi-coherent in the sense of [EGA I, 5.1.3] since it is locally free.

Appendix B

Quasi-coherent GAGA

B.1 Introduction

Serre's work [Ser56] (known as GAGA) reveals the close relation between algebraic geometry and analytic geometry. An algebraic variety means a finite type, separated scheme over a field. Let X be a complex algebraic variety. Then the set of complex points $X(\mathbb{C})$ underlies a natural complex analytic space (in the sense of [Ser56, Déf. 1]) structure, denoted by X^{an}. When X is a projective variety, Serre [Ser56, Théorèmes 2 et 3] proves that the abelian category of (algebraic) coherent sheaves on X is naturally equivalent to that of (analytic) coherent sheaves on X^{an}. By [Ser56, Thm. 1], the equivalence leaves cohomology groups invariant. Hall [Hal23] extends the equivalence to the bounded derived category of coherent sheaves (Fact B.2.0.2).

A natural question is to find analogous equivalences for the larger category of quasi-coherent sheaves on X. We show that Kashiwara's notion of *good modules* (Definition A.1.4.1) is an analytic counterpart of quasi-coherent sheaves on algebraic varieties.

For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the abelian category of O_X-modules. Let $D(X)$ be the unbounded derived category of $\mathrm{Mod}(O_X)$. For an algebraic variety (resp. a complex analytic space) X, let $\mathrm{Qch}(X) \subset \mathrm{Mod}(O_X)$ (resp. $\mathrm{Good}(X) \subset \mathrm{Mod}(O_X)$) be the full subcategory of quasi-coherent (resp. good) modules. Let $D_{\mathrm{qc}}(X)$ (resp. $D_{\mathrm{good}}(X)$) be the full subcategory of $D(X)$ comprised of objects with quasi-coherent (resp. good) cohomologies.

Theorem (Proposition B.3.0.2, Theorem B.4.0.2)**.** *Let X be a proper scheme over $\mathbb{C}$. Then the analytification functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{good}}(X^{\mathrm{an}})$ is an equivalence of triangulated categories. It restricts to an equivalence of abelian categories $\mathrm{Qch}(X) \to \mathrm{Good}(X^{\mathrm{an}})$.*

B.2 Review

We recall the work of Serre [Ser56], which gives an equivalence of algebraic coherent sheaves and analytic coherent sheaves on complex projective varieties. The theory is extended to complex, proper algebraic varieties in [SGA 1, Exp. XII].

Let X be a complex algebraic variety. Let An (resp. Set) be the category of complex analytic spaces (resp. sets). Let Ψ_X be the functor An $\to$ Set sending a complex analytic space Y to the set $\mathrm{Hom}_{\mathbb{C}}(Y, X)$ of morphisms of spaces with a sheaf of $\mathbb{C}$-algebras. By [SGA 1, Exp. XII, Thm. 1.1], the functor Ψ_X is represented by a complex analytic space X^{an} (called the *analytification* of X) and a *flat* morphism $\psi_X \in \mathrm{Hom}_{\mathbb{C}}(X^{\mathrm{an}}, X)$. From [SGA 1, Exp. XII, Prop. 2.1 (viii)], because X is of finite type over $\mathbb{C}$, the dimension of X^{an} is finite.

Remark B.2.0.1. Strictly speaking, complex analytic spaces are allowed to be non-Hausdorff in [SGA 1, Exp. XII]. In our case, the algebraic variety X is assumed to be separated over $\mathbb{C}$. By [SGA 1, Exp. XII, Prop. 3.1 (viii)], the topology of X^{an} is Hausdorff.

By [SGA 1, Exp. XII, 1.2], for every morphism $f : X \to Y$ of complex algebraic varieties, there is a commutative square

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "\psi_X"] \arrow[d, "f^{\mathrm{an}}"] & X \arrow[d, "f"] \\ Y^{\mathrm{an}} \arrow[r, "\psi_Y"] & Y \end{tikzcd} \tag{B.1}$$

in the category of ringed spaces. In other words, the analytification induces a functor $(\cdot)^{\mathrm{an}}$ from the category of complex algebraic varieties to An.

For a ringed space (Y, O_Y), let $\mathrm{Coh}(Y) \subset \mathrm{Mod}(O_Y)$ be the full subcategory comprised of coherent sheaves in the sense of [Sta24, Tag 01BV]. Let $D_c(Y) \subset D(Y)$ and $D^b_c(Y) \subset D^b(Y)$ be the full subcategories of objects with coherent cohomologies. As ψ_X is flat, the pullback functor

$$\psi_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X^{\mathrm{an}}}), \quad F \mapsto F^{\mathrm{an}} \tag{B.2}$$

is exact, admits a right adjoint and commutes with colimits. It extends to a functor $D(X) \to D(X^{\mathrm{an}})$, which is t-exact relative to the standard t-structures. From [SGA 1, Exp. XII, 1.3], it restricts to a functor $D^b_c(X) \to D^b_c(X^{\mathrm{an}})$ and $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$.

Fact B.2.0.2 can be retracted from [Hal23, Remark 1.1 and the proof of Theorem A]. Neeman [Nee21, Example A.2] modifies Hall's proof to some extent.

Fact B.2.0.2. *Assume that the complex algebraic variety X is proper. Then the functor* (B.2) *induces an equivalence $D^b_c(X) \to D^b_c(X^{\mathrm{an}})$ of triangulated* *categories. In particular, it restricts to an equivalence of abelian categories* $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$*.*

By Lemma A.1.4.3, $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$, and $D_{\mathrm{good}}(X)$ is a triangulated subcategory of $D(X)$.

Lemma B.2.0.3. *For the complex algebraic variety X, the functor (B.2) restricts to a functor*

$$\mathrm{Qch}(X) \to \mathrm{Good}(X^{\mathrm{an}}) \tag{B.3}$$

and induces a functor

$$D_{\mathrm{qc}}(X) \to D_{\mathrm{good}}(X^{\mathrm{an}}). \tag{B.4}$$

Proof. By Fact B.2.0.4, every quasi-coherent O_X-module F can be written as the sum of a direct family of coherent O_X-submodules

$$F = \sum_{i \in I} F_i. \tag{B.5}$$

As ψ_X^* commutes with colimits, one has

$$\psi_X^* F = \mathrm{colim}_{i \in I} \psi_X^* F_i \tag{B.6}$$

in the category $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Since ψ_X^* is exact, each $\psi_X^* F_i$ is a coherent $O_{X^{\mathrm{an}}}$-*submodule* of $\psi_X^* F$. Therefore, the $O_{X^{\mathrm{an}}}$-module $\psi_X^* F$ is good.

For every $G \in D_{\mathrm{qc}}(X)$ and every integer n, because (B.2) is an exact functor, the $O_{X^{\mathrm{an}}}$-module $H^n(\psi_X^* G) = \psi_X^*(H^n G)$ is good by last paragraph. Hence $\psi_X^* G \in D_{\mathrm{good}}(X^{\mathrm{an}})$. □

Fact B.2.0.4 ([EGA I, Cor. 9.4.9], [Sta24, Tag 01PG])**.** *On a Noetherian scheme, every quasi-coherent sheaf is the sum of the directed family of all coherent submodules.*

B.3 GAGA for quasi-coherent sheaves

Using Fact B.2.0.4 and that ψ_X^* commutes with colimits, we extend GAGA from coherent sheaves to quasi-coherent sheaves.

When $Y = \operatorname{Spec} \mathbb{C}$, Proposition B.3.0.1 generalizes [Ser56, Thm. 1].

Proposition B.3.0.1. *Let $f : X \to Y$ be a* proper *morphism of complex algebraic varieties. Then the base change natural transformation $(Rf_* \cdot)^{\mathrm{an}} \to Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ (induced by the commutative square* (B.1)*) induces an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{good}}(Y^{\mathrm{an}})$.*

Proof. By [Lip60, Prop. 3.9.2], for every $F \in D_{\mathrm{qc}}(X)$, one has $Rf_*F \in D_{\mathrm{qc}}(Y)$. By Lemma B.2.0.3, one has $F^{\mathrm{an}} \in D_{\mathrm{good}}(X^{\mathrm{an}})$ and $(Rf_*F)^{\mathrm{an}} \in D_{\mathrm{good}}(Y^{\mathrm{an}})$. From [SGA 1, Exp. XII, Prop. 3.2 (v)], since f is proper, the morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper. By Theorem 5.3.1.7, as X^{an} has finite dimension, one has $Rf_*^{\mathrm{an}}F^{\mathrm{an}} \in D_{\mathrm{good}}(Y^{\mathrm{an}})$. Therefore, both functors $(Rf_*\cdot)^{\mathrm{an}}$ and $Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ restrict to functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{good}}(Y^{\mathrm{an}})$.

We prove that the morphism $(Rf_*F)^{\mathrm{an}} \to Rf_*^{\mathrm{an}}F^{\mathrm{an}}$ is an isomorphism. By Lemma 5.3.1.11 (resp. [Lip60, Prop. 3.9.2]), the functor $Rf_*^{\mathrm{an}} : D(X^{\mathrm{an}}) \to D(Y^{\mathrm{an}})$ (resp. $Rf_* : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$) is bounded. From [Sta24, Tag 06YZ], the inclusion functor $\mathrm{Qch}(X) \to \mathrm{Mod}(O_X)$ exhibits $\mathrm{Qch}(X)$ as a weak Serre subcategory of $\mathrm{Mod}(O_X)$ in the sense of [Sta24, Tag 02MO]. Then by the way-out argument [Har66, I, Prop. 7.1 (iii)], one may assume $F \in \mathrm{Qch}(X)$. By [KS06, Prop. 13.1.5 (ii), p.320], it suffices to check that for every integer $n \geq 0$, the natural morphism $(R^nf_*F)^{\mathrm{an}} \to R^nf_*^{\mathrm{an}}(F^{\mathrm{an}})$ in $\mathrm{Mod}(O_{Y^{\mathrm{an}}})$ is an isomorphism.

One can write F as in (B.5). By [Sta24, Tag 07TB], one has

$$\mathrm{colim}_{i\in I} R^nf_*F_i \xrightarrow{\sim} R^nf_*F.$$

The analytification commutes with colimits, so

$$\mathrm{colim}_{i\in I}(R^nf_*F_i)^{\mathrm{an}} \xrightarrow{\sim} (R^nf_*F)^{\mathrm{an}}.$$

By [SGA 1, XII, Thm. 4.2], the natural morphisms $(R^nf_*F_i)^{\mathrm{an}} \to R^nf_*^{\mathrm{an}}(F_i^{\mathrm{an}})$ are isomorphisms. By Lemma 5.3.1.9, the natural morphism

$$\mathrm{colim}_{i\in I} R^nf_*^{\mathrm{an}}(F_i^{\mathrm{an}}) \to R^nf_*^{\mathrm{an}}(F^{\mathrm{an}})$$

is an isomorphism. □

Proposition B.3.0.2 shows that goodness on complex analytic spaces is an analytic counterpart of quasi-coherence on complex algebraic varieties.

Proposition B.3.0.2. *Suppose that the complex algebraic variety X is proper. Then (B.3) is an equivalence of abelian categories.*

Proof.

- The functor (B.3) is essentially surjective: Indeed, by properness of X and [SGA 1, Exp. XII, Prop. 3.2 (v)], X^{an} is compact. Then for every good $O_{X^{\mathrm{an}}}$-module G, one can write $G = \sum_{i\in I} G_i$ as the sum of a directed family of coherent $O_{X^{\mathrm{an}}}$-submodules. From the equivalence $\psi_X^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ in [SGA 1, XII, Thm. 4.4], there is a filtered inductive system $\{H_i\}_{i\in I}$ in $\mathrm{Coh}(X)$ whose analytification is the filtered inductive system $\{G_i\}_{i\in I}$. By [Sta24, Tag 01LA (4)], the colimit H of $\{H_i\}_{i\in I}$ in $\mathrm{Mod}(O_X)$ exists and lies in $\mathrm{Qch}(X)$. Because ψ_X^* commutes with colimits, one has $H^{\mathrm{an}} = \mathrm{colim}_{i\in I} G_i$. In particular, H^{an} is isomorphic to G in $\mathrm{Good}(X^{\mathrm{an}})$.

- The functor (B.3) is fully faithful: For any quasi-coherent O_X-modules F and G, we have to show that the canonical morphism

$$\mathrm{Hom}_{O_X}(F,G) \to \mathrm{Hom}_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}) \tag{B.7}$$

  is an isomorphism. Assume first that F is coherent.

  - From [GW20, Exercise 7.20 (b)], one has

$$[\mathcal{H}om_{O_X}(F,G)]^{\mathrm{an}} = \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}).$$

  - As F is of finite presentation, the O_X-module $\mathcal{H}om_{O_X}(F,G)$ is quasi-coherent.

  Therefore, by Proposition B.3.0.1, the canonical morphism

$$H^0(X, \mathcal{H}om_{O_X}(F,G)) \to H^0(X^{\mathrm{an}}, \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}))$$

  is an isomorphism, which is exactly (B.7).

  By (B.5) and (B.6), the general case follows.

□

B.4 Derived category of quasi-coherent sheaves

Theorem B.4.0.2 extends GAGA further to the derived category of quasi-coherent sheaves.

By [Sta24, Tag 0BKN], for every ringed space Y, the derived category $D(Y)$ has products and derived limits. This together with Definition B.4.0.1 plays an essential role in the step 4 of the proof of Theorem B.4.0.2.

Definition B.4.0.1.

- ([Sta24, Tag 07LS]) Let $\mathcal{A}$ be an additive category with arbitrary direct sums. An object $K \in \mathcal{A}$ is called *compact*, if $\mathrm{Hom}_{\mathcal{A}}(K, \cdot) : \mathcal{A} \to \mathrm{Ab}$ preserves direct sums.

- ([Sta24, Tag 090Z]) Let $\mathcal{D}$ be a triangulated category. Let $(K_n, f_n)_{n>0}$ be a system of objects of $\mathcal{D}$. An object K is a *homotopy colimit* of the system if the direct sum $\oplus_{n>0} K_n$ exists and there is an exact triangle $\oplus_{n>0} K_n \overset{(1-f_n)_n}{\to} \oplus_{n>0} K_n \to K \overset{+1}{\to} \oplus_{n>0} K_n[1]$. In this case, we write $K = \mathrm{hocolim}_n K_n$.

A triangulated category $\mathcal{D}$ with arbitrary direct sums is *compactly generated*, if there is a family of compact objects $\{E_i\}_{i \in I}$ such that $\oplus_{i \in I} E_i$ generates $\mathcal{D}$. Let S be an algebraic variety. By [Nee96, Prop. 2.5], $D_{\mathrm{qc}}(S)$ is compactly generated. By [Sta24, Tag 09M1], the compact objects of $D_{\mathrm{qc}}(S)$ are exactly the perfect complexes in $D(S)$. By [Orl06, p.1827], when S is smooth, the full subcategory of perfect complexes coincides with $D^b_c(S)$.

Theorem B.4.0.2. *If the complex algebraic variety X is proper, then the functor (B.4) is an equivalence of triangulated categories.*

Proof. By compactness of X^{an} and Lemma B.4.0.6, the perfect complex $O_{X^{\mathrm{an}}}$ is a compact object of $D(X^{\mathrm{an}})$. Then from the proof of [Hal23, Lem. 4.3], the functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ admits a right adjoint functor $R\psi_{\mathrm{qc},*} : D(X^{\mathrm{an}}) \to D_{\mathrm{qc}}(X)$ which preserves small coproducts.

1. The functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ is fully faithful.

The unit of the adjunction $\eta : \mathrm{Id} \to R\psi_{\mathrm{qc},*}\psi_X^*$ is a natural transformation of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$. From Fact B.2.0.2, it restricts to an isomorphism of functors $D_c^b(X) \to D_c^b(X)$. By [BB03, Thm. 3.1.1 1], the compact objects of $D_{\mathrm{qc}}(X)$ are precisely the perfect complexes. From [Nee96, Prop. 2.5], $D_{\mathrm{qc}}(X)$ is generated by a family of perfect complexes $\{E_i\}_{i\in I}$. By [Sta24, Tag 0FXU (1)], every perfect complex in $D(X)$ belongs to $D_c^b(X)$, so the η_{E_i} are isomorphisms. From Lemma B.4.0.5, the unit η is an isomorphism. Thus, 1 is proved.

2. The functor (B.4) restricts to an equivalence $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{good}}(X^{\mathrm{an}})$.

We prove that every object $F \in D^b_{\mathrm{good}}(X^{\mathrm{an}})$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{good}}(X^{\mathrm{an}})$. Induction on the cohomological length of F. By Proposition B.3.0.2, it holds when F has length zero. Suppose that it is true for objects of length $\leq n$ and F has length $n+1$. There is an integer i such that the truncations $\tau^{\leq i}F$ and $\tau^{>i}F$ have length $\leq n$. There is a canonical exact triangle

$$\tau^{\leq i}F \to F \to \tau^{>i}F \xrightarrow{+1} \tau^{\leq i}F[1]$$

in $D^b_{\mathrm{good}}(X^{\mathrm{an}})$. By 1 and the inductive hypothesis, the morphism $+1 : \tau^{>i}F \to \tau^{\leq i}F[1]$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{good}}(X^{\mathrm{an}})$. Then so is F. The essential surjectivity together with 1 proves 2.

3. The functor $\psi_X^* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{good}}(X^{\mathrm{an}})$ is an equivalence.

By Lemma B.4.0.3, for every $F \in D^+_{\mathrm{good}}(X^{\mathrm{an}})$, one has $\mathrm{hocolim}_{n>0}\, \tau^{\leq n}F \xrightarrow{\sim} F$. Every $\tau^{\leq n}F$ is in $D^b_{\mathrm{good}}(X^{\mathrm{an}})$. From 2, there is a system $(K_n)_{n>0}$ of objects of $D^b_{\mathrm{qc}}(X)$, whose image under ψ_X^* is isomorphic to the system $(\tau^{\leq n}F)_{n>0}$. Since (B.4) respects coproducts, it respects homotopy colimits. Since $\mathrm{Qch}(X)$ is closed under filtered colimits in $\mathrm{Mod}(O_X)$, the subcategory $D_{\mathrm{qc}}(X)$ is closed under homotopy colimits in $D(X)$. Then F is isomorphic to the image of $K := \mathrm{hocolim}_{n>0} K_n \in D_{\mathrm{qc}}(X)$ under ψ_X^*. There is an integer q, such that $H^i(F) = 0$ for every integer $i < q$. Then $\psi_X^* H^i(K) = H^i(\psi_X^* K) = 0$. By Proposition B.3.0.2, one has $H^i(K) = 0$. Hence $K \in D^+_{\mathrm{qc}}(X)$. Thus, the functor $\psi_X^* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{good}}(X^{\mathrm{an}})$ is essentially surjective. By 1, it is an equivalence.

4. Every $Z \in D_{\mathrm{good}}(X^{\mathrm{an}})$ is in the essential image of (B.4).

By Lemma B.4.0.4, the canonical morphism $Z \to \mathrm{Rlim}_{n>0}\, \tau^{\geq -n} Z$ is an isomorphism in $D(X^{\mathrm{an}})$. By 3, there is an inverse system $(Y^{-n})_{n>0}$ of objects of $D^+_{\mathrm{qc}}(X)$, whose image is isomorphic to the inverse system $(\tau^{\geq -n}Z)_{n>0}$. Let Y be $\mathrm{Rlim}_{n>0}\, Y^{-n}$ in $D(X)$. For any integers $n \geq 1$ and q, the functor ψ_X^* transforms the morphism $H^q(Y^{-n-1}) \to H^q(Y^{-n})$ in $\mathrm{Qch}(X)$ to $H^q(\tau^{\geq -n-1}Z) \to H^q(\tau^{\geq -n}Z)$ in $\mathrm{Good}(X^{\mathrm{an}})$.

The morphism $H^q(\tau^{\geq -n-1}Z) \to H^q(\tau^{\geq -n}Z)$ is a surjection, and when $n \geq -q$, it is an isomorphism. By Proposition B.3.0.2, so is $H^q(Y^{-n-1}) \to H^q(Y^{-n})$. Then by [Sta24, Tag 0A0J (1)], the canonical morphism $H^q(Y) \to H^q(Y^{\min(q,-1)})$ is an isomorphism. In particular, the O_X-module $H^q(Y)$ is quasi-coherent. Hence, Y is in $D_{\mathrm{qc}}(X)$.

For every integer $m > 0$, the functor ψ_X^* transforms the projection $\prod_{n>0} Y^{-n} \to Y^{-m}$ to $\psi_X^*(\prod_{n>0} Y^{-n}) \to \tau^{\geq -m}Z$. Thus, one has a morphism $\psi_X^*(\prod_{n>0} Y^{-n}) \to \prod_{n>0} \tau^{\geq -n}Z$ in $D(X^{\mathrm{an}})$. It fits to a commutative diagram

$$\begin{tikzcd}
\psi_X^*(\prod_{n>0} Y^{-n})[-1] \arrow[r] \arrow[d] & \psi_X^*(\prod_{n>0} Y^{-n})[-1] \arrow[r] \arrow[d] & \psi_X^* Y \arrow[r] \arrow[d, dashed] & \psi_X^*(\prod_{n>0} Y^{-n}) \arrow[d] \\
\prod_{n>0} \tau^{\geq -n}Z[-1] \arrow[r] & \prod_{n>0} \tau^{\geq -n}Z[-1] \arrow[r] & Z \arrow[r] & \prod_{n>0} \tau^{\geq -n}Z
\end{tikzcd}$$

in $D(X^{\mathrm{an}})$, where the rows are exact triangles. By TR3, it induces a morphism of triangles. Hence, one has a commutative square

$$\begin{tikzcd}
H^q(\psi_X^* Y) \arrow[r, "\sim"] \arrow[d, dashed] & H^q(\psi_X^* Y^{\min(q,-1)}) \arrow[d, "\sim"] \\
H^q(Z) \arrow[r, "\sim"] & H^q(\tau^{\geq \min(q,-1)} Z)
\end{tikzcd}$$

in $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Therefore, for every integer q, the induced morphism $H^q(\psi_X^* Y) \to H^q(Z)$ is an isomorphism. Consequently, the morphism $\psi_X^* Y \to Z$ is an isomorphism in $D_{\mathrm{good}}(X^{\mathrm{an}})$. Thus, 4 is proved.

By 4 and 1, the functor (B.4) is an equivalence. □

Lemma B.4.0.3. *Let $\mathcal{A}$ be an abelian category, where colimits over $\mathbb{Z}_{>0}$ exist and are exact. Then the natural transformation* $\mathrm{hocolim}_{n>0}\, \tau^{\leq n} \cdot \to \mathrm{Id}$ *is an isomorphism of functors $\mathcal{A} \to \mathcal{A}$.*

Proof. It follows from [Sta24, Tag 0949] and the construction of canonical truncations. □

Lemma B.4.0.4. *Let X be a complex analytic space. Then the natural transformation* $\mathrm{Id} \to \mathrm{Rlim}_{n>0}\, \tau^{\geq -n} \cdot$ *is an isomorphism of functors $D(X) \to D(X)$.*

Proof. For every $x \in X$, there is an integer $d_x \geq 0$, and a fundamental system $\mathcal{U}_x$ of open neighborhoods of x, such that every $U \in \mathcal{U}_x$ is a closed complex subspace of a domain in $\mathbb{C}^{d_x}$. By Fact 5.3.1.10, for every $E \in D(X)$, any integers $p > 2d_x$ and q, one has $H^p(U, H^q(E)) = 0$. Then by [Sta24, Tag 0D63], the canonical morphism $E \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} E$ is an isomorphism in $D(X)$. $\square$

Lemma B.4.0.5. *Let $\mathcal{C}, \mathcal{D}$ be triangulated categories. Assume that $\mathcal{C}$ has direct sums. Let $\{E_i\}_{i \in I}$ be a family of compact objects of $\mathcal{C}$ such that $\oplus_{i \in I} E_i$ generates $\mathcal{C}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be triangulated functors preserving direct sums. Let $\eta : F \to G$ be a natural transformation. If for every $i \in I$, the morphism $\eta_{E_i} : F(E_i) \to G(E_i)$ is an isomorphism in $\mathcal{D}$, then η is an isomorphism.*

Proof. From [Sta24, Tag 09SN], every object $X \in \mathcal{C}$ can be written as $X = \mathrm{hocolim}_{n>0} X_n$, where

- X_1 is a direct sum of shifts of the E_i,
- each transition morphism $X_n \to X_{n+1}$ fits into an exact triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$,
- and Y_n is a direct sum of shifts of the E_i.

Since F, G preserve direct sums, and the η_{E_i} are isomorphisms, so are the $\{\eta_{Y_n}\}_{n>0}$ and η_{X_1}. By [Sta24, Tag 014A] and induction on $n > 0$, one proves that the η_{X_n} are isomorphisms. By [BN93, Lem. 4.1], $F, G : \mathcal{C} \to \mathcal{D}$ preserve homotopy colimits. Therefore, η_X is an isomorphism. $\square$

Lemma B.4.0.6. *Let X be a compact complex analytic space. Then every perfect object of $D(X)$ belongs to $D^b_c(X)$. It is a compact object of $D(X)$ and of $D_{\mathrm{good}}(X)$.*

Proof. Let $E \in D(X)$ be a perfect object. By definition, there is an open covering $X = \cup_{i \in I} U_i$, such that for every $i \in I$, there is a morphism of complexes $E_i^\bullet \to E|_U$ which is a quasi-isomorphism, with $E_i^j = 0$ for all but finite many integers j, and every E_i^j is a direct summand of a finite free O_X-module. Since X is compact, one has $E \in D^b(X)$. By [Sta24, Tag 01BY (1)], every E_i^j is coherent. Therefore, every $H^j(E)|_{U_i}$ is coherent over O_{U_i}. Thus, $H^j(E)$ is coherent over O_X for all j. Hence $E \in D^b_c(X)$. In particular, E is in $D_{\mathrm{good}}(X)$.

Let $E^\vee := R\mathcal{H}om(E, O_X) \in D(X)$. From [Sta24, Tag 08DQ], there is a natural isomorphism $\mathrm{Hom}_{D(X)}(E, \cdot) \to H^0(X, E^\vee \otimes^L_{O_X} \cdot)$ of functors $D(X) \to \mathrm{Ab}$. The functor $E^\vee \otimes^L_{O_X} \cdot : D(X) \to D(X)$ commutes with direct sums. Since X is compact, $\dim X$ is finite. Then by Lemma B.4.0.7, the functor $H^0(X, \cdot) : D(X) \to \mathrm{Ab}$ also commutes with direct sums. Therefore,

E is a compact object of $D(X)$. By Lemma A.1.4.3 2, $D_{\mathrm{good}}(X)$ is closed under direct sums in $D(X)$. Then E is also a compact object of $D_{\mathrm{good}}(X)$. □

Lemma B.4.0.7. *Let $f : X \to Y$ be a proper morphism of complex analytic spaces. If $\dim X$ is finite, then the functor $Rf_* : D(X) \to D(Y)$ commutes with direct sums.*

Proof. First, we prove that for every integer q, there is a natural isomorphism

$$R^q f_* \xrightarrow{\sim} R^q f_* \tau_{\geq q-2\dim X} : D(X) \to \mathrm{Mod}(O_Y). \tag{B.8}$$

Indeed, by [Sta24, Tag 08J5], for every object $E \in D(X)$, there is an exact triangle $\tau_{\leq q-2\dim X-1}E \to E \to \tau_{\geq q-2\dim X}E \to (\tau_{\leq q-2\dim X-1}E)[1]$. It induces an exact sequence

$$R^q f_* \tau_{\leq q-2\dim X-1}E \to R^q f_* E \to R^q f_* \tau_{\geq q-2\dim X}E \to R^{q+1} f_* \tau_{\leq q-2\dim X-1}E$$

in $\mathrm{Mod}(O_Y)$. From Lemma 5.3.1.11, one has

$$R^q f_* \tau_{\leq q-2\dim X-1}E = R^{q+1} f_* \tau_{\leq q-2\dim X-1}E = 0.$$

Hence, one has an isomorphism $R^q f_* E \to R^q f_* \tau_{\geq -q-2\dim X}E$ which is functorial in E.

Let $\{E_i\}_{i\in I}$ be a family of objects of $D(X)$. Set $E = \oplus_{i\in I} E_i$. To prove that the canonical morphism $\oplus_{i\in I} Rf_* E_i \to Rf_* E$ in $D(Y)$ is an isomorphism, it suffices to show that for every integer q, the induced morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Since $\tau_{\geq q-2\dim X}E = \oplus_{i\in I} \tau_{\geq q-2\dim X} E_i$, by (B.8), one may assume that E and all the E_i are in $D^{\geq q-2\dim X}(X)$. Then from [Sta24, Tag 015J], one has canonical spectral sequences

$$R^s f_* H^t(E) \Rightarrow R^{s+t} f_* E, \quad R^s f_* H^t(E_i) \Rightarrow R^{s+t} f_* E_i.$$

By Lemma 5.3.1.9, for any integers s and t, the canonical morphism $\oplus_{i\in I} R^s f_* H^t(E_i) \to R^s f_* H^t(E)$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Consequently, the canonical morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ is an isomorphism. □

Corollary B.4.0.8. *If the complex algebraic variety X is proper, then the functor $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is an equivalence of triangulated categories.*

Proof. For every $F \in D_c(X)$ and every integer i, the $O_{X^{\mathrm{an}}}$-module $H^i(\psi_X^* F) = \psi_X^* H^i(F)$ is coherent. Thus, the functors $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is well-defined. By Theorem B.4.0.2, the functor $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is fully faithful. By Theorem B.4.0.2, for every $F \in D_c(X^{\mathrm{an}})$, there is $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. Then the $O_{X^{\mathrm{an}}}$-module $\psi_X^* H^i(G) = H^i(\psi_X^* G) \xrightarrow{\sim} H^i(F)$ is coherent. By Fact B.2.0.2 and Proposition B.3.0.2, the O_X-module $H^i(G)$ is coherent, so $G \in D_c(X)$. Therefore, $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is essential surjective and hence an equivalence. □

B.5 Compact objects

We determine the compact objects of the derived category of an algebraic compact complex manifold.

Corollary B.5.0.1. *Let X be a proper complex algebraic variety. Then the compact objects of $D_{\mathrm{good}}(X^{\mathrm{an}})$ are precisely the perfect complexes in $D(X^{\mathrm{an}})$.*

Proof. By compactness of X^{an} and Lemma B.4.0.6, prefect complexes are compact objects of $D_{\mathrm{good}}(X^{\mathrm{an}})$. Conversely, let F be a compact object of $D_{\mathrm{good}}(X^{\mathrm{an}})$. By Theorem B.4.0.2, there is a compact object $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. By [Sta24, Tag 09M1], G is a perfect complex in $D(X)$. By definition, F is a perfect complex in $D(X^{\mathrm{an}})$. $\square$

Let X be a compact complex manifold.

Question B.5.0.2. Does the full subcategory of $D_{\mathrm{good}}(X)$ of compact objects coincide with $D^b_c(X)$?

Question B.5.0.3. Is the category $D_{\mathrm{good}}(X)$ compactly generated?

When X is the analytification of a smooth proper complex algebraic variety, Corollary B.5.0.1 (resp. Theorem B.4.0.2) answers Questions B.5.0.2 (resp. B.5.0.3) affirmatively.

Appendix C

Quasi-coherent sheaves on complex analytic spaces

C.1 Introduction

Let (X, O_X) be a ringed space. The category of O_X-modules is denoted by $\mathrm{Mod}(O_X)$.

Definition C.1.0.1. An O_X-module F is called *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$ of x, two sets I, J and a morphism $O_U^{\oplus J} \to O_U^{\oplus I}$ with cokernel isomorphic to $F|_U$. The full subcategory of $\mathrm{Mod}(O_X)$ of quasi-coherent modules is denoted by $\mathrm{Qch}(X)$.

According to [Sta24, Tag 01BD], in general $\mathrm{Qch}(X)$ is not an abelian category. By [Sta24, Tag 06YZ], if X is a scheme, then $\mathrm{Qch}(X)$ is a weak Serre subcategory (in the sense of [Sta24, Tag 02MO (2)]) of $\mathrm{Mod}(O_X)$. We show a complex analytic analog of this result, contrary to a guess made in [m)].

Theorem C.1.0.2. *If X is a complex analytic space, then* $\mathrm{Qch}(X) \subset \mathrm{Mod}(O_X)$ *is a weak Serre subcategory. In particular, it is an abelian subcategory.*

C.2 Preliminaries

Let (X, O_X) be a ringed space. Every $O_X(X)$-module naturally induces a quasi-coherent O_X-module.

Example C.2.0.1 ([Sta24, Tag 01BI])**.** Let $f : (X, O_X) \to (\{*\}, O_X(X))$ be the morphism of ringed spaces, with $f : X \to \{*\}$ the unique map and $f_*^\natural : O_X(X) \to O_X(X)$ the identity. For an $O_X(X)$-module M, its pullback f^*M is called the sheaf associated with M. This O_X-module is quasi-coherent. The functor $f^* : \mathrm{Mod}(O_X(X)) \to \mathrm{Mod}(O_X)$ is called *localization* and denoted by $\tilde{\cdot}$.

From [EGA I, 4.1.1], on a scheme the direct sum of any family of quasi-coherent modules is quasi-coherent. It fails for complex manifolds, shown by Example C.2.0.2.

Example C.2.0.2. [sta] Let $X \subset \mathbb{C}$ be the unit open disk. For every integer $n \geq 2$, Gabber ([Con06, Eg. 2.1.6], see also Example A.2.0.5) constructs a locally free (hence quasi-coherent) O_X-module F_n of infinite rank, such that for every open subset $U \subset X$ containing $\{\pm 1/n\}$, one has $\Gamma(U, F_n) = 0$. We prove that $F := \oplus_{n\geq 2} F_n$ is not quasi-coherent.

Assume the contrary. Then there is an open neighborhood V of $0 \in X$, a set I and a quotient morphism $q : O_V^{\oplus I} \to F|_V$. There is an integer $N \geq 2$ with $\{\pm 1/N\} \subset V$. Let $p : F|_V \to F_N|_V$ be the quotient morphism. Because $\mathrm{Hom}_{\mathrm{Mod}(O_V)}(O_V, F_N|_V) = \Gamma(V, F_N) = 0$, the morphism $pq = 0$. However, it contradicts $F_N|_V \neq 0$.

Let X be a complex analytic space in the sense of [GR04, p.18]. For an inclusion $i : K \to X$ of a compact subset, let $O_K = i^{-1}O_X$. Then O_K is naturally a sheaf of rings on K.

Definition C.2.0.3. A compact subset K of X is a *Stein compactum*, if K has a fundamental system of open neighborhoods that are Stein subspaces of X. A Stein compactum K is *Noetherian* if $O_K(K)$ is a Noetherian ring.

Fact C.2.0.4 ([Fri67, Thm. I, 9, Rem. I, 10])**.** *Every $x \in X$ admits a neighborhood which is a Noetherian Stein compactum in X.*

Lemma C.2.0.5. *Let F be an O_X-module. Then the following conditions are equivalent:*

1. ([BBP07, Def. 5.1]) *Every $x \in X$ admits a neighborhood K which is a Noetherian Stein compactum, such that $F|_K$ is associated with an $O_K(K)$-module.*

2. *The O_X-module F is quasi-coherent.*

Proof.

