Integral points of Shimura varieties: an “all or nothing" principle

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Abstract

For algebraic varieties over number fields, we define a locus that measures the infiniteness of integral points (of chosen integral models). For a Shimura variety S, Lang's conjecture predicts that the locus of S is empty when the level structure is high, and we prove that this locus is either empty or S itself.

1 Introduction

A complex analytic space X is called *Brody hyperbolic*, if every morphism $\mathbb{C} \to X$ is constant. For example, by [Cos05, p.78], a genus g connected compact Riemann surface is Brody hyperbolic if and only if $g \geq 2$. Conjecture 1.1 predicts that hyperbolicity restricts the behavior of rational points.

Conjecture 1.1 (Lang, [Lan74, (1.3)], [Lan86, p.160])**.** *Let X be an integral projective variety over a number filed $F(\subset \mathbb{C})$. If the complex analytification $X(\mathbb{C})$ is Brody hyperbolic, then the set of rational points $X(F)$ is finite.*

Ullmo and Yafaev [UY10] study Conjecture 1.1 in the case of Shimura varieties. Let (G, X) be a Shimura datum in the sense of [Mil17, Def. 5.5]. Let K be a sufficiently small compact open subgroup of $G(\mathbb{A}_f)$. For every connected component S of $\mathrm{Sh}_K(G, X)$, denote the Baily-Borel compactification of S by S^*. In [UY10, p.691], Fact 1.2 is derived from [Nad89, Thm. 0.2].

Fact 1.2 (Nadel)**.** *There exists an open subgroup K' of K such that for every connected component S' of $\mathrm{Sh}_{K'}(G, X)$, the Baily-Borel compactification $S'^*(\mathbb{C})$ is Brody hyperbolic.*

For one thing, by Fact 1.2, shrinking K to a sufficiently small open subgroup, one may assume that the Shimura variety $S(\mathbb{C})$ is Brody hyperbolic. For another, S^* has a natural structure of projective variety over a number field $F(\subset \mathbb{C})$. Then Conjecture 1.1 predicts $S(F')$ to be finite for every finite extension F'/F. Ullmo and Yafaev [UY10] introduce the “Lang locus" (recalled in Example 2.2) for algebraic varieties over $\bar{\mathbb{Q}}$ to measure the possible failure of Conjecture 1.1. By construction, the Lang locus of an algebraic variety X is empty if and only if X has only finitely many rational points over every number field on which X can be defined. The Lang locus of Shimura varieties satisfies an alternative principle.

Fact 1.3 ([UY10, Thm. 1.1]). *Let S be a Shimura variety of sufficiently high level. Then its Lang locus is either empty or all of S.*

As Ullmo and Yafaev put it, Fact 1.3 means that for Shimura varieties, Conjecture 1.1 is either true or very false.

Shimura varieties may not be proper, so it is natural to consider integral points. Conjecture 1.1 predicts that S has only finitely many integral points. We define a notion of “integral Lang locus" (Definition 6.1) for algebraic varieties over $\bar{\mathbb{Q}}$ that measures the failure of finiteness of integral points. Then we derive an analogue of Fact 1.3 for this locus.

Theorem 1.4 (Theorem 6.13). *The integral Lang locus of a Shimura variety S is either empty or S itself.*

A minor difference is that by the Chevalley-Weil theorem, we don’t need the level structure of S to be high. As Example 6.5 shows, Theorem 1.4 is not a direct corollary of Fact 1.3.

Notation and conventions

Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $\mathbb{A}_f$ be the ring of finite adèles of $\mathbb{Q}$. For a topological space X, let $X^{>0}$ be the union of irreducible components of X of positive Krull dimension. Then for every subspace $Y \subset X$, one has $Y^{>0} \subset X^{>0}$.

An algebraic variety over a field k means a reduced scheme separated of finite type over k. The closure of a subset of an algebraic variety is taken in the Zariski topology. A subvariety is assumed to be Zariski closed. By an *étale cover* $X \to Y$, we mean a finite étale morphism between integral algebraic varieties. In particular, it is surjective. If the automorphism group $\mathrm{Aut}(X/Y)$ of the étale cover acts transitively on each fiber, then $X \to Y$ is called a *Galois cover*, of Galois group $\mathrm{Aut}(X/Y)$.

2 Locus formation

We shall show that an alternative principle (Corollary 5.4) for an abstract locus is a consequence of some axioms listed in Section 2.