- Assume Condition 1. For every $x \in X$, take such a K and suppose that $F|_K$ is associated with an $O_K(K)$-module M. There are sets I, J and an exact sequence $O_K(K)^{\oplus I} \to O_K(K)^{\oplus J} \to M \to 0$ in the category of $O_K(K)$-modules. By [Sta24, Tag 01BH], it induces an exact sequence $O_K^{\oplus I} \to O_K^{\oplus J} \to F|_K \to 0$ in $\mathrm{Mod}(O_K)$. Then the O_{K°-module $F|_{K^\circ}$ is quasi-coherent. Thus, Condition 2 is proved.

- Assume Condition 2. By [Sta24, Tag 01BK], because X is locally compact Hausdorff, for every $x \in X$, there is an open neighborhood U of x in X such that $F|_U$ is associated with a $\Gamma(U, O_X)$-module.

From Fact C.2.0.4, there is a neighborhood K of $x \in U$ which is a Noetherian Stein compactum. By [Sta24, Tag 01BJ] applied to the morphism $(K, O_K) \to (U, O_U)$ of ringed spaces, $F|_K$ is associated with an $O_K(K)$-module. Thus, Condition 1 is proved.

$\square$

Lemma C.2.0.6. *Let K be a Noetherian Stein compactum in X.*

1. *The natural transformation* $\mathrm{Id} \to \Gamma(K, \tilde{\cdot})$ *of functors* $\mathrm{Mod}(O_K(K)) \to \mathrm{Mod}(O_K(K))$ *is an isomorphism.*
2. *The localization functor* $\tilde{\cdot} : \mathrm{Mod}(O_K(K)) \to \mathrm{Mod}(O_K)$ *is exact, fully faithful.*
3. *For every $O_K(K)$-module M and every integer $q > 0$, one has $H^q(K, \tilde{M}) = 0$.*

Proof.

1. Let M be an $O_K(K)$-module. We prove that the morphism $M \to \Gamma(K, \tilde{M})$ is an isomorphism. Assume first that M is finitely generated. Then the result follows from [Tay02, p.299]. Assume now that M is arbitrary. Let $\{M_i\}_{i \in I}$ be the family of all finitely generated submodules of M. This family is directed in the inclusion relation and
$$M = \sum_{i \in I} M_i. \tag{C.1}$$
By [Sta24, Tag 01BH (4)], the localization functor preserves colimits. Therefore,
$$\tilde{M} = \mathrm{colim}_{i \in I} \tilde{M}_i. \tag{C.2}$$
By [God58, Thm. 4.12.1], one has
$$\Gamma(K, \tilde{M}) = \mathrm{colim}_{i \in I} \Gamma(K, \tilde{M}_i) = \mathrm{colim}_{i \in I} M_i = M.$$

2. The exactness is proved in [Tay02, Prop. 11.9.3 (ii)]. For any $M, N \in \mathrm{Mod}(O_K(K))$, we prove that the natural morphism
$$\mathrm{Hom}_{O_K(K)}(M, N) \to \mathrm{Hom}_{O_K}(\tilde{M}, \tilde{N}) \tag{C.3}$$
is an isomorphism.

   Assume first that M is finitely generated. As the ring $O_K(K)$ is Noetherian, the $O_K(K)$-module M is of finite presentation. Then by [GW20, Exercise 7.20 (b)], one has $\widetilde{\mathrm{Hom}_{O_K(K)}(M, N)} = \mathcal{H}om_{O_K}(\tilde{M}, \tilde{N})$. By Part 1, the morphism (C.3) is an isomorphism. Assume now that M is arbitrary. By (C.1) and (C.2), the morphism (C.3) is the inverse limit of the morphisms $\mathrm{Hom}_{O_K(K)}(M_i, N) \to \mathrm{Hom}_{O_K}(\tilde{M}_i, \tilde{N})$, each of which is an isomorphism.

3. When M is finitely generated, it follows from [Tay02, Prop. 11.9.2] and [Car57, Thm. 1 (B)]. Assume now that M is arbitrary. By (C.2) and [God58, Thm. 4.12.1], one has $H^q(K,\tilde{M}) = \mathrm{colim}_i H^q(K,\tilde{M}_i) = 0$.

$\square$

C.3 Proof of Theorem C.1.0.2

1. For every morphism $f : F \to G$ in $\mathrm{Qch}(X)$, we prove that $\ker(f), \mathrm{coker}(f)$ in $\mathrm{Mod}(O_X)$ lie in $\mathrm{Qch}(X)$.

By Lemma C.2.0.5, for every $x \in X$, there is a neighborhood A (resp. B) of $x \in X$ which is a Noetherian Stein compactum and an $O_A(A)$-module M (resp. $O_B(B)$-module N), such that $F|_A$ (resp. $G|_B$) is associated with M (resp. N). By Fact C.2.0.4, there is a neighborhood C of x in $A^\circ \cap B^\circ$ which is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F|_C$ (resp. $G|_C$) is associated with $M \otimes_{O_A(A)} O_C(C)$ (resp. $N \otimes_{O_B(B)} O_C(C)$). By Lemma C.2.0.6 2, there is a morphism

$$\phi : M \otimes_{O_A(A)} O_C(C) \to N \otimes_{O_B(B)} O_C(C)$$

in $\mathrm{Mod}(O_C(C))$ whose localization is $f|_C : F|_C \to G|_C$. The restriction functor $\mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{C^\circ})$ is exact, so $\ker(f)|_{C^\circ}$ (resp. $\mathrm{coker}(f)|_{C^\circ}$) is the localization of $\ker(\phi \otimes_{O_C(C)} \mathrm{Id}_{O_X(C^\circ)})$ (resp. $\mathrm{coker}(\phi \otimes_{O_C(C)} \mathrm{Id}_{O_X(C^\circ)})$) in $\mathrm{Mod}(O_X(C^\circ))$. Therefore, the O_X-modules $\ker(f)$ and $\mathrm{coker}(f)$ are quasi-coherent.

2. Let
$$0 \to F' \to F \to F'' \to 0 \tag{C.4}$$
be a short exact sequence in $\mathrm{Mod}(O_X)$, with F' and F'' quasi-coherent. We prove that F is quasi-coherent.

By Lemma C.2.0.5, for every $x \in X$, there is a neighborhood K' (resp. K'') of x which is a Noetherian Stein compactum, and an $O_{K'}(K')$-module M' (resp. $O_{K''}(K'')$-module M'') whose localization is $F'|_{K'}$ (resp. $F''|_{K''}$). By Fact C.2.0.4, there is a neighborhood K of $x \in K'^\circ \cap K''^\circ$ that is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F'|_K$ (resp. $F''|_K$) is associated with the $O_K(K)$-module $M' \otimes_{O_{K'}(K')} O_K(K)$ (resp. $M'' \otimes_{O_{K''}(K'')} O_K(K)$).

Let $P = \Gamma(K,F)$. By Lemma C.2.0.6 1 and 3, the sequence (C.4) induces a short exact sequence in $\mathrm{Mod}(O_K(K))$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to P \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0.$$

From Lemma C.2.0.6 2, by localization it induces a shot exact sequence in $\mathrm{Mod}(O_K)$:

$$0 \to M' \otimes_{\widetilde{O_{K'}(K')}} O_K(K) \to \tilde{P} \to M'' \otimes_{\widetilde{O_{K''}(K'')}} O_K(K) \to 0.$$

By restricting to K° and [Sta24, Tag 01BJ], one has a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M' \otimes_{O_{K'}(K')} O_X(K^\circ)} \arrow[r] \arrow[d] & \widetilde{P \otimes_{O_K(K)} O_X(K^\circ)} \arrow[r] \arrow[d] & \widetilde{M'' \otimes_{O_{K''}(K'')} O_X(K^\circ)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'|_{K^\circ} \arrow[r] & F|_{K^\circ} \arrow[r] & F''|_{K^\circ} \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{Mod}(O_{K^\circ})$. The vertical morphisms are given by the canonical morphism $P \otimes_{O_K(K)} O_X(K^\circ) \to \Gamma(K^\circ, F)$ in $\mathrm{Mod}(O_X(K^\circ))$, and the adjunction between functors $\tilde{\ } : \mathrm{Mod}(O_X(K^\circ)) \to \mathrm{Mod}(O_{K^\circ})$ and $\Gamma(K^\circ, \cdot) : \mathrm{Mod}(O_{K^\circ}) \to \mathrm{Mod}(O_X(K^\circ))$. The rows are exact, and the two outside vertical arrows are isomorphisms. By the five lemma, the middle vertical morphism is an isomorphism. By Example C.2.0.1, the O_{K°-module $F|_{K^\circ}$ is quasi-coherent. Consequently, F is quasi-coherent.

By 1, 2 and [Sta24, Tag 0754], $\mathrm{Qch}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$.

Appendix D

Complex analytic geometry

D.1 Dimension of the fiber product

Section D.1 aims at understanding the dimension of the fiber product of algebraic varieties/complex analytic spaces. The proof in the analytic case is inspired by that in the algebraic case. Therefore, we begin with the algebraic situation.

D.1.1 Algebraic case

Fix a field k. Under flatness condition, the dimension of the fiber product behaves well.

Lemma D.1.1.1. *Let X, Y, Z be three schemes of finite type over k and $f : X \to Z$, $g : Y \to Z$ be k-morphisms. Assume that the schemes X, Z are irreducible, Y is equidimensional, and g is flat. Put $W = X \times_Z Y$. If W is nonempty, then W is equidimensional of dimension $\dim X + \dim Y - \dim Z$.*

Proof. Applying [Har77, Ch. III, Corollary 9.6] to the flat morphism g, we find that g is of relative dimension $\dim Y - \dim Z$. By virtue of [Sta24, Tag 02NK], its base change $W \to X$ is also flat of relative dimension $\dim Y - \dim Z$. Then the reverse direction of the cited [Har77, Ch. III, Corollary 9.6] shows that W is equidimensional of dimension $\dim Y - \dim Z + \dim X$. $\square$

In the proof of Proposition D.1.1.2, the general case is reduced to the case of a flat morphism.

Proposition D.1.1.2. *Let $X, Y, Z/k$ be three finite type schemes, $f : X \to Z$, $g : Y \to Z$ be dominant k-morphisms. Assume that X, Z are irreducible and Y is equidimensional, and put $W = X \times_Z Y$, then $\dim W + \dim Z \geq \dim X + \dim Y$.*

Proof. Since the reduction $Z_{\text{red}} \to Z$ is a universal homeomorphism, we may assume that Z is an integral scheme. By generic flatness [Sta24, Tag 052A], there is a nonempty affine open subset $U \subset Z$ such that the restriction $g^{-1}(U) \to U$ is flat. By [Sta24, Tag 01UA], the morphism $g^{-1}(U) \to U$ is open. By shrinking U, we may assume further that $g^{-1}(U) \to U$ is surjective.

Because f is dominant, $f^{-1}(U)$ is a nonempty open subset of X. Therefore, by [Har77, Ch. II, Exercise 3.20 (e)] we have $\dim U = \dim Z$, $\dim f^{-1}(U) = \dim X$ and $g^{-1}(U)$ is equidimensional of dimension $\dim Y$. Hence, we may base change everything along $U \to Z$ which does not increase $\dim W$. In particular, we can assume that g is flat surjective. Then $W \to X$ is also flat surjective. In particular, $W \neq \emptyset$. We conclude by Lemma D.1.1.1. $\square$

Example D.1.1.3 shows that the inequality in Proposition D.1.1.2 can be strict.

Example D.1.1.3. If $f : X \to P^3_k$ is the blow up at a point $p \in P^3(k)$, then the morphism f is projective surjective, $\dim X = 3$, $\dim X \times_{P^3_k} X = 4$ and the defect of semismallness $r(f) = 1$.

Corollary D.1.1.4. *Let $X, Y/k$ be two finite type schemes and $f : X \to Y$ be a k-morphism. If the scheme X is irreducible, then $\dim X \times_Y X \geq 2\dim X - \dim \overline{f(X)}$, where $\overline{f(X)}$ is the Zariski closure of $f(X)$ in Y.*

Proof. Because the reduction $X_{\mathrm{red}} \to X$ is a universal homeomorphism, we may assume that X is reduced. Let $Z \to Y$ be the scheme theoretic image of f. By [Har77, Ch. II, Exercise 3.11 (d)], the induced morphism $X \to Z$ is dominant and the underlying topological space of Z is $\overline{f(X)}$. Therefore, Z is also irreducible. By magic square [Vak23, 1.3.S], the natural morphism $X \times_Z X \to X \times_Y X$ is the base change of the diagonal isomorphism $Z \to Z \times_Y Z$, hence also an isomorphism. By Proposition D.1.1.2, $\dim X \times_Y X = \dim X \times_Z X \geq 2\dim X - \dim Z$. $\square$

D.1.2 Analytic case

The contents of this section is parallel to those of Section D.1.1. Lemma D.1.2.1 is an analogue of [Har77, III, Corollary 9.6], whose proof is also a direct adaptation. A complex analytic space is called equidimensional if every irreducible component is of same dimension.

Lemma D.1.2.1. *Let $f : X \to Y$ be a flat morphism of complex analytic spaces, and assume that Y is irreducible. Then the following conditions are equivalent:*

1. *X is equidimensional of dimension $n + \dim Y$;*
2. *for every $y \in f(X)$, the fiber X_y is equidimensional of dimension n.*

In that case, we say f is flat of relative dimension n.

Proof. Assume 1. Given $y \in f(X)$, let Z be an irreducible component of X_y. Because the set of irreducible components of a complex analytic space is locally finite, there is $x \in Z$ which is not in any other irreducible component of X_y. Applying [CD94, Proposition 2.11, p.113], we have $\dim_x Z + \dim_y Y = \dim_x X$. As Y, Z are irreducible hence pure dimensional, we have $\dim_y Y = \dim Y$ and $\dim_x Z = \dim Z$. Now that $\dim_x X = \dim Y + n$, we have $\dim Z = n$.

Conversely, assume 2. Let W be an irreducible component of X. Let $x \in W$ be a point which is not contained in any other irreducible component of X and $y = f(x)$. Then we have $\dim_x X = \dim W$ and $\dim_y Y = \dim Y$. Applying [CD94, Proposition 2.11, p.113], we obtain

$$\dim_x(X_y) + \dim_y Y = \dim_x X.$$

By assumption, $\dim_x(X_y) = n$. Thus $\dim W = \dim Y + n$ as required. $\square$

Lemma D.1.2.2 is similar to Lemma D.1.1.1.

Lemma D.1.2.2. *Let $f : X \to Z$, $g : Y \to Z$ be complex analytic space morphisms. Assume that X, Z are irreducible, Y is equidimensional, and g is flat. Put $W = X \times_Z Y$. If W is nonempty, then W is equidimensional of dimension* $\dim X + \dim Y - \dim Z$.

Proposition D.1.2.3 is the main result of Section D.1.

Proposition D.1.2.3. *Let X, Y, Z be irreducible complex analytic spaces. Let $f : X \to Z$, $g : Y \to Z$ be morphisms and put $W = X \times_Z Y$. If f is surjective and the (Euclidean) topology of X is second-countable, then* $\dim W + \dim Z \geq \dim X + \dim Y$.

Proof. Because reduction does not change the dimension [CAS, p.96], we may assume that X, Y, Z are reduced. Let $A = \{x \in X : f \text{ is not flat at } x\}$. By Frisch's theorem [CD94, Theorem 2.8, p.112], A is an analytic subset of X and $f(A) \neq Z$. Then $X \setminus f^{-1}(f(A)) \to Z \setminus f(A)$ is a surjective flat morphism. By shrinking X, Y, Z suitably, we may assume further that f is flat surjective. Then W is nonempty and we conclude by Lemma D.1.2.2. $\square$

The invariant $\dim X \times_Y X$ considered in Corollary D.1.2.4 appears in the definition of defect of semismallness (Definition 4.5.2.1).

Corollary D.1.2.4. *Let $f : X \to Y$ be a* proper *morphism of irreducible complex analytic spaces. If the (Euclidean) topology of X is second-countable, then* $\dim X \times_Y X \geq 2 \dim X - \dim f(X)$.

Proof. The image $Z := f(X)$ is an analytic subset of Y. Endow Z with the reduced structure of complex analytic space. Then Z is also irreducible and the morphism $f : X \to Z$ is surjective. The natural morphism $X \times_Z X \to X \times_Y X$ is an isomorphism. Then we conclude by Proposition D.1.2.3. $\square$

D.2 Connection on line bundles

The purpose of Section D.2 is to show Lemma D.2.0.4. For one thing, it is closely related to Corollary 4.4.2.2. For another, it implies the possibility to extend the Donaldson-Uhlenbeck-Yau theorem and nonabelian Hodge theory to manifolds more general than Kähler ones (Remarks D.2.0.6 and D.2.0.7). For work towards this direction, see [BD23], which extends nonabelian Hodge theory to Fujiki class $\mathcal{C}$ manifolds.

We begin the proof with a variation of the classical maximum principle.

Proposition D.2.0.1. *Let $U \subset \mathbb{R}^n$ be a nonempty connected open subset, $f : U \to \mathbb{C}$ be a harmonic function. If $|f|$ attains its maximum in U, then f is constant.*

Lemma D.2.0.2 concerns the uniqueness of solution to $\bar{\partial}\partial$-equation.

Lemma D.2.0.2. *Let $f : X^n \to \mathbb{C}$ be a smooth function on a compact connected complex manifold X with $\bar{\partial}\partial f = 0$, then f is constant.*

Proof. Since X is compact, the subset $A = \{x \in X : |f(x)| = \max_{t \in X} |f(t)|\}$ is nonempty closed in X. For any $p \in A$, there exists a local holomorphic coordinate $(U; z_1, \ldots, z_n)$, where U is a connected open neighborhood of p in X. With this chart, we identify U as an open subset of $\mathbb{C}^n$. Since $\bar{\partial}\partial f = 0$, we have $\frac{\partial^2 f}{\partial \bar{z}_j \partial z_l} = 0$ for all $1 \leq j, l \leq n$. In particular, $\sum_{j=1}^n \frac{\partial^2 f}{\partial \bar{z}_j \partial z_j} = 0$, or equivalently, f is a harmonic function on U. By Proposition D.2.0.1, f is constant on U and so $U \subset A$. Therefore, A is open in X. By connectedness of X, $A = X$. So for any $p \in X$, f is locally constant near p. By connectedness of X again, f is constant. $\square$

Let X be a regular manifold for the rest of Section D.2.

We need a comparison between the Atiyah class (in the sense of [Huy05, Def. 4.2.18]) and the first Chern class. For Kähler manifolds, it is [Ati57a, Prop. 12].

Lemma D.2.0.3. *Let X be a regular manifold. Let $L \to X$ be a holomorphic line bundle. Let $A(L) \in H^1(X, \Omega_X^1)$ be the Atiyah class of L. Then*

$$\frac{i}{2\pi} A(L) = c_1^{\mathbb{R}}(L)$$

in $H^2(X, \mathbb{R})$. In particular, L admits a holomorphic connection if and only if $L \in \mathrm{Pic}^\tau(X)$.

Proof. By Corollary 4.3.1.4, we have a commutative diagram

$$\begin{tikzcd} Z^{1,1}(X) \arrow[r, hook] \arrow[d, "\phi"] & Z^2(X) \arrow[d, "\psi"] \\ H^1(X, \Omega_X^1) \arrow[r, "\iota^{1,1}"] & H^2_{dR}(X; \mathbb{C}), \end{tikzcd}$$

where ϕ is taking Dolbeault cohomology class and ψ is taking de Rham cohomology class. Take a hermitian metric h on L. Let R be the corresponding Chern curvature form. By [Huy05, Corollary 4.4.5], $R \in Z^{1,1}(X)$. Then by [Huy05, Proposition 4.3.10], $A(L) = \phi(R)$ and $c_1^{\mathbb{R}}(L) = \frac{i}{2\pi}\psi(R)$. The equality follows. The second part follows from [Huy05, Proposition 4.2.19]. $\square$

Lemma D.2.0.4. *Let X be a regular manifold, $L \in \mathrm{Pic}^{\tau}(X)$, then:*

1. *L admits a unique (up to a positive scalar) hermitian metric whose Chern connection is flat;*

2. *Every holomorphic connection on L is flat.*

Proof.

1. We begin with the existence. By Corollary 4.4.2.2 2, there is a unitary local system $\mathcal{L} \in \mathrm{Loc}^{u,1}(X)$ on X with $\mathcal{L} \otimes_{\mathbb{C}} O_X = L$. Applying Theorem 4.2.3.1 the existence of such metric follows.

   Now for uniqueness. Let h, h' be two hermitian metrics whose respective Chern connections ∇, ∇' are flat holomorphic connections. By Theorem 4.2.3.1, $\ker(\nabla), \ker(\nabla') \in \mathrm{Loc}^{u,1}(X)$ have the same induced line bundle. By Corollary 4.4.2.2 2, $\ker(\nabla) = \ker(\nabla')$ in $\mathrm{Loc}^{u,1}(X)$. The hermitian metrics h, h' restrict to two monodromy invariant hermitian forms on the common local system $\ker(\nabla)$. Moreover, by Theorem 4.2.3.1 one can recover the hermitian metric on the line bundle L from the restricted hermitian form on the local system. Since this local system is of rank 1, at one stalk these two hermitian forms differ by a scalar. Globally they differ by this scalar as they are monodromy invariant. Then the metrics h, h' also differ by a scalar.

2. By Lemma D.2.0.3 and [Huy05, Prop. 4.2.19], L admits a holomorphic connection. We show that the curvature forms (which are global holomorphic 2 forms) of different holomorphic connections on L are the same. In fact, for two such connections D, D' on L, by [Huy05, p.179], $D' - D \in H^0(X, \Omega_X^1)$. This form is d-closed by [Uen06, Corollary 9.5, p.101]. By [Huy05, Lemma 4.3.4], the curvature of D' equals that of D.

   We adopt the argument in [BK09, Footnote (6), p.388]. By Cartan-Serre theorem [Car53, Théorème], the complex vector space $H^0(X, \Omega_X^2)$ is finite dimensional. Taking the curvature form of one (hence every) holomorphic connection on elements of $\mathrm{Pic}^0(X)$, we get a holomorphic map $\mathrm{Pic}^0(X) \to H^0(X, \Omega_X^2)$. As the complex torus $\mathrm{Pic}^0(X)$ is compact connected, this map is constant. The canonical connection on the trivial line bundle $O_X(\in \mathrm{Pic}^0(X))$ is flat, so this map is constantly zero. In other words, for every $K \in \mathrm{Pic}^0(X)$, any holomorphic connection on K is flat.

As $L \in \mathrm{Pic}^\tau(X)$, there is an integer $n \geq 1$ such that $L^{\otimes n} \in \mathrm{Pic}^0(X)$. Take a holomorphic connection on L of curvature form R, then it induces a holomorphic connection on $L^{\otimes n}$ of curvature form nR. As $nR = 0$, it holds that $R = 0$. The flatness follows from the first paragraph and the existence in Point 1.

□

Remark D.2.0.5. Here is a second proof of Lemma D.2.0.4 1. Take a hermitian metric h on L. Locally its Chern curvature is given by $\nabla = d + h^{-1}\partial h$. More precisely, let s be a local holomorphic frame for L, and by abuse of notation let h be the local (smooth positive) function $h(s,s)$. Then $\nabla(s) = (h^{-1}\partial h) \otimes s$ and the Chern curvature form $R = \bar{\partial}(h^{-1}\partial h)$ is a d-closed smooth $(1,1)$-form whose de Rham class is 0. Moreover iR is a real form. (This is part of Chern-Weil theory, see [Huy05, Proposition 4.3.8 (iii), 4.3.10 and p.196].) Therefore, by Fact 4.3.1.2, there is a smooth function $f : X \to \mathbb{R}$ with

$$R + \bar{\partial}\partial f = 0. \tag{D.1}$$

Define a new hermitian metric h' by

$$h'(s,s) = e^f h(s,s). \tag{D.2}$$

Then the new Chern connection is given by $\nabla'(s) = \nabla(s) + (\partial f) \otimes s$. The new curvature form $R' = R + \bar{\partial}\partial f = 0$, *i.e.*, the new Chern connection is flat and compatible with the holomorphic structure, hence a holomorphic connection.

So far we have established the existence of such metric. As for uniqueness, any hermitian metric h' with flat Chern connection is in the form of (D.2) where f is a solution to (D.1). Lemma D.2.0.2 shows that such a solution f is unique up to addition by constant. So such metric h' is unique up to a positive scalar.

Remark D.2.0.6. When X is a compact Kähler manifold, Lemma D.2.0.4 1 is a consequence of known results. In fact [Kob87, Proposition 5.7.7 (a)] shows a holomorphic line bundle is slope stable. By Donaldson-Uhlenbeck-Yau theorem [UY86, Corollary 8.1, p.292], there is $\mathcal{L} \in \mathrm{Loc}^{u,1}(X)$ such that $L = \mathcal{L} \otimes_{\mathbb{C}} O_X$, and $\mathcal{L}$ induces such a metric *via* Theorem 4.2.3.1. For any such hermitian metric, its Chern connection is a Hermitian-Yang-Mills connection. The uniqueness of such metric is mentioned in [Bea92, (3.2) c)] and follows from [UY86, Theorem, p.262] and [Che22, Corollary 2.18].

Remark D.2.0.7. Lemma D.2.0.4 can be viewed as a step toward nonabelian Hodge theory on regular manifolds. In fact, by [Sim92, Example, p.21], a semisimple local system on a compact Kähler manifold is unitary if and only if the associated Higgs bundle (E, θ) has $\theta = 0$. The metric given by Lemma D.2.0.4 is exactly the harmonic metric provided by Corlette Theorem [GRR15, Theorem 1, p.151].

D.3 Jacobi inversion theorem

In this section, we give a refinement of Proposition 4.4.1.2 3.

Lemma D.3.0.1. *For a pointed regular manifold (X, x_0), for every $n \geq h^{1,0}(X)$, the holomorphic map $f_n : X^n \to \mathrm{Alb}(X)$ defined by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \alpha_{x_0}(x_i)$ is surjective.*

When X is a compact Riemann surface, then Lemma D.3.0.1 reduces to (part of) Jacobi inversion theorem in [GH78, p.235].

Two proofs are provided. They are inspired by [Voi02, Lemma 12.11] and [BL04, Proposition 11.11.8] respectively, but with an extra attention to the feasible range of n. The first proof is shorter, while the second proof provides a stronger result, Lemma D.3.2.1.

D.3.1 First proof

Lemma D.3.1.1. *Let X be a compact complex manifold. Then there there is subset $S \subset X$ with $\#S \leq h^{1,0}(X)$ such that, for any $\eta \in H^0(X, \Omega_X^1)$ with $\eta(x) = 0$ in the (holomorphic) cotangent space $(T_x^h X)^\vee$, we have $\eta = 0$.*

Proof. For every $x \in X$, let V_x be the subspace $\{\eta \in H^0(X, \Omega_X^1) : \eta(x) = 0\}$ of $H^0(X, \Omega_X^1)$. Then $\cap_{x \in X} V_x = \{0\}$. Hence, there is a subset $S \subset X$ with $\#S \leq h^{1,0}(X)$ and $\cap_{x \in S} V_x = \{0\}$. □

Here is the first proof.

First proof of Lemma D.3.0.1. Consider the cotangent map $(d_p f_n)^* : (T^h_{f_n(p)} \mathrm{Alb}(X))^\vee \to (T_p^h X^n)^\vee$ at $p = (p_1, \ldots, p_n) \in X^n$. Since the cotangent bundle $\Omega^1_{\mathrm{Alb}(X)}$ is trivial, this map is identified with the composition

$$H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) \to (T^h_{f_n(p)} \mathrm{Alb}(X))^\vee \to \prod_{i=1}^n (T^h_{p_i} X)^\vee.$$

By Proposition 4.4.1.2 4, it is further identified with the natural map

$$H^0(X, \Omega_X^1) \to \prod_{i=1}^n (T^h_{p_i} X)^\vee. \tag{D.3}$$

By Lemma D.3.1.1, there exist $n_0 \leq h^{1,0}(X)$ and $x = (x_1, \ldots, x_{n_0}) \in X^{n_0}$ such that for any $\eta \in H^0(X, \Omega_X^1)$ with $\eta(x_i) = 0$ for all i, we have $\eta = 0$. Then for every $n \geq n_0$, the map (D.3) is injective when $p = (x, x_0, \ldots, x_0)$. Or equivalently, f_n is a submersion of smooth manifolds near p. From local normal form theorem, the image $f_n(X^n)$ contains a nonempty open subset of $\mathrm{Alb}(X)$. By Remmert theorem [Whi72, Theorem 4A, p.150], $f_n(X^n)$ is an analytic subset of $\mathrm{Alb}(X)$. By [CAS, Theorem, p.168], $f_n(X^n) = \mathrm{Alb}(X)$, i.e, f_n is surjective. □

D.3.2 Second proof

To certain extent, Lemma D.3.2.1 shows that a generating subset of a complex torus generates the complex torus "uniformly".

Lemma D.3.2.1. *Let A be a g-dimensional commutative complex Lie group. Let M be a compact irreducible analytic subset of A containing* 0*. If the complex Lie subgroup of A generated by M is A, then for every integer $n \geq g$, the map $f_n : M^n \to A$ defined by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$ is surjective. In particular, A is a complex torus.*

Proof. Since M is connected, the identity component of A contains M. Therefore, A is connected.

The statement is true when $g = 0$. So we assume $g > 0$, then $M \neq \{0\}$ and hence $\dim M \geq 1$. For every $n \geq 1$, let $A_n = f_n(M^n)$, which is an analytic subset of A by Remmert theorem [Whi72, Theorem 4A, p.150]. Since $f_1 : M \to A$ is the inclusion, we find $A_1 = M \ni 0$. For every $x \in M^n$, $f_{n+1}(x, 0) = f_n(x)$, so $A_n \subset A_{n+1}$, hence an increasing sequence of analytic subsets of A:

$$A_1 \subset A_2 \subset \ldots$$

Consider the integer sequence of analytic dimensions $\{\dim_0 A_n\}_{n\geq 1}$. By [CAS, p.96], this sequence is non-decreasing and bounded above by $\dim_0 A = g$. Therefore, there is $n_0 \leq g$ such that $\dim_0 A_{n_0} = \dim_0 A_{n_0+1}$.

By assumption, M^n is an irreducible complex analytic space. By [CD94, (14.14), p.89], the complex analytic space A_n is irreducible and pure dimensional for every $n \geq 1$ and $A_{n_0} = A_{n_0+1}$.

We claim that for every $m > n_0$, $A_{n_0} = A_m$.

We prove the claim by induction on m. It holds when $m = n_0 + 1$. If it is true for $m - 1$ with $m \geq n_0 + 2$, then for every $(x_1, \ldots, x_m) \in M^m$, $\sum_{i=1}^{m-1} x_i \in A_{m-1} = A_{n_0}$, so there is $(p_1, \ldots, p_{n_0}) \in M^{n_0}$ with $\sum_{j=1}^{n_0} p_j = \sum_{i=1}^{m-1} x_i$. Then

$$\sum_{i=1}^{m} x_i = x_m + \sum_{j=1}^{n_0} p_j \in A_{n_0+1} = A_{n_0}.$$

Therefore, $A_m = A_{n_0}$. The induction is completed.

For every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in M^n$, we have $f_n(x) + f_n(y) = f_{2n}(x, y)$, so $A_n + A_n \subset A_{2n}$. In particular, $A_{n_0} + A_{n_0} \subset A_{2n_0} = A_{n_0}$. This shows A_{n_0} is closed under addition.

We are going to show that $-A_{n_0} = A_{n_0}$ and then A_{n_0} would be a subgroup of A.

As A is commutative connected, by [AK01, Proposition 1.1.2], its universal covering is in the form of $\pi : \mathbb{C}^g \to A$ and the lattice $\ker(\pi)$ is identified with the fundamental group $\pi_1(A, 0)$. As π is locally biholomorphic and every

A_n is irreducible, the preimage $\pi^{-1}(A_n)$ is an analytic subset of $\mathbb{C}^g$, every irreducible component of whom is of dimension $\dim A_n$. Any two different irreducible components are disjoint and differ by a translation by an element of $\ker(\pi)$.

Let V_n be the unique irreducible component of $\pi^{-1}(A_n)$ containing 0, then $\pi(V_n) = A_n$. Fix an integer $k \geq 1$ and let $[k] : A \to A$ be the multiplication by k. As A_{n_0} is closed under addition, we get $[k]A_{n_0} \subset A_{n_0}$. As π is a group morphism, we have $k \cdot \pi^{-1}(A_{n_0}) \subset \pi^{-1}(A_{n_0})$. As $k : \mathbb{C}^g \to \mathbb{C}^g$ is biholomorphic, kV_{n_0} is an irreducible analytic subset of $\mathbb{C}^g$ isomorphic to V_{n_0}. As $0 \in kV_n$, we have $kV_{n_0} \subset V_{n_0}$. As $\dim kV_{n_0} = \dim V_{n_0}$, by [CD94, (14.14), p.89] again, we get $kV_{n_0} = V_{n_0}$, *i.e.*, the morphism $k : V_{n_0} \to V_{n_0}$ is biholomorphic.

For every $x(\neq 0) \in A_{n_0}$, we check that $-x \in A_{n_0}$. In fact, take $v \in V_{n_0} \cap \pi^{-1}(x)$. Then $v \neq 0$. By last paragraph, $v/k \in V_{n_0}$ for every $k \geq 1$. Let l be the complex line in $\mathbb{C}^g$ spanned by v. By the identity theorem for holomorphic functions on l, the smallest analytic subset of l containing $\{v/k\}_{k \geq 1}$ is l. Now that $V_{n_0} \cap l$ is an analytic subset of l containing $\{v/k\}_{k \geq 1}$, we get $l = V_{n_0} \cap l \subset V_{n_0}$. In particular, $-v \in V_{n_0}$ and then $-x \in A_{n_0}$ as desired.

So far we have shown that A_{n_0} is a subgroup of A that is a complex analytic subset. By Corollary F.2.0.5, A_{n_0} is an embedded complex Lie subgroup of A. By assumption, $A_{n_0} = A$. From the claim we get the surjectivity of f_n for every $n \geq n_0$. In particular, A is compact, hence a complex torus. □

Example D.3.2.2. In Lemma D.3.2.1, we cannot remove the condition that $0 \in M$. For example, consider $A = \mathbb{C}^*$ and $M = \{2\}$. The irreducibility of M is also necessary. For instance, take A to be the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$, $M = \{0, x\}$, where $x \in A \setminus A_{\mathrm{tor}}$. Then f_n is not surjective for all integers $n \geq 1$.

Second proof of Lemma D.3.0.1. Let $M = \alpha_{x_0}(X)$, which is an irreducible analytic subset of $\mathrm{Alb}(X)$ by Remmert theorem [Whi72, Theorem 4A, p.150] and [CD94, (14.14), p.89]. In addition, $0 \in M$. The proof is completed by citing Proposition 4.4.1.2 3 and Lemma D.3.2.1. □

Appendix E

D-modules

E.1 Unbounded Bernstein's equivalence

In Section E.1, let X be a smooth algebraic variety over be an algebraically closed field k of characteristic zero. Let $\mathrm{Qch}(O_X) \subset \mathrm{Mod}(O_X)$ and $\mathrm{Mod}_{\mathrm{qc}}(D_X) \subset \mathrm{Mod}(D_X)$ be the full subcategories of objects quasi-coherent over O_X. They are weak Serre subcategories.

Fact E.1.0.1 (Bernstein, [Bor+87, VI, Thm. 2.10])**.** *The natural functor*

$$\iota'_X : D^b(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D^b_{\mathrm{qc}}(D_X)$$

is an equivalence.

Remark E.1.0.2. The first sentence of the proof in [Bor+87] needs (implicitly) [Mur07, Remark 64] and Fact E.1.0.3.

Fact E.1.0.3 can be proved as [Bor+87, I, Prop. 12.8, VI, Prop. 1.14].

Fact E.1.0.3. *Let B be an weak Serre subcategory of an abelian category A. Then the full class* $\mathrm{Ob}(B)$ *of objects in B is a generating class of* $D^b_B(A)$ *(defined in [Sta24, Tag 06UP]) in the sense of [Bor+87, I, Def. 12.4].*

Theorem E.1.0.4 is an unbounded generalization of Fact E.1.0.1. It is left "to the reader to state and prove" in [Nee96, p.207]. We follow the strategy pointed out in [gdb], and do not claim originality here.

Theorem E.1.0.4. *The functor*

$$\iota'_X : D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D_{\mathrm{qc}}(D_X) \tag{E.1}$$

induced by the inclusion $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X)$ *is an equivalence of categories.*

We need a series of lemmas for the proof of Theorem E.1.0.4. For an open immersion $j : U \to X$, we have a natural morphism of ringed spaces $j : (U, D_U) \to (X, D_X)$. From [Bor+87, VI, 5.2] and [HT07, Prop. 1.5.29], the functor $j_+ : D(D_U) \to D(D_X)$ is the right derived functor of the corresponding (left exact) direct image $j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}(D_X)$. By [Ber83, 2, p.12] and [Sta24, Tag 0096], the inverse image $j^* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_U)$ is left adjoint to j_*. Lemma E.1.0.5 2 helps to construct a quasi-inverse to (E.1).

Lemma E.1.0.5.

1. *The category $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is a locally noetherian Grothendieck category.*
2. *The inclusion functor $\iota' : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X)$ admits a right adjoint $Q' = Q'_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$. The unit natural transform $\eta' : \mathrm{Id}_{\mathrm{Mod}_{\mathrm{qc}}(D_X)} \to Q'\iota'$ is an isomorphism.*

Proof. By [Sta24, Tag 01LA (4)], $\mathrm{Qch}(O_X) \subset \mathrm{Mod}(O_X)$ is an abelian subcategory closed under colimits. Then so is $\mathrm{Mod}_{\mathrm{qc}}(D_X) \subset \mathrm{Mod}(D_X)$.

1. When X is affine, by [HT07, Prop. 1.4.4 (ii)], the functor $\Gamma(X, \cdot) : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X(X))$ is an equivalence of abelian categories. As the ring $D_X(X)$ is left noetherian, the category $\mathrm{Mod}(D_X(X))$ is locally noetherian by the last paragraph of [Gab62, p.402].

   For a general X, one may assume that there exists an open covering $X = U \cup V$, such that the statement holds for U and V. Arguing as in [Gab62, Prop. 2, p.441], one can prove that $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is the gluing of $\mathrm{Mod}_{\mathrm{qc}}(D_U)$ and $\mathrm{Mod}_{\mathrm{qc}}(D_V)$ along $\mathrm{Mod}_{\mathrm{qc}}(D_{U\cap V})$ in the sense of [Gab62, VI. 1]. Let $j : U \to X$ be the inclusion. Then

   $$j^* : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_U)$$

   is exact and left adjoint to

   $$j_* : \mathrm{Mod}_{\mathrm{qc}}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X).$$

   The (counit) natural transformation $\epsilon : j^*j_* \to \mathrm{Id}_{\mathrm{Mod}_{\mathrm{qc}}(D_U)}$ is an isomorphism of functors $\mathrm{Mod}_{\mathrm{qc}}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_U)$. From [Gab62, Prop. 5, p.374], the subcategory $\ker(j^*)$ of $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is localizing (in the sense of [Gab62, p372]), and j^* induces an equivalence

   $$\mathrm{Mod}_{\mathrm{qc}}(D_X)/\ker(j^*) \to \mathrm{Mod}_{\mathrm{qc}}(D_U).$$

   A similar result holds for V. Then by [Gab62, Lem. 2, p.442], the gluing category $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is locally noetherian.

2. It follows from 1 and Lemma E.1.0.7.

$\square$

Remark E.1.0.6. For an affine (possibly singular) variety V, by [GR14, 4.7.1, 5.5], the abelian category $\mathrm{Mod}_{\mathrm{qc}}(D_V)$ is still Grothendieck.

Lemma E.1.0.7. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor preserving all colimits.*

1. *Then F admits a right adjoint $G : \mathcal{B} \to \mathcal{A}$.*
2. *If further F is fully faithful, then the unit natural transformation $\eta : \mathrm{Id}_{\mathcal{A}} \to GF$ is an isomorphism.*

Proof.

1. Let Set be the category of sets. For each object $Y \in \mathcal{B}$, consider the functor
$$\mathrm{Hom}_{\mathcal{B}}(F(\cdot), Y) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}.$$
It transforms colimits into limits. Then by [Sta24, Tag 07D7], it is representable. From [ML78, Cor. 2, p.85], the functor F admits a right adjoint.
2. If follows from Yoneda's lemma.

$\square$

By [Sta24, Tag 077P (2)], the inclusion $\iota = \iota_X : \mathrm{Qch}(O_X) \to \mathrm{Mod}(O_X)$ admits a right adjoint $Q_X = Q : \mathrm{Mod}(O_X) \to \mathrm{Qch}(O_X)$, called the *coherator* of X. To reduce the problem to the study of O_X-modules, consider the square

$$\begin{array}{ccc} \mathrm{Mod}(D_X) & \xrightarrow{Q'_X} & \mathrm{Mod}_{\mathrm{qc}}(D_X) \\ \downarrow{\scriptstyle \mathrm{for}_X} & & \downarrow{\scriptstyle \mathrm{for}_X} \\ \mathrm{Mod}(O_X) & \xrightarrow{Q_X} & \mathrm{Qch}(O_X), \end{array} \tag{E.2}$$

where the vertical functors are forgetful.

Lemma E.1.0.8. *Suppose that X is affine. Write $R = \Gamma(X, D_X)$. Then:*

1. *The functor $\tilde{\cdot} := D_X \otimes_R \cdot : \mathrm{Mod}(R) \to \mathrm{Mod}(D_X)$ is left adjoint to the global section functor $\Gamma(X, \cdot) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$;*
2. *The square (E.2) is commutative.*

Proof.

1. Let $(\sigma, \sigma^{\#}) : (X, D_X) \to (\{*\}, R)$ be the morphism of ringed spaces, with $\sigma : X \to \{*\}$ the unique map and $\sigma^{\#}$ given by Id_R. Then $\Gamma(X, \cdot) = \sigma_* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$. By [Sta24, Tag 01BH], the functor $\tilde{\cdot} = \sigma^*$. The adjunction follows from [Sta24, Tag 0096].

2. From 1 and [HT07, Prop. 1.4.4 (ii)], the functor $Q' : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ is the composition of $\Gamma(X, \cdot) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$ with $\tilde{\cdot} : \mathrm{Mod}(R) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$. The largest rectangle in the following diagram

$$\begin{tikzcd}
\mathrm{Mod}(D_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[d] \arrow[rr, bend left, "Q'"] & \mathrm{Mod}(R) \arrow[r, "D_X \otimes_R \cdot"] \arrow[d] & \mathrm{Mod}_{\mathrm{qc}}(D_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[d] & \mathrm{Mod}(R) \arrow[d] \\
\mathrm{Mod}(O_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[rr, bend right, "Q"'] & \mathrm{Mod}(O_X(X)) \arrow[r, "O_X \otimes_{O_X(X)} \cdot"] & \mathrm{Qch}(O_X) \arrow[r, "\Gamma(X,\cdot)"] & \mathrm{Mod}(O_X(X))
\end{tikzcd}$$

is same as the small square on the left, hence commutative. Moreover, the two horizontal functors $\Gamma(X, \cdot)$ on the right are equivalences, so Q' is compatible with Q.

□

The abelian categories $\mathrm{Mod}(D_X)$ and $\mathrm{Mod}(O_X)$ are Grothendieck. By [Sta24, Tag 079P] and [Sta24, Tag 070K], the functor $Q' : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ and $Q : \mathrm{Mod}(O_X) \to \mathrm{Qch}(O_X)$ admit right derived functors $RQ' : D(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ and $RQ : D(O_X) \to D(\mathrm{Qch}(O_X))$.

Lemma E.1.0.9. *1. The square (E.2) is commutative.*

2. The square

$$\begin{tikzcd}
D(D_X) \arrow[r, "RQ'_X"] \arrow[d, "\mathrm{for}_X"] & D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \arrow[d, "\mathrm{for}_X"] \\
D(O_X) \arrow[r, "RQ_X"] & D(\mathrm{Qch}(O_X)),
\end{tikzcd}$$

is commutative.

Proof.

1. We deduce a formula for Q'_X. Since X is quasi-compact, there is a finite cover $\{U_\alpha\}_{\alpha \in I}$ of X by affine opens. For any $\alpha \neq \beta$ in I, since X is separated over k, the scheme $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is affine. Denote all the various open immersions $U_{\alpha\beta} \to X$ and $U_\alpha \to X$ as j. For

every D_X-module F, the sheaf axiom gives an equalizer diagram in $\mathrm{Mod}(D_X)$:

$$0 \to F \to \oplus_\alpha j_*(F|_{U_\alpha}) \rightrightarrows \oplus_{(\alpha,\beta)} j_*(F|_{U_{\alpha\beta}}),$$

where the two right morphisms are induced by the inclusions $U_{\alpha\beta} \to U_\alpha$ and $U_{\alpha\beta} \to U_\beta$. By Lemma E.1.0.10, it induces another equalizer diagram in $\mathrm{Mod}_{\mathrm{qc}}(D_X)$:

$$0 \to Q'_X F \to \oplus_\alpha j_* Q'_{U_\alpha}(F|_{U_\alpha}) \rightrightarrows \oplus_{(\alpha,\beta)} j_* Q'_{U_{\alpha\beta}}(F|_{U_{\alpha\beta}}). \tag{E.3}$$

There is a natural transformation $\iota' Q'_X \to \mathrm{Id}_{\mathrm{Mod}(D_X)} : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X)$. Applying $\mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$, one gets a natural transformation $\mathrm{for}_X \circ \iota' \circ Q'_X \to \mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$. Since $\mathrm{for}_X \circ \iota' = \iota \circ \mathrm{for}_X : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(O_X)$ and Q_X is right adjoint to ι, there is a natural transformation

$$\mu_X : \mathrm{for}_X \circ Q'_X \to Q_X \circ \mathrm{for}_X$$

of functors $\mathrm{Mod}(D_X) \to \mathrm{Qch}(O_X)$. By Lemma E.1.0.8 2, it is an isomorphism when X is affine.

For a general X, by (E.3) and [TT90, (B.14.2)], there is a commutative diagram of functors $\mathrm{Mod}(D_X) \to \mathrm{Qch}(O_X)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{for}_X Q'_X \arrow[r] \arrow[d, "\mu_X"] & \oplus_\alpha j_* \mathrm{for}_{U_\alpha} Q'_{U_\alpha}(\cdot|_{U_\alpha}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \oplus_{(\alpha,\beta)} j_* \mathrm{for}_{U_{\alpha\beta}} Q'_{U_{\alpha\beta}}(\cdot|_{U_{\alpha\beta}}) \arrow[d] \\
0 \arrow[r] & Q_X \mathrm{for}_X \arrow[r] & \oplus_\alpha j_* Q_{U_\alpha} \mathrm{for}_{U_\alpha}(\cdot|_{U_\alpha}) \arrow[r, shift left] \arrow[r, shift right] & \oplus_{(\alpha,\beta)} j_* Q_{U_{\alpha\beta}} \mathrm{for}_{U_{\alpha\beta}}(\cdot|_{U_{\alpha\beta}}),
\end{tikzcd}$$

where the two vertical arrows on the right are isomorphisms. Therefore, μ_X is an isomorphism.

2. The morphism $(X, D_X) \to (X, O_X)$ of ringed spaces is flat, and the direct image functor is the forgetful functor $\mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$. By [Sta24, Tag 08BJ], it preserves K-injective complexes. The conclusion follows from Point 1, Lemma E.1.0.11 and [Sta24, Tag 070K].

□

Lemma E.1.0.10. *Let $j : U \to X$ be an open immersion. Then the natural transformation $j_* \circ Q'_U \to Q'_X \circ j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ is an isomorphism.*

Proof. As $j^* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_U)$ restricts to a functor $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_U)$, one has $\iota'_U j^* = j^* \iota'_X$ as functors $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_U)$. The functor $j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}(D_X)$ regards the direct image $j_* : \mathrm{Mod}(O_U) \to \mathrm{Mod}(O_X)$, so it also restricts to a functor $\mathrm{Mod}_{\mathrm{qc}}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$. As Q' is right adjoint to ι' and j_* is right adjoint to j^*, the isomorphism follows. □

Lemma E.1.0.11. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories. Assume that $\mathcal{A}$, $\mathcal{B}$ are Grothendieck. If for ever K-injective complex I over $\mathcal{A}$, the natural morphism $GF(I) \to RG(F(I))$ in $D(\mathcal{C})$ is an isomorphism,*[1] *then the canonical natural transformation (constructed in [Sta24, Tag 05T2 (1)]) $t : R(G \circ F) \to RG \circ RF$ is an isomorphism of functors from $D(\mathcal{A}) \to D(\mathcal{C})$.*

Proof. Let A be a complex over $\mathcal{A}$. As $\mathcal{A}$ is Grothendieck, by [Sta24, Tag 079P], there is a quasi-isomorphism $A \to I$ such that I is a K-injective complex. By [Sta24, Tag 070K], the morphism t_A is the composition of isomorphisms

$$R(G \circ F)(A) = GF(I) \to RG(F(I)) = RG(RF(A)).$$

□

Proof of Theorem E.1.0.4. By [Sta24, Tag 09T5], $RQ' : D(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ is right adjoint to $L\iota' = \iota' : D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D(D_X)$. Let $\Psi' : D_{\mathrm{qc}}(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ (resp. $\Psi : D_{\mathrm{qc}}(O_X) \to D(\mathrm{Qch}(O_X))$) be the restriction of RQ' (resp. RQ). By Lemma E.1.0.9 2, there are natural commutative squares

$$\begin{array}{ccc} D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) & \xrightarrow{L\iota'} & D_{\mathrm{qc}}(D_X) \\ \downarrow{\scriptstyle \mathrm{for}} & & \downarrow{\scriptstyle \mathrm{for}} \\ D(\mathrm{Qch}(O_X)) & \xrightarrow{L\iota} & D_{\mathrm{qc}}(O_X), \end{array} \qquad \begin{array}{ccc} D_{\mathrm{qc}}(D_X) & \xrightarrow{\Psi'} & D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \\ \downarrow{\scriptstyle \mathrm{for}} & & \downarrow{\scriptstyle \mathrm{for}} \\ D_{\mathrm{qc}}(O_X) & \xrightarrow{\Psi} & D(\mathrm{Qch}(O_X)), \end{array}$$

where $L\iota$ is induced by the inclusion $\iota : \mathrm{Qch}(O_X) \to \mathrm{Mod}(O_X)$.

Since Ψ is right adjoint to ι, the counit $\epsilon' : \iota'\Psi' \to \mathrm{Id}_{D_{\mathrm{qc}}(D_X)}$ (resp. unit $\eta' : \mathrm{Id}_{D(\mathrm{Mod}_{\mathrm{qc}}(D_X))} \to \Psi'\iota'$) is compatible with the counit $\epsilon : \iota\Psi \to \mathrm{Id}_{D_{\mathrm{qc}}(O_X)}$ (resp. unit $\eta : \mathrm{Id}_{D(\mathrm{Qch}(O_X))} \to \Psi\iota$). The functor for : $D(D_X) \to D(O_X)$ is conservative. By [Sta24, Tag 09T4], the counit ϵ and the unit η are isomorphisms, so are the counit ϵ' and the unit η'. In particular, the functor (E.1) is an equivalence with a quasi-inverse Ψ'. □

Proposition E.1.0.12. *Let X be a complex manifold. Let F be an O_X-module. Then the following conditions are equivalent:*

[1] *i.e.*, $F(I)$ computes RG in the sense of [Sta24, Tag 05SX (1)]

1. *the induced module $D_X \otimes_{O_X} F$ is holonomic;*

2. *F is coherent with* $\mathrm{Supp}(F)$ *discrete.*

Corollary E.1.0.13. *Let X be a connected complex manifold. Let (E, ∇) be a nonzero flat vector bundle on X. If the corresponding D_X-module is induced by some O_X-module, then X is a point.*

Proof. Let F be an O_X-module with $D_X \otimes_{O_X} F = (E, \nabla)$. As D_X is a locally free O_X-module, one has $\mathrm{Supp}(F) = \mathrm{Supp}(E) = X$. By Proposition E.1.0.12, since (E, ∇) is a holonomic D_X-module, $X = \mathrm{Supp}(F)$ is discrete and hence a point. □

Lemma E.1.0.14 and Lemma E.1.0.15 are needed for the proof of Proposition E.1.0.12.

Lemma E.1.0.14. *Let A be a Gorenstein local ring* (in the sense of [Sta24, Tag 0DW7 (1)]) *of Krull dimension n. Let M be a finite A-module. Then the following conditions are equivalent:*

1. *For all integers $i \neq n$, one has* $\mathrm{Ext}^i(M, A) = 0$*;*

2. *the length of M is finite.*

Proof. Let k be the residue field of A.

- Assume Condition 1. To prove 2, one may assume $M \neq 0$. As A is Gorenstein, $A[0]$ is a dualizing complex of A. By [Mat87, Thm. 18.1, p.141], one has $R\mathcal{H}om_A(k, A[n]) = k[0]$, so $A[n]$ is the normalized dualizing complex of A (in the sense of [Sta24, Tag 0A7M]). Let d be the depth of M. By [Sta24, Tag 0B5A], the module M is Cohen-Macaulay and
$$M = \mathrm{Ext}_A^{n-d}(\mathrm{Ext}_A^{n-d}(M, A), A).$$
Thus, $\mathrm{Ext}_A^{n-d}(M, A) \neq 0$. By Condition 1, one has $n - d = n$. Hence $\dim \mathrm{Supp}(M) = d = 0$. By [Ati69, Exercise 19 v), p.46], one has $\dim A/\mathrm{Ann}(M) = 0$. Then $A/\mathrm{Ann}(M)$ is an artinian ring. From [Eis95, Cor. 2.17], the length of M is finite.

- Assume Condition 2. Induction on the length $l(M)$ of M. When $l(M) = 0$, one has $M = 0$ and Condition 1 holds. Now assume $l(M) > 0$ and the statement holds for all modules of length less than $l(M)$. There is a submodule N of M such that M/N is a simple module and $l(N) < l(M)$. By [Sta24, Tag 00J2], the module M/N is isomorphic to k. For every integer $i \neq n$, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces an exact sequence $\mathrm{Ext}^i(M/N, A) \to \mathrm{Ext}^i(M, A) \to \mathrm{Ext}^i(N, A)$. By the inductive hypothesis, $\mathrm{Ext}^i(N, A) = 0$. By [Mat87, Thm. 18.1, p.141], one has $\mathrm{Ext}^i(M/N, A) = 0$. Hence $\mathrm{Ext}^i(M, A) = 0$.

$\square$

Lemma E.1.0.15. *Let X be a complex analytic space. Let F be a coherent O_X-module. Then the length of the $O_{X,x}$-module F_x is finite for all $x \in X$ if and only if the subspace* $\mathrm{Supp}(F) \subset X$ *is discrete.*

Proof. The "if" part follows from Lemma 5.5.2.8 1. We prove the "only if" part. By coherence of F and [CAS, p.76], $\mathrm{Supp}(F)$ is a closed analytic set of X. Assume to the contrary that $\mathrm{Supp}(F)$ is not discrete. Then $\dim \mathrm{Supp}(F) > 0$. Let C be an irreducible component of $\mathrm{Supp}(F)$ of maximal dimension. Endow C with the reduced induced closed subspace structure. Let $i : C \to X$ be the closed embedding of complex analytic spaces.

For every $x \in C$, the morphism $O_{X,x} \to O_{C,x}$ is surjective. Then by [Sta24, Tag 00IX], one has $l_{O_{C,x}}(i^*F)_x = l_{O_{X,x}}(i^*F)_x$. The morphism $F_x \to (i^*F)_x$ of $O_{X,x}$-modules is surjective, so $l_{O_{X,x}}(i^*F)_x \leq l_{O_{X,x}}F_x$. In particular, the length of $(i^*F)_x$ over $O_{C,x}$ is finite. By [GD71, Cor. 5.2.4.1], the support of i^*F is C. Replacing (X, F) by (C, i^*F), one may assume further that X is irreducible with $\dim X > 0$.

By the generic freeness [Ros68, Prop. 3.1], there is $x_0 \in X$ such that F_{x_0} is a free O_{X,x_0}-module. As the support of F is X, from [RS17, p.238], F is not a torsion sheaf. Then by irreducibility of X and [Ros68, p.69], the O_{X,x_0}-module F_{x_0} has positive rank. Thus, O_{X,x_0} has finite length over itself, hence an artinian ring. The dimension formula in [CAS, p.96] and [CD94, (14.14), p.89] yield $\dim X = \dim_{x_0} X = \dim O_{X,x} = 0$, a contradiction. $\square$

Proof of Proposition E.1.0.12. Let $M = D_X \otimes_{O_X} F$ and $\hat{F} = R\mathcal{H}om_{O_X}(F, O_X)$. By [Sta24, Tag 08DJ], one has

$$\mathcal{H}om_{O_X}(\omega_X, \hat{F}) = R\mathcal{H}om_{O_X}(\omega_X \otimes_{O_X} F, O_X). \tag{E.4}$$

Provided that F is *coherent*, [Bjö93, (ii) p.122] gives

$$\Delta^{D_X} M = D_X \otimes_{O_X} \mathcal{H}om_{O_X}(\omega_X, \hat{F})[\dim X]. \tag{E.5}$$

Plugging (E.4) into (E.5), one gets

$$\Delta^{D_X} M = D_X \otimes_{O_X} R\mathcal{H}om_{O_X}(\omega_X \otimes_{O_X} F, O_X)[\dim X].$$

For every nonzero integer i, one has

$$H^i(\Delta^{D_X} M) = D_X \otimes_{O_X} \mathcal{E}xt^{i+\dim X}_{O_X}(\omega_X \otimes_{O_X} F, O_X).$$

By [Sta24, Tag 01CB] and [GH78, 1. p.700], its stalk at $x \in X$ is isomorphic to

$$D_{X,x} \otimes_{O_{X,x}} \mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}).$$

- Assume Condition 2. By [Bjö93, 1.5.1], the D_X-module M is coherent. By Lemma E.1.0.15, the $O_{X,x}$-module F_x has finite length. As $O_{X,x}$ is a noetherian regular local ring of Krull dimension $\dim_x X$, by Lemma E.1.0.14, one has $\mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0$ for all $x \in X$. Hence $H^i(\Delta^{D_X} M) = 0$. From Fact 6.7.1.2 2, the D_X-module M is holonomic.

- Assume Condition 1. From [SS94, p.55], the O_X-module F is coherent. From Fact 6.7.1.2 2, for every nonzero integer i, one has $H^i(\Delta^{D_X} M) = 0$. As $D_{X,x}$ is a nonzero free $O_{X,x}$-module, one gets $\mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0$. By Lemma E.1.0.14, the $O_{X,x}$-module F_x has finite length for every $x \in X$. From Lemma E.1.0.15, the support of F is discrete.

□

The proof of Proposition E.1.0.16 is similar to that of Proposition E.1.0.12.

Proposition E.1.0.16. *Let k be an algebraically closed field of characteristic zero. Let X be a smooth separated irreducible scheme over k of finite type. Let F be an O_X-module. Then the following conditions are equivalent:*

1. *the induced module $D_X \otimes_{O_X} F$ is holonomic;*

2. *F is coherent with $\mathrm{Supp}(F)$ finite.*

Appendix F

Group extensions of complex Lie groups

F.1 Introduction

In the history of cohomology theory of abelian varieties over positive characteristic fields, the study of group extension problem played an important role. For instance, Rosenlicht obtains Fact F.1.0.1 through considering vectorial extensions of abelian varieties. Let k be an algebraically closed field. Let A be an abelian variety over k of dimension g. The dual abelian variety of A is denoted by $A^\vee$. Let $A^\natural$ be the moduli space of line bundles equipped with an integrable connection on $A^\vee$.

Fact F.1.0.1 ([Ros58, Theorems 1 and 2]). *The dimension of the k-vector space $H^1(A, O_A)$ is g.*

A notable byproduct of Rosenlicht's work is the existence of *universal vectorial extension.* Short exact sequence of algebraic groups are defined in [Ros58, Sec. 2, p.691].

Fact F.1.0.2 ([Ros58, Prop. 11]). *There is a canonical short exact sequence $0 \to H^0(A^\vee, \Omega^1) \to A^\natural \to A \to 0$ of commutative algebraic groups over k. It is universal among vectorial extensions in the following sense: For another such exact sequence $0 \to V \to E \to A \to 0$ with V a vector group, the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & H^0(A^\vee, \Omega^1) \arrow[r] \arrow[d, dashed] & A^\natural \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & V \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

can be completed to be commutative making the left square a pushout.

For a smooth algebraic variety X over k, let $\mathrm{Qch}(O_X)$ (resp. $\mathrm{Qch}(D_X)$) be the category of quasi-coherent O_X (resp. left D_X) modules. Laumon [Lau96, Thm. 3.2.1] and Rothstein [Rot96, (1.17)] independently proves that the Fourier-Mukai transform $D^b(\mathrm{Qch}(O_A)) \to D^b(\mathrm{Qch}(O_{A^\vee}))$ lifts to a canonical equivalence $D^b(\mathrm{Qch}(O_{A^\natural})) \to D^b(\mathrm{Qch}(D_{A^\vee}))$.

The cohomology theory of complex analytic analogue of abelian varieties, namely complex tori, is elementary. By contrast, as far as we know, the existence of universal vectorial extension in the analytic setting is not covered in the literature. For a Lie group G, its identity component is denoted by G_0. Let $\pi_0(G) := G/G_0$. The main results are summarized as follows.

Proposition (Proposition F.4.3.1)**.** *For two commutative complex Lie groups A, B, the* commutative *extensions of A by B are classified by the abelian group*

$$\mathrm{Ext}^1_{\mathbb{Z}}(\pi_0(A), \pi_0(B)) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), \pi_0(B)) \oplus \mathrm{coker}(s).$$

Here s is the restriction morphism $\mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B_0)) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0)$, *$A_0$ (resp. B_0) signifies the identity component of A (resp. B), the notation $\pi_1(*)$ refers to the fundamental group, and $A/A_0 = \pi_0(A)$ denotes the* 0*-th homotopy group of A and similar for B.*

Theorem. *Let A be a complex torus of dimension g. Then:*

- (Theorem F.5.2.4 (resp. F.5.3.2)) *The dual torus* $\mathrm{Pic}^0(A)$ *(resp. tangent space $T_0A = H^1(A, O_A)$) naturally classifies the extensions of A by the multiplicative group $\mathbb{C}^*$ (resp. additive group $\mathbb{C}$).*
- (Propositions F.5.4.5 1 and F.5.4.7) *There is an extension*

  $$0 \to H^0(A^\vee, \Omega^1_{A^\vee}) \to (\mathbb{C}^*)^{2g} \to A \to 0$$

  that is universal among all vectorial extensions of A.

We emphasis some differences between the analytic case and the algebraic case. For a complex torus A, let $\mathrm{Div}(A)$ be the group of analytic divisors on A modulo linear equivalence. Let $\mathrm{Pic}(A)$ be the group of isomorphic classes of line bundles on A. By [Deb05a, Sec. 4.3, Cor. 4], the natural map $\mathrm{Div}(A) \to \mathrm{Pic}(A)$ is surjective if and only if A is an abelian variety. This is why the Picard group is used in Theorem F.5.2.4, while the divisor group appears in its algebraic analogue [Wei49, no. 2] and [Ser88, Thm. 6]. Discrete groups like $\mathbb{Z}$ are not (finite type) algebraic groups, but there is no reason to exclude them as complex Lie groups. Plenty of important analytic morphisms are not algebraic, like the universal covering (exponential map) $\exp : \mathbb{C} \to \mathbb{C}^*$.

The organization is as follows. The main goal of this text is to classify extensions of complex Lie groups. Section F.2 contains preliminaries about complex Lie groups. In Section F.3 we define complex Lie group extensions and give several first results about the classification. Then we focus on commutative extensions in Section F.4. Commutative extensions of complex tori deserve extra attention, and they are discussed in Section F.5. Some extensions with complex-tori base are automatically commutative, as Section F.6 shows. Noncommutative extensions are treated superficially in Section F.7.

Convention and notation

A statement about Lie groups is understood to hold for both real and complex Lie groups. The topology underlying a Lie group is always assumed to be second countable.[1]

The Lie algebra of G is written as $L(G)$. And $Z(G)$ denotes the center of G. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. Let $\mathrm{Inn} : G \to \mathrm{Aut}(G)$ be the group morphism defined by taking conjugation $g \mapsto g \bullet g^{-1}$. Then the subgroup $\mathrm{Inn}(G)$ of inner automorphisms is normal in $\mathrm{Aut}(G)$. Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ be the group of outer automorphisms. Let G^{op} be the Lie group opposite to G. (If G is complex, then so is G^{op}.) There is a natural identification of real/complex manifolds $G \to G^{\mathrm{op}}$ denoted by $g \mapsto g^*$. If G is connected, then the universal covering group of G is denoted by $\tilde{G}$ and the fundamental group of G with the identity e_G as the base point is denoted by $\pi_1(G)$.

Complex Lie subgroups refer to embedded closed complex Lie subgroups. If G is a complex Lie group and $S \subset G$ is a subset, by [HN11, Exercise 15.1.3 (b)] there is a smallest complex Lie subgroup of G containing S, called the complex Lie subgroup generated by S.

Let Vec (resp. Ab, resp. Set) be the category of finite dimensional complex vector spaces (resp. abelian groups, resp. commutative complex Lie groups, resp. sets). For a complex manifold X and a commutative complex Lie group B, let $\mathcal{B}_X$ be the abelian sheaf on X of germs of holomorphic maps from X to B.

F.2 Generalities on complex Lie groups

Two fundamental facts about complex Lie groups are recalled.

Fact F.2.0.1 ([Bou72, Ch. III, §3, no. 8, Prop. 28])**.** *Let $f : G \to H$ be a morphism of complex Lie groups. Then:*

1. $\ker(f)$ *is a normal complex Lie subgroup of* G *and* $L(\ker(f)) = \ker(d_e f : L(G) \to L(H))$.

[1] A partial reason for such restriction is that, in this case, Condition (2) of [Hoc51b, Definition 1.1] is implied by Condition (1), showed in p.542 *loc.cit.*

2. *If $f(G)$ is* closed *in H, then $f(G)$ is a complex Lie subgroup of H, and f induces a complex Lie group isomorphism $G/\ker(f) \to f(G)$. In particular, if f is surjective, then $d_e f : L(G) \to L(H)$ is surjective. If f bijective, then f is an isomorphism.*

Remark F.2.0.2. Fact F.2.0.1 2 fails if the topology of G is not assumed to be second countable. For example, let τ (resp. τ') be the discrete topology (resp. the Euclidean topology) of $\mathbb{C}$, then $\mathrm{Id} : (\mathbb{C}, \tau) \to (\mathbb{C}, \tau')$ is a bijective morphism but not open.

Right principal bundle is defined in [Bou07, 6.2.1]. Left principal bundle can be defined similarly.

Fact F.2.0.3 ([HBS66, Thm. 3.4.3], [Bou72, Ch. III, §1, Propositions 10 and 11])**.** *Suppose G is a complex Lie group and K is a normal complex Lie subgroup of G. Then the group G/K has a unique structure of complex manifold, such that the quotient map $\pi : G \to G/K$ is a submersion.*[2] *With this structure, G/K is a complex Lie group and p is a left principal K-bundle under the natural left group action $K \times G \to G$ defined by $(k, g) \mapsto kg$. In particular, every surjective morphism of complex Lie groups is open.*

We recall that principal bundles are classified by the first sheaf cohomology, in the following way. Let X (resp. B) be a complex manifold (resp. commutative complex Lie group). Let S be the set of isomorphism classes of principal B-bundles[3] over X. Define a map

$$\Psi : S \to H^1(X, \mathcal{B}_X) \tag{F.1}$$

as follows. For every $[p : P \to X] \in S$, there exists an open cover $\{U_i\}_{i \in I}$ of X and a family of local trivializations $f_i : U_i \times B \to p^{-1}(U_i)$ for every $i \in I$. For any indices $i, j \in I$ and every $x \in U_i \cap U_j$, there exists a unique element $b_{ij}(x) \in B$ such that $b_{ij}(x) \cdot f_i(y) = f_j(y)$ for all $y \in p^{-1}(x)$. Hence a morphism $b_{ij} : U_i \cap U_j \to B$ of complex manifolds. Moreover, for any indices $i, j, k \in I$ and every $x \in U_i \cap U_j \cap U_k$, they satisfy the 1-cocycle relation $b_{ij}(x) + b_{jk}(x) + b_{ki}(x) = 0$. Thus, the family $\{b_{ij}\}_{i,j \in I}$ defines an element $\Psi(p)$ of $H^1(X, \mathcal{B}_X)$.

As per [HBS66, 3.2 b), p.41], the map Ψ is bijective. The structure of abelian group on $H^1(X, \mathcal{B}_X)$ is translated to S *via* Ψ. The zero element of S is the class of the trivial principal B-bundle. For every pair $[p_1 : P_1 \to X]$ and $[p_2 : P_2 \to X]$ in S, by taking a family of trivialization for each p_i, we can define a morphism $\phi : P_1 \times_X P_2 \to P_1 + P_2$ of principal B-bundles on X such that or every $b, b' \in B$, $u \in P_1$, $v \in P_2$ with $p_1(u) = p_2(v)$, one has

$$\phi(b \cdot u, b' \cdot v) = (b + b') \cdot \phi(u, v). \tag{F.2}$$

[2] in the sense of [Bou07, 5.9.1]

[3] Here B is commutative, so it is unnecessary to specify the principal bundle to be left or right.

In particular, ϕ is surjective. Restricted to the fiber at some $x \in X$, ϕ is induced by the group law of B and the chosen trivializations.

We need a complex version of Cartan's subgroup theorem. Notice that a real analytic closed subgroup of a complex Lie group may not be a complex analytic subset.