Suppose that for every integral algebraic variety X over $\bar{\mathbb{Q}}$, one has a subvariety $X^L \subset X$. For a reducible algebraic variety Z over $\bar{\mathbb{Q}}$, let $Z = \cup_{i=1}^n Z_i$ be the irreducible decomposition. Set $Z^L := \cup_{i=1}^n Z_i^L$. Suppose that the formation $(\cdot)^L$ satisfies Assumption 2.1.

Assumption 2.1. For any integral algebraic varieties X, Y over $\bar{\mathbb{Q}}$:

1. Every irreducible component of X^L has positive dimension;
2. For every closed immersion $i : X \to Y$ over $\bar{\mathbb{Q}}$, one has $i(X^L) \subset Y^L$;
3. For every étale cover $f : X \to Y$ over $\bar{\mathbb{Q}}$, one has $f(X^L) \subset Y^L$;
4. One has $X^L \subset (X^L)^L$;
5. For every finite birational morphism $f : X \to Y$ over $\bar{\mathbb{Q}}$, one has $f(X^L) \subset Y^L$.

By Assumption 2.1 2, for every integral algebraic variety X over $\bar{\mathbb{Q}}$, one has $X^L \supset (X^L)^L$. Together with Assumption 2.1 4, it implies $X^L = (X^L)^L$.

Example 2.2. We recall the Lang locus defined in [UY10, Sec. 2.2] and the reason why it satisfies Assumption 2.1. By [Sta25, Tag 01ZM (1), Tag 01ZQ], for every integral algebraic variety X over $\bar{\mathbb{Q}}$, there exists a number field F, an algebraic variety X_F over F called a model of X, and an isomorphism $\phi : X_F \otimes_F \bar{\mathbb{Q}} \to X$ over $\bar{\mathbb{Q}}$. For every finite subextension M/F inside $\bar{\mathbb{Q}}$, let $X(X_F, M)$ be the image of the injection[1] $\phi_* : X_F(M) \to X(\bar{\mathbb{Q}})$. The *Lang locus* X^{UY} of X relative to (X_F, ϕ) is defined to be the Zariski closure of

$$\cup_M \overline{X(X_F, M)}^{>0}$$

in X, where M runs through all finite subextensions of $\bar{\mathbb{Q}}/F$. By Lemma 2.3, the Lang locus X^{UY} depends only on X. It measures the failure of finiteness of rational points, since $X^{\mathrm{UY}} = \emptyset$ if and only if $X_F(M)$ is finite for every finite subextension M/F.

Every irreducible component of X^{UY} of dimension 0 is an isolated point p. Then p is contained in $\overline{X(X_F, M)}^{>0}$ for some finite extension M/F, so there is a positive dimensional irreducible component C of $\overline{X(X_F, M)}$ containing p. Since $C \subset X^{\mathrm{UY}}$, one has a contradiction. This contradiction proves Assumption 2.1 1. From [UY10, Lemme 2.3 (resp. 2.5)], the formation $(\cdot)^{\mathrm{UY}}$ satisfies Assumption 2.1 3 and 5 (resp. 2 and 4).

Lemma 2.3. *In the notation of Example 2.2, the Lang locus X^{UY} of X is independent of the choice of (X_F, ϕ).*

Proof. Take another model $X_{F'}$ over a number field F' and an isomorphism $\phi' : X_{F'} \otimes_{F'} \bar{\mathbb{Q}} \to X$. Then $\phi'^{-1} \circ \phi : X_F \otimes_F \bar{\mathbb{Q}} \to X_{F'} \otimes_{F'} \bar{\mathbb{Q}}$ is an isomorphism over $\bar{\mathbb{Q}}$. By [EGA IV 2, Prop. 4.8.13], because $X_{F'}$ is separated over F', the

[1]The natural map $X_F(M) \to X_F$ may not be injective.

morphism is defined over a number field F'' containing both F and F'. For every finite extension M/F, there is a number field M' containing M and F''. Then $X(X_F, M) \subset X(X_{F'}, M')$, so $\overline{X(X_F, M)}^{>0} \subset \overline{X(X_{F'}, M')}^{>0}$ and hence the Lang locus relative to (X_F, ϕ) is contained in that relative to $(X_{F'}, \phi')$. The reverse inclusion follows by symmetry. $\square$

Remark 2.4.