Lemma F.2.0.4 ([Bjö93, p.513])**.** *Let X be a complex manifold, $Y \subset X$ be a complex analytic subset. If $p \in Y$ is a smooth point of Y, then near p, the subset Y is an embedded complex submanifold of X.*

Proof. As the problem is local, we may assume that X is an open subset $\mathbb{C}^n$, there exist $f_1, \ldots, f_m \in O_X(X)$ with $O_{X,p}/(f_1, \ldots, f_m) = O_{Y,p}$ and $Y = Z(f_1, \ldots, f_m)$. Let $r = \mathrm{rank}_p(f_1, \ldots, f_m)$. By reordering subscripts, one may assume

$$\det(\frac{\partial f_i}{\partial z_j})_{1 \leq i,j \leq r} \neq 0.$$

Then $(f_1, \ldots, f_r) : X \to \mathbb{C}^r$ is a holomorphic submersion near p. Therefore, near p, the subset $Z(f_1, \ldots, f_r)$ is an embedded complex submanifold of X of dimension $n - r$. By the Jacobian criterion (see, *e.g.*, [CAS, p.114]), one has $\mathrm{emb}_p Y = n - r$. By the criterion of smoothness [CAS, p.116], one has $\dim_p Y = n - r$. Now that $Y \subset Z(f_1, \ldots, f_r)$, near p the subset Y is an irreducible component of $Z(f_1, \ldots, f_r)$, hence also an embedded complex submanifold of X. □

Corollary F.2.0.5 contains [Lee01, Prop. 1.23] as a special case.

Corollary F.2.0.5 (Complex Cartan subgroup theorem)**.** *Let G be a complex Lie group, and let H be a subgroup that is a complex analytic subset of G. Then H is a complex Lie subgroup of G.*

Proof. Endow H with the induced structure of reduced complex analytic space. By [CAS, p.117], the complex analytic space H has a smooth point p. For every $q \in H$, the left multiplication by qp^{-1} gives a biholomorphic map $G \to G$, which sends H to H and maps p to q. Therefore, q is also a smooth point of H. By Lemma F.2.0.4, H is a complex submanifold of G near q for all $q \in H$. Thus, H is a complex submanifold of G and hence a complex Lie subgroup. □

In Lemma F.2.0.6, if G is furthermore connected, then the result of is contained in [Bou72, Ch.III, Sec. 6, no. 4, Cor. 4].

Lemma F.2.0.6. *Let G be a complex Lie group. Then the center $Z(G)$ is a complex Lie subgroup of G.*

Proof. The holomorphic map $G \times G \to G$ defined by $(x, y) \mapsto yxy^{-1}$ is a group action of G on itself. By [Bou72, Ch. III, Sec. 1, no. 7, Prop. 14], for every $x \in G$, the stabilizer $C_G(x)$ of $x \in G$ is a complex Lie subgroup of G. Therefore, so is $Z(G) = \cap_{x \in G} C_G(x)$ by [HN11, Exercise 15.1.3 (a)]. □

A complex Lie group isomorphic to a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some integer $n \geq 1$ is called *linear*. Proposition F.2.0.7, due to Matsushima and Morimoto, is a characterization of commutative linear complex Lie groups.

Proposition F.2.0.7. *Let B be a connected commutative complex Lie group. Then the following conditions are equivalent:*

1. *B is isomorphic to $\mathbb{C}^m \times (\mathbb{C}^*)^n$ for some integers $m, n \geq 0$;*
2. *the complex Lie group B is linear;*
3. *B is a Stein group (i.e., the underlying complex manifold is a Stein manifold).*

In that case, the pair (m, n) is unique.

Proof. See [HN11, Exercise 15.3.1] for the fact that 1 implies 2. Since $\mathrm{GL}_n(\mathbb{C})$ is a Stein manifold, 2 implies 3. As per [MM60, Proposition 4], 3 implies 1. The uniqueness is contained in the Remmert-Morimoto decomposition (see, *e.g.*, [AK01, Thm. 1.1.5]). □

Remark F.2.0.8. The commutativity of B in Proposition F.2.0.7 is important. In fact, by [Ari19, Sec. 1], there is a connected Stein group that is not linear. This differs from the algebraic case. By [Mil17a, Cor. 4.10], every affine algebraic group is linear.

In some sense, Definition F.2.0.9 is an antipode to Stein groups.

Definition F.2.0.9. A connected complex Lie group on which every holomorphic function is constant is called a toroidal group.[4]

Complex tori are compact toroidal groups. From [AK01, p.1], there exist toroidal groups that are not compact. Every toroidal group is a semi-torus in the sense of [NW13, Def. 5.1.5].

By [AK01, 1.1.5], every connected commutative complex Lie group G is uniquely isomorphic to $\mathbb{C}^l \times (\mathbb{C}^*)^m \times X$ with a toroidal group X. In particular, G can be presented as an extension of a complex torus by a connected linear group. (From [NW13, pp.169-170], a semi-torus can admit nonequivalent presentations, while semiabelian varieties admit exactly one algebraic presentation.)

[4]also known as a Cousin group

F.3 Group extensions

Given a surjective Lie group morphism $p : E \to Q$, by Fact F.2.0.1, $K := \ker(p)$ is a normal Lie subgroup of E and the induced morphism $E/K \to Q$ is an isomorphism. We write it as

$$1 \to K \overset{i}{\to} E \overset{p}{\to} Q \to 1 \tag{F.3}$$

and call it a short exact sequence. In that case, E is called an *extension* of the base Q by the extension kernel K. Moreover, $d_e p : L(E) \to L(Q)$ is surjective of kernel $L(K)$, hence an extension of Lie algebras

$$0 \to L(K) \to L(E) \overset{d_e p}{\to} L(Q) \to 0.$$

When $K \subset Z(E)$, such an extension is called *central.* If (F.3) is a central extension with Q commutative, as in [MRM74, p.222], using Fact F.2.0.3 one can construct a skew-symmetric bimorphism

$$e : Q \times Q \to K, \tag{F.4}$$

to measure the deviation of E from commutativity. Indeed, the group E is commutative if and only if e is constant.

Several topological properties of Lie groups are preserved by extensions.

Fact F.3.0.1. *If K, Q in (F.3) are compact (resp.connected, resp. discrete), then so is E.*

Proof. The statement concerning connectedness is in [Che46, Prop. 2, p.36]. The others are consequences of Fact F.2.0.3. □

Fact F.3.0.2 ([HN11, Cor. 16.3.9])**.** *If (F.3) is a central extension of complex Lie groups, where K is finite and E is connected, then Q is linear if and only if E is linear.*

The finiteness of K in Fact F.3.0.2 is necessary. Consider the exact sequence $0 \to \mathbb{Z}^2 \to \mathbb{C} \to A \to 0$ defining a complex torus A. Here $\mathbb{Z}^2$ and $\mathbb{C}$ are linear, while A is not.

Similarly, an extension E of a finite group Q by a linear group K is linear. Indeed, let $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ be a faithful representation, then the induced representation $\mathrm{Ind}_K^E \rho : E \to \mathrm{GL}_{mn}(\mathbb{C})$ is also faithful, where $m = \#Q$. Again, the finiteness of Q is essential here. Example F.3.0.3 shows the statement fails when Q is only discrete and linear but infinite.

Example F.3.0.3. Deligne [Del78] (see also [KRW20, p.470]) shows that for any integers $g \geq 2, n \geq 3$, there is a central extension $1 \to \mathbb{Z}/n \to G \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1$ for which G is not residually finite. By Malcev's theorem ([Mal40, Thm. VII]; see also [Nic13, p.1]), the discrete complex Lie group G is not linear, even though $\mathbb{Z}/n$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$ are linear.

We turn to the classification of extensions. Two Lie group extensions C and C' of B by A are called equivalent if there exists a morphism $f : C \to C'$ making a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & C' \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

In this case, f is bijective, hence an isomorphism by Fact F.2.0.1. The *trivial* extension of Q by K refers to the equivalence class of the obvious sequence

$$1 \to K \to K \times Q \to Q \to 1.$$

Fact F.3.0.4 ([Bou72, Ch.III, no.4, Prop. 8])**.** *The Lie group extension (F.3) is trivial if and only if there is a morphism $r : E \to K$ with $ri = \mathrm{Id}_K$. The extension is a semidirect product if and only if there is a morphism $s : Q \to E$ with $ps = \mathrm{Id}_Q$.*

The extension (F.3) defines a group morphism $\psi : Q \to \mathrm{Out}(K)$, called the *outer action* corresponding to the extension. We call (K, ψ) the extension kernel of (F.3). Equivalent extensions induce the same outer action. For two complex Lie groups Q, K and a group morphism $\psi : Q \to \mathrm{Out}(K)$, denote by $\mathrm{Ext}(Q, K, \psi)$ the set of equivalence classes of extensions of Q by K with outer action ψ.

Since the center $Z(K)$ is a characteristic complex Lie subgroup of K by Lemma F.2.0.6, there is a canonical group morphism $\mathrm{Aut}(K) \to \mathrm{Aut}(Z(K))$ which passes to another group morphism $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$. Hence a group morphism

$$\psi_0 : Q \to \mathrm{Aut}(Z(K)) \tag{F.5}$$

induced by ψ. When K is *commutative*, $\psi = \psi_0$ and the construction of Baer sum ((F.41) and [FLA19, p.444]) makes $\mathrm{Ext}(Q, K, \psi)$ an abelian group.

F.3.1 Pullback and pushout

Extensions can be pulled back.

Example F.3.1.1 (Pullback)**.** Given a morphism $g : Q' \to Q$ of complex Lie groups, pulling (F.3) back along g gives an extension of Q' by K as follows.

The map $E \times Q' \to Q$ defined by $(x, h') \mapsto p(x)^{-1}g(h')$ is holomorphic, so the preimage E' of the identity element $e_Q \in Q$ is an analytic subset of $E \times Q'$. As $E' = \{(x, h') \in E \times Q' : p(x) = g(h')\}$ is a subgroup of $E \times Q'$, by Corollary F.2.0.5, E' is a complex Lie subgroup of $E \times Q'$ (which is the extension group). Let $p' : E' \to Q'$ and $\epsilon : E' \to E$ be the projections. Then the triple (E', ϵ, p') is the fiber product $E \times_Q Q'$ in the category of complex Lie groups.

For every $h' \in Q'$, by surjectivity of p, there is $x \in E$ with $p(x) = g(h')$. Then $(x, h') \in E'$ with $p'(x, h') = h'$. Hence p' is surjective.

Define a morphism $i' : K \to E'$ by $i'(k) = (k, e_{Q'})$. Then i' is injective. Since $p'i'$ is trivial, $i'(K) \subset \ker(p')$. Conversely, for every $(x, h') \in \ker(p')$, $h' = e_{Q'}$ and $p(x) = g(e_{Q'}) = e_Q$. Thus, $x \in K$ and $(x, h') = i'(x) \in i'(K)$. Hence a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "i'"] \arrow[d, "\mathrm{Id}"] & E' \arrow[r, "p'"] \arrow[d, "\epsilon"] & Q' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & K \arrow[r, "i"] & E \arrow[r, "p"'] & Q \arrow[r] & 1.
\end{tikzcd}$$

The first row is called the pullback extension of (F.3) along g. Its outer action is $\psi g : Q' \to \mathrm{Out}(K)$. Hence a map $\mathrm{Ext}(Q, K, \psi) \to \mathrm{Ext}(Q', K, \psi g)$. As in [Hoc51a, p.99], it is a group morphism when K is commutative.

The universal property of pullback shows that the first row of every such commutative diagram is determined by the second row and $g : Q' \to Q$. By construction, the pullback of a central extension is also central.

A pushout extension along a morphism $f : K \to K'$ of complex Lie groups may not exist. When it exists, it satisfies a universal property.

Lemma F.3.1.2. *Consider a commutative diagram of complex Lie groups, where each row is exact*

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "m"] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & \text{‘} \\
1 \arrow[r] & K' \arrow[r, "\iota"] & E' \arrow[r, "\pi"] & Q \arrow[r] & 1.
\end{tikzcd} \tag{F.6}$$

Then the triple (E', m, ι) has the following universal property: For every commutative diagram of complex Lie groups

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "f"] & E \arrow[d, "m"'] \arrow[ddr, "\phi"] & \\
K' \arrow[r, "\iota"] \arrow[drr, "\psi"'] & E' \arrow[dr, dashed, "\eta"] & \\
 & & H
\end{tikzcd}$$

with $\psi(m(c)^{-1} b m(c)) = \phi(c)^{-1} \psi(b) \phi(c)$ for every $c \in E$ and $b \in K'$, there exists a unique morphism $\eta : E' \to H$ keeping the diagram commutative.

In particular, up to a unique equivalence, the second row of (F.6) has at most one choice when the first row and $f : K \to K'$ are given.

Proof. We construct a map $\eta : E' \to H$ as follows. For every $c' \in E'$, there exists $c \in E$ with $p(c) = \pi(c')$. Let $b' = m(c)^{-1}c'$. Then $\pi(b') = p(c)^{-1}\pi(c') = e_Q$, so $b' \in K'$. Define $\eta(c') = \phi(c)\psi(b')$.

To show that η is well-defined, we claim that $\eta(c')$ is independent of the choice of c. Indeed, take another $c_1 \in E$ with $p(c_1) = \pi(c')$, then $p(c^{-1}c_1) = e_Q$, hence $c^{-1}c_1 \in K$. This time the element in K' is $b'_1 = m(c_1)^{-1}c'$, so $b' = f(c^{-1}c_1)b'_1$ in K' and hence $\psi(b') = \phi(c^{-1}c_1)\psi(b'_1)$. Therefore, $\phi(c)\psi(b') = \phi(c_1)\psi(b'_1)$ in H as claimed.

We check that η is holomorphic near $c' \in E'$. Indeed, by Fact F.2.0.3, there is an open neighborhood U of $\pi(c') \in Q$, and a holomorphic map $s : U \to E$ with $ps = \mathrm{Id}_U$. The map

$$\pi^{-1}(U) \to U \times K', \quad x \mapsto (\pi(x), [ms\pi(x)]^{-1}x)$$

is biholomorphic. The map

$$U \times K' \to H, \quad (u, b') \mapsto \phi(s(u))\psi(b')$$

is holomorphic. The composition is exactly $\eta|_{\pi^{-1}(U)}$.

We check that η is a group morphism. For $c'_i \in E'$ $(i = 1, 2)$, choose $c_i \in E$ with $p(c_i) = \pi(c'_i)$. Then for $c'_1c'_2$ we can choose c_1c_2. Let $b'_1 = m(c_1)^{-1}c'_1$ and $b'_2 = m(c_2)^{-1}c'_2$. Then

$$b' := m(c_1c_2)^{-1}c'_1c'_2 = m(c_2)^{-1}b'_1m(c_2)b'_2.$$

By the construction of η, one has

$$\begin{aligned}\eta(c'_1c'_2) &= \phi(c_1c_2)\psi(b')\\ &=\phi(c_1)\phi(c_2)\psi[m(c_2)^{-1}b'_1m(c_2)]\psi(b'_2)\\ &=\phi(c_1)\psi(b'_1)\phi(c_2)\psi(b'_2) = \eta(c'_1)\eta(c'_2)\end{aligned}$$

Then η is a morphism of complex Lie groups. By construction, η is the unique group morphism keeping the diagram commutative. □

Example F.3.1.3. Assume that Q is connected. As the map $p : E \to Q$ in (F.3) is open by Fact F.2.0.3, $p(E_0)$ is a nonempty open subgroup of Q and hence $p(E_0) = Q$ by the connectedness of Q. Then the following diagram is commutative and each row is exact

$$\begin{tikzcd} 1 \arrow[r] & K \cap E_0 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, hook] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\ 1 \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 1 \end{tikzcd}$$

By Lemma F.3.1.2, the second row is determined by the inclusion $K \cap E_0 \to K$ (an open normal subgroup) and the first row.

F.3.2 Rudimentary classification

Let K, Q be complex Lie groups, where Q is discrete. Consider an abstract group extension $1 \to K \to E \to Q \to 1$. Then as a set $E = \sqcup_x xK$, where x runs through a set of left representatives of E/K. Thus E admits a unique complex manifold structure making the maps holomorphic. However, the group law of E needs not to be holomorphic in this complex structure. The semidirect product sequence $1 \to \mathbb{C} \to \mathbb{C} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1$ serves as an example, where $\mathbb{Z}/2$ acts on $\mathbb{C}$ by complex conjugation. But when the base is discrete and the outer action is trivial, the Lie group extension problem reduces to the abstract group extension problem.

Proposition F.3.2.1. *Let K, Q be complex Lie groups. If Q is discrete, then the natural forgetful map $\phi : \mathrm{Ext}(Q, K, 1) \to \mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$ is bijective, where $\mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$ denotes the set of isomorphism classes of* abstract *group extensions of Q by K with trivial outer action. In fact, for every abstract group extension $1 \to K \to E \to Q \to 1$, E admits a unique complex manifold structure making the sequence an extension of complex Lie groups.*

Proof. We prove that ϕ is injective. Consider $E_1, E_2 \in \mathrm{Ext}(Q, K, 1)$ with $\phi(E_1) = \phi(E_2)$. Then there is an abstract group isomorphism $f : E_1 \to E_2$ making a commutative diagram

$$\begin{tikzcd} K \arrow[r, hook] \arrow[dr, hook] & E_1 \arrow[d, "f"] \\ & E_2. \end{tikzcd}$$

For every $x \in E_1$, the restriction $xK \to f(x)K$ of f is holomorphic, since the left multiplication $K \to xK$ (resp. $K \to f(x)K$) by x (resp. $f(x)$) in E_1 (resp. E_2) is biholomorphic. Thus, f is holomorphic and hence an equivalence of complex Lie group extensions.

We prove that ϕ is surjective. Given an abstract group extension $1 \to K \to E \to Q \to 1$ in $\mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$, we endow E with the complex structure making the maps holomorphic. We show the group law $m : E \times E \to E$ is holomorphic. Choose a set-theoretic section $s : Q \to E$. Then the map $K \times Q \to E$ defined by $(a, b) \mapsto as(b)$ is biholomorphic. With this identification, m becomes the map

$$\mu : K \times Q \times K \times Q \to K \times Q, \quad (a, b, a', b') \mapsto (as(b)a's(b')s(bb')^{-1}, bb') = (a\rho(a')s(b)s(b'))s(bb')^{-1}, b$$

where $\rho : K \to K$ is $x \mapsto s(b)xs(b)^{-1}$. Since the outer action is trivial, $\rho \in \mathrm{Inn}(K)$. Therefore, the map $K \times K \to K$ defined by $(a, a') \mapsto a\rho(a')$ is holomorphic. Because Q is discrete, μ (and hence m) is holomorphic. Then E is a complex Lie group and the abstract extension lifts to $\mathrm{Ext}(Q, K, 1)$. $\square$

Corollary F.7.2.6 below is a result about discrete base with nontrivial outer action. We turn to two other simple cases.

Proposition F.3.2.2. *Every extension of $\mathbb{C}$ is a semidirect product. In particular, every central extension of $\mathbb{C}$ trivial.*

Proof. Let $0 \to B \to C \xrightarrow{p} \mathbb{C} \to 0$ be an extension. Then $0 \to L(B) \to L(C) \xrightarrow{d_e p} L(\mathbb{C}) \to 0$ is an exact sequence of Lie algebras. Take a $\mathbb{C}$-linear map $ds : L(\mathbb{C}) \to L(C)$ with $d_e p \circ ds = \mathrm{Id}_{L(\mathbb{C})}$. Because $\dim_{\mathbb{C}} L(\mathbb{C}) = 1$, ds is a Lie algebra morphism. As $\mathbb{C}$ is simply connected, there is a unique morphism $s : \mathbb{C} \to C$ with $d_e s = ds$. Since $d_e(ps) = \mathrm{Id}_{L(\mathbb{C})}$, one has $ps = \mathrm{Id}_{\mathbb{C}}$. Therefore, this extension is a semidirect product by Fact F.3.0.4. □

Proposition F.3.2.3. *Let B be a connected commutative complex Lie group. Then every central extension of $\mathbb{C}^*$ by B is trivial.*

Proof. Let C be a central extension of $\mathbb{C}^*$ by B. Consider the pullback extension along $\exp(2\pi i \bullet) : \mathbb{C} \to \mathbb{C}^*$. By Proposition F.3.2.2, there is a morphism $\rho : \mathbb{C} \to C'$ with $p'\rho = \mathrm{Id}_{\mathbb{C}}$. Then $p\epsilon\rho(1) = \exp(2\pi i) = 1$, so $\epsilon\rho(1) \in B$. As B is connected commutative, its exponential map $\exp_B : L(B) \to B$ is surjective. Take $v \in L(B)$ with $\exp_B(-v) = \epsilon\rho(1)$.

$$\begin{tikzcd}
 & & & \mathbb{Z} \arrow[d, hook] & \\
1 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C' \arrow[r, "p'"'] \arrow[d, "\epsilon"] & \mathbb{C} \arrow[r] \arrow[l, bend right, "\rho"'] \arrow[d, "\exp(2\pi i \bullet)"] & 0 \\
1 \arrow[r] & B \arrow[r] & C \arrow[r, "p"] & \mathbb{C}^* \arrow[r] & 1
\end{tikzcd}$$

Define a holomorphic map

$$\rho' : \mathbb{C} \to C', \quad \rho'(z) = \exp_B(zv)\rho(v).$$

We check that ρ' is a group morphism. For every $z, w \in \mathbb{C}$,

$$\begin{aligned} \rho'(z+w) &= \exp_B((z+w)v)\rho(z+w) = \exp_B(zv)\exp_B(wv)\rho(z)\rho(w) \\ &= \exp_B(zv)\rho(z)\exp_B(wv)\rho(w) = \rho'(z)\rho'(w), \end{aligned}$$

where the last but one equality uses $B \subset Z(C)$.

Therefore, ρ' is a complex Lie group morphism. Moreover, $\rho'(1) = \exp_B(v)\rho(1) = \epsilon\rho(-1)\rho(1)$. Then $\epsilon\rho'(1) = e_C$. Therefore, $\rho'(\mathbb{Z}) \subset \ker(\epsilon)$. Thus, ρ' induces a morphism $s : \mathbb{C}^* \to C$ making a commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\rho'"] \arrow[d, "\exp(2\pi i \bullet)"] & C' \arrow[r, "p'"] \arrow[d, "\epsilon"] & C \arrow[d, "\exp(2\pi i \bullet)"] \\
\mathbb{C}^* \arrow[r, dashed, "s"] & C \arrow[r, "p"] & \mathbb{C}^*
\end{tikzcd}$$

Since $p'\rho' = \mathrm{Id}_{\mathbb{C}}$ and $\exp(2\pi i \bullet) : \mathbb{C} \to \mathbb{C}^*$ is surjective, $ps = \mathrm{Id}_{\mathbb{C}^*}$. From Fact F.3.0.4, the extension C is trivial. □

Example F.7.1.7 gives a result about non-central extensions of $\mathbb{C}^*$.

Now assume that the Lie group K is discrete and *commutative.* We recall results[5] from [Hoc51b, Sec. 3].

Fact F.3.2.4 ([Hoc51b, p.545])**.** *Let K, Q be Lie groups. If K is discrete commutative and Q is connected, then the extension (F.3) of Lie groups is central.*

Corollary F.3.2.5. *Let K, Q be commutative Lie groups. If Q is connected and K is discrete, then every extension of Q by K is commutative.*

Proof. Let (F.3) be such an extension. By Fact F.3.2.4, this extension is central. Then consider the induced continuous map (F.4). Since Q is connected and K is discrete, this map is constant, or equivalently, E is commutative. $\square$

Let Ab_c be the abelian category of abelian groups that are at most countable. Fact F.3.2.6 shows that the universal cover of a connected Lie group is "universal" among all the extensions with discrete commutative kernels.

Fact F.3.2.6 (Hochschild, [Hoc51b, Thm. 3.2 and Cor.])**.** *Let Q be a connected Lie group. Then the functor* $\mathrm{Ext}(Q, \cdot, 1) : \mathrm{Ab}_c \to \mathrm{Ab}$ *is represented by* $\pi_1(Q)$ *and the class of the universal cover sequence* $1 \to \pi_1(Q) \to \tilde{Q} \to Q \to 1$ *in* $\mathrm{Ext}(Q, \pi_1(Q), 1)$*. Hence an isomorphism* $\Gamma_K : \mathrm{Ext}(Q, K, 1) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(Q), K)$ *functorial in* $K \in \mathrm{Ab}_c$*. Moreover,* $E \in \mathrm{Ext}(Q, K, 1)$ *is connected if and only if* $\Gamma_K(E)$ *is surjective.*

F.4 Commutative Extensions

F.4.1 Generalities

Let $\mathcal{C}$ be the category of commutative complex Lie groups.

Lemma F.4.1.1. *The category $\mathcal{C}$ is naturally an additive category with finite direct products.*

Proof. The Hom sets are commutative groups, and composition of morphisms is bilinear. Moreover, the product $G_1 \times G_2$ of two commutative complex Lie groups is both a product and a coproduct of G_1 and G_2 in $\mathcal{C}$. $\square$

By [Mil17a, Thm. 5.62], the category Alg of commutative complex algebraic groups is an abelian category. By contrast, as Example F.4.1.2 and Example F.4.1.3 show, $\mathcal{C}$ is NOT an abelian category.

[5]They are stated for real Lie groups, but the proofs extend to the complex setting.

Example F.4.1.2. The morphism $i : \mathbb{Z}^2 \to \mathbb{C}, \quad (a,b) \mapsto a + b\sqrt{2}$ is injective. The image is dense in $\mathbb{R}$ and not closed in $\mathbb{C}$. For every morphism $f : \mathbb{C} \to X$ in $\mathcal{C}$ with $fi = 0$, one has $f = 0$ by the identity theorem for holomorphic maps. Thus i is a monomorphism and epimorphism in $\mathcal{C}$, but not an isomorphism.

Example F.4.1.3. Let $p : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^4$ be the natural projection. Let $i : \mathbb{C} \to \mathbb{C}^2$ be the closed embedding defined by $z \mapsto (z, \sqrt{2}z)$. Then the composition $pi : \mathbb{C} \to \mathbb{C}^2/\mathbb{Z}^4$ is an injective morphism (hence a monomorphism) in $\mathcal{C}$. By [Lee13, Example 7.19], $pi(C)$ is a connected dense subset of $\mathbb{C}^2/\mathbb{Z}^4$. In particular, pi is an epimorphism in $\mathcal{C}$. The cokernel of pi is the zero morphism $\mathbb{C}^2/\mathbb{Z}^4 \to 0$. However, pi is not an isomorphism in $\mathcal{C}$.

Proposition F.4.1.4 3 is a special case of [Con14, Prop. D.2.1]. An elementary proof is given.

Proposition F.4.1.4.

1. *$\mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^*, \mathbb{C}) = 0$.*

2. *For $A \in \mathcal{C}$, the map*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^n, A) \to \mathrm{Hom}_{\mathrm{Vec}}(L(\mathbb{C}^n), L(A)), \quad f \mapsto d_e f$$

   is a group isomorphism.

3. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a morphism in $\mathcal{C}$. Then there is an integer k such that $f(z) = z^k$ for every $z \in \mathbb{C}^*$. Hence an isomorphism $\mathbb{Z} = \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^*, \mathbb{C}^*)$.*

Proof. The Lie algebra of $\mathbb{C}^*$ is $\mathbb{C}$. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is normalized as $w \mapsto e^{2\pi i w}$.

1. Let $f : \mathbb{C}^* \to \mathbb{C}$ be a morphism. Then $d_e f : \mathbb{C} \to \mathbb{C}$ is linear. There is $a \in \mathbb{C}$ with $d_e f(v) = av$ for all $v \in \mathbb{C}$. Since $1 \in \mathbb{C} = L(\mathbb{C}^*)$ is mapped to $1 \in \mathbb{C}^*$ under the exponential map $\exp(2\pi i \bullet)$, one has $0 = f(1) = d_e f(1) = a$. Then $d_e f = 0$ and $f = 0$.

2. It follows from the fact that $\mathbb{C}^n$ is simply connected and both groups are commutative.

3. Consider the induced linear map on Lie algebras $df : \mathbb{C} \to \mathbb{C}$. There is a unique complex number k such that $df(w) = kw$ for all $w \in \mathbb{C}$. Then
$$e^{2\pi i k} = \exp(df(1)) = f \exp(1) = f(1) = 1.$$
   Therefore, k is an integer. For every $z \in \mathbb{C}^*$, there is $w \in \mathbb{C}$ with $\exp(w) = z$. Then $f(z) = f(\exp(w)) = \exp df(w) = \exp(kw) = z^k$.

$\square$

Given a surjective morphism $p : C \to A$ in $\mathcal{C}$, let $i : B \to C$ be its kernel. We write it as $1 \to B \xrightarrow{i} C \xrightarrow{p} A \to 1$ and call it a *short exact sequence* in $\mathcal{C}$. In that case, C is called an *extension* of A by B. Two such extensions C and C' are called *equivalent* if there exists a morphism $f : C \to C'$ making a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

In this case, $f : C \to C'$ is an isomorphism in $\mathcal{C}$. For $A, B \in \mathcal{C}$, the set of isomorphism classes of extensions of A by B is denoted by $\mathrm{Ext}(A, B)$.

Proposition F.4.1.5.

1. *The formation* $\mathrm{Ext}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ *is a covariant functor.*

2. *Let* $\mathcal{E}$ *be the collection of extensions in* $\mathcal{C}$. *Then the pair* $(\mathcal{C}, \mathcal{E})$ *is an exact category* in the sense of [Sta24, Tag 05SF].

Proof. 1. Fix $A, B \in \mathcal{C}$ and an element of $\mathrm{Ext}(A, B)$: $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$.

(a) If $f : B \to B'$ is a morphism in $\mathcal{C}$, then

$$g : B \to C \times B', \quad b \mapsto (-b, f(b))$$

is a morphism in $\mathcal{C}$. It is injective and the (set-theoretic) image is *closed* in $C \times B'$. By Fact F.2.0.1 2, g identifies B as a complex Lie subgroup of $C \times B'$. Let f_*C be the quotient $(C \times B')/B$ provided by Fact F.2.0.3. The canonical map $B' \to C \times B'$ induces an injective morphism $f_*i : B' \to f_*C$ since $B \cap (\{0\} \times B') = \{0\}$. Moreover, B is in the kernel of the composition $C \times B' \to A$ by $(c, \beta) \mapsto p(c)$, hence a surjective morphism $f_*p : f_*C \to A$.

Note that $f_*p \circ f_*i = 0$, so $f_*i(B') \subset \ker(f_*p)$. For every element x of $\ker(f_*p)$, take a representative $(c, \beta) \in C \times B'$. As $p(c) = 0$, $c \in B$. Then $(0, \beta + f(c)) - (c, \beta) = g(c)$. Therefore,

$$x = [(0, \beta + f(c))] = f_*i(\beta + f(c)) \in f_*(B').$$

Thus, $f_*i(B') = \ker(f_*p)$

Therefore, the sequence

$$0 \to B' \xrightarrow{f_*i} f_*C \xrightarrow{f_*p} A \to 0$$

is exact and $f_*C \in \mathrm{Ext}(A, B')$. Hence a morphism $f_* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B')$ in the category Set.

Let F be the canonical morphism $C \to f_*C$. By construction, the extension $f_*C \in \mathrm{Ext}(A, B')$ has the following universal property: for every morphism $h : A \to A'$ in C, every $C' \in \mathrm{Ext}(A', B')$, every morphism $G : C \to C'$ making the diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "F"'] \arrow[dd, bend left, "G"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d, "\mathrm{Id}"] & f_*C \arrow[r] \arrow[d, dashed, "u"'] & A \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & 0,
\end{tikzcd} \tag{F.7}$$

there exists a unique morphism $u : f_*C \to C'$ keeping the diagram commutative.

(b) If $h : A' \to A$ is a morphism in $\mathcal{C}$, by Example F.3.1.1, we get a morphism $h^* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ in the category Set. Let F be the canonical projection $h^*C \to C$. By construction, the extension g^*C has the following universal property: for every morphism $g : B' \to B$, every extension $C' \in \mathrm{Ext}(A', B')$, every morphism $G : C' \to C$ making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, dashed, "v"] \arrow[dd, bend left, "G"] & A' \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & h^*C \arrow[r] \arrow[d, "F"'] & A' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0,
\end{tikzcd} \tag{F.8}$$

there exits a unique morphism $v : C' \to h^*C$ keeping the diagram commutative.

(c) Let $f : B \to B'$, $g : A \to A'$ be morphisms in $\mathcal{C}$, $C \in \mathrm{Ext}(A, B)$, and $C' \in \mathrm{Ext}(A', B')$. Then the relation $f_*C = g^*C'$ in $\mathrm{Ext}(A, B')$ is equivalent to the existence of a morphism $F : C \to C'$ making a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "F"] & A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & 0.
\end{tikzcd}$$

Indeed, it follows from the universal properties in Points (1a) and (1b). For every $X \in \mathrm{Ext}(A', B)$, in view of the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & g^*X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B' \arrow[r] & f_*X \arrow[r] & A' \arrow[r] & 0,
\end{tikzcd}$$

one has $f_*g^*X = g^*f_*X$. This completes the proof.

2. It follows from Part 1 and Lemma F.4.1.1.

□

Example F.4.1.6. If A is a complex torus with $\dim A = g$, B is the discrete group $\mathbb{Q}/\mathbb{Z}$, then $\mathrm{Ext}(A, B)$ is isomorphic to B^{2g} by Fact F.3.2.6. Even though B is an injective object of Ab, the functor $\mathrm{Ext}(\cdot, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is nonzero.

Example F.4.1.7. For an extension $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$ in $\mathcal{C}$, the pushout $i_*C \in \mathrm{Ext}(A, C)$ is the trivial extension. In fact, $i_*C = C \times C/B$ with the embedding

$$B \to C \times C, \quad b \mapsto (-b, b).$$

The group law $C \times C \to C$ descents to a morphism $r : i_*C \to C$. Then $r \circ i_*(i) = \mathrm{Id}_C$. By Fact F.3.0.4, i_*C is trivial.

Similarly, as the diagonal inclusion $C \to C \times C$ factors through a morphism $s : C \to p^*C$ and $p^*(p) \circ s = \mathrm{Id}_C$, the pullback $p^*C \in \mathrm{Ext}(C, B)$ is also trivial.

Fact F.4.1.8 follows from Proposition F.4.1.5.

Fact F.4.1.8 ([Ros58, Prop. 5], [Ser88, Prop. 1, p.163])**.** *1. For every $A, B \in \mathcal{C}$, under the Baer sum $\mathrm{Ext}(A, B)$ is an abelian subgroup of $\mathrm{Ext}(A, B, 1)$.*

2. *The functor $\mathrm{Ext} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ is additive.*

3. *For any $C, C' \in \mathrm{Ext}(A, B)$, the product $C \times C'$ is naturally an element of $\mathrm{Ext}(A \times A, B \times B)$.*

4. *Let $d : A \to A \times A$ the diagonal map of A and $s : B \times B \to B$ the group law of B. Then $C + C' = d^*s_*(C \times C')$ in $\mathrm{Ext}(A, B)$.*

Corollary F.4.1.9. *For every commutative complex Lie group A, the restriction $\mathrm{Ext}(A, \cdot) : \mathrm{Vec} \to \mathrm{Ab}$ factors through a functor from Vec to the category of all complex vector spaces.*

By Example F.4.3.3 below, for every $V \in \mathrm{Vec}$, $\dim_{\mathbb{C}} \mathrm{Ext}(A, V)$ is finite. Hence an additive functor $\mathrm{Ext}(A, \cdot) : \mathrm{Vec} \to \mathrm{Vec}$.

Example F.4.1.10. Endowing each object of Ab_c the discrete topology gives a functor $\mathrm{Ab}_c \to \mathcal{C}$. This functor identifies Ab_c as a full subcategory of $\mathcal{C}$. The subcategory Ab_c is closed under extension by Fact F.3.0.1. From Proposition F.3.2.1, the forgetful natural transformation $\mathrm{Ext}(-,+) \to \mathrm{Ext}^1_{\mathbb{Z}}(-,+)$ is an isomorphism of functors $\mathrm{Ab}_c^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$.

Example F.4.1.11. Analytification functor $(\bullet)^{\mathrm{an}} : \mathrm{Alg} \to \mathcal{C}$ identifies Alg as a subcategory of $\mathcal{C}$ (which is not full). The extension problem within the subcategory Alg is discussed by Rosenlicht [Ros58] and Serre [Ser88, Ch. VII]. They define a similar additive functor $\mathrm{Ext}_{\mathrm{Alg}} : \mathrm{Alg}^{\mathrm{op}} \times \mathrm{Alg} \to \mathrm{Ab}$. For every $A, B \in \mathrm{Alg}$, there is a natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(A,B) \to \mathrm{Ext}(A^{\mathrm{an}}, B^{\mathrm{an}})$. In general, this morphism is neither injective nor surjective.

By [MM66, Introduction, 1.], when A is a complex abelian variety, one has$\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_a, A) = 0$ while $\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_m, A)$ is (non-canonically) isomorphic to the torsion subgroup A_{tor} of A. But $\mathrm{Ext}(\mathbb{C}^*, A^{\mathrm{an}}) = 0$ by Proposition F.3.2.3, so the natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_m, A) \to \mathrm{Ext}(\mathbb{C}^*, A^{\mathrm{an}})$ is not injective.

For any two complex abelian varieties X_i $(i = 1, 2)$ of positive dimension, $\mathrm{Ext}_{\mathrm{Alg}}(X_2, X_1)$ is countable while $\mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$ is uncountable. In fact, by [BL99, Ch. 1, Prop. 6.1, Cor. 6.3], the natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(X_2, X_1) \to \mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$ is an embedding onto the torsion subgroup of $\mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$.

Lemma F.4.1.12 is mentioned at the bottom of [Hoc51b, p.546].

Lemma F.4.1.12. *If G is a commutative connected Lie group, then G is a divisible $\mathbb{Z}$-module.*

Proof. The exponential map $\exp : L(G) \to G$ is surjective. For every $x \in G$, there is $v \in L(G)$ with $\exp(v) = x$. For every integer $n \geq 1$, $\exp(v/n) \in G$ and $n(\exp(v/n)) = x$. □

Corollary F.4.1.13. *An extension $0 \to B \to C \to A \to 0$ in $\mathcal{C}$ with B connected and A discrete is trivial. In particular, for every $G \in \mathcal{C}$, the natural exact sequence*

$$0 \to G_0 \to G \to G/G_0 \to 0$$

is a trivial extension, hence a non-canonical isomorphism $G \to G_0 \times G/G_0$ in $\mathcal{C}$.

Proof. By Lemma F.4.1.12, the $\mathbb{Z}$-module B is divisible, so the functor $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, B) : \mathrm{Ab} \to \mathrm{Ab}$ is zero. Since A is discrete, the result follows from Example F.4.1.10. □

Example F.4.1.14. The abelian group underlying a complex torus B is divisible by Lemma F.4.1.12, hence an injective object of Ab and $\mathrm{Ext}^1_{\mathbb{Z}}(\bullet, B) : \mathrm{Ab} \to \mathrm{Ab}$ is zero. However, $\mathrm{Ext}(\bullet, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ can be nonzero. In fact, [BL04, (8) b), p.68] gives an example of a nontrivial exact sequence of complex tori

$$0 \to B \to C \to A \to 0$$

with $\dim A = \dim B = 1$.

F.4.2 Exact sequences of Ext

Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be an exact sequence in $\mathcal{C}$, *i.e.*, $A \in \mathrm{Ext}(A'', A')$. For $f \in \mathrm{Hom}(A', B)$, there is $f_* A \in \mathrm{Ext}(A'', B)$. Hence a map

$$d : \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B), \quad d(f) = f_* A.$$

Then d is a group morphism. The formation of d is functorial in B.

Proposition F.4.2.1. *Let $B \in \mathcal{C}$. The sequence in* Ab *with canonical morphisms*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{C}}(A'', B) \to \mathrm{Hom}_{\mathcal{C}}(A, B) &\xrightarrow{i_*} \mathrm{Hom}_{\mathcal{C}}(A', B) \\ \xrightarrow{d} \mathrm{Ext}(A'', B) \xrightarrow{p^*} \mathrm{Ext}(A, B) &\to \mathrm{Ext}(A', B) \end{aligned}$$

is exact and functorial in B.

Proof.

- Exactness at $\mathrm{Hom}(A, B)$ follows from Fact F.2.0.1.
- Exactness at $\mathrm{Hom}(A', B)$: By Example F.4.1.7, the composition

  $$\mathrm{Hom}(A, B) \xrightarrow{i_*} \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B)$$

  is zero. Now take $\phi \in \ker(d)$. By Fact F.3.0.4, there is a morphism $r : \phi_* A \to B$ with $r\phi_*(i) = \mathrm{Id}_B$. Let $F : A \to \phi_* A$ be the canonical morphism. Then $rFi = r\phi_*(i)\phi = \phi$ as morphism $A' \to B$. Hence $\phi \in \mathrm{im}(i_*)$.
- Exactness at $\mathrm{Ext}(A'', B)$: By Example F.4.1.7, for every $\phi \in \mathrm{Hom}(A', B)$, one has $p^* d(\phi) = p^* \phi_* A = \phi_* p^* A = 0$, *i.e.*, $p^* \circ d = 0$.

  Take an extension $0 \to B \xrightarrow{f} C \xrightarrow{g} A'' \to 0$ whose class in $\mathrm{Ext}(A'', B)$ belongs to $\ker(p^*)$. By Fact F.3.0.4, there is a morphism $s : A \to p^* C$ with $p^*(g) \circ s = \mathrm{Id}_A$. Let $F : p^* C \to C$ be the canonical morphism. Define $\psi : A \to C$ by $\psi = F \circ s$. Then $0 = p|_{A'} = p \circ p^*(g) \circ s|_{A'} = g \circ \psi|_{A'}$. Thus, there is a factorization $\phi : A' \to B$ with $f \circ \phi = \psi|_{A'}$.

  By construction, the extension group of $d(\phi) = \phi_* A \in \mathrm{Ext}(A'', B)$ is $(A \times B)/D$, with $D = \{(-a', \phi(a')) : a' \in A'\}$. Define

  $$A \times B \to C, \quad (a, b) \mapsto \psi(a) + f(b).$$

  Since $\psi(-a') + f(\phi(a')) = 0$ for all $a' \in A'$, this map induces a factorization $\phi_* A \to C$ in the middle

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, dashed, "\phi"] & A \arrow[r, "p"] \arrow[d] & A'' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "\phi_* i"] \arrow[d, "\mathrm{Id}"] & \phi_* A \arrow[r] \arrow[d, dashed] & A'' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & A'' \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, "\mathrm{Id}"] & p^*C \arrow[r, "p^*(g)"] \arrow[u, "F"] & A \arrow[r] \arrow[u, "p"] \arrow[ul, dashed, "\psi"] \arrow[l, dashed, bend left, "s"] & 0.
\end{tikzcd}$$

keeping the diagram commutative. Then $C = \phi_* A = d(\phi)$ in $\mathrm{Ext}(A'', B)$. One has $\ker(p^*) = \mathrm{im}(d)$.

- Exactness at $\mathrm{Ext}(A, B)$: As the composition $A' \to A \to A''$ is zero and $\mathrm{Ext}(\cdot, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is an additive functor, the composition $\mathrm{Ext}(A'', B) \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ is zero.

  Take an extension $0 \to B \xrightarrow{f} C_1 \xrightarrow{g} A \to 0$ with $i^*(C_1) = 0$ in $\mathrm{Ext}(A', B)$. Then there is a morphism $s : A' \to i^*C_1$ with $i^*(g) \circ s = \mathrm{Id}_{A'}$. The composition $\phi : A' \to C_1$ is injective. Indeed, if $a' \in \ker(\phi)$, then $s(a') = (a', 0)$ in $A' \times C_1$. Thus, $i(a') = 0$ by the construction of pullback extension. Since i is injective, one has $a' = 0$.

  Let $C := C_1/\phi(A')$ and $C_1 \to C$ be the quotient morphism. Let $f_0 : B \to C$ be the induced morphism. Then f_0 is injective. Indeed, if $b \in \ker(f_0)$, then $f(b) = \phi(a')$ for some $a' \in A'$. One has $(a', f(b)) \in i^*C_1$, so $i(a') = gf(b) = 0$. Hence $a' = 0$ and $f(b) = 0$. Since f is injective, one has $b = 0$.

  Because $pg\phi = p \circ i = 0$, the morphism $pg : C_1 \to A''$ descends to a surjective morphism $g_0 : C \to A''$. We prove that the bottom row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & i^*C_1 \arrow[r, "i^*(g)"'] \arrow[d] & A' \arrow[r] \arrow[d, "i"] \arrow[l, dashed, bend right, "s"'] \arrow[dl, dashed, "\phi"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] \arrow[d, "\mathrm{Id}"] & C_1 \arrow[r, "g"] \arrow[d] & A \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & B \arrow[r, "f_0"] & C \arrow[r, dashed, "g_0"] & A'' \arrow[r] & 0.
\end{tikzcd}$$

  is exact.

  Since $gf = 0$, one has $f_0(B) \subset \ker(g_0)$. Conversely, for $c \in \ker(g_0)$, there is $c_1 \in C_1$ with $[c_1] = c$. Since $pg(c_1) = g_0(c) = 0$, one gets

$g(c_1) \in A'$. Then $g\phi g(c_1) = g(c_1)$. So $c_1 - \phi g(c_1) \in \ker(g) = B$ and

$$f_0(c_1 - \phi g(c_1)) = [c_1 - \phi g(c_1)] = c.$$

Therefore, $\ker(g_0) = f_0(B)$. In particular, one has $C \in \operatorname{Ext}(A'', B)$. By the universal property showed in the diagram (F.8), one has $C_1 = p^*C$.

$\square$

Example F.4.2.2. Let A be a complex torus, and let B be a finite abelian group. Then $\operatorname{Hom}_{\mathcal{C}}(A, B) = 0$. Let integer $n(\geq 1)$ be a multiple of $\#B$. Applying Proposition F.4.2.1 to the exact sequence in $\mathcal{C}$

$$0 \to A[n] \to A \overset{[n]_A}{\to} A \to 0,$$

one gets an exact sequence in Ab:

$$0 \to \operatorname{Hom}(A[n], B) \to \operatorname{Ext}(A, B) \overset{f}{\to} \operatorname{Ext}(A, B).$$

Since the morphism $[n]_B : B \to B$ is zero in $\mathcal{C}$, by Fact F.4.1.8, $f = ([n]_B)_* = 0$. Hence an isomorphism $\operatorname{Hom}(A[n], B) \to \operatorname{Ext}(A, B)$ that is functorial in B, which is also confirmed by Fact F.3.2.6.

Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{C}$. If $A \in \mathcal{C}$ and $\phi \in \operatorname{Hom}(A, B'')$, then $\phi^*B \in \operatorname{Ext}(A', B)$. Define a map $d : \operatorname{Hom}(A, B'') \to \operatorname{Ext}(A, B')$ by $d(\phi) = \phi^*B$.

Proposition F.4.2.3. *Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{C}$ and $A \in \mathcal{C}$. Then the sequence*

$$0 \to \operatorname{Hom}(A, B') \to \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B'') \overset{d}{\to} \operatorname{Ext}(A, B') \to \operatorname{Ext}(A, B) \to \operatorname{Ext}(A, B'')$$

in Ab *is exact and functorial in A.*

The proof is analogous to that of Proposition F.4.2.1 and is thereby omitted.

Consider the extension problem with connected bases. Corollary F.4.2.4 should be compared to [Sha49, Thm. 1]: for two compact connected real Lie groups G, H, the cokernel of the restriction morphism $\operatorname{Hom}(\tilde{H}, Z(G)) \to \operatorname{Hom}(\pi_1(H), Z(G))$ is isomorphic to the group of extensions of H by G.

Corollary F.4.2.4. *Let A, B be commutative complex Lie groups. Assume that A is connected with universal cover $\omega : \tilde{A} \to A$. Then there is a canonical exact sequence in* Ab*:*

$$0 \to \operatorname{Hom}_{\mathcal{C}}(A, B) \overset{\cdot\circ\omega}{\to} \operatorname{Hom}_{\mathcal{C}}(\tilde{A}, B) \overset{r}{\to} \operatorname{Hom}_{\mathrm{Ab}}(\pi_1(A), B) \to \operatorname{Ext}(A, B) \to 0, \tag{F.9}$$

where r is induced by restriction.

Proof. By Proposition F.3.2.2, Fact F.3.2.6 and Corollary F.4.1.13, the functor $\mathrm{Ext}(\mathbb{C}, \bullet) : \mathcal{C} \to \mathrm{Ab}$ is zero. By Fact F.4.1.8, one has

$$\mathrm{Ext}(\mathbb{C}^n, \bullet) = 0. \tag{F.10}$$

The proof is concluded by Proposition F.4.2.1. □

Remark F.4.2.5. In Corollary F.4.2.4, if B discrete, then $\mathrm{Hom}_{\mathcal{C}}(\tilde{A}, B) = 0$ and the natural morphism $\mathrm{Hom}(\pi_1(A), B) \to \mathrm{Ext}(A, B)$ is an isomorphism, which agrees with Fact F.3.2.6.

F.4.3 Determination of commutative extension group

The commutative extension problem of complex Lie groups is answered by Proposition F.4.3.1. Fix two commutative complex Lie groups A, B.

Proposition F.4.3.1. *There is a non-canonical isomorphism in* Ab*:*

$$\mathrm{Ext}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0) \oplus \mathrm{Ext}(A_0, B_0),$$

and $\mathrm{Ext}(A_0, B_0)$ *is the cokernel of the restriction morphism*

$$s : \mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B)) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0).$$

Proof. By Corollary F.4.1.13, there are non-canonical isomorphisms $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$ in $\mathcal{C}$. Using Fact F.4.1.8, one gets an isomorphism in Ab:

$$\mathrm{Ext}(A, B) \to \mathrm{Ext}(A/A_0, B_0) \oplus \mathrm{Ext}(A/A_0, B/B_0) \oplus \mathrm{Ext}(A_0, B/B_0) \oplus \mathrm{Ext}(A_0, B_0).$$

By Corollary F.4.1.13, one has $\mathrm{Ext}(A/A_0, B_0) = 0$. By Example F.4.1.10, the natural morphism $\mathrm{Ext}(A/A_0, B/B_0) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0)$ is an isomorphism. Fact F.3.2.6 gives a natural isomorphism $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0) \to \mathrm{Ext}(A_0, B/B_0)$. Corollary F.4.2.4 identifies $\mathrm{Ext}(A_0, B_0)$ with the cokernel of the restriction map $r : \mathrm{Hom}_{\mathcal{C}}(\tilde{A}_0, B_0) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0)$. By Proposition F.4.1.4 2, the group morphism

$$t : \mathrm{Hom}_{\mathcal{C}}(\tilde{A}_0, B_0) \to \mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B)), \quad \phi \mapsto d_e\phi$$

is an isomorphism. The proof is finished by setting $s = rt^{-1}$. □

For every $C \in \mathrm{Ext}(A, B)$, by Fact F.2.0.3, the morphism $C \to A$ is a principal B-bundle. The bijection (F.1) gives rise to a canonical map

$$\pi : \mathrm{Ext}(A, B) \to H^1(A, \mathcal{B}_A). \tag{F.11}$$

Fact F.4.3.2 is taken from [Ros58, pp.698-699] and the proof of [Ser88, Ch. VII, no. 5, Prop. 5].

Fact F.4.3.2. *The map (F.11) is a group morphism and the formation of π is functorial, in the sense that it commutes with the morphisms $f_* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B')$ defined by $f : B \to B'$ and $g^* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ defined by $g : A' \to A$. When B is a vector group, the map π is $\mathbb{C}$-linear.*

Example F.4.3.3. Let X be a toroidal group, and let $\omega : \tilde{X} \to X$ be the universal covering of kernel F. Then F is a discrete subgroup of the vector space $\tilde{X}$. By Proposition F.4.2.1,

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{C}) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \mathbb{C}) \to \mathrm{Hom}_{\mathcal{C}}(F, \mathbb{C}) \to \mathrm{Ext}(X, \mathbb{C}) \to \mathrm{Ext}(\tilde{X}, \mathbb{C})$$

is an exact sequence in Ab. From Definition F.2.0.9, $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{C}) = 0$. By Proposition F.10, $\mathrm{Ext}(\tilde{X}, \mathbb{C}) = 0$. Hence the first exact row of Diagram (F.12).

According to [AK01, p.48], there is a $\mathbb{C}$-linear isomorphism $\mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \mathbb{C}) \to H^0(X, \Omega^1_X)$ and every global holomorphic 1-form on X is d-closed. So taking de Rham cohomology class results in a linear map $H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$. The inclusion $\mathbb{C}_X \to O_X$ induces a linear map $H^1(X, \mathbb{C}) \to H^1(X, O_X)$. By universal coefficient theorem (see, *e.g.*, [Hat05, Thm. 3.2]), the natural morphism $\mathrm{Hom}_{\mathcal{C}}(F, \mathbb{C}) \to H^1(X, \mathbb{C})$ is an isomorphism. Hence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \mathbb{C}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(F, \mathbb{C}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}(X, \mathbb{C}) \arrow[r] \arrow[d, "(F.11)"] & 0 \\
0 \arrow[r] & H^0(X, \Omega^1_X) \arrow[r] & H^1(X, \mathbb{C}) \arrow[r] & H^1(X, O_X). &
\end{tikzcd} \tag{F.12}$$

Let $b_1(X) := \dim_{\mathbb{C}} H^1(X, \mathbb{C})$ be the first Betti number of X, *i.e.*, the $\mathbb{Z}$-rank of F. From [AK01, p.48], as a $\mathbb{C}$-vector space

$$\mathrm{Ext}(X, \mathbb{C}) = \frac{H^1(X, \mathbb{C})}{H^0(X, \Omega^1_X)} \tag{F.13}$$

is of dimension $b_1(X) - \dim X$.

If X is a toroidal theta group,[6] then $\pi : \mathrm{Ext}(X, \mathbb{C}) \to H^1(X, O_X)$ is a $\mathbb{C}$-linear isomorphism by [AK01, Thm. 2.2.6 b)]. Otherwise, X is a toroidal wild group[6]and $H^1(X, O_X)$ is infinite dimensional by [AK01, Prop. 2.2.7].

A seemingly different way to compute the last factor in Proposition F.4.3.1, *i.e.*, the group of commutative extensions of two connected commutative complex Lie groups, is given in Example F.4.3.4.

[6]in the sense of [AK01, Def. 2.2.1]

Example F.4.3.4. Start by the special case that X is a toroidal group and B is a connected commutative complex Lie group. Denote the kernel of the universal cover of B (resp. X) by $\iota : K \to \tilde{B}$ (resp. $F \to \tilde{X}$). By (F.10) and Proposition F.4.1.4 2, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, K) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{B}) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, B) \to 0$$

is exact in Ab. As F is a free $\mathbb{Z}$-module,

$$0 \to \mathrm{Hom}_{\mathcal{C}}(F, K) \to \mathrm{Hom}_{\mathcal{C}}(F, \tilde{B}) \to \mathrm{Hom}_{\mathcal{C}}(F, B) \to 0$$

in Ab is also exact. Applying Proposition F.4.2.1 and the snake lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, K) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{B}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, K) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, \tilde{B}) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F, B) \arrow[r] & 0,
\end{tikzcd}$$

one gets an exact sequence in Ab:

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{j} \mathrm{Ext}(X, K) \xrightarrow{\iota_*} \mathrm{Ext}(X, \tilde{B}) \to \mathrm{Ext}(X, B) \to 0. \quad \text{(F.14)}$$

Since K is a free $\mathbb{Z}$-module, by Fact F.3.2.6, $\mathrm{Ext}(X, K) = H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} K$. By Fact F.4.1.8 and (F.13), one has

$$\mathrm{Ext}(X, \tilde{B}) = \frac{H^1(X, \mathbb{C})}{H^0(X, \Omega^1_X)} \otimes_{\mathbb{C}} \tilde{B}.$$

The group morphism ι_* is induced by the $\mathbb{Z}$-bilinear map

$$H^1(X, \mathbb{Z}) \times K \to (\frac{H^1(X, \mathbb{C})}{H^0(X, \Omega^1_X)}) \otimes_{\mathbb{C}} \tilde{B}, \quad (\eta, x) \mapsto [\eta] \otimes \iota(x).$$

Thus we can compute $\mathrm{Ext}(X, B)$ from (F.14).

For a general connected commutative complex Lie group A, by [AK01, 1.1.5], $A = \mathbb{C}^l \times (\mathbb{C}^*)^m \times X_0$ for some integers $l, m \geq 0$ and a toroidal group X_0. By Fact F.4.1.8, Proposition F.3.2.2 and Proposition F.3.2.3, $\mathrm{Ext}(A, B) = \mathrm{Ext}(X_0, B)$, reducing to the previous case.

F.5 Commutative extensions of complex tori

F.5.1 Primitive cohomology classes

Every central extension of a compact real Lie group by a vector group is trivial, shown by Fact F.5.1.1.

Fact F.5.1.1 (Iwasawa, [Iwa49, Lem. 3.7], [Hoc51a, Footnote 10, p.107])**.** *Let (F.3) be an exact sequence of* real *Lie groups. If K is a vector group and Q is compact, then this extension is a semidirect product. In particular, if this extension is central, then it is trivial.*

Contrary to the real case, Example F.5.1.2 shows a commutative extension of a complex torus by a vector group can be nontrivial.

Example F.5.1.2 ([MM60, p.145, Exemple], [Har76, Sec. I.3])**.** Set $C = \mathbb{C}^* \times \mathbb{C}^*$. Then $B = \{(e^z, e^{iz}) : z \in \mathbb{C}\}$ is a complex Lie subgroup of C (but not an algebraic subgroup of $\mathbb{G}_m \times \mathbb{G}_m$) isomorphic to $\mathbb{C}$. The quotient $A = C/B$ is an elliptic curve. The exact sequence $0 \to B \to C \to A \to 0$ is a nontrivial extension, as C is not biholomorphic to $B \times A$.

In the remainder of Section F.5, unless otherwise specified, let A be a complex torus of dimension g and B be a commutative complex Lie group. Let $s_A : A \times A \to A$ be the group law of A. The dual of A is $A^\vee = \mathrm{Pic}^0(A)$.

The analogue of Proposition F.5.1.3 for abelian varieties is [Ros58, Prop. 9].

Proposition F.5.1.3. *The morphism (F.11) is injective.*

Proof. Let $C \in \ker(\pi)$. The principal bundle $C \to A$ is trivial, so there is a morphism $s : A \to C$ of complex manifolds with $ps = \mathrm{Id}_A$. Then there exists a unique $b \in B$ with $b \cdot s(e_A) = e_C$, where dot signifies the action of B on the fiber $p^{-1}(e_A)$. Define

$$s' : A \to C, \quad s(a) = b \cdot s(a).$$

Then s' is a complex manifold morphism with $ps' = \mathrm{Id}_A$. Replacing s by s', we may suppose that $s(e_A) = e_C$. By [NW13, Thm. 5.1.36], s is a morphism in $\mathcal{C}$. By Fact F.3.0.4, $C = 0$ in $\mathrm{Ext}(A, B)$. Therefore, π is injective. $\square$

We propose to determine the image of (F.11). Let Mfd be the category of complex manifolds. Define a functor

$$T : \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Ab}, \quad T(X) = H^1(X, \mathcal{B}_X).$$

When X is a point, $T(X) = 0$. Let $X_1, X_2 \in \mathrm{Mfd}$, and let $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection to the i-th factor. There is a morphism $p_1^* \oplus p_2^* : T(X_1) \times T(X_2) \to T(X_1 \times X_2)$.

Definition F.5.1.4. [Ser88, (29), no.14, Ch. VII] For $A \in \mathcal{C}$, an element $x \in T(A) = H^1(A, \mathcal{B}_A)$ is called primitive if $s_A^*(x) = p_1^*(x) + p_2^*(x)$ in $T(A \times A)$. Denote by $\mathrm{PT}(A)$ the subgroup of $T(A)$ formed by the primitive elements.

Fact F.5.1.5 ([Ser88, Lem. 8, p.181])**.** *The functor* $\mathrm{PT} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *is additive.*

Theorem F.5.1.6 is an analytic analog of [Ser88, Thm. 5, p.181].

Theorem F.5.1.6. *Assume that B_0 is linear. Then the image of the morphism (F.11) is the set of primitive elements of $H^1(A, \mathcal{B}_A)$.*

Proof. Take $C \in \operatorname{Ext}(A, B)$ and put $x = \pi(C)$. By Facts F.4.1.8 and F.4.3.2,

$$s_A^*(x) = s_A^*\pi(C) = \pi s_A^*(C) = \pi(p_1^*C + p_2^*C) = p_1^*\pi(C) + p_2^*\pi(C) = p_1^*x + p_2^*x,$$

so x is primitive.

Conversely, let $x \in H^1(A, \mathcal{B}_A)$ be a primitive element and let $p : C \to A$ be the corresponding principal B-bundle. We show that there exists a structure of commutative complex Lie group on C which makes it an extension of A by B.

By Corollary F.4.1.13, every morphism of complex manifolds $A \to B$ is constant. Let $C' \to A \times A$ be the pull-back of $C \to A$ along $s_A : A \times A \to A$. As x is primitive, $C' = p_1^*C + p_2^*C$ in $T(A \times A)$. Choose a surjection $p_1^*C \times_{A\times A} p_2^*C \to C'$ satisfying (F.2). Since $p_1^*C = C \times A$ and $p_2^*C = A \times C$, as a complex manifold $p_1^*C \times_{A\times A} p_2^*C$ is isomorphic to $C \times C$. Hence a morphism $g : C \times C \to C$ of complex manifolds:

$$\begin{tikzcd} C \times C = p_1^*C \times_{A\times A} p_2^*C \arrow[r] \arrow[rr, bend left, "g"] & p_1^*C + p_2^*C = C' \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ & A \times A \arrow[r, "s_A"] & A. \end{tikzcd} \tag{F.15}$$

By construction, it satisfies

$$g(b \cdot c, b' \cdot c') = (b + b') \cdot g(c, c') \tag{F.16}$$

for every $c, c' \in C$ and $b, b' \in B$.

Choose a point $e \in p^{-1}(e_A)$. Since $p(g(e, e)) = s_A(e_A, e_A) = e_A$, there exists a unique $b \in B$ with $b \cdot g(e, e) = e$. Replacing e by $b \cdot e$, we can suppose that

$$g(e, e) = e. \tag{F.17}$$

We verify that (C, e, g) is a group.

Identity According to (F.15), there is a morphism $h : C \to B$ of complex manifolds with $g(c, e) = h(c) \cdot c$ for all $c \in C$. By (F.17), $h(e) = e_B$. Furthermore, (F.16) shows that $h(b \cdot c) = h(c)$ for all $b \in B$. Therefore, h factors as $C \xrightarrow{p} A \xrightarrow{\bar{h}} B$. The morphism $\bar{h}$ of complex manifolds is constant, so $g(c, e) = c$ for all $c \in C$. The formula $g(e, c) = c$ is proved similarly.

Associativity According to (F.15), there is a complex manifold morphism $u : C \times C \times C \to B$ with

$$g(c, g(c', c'')) = u(c, c', c'') \cdot g(g(c, c'), c'')$$

for all $c, c', c'' \in C$. Then $u(e, e, e) = e_B$. Equation (F.16) shows that u factors through a morphism $\bar{u} : A \times A \times A \to B$ of complex manifolds. Then $\bar{u}$ is of constant value e_B. Therefore, $g(c, g(c', c'')) = g(g(c, c'), c'')$ for all $c, c', c'' \in C$.

Inverse Denote by $i_A : A \to A$ (resp. $i_B : B \to B$) the inverse map of A (resp. B). Let $C^- \to A$ be the principal B-bundle corresponding to $-x \in H^1(A, \mathcal{B}_A)$. There is a morphism $f : C \to C^-$ of principal B-bundles over A, such that for every $b \in B$, $c \in C$, $f(b \cdot c) = (-b) \cdot c$. Since $0_A = i_A + \mathrm{Id}_A$, by Fact F.5.1.5, $0 = 0_A^* x = i_A^* x + x$, hence $i_A^* x = -x$. In other words, the pullback of $p : C \to A$ along i_A is $C^- \to A$.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[rr, bend left, "i"] & i_A^* C \arrow[r] \arrow[d] & C \arrow[d, "p"] \\
& A \arrow[r, "i_A"] & A
\end{tikzcd}$$

The induced morphism $i : C \to C$ of complex manifolds is such that for every $c \in C$, $b \in B$,

$$i(b \cdot c) = (-b) \cdot i(c). \tag{F.18}$$

Since $i(e) \in p^{-1}(e_A)$, there is $b \in B$ with $b \cdot i(e) = e$. Define $i' : C \to C$ by $i'(x) = b \cdot i(x)$ and replace i by i'. Then we may further assume that $i(e) = e$. Because

$$p(g(c, i(c))) = s_A(p(c), pi(c)) = s_A(p(c), i_A(p(c))) = e_A,$$

there exists a morphism $v : C \to B$ of complex manifolds such that $g(c, i(c)) = v(c) \cdot e$ and $v(e) = e_B$. By (F.16) and (F.18), v factors through $\bar{v} : A \to B$, which is of constant value e_B. Therefore, $g(c, i(c)) = e$ for all $c \in C$.

In conclusion, (C, e, g, i) is a complex Lie group and (F.15) shows that $p : C \to A$ is a morphism. Define an injective map $\iota : B \to C$ by $b \mapsto b \cdot e$. By (F.16), then ι is a morphism. Since $\iota(B) = p^{-1}(e)$, the sequence

$$0 \to B \xrightarrow{\iota} C \xrightarrow{p} A \to 0$$

is exact. By Proposition F.6.0.2 2 below, C is commutative and hence $C \in \mathrm{Ext}(A, B)$. (The commutativity of C can also be proved using an argument of similar type.) Therefore, $x = \pi(C)$ is in the image of π. $\square$

F.5.2 The case $B = \mathbb{C}^*$

We review some basics about (holomorphic) line bundles on complex tori.

Definition F.5.2.1. [Wei48, Ch.VIII, n.58] Let $L \to A$ be a line bundle on a complex torus. If for every $a \in A$, the pullback line bundle $T_a^* L$ is isomorphic to L, then we write $L \equiv O_A$. Here $T_a : A \to A$ is defined by $T_a(x) = x + a$.

By [BL04, p.36], L induces a morphism

$$\phi_L : A \to A^\vee, \quad a \mapsto T_a^* L \otimes L^{-1}.$$

Then $L \equiv O_A$ is equivalent to $\phi_L = 0$. Then [BL04, Prop. 2.5.3] becomes Fact F.5.2.2.

Fact F.5.2.2. *Let $L \to A$ be a line bundle on a complex torus. The following conditions are equivalent:*

1. *L is analytically equivalent to O_A;*
2. *$L \in \mathrm{Pic}^0(A)$;*
3. *$L \equiv O_A$.*

Proposition F.5.2.3. *Let $L \to A$ be a line bundle on complex torus. Then $L \equiv O_A$ if and only if $s_A^* L = p_1^* L \otimes p_2^* L$.*

Proof. If $s_A^* L = p_1^* L \otimes p_2^* L$, then for every $a \in A$, the line bundle $T_a^* L = (s_A^* L)|_{A\times a} = (p_1^* L \otimes p_2^* L)|_{A \times a} = L$, *i.e.*, $L \equiv O_A$.

Conversely, if $L \equiv O_A$, then for every $a \in A$, $(s_A^* L)|_{A\times a} = T_a^* L = L = (p_1^* L)|_{A\times a}$. Therefore, $s^* L \otimes p_1^* L^{-1} \to A \times A$ is a line bundle, whose restriction to $A \times a$ is trivial for all $a \in A$. By seesaw theorem [BL04, A.8], there is a line bundle $M \to A$ such that $s^* L \otimes p_1^* L^{-1} = p_2^* M$. Then $s^* L = p_1^* L \otimes p_2^* M$. Hence, $L = s^* L|_{0\times A} = (p_1^* L \otimes p_2^* M)|_{0 \times A} = M$. Therefore, $s^* L = p_1^* L \otimes p_2^* L$. □

Theorem F.5.2.4 is mentioned without proof in [KKN08, Sec. 1.2]. The analogue for abelian varieties is in [Wei49, no. 2].

Theorem F.5.2.4 (Weil)**.** *If A is a complex torus, then $\pi : \mathrm{Ext}(A, \mathbb{C}^*) \to \mathrm{Pic}^0(A)$ is an isomorphism.*

Proof. For $B = \mathbb{C}^*$, the sheaf $\mathcal{B}_A = O_A^*$ and $H^1(A, \mathcal{B}_A) = \mathrm{Pic}(A)$. The class of a line bundle $L \to A$ is primitive means the line bundle $s_A^* L$ is isomorphic to $p_1^* L \otimes p_2^* L$ on $A \times A$. By Proposition F.5.2.3 and Fact F.5.2.2, it is equivalent to $[L] \in \mathrm{Pic}^0(A)$. Then Proposition F.5.1.3 and Theorem F.5.1.6 complete the proof. □

With the identifications provided by Theorem F.5.2.4 and Proposition F.4.1.4 3, [AK01, Remark 1.1.16] can be rephrased in a coordinate-free way as follows. It is a criterion telling whether a semi-torus is a toroidal group.

Fact F.5.2.5. *Let $r \geq 1$ be an integer, and let $0 \to (\mathbb{C}^*)^r \to X \to A \to 0$ be an extension in $\mathcal{C}$. Denote by $(L_1, \dots, L_r) \in (A^\vee)^r$ the point corresponding to the equivalent class $[X] \in \operatorname{Ext}(A, (\mathbb{C}^*)^r)$. Then the following are equivalent:*

- *X is a toroidal group;*
- *for all $\sigma \in \mathbb{Z}^r \setminus \{0\}$, $\sum_{i=1}^r \sigma_i L_i \neq 0$ in $A^\vee$;*
- *for every nontrivial morphism $f : (\mathbb{C}^*)^r \to \mathbb{C}^*$, the pushout extension $f_* X$ of A by $\mathbb{C}^*$ is nontrivial.*

F.5.3 The case $B = \mathbb{C}$

When $B = \mathbb{C}$, the sheaf $\mathcal{B}_A = O_A$. Fact F.5.3.1 can be found in, *e.g.*, [Men20, (3.1)].

Fact F.5.3.1 (Künneth formula)**.** *Let X, Y be connected complex manifolds. Assume that Y is compact. Then there is a canonical decomposition $H^1(X \times Y, O_{X \times Y}) = H^1(X, O_X) \oplus H^1(Y, O_Y)$.*

The analogue of Theorem F.5.3.2 for abelian varieties is [Ros58, Theorem 1].