1. The Lang locus X^{UY} in Example 2.2 is slightly different from the Zariski closed subset X_F^L of X_F defined by [UY10, (1)]. They are related as follows. Let $\varphi : X \to X_F$ be the morphism of schemes induced by the isomorphism $\phi : X_F \otimes_F \bar{\mathbb{Q}} \to X$. Then $\varphi(X^{\mathrm{UY}}) = X_F^L$. To prove it, for every finite extension M/F, let $X_F[M]$ be the image of the natural map $X_F(M) \to X_F$. Then $\varphi(X(X_F, M)) = X_F[M]$. Because surjective integral morphisms preserve the dimension, and because φ is integral, one has $\varphi(\overline{X(X_F, M)}) = \overline{X_F[M]}$ and $\varphi(\overline{X(X_F, M)}^{>0}) = \overline{X_F[M]}^{>0}$. Therefore, $\varphi(X^{\mathrm{UY}}) = X_F^L$.

2. For a finite birational morphism $f : X \to Y$ of integral algebraic varieties over $\bar{\mathbb{Q}}$, we do not know whether the induced map $X^{\mathrm{UY}} \to Y^{\mathrm{UY}}$ is surjective, even if the surjectivity is stated in [UY10, p.697]. That is why we require only inclusion but not equality in Assumption 2.1 5.

We gather some consequences of Assumption 2.1.

Lemma 2.5. *Let X be an algebraic variety over $\bar{\mathbb{Q}}$.*

1. *If $X = \cup_{i=1}^r Z_i$, where each Z_i is a subvariety of X, then $X^L = \cup_{i=1}^r Z_i^L$. In particular, one has $X^L = (X^L)^L$.*

2. *Let Z be an irreducible component of X^L. Then $Z^L = Z$.*

3. *Let $f : X \to Y$ be a Galois cover over $\bar{\mathbb{Q}}$. Then $f^{-1}(f(X^L)) = X^L$. Let $Z \subset Y$ be an irreducible subvariety. Let Z' be an irreducible component of $f^{-1}(Z)$. Then $f(f^{-1}(Z)^L) = f(Z'^L)$.*

Proof.

1. By Assumption 2.1 2, one has $\cup_{i=1}^r Z_i^L \subset X^L$. Let C be an irreducible component of X. Then there exists an index i with $C \subset Z_i$. From Assumption 2.1 2, one has $C^L \subset Z_i^L$ and hence $X^L \subset \cup_{i=1}^r Z_i^L$. Let $X = \cup_{i=1}^n X_i$ be the decomposition into irreducible components. By definition, one has $X^L = \cup_{i=1}^n X_i^L$. Every X_i is integral, so $X_i^L = (X_i^L)^L$. Then one has $(X^L)^L = \cup_{i=1}^n (X_i^L)^L = \cup_{i=1}^n X_i^L = X^L$.

2. Write $X^L = \cup_{i=1}^n Z_i$ for the irreducible decomposition with $Z_1 = Z$. One has
$$Z \subset X^L \overset{\text{(a)}}{\subset} (X^L)^L = \cup_{i=1}^n Z_i^L,$$
where (a) uses Assumption 2.1 4. As Z is irreducible, there is an index i with $Z \subset Z_i^L \subset Z_i$. As $Z = Z_1$ is an irreducible component of X^L, one has $i = 1$ and $Z = Z^L$.

3. For every $x \in f^{-1}(f(X^L))$, there is $x' \in X^L$ with $f(x') = f(x)$. Let Θ be the Galois group of $f : X \to Y$. There is $\theta \in \Theta$ with $\theta(x') = x$. Applying Assumption 2.1 2 to the isomorphism $\theta : X \to X$, one has $x \in X^L$. Therefore, $f^{-1}(f(X^L)) = X^L$. Since Θ permutes transitively the irreducible components of $f^{-1}(Z)$, one has $f^{-1}(Z) = \Theta \cdot Z'$. By Part 1, one has $f^{-1}(Z)^L = \Theta \cdot Z'^L$ and hence $f(f^{-1}(Z)^L) = f(Z'^L)$.

$\square$

3 Shimura varieties

We review some basic facts about Shimura varieties, the main objects of interest in this note. We use essentially results on the geometry of Hecke correspondences and special subvarieties from [UY10; UY14].

Complex structure

Let G be an affine algebraic group over $\mathbb{Q}$.

Definition 3.1 ([Pin90, Sec. 0.1, p.13])**.** For every prime number p, choose an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$.

1. For an element $g = (g_p)_p \in \mathrm{GL}_n(\mathbb{A}_f)$, let $\Gamma_p \leq \bar{\mathbb{Q}}_p^\times$ be the subgroup generated by all the eigenvalues of $g_p \in \mathrm{GL}_n(\mathbb{Q}_p)$. If the intersection of torsion subgroups
$$\cap_p(\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\mathrm{tor}}$$
for p running through all primes is trivial, then g is called *neat*.