Theorem F.5.3.2 (Rosenlicht, Serre)**.** *If A is a complex torus, then the canonical morphism $\pi : \operatorname{Ext}(A, \mathbb{C}) \to H^1(A, O_A)$ is a $\mathbb{C}$-linear isomorphism. In particular, $\dim_{\mathbb{C}} \operatorname{Ext}(A, \mathbb{C}) = \dim A$.*

Proof. Let m_1 (resp. m_2) be the injection $A \to A \times A$ defined by $a \mapsto (a, 0)$ (resp. $a \mapsto (0, a)$). Let $p_i : A \times A \to A$ $(u = 1, 2)$ be the two projections. By Fact F.5.3.1, p_1^* and p_2^* identify $T(A \times A)$ as the direct sum $T(A) \oplus T(A)$. The projection to ith factor is m_i^*. Because $s_A \circ m_i = \mathrm{Id}_A$, one has $s_A^*(x) = p_1^* x + p_2^* x$ for every $x \in T(A)$, *i.e.*, x is primitive. Then Proposition F.5.1.3 and Theorem F.5.1.6 conclude the proof. □

Remark F.5.3.3. Another way to prove Theorem F.5.3.2 is to use (F.13). In this case, the diagram (F.12) can be completed into a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(\tilde{A}, \mathbb{C}) \arrow[r] \arrow[d] & \operatorname{Hom}(\pi_1(A), \mathbb{C}) \arrow[r] \arrow[d] & \operatorname{Ext}(A, \mathbb{C}) \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & H^0(A, \Omega_A^1) \arrow[r] & H^1(A, \mathbb{C}) \arrow[r] & H^1(A, O_A) \arrow[r] & 0. \end{tikzcd} \tag{F.19}$$

The bottom row comes from the Hodge structure on $H^1(A, \mathbb{C})$ ([Huy05, Lem. 3.3.1]).

Corollary F.5.3.4. *Let A be a complex abelian variety, and let $n(\geq 0)$ be an integer. Then the natural morphism* $\mathrm{Ext}_{\mathrm{Alg}}(A, \mathbb{G}_a^n) \to \mathrm{Ext}(A^{\mathrm{an}}, \mathbb{C}^n)$ *is an isomorphism.*

Proof. It is a combination of [Ser88, Thm. 7, p.185], Theorem F.5.3.2 and [Ser56, Thm. 1]. □

F.5.4 Universal vectorial extension

Definition F.5.4.1. [Ros58, p.705] Let H be a vector group. An extension

$$0 \to H \to G \to A \to 0 \tag{F.20}$$

in $\mathcal{C}$ is called decomposable if there exists an extension

$$0 \to H_1 \to G_1 \to A \to 0$$

in $\mathcal{C}$ of A by a vector subgroup H_1 of H, and H' is a vector subgroup of H of positive dimension with an isomorphism $f : G_1 \oplus H' \to G$ such that the maps $H_1 \to H \to G$ and $H_1 \to G_1 \stackrel{f|_{G_1}}{\to} G$ coincide. Otherwise, the extension G is called indecomposable.

Proposition F.5.4.2. *The extension (F.20) is decomposable if and only if there is a strict vector subgroup H_1 of H and an extension $0 \to H_1 \to G_1 \stackrel{p_1}{\to} A \to 0$ with $\iota_* G_1 = G$, where $\iota : H_1 \to H$ is the inclusion.*

Proof. If G is decomposable, by definition, we can write $G = G_1 \oplus H'$, where $H' \subset H$ is a positive-dimensional vector subgroup and $0 \to H_1 \to G_1 \to A \to 0$ is an extension in $\mathcal{C}$ of A by a vector subgroup $H_1 \subset H$ making a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

By the universal property (F.7), $G = \iota_* G_1$. Moreover,

$$\dim H_1 = \dim G_1 - \dim A = \dim G - \dim H' - \dim A = \dim H - \dim H' < \dim H.$$

Conversely, assume that $\iota_* G_1 = G$. Choose a vector subspace H' of H with $H = H' \oplus H_1$, then $\dim H' = \dim H - \dim H_1 > 0$. The composed morphism $G_1 \oplus H' \stackrel{pr_1}{\to} G_1 \stackrel{p_1}{\to} A$ is surjective of kernel $H_1 \oplus H' = H$, hence a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & H & \longrightarrow & G_1 \oplus H' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

with exact rows. By the universal property (F.7), $G = \iota_* G_1 = G_1 \oplus H'$. This identification makes the maps $H_1 \to H \to G$ and $H_1 \to G_1 \to G$ coincide. Therefore, G is decomposable. $\square$

Proposition F.5.4.3. *Let $0 \to \mathbb{C}^n \to G \to A \to 0$ be an extension in $\mathcal{C}$. Let $q_i : \mathbb{C}^n \to \mathbb{C}$ be the i-th coordinate function. Then G is indecomposable if and only if the family $\{q_{i,*}G\}_{1\le i\le n}$ of vectors in* $\operatorname{Ext}(A,\mathbb{C})$ *is linearly independent.*

Proof. Assume that $\{q_{i,*}G\}$ is linearly dependent. By changing of coordinate, one may assume that $q_{n,*}G = 0$ in $\operatorname{Ext}(A,\mathbb{C})$. By Fact F.3.0.4, there is a morphism $r : q_{n,*}G \to \mathbb{C}$ with $i_n r = \mathrm{Id}$ on $q_{n,*}G$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}^n \arrow[r, "i"] \arrow[d, "q_n"] & G \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r, "i_n"'] & q_{n,*}G \arrow[r] \arrow[l, dashed, bend right, "r"'] & A \arrow[r] & 0 \end{tikzcd}$$

Then $i_n r \alpha i = \alpha i = i_n q_n$. Since i_n is injective, one has

$$r\alpha i = q_n. \tag{F.21}$$

Let $q : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection to the first $(n-1)$ coordinates. Let $\beta : G \to q_* G$ be the canonical morphism. Define a morphism

$$\epsilon : G \to q_*G \oplus \mathbb{C}, \quad g \mapsto (\beta(g), r\alpha(g)).$$

Then the right square of the following diagram is commutative.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}^n \arrow[r, "i"] \arrow[d, "q\oplus q_n"] & G \arrow[r] \arrow[d, "\epsilon"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C}^{n-1} \oplus \mathbb{C} \arrow[r] & q_*G \oplus \mathbb{C} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

By (F.21), the left square of the above diagram is commutative. Therefore, ϵ is an equivalence of extensions and $G = q_*G \oplus \mathbb{C}$ is decomposable.

Conversely, assume that G is decomposable. By Proposition F.5.4.2, there is a vector subgroup $\iota : H_1 \to \mathbb{C}^n$ with $\dim H_1 < n$ and an extension $0 \to H_1 \to G_1 \to A \to 0$ with $\iota_* G_1 = G$. There is a linear combination $f = \sum_{i=1}^n a_i q_i : \mathbb{C}^n \to \mathbb{C}$, where $a_1, \dots, a_n \in \mathbb{C}$ are not all zero, such that $f\iota = 0$. Then $\sum_{i=1}^m a_i q_{i,*} G = f_* G = (f\iota)_* G_1 = 0$. Thus, the family $\{q_{i,*}G\}_i$ is linearly dependent. $\square$

Corollary F.5.4.4 follows from Proposition F.5.4.3 and Theorem F.5.3.2.

Corollary F.5.4.4. *Let $0 \to V \to G \to A \to 0$ be an extension in $\mathcal{C}$ by a vector group V. If $\dim_{\mathbb{C}} V > g$, then G is decomposable.*

Proposition F.5.4.5 is an analytic analogue of [Ros58, Prop. 11].

Proposition F.5.4.5.

1. *There is a* $\mathbb{C}$*-vector group* H *with* $\dim_{\mathbb{C}} H = g$ *and an indecomposable extension*
$$0 \to H \to G \to A \to 0 \tag{F.22}$$
such that for every $V \in \mathrm{Vec}$*, the map*
$$\phi_V : \mathrm{Hom}_{\mathrm{Vec}}(H, V) \to \mathrm{Ext}(A, V), \quad l \mapsto l_* G \tag{F.23}$$
is a linear isomorphism. In other words, H *together with the extension (F.22) represents the functor* $\mathrm{Ext}(A, \bullet) : \mathrm{Vec} \to \mathrm{Vec}$*.*

2. *A* $G' \in \mathrm{Ext}(A, V)$ *is indecomposable if and only if the corresponding linear map* $\phi_V^{-1}(G') : H \to V$ *is surjective.*

Proof.

1. By Theorem F.5.3.2, $\dim_{\mathbb{C}} \mathrm{Ext}(A, \mathbb{C}) = g$. Take a $\mathbb{C}$-basis $\{G_1, \ldots, G_g\}$ of $\mathrm{Ext}(A, \mathbb{C})$. By Fact F.4.1.8, $\mathrm{Ext}(A, \mathbb{C}^g) = \oplus_{i=1}^g \mathrm{Ext}(A, \mathbb{C})$, so there is an element $G \in \mathrm{Ext}(A, \mathbb{C}^g)$ corresponding to $(G_1, \ldots, G_g) \in \oplus_{i=1}^g \mathrm{Ext}(A, \mathbb{C})$. Hence an extension $0 \to H \to G \to A \to 0$, where $H = \mathbb{C}^g$. By Proposition F.5.4.3, G is indecomposable.

   When $l \in H^\vee$ is taking the i-th coordinate of $H = \mathbb{C}^g$, $l_* G = G_i$. Therefore, the image of the linear map $\phi_{\mathbb{C}}$ contains a basis of $\mathrm{Ext}(A, \mathbb{C})$. Thus, $\phi_{\mathbb{C}}$ is surjective. Since $\dim_{\mathbb{C}} H^\vee = \dim_{\mathbb{C}} \mathrm{Ext}(A, \mathbb{C})$, $\phi_{\mathbb{C}}$ is a linear isomorphism. Since every $V \in \mathrm{Vec}$ is the direct sum of finitely many copies of $\mathbb{C}$ and the formation of ϕ_V is functorial in V, ϕ_V is also a linear isomorphism.

2. By Proposition F.5.4.2, G' is decomposable iff there is a proper linear subspace $\iota : V_1 \to V$ with G' in the image of the map $\iota_* : \mathrm{Ext}(A, V_1) \to \mathrm{Ext}(A, V)$ iff there is a proper linear subspace $\iota : V_1 \to V$ with $\phi_V^{-1}(G')$ in the image of the map $\iota_* : \mathrm{Hom}_{\mathrm{Vec}}(H, V_1) \to \mathrm{Hom}_{\mathrm{Vec}}(H, V)$ iff $\phi_V^{-1}(G') : H \to V$ factors through a proper linear subspace $\iota : V_1 \to V$ iff $\phi_V^{-1}(G') : H \to V$ is not surjective.

□

The extension (F.22) is called the *universal vectorial extension* of A. (As a representing object, such an extension is unique up to equivalence.) By (F.23) and Theorem F.5.3.2, $H = H^0(A^\vee, \Omega^1_{A^\vee})$.

Example F.5.1.2 (continued). Since $\dim \mathrm{Ext}(A, C) = 1$, this nontrivial extension is equivalent to the universal vectorial extension.

We proceed to give an explicit construction of the universal vectorial extension.

Proposition F.5.4.6. *Let $B^{\natural 1}$ be the group of isomorphic classes of rank* 1 *local systems on A. Let $B^\natural$ be the group of isomorphic classes of pairs (L, ∇), where $L \to A$ is a holomorphic line bundle and ∇ is a flat holomorphic connection on L. Then there exist natural identifications of groups*

$$B^\natural = B^{\natural 1} = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*) = H^1(A, \mathbb{C}^*) = \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})}.$$

They are isomorphic to $(\mathbb{C}^*)^{2g}$.

Proof. By the Riemann-Hilbert correspondence [Del70, Théorème 2.17, p.12], the map $B^\natural \to B^{\natural 1}$ defined by $(L, \nabla) \mapsto \ker(\nabla)$ is a group isomorphism. By [Del70, Corollaire 1.4, p.4], there is an isomorphism $B^{\natural 1} \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*)$. By the universal coefficient theorem [Hat05, Thm. 3.2], there is a natural isomorphism $H^1(A, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(A), \mathbb{C}^*)$. The exact sequences $0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp(2\pi i \bullet)}{\to} \mathbb{C}^* \to 0$ of constant sheaves on A gives rise to an exact sequence

$$H^0(A, \mathbb{C}) \to H^0(A, \mathbb{C}^*) \to H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}) \to H^1(A, \mathbb{C}^*) \to H^2(A, \mathbb{Z}) \to H^2(A, \mathbb{C}).$$

Since the first map is surjective and the last map is injective, it breaks into a short exact sequence

$$0 \to H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}) \to H^1(A, \mathbb{C}^*) \to 0$$

and hence an isomorphism $H^1(A, \mathbb{C})/H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}^*)$ functorial in A. Moreover, there is a non-canonical isomorphism $H^1(A, \mathbb{C}^*) \to (\mathbb{C}^*)^{2g}$. □

By [Dem12, Ch. V, §9], every $(L, \nabla) \in B^\natural$ has $[L] \in \mathrm{Pic}^0(A) = A^\vee$. The bottom row of (F.19) induces an exact sequence in $\mathcal{C}$:

$$0 \to H^0(A, \Omega^1_A) \to \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})} \to \frac{H^1(A, O_A)}{H^1(A, \mathbb{Z})} \to 0. \qquad \text{(F.24)}$$

Using the identifications $B^\natural \cong \frac{H^1(A,\mathbb{C})}{H^1(A,\mathbb{Z})}$ from Proposition F.5.4.6 and $A^\vee = \mathrm{Pic}^0(A) = H^1(A, O_A)/H^1(A, \mathbb{Z})$, (F.24) is an extension of $A^\vee$ by $H^0(A, \Omega^1_A)$ and gives a morphism $B^\natural \to \mathrm{Pic}^0(A)$, which sends (L, ∇) to L. Hence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\tilde{A}, \mathbb{C}^*) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*) \arrow[r] & \mathrm{Ext}(A, \mathbb{C}^*) \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & H^0(A, \Omega^1_A) \arrow[r] & B^\natural \arrow[r, "u"] \arrow[u] & \mathrm{Pic}^0(A) \arrow[r] & 0,
\end{tikzcd}$$

where the first exact row is (F.9) and the second comes from (F.24). The left vertical isomorphism uses Proposition F.4.1.4 2 and the isomorphism

$L(A)^\vee \to H^0(A, \Omega^1_A)$ given by [BL04, Thm. 1.4.1 b)]. The middle vertical isomorphism is contained in Proposition F.5.4.6.

When A is an abelian variety, it is proved in [Mes73, p.260] that (F.24) is the universal vectorial extension of $A^\vee$. The proof is based on [Ros58, Thm. 1]. In a similar manner, Proposition F.5.4.7 follows from Theorem F.5.3.2.

Proposition F.5.4.7. *The extension (F.24) is the universal vectorial extension of $A^\vee = \mathrm{Pic}^0(A)$. In particular, the extension group is isomorphic to $(\mathbb{C}^*)^{2g}$ (as a complex Lie group).*

Proof. Let $U = H^0(A, \Omega^1_A)$. Pushing out the extension (F.24) defines a natural transformation $\psi : \mathrm{Hom}_{\mathrm{Vec}}(U, \bullet) \to \mathrm{Ext}(A^\vee, \bullet)$ between two functors on Vec.

We claim that $\psi_{\mathbb{C}}$ is an isomorphism. Choose $u \in \ker(\psi_{\mathbb{C}}) \subset \mathrm{Hom}_{\mathrm{Vec}}(U, \mathbb{C})$. As the push-out along u is trivial, by Fact F.3.0.4, there is a morphism $r : E \to \mathbb{C}$ with $ir = \mathrm{Id}_E$. Let $u' : H^1(A, \mathbb{C}) \to \mathbb{C}$ be the morphism in $\mathcal{C}$ induced by r. Then $u' = d_e u'$ is $\mathbb{C}$-linear. Now that $u'(H^1(A, \mathbb{Z})) = 0$ and $H^1(A, \mathbb{Z})$ contains a $\mathbb{C}$-basis of $H^1(A, \mathbb{C})$, one has $u' = 0$. As the diagram commutes, $u = 0$.

$$\begin{tikzcd}
 & & H^1(A,\mathbb{C}) \arrow[d] \arrow[ddl, dashed, "u'"'] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d, "u"'] & \frac{H^1(A,\mathbb{C})}{H^1(A,\mathbb{Z})} \arrow[r] \arrow[d] & A^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r, "i"'] & E \arrow[r] \arrow[l, dashed, bend right, "r"'] & A^\vee \arrow[r] & 0
\end{tikzcd}$$

Therefore, $\psi_{\mathbb{C}}$ is injective. By Theorem F.5.3.2, $\dim_{\mathbb{C}} \mathrm{Ext}(A^\vee, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Vec}}(U, \mathbb{C})$. Therefore, $\psi_{\mathbb{C}}$ is a linear isomorphism. Similar to the proof of Proposition F.5.4.5 1, ψ is a natural isomorphism of the two functors. □

Another construction of the universal vectorial extension is in [Nak94, Prop. 2.4][7].

Remark F.5.4.8. The real Lie group extension underlying (F.24) is trivial by Fact F.5.1.1. Indeed, consider the real analytic group morphism $A^\vee \to B^\natural$ defined by $L \mapsto (L, \nabla^L)$, where ∇^L is the unique flat Chern connection on L given by Lemma D.2.0.4 1. This map is a real Lie group section to (F.24), but not holomorphic.

Remark F.5.4.9. Let A be a complex abelian variety of dimension g. By Corollary F.5.3.4, the extension (F.22) is equivalent to an algebraic one. Thus, the analytification of the algebraic universal vectorial extension $0 \to$

[7] stated for complex abelian varieties but the proof extends to complex tori.

$\mathbb{G}_a^g \to E \to A \to 0$ is exactly the analytic universal vectorial extension . From [Bri09, Prop. 2.3 (i)] and the footnote in [MRM74, p.34], the algebraic variety E is anti-affine, *i.e.*, every morphism $E \to A^1_{\mathbb{C}}$ of algebraic varieties is constant. On the other hand, by Proposition F.5.4.7, E^{an} is isomorphic to $(\mathbb{C}^*)^{2g}$ as a complex Lie group, so E^{an} is not a toroidal group. Although E is not an affine variety, E^{an} is a Stein manifold. See also Serre's example [Har70, Exampe 3.2, p.232].

Remark F.5.4.10. Universal vectorial extensions can be defined for not only complex tori but also toroidal groups. Consider a toroidal group X of dimension n. Similar to Proposition F.5.4.5, the functor $\mathrm{Ext}(X, \cdot) : \mathrm{Vec} \to \mathrm{Vec}$ is represented by $\mathrm{Ext}(X, \mathbb{C})^\vee$, which is the kernel of the natural linear map $H_1(X, \mathbb{C}) \to H^0(X, \Omega^1_X)^\vee$ by (F.13).

An extrinsic description is possible. Choose a presentation

$$0 \to (\mathbb{C}^*)^{n-q} \to X \to T \to 0 \tag{F.25}$$

according to [AK01, 1.1.14], where T is a complex torus of dimension q. For every $V \in \mathrm{Vec}$, by Proposition F.4.2.1, the induced sequence

$$\mathrm{Hom}_{\mathcal{C}}((\mathbb{C}^*)^{n-q}, V) \to \mathrm{Ext}(T, V) \to \mathrm{Ext}(X, V) \to \mathrm{Ext}((\mathbb{C}^*)^{n-q}, V)$$

is exact in Vec. By Proposition F.4.1.4 1, $\mathrm{Hom}_{\mathcal{C}}((\mathbb{C}^*)^{n-q}, V) = 0$. By Proposition F.3.2.3, $\mathrm{Ext}((\mathbb{C}^*)^{n-q}, V) = 0$. Thus, the morphism $\mathrm{Ext}(T, V) \to \mathrm{Ext}(X, V)$ is a $\mathbb{C}$-linear isomorphism. In other words, the natural transformation $\mathrm{Ext}(T, \cdot) \to \mathrm{Ext}(X, \cdot)$ between the two functors on Vec is an isomorphism. In this way, the case of toroidal groups is reduced to the case of complex tori.

F.5.5 Application to the functor $\mathrm{Ext}(A, \bullet)$

Analogue of Proposition F.5.5.1 for abelian varieties is [Ros58, Cor., p.711].

Proposition F.5.5.1. *If B is a complex Lie subgroup (not necessarily connected) of A, then there is a natural exact sequence in* Ab*:*

$$0 \to \mathrm{Ext}(A/B, \mathbb{C}) \to \mathrm{Ext}(A, \mathbb{C}) \to \mathrm{Ext}(B, \mathbb{C}) \to 0.$$

Proof. By Corollary F.4.1.13, there is an isomorphism $B \to B_0 \times B/B_0$ in $\mathcal{C}$ and $\mathrm{Ext}(B/B_0, \mathbb{C}) = 0$. By Fact F.4.1.8, $\mathrm{Ext}(B, \mathbb{C}) = \mathrm{Ext}(B_0, \mathbb{C})$. Since B is compact and B_0 is open in B, the quotient B/B_0 is finite, thus $\mathrm{Hom}_{\mathrm{Ab}}(B/B_0, \mathbb{C}) = 0$. By the compactness of B_0, $\mathrm{Hom}_{\mathcal{C}}(B_0, \mathbb{C}) = 0$. Then $\mathrm{Hom}(B, \mathbb{C}) = 0$. Now that $A, B_0, A/B$ are complex tori, Theorem F.5.3.2 implies $\dim \mathrm{Ext}(A, \mathbb{C}) = \dim \mathrm{Ext}(A/B, \mathbb{C}) + \dim \mathrm{Ext}(B, \mathbb{C})$. This together with Proposition F.4.2.1 proves the stated exactness. $\square$

The proof of Theorem F.5.5.2 is shorter than that of its algebraic analogue [Ser88, Thm. 12, p.195].

Theorem F.5.5.2. *If* $0 \to B' \to B \xrightarrow{\phi} B'' \to 0$ *is an exact sequence in* $\mathcal{C}$, *then the sequence*[8] *in* Ab

$$\operatorname{Ext}(A, B') \to \operatorname{Ext}(A, B) \xrightarrow{\phi_*} \operatorname{Ext}(A, B'') \to 0 \tag{F.26}$$

is exact. If B_0'' *is linear, then the first map in (F.26) is injective.*

Proof. By Proposition F.4.2.3, it suffices to prove that $\phi_* : \operatorname{Ext}(A, B) \to \operatorname{Ext}(A, B'')$ is surjective. From (F.10) and Proposition F.4.2.1, one obtains a commutative square

$$\begin{tikzcd}
\operatorname{Hom}(\pi_1(A), B) \arrow[r] \arrow[d] & \operatorname{Hom}(\pi_1(A), B'') \arrow[d] \\
\operatorname{Ext}(A, B) \arrow[r, "\phi_*"] & \operatorname{Ext}(A, B''),
\end{tikzcd}$$

where the vertical maps are surjective. Since $\pi_1(A)$ is a free $\mathbb{Z}$-module, the top row is surjective, then so is the bottom.

Now assume that B_0'' is linear, then $\operatorname{Hom}_{\mathcal{C}}(A, B'') = 0$. By Proposition F.4.2.3, the first map is injective. □

Remark F.5.5.3. The linearity of B_0'' is necessary to guarantee the injectivity in Theorem F.5.5.2. For instance, let $0 \to \mathbb{C}^g \to (\mathbb{C}^*)^{2g} \to A \to 0$ be the universal vectorial extension of A and assume $g \geq 1$. By Proposition F.4.2.3, the natural sequence $0 \to \operatorname{Hom}_{\mathcal{C}}(A, A) \to \operatorname{Ext}(A, \mathbb{C}^g) \to \operatorname{Ext}(A, (\mathbb{C}^*)^{2g})$ is exact. Thus, Id_A is a nonzero element in the kernel of the first map of (F.26).

Example F.5.5.4. Applying Theorem F.5.5.2 to the exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp(2\pi i \bullet)}{\to} \mathbb{C}^* \to 1$, and using Fact F.3.2.6, Theorems F.5.2.4 and F.5.3.2, one gets an exact sequence

$$0 \to \operatorname{Hom}(\pi_1(A), \mathbb{Z}) \to H^1(A, O_A) \to \operatorname{Pic}^0(A) \to 0. \tag{F.27}$$

In particular, $\operatorname{Ext}(A, \cdot)$ tuns the exponential map to the universal cover of the complex torus $A^\vee$. Identifying $\operatorname{Hom}(\pi_1(A), \mathbb{Z})$ with the sheaf cohomology $H^1(A, \mathbb{Z})$, th sequence (F.27) is also induced by the exponential sequence of sheaves on A:

$$0 \to \mathbb{Z}_A \to O_A \overset{\exp(2\pi i)}{\to} O_A^* \to 1.$$

Theorem F.5.5.5 is an analytic version of [Ser88, Thm. 13, p.196]

Theorem F.5.5.5. *If* $0 \to L \xrightarrow{i} C \to A \to 0$ *is an exact sequence in* $\mathcal{C}$ *with* L *connected and* $G \in \mathrm{Ab}_c$. *Then there is a natural exact sequence*

$$0 \to \operatorname{Ext}(A, G) \to \operatorname{Ext}(C, G) \xrightarrow{i^*} \operatorname{Ext}(L, G) \to 0.$$

[8] induced by Proposition F.4.2.3

Proof. As L is connected and G is discrete, $\mathrm{Hom}_{\mathcal{C}}(L, G) = 0$. By Proposition F.4.2.1, it suffices to show that $i^* : \mathrm{Ext}(C, G) \to \mathrm{Ext}(L, G)$ is surjective. For every $L' \in \mathrm{Ext}(L, G)$, by Theorem F.5.5.2, the map $\mathrm{Ext}(A, L') \to \mathrm{Ext}(A, L)$ is surjective. Thus, there exists $C' \in \mathrm{Ext}(A, L')$ having image $C \in \mathrm{Ext}(A, L)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G \arrow[r, "\beta"] \arrow[d] & \ker(\alpha) \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & C \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

By the snake lemma, α is surjective and β is an isomorphism. Therefore, $C' \in \mathrm{Ext}(C, G)$ and $i^*C' = L'$ in $\mathrm{Ext}(L, G)$. $\square$

In Example F.5.5.6, we give another proof of [BL99, Prop. 5.7, p.21], which computes the extension group of two complex tori.

Example F.5.5.6. Let $X_i = \mathbb{C}^{g_i}/\Pi_i\mathbb{Z}^{2g_i}$ ($i = 1, 2$) be two complex tori, where the chosen period matrix is of the form $\Pi_i = (\tau_i, I_{g_i})$ with $\tau_i \in M_{g_i}(\mathbb{C})$ and $\det(\mathrm{Im}(\tau_i)) \neq 0$. Define $\xi : M(2g_1 \times 2g_2, \mathbb{Z}) \to M(g_1 \times g_2, \mathbb{C})$ by $\xi(P) = \Pi_1 P\binom{I_{g_2}}{\tau_2}$.

Define a map $\rho : M(g_1 \times g_2, \mathbb{C}) \to \mathrm{Ext}(X_2, \tilde{X}_1)$ as follows. For every $\alpha \in M(g_1 \times g_2, \mathbb{C})$, let $\alpha' = (\alpha, 0) \in M(g_1 \times 2g_2, \mathbb{C})$. Consider the sequence

$$0 \to \mathbb{C}^{g_1} \xrightarrow{i} \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \xrightarrow{p} X_2 \to 0,$$

where i is induced by $\mathbb{C}^{g_1} \to \mathbb{C}^{g_1+g_2}$ defined by $x \mapsto (x, 0)$ and p is induced by the second projection $\mathbb{C}^{g_1+g_2} \to \mathbb{C}^{g_2}$. It is an exact sequence. Denote its class by $\rho(M) \in \mathrm{Ext}(X_2, \tilde{X}_1)$. This sequence fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{X}_1 \arrow[r] \arrow[d] & \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] & X_2 \arrow[r] & 0,
\end{tikzcd}$$

where the second row is $\psi_{\Pi_1,\Pi_2}(\alpha') \in \mathrm{Ext}(X_2, X_1)$ defined in [BL99, p.20], and

$$X = \frac{\mathbb{C}^{g_1+g_2}}{\{(\Pi_1 u + \alpha' v, \Pi_2 v) : u \in \mathbb{Z}^{2g_1}, v \in \mathbb{Z}^{2g_2}\}}.$$

Then ρ is a linear isomorphism by Theorem F.5.3.2.

Define a map $\phi : M(2g_1 \times 2g_2, \mathbb{Z}) \to \mathrm{Ext}(X_2, \pi_1(X_1))$ as follows. Given $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in M(2g_1 \times 2g_2, \mathbb{Z})$, with each $P_i \in M(g_1 \times g_2, \mathbb{Z})$, we set $A = \tau_1 P_2 + P_4 \in M(g_1 \times g_2, \mathbb{C})$ and $\alpha = \xi(P)$. The linear map $\mathbb{C}^{g_1+g_2} \overset{(I,-A)}{\to} \mathbb{C}^{g_1}$ sends $(u, 0)$ to u for all $u \in \mathbb{C}^{g_1}$ and sends $(\alpha' v, \Pi_2 v)$ to $\Pi_1 \begin{pmatrix} P_1 & -P_2 \\ P_3 & -P_4 \end{pmatrix} v \in \Pi_1 \mathbb{Z}^{2g_1}$ for all $v \in \mathbb{Z}^{2g_2}$. Thus it descents to the vertical morphism in the middle of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^{g_1} \arrow[r] \arrow[d] & \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \arrow[r] \arrow[d, "{(I_{g_1}, -A)}"] & X_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X_1 \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd} \tag{F.28}$$

where the first row is of class $\rho(\alpha) = \rho(\xi(P))$. The snake lemma gives an extension of X_2 by $\pi_1(X_1)$, whose class is denoted by $\phi(P)$.

The image of $\phi(P)$ under the pushout map $\mathrm{Ext}(X_2, \pi_1(X_1)) \to \mathrm{Ext}(X_2, \tilde{X}_1)$ is exactly the first row of (F.28), *i.e.*, $\rho(\xi(P))$. Then ϕ is a group isomorphism by Fact F.3.2.6. And there is a commutative diagram

$$\begin{tikzcd}
 & & & & M(g_1 \times 2g_2, \mathbb{C}) \arrow[ddl, "\psi_{\Pi_1,\Pi_2}"] \\
M(2g_1 \times 2g_2, \mathbb{Z}) \arrow[r, "\xi"] \arrow[d, "\phi"] & M(g_1 \times g_2, \mathbb{C}) \arrow[r] \arrow[d, "\rho"] \arrow[urrr, hook, "\alpha \mapsto \alpha'"] & \frac{M(g_1 \times g_2, \mathbb{C})}{\mathrm{Im}(\xi)} \arrow[d, dashed] & & \\
\mathrm{Ext}(X_2, \pi_1(X_1)) \arrow[r] & \mathrm{Ext}(X_2, \tilde{X}_1) \arrow[r] & \mathrm{Ext}(X_2, X_1) \arrow[r] & 0 &
\end{tikzcd}$$

where the second row is from (F.14) and the induced dotted isomorphism is exactly the content of [BL99, Proposition 5.7, p.21].

To conclude Section F.5.5, we show that the groups of commutative extensions of complex tori by linear groups are naturally complex Lie groups. Let $\mathcal{T}$ (resp. $\mathcal{S}$) be the full subcategory of $\mathcal{C}$ comprised of complex tori (resp. objects whose identity component is linear). Then $\mathrm{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathrm{Ab}$ is an additive functor by Fact F.4.1.8. Theorem F.5.5.7, an analytic analogue of [Wu86, Theorem 5], lifts this functor.

Theorem F.5.5.7 (Wu)**.** *There is a natural way to lift* $\mathrm{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathrm{Ab}$ *to an additive functor* $\mathrm{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{C}$*.*

Proof. First we define a complex Lie group structure on $\mathrm{Ext}(A, H)$, where $A \in \mathcal{T}$ and $H \in \mathcal{S}$. Let $g = \dim A$.

If there is an isomorphism $f : H \to (\mathbb{C}^*)^n$ in $\mathcal{S}$, then by Theorem F.5.2.4, f gives rise to an isomorphism $\mathrm{Ext}(A, H) \to (A^\vee)^n$ making $\mathrm{Ext}(A, H)$ a complex torus. The complex structure on $\mathrm{Ext}(A, H)$ is independent of the choice of the isomorphism f.

If H is connected, by Proposition F.2.0.7, there is an isomorphism $u : H \to V \times H_m$, where $V \in \mathrm{Vec}$ and H_m is a power of $\mathbb{C}^*$. Then $u_* : \mathrm{Ext}(A, H) \to \mathrm{Ext}(A, V) \times \mathrm{Ext}(A, H_m)$ is an isomorphism. By Theorem F.5.3.2, the vector space $\mathrm{Ext}(A, V)$ is *finite dimensional.* Together with last paragraph, $\mathrm{Ext}(A, H)$ inherits a complex Lie group structure, which is independent of the choice of u.

For a general object $H \in \mathcal{S}$, the natural exact sequence $0 \to H_0 \to H \to H/H_0 \to 0$ in $\mathcal{C}$ is trivial by Corollary F.4.1.13. Thus, the resulting exact sequence $0 \to \mathrm{Ext}(A, H_0) \to \mathrm{Ext}(A, H) \to \mathrm{Ext}(A, H/H_0) \to 0$ in Ab is also trivial. Now that $\mathrm{Ext}(A, H/H_0) = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), H/H_0)$ by Fact F.3.2.6, one regards it as a discrete group. From the complex structure on $\mathrm{Ext}(A, H_0)$, the group $\mathrm{Ext}(A, H)$ has a unique complex Lie group structure, such that the identity component is $\mathrm{Ext}(A, H_0)$.

It remains to show:

1. If $A \in \mathcal{T}$ is fixed, then $\mathrm{Ext}(A, \cdot)$ sends morphisms in $\mathcal{S}$ to morphisms in $\mathcal{C}$.

2. If $H \in \mathcal{S}$ is fixed, then $\mathrm{Ext}(\cdot, H)$ sends morphisms in $\mathcal{T}$ to morphisms in $\mathcal{C}$.

To show 1, let $h : H \to H'$ be a morphism in $\mathcal{S}$. By decomposing H, H' according to Corollary F.4.1.13 and Proposition F.2.0.7, one may assume that each of H and H' is either discrete, $\mathbb{C}$ or $\mathbb{C}^*$.

- If H is discrete, then so is $\mathrm{Ext}(A, H)$, hence $\mathrm{Ext}(A, h)$ is a morphism in $\mathcal{C}$.

- If $H = H' = \mathbb{C}$, by Proposition F.4.1.4 2, h is a linear map. By Corollary F.4.1.9, so is $\mathrm{Ext}(A, h)$.

- If $H = \mathbb{C}$, $H' = \mathbb{C}^*$. By Proposition F.4.1.4 2, h is the composition of a linear map $\mathbb{C} \to \mathbb{C}$ followed by the exponential map $\exp(2\pi i \cdot) : \mathbb{C} \to \mathbb{C}^*$. By Example F.5.5.4, $\mathrm{Ext}(A, h)$ is the composition of a linear map $H^1(A, O_A) \to H^1(A, O_A)$ followed by the universal cover $H^1(A, O_A) \to A^\vee$. Thus, $\mathrm{Ext}(A, h)$ is a morphism in $\mathcal{C}$.