2. An element of $G(\mathbb{A}_f)$ is *neat* if its image under some faithful algebraic representation of $G \to \mathrm{GL}_{n/\mathbb{Q}}$ is neat.

3. A subgroup of $G(\mathbb{A}_f)$ is *neat* if all of its elements are neat.

Fact 3.2 ([Pin90, p.13])**.**

1. *Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then there is a normal open subgroup K' of K that is neat.*

2. *Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Then $K \cap G(\mathbb{Q})$ is a neat subgroup of $G(\mathbb{Q})$* (in the sense of [Mil17, p.34]).

Let (G, X) be a Shimura datum. The set $G(\mathbb{R})$ is naturally a real Lie group. For a Lie group L, let L^+ be its identity component. Let G^{ad} be the quotient of G by its center. Set $G(\mathbb{R})_+$ to be the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the natural morphism $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$ of Lie groups. By [Noo06, p.168] and [Mil17, Prop. 5.9], X is naturally a finite disjoint union of isomorphic hermitian symmetric domains. Let X^+ be a connected component of X. By [Mil17, Prop. 5.7 (b)], the stabilizer of X^+ in $G(\mathbb{Q})$ is $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Set $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$. Then $G(\mathbb{Q})^+ \subset G(\mathbb{Q})_+ \subset G(\mathbb{Q})$.

Let K be a compact open subgroup of $G(\mathbb{A}_f)$. From Fact 3.2 1, by passing to an open subgroup of K, we may and *always* assume that K is neat. Then by [Pin90, Prop. 3.3 (b)], $\mathrm{Sh}_K(G,X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is naturally a complex manifold. For every $g \in G(\mathbb{A}_f)$, put $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$ and $S_g := \Gamma_g\backslash X^+$. By Fact 3.2 2 and [Mil17, Prop. 4.1], Γ_g is a neat (hence torsion-free) arithmetic subgroup of $G(\mathbb{Q})$ in the sense of [Mil17, p.33]. From [Mil17, Prop. 3.1], $S_g = [X^+, g]_K$ is naturally a connected complex submanifold of $\mathrm{Sh}_K(G,X)$. Let C be a set of representatives for the double coset space $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. From [Mil17, Lemmas 5.12 and 5.13], the set C is finite, and as a complex manifold $\mathrm{Sh}_K(G,X) = \sqcup_{g\in C} S_g$.

By [Mil17, Thm. 3.12, Cor. 3.16], the complex manifold S_g has a canonical structure of a complex algebraic variety. The algebraic variety S_g is an irreducible, *smooth*, arithmetic locally symmetric variety in the sense of [Mil17, p.58]. From [Mil17, p.40], it is Zariski open in its Baily-Borel compactification S_g^*, which is a projective variety. Thus, $\mathrm{Sh}_K(G,X)$ is naturally a smooth quasi-projective (possibly disconnected) complex algebraic variety.

Lemma 3.3. *Let (H, X_H) be a Shimura subdatum of (G,X)* in the sense of [CLZ16, p.894]. *Then there is a compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup K of K_0, the induced morphism $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G,X)$ is a closed immersion.*

Proof. By [Mil17, Aside, p.58], there is a compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup K of K_0, there is an open subgroup K' of $K\cap H(\mathbb{A}_f)$, such that the induced morphism $\mathrm{Sh}_{K'}(H,X_H) \to \mathrm{Sh}_K(G,X)$ is a closed immersion. The morphism $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H,X_H) \to \mathrm{Sh}_K(G,X)$ is separated, so the morphism $p_{K',K\cap H(\mathbb{A}_f)} : \mathrm{Sh}_{K'}(H,X_H) \to \mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H,X_H)$ is a closed immersion. By [Sta25, Tag 025G], the étale surjective morphism $p_{K',K\cap H(\mathbb{A}_f)} : \mathrm{Sh}_{K'}(H,X_H) \to \mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H,X_H)$ is an isomorphism. Therefore, the natural morphism
$$\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H,X_H) \to \mathrm{Sh}_K(G,X)$$
is a closed immersion. $\square$

Canonical model

Let $E(G,X) \subset \bar{\mathbb{Q}}$ be the reflex field in the sense of [Mil17, Def. 12.2] of the Shimura datum (G,X). By [Mil17, Rk. 12.3 (a)], $E(G,X)$ is a number field. From [Mil99, Rk. 2.4 (b)] and [Mil17, p.128], $\mathrm{Sh}_K(G,X)$ admits a unique (up to a unique isomorphism) *canonical model* over $E(G,X)$ in the sense of [Mil17, Def. 12.8]. Hence, one obtains a *smooth* quasi-projective variety $\mathrm{Sh}_K(G,X)$ over $E(G,X)$. By [Moo98b, p.282] and [GN20, Remark (3), p.56], every connected component S of $\mathrm{Sh}_K(G,X)$ and its inclusion $S \to S^*$ to the Baily-Borel compactification are defined over a finite abelian extension of $E(G,X)$. Such an S is called a *Shimura variety* associated with (G,X,K). By

[Moo98b, Prop. 2.9] and [Mil17, Thm. 5.17], when K is sufficiently small, S is a connected Shimura variety in the sense of [Mil17, Def. 4.10].

By [Del71, Cor. 5.4], for every morphism of Shimura data $\phi : (G', X') \to (G, X)$ and every compact open subgroup $K \leq G(\mathbb{A}_f)$ containing $\phi(K')$, the induced morphism $\mathrm{Sh}_{K'}(G', X') \to \mathrm{Sh}_K(G, X)$ is defined over a number field. Assume that ϕ is the identity of (G, X). Then the induced morphism $p_{K',K} : \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$ is surjective finite étale and defined over $E(G, X)$. For every $g \in G(\mathbb{A}_f)$, the image of the irreducible component $[X^+, g]_{K'}$ of $\mathrm{Sh}_{K'}(G, X)$ is the irreducible component $[X^+, g]_K$ of $\mathrm{Sh}_K(G, X)$. The restriction $[X^+, g]_{K'} \to [X^+, g]_K$ is an étale cover defined over a finite extension of $E(G, X)$, called an *étale cover of Shimura varieties*. From [Mil17, Remark 5.29 (c)], when K' is normal in K, K/K' acts on the complex algebraic variety $\mathrm{Sh}_{K'}(G', X')$ with quotient $\mathrm{Sh}_K(G, X)$. This action permutes the irreducible components of $p^{-1}_{K',K}([X^+, g]_K)$. So $[X^+, g]_{K'} \to [X^+, g]_K$ is a Galois cover. By [CK16, p.1901], if moreover (G, X) is an irreducible Shimura datum, then the Galois group is

$$(gKg^{-1} \cap G(\mathbb{Q})_+)/(gK'g^{-1} \cap G(\mathbb{Q})_+).$$

Remark 3.4. The family of étale covers of Shimura varieties over S may not be directed, in the sense that for such covers $S_1 \to S$ and $S_2 \to S$ associated with open subgroups K_1 and K_2 of K respectively, there may not be any étale cover of Shimura varieties $S_{1,2} \to S$ that factors through both S_1 and S_2 naturally. In fact, for an open subgroup $K_{1,2}$ of $K_1 \cap K_2$, there may not exist any connected component of $\mathrm{Sh}_{K_{1,2}}(G, X)$ lying over S_1 and S_2 at the same time. Still, one may take an open subgroup $K_{1,2}$ of $K_1 \cap K_2$ that is normal in K. For $i = 1, 2$, take a connected component S'_i of $p^{-1}_{K_{1,2},K_i}(S_i)$. Then S'_1 and S'_2 are connected components of $\mathrm{Sh}_{K_{1,2}}(G, X)$, so they are isomorphic as étale covers of S. Therefore, the last paragraph of [UY10, p.694] should be understood up to isomorphism.

Hecke correspondences

For every $g \in G(\mathbb{A}_f)$, let $T(g)$ be the morphism of complex manifolds

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{g^{-1}Kg}(G, X), \quad [x, a]_K \mapsto [x, ag]_{g^{-1}Kg}.$$

By [Mil17, Thm. 13.6], it descends to an isomorphism of algebraic varieties over $E(G, X)$. For every $h \in G(\mathbb{A}_f)$, the morphism $T(g)$ sends the connected component $[X^+, h]_K$ of $\mathrm{Sh}_K(G, X)$ isomorphically to the connected component $[X^+, hg]_{g^{-1}Kg}$ of $\mathrm{Sh}_{g^{-1}Kg}(G, X)$.