- If H' is discrete and H is connected, then h is trivial and so is $\mathrm{Ext}(A, h)$.

- If $H = \mathbb{C}^*$ and $H' = \mathbb{C}$, then h is trivial by Proposition F.4.1.4 1 and so is $\mathrm{Ext}(A, h)$.
- If $H = H' = \mathbb{C}^*$, then h is a power map by Proposition F.4.1.4 3. Then $\mathrm{Ext}(A, h)$ is a power map of $A^\vee$, hence a morphism in $\mathcal{C}$.

This proves 1.

To show 2, let $g : A \to A'$ be a morphism in $\mathcal{T}$. By decomposing H again, we may divide the proof into three cases.

- $H = \mathbb{C}^*$. By pulling back line bundles, g induces the dual morphism $g^* : \mathrm{Pic}^0(A') \to \mathrm{Pic}^0(A)$. It is identified with $\mathrm{Ext}(g, H)$ by Fact F.4.3.2 and Theorem F.5.2.4.
- H is discrete. Then so is $\mathrm{Ext}(A', H)$ and thus $\mathrm{Ext}(g, H)$ is a morphism in $\mathcal{C}$.
- $H = \mathbb{C}$. By pulling back, g induces a $\mathbb{C}$-linear map $H^1(A', O_{A'}) \to H^1(A, O_A)$. It is identified with $\mathrm{Ext}(g, H)$ by Fact F.4.3.2 and Theorem F.5.3.2.

This proves 2. □

Remark F.5.5.8. In Theorem F.5.5.7, we cannot generalize from complex tori to toroidal groups, nor remove the linear restriction.

Let X be a toroidal group. Then $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{C}^*) = 0$, hence (F.14) specializes to

$$0 \to \mathrm{Ext}(X, \mathbb{Z}) \xrightarrow{i} \mathrm{Ext}(X, \mathbb{C}) \to \mathrm{Ext}(X, \mathbb{C}^*) \to 0. \tag{F.29}$$

Note that $\mathrm{Ext}(X, \mathbb{Z}) = H^1(X, \mathbb{Z})$ (Fact F.3.2.6), and by (F.13) the injection i is the composition of the inclusion $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})$ with the projection $H^1(X, \mathbb{C}) \to \frac{H^1(X,\mathbb{C})}{H^0(X,\Omega^1_X)}$.

When X is compact, the sequence (F.29) lifts to an exact sequence in $\mathcal{C}$ by Theorem F.5.5.7. As opposed to the compact case, when X is not compact and consider the presentation (F.25), one has $1 \le q < n$, so

$$\mathrm{rank}_Z \mathrm{Ext}(X, \mathbb{Z}) = n + q > 2q = \dim_{\mathbb{R}} \mathrm{Ext}(X, \mathbb{C}).$$

Therefore, the image of i is not closed in the vector space $\mathrm{Ext}(X, \mathbb{C})$ (a phenomenon seen in Example F.4.1.2). In particular, the sequence (F.29) has no lift to an exact sequence in $\mathcal{C}$.

Let A, B be two complex tori, $g = \dim A$, $g' = \dim B$ and reconsider (F.14):

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{j} \mathrm{Ext}(A, \pi_1(B)) \to \mathrm{Ext}(A, \tilde{B}) \to \mathrm{Ext}(A, B) \to 0.$$

Here, $\mathrm{Ext}(A, \tilde{B})$ is a $\mathbb{C}$-vector space of dimension gg' by Theorem F.5.3.2. Identifying $\mathrm{Ext}(A, \pi_1(B))$ with $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \pi_1(B))$ *via* Fact F.3.2.6, j is the map ρ_r in [BL04, p.10]. The quotient $\frac{\mathrm{Ext}(A,\pi_1(B))}{\mathrm{Hom}_{\mathcal{C}}(A,B)}$ is a free abelian group of rank $4gg' - \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, B)$. As long as $\mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, B) < 2gg'$ (say, when $A = B$ is an elliptic curve without complex multiplication, then $\mathbb{Z} = \mathrm{Hom}_{\mathcal{C}}(A, B)$), the image of the induced injection $\frac{\mathrm{Ext}(A,\pi_1(B))}{\mathrm{Hom}_{\mathcal{C}}(A,B)} \to \mathrm{Ext}(A, \tilde{B})$ is not closed. In particular, $\mathrm{Ext}(A, B)$ has no structure of complex Lie group making this sequence exact in $\mathcal{C}$.

F.6 Extensions of complex tori are often commutative

In Section F.6, we prove that under suitable hypotheses, an extension of a complex torus is commutative.

Proposition F.6.0.1. *If* $1 \to B \to C \xrightarrow{p} A \to 1$ *is a* central *extension of complex Lie groups, where A is a toroidal group, then C is commutative. Or equivalently, for every* $B \in \mathcal{C}$*, the natural injection* $\mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B, 1)$ *is an isomorphism.*

Proof. Consider the holomorphic map $A \times A \to B$ given by (F.4). By [NW13, Thm. 5.1.36], it is a group morphism, so constant. Thus, C is commutative. □

An algebraic analogue of Proposition F.6.0.2 is [Wu86, Cor. 2, p.370].

Proposition F.6.0.2. *Let* $1 \to K \to E \to A \to 1$ *be an extension of complex Lie groups, where A is a complex torus.*

1. *If* $Z(K)_0$ *is Stein, then* $Z(K) = Z(E) \cap K$.
2. *If K is commutative and* K_0 *is Stein, then E is commutative.*

Proof.

1. Since $Z(E) \cap K \subset Z(K)$, it suffices to prove that $Z(K) \subset Z(E)$. Consider the group morphism (F.5): $\theta : A \to \mathrm{Aut}(Z(K))$. For every $x \in Z(K)$, the map
$$\phi : A \to Z(K), \quad a \mapsto \theta_a(x)x^{-1}$$
is continuous. Moreover, $\phi(0) = e_K$. By the connectedness of A, $\phi(A) \subset Z(K)_0$. As $Z(K)_0$ is Stein and A is compact, $\phi(A)$ is the singleton $\{e_K\}$. Therefore, $\theta_a(x) = x$ for every $x \in Z(K)$, which proves $Z(K) \subset Z(E)$.
2. By 1, $K \subset Z(E)$. By Proposition F.6.0.1, E is commutative.

□

In Proposition F.6.0.3, when B is isomorphic to $\mathbb{C}^n$ for some integer $n \geq 0$ or to $\mathbb{C}^*$, we recover [BZ23a, Lem. 2.10].

Proposition F.6.0.3. *Let* $1 \to B \to C \xrightarrow{p} A \to 1$ *be an exact sequence of complex Lie groups, where* A *is a complex torus and* B *is commutative. If the group* B/B_0 *is torsion (i.e., every element of* B/B_0 *has finite order), then* C *is commutative.*

Proof. Let Z be the center of C. By Proposition F.6.0.1, it suffices to check $B \subset Z$.

The outer action induces a morphism $A \to \mathrm{Aut}(B_0)(\leq \mathrm{GL}(L(B)))$. It is trivial by the compactness of A, *i.e.*, $B_0 \leq Z$. By Corollary F.4.1.13, one may assume $B = B_0 \times D$, where D is a discrete subgroup of B isomorphic to B/B_0 and $D \cap B_0 = \{e_B\}$. Let $q : B \to D$ and $r : B \to B_0$ be the corresponding projections.

It remains to show that $0 \times D(\leq B)$ is contained in Z. Fix $d \in D$ and put $b = (0, d) \in B$. The map

$$\nu : C \to C, \quad c \mapsto cbc^{-1}$$

is holomorphic and $\nu(e) = b$. For every $b' \in B$, one has

$$\nu(cb') = cb'bb'^{-1}c^{-1} = cbc^{-1} = \nu(c).$$

The right multiplication action of B on the complex manifold C has quotient A by Fact F.2.0.3, so ν factors through a morphism $u : A \to B$ of complex manifolds. Then $qu : A \to D$ is continuous. Since A is connected, qu is constant. Since $qu(e_A) = d$, one gets $qu \equiv d$.

On the other hand, the map $ru : A \to B_0$ is holomorphic. By assumption, there is an integer $n \geq 1$ (depending on d) such that $d^n = e_D$ in D. Thus, $b^n = e_B$. For every $c \in C$, one has $\nu(c)^n = (cbc^{-1})^n = cb^nc^{-1} = e_B$. Therefore, $ru(A)$ is contained in the torsion subgroup $B_{0,\mathrm{tor}}$ of B_0. In view of [AK01, Prop. 1.1.2], $B_{0,\mathrm{tor}}$ is totally disconnected. Since A is connected, ru is constant.

Since $ru(e_A) = 0$, one has $ru \equiv 0$. Therefore, $u \equiv b$, *i.e.*, $b \in Z$. Therefore, $0 \times D \subset Z$ and the proof is completed. □

Corollary F.6.0.4 follows immediately from Proposition F.6.0.3.

Corollary F.6.0.4. *Given an extension*

$$0 \to (\mathbb{C}^*)^n \to G \to A \to 0 \tag{F.30}$$

of complex Lie groups, where A *is a complex tours and* $n(\geq 1)$ *is an integer, then* G *is a semi-torus.*

Corollary F.6.0.5. *In Corollary F.6.0.4, if A is algebraic, then G admits a unique structure of semiabelian variety such that (F.30) defines a commutative extension of algebraic groups.*

Proof. From Corollary F.6.0.4, (F.30) defines an element of $\mathrm{Ext}(A^{\mathrm{an}}, (\mathbb{C}^*)^n)$. By [Ser88, Thm. 6, p.184] and Theorem F.5.2.4, the natural map $\mathrm{Ext}_{\mathrm{Alg}}(A, \mathbb{G}_m^n) \to \mathrm{Ext}(A^{\mathrm{an}}, (\mathbb{C}^*)^n)$ is identified with the analytification map $[\mathrm{Pic}^0(A)]^n \to [\mathrm{Pic}^0(A^{\mathrm{an}})]^n$, hence a group isomorphism. In particular, there is a unique exact sequence $0 \to \mathbb{G}_m^n \to C \to A \to 0$ in Alg whose analytification is equivalent to (F.30). $\square$

Lemma F.6.0.6 is used in the proof of Proposition F.6.0.7.

Lemma F.6.0.6. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$.*

1. *If $X, Y \in \mathfrak{g}$ are such that $[X, [X, Y]] = 0$ and $[Y, [X, Y]] = 0$, then $\exp(X)\exp(Y)\exp(-X)\exp(-Y) = \exp([X, Y])$.*
2. *If $X \in \mathfrak{g}$ satisfies that $\exp(X)$ commutes with every element of G_0 and $[X, \mathfrak{g}] \subset Z(\mathfrak{g})$, then $X \in Z(\mathfrak{g})$.*

Proof.

1. According to Baker-Campbell-Hausdorff formula (see, *e.g.*, [Far08, Cor. 3.4.5]), there is a symmetric open neighborhood U of $0 \in \mathfrak{g}$ such that for every $A, B \in U$, $\exp(A)\exp(B) = \exp(Z)$, where

   $$Z = Z(A, B) = A + B + [A, B]/2 + \dots$$

   and "..." indicates terms involving higher commutators of A and B. There is a symmetric open neighborhood V of $0 \in U$ such that $Z(A, B) \in U$ for every $A, B \in V$.

   Define $f : \mathbb{R} \to G$ by

   $$f(t) = \exp(tX)\exp(tY)\exp(-tX)\exp(-tY)\exp(-t^2[X, Y]).$$

   Then f is real analytic. There is $\epsilon > 0$ such that $tX, tY \in V$ for all $t \in (-\epsilon, \epsilon)$. By assumption, $[Z(tX, tY), Z(-tX, -tY)] = 0$ and $Z(tX, tY) + Z(-tX, -tY) = t^2[X, Y]$. Then

   $$f(t) = \exp(Z(tX, tY))\exp(Z(-tX, -tY))\exp(-t^2[X, Y]) = e_G$$

   for all $t \in (-\epsilon, \epsilon)$ (see [Laz54, p.144]). By [Azz+23, Cor. A.5], $f(1) = e_G$.
2. Let $D = \exp^{-1}(e_G)$. There is an open neighborhood W of $0 \in \mathfrak{g}$ such that $\exp(W)$ is open in G and $\exp : W \to \exp(W)$ is a diffeomorphism. Then $D \cap W = \{0\}$. For every $Y \in \mathfrak{g}$, there is

$k > 0$ with $[X, Y/k] \in W$. By assumption, $[X, Y/k] \in Z(\mathfrak{g})$, so $[X, [X, Y/k]] = 0$ and $[Y/k, [X, Y/k]] = 0$. Since $\exp(Y/k) \in G_0$, it commutes with $\exp(X)$. By 1, one has $\exp([X, Y/k]) = e_G$. Then $[X, Y/k] \in D \cap W$. Therefore, $[X, Y] = 0$. Thus, $X \in Z(\mathfrak{g})$.

□

An algebraic analogue of Proposition F.6.0.7 is [Ros56, Cor. 2, p.433].

Proposition F.6.0.7. *Let $1 \to B \to C \xrightarrow{p} A \to 1$ be an exact sequence of complex Lie groups, with A complex torus and B commutative. Then C_0 is commutative.*

Proof. We may assume that C is connected by replacing C (resp. B) with C_0 (resp. $B \cap C_0$). Let $\omega : \mathbb{C}^g \to A$ be the universal covering of A. Denote by $\mathfrak{b}$ (resp. $\mathfrak{c}$) the Lie algebra of B (resp. C). Let $\eta : A \to \mathrm{Aut}(B)$ be the outer action. Then η induces a holomorphic morphism $\eta_0 : A \to \mathrm{Aut}(B_0)$. Because $\mathrm{Aut}(B_0)$ is complex Lie subgroup of $\mathrm{GL}(\mathfrak{b})$, η_0 is trivial.

Consider the pullback extension along ω.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker(\epsilon) \arrow[r, "\pi|_{\ker(\epsilon)}"] \arrow[d] & \ker(\omega) \arrow[d] \arrow[dddll, dashed] & \\
1 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & E \arrow[r, "\pi"'] \arrow[d, "\epsilon"'] & \mathbb{C}^g \arrow[r] \arrow[d, "\omega"] & 1 \\
1 \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r, "p"] & A \arrow[r] & 1 \\
 & 0 & & &
\end{tikzcd}$$

By the snake lemma, ϵ is surjective and π restricts to an isomorphism $\ker(\pi) \to \ker(\omega)$. In particular, $d_e\epsilon : L(E) \to L(C)$ is an isomorphism. By Fact F.2.0.3, the morphism ϵ is open. Since E_0 is open in E, $\epsilon(E_0)$ is an open subgroup of C. By connectedness of C, $\epsilon(E_0) = C$. Similarly, $\pi(E_0) = \mathbb{C}^g$. By Fact F.7.2.7 1 below, $B \cap E_0$ is connected. Therefore, $B \cap E_0 \subset B_0$. Since $B_0 \subset B \cap E_0$, one has $B_0 = B \cap E_0$. Hence an extension $1 \to B_0 \to E_0 \to \mathbb{C}^g \to 1$. The outer action is $\eta_0\omega : \mathbb{C}^g \to \mathrm{Aut}(B_0)$, so it is a central extension. Then

$$0 \to \mathfrak{b} \to \mathfrak{c} \to \mathbb{C}^g \to 0 \tag{F.31}$$

is a central extension of Lie algebras. In particular, $\mathfrak{b} \subset Z(\mathfrak{c})$. We shall prove the extension (F.31) is trivial.

We show that $\exp_E : \mathfrak{c} \to E_0$ is surjective. Indeed, for every $x \in E_0$, there is $v \in \mathfrak{c}$ with $d_e p(v) = \pi(x)$. Then $\pi(\exp_E(v)) = \pi(x)$, so $\pi(x \exp_E(-v)) = 0$ and hence $x \exp_E(-v) \in B_0$. As B_0 is connected commutative, there is $u \in \mathfrak{b}$ with $\exp_B(u) = x\exp_E(-v)$. Since $u \in Z(\mathfrak{c})$, one gets $x = \exp_E(u)\exp_E(v) = \exp_E(u+v)$.

By Corollary F.4.1.13, there is a decomposition $B = B_0 \times D$, where $D \in \mathrm{Ab}_c$ is discrete. The natural morphism $E_0 \times D \to E_0 \to \mathbb{C}^g$ is surjective of kernel $B_0 \times D$, hence the first row of the diagram

$$\begin{tikzcd}
1 \arrow[r] & B \arrow[r] & E_0 \times D \arrow[r] & \mathbb{C}^g \arrow[r] & 1 \\
1 \arrow[r] & B_0 \arrow[r] \arrow[u, hook] \arrow[d, hook] & E_0 \arrow[r] \arrow[u] \arrow[d] & \mathbb{C}^g \arrow[r] \arrow[u, "\mathrm{Id}"'] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & B \arrow[r] & E \arrow[r] \arrow[uu, bend right, dashed, "\phi"'] & \mathbb{C}^g \arrow[r] & 1
\end{tikzcd}$$

By Lemma F.3.1.2, there is an equivalence of extensions $\phi : E \to E_0 \times D$.

Fix $x \in \ker(\epsilon)$, let $\phi(x) = (\phi_1(x), \phi_2(x)) \in E_0 \times D$. For every $y \in E_0$,

$$(y,1)\phi(x)(y,1)^{-1} = (y\phi_1(x)y^{-1}, \phi_2(x)) \in \phi(\ker(\epsilon)).$$

Hence, $\phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x))) \in \ker(\epsilon)$. The map

$$E_0 \to \ker(\epsilon), \quad y \mapsto \phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x)))$$

is continuous. As E_0 is connected and $\ker(\epsilon)$ is discrete, this map is constantly x. Thus, $y\phi_1(x)y^{-1} = \phi_1(x)$. Therefore, $\phi_1(x)$ commutes with every element of E_0. As $\exp_E : \mathfrak{c} \to E_0$ is surjective, there is $X \in \mathfrak{c}$ with $\exp_E(X) = \phi_1(x)$. Since $\mathbb{C}^g$ is an abelian Lie algebra, $[\mathfrak{c}, \mathfrak{c}]$ is contained in the kernel of $d_ep : \mathfrak{c} \to \mathbb{C}^g$, which is $\mathfrak{b}$. Then $[\mathfrak{c}, \mathfrak{c}] \subset Z(\mathfrak{c})$, *i.e.*, $[\mathfrak{c}, [\mathfrak{c}, \mathfrak{c}]] = 0$. By Lemma F.6.0.6 2, $X \in Z(\mathfrak{c})$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{c} \arrow[r, "d_ep"'] \arrow[d, "\exp_E"] & \mathbb{C}^g \arrow[d, "\mathrm{Id}"] \\
E \arrow[r, "\pi"] & \mathbb{C}^g
\end{tikzcd}$$

Then $\pi(x) = \pi(\phi_1(x)) = d_ep(X) \in d_ep(Z(\mathfrak{c}))$. Therefore, $\ker(\omega) = \pi(\ker(\epsilon)) \subset d_ep(Z(\mathfrak{c}))$. Since d_ep is $\mathbb{C}$-linear and $\ker(\omega)$ contains a $\mathbb{C}$-basis of $\mathbb{C}^g$, one has $d_ep(Z(\mathfrak{c})) = \mathbb{C}^g$. Consequently, there is a $\mathbb{C}$-linear map $s : \mathbb{C}^g \to Z(\mathfrak{c})$ with $d_ep \circ s = \mathrm{Id}_{\mathbb{C}^g}$. As $s : \mathbb{C}^g \to \mathfrak{c}$ is a Lie algebra morphism, the central extension (F.31) is trivial and $\mathfrak{c}$ is the direct sum of $\mathfrak{b}$ and $\mathbb{C}^g$. In particular, $\mathfrak{c}$ is abelian. As C is connected and its Lie algebra is abelian, C is commutative. $\square$

Example F.6.0.8 shows that the the condition that B/B_0 is torsion (resp. K_0 is Stein) in Proposition F.6.0.3 (resp. Proposition F.6.0.2 2) is necessary. Moreover, in Proposition F.6.0.7, the commutativity of C fails in general.

Example F.6.0.8. Let A be a complex torus and $B = A \times \mathbb{Z}$ be the product group. Consider the complex manifold morphism $A \times B \to B$ defined by $(a, a', k) \mapsto (a' + ka, k)$. It is a non trivial group action of A on B. Let C be the corresponding semidirect product (see [Bou72, Ch.III, no. 4, Prop. 7]), then the resulting complex Lie group extension $1 \to B \to C \to A \to 1$ is not central.

F.7 Noncommutative extensions

F.7.1 Lifted extensions

The real Lie group extension problem is studied by G. Hochschild in [Hoc51a] and [Hoc51b]. As Example F.7.1.1 shows, the case of real Lie groups is different from the case of complex Lie groups.

Example F.7.1.1. Let $G = \mathbb{C}$. The morphism of real Lie groups $\rho : \mathbb{C} \to \mathbb{C}^* = \mathrm{Aut}(G)$ defined by $z \mapsto e^{\bar{z}}$ is an action of G on itself which is real analytic but not holomorphic. Hence an exact sequence of real Lie groups $1 \to G \to G \rtimes_\rho G \to G \to 1$ by [Bou72, Ch. III, no. 4, Prop. 7]. However, the middle term has no structure of complex Lie group making the maps holomorphic. Therefore, [Iwa49, Theorem 7] fails for complex Lie groups. Besides, this shows that the real Lie group extension problem and the complex one are different.

In Section F.7, we review Hochschild's work, but in the context of complex Lie groups. References to the original statement are given when the proofs are similar modulo slight modifications. All results in the sequel are essentially known.

In Section F.7.1, the goal is to derive Corollary F.7.1.6, a result about the extensions of a commutative group by a connected group.

Let L be a complex Lie group and $K \in \mathcal{C}$. For a fixed holomorphic group action $L \times K \to K$, let $\phi : L \to \mathrm{Aut}(K)$ denote the induced group morphism. Let $Z(L, K, \phi)$ denote the set of crossed morphisms, *i.e.*, morphisms $\rho : L \to K$ of complex manifolds such that $\rho(l_1 l_2) = \rho(l_1)\phi_{l_1}(\rho(l_2))$ for all $l_1, l_2 \in L$. Then $Z(L, K, \phi)$ is an abelian group under addition. (When ϕ is trivial, $Z(L, K, \phi) = \mathrm{Hom}(L, K)$.)

For a *normal* complex Lie subgroup H of L, define

$$\mathrm{Ophom}_L(H, K, \phi) = \{\psi \in \mathrm{Hom}(H, K) : \psi(lhl^{-1}) = \phi_l(\psi(h)), \forall l \in L, h \in H\}.$$

Then $\mathrm{Ophom}_L(H, K, \phi)$ is a subgroup of $\mathrm{Hom}(H, K)$. When $H \subset Z(L)$, one has

$$\mathrm{Ophom}_L(H, K, \phi) = \mathrm{Hom}_{\mathcal{C}}(H, K^{\phi(L)}), \tag{F.32}$$

where $K^{\phi(L)} = \cap_{l \in L}\{x \in K : \phi_l(x) = x\}$ is the set of elements fixed by $\phi(L) (\leq \mathrm{Aut}(K))$. Here $K^{\phi(L)}$ is indeed a complex Lie subgroup of K

by Corollary F.2.0.5. When ϕ is trivial, $\mathrm{Ophom}_L(H, K, \phi)$ is the set of morphisms $H \to K$ invariant under the conjugation action of L.

Proposition F.7.1.2. *Assume that H is a normal complex Lie subgroup of L contained in* $\ker(\phi)$. *For every $\rho \in Z(L, K, \phi)$, $\rho|_H \in \mathrm{Ophom}_L(H, K, \phi)$, hence a group morphism $Z(L, K, \phi) \to \mathrm{Ophom}_L(H, K, \phi)$, whose image is denoted by $Z_H(L, K, \phi)$.*

Proof. For every $h, h' \in H$, $\rho(hh') = \rho(h)\phi_h(\rho(h')) = \rho(h)\rho(h')$ since $h \in \ker(\phi)$. Thus $\rho|_H \in \mathrm{Hom}(H, K)$. In particular, $\rho(e_L) = e_K$. For every $l \in L$,

$$e_K = \rho(e_L) = \rho(ll^{-1}) = \rho(l)\phi_l(\rho(l^{-1})),$$

so $\rho(l)^{-1} = \phi_l(\rho(l^{-1}))$. Then

$$\begin{aligned}\rho(lhl^{-1}) &= \rho(lh)\phi_{lh}(\rho(l^{-1}))\\ &=\rho(lh)\phi_l(\rho(l^{-1})) = \rho(lh)\rho(l)^{-1}\\ &=\rho(l)\phi_l(\rho(h))\rho(l)^{-1} = \phi_l(\rho(h)).\end{aligned}$$

The last equality uses the commutativity of K. Therefore, $\rho|_H \in \mathrm{Ophom}_L(H, K, \phi)$. □

Let $\omega : Q' \to Q$ be a surjective morphism of connected complex Lie groups with kernel F. Let $\eta : Q \to \mathrm{Aut}(K)$ be a group morphism such that the induced group action $Q \times K \to K$ is *holomorphic.* As K is commutative, the pulling back map $\omega^* : \mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(Q', K, \eta\omega)$ is a group morphism. Fact F.7.1.3 gives a description of $\ker(\omega^*)$.

Define a map $\sigma : \mathrm{Ophom}_{Q'}(F, K, \eta\omega) \to \mathrm{Ext}(K, Q, \eta\omega)$ as follows. As the group action defined by η is holomorphic, the semidirect complex Lie group $K \rtimes_{\eta\omega} Q'$ exists by [Bou72, Ch.III, no.4, Prop. 7]. For $\psi \in \mathrm{Ophom}_{Q'}(F, K, \eta\omega)$, the morphism $F \to K \rtimes_{\eta\omega} Q'$ defined by $k \mapsto (\psi(k), k)$ identifies F as a normal complex Lie subgroup of $K \rtimes_{\eta\omega} Q'$. Let $E = K \rtimes_{\eta\omega} Q'/F$. The projection $K \rtimes_{\eta\omega} Q' \to Q'$ descends to a morphism $E \to Q$. The injection $K \to K \rtimes_{\eta\omega} Q'$ induces a morphism $K \to E$. Then the resulting sequence $1 \to K \to E \to Q \to 1$ is exact with outer action $\eta\omega$, whose equivalence class is denoted by $\sigma(\psi)$.

Fact F.7.1.3 ([Hoc51a, Thm. 1.1])**.** *The map σ is a group morphism and the sequence*

$$Z(Q', K, \eta\omega) \to \mathrm{Ophom}_{Q'}(F, K, \eta\omega) \xrightarrow{\sigma} \mathrm{Ext}(Q, K, \eta) \xrightarrow{\omega^*} \mathrm{Ext}(Q', K, \eta\omega)$$

is exact.

The use of Fact F.7.1.3 is based on the existence of $\omega : Q' \to Q$ such that every extension in $\mathrm{Ext}(Q, K, \eta)$ becomes a semidirect product when pulled back to $\mathrm{Ext}(Q', K, \eta\omega)$ along ω.

Fact F.7.1.4 ([Hoc51a, Thm. 2.1]). *Let Q be a connected complex Lie group. Assume that $\eta : Q \to \mathrm{Aut}(K)$ is a group morphism such that the induced group action is holomorphic. Then there exists a simply connected complex Lie group Q' and a surjective morphism $\omega : Q' \to Q$ such that the pullback morphism $\omega^* : \mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(Q', K, \eta\omega)$ is zero.*

Remark F.7.1.5. The connectedness condition of the extension kernel in [Hoc51a, Theorems 1.1 and 2.1] is in fact unnecessary.

Corollary F.7.1.6 follows from Fact F.7.1.3 and Fact F.7.1.4.

Corollary F.7.1.6 ([Hoc51a, Cor. 2.1]). *In the notation of Fact F.7.1.4,* $\mathrm{Ext}(Q, K, \eta) = \mathrm{Ophom}_{Q'}(F, K, \eta\omega)/Z_F(Q', K, \eta\omega)$, *where* $F = \ker(\omega)$.

Example F.7.1.7. Let $Q = \mathbb{C}^*$, $L = \mathbb{C}$ and $\omega : L \to \mathbb{Q}$ be defined by $\omega(z) = e^{2\pi i z}$. Then $F = \ker(\omega) = \mathbb{Z}$. Let $\mathbb{C}^* \times K \to K$ be a holomorphic group action and $\eta : \mathbb{C}^* \to \mathrm{Aut}(K)$ be the induced group morphism. Then $\mathrm{Ophom}_L(F, K, \eta\omega) = \mathrm{Hom}(\mathbb{Z}, K^{\eta(\mathbb{C}^*)}) = K^{\eta(\mathbb{C}^*)}$. By Proposition F.3.2.2 and Corollary F.7.1.6, one has $\mathrm{Ext}(\mathbb{C}^* K, \eta) = K^{\eta(\mathbb{C}^*)}/Z_{\mathbb{Z}}(\mathbb{C}, K, \eta\omega)$.

F.7.2 Factor systems

It is well-known that extensions of abstract groups can be classified in terms of factor systems, see [CE99, Ch. XIV, Sec. 4]. This description relies on the existence of set-theoretical cross sections. In general, nevertheless, it is not possible to find a continuous cross section to a surjective morphism of topological groups.

Consider the extension (F.3) of complex Lie groups with outer action $\psi : Q \to \mathrm{Out}(K)$.

Example F.7.2.1. Assume that there is a cross section to (F.3), *i.e.*, a morphism $s : Q \to E$ of complex manifolds with $ps = \mathrm{Id}_Q$. Replacing s by $s(e_Q)^{-1}s$ when necessary, one may assume that s is normalized as $s(e_Q) = e_E$. Define

$$f : Q \times Q \to E, \quad f(g, h) = s(g)s(h)s(gh)^{-1}.$$

Then f is holomorphic. Since $p(f(g, h)) = e_Q$, $f(g, h) \in K$, so f factors through K. The map f measures the failure of s to be a morphism. If E is commutative, then additionally f is symmetric in the sense of [Ser88, (16), p.166]:

$$f(x, y) = f(y, x) \quad \forall x, y \in Q. \tag{F.33}$$

Define $\phi : Q \to \mathrm{Aut}(K)$ by $\phi_g = \mathrm{Inn}_{s(g)}|_K$. Then ϕ is a map (but not necessarily a group morphism) lifting ψ, and the induced map

$$Q \times K \to K, \quad (g, x) \mapsto \phi_g(x) \tag{F.34}$$

is holomorphic. When K is commutative, $\phi = \psi$ is a group morphism independent of the choice of s. When (F.3) is a central extension, ϕ is constantly Id_K.

Moreover, f and ϕ satisfy the following relations:

$$\begin{aligned}
&f(e_Q, h) = f(g, e_Q) = e_K;\\
&\phi_e = \mathrm{Id}_K;\\
&\phi_g\phi_h = \mathrm{Inn}_{f(g,h)}\,\phi_{gh};\\
&f(g,h)f(gh,k) = \phi_g(f(h,k))f(g,hk).
\end{aligned} \tag{F.35}$$

Example F.7.2.1 motivates Definition F.7.2.2.

Definition F.7.2.2 (Factor system)**.** If a morphism $f : Q \times Q \to K$ of complex manifolds and a map $\phi : Q \to \mathrm{Aut}(K)$ making (F.34) holomorphic satisfy the relations (F.35), then f is called a ϕ-factor system (and simply a factor system when ϕ is trivial, in which case the last relation in (F.35) is $f(g,h)f(gh,k) = f(h,k)f(g,hk)$.) A factor system f is called symmetric if (F.33) holds.

When K is commutative, the set of ϕ-factor systems is an abelian group under addition.

We examine how the ϕ-factor system f induced by s in Example F.7.2.1 depends on the choice of the cross section s.

Example F.7.2.3. Let $s' : Q \to E$ be another normalized cross section still inducing ϕ. Define

$$g : Q \to E, \quad g(x) = s(x)^{-1}s'(x).$$

Then $g(e_Q) = e_E$ as s, s' are normalized and g is holomorphic. For every $x \in Q$, $p(g(x)) = e_Q$, so $g(x) \in K$. For every $k \in K$, $\mathrm{Inn}_{s(x)}\,k = \phi_x(k) = \mathrm{Inn}_{s'(x)}\,k$, so $g(x) \in Z(K)$, *i.e.*, g factors through $Z(K)$. Then $s'(x) = s(x)g(x)$. Let f' be the factor system induced by s'. Then

$$\begin{aligned}
f'(x,y) &= s'(x)s'(y)s'(xy)^{-1}\\
&= s(x)g(x)s(y)g(y)[s(xy)g(xy)]^{-1}\\
&= \phi_x(g(x))s(x)s(y)g(y)g(xy)^{-1}s(xy)^{-1}\\
&= \phi_x(g(x))f(x,y)s(xy)g(y)g(xy)^{-1}s(xy)^{-1}\\
&= \phi_x(g(x))f(x,y)\phi_{xy}(g(y)g(xy)^{-1})\\
&= g^{\phi}(x,y)f(x,y),
\end{aligned}$$

where $g^{\phi} : Q \times Q \to K$ is a morphism of complex manifolds defined by

$$g^{\phi}(x,y) = \phi_x(g(x))\phi_{xy}(g(y)g(xy)^{-1}). \tag{F.36}$$

When (F.3) is a central extension, ϕ is trivial, then (F.36) reduces to [Ser88, (15), p.166]: $g^{\phi}(x,y) = g(x)g(y)g(xy)^{-1}$.

Example F.7.2.3 motivates Definition F.7.2.4.

Definition F.7.2.4. Let f, f' be two ϕ-factors systems. If there is a holomorphic map $g : Q \to Z(K)$ with $g(e_Q) = e_E$ such that $f' = g^\phi f$ with g^ϕ defined by (F.36), then f and f' are called ϕ-equivalent, denoted by $f \sim_\phi f'$.

In Definition F.7.2.4, $\sim_\phi$ is an equivalent relation on the set of ϕ-factor systems. When K is commutative, inside the group of all ϕ-factor systems, the elements ϕ-equivalent to the zero form a subgroup. A result similar to Proposition F.7.2.5 for algebraic groups is in [Ser88, Ch. VII, Sec. 1, no.4].

Proposition F.7.2.5. *Let K, Q be complex Lie groups with a map $\phi : Q \to \mathrm{Aut}(K)$ such that (F.34) is holomorphic and the induced map $\psi : Q \to \mathrm{Out}(K)$ is a group morphism. Then:*

1. *The set $\mathcal{F}$ of $\sim_\phi$-equivalence classes of ϕ-factor systems is canonically identified with the subset $\mathcal{E} \subset \mathrm{Ext}(Q, K, \psi)$ of equivalence classes of extensions of Q by K which admit at least one normalized cross section inducing ϕ.*

2. *When K is commutative, the identification in 1 is a group isomorphism.*

3. *If further Q is also commutative and $\phi = \psi = 1$ is trivial, then the subgroup of equivalence classes of symmetric factor systems corresponds to the subgroup of equivalence classes of* commutative *extensions.*

Proof. We only prove 1. Examples F.7.2.1 and F.7.2.3 construct a map $\Phi : \mathcal{E} \to \mathcal{F}$. (Note that equivalent extensions induces the same ϕ-equivalence class.)