Let $S = (K \cap G(\mathbb{Q})_+)\backslash X^+$. For every $q \in G(\mathbb{Q})_+$, let

$$S_q := (K \cap q^{-1}Kq \cap G(\mathbb{Q})_+)\backslash X^+.$$

Then S_q (resp. S) is a connected component of $\mathrm{Sh}_{K\cap q^{-1}Kq}(G, X)$ (resp. $\mathrm{Sh}_K(G, X)$). The map $\mathrm{Id}_{X^+} : X^+ \to X^+$ (resp. the map $X^+ \to X^+, \quad x \mapsto q \cdot x$) induces an étale cover $\alpha_q : S_q \to S$ (resp. $\beta_q : S_q \to S$). There is a commutative diagram

$$\begin{tikzcd}
S & S_q \arrow[l, "\alpha_q"'] \arrow[r, "\beta_q"] & S \\
\mathrm{Sh}_K(G,X) \arrow[from=u, hook] & \mathrm{Sh}_{K\cap q^{-1}Kq}(G,X) \arrow[from=u, hook] \arrow[l, "p_{K\cap q^{-1}Kq,K}"] \arrow[r, "T(q^{-1})\circ p_{K\cap q^{-1}Kq,q^{-1}Kq}"'] & \mathrm{Sh}_K(G,X) \arrow[from=u, hook]
\end{tikzcd}$$

of complex manifolds. Therefore, the correspondence $S \xleftarrow{\alpha_q} S_q \xrightarrow{\beta_q} S$ is algebraic and defined over a number field. It is called the *(rational) Hecke correspondence* induced by q, and denoted by T_q.

Special subvarieties

Definition 3.5. [Moo98a, Def. 2.5] An irreducible subvariety Z of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is called *special*, if there exists a connected, reductive algebraic subgroup H of G defined over $\mathbb{Q}$, an element $g \in G(\mathbb{A}_f)$ and a connected component D_H^+ of

$$D_H := \{x \in X | h_x : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}} \text{ factors through } H_{\mathbb{R}}\},$$

such that $Z(\mathbb{C})$ is the image of $D_H^+ \times gK$ in $\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$.

By [Moo98a, 2.4], D_H is a finite union of $H(\mathbb{R})$-conjugacy classes. Let C be the $H(\mathbb{R})$-conjugacy class containing D_H^+. Then (H,C) is a Shimura subdatum of (G,X). From [Del71, Cor. 5.4] and [Moo98a, Rk. 2.6], the inclusion of a special subvariety into $\mathrm{Sh}_K(G,X)$ is defined over a number field.

For every $g \in G(\mathbb{A}_f)$, an irreducible subvariety Z of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is special if and only if $T(g)(Z)$ is special in $\mathrm{Sh}_{g^{-1}Kg}(G,X)$. By [Moo98a, Sec. 2.9], an irreducible component of the intersection of a family of special subvarieties of $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$ is again special. Therefore, for a complex irreducible subvariety Y of $\mathrm{Sh}_K(G,X)$, there is a smallest special subvariety Z_Y of $\mathrm{Sh}_K(G,X)$ containing Y. We say that Y is *Hodge generic* in Z_Y. Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ be the Deligne torus.

Definition 3.6. The *generic Mumford-Tate group* of the Shimura datum (G,X), denoted by $\mathrm{MT}(X)$, is the smallest closed subgroup H of G over $\mathbb{Q}$, such that every $h : \mathbb{S} \to G_{\mathbb{R}}$ in X factors through $H_{\mathbb{R}}$. If $\mathrm{MT}(X) = G$, then the Shimura datum (G,X) is called *irreducible*.

The subgroup $\mathrm{MT}(X)$ is normal in G, connected and reductive. By [Che09, Def. 1.3.3], $(\mathrm{MT}(X), X)$ is a Shimura subdatum of (G,X). Fact 3.7 characterizes special subvarieties as Hecke image of irreducible components of a Shimura subvariety. Recall that $K \leq G(\mathbb{A}_f)$ is a *neat*, compact open subgroup. For $g \in G(\mathbb{A}_f)$, the quotient $S_g = \Gamma_g \backslash X^+$ is an irreducible component of $\mathrm{Sh}_K(G,X)$, and $gKg^{-1} \cap H(\mathbb{A}_f)$ is neat.

Fact 3.7 ([UY10, Lem. 2.7]). *Let $(H, X_H) \subset (G,X)$ be an* irreducible *Shimura subdatum. Let X_H^+ be a connected component of X_H contained in X^+. Set $\Gamma_{H,g} =$*

$gKg^{-1} \cap H(\mathbb{Q})_+$. *Let* $\tilde{Z}_g := \Gamma_{H,g}\backslash X_H^+$, *which is an irreducible component of* $\mathrm{Sh}_{gKg^{-1}\cap H(\mathbb{A}_f)}(H, X_H)$. *Then the image* Z_g *of* $\tilde{Z}_g$ *under the morphism over* $\mathbb{C}$

$$\mathrm{Sh}_{gKg^{-1}\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G, X), \quad [x, h] \mapsto [x, hg]$$

is a special subvariety of S_g. *The induced morphism* $\pi : \tilde{Z}_g \to Z_g$ *is finite and birational. Conversely, every special subvariety of* S_g *arises in this way.*

Remark 3.8. If the special subvariety Z_g in Fact 3.7 is normal, then by Zariski's main theorem (see, *e.g.*, [Liu06, Ch. 4, Cor. 4.6]), $\pi : \tilde{Z}_g \to Z_g$ is an isomorphism.

4 Locus at infinite level

Let $(\cdot)^L$ be a locus formation satisfying Assumption 2.1. Given an étale cover $f : X \to Y$ over $\bar{\mathbb{Q}}$, the induced map $f : X^L \to Y^L$ may not be surjective. Ullmo and Yafaev introduce a sublocus that lifts along étale covers of Shimura varieties.

Definition 4.1. For a Shimura variety S, its *locus at infinite level* is

$$S^{L_\infty} := \cap_{f:S' \to S} f(S'^L),$$

where $f : S' \to S$ runs through all étale covers of Shimura varieties over $\bar{\mathbb{Q}}$.

Remark 4.2. By Assumption 2.1 3, one has $S^{L_\infty} \subset S^L$. For any $g, h \in G(\mathbb{A}_f)$, the Hecke isomorphism $T(g) : [X^+, h]_K \to [X^+, hg]_{g^{-1}Kg}$ identifies the loci at infinite level.

Remark 4.3. Given a Shimura variety S and an étale cover $f : T \to S$ over $\bar{\mathbb{Q}}$, we don't know whether $f(T^L)$ contains S^{L_∞}. The following example shows that $f : T \to S$ may not be dominated by any étale cover of Shimura varieties over S.

Let $G = \mathrm{GL}_2$. For every integer $n > 1$, let $K(n)$ be the compact open subgroup of $G(\mathbb{A}_f)$ defined in [Mil17, Proof of Prop. 4.1]. Then $K(n) \cap G(\mathbb{Q}) = \Gamma(n)$ is a principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $X = \mathcal{H}^\pm \subset \mathbb{C}$ be the union of the upper and the lower upper half-planes. Let (G, X) be the Siegel Shimura datum in the sense of [Mil17, p.69]. Let X^+ be the upper half-plane. Then the modular curve $Y(n) := \Gamma(n)\backslash X^+$ coincides with the Shimura variety $[X^+, 1]_{K(n)}$. By Fact 3.2 1, there is $n_0 > 1$ such that $K(n)$ is neat for every $n \geq n_0$.

The congruence subgroup problem of $\mathrm{SL}_2(\mathbb{Z})$ has a negative solution. Take a finite-index, noncongruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Replacing Γ with $\Gamma \cap \Gamma(n_0)$, one may assume $\Gamma \subset \Gamma(n_0)$. Let $Y := \Gamma\backslash X^+$. Then the projection $f : Y \to Y(n_0)(\mathbb{C})$ is a finite-sheeted cover of complex manifolds. By [Har77, Appendix B, Thm. 3.2], Y has a unique structure of normal complex algebraic variety making $f : Y \to Y(n_0)$ a finite morphism of schemes. From [SGA 1, XII, Prop. 3.1 (iii)], f is an étale cover over $\mathbb{C}$. By [Ser07, Thm. 6.3.3], f descends to an étale cover over $\bar{\mathbb{Q}}$.

There is no étale cover of Shimura varieties $f' : Y' \to Y(n_0)$ that covers $f : Y \to Y(n_0)$. Assume the contrary. Let K' be the open subgroup of $K(n_0)$ corresponding to f'. As f' factors through a finite-sheeted cover $Y'(\mathbb{C}) \to Y(\mathbb{C})$, taking fundamental groups, one has $K' \cap G(\mathbb{Q}) \subset \Gamma$. By [Mil17, Prop. 4.1], $K' \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$, which contradicts the choice of Γ.

Let S be a Shimura variety associated with (G, X, K). Let $\mathrm{Sh}_K(G, X)^{L_\infty}$ be the union of the loci at infinite level of the connected components of $\mathrm{Sh}_K(G, X)$.