Conversely, we define a map $\Psi : \mathcal{F} \to \mathcal{E}$ by the following construction. Given a ϕ-factor system f, one can construct an exact sequence $1 \to K \to E_{f,\phi} \to Q \to 1$ of complex Lie groups with a (holomorphic) normalized cross section $s : Q \to E_{f,\phi}$ as follows. Let $E_{f,\phi} = K \times Q$ as a complex manifold. Define a map

$$g : E_{f,\phi} \times E_{f,\phi} \to E_{f,\phi}, \quad g((k, x), (l, y)) = (k\phi_x(l)f(x, y), xy).$$

As f and the map (F.34) are holomorphic, so is g. Moreover, (F.35) shows g defines an associative multiplication. The pair $(1, 1) \in E_{f,\phi}$ is the identity, and the inverse of (k, x) is

$$(\phi_x^{-1}[k^{-1}f(x, x^{-1})^{-1}], x^{-1}).$$

Hence $(E_{f,\phi}, g)$ is a complex Lie group. The projection $p : E_{f,\phi} \to Q$ is a surjective morphism. The map $i : K \to E_{f,\phi}$ by $k \mapsto (k, 1)$ is the kernel of p. Moreover, define $s : Q \to E_{f,\phi}$ by $s(g) = (1, g)$, then s is normalized cross section. Put $\Psi(f) = E_{f,\phi}$.

We check that $\Psi\Phi = \mathrm{Id}_{\mathcal{E}}$. Indeed, the map $E_{f,\phi} \to E$ defined by $(k, x) \mapsto ks(x)$ is an equivalence of extensions. We check that $\Phi\Psi = \mathrm{Id}_{\mathcal{F}}$, or equivalently s induces f and ϕ. In fact, for every $x \in Q$, $k \in K$, one has

$$\phi_x(k)s(x) = (\phi_x(k), 1)(1, x) = (\phi_x(k), x) = (1, x)(k, 1) = s(x)k,$$

so $\phi_x = \mathrm{Inn}_{s(x)}|_K$, *i.e.*, s induces ϕ. For every $y \in Q$,

$$\begin{aligned} &s(x)s(y)s(xy)^{-1} = (1, x)(1, y)(1, xy)^{-1} \\ =&(f(x, y), xy)(\phi_{xy}^{-1}[f(xy, y^{-1}x^{-1})^{-1}], y^{-1}x^{-1}) \\ =&(f(x, y)\phi_{xy}\phi_{xy}^{-1}(f(xy, y^{-1}x^{-1})^{-1})f(xy, y^{-1}x^{-1}), 1) \\ =&(f(x, y), 1). \end{aligned}$$

Therefore, s induces f. □

When the base Q of (F.3) is discrete, then a set-theoretic cross section is automatically holomorphic.

Corollary F.7.2.6. *Let Q be a discrete complex Lie groups, and let $\eta : Q \to \mathrm{Aut}(K)$ be a group morphism. Then the group $\mathrm{Ext}(Q, K, \eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of η-factor systems. Furthermore, if Q is also commutative, then $\mathrm{Ext}(Q, K)$ is isomorphic to the group of $\sim$-equivalence classes of symmetric factor systems.*

Proof. Since Q is discrete, the group action $Q \times K \to K$ induced by η is holomorphic. The first (resp. second) half follows from Proposition F.7.2.5 2 (resp. 3). □

Another important case where a cross section exists is with simply connected bases. For this, we need a holomorphic version of Malcev's theorem ([Mal42, (E), p.12], [Hoc51a, Lemma 3.1], [Mac60, Theorem 3.2]).

Fact F.7.2.7 (Malcev, [Bou72, Ch. III, § 6, no. 6, Prop. 14, Cor. 2])**.** *Let L be a connected complex Lie group, N be a normal immersed complex Lie subgroup of L.*

1. *If N is closed in L and L/N is simply connected, then N is connected.*

2. *If L is simply connected, N is connected, then N is closed in L and there exists a biholomorphic map $f : L \to N \times L/N$ making a commutative diagram*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, "q"'] & N \times L/N \arrow[d, "p_2"] \\ & L/N, \end{tikzcd}$$

where p_2 is the projection to the second factor and $q : L \to L/N$ is the quotient morphism.

In the same way that [Hoc51a, Theorem 3.1] follows from [Hoc51a, Lemma 3.1], Fact F.7.2.8 can be deduced from Fact F.7.2.7.

Fact F.7.2.8. *Let (F.3) be an exact sequence of complex Lie groups, where E is connected and Q is simply connected. Then there exists a cross section, i.e., a holomorphic map $s : Q \to E$ with $ps = \mathrm{Id}_Q$. In particular, the principal K-bundle $p : E \to Q$ is trivial.*

Example F.7.2.9. Let A be a complex elliptic curve. Take a nonzero element of $A^\vee$, which induces a nontrivial extension E of A by $\mathbb{C}^*$ *via* Theorem F.5.2.4. By Proposition F.5.1.3, the principal $\mathbb{C}^*$-bundle $E \to A$ is nontrivial. Therefore, Fact F.7.2.8 fails if the base is not simply connected.

Corollary F.7.2.10 follows immediately from Fact F.7.2.8 and Proposition F.7.2.5.

Corollary F.7.2.10. *Let K, Q be complex Lie groups, where K is connected commutative and Q is simply connected. Let $\eta : Q \to \mathrm{Aut}(K)$ be a complex Lie group morphism[9]. Then $\mathrm{Ext}(Q, K, \eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of η-factor systems.*

Similar to [Hoc51a, Theorem 3.2], Fact F.7.2.11 can be proved using Fact F.7.2.7 and Fact F.7.2.8,

Fact F.7.2.11. *Let K, Q be complex Lie groups, where K is connected and Q is simply connected. Then the map (on the set of equivalence classes) which associates with each extension of Q by K the induced extension of $L(Q)$ by $L(K)$ is injective. The image is the set of classes of those extensions $0 \to L(K) \to \mathfrak{E} \to L(Q) \to 0$ in which the derivation*

$$[x, \bullet]_{\mathfrak{E}}|_{L(K)} \in \mathrm{Der}(L(K)) = L(\mathrm{Aut}(L(K)))$$

belongs to $L(\mathrm{Aut}(K))$ for every $x \in \mathfrak{E}$. Furthermore, if K is commutative and $\eta : Q \to \mathrm{Aut}(K)$ is a morphism, then the resulting map

$$\mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(L(Q), L(K), d_e\eta)$$

is a group isomorphism.

A connected Lie group is called semisimple if its Lie algebra is semisimple. Analogue of Fact F.7.2.12 for semisimple real Lie groups H and real vector groups G is contained in the proof of [Hoc51b, Theorem 5.1]. Fact F.7.2.12 can be proved in a similar way.

[9] Here $\mathrm{Aut}(K)$ is a complex Lie subgroup of $\mathrm{GL}(L(K))$ by [Lee01, Propositions 1.26 and 1.27].

Fact F.7.2.12. *Let G, H be connected complex Lie groups, where G is commutative and H is* semisimple. *Let $\eta : H \to \mathrm{Aut}(G)$ be a morphism of complex Lie groups. If $\phi \in Z(H, G, \eta)$ is a crossed morphism, then there exists $g \in G$ such that $\phi(x) = \eta_x(g)g^{-1}$ for all $x \in H$. In particular, $\phi \equiv e_G$ on $\ker(\eta)$.*

Theorem F.7.2.13 is a complex version of [Hoc51a, Theorem 4.4].

Theorem F.7.2.13. *In Fact F.7.2.12, $\mathrm{Ext}(H, G, \eta)$ is canonically isomorphic to $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(H), G^{\eta(H)})$.*

Proof. Let $\omega : \tilde{H} \to H$ be the universal covering of H. Then $\ker(\omega) = \pi_1(H)$ is a discrete subgroup of $\tilde{H}$. By Fact F.3.2.4, $\pi_1(H) \subset Z(\tilde{H})$. Then (F.32) gives

$$\mathrm{Ophom}_{\tilde{H}}(\ker(\omega), G, \eta\omega) = \mathrm{Hom}(\pi_1(H), G^{\eta(H)}).$$

By Fact F.7.2.12, for every $\rho \in Z(\tilde{H}, G, \eta\omega)$, $\rho|_{\ker\omega} = 1$, *i.e.*, $Z_{\ker(\omega)}(\tilde{H}, G, \eta\omega) = 0$. By Fact F.7.2.11, the natural map $\mathrm{Ext}(\tilde{H}, G, \eta\omega) \to \mathrm{Ext}(L(H), L(G), d_e\eta)$ is a group isomorphism. Since $L(H)$ is a semisimple complex Lie algebra, Levi's theorem [Ser92, Theorem 4.1, p.48] affirms that $\mathrm{Ext}(L(H), L(G), d_e\eta) = 0$. By Fact F.7.1.3, $\mathrm{Ext}(H, G, \eta) = \mathrm{Hom}(\pi_1(H), G^{\eta(H)})$. $\square$

F.7.3 Non-abelian kernels and extensions of the center

For two complex Lie groups K, Q and a group morphism $\theta : Q \to \mathrm{Out}(K)$, if θ is induced by some extension of Q by K, then the extension kernel (K, θ) is called *extendible*. The problem to determine the extendibility of a given extension kernel is more difficult than that for abstract groups treated in [EM47, Theorem 8.1], because of the obstruction to the existence of a cross section. For extendible kernels, Corollary F.7.3.8 shows that the problem for extensions by K can be reduced to that with an *abelian* kernel, namely $Z(K)$.

Let $1 \to K \to E \xrightarrow{p} Q \to 1$ and $1 \to K' \to E' \xrightarrow{p'} Q \to 1$ be two extension of complex Lie groups. Denote their outer action by $\theta : Q \to \mathrm{Out}(K)$ and $\theta' : Q \to \mathrm{Out}(K')$ respectively. Assume that $Z(K) = Z(K') := C$ and θ, θ' induce a common center action[10] $\theta_0 : Q \to \mathrm{Aut}(C)$. Hence a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Out}(K) \arrow[dr] & \\
Q \arrow[ur, "\theta"] \arrow[rr, "\theta_0"] \arrow[dr, "\theta'"'] & & \mathrm{Aut}(C) \\
 & \mathrm{Out}(K') \arrow[ur] &
\end{tikzcd} \tag{F.37}$$

[10] see (F.5)

We recall the multiplication of kernels defined in [EM47, Sec. 4]. The group law $C \times C \to C$ is holomorphic, so the subset

$$C^* := \{(x, x^{-1}) : x \in C\} \tag{F.38}$$

is analytic in $C \times C$. By Lemma F.2.0.6, $C \times C$ is an analytic subset of $K \times K'$. As C^* is a central subgroup of $K \times K'$, it is also a complex Lie subgroup of $K \times K'$ by Corollary F.2.0.5. Let $K'' = K \times K'/C^*$. From [EM47, p.328], the morphism $C \to K''$ by $g \mapsto [(g, 1)]$ identifies C as the center of K''.

For every $x \in Q$, select automorphisms $\alpha \in \theta(x)(\subset \mathrm{Aut}(K))$ and $\alpha' \in \theta'(x)(\subset \mathrm{Aut}(K'))$. Because the diagram (F.37) is commutative, $\alpha \times \alpha'$ is an automorphism of $K \times K'$ sending C^* into itself. It thus determines an automorphism α'' of K''. The class $[\alpha''] \in \mathrm{Out}(K'')$ depends only on θ, θ', but not the choices of α, α'. Hence a group morphism

$$\theta'' : Q \to \mathrm{Out}(K'') \tag{F.39}$$

that also induces $\theta_0 : Q \to \mathrm{Aut}(C)$.

Definition F.7.3.1. The pair (K'', θ'') constructed above is called the C-product of the two given extension kernels (K, θ) and (K', θ').

Example F.7.3.2. If $K' = C$ is commutative, it is asserted in [EM47, (4.4)] that K' acts as an identity for the C-product. To make it explicit, we define a surjective morphism $\phi : K \times C \to K$ of complex manifolds by $\phi(k, k') = k'k$. Then ϕ is a morphism and $C^* = \ker(\phi)$. Thus, ϕ induces an isomorphism $\sigma : K'' \to K$ satisfying [EM47, (4.2), (4.3)].

Then we review the multiplication of the given two extensions, contained the proof of [EM47, Lem. 5.1].

As the map $E \times E' \to Q$ by $(x, x') \mapsto p'(x')p(x)^{-1}$ is holomorphic, the preimage of e_Q

$$D = D_{p,p'}(E, E') = \{(x, x') \in E \times E' : p(x) = p'(x')\}, \tag{F.40}$$

is analytic in $E \times E'$. Since D is a subgroup of $E \times E'$, by Corollary F.2.0.5, D is a complex Lie subgroup of $E \times E'$.

For every $(x, x') \in D$ with $y = p(x) = p(x')$, every $g \in C$, the element

$$(x, x')(g, g^{-1})(x^{-1}, x'^{-1}) = (\theta_0(y)(g), \theta_0(y)(g)^{-1})$$

is in C^*. Therefore, C^* defined by (F.38) is normal in D.

As C^* is a normal complex Lie subgroup of D, we can set $E'' = D/C^*$. The inclusion $K \times K' \to D$ descends to an injective morphism $K'' \to E''$. The map $D \to Q$ defined by $(x, x') \mapsto p(x)$ induces a surjective morphism $p'' : E'' \to Q$ whose kernel is K''. Hence an extension $1 \to K'' \to E'' \to$

$Q \to 1$. The induced outer action $Q \to \mathrm{Out}(K'')$ is (F.39). We call (E'', p'') the C-product of the two given extensions (E, p) and (E', p'), written as $(E'', p'') = (E, p) \otimes (E', p')$. Thus, [EM47, Lemmas 5.1 and 5.2] hold for complex Lie groups.

Fact F.7.3.3. *The C-product of two extendible kernels is extendible. The kernel of the C-product $(E, p) \otimes (E', p')$ of two extensions is the C-product of the two kernels.*

Proposition F.7.3.4. *When $K' = C$, (E', p') is the semidirect product $C \rtimes_{\theta_0} Q$, then (E'', p'') is naturally equivalent to (E, p).*

Proof. Consider the subgroup $D \leq E \times E' = E \times (C \rtimes_{\theta_0} Q)$ defined in (F.40). Define a map $\psi : D \to E$ by $(x, c, q) \mapsto cx$ for $x \in E$ and $(c, q) \in C \rtimes_{\theta_0} Q$. Then ψ is holomorphic.

We check that ψ is a group morphism. Take another $(x, c', q') \in D$. Since $\theta_{0,q}(c') = \theta_{p(x)}(c') = xc'x^{-1}$, one has

$$\begin{aligned} &\psi((x, c, q)(x', c', q')) = \psi(xx', c\theta_{0,q}(c'), qq') \\ =&c\theta_{0,q}(c')xx' = cxc'x' = \psi(x, c, q)\psi(x', c', q'). \end{aligned}$$

For every $g \in C$, $\psi(g, g^{-1}) = e_E$, so $C^* \subset \ker \psi$. Thus, ψ induces a morphism $\epsilon : E'' \to E$. Together with σ defined in Example F.7.3.2, ϵ fits into a commutative diagram.

$$\begin{tikzcd} 1 \arrow[r] & K'' \arrow[r] \arrow[d, "\sigma"] & E'' \arrow[r] \arrow[d, "\epsilon"] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\ 1 \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 1 \end{tikzcd}$$

Therefore, ϵ is an equivalence of extensions. $\square$

By construction, C-product defines a map $\mathrm{Ext}(Q, K, \theta) \times \mathrm{Ext}(Q, K', \theta') \to \mathrm{Ext}(Q, K'', \theta'')$. When $K' = C$, it specializes to

$$\mathrm{Ext}(Q, K, \theta) \times \mathrm{Ext}(Q, C, \theta_0) \to \mathrm{Ext}(Q, K, \theta), \tag{F.41}$$

which defines an action of the abelian group $\mathrm{Ext}(Q, C, \theta_0)$ on the set $\mathrm{Ext}(Q, K, \theta)$. If further K is also commutative, by [Hoc51a, p.97], (F.41) is exactly the group law defined by the Baer sum on $\mathrm{Ext}(Q, C, \theta_0)$.

Definition F.7.3.5. [EM47, p.329] For every extension kernel (K, θ), let θ^* be the composition of $\theta : Q \to \mathrm{Out}(K)$ with the natural group isomorphism $\mathrm{Out}(K) \to \mathrm{Out}(K^{\mathrm{op}})$. Then the extension kernel $(K^{\mathrm{op}}, \theta^*)$ is called the inverse of (K, θ).

For every $(E, p) \in \mathrm{Ext}(Q, K, \theta)$, define $p^* : E^{\mathrm{op}} \to Q$ by $p^*(x^*) = p(x^{-1})$, then it is a surjective morphism. Since $\ker(p^*) = K^{\mathrm{op}}$, $1 \to K^{\mathrm{op}} \to E^{\mathrm{op}} \xrightarrow{p^*} Q \to 1$ is an extension. The associated outer action is θ^*. Thus, we get an element $(E^{\mathrm{op}}, p^*) \in \mathrm{Ext}(Q, K^{\mathrm{op}}, \theta^*)$ of (E, p). It is called the inverse of (E, p) and its extension kernel is the inverse of (K, θ).

It is a classical result that the group action (F.41) is simple transitive. For abstract groups, see [EM47, Lem. 11.2 and 11.3]. For algebraic groups, see [FLA19, Thm. 1.1]. It remains true for complex Lie groups. The first half, Fact F.7.3.6, can be proved in the same way as in [Hoc51b, Thm. 1.1], using the inverse in the group $\mathrm{Ext}(Q, C, \theta_0)$ and Proposition F.7.3.4.

Fact F.7.3.6. *Let K, Q be complex Lie groups, $C = Z(K)$. Let $\theta : Q \to \mathrm{Out}(K)$ be a group morphism that induces $\theta_0 : Q \to \mathrm{Aut}(C)$. Then the action of $\mathrm{Ext}(Q, C, \theta_0)$ on $\mathrm{Ext}(Q, K, \theta)$ defined by (F.41) is free.*

Theorem F.7.3.7 is analogue to [EM47, Lemma 11.2].

Theorem F.7.3.7. *In the notation of Fact F.7.3.6, if $\mathrm{Ext}(Q, K, \theta)$ is nonempty (i.e., the extension kernel (K, θ) is extendible), then its $\mathrm{Ext}(Q, C, \theta_0)$-action defined by (F.41) is transitive. Equivalently, for every $(E, p), (E_1, p_1) \in \mathrm{Ext}(Q, K, \theta)$, there exits $F \in \mathrm{Ext}(Q, C, \theta_0)$ with $F \otimes E$ equivalent to E_1.*

Proof. Define $D_{p_1, p^*}(E_1, E^{\mathrm{op}})$ like (F.40). Set

$$S = \{(x_1^{-1}, x^*) \in D_{p_1, p^*}(E_1, E^{\mathrm{op}}) : x_1 k x_1^{-1} = x k x^{-1}, \forall k \in K\}.$$

Then S is a subgroup of $E_1 \times E^{\mathrm{op}}$. For every $k \in K$, the map

$$\phi_k : E_1 \times E^{\mathrm{op}} \to K \quad (x_1, x^*) \mapsto x_1^{-1} k x_1 x k^{-1} x^{-1}$$

is holomorphic, so $\phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\mathrm{op}}$. Then $S = D_{p_1, p^*}(E_1, E^{\mathrm{op}}) \cap \cap_{k \in K} \phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\mathrm{op}}$, by [Whi72, Theorem 9C, p.100]. By Corollary F.2.0.5, S is a complex Lie subgroup of $E_1 \times E^{\mathrm{op}}$.

The map $K \times K^{\mathrm{op}} \to K$ by $(k, k'^*) \mapsto kk'$ is holomorphic, so $K^* = \{(k^{-1}, k^*) : k \in K\}$ is an analytic subset of $K \times K^{\mathrm{op}}$. It is a subgroup of S, hence a complex Lie subgroup of S by Corollary F.2.0.5.

For every $(x_1^{-1}, x^*) \in S$, $k \in K$, one has

$$\begin{aligned}&(x_1^{-1}, x^*)(k^{-1}, k^*)(x_1, (x^*)^{-1}) = (x_1^{-1} k^{-1} x_1, x^* k^* (x^{-1})^*)\\ =&(x^{-1} k^{-1} x, (x^{-1} k x)^*) \in K^*,\end{aligned}$$

so K^* is a normal subgroup of S. Let $F = S/K^*$ and $\nu : S \to F$ be the quotient morphism. The map $i : C \to F$ defined by $c \mapsto [(c, 1)]$ is an injective morphism.

The map $\bar{\phi} : S \to Q$ defined by $\bar{\phi}(x_1^{-1}, x^*) = p(x^{-1})$ is a morphism with K^* contained in the kernel. We check that $\bar{\phi}$ is surjective. For every

$h \in Q$, there exist $x \in E$ and $x_1 \in E_1$ with $p(x) = p_1(x_1) = h^{-1}$. Since the two automorphisms of K, $\mathrm{Inn}_x|_K$ and $\mathrm{Inn}_{x_1}|_K$ have the same class $\theta_{h^{-1}}$ in $\mathrm{Out}(K)$, there exists $k_0 \in K$ such that $\mathrm{Inn}_{x_1}|_K = \mathrm{Inn}_x|_K \mathrm{Inn}_{k_0}$. Then $(x_1^{-1}, (xk_0)^*) \in S$ and $\bar{\phi}(x_1^{-1}, (xk_0)^*) = h$.

If $(x_1^{-1}, x^*) \in \ker \bar{\phi}$, then $p_1(x_1) = p(x_1) = e_Q$, so $x_1, x \in K$. Moreover, $x_1 k x_1^{-1} = xkx^{-1}$ for all $k \in K$. Then $x_1^{-1}x \in C$, so $(x_1^{-1}, x^*) = (x_1^{-1}x, 1^*)(x^{-1}, x^*)$. Thus, $[(x_1^{-1}, x^*)] = i(x_1^{-1}x) \in i(C)$.

Thus $\bar{\phi}$ induces a surjective morphism $\phi : F \to Q$ with $i(C) \supset \ker\phi$. In addition, ϕi is trivial, so $i(C) \subset \ker(\phi)$. Hence an extension $1 \to C \xrightarrow{i} F \xrightarrow{\phi} Q \to 1$ with the induced action $Q \to \mathrm{Aut}(C)$ coinciding with θ_0.

It remains to show that the C-product extension $F \otimes E$ is equivalent to E_1. By construction, $F \otimes E$ is represented by $G = D_{\phi,p}(F, E)/C^*$, where $C^* = \{(c, c^{-1}) \in F \times E : c \in C\}$. The pullback of $D_{\phi,p}(F, E)$ along the natural surjection $S \times E \to F \times E$ is $D_{\phi\nu,p}(S, E)$.

For every $(a, b^*, x) \in D_{\phi\nu,p}(S, E) \subset E_1 \times E^{\mathrm{op}} \times E$, one has $p_1(a) = p(b^{-1}) = p(x)$, whence $bx \in K$ and $a \cdot (bx) \in E_1$. Define a holomorphic map $\tau : D_{\phi\nu,p}(S, E) \to E_1$ by $\tau(a, b^*, x) = a \cdot (bx)$.

$$\begin{tikzcd}
E_1 & & \\
D_{\phi\nu,p}(S,E) \arrow[u, "\tau"] \arrow[r] \arrow[rr, bend left, "\nu^*"] \arrow[d, hook] & D_{\phi,p}(F,E) \arrow[r] \arrow[d, hook] & G \arrow[llu, dashed] \\
S \times E \arrow[r, "\nu\times\mathrm{Id}_E"] \arrow[d, hook] & F \times E & \\
E_1 \times E^{\mathrm{op}} \times E & &
\end{tikzcd}$$

We check that τ is a group morphism. For every $(a, b^*, x), (a', b'^*, x') \in D_{\phi\nu,p}(S, E)$, since $(a', b'^*) \in S$ and $bx \in K$, one has $a'^{-1}(bx)a' = b'(bx)b'^{-1}$. Hence,

$$\begin{aligned}
&\tau(a, b^*, x)\tau(a', b'^*, x') = [a(bx)][a'(b'x')] \\
=&aa'[a'^{-1}(bx)a'](b'x') = aa'[b'(bx)b'^{-1}](b'x') \\
=&aa'(b'bxx') = \tau(aa', (b'b)^*, xx') = \tau(aa', b^*b'^*, xx').
\end{aligned}$$

We check that τ is surjective. For every $x_1 \in E_1$, $p_1(x_1) \in Q$. As $\phi\nu : S \to Q$ is surjective, there is $(a, b^*) \in S$ with $\phi\nu(a, b^*) = p_1(x_1)$. Then $p_1(a) = p_1(x_1)$. Thus, $a^{-1}x_1 \in K$. Let $x = b^{-1}(a^{-1}x_1) \in E$. Then $p(x) = p(b^{-1}) = \phi\nu(a, b^*)$, so $(a, b^*, x) \in D_{\phi\nu,p}(S, E)$ and $\tau(a, b^*, x) = a(bx) = a(a^{-1}x_1) = x_1$.

We check that $\ker(\nu^*) \subset \ker(\tau)$. For every $(x_1, x^*, y) \in \ker(\nu^*) \subset E_1 \times E^{\mathrm{op}} \times E$, there is $c \in C$ with $([(x_1, x^*)], y) = (c, c^{-1})$ in $F \times E$. Equivalently, $y = c^{-1}$ in E and $[(x_1, x^*)] = [(c, 1^*)]$ in $F = S/K^*$. Whence, $(x_1c^{-1}, x^*) \in$

K^*, *i.e.*, $x \in K$ and $x_1 = x^{-1}c$. Therefore, $(x_1, x^*, y) = (x^{-1}c, x^*, c^{-1})$ with $x \in K, c \in C$. Thus, $\tau(x_1, x^*, y) = x^{-1}c(xc^{-1}) = e_{E_1}$ and $(x_1, x^*, y) \in \ker(\tau)$.

Conversely, we check $\ker(\tau) \subset \ker(\nu^*)$. For every $(a, b^*, x) \in \ker(\tau)$, one has $a(bx) = e_{E_1}$, so $a \in K$. Because $(a, b^*) \in D_{p_1,p^*}(E_1, E^{\mathrm{op}})$, we obtain $p(b^{-1}) = p(a) = e_Q$ and hence $b \in K$. Since $\mathrm{Inn}_{a^{-1}} = \mathrm{Inn}_b \in \mathrm{Aut}(K)$, one has $ab \in C$. Therefore, $[(a, b^*)] = [(ab, 1^*)] = i(ab)$ in $F = S/K^*$ and $(a, b^*, x) = (ab, (ab)^{-1}) \in C^* \leq F \times E$. Then $(a, b^*, x) \in \ker(\nu^*)$.

Therefore, $\ker(\tau) = \ker(\nu^*)$, so τ induces an isomorphism $G \to E_1$ that establishes an equivalence between the two elements of $\mathrm{Ext}(Q, K, \theta)$. □

Fact F.7.3.6 and Theorem F.7.3.7 yield Corollary F.7.3.8.

Corollary F.7.3.8. *Let K, Q be complex Lie groups, $C = Z(K)$, $\theta : Q \to \mathrm{Out}(K)$ be a group morphism. Let $\theta_0 : Q \to \mathrm{Aut}(C)$ be the induced group morphism. If $\mathrm{Ext}(Q, K, \theta)$ is nonempty, then $\mathrm{Ext}(Q, K, \theta)$ is in (non-canonical) bijection with $\mathrm{Ext}(Q, C, \theta_0)$.*

F.8 Maximal morphisms

A result stronger than Proposition F.5.1.3 holds.

Definition F.8.0.1. [Ser88, Definition 1, p.125]. Let X be a complex manifold, A be a complex torus. A morphism $f : X \to A$ is called maximal if whenever f factors as $X \xrightarrow{g} A' \xrightarrow{h} A$, where $A' \in \mathcal{C}$ is connected and $h - h(0) : A' \to A$ is a finite morphism, it holds that $h - h(0)$ is an isomorphism.

Proposition F.8.0.2. *If X is a regular manifold*[11]*, then the Albanese morphism $f : X \to \mathrm{Alb}(X)$ associated to some base point $x \in X$ is maximal.*

Proof. Assume that f factors as $X \xrightarrow{g} A' \xrightarrow{h} \mathrm{Alb}(X)$, where $A' \in \mathcal{C}$ is a connected and $h - h(0)$ is a finite morphism. Then A' is compact, hence a complex torus. Choosing $g(x)$ as the new zero element of A', we get a new structure of complex torus on A', to which we stick from now on. Then h is a finite morphism. By Proposition 4.4.1.2 3, there is a morphism $\phi : \mathrm{Alb}(X) \to A'$ with $\phi f = g$ and the complex Lie subgroup of $\mathrm{Alb}(X)$ generated by $f(X)$ is $\mathrm{Alb}(X)$ itself. Then $h\phi f = f$ and hence $h\phi = \mathrm{Id}_{\mathrm{Alb}(X)}$. In particular, h is surjective. By Fact F.3.0.4, the exact sequence $0 \to \ker(h) \to A' \xrightarrow{h} A \to 0$ defines a trivial extension, so A' is isomorphic to $\ker(h) \times A$. By connectedness of A', $\ker(h) = 0$ and h is an isomorphism. □

When $f = \mathrm{Id}_A$, Proposition F.8.0.3 reduces to Proposition F.5.1.3.

[11] in the sense of [Var86, p.233]

Proposition F.8.0.3 ([Ser88, Prop. 14, p.188]). *Let X be a connected compact complex manifold, A be a complex torus, $B \in \mathcal{C}$. Let $f : X \to A$ be a maximal morphism. If B_0 is linear, then the composed morphism*

$$\operatorname{Ext}(A, B) \xrightarrow{\pi} H^1(A, \mathcal{B}_A) \xrightarrow{f^*} H^1(X, \mathcal{B}_X) \tag{F.42}$$

is injective.

Proof. Let $C \in \ker(f^* \circ \pi)$. Then the principal fiber bundle $f^*p : f^*C \to X$ is trivial. Fix a point $c \in f^*C$ lying over $0 \in C$. Then there is a morphism $s : X \to f^*C$ with $f^*p \circ s = \mathrm{Id}_X$ and $s(f^*p(c)) = c$. Let $t : X \to C$ be the morphism induced by s.

$$\begin{tikzcd}
 & & f^*C \arrow[r, dashed, "f^*p"] \arrow[d] & X \arrow[l, dashed, bend right, "s"'] \arrow[d, "f"] \arrow[dl, dashed, "t"'] & \\
0 \arrow[r] & B \arrow[r] & C \arrow[r, "p"] & A \arrow[r] \arrow[l, dashed, bend left] \arrow[dl, dashed, "h^{-1}"'] & 0 \\
0 \arrow[r] & B \cap A' \arrow[r] \arrow[u] & A' \arrow[r, "h"] \arrow[u, hook] & A \arrow[r] \arrow[u, "\mathrm{Id}"] & 0
\end{tikzcd}$$

By Remmert's theorem [Whi72, Theorem 4A, p.150], $t(X)$ is an analytic subset of C. By [CD94, (14.14), p.89], the analytic space $t(X)$ is irreducible. Moreover, $t(X)$ is compact and $0 = t(f^*p(c)) \in t(X)$. Let A' be the complex Lie subgroup of C generated by $t(X)$. By Lemma D.3.2.1, A' is a complex torus. Then $(A' \cap B)_0$ is a compact. As a closed complex submanifold of B_0, $(A' \cap B)_0$ is also a Stein manifold, hence a point. Thus, $A' \cap B$ is discrete and compact, hence finite. Therefore, $h : A' \to A$ is a finite morphism. As the maximal morphism f factors as $X \xrightarrow{t} A' \xrightarrow{h} A$, h is an isomorphism. Then $h^{-1} : A \to C$ is a morphism and $ph^{-1} = \mathrm{Id}_A$. By Fact F.3.0.4, $C = 0$ in $\operatorname{Ext}(A, B)$. $\square$

Example F.8.0.4. Let X be a regular manifold, $f : X \to A$ be the Albanese morphism associated to some base point $x \in X$. When $B = \mathbb{C}$, the composed morphism (F.42) is a linear isomorphism $f^* : H^1(A, O_A) \to H^1(X, O_X)$. When $B = \mathbb{C}^*$, it is the inclusion of the identity component $\mathrm{Pic}^0(A) \to \mathrm{Pic}(X)$.

F.9 Commutative extensions of real Lie groups

Let $\mathcal{R}$ be the category of commutative real Lie groups. The solution to the extension problem within $\mathcal{R}$ is summarized in Proposition F.9.0.2. Similar to Lemma F.4.1.1, the category $\mathcal{R}$ is additive but not abelian. Parallel to the construction in Section F.4, we can define an additive functor $\operatorname{Ext}_{\mathcal{R}} : \mathcal{R}^{\mathrm{op}} \times \mathcal{R} \to \mathrm{Ab}$ by considering commutative extensions.

Proposition F.9.0.1 generalizes [Har76, Proposition 5, p.110] (which says that C is isomorphic to $A \times B$) and [HN11, Lemma 15.3.2] (which is for real tori). The similar statement for complex tori is false, shown by Example F.4.1.14.

Proposition F.9.0.1. *Let* $0 \to B \to C \to A \to 0$ *be an extension of commutative real Lie groups. If* A, B *are connected, this extension is trivial.*

Proof. Similar to Proposition F.3.2.2, every extension of $\mathbb{R}$ is a semidirect product, hence $\mathrm{Ext}_{\mathcal{R}}(\mathbb{R}, \bullet) = 0$ on $\mathcal{R}$. Similar to Proposition F.3.2.3, $\mathrm{Ext}_{\mathcal{R}}(S^1, B) = 0$. According to [Har76, Proposition 4, p.109], A is isomorphic to $(S^1)^n \times \mathbb{R}^m$ for some $m, n \in \mathbb{N}$. As the functor $\mathrm{Ext}_{\mathcal{R}}(\bullet, B) : \mathcal{R} \to \mathrm{Ab}$ is additive, we get $\mathrm{Ext}_{\mathcal{R}}(A, B) = 0$. □

Proposition F.9.0.2. *For every* $A, B \in \mathcal{R}$*, there is a non canonical isomorphism in* Ab*:*

$$\mathrm{Ext}_{\mathcal{R}}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0).$$

Proof. By a real version of Corollary F.4.1.13, there are non canonical isomorphisms in $\mathcal{R}$: $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$. By additivity of the bifunctor $\mathrm{Ext}_{\mathcal{R}}$, we get an isomorphism in Ab:

$$\mathrm{Ext}_{\mathcal{R}}(A, B) \to \mathrm{Ext}_{\mathcal{R}}(A/A_0, B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A/A_0, B/B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A_0, B/B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A_0, B_0).$$

Using Lemma F.4.1.12, one can prove that $\mathrm{Ext}_{\mathcal{R}}(A/A_0, B_0) = 0$. Identical to Example F.4.1.10, $\mathrm{Ext}_{\mathcal{R}}(A/A_0, B/B_0) = \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0)$. Similar to Corollary F.3.2.5 and [Hoc51b, Thm. 3.2], $\mathrm{Ext}_{\mathcal{R}}(A_0, B/B_0) = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0)$. By Proposition F.9.0.1, $\mathrm{Ext}_{\mathcal{R}}(A_0, B_0) = 0$. The proof is completed. □

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