Lemma 4.4. *1. One has $\mathrm{Sh}_K(G, X)^{L_\infty} = \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L)$, where K' runs through the normal open subgroups of K.*

2. *There exists an open subgroup K_1 of K with the following property. For every connected component S_1 of $\mathrm{Sh}_{K_1}(G, X)$ with $p_{K_1,K}(S_1) = S$, the étale cover of Shimura varieties $f_1 : S_1 \to S$ induced by $p_{K_1,K}$ satisfies $f_1(S_1^L) = S^{L_\infty}$.*

3. *Every irreducible component of S^{L_∞} is positive dimensional.*

Proof. 1. For $x \in \mathrm{Sh}_K(G, X)$, let S be the irreducible component of $\mathrm{Sh}_K(G, X)$ containing x. Assume first $x \in \mathrm{Sh}_K(G, X)^{L_\infty}$. Then $x \in S^{L_\infty}$. For every open normal subgroup K' of K, choose an irreducible component S' of $\mathrm{Sh}_{K'}(G, X)$ with $p_{K',K}(S') = S$. As $S' \to S$ is an étale cover of Shimura varieties, one has $x \in p_{K',K}(S'^L) \subset p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L)$. One obtains

$$\mathrm{Sh}_K(G, X)^{L_\infty} \subset \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L).$$

Conversely, assume $x \in \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L)$. For every étale cover of Shimura varieties $f : S_1 \to S$, there is an open subgroup K_1 of K, such that S_1 is an irreducible component of $\mathrm{Sh}_{K_1}(G, X)$. There is a normal open subgroup K_2 of K with $K_2 \subset K_1$. Choose an irreducible component S_2 of $\mathrm{Sh}_{K_2}(G, X)$ with $p_{K_2,K_1}(S_2) = S_1$. By assumption, there is an irreducible component S_2' of $p_{K_2,K}^{-1}(S)$ with $p_{K_2,K}(S_2'^L) \ni x$. As K_2 is normal in K, the quotient group K/K_2 permutes the irreducible components of $p_{K_2,K}^{-1}(S)$. Thus, there is $g \in K$ such that $T(g) : \mathrm{Sh}_{K_2}(G, X) \to \mathrm{Sh}_{K_2}(G, X)$ restricts to an isomorphism $S_2' \to S_2$ over $\bar{\mathbb{Q}}$. By Assumption 2.1 3, one has $p_{K_2,K_1}(T(g)S_2'^L) \subset S_1^L$. Then $x \in f(p_{K_2,K_1}(T(g)S_2'^L)) \subset f(S_1^L)$. Thus, one has $x \in S^{L_\infty} \subset \mathrm{Sh}_K(G, X)^{L_\infty}$, which shows

$$\mathrm{Sh}_K(G, X)^{L_\infty} \supset \cap_{K'} p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L).$$

2. The algebraic variety $\mathrm{Sh}_K(G, X)_{\bar{\mathbb{Q}}}$ is topologically Noetherian, so the family of closed subsets $\{p_{K',K}(\mathrm{Sh}_{K'}(G, X)^L)\}$ for normal open subgroups K' of K has a minimal member $p_{K_1,K}(\mathrm{Sh}_{K_1}(G, X)^L)$. Let $f_1 : S_1 \to S$ be an étale cover of Shimura varieties induced by $p_{K_1,K} : \mathrm{Sh}_{K_1}(G, X) \to \mathrm{Sh}_K(G, X)$. For every étale cover of Shimura varieties $f_2 : S_2 \to S$ associated with an open subgroup K_2 of K, there is a normal open subgroup

$K_{1,2}$ of K with $K_{1,2} \subset K_1 \cap K_2$. Let S'_2 be an irreducible component of $\mathrm{Sh}_{K_{1,2}}(G,X)$ with $p_{K_{1,2},K_2}(S'_2) = S_2$. One has

$$\begin{aligned}
&f_1(S_1^L) \subset p_{K_1,K}(\mathrm{Sh}_{K_1}(G,X)^L) \overset{\text{(a)}}{=} p_{K_{1,2},K}(\mathrm{Sh}_{K_{1,2}}(G,X)^L) \\
\overset{\text{(b)}}{=}& p_{K_{1,2},K}(\frac{K}{K_{1,2}} \cdot S_2'^L) = p_{K_{1,2},K}(S_2'^L) \overset{\text{(c)}}{\subset} f_2(S_2^L),
\end{aligned}$$

where (a) uses the choice of K_1, (b) is because $K/K_{1,2}$ permutes the irreducible components of $\mathrm{Sh}_{K_{1,2}}(G,X)$, and (c) uses Assumption 2.1 3.

3. Choose $f_1 : S_1 \to S$ as in Part 2. For every $x \in S^{L\infty}$, there is $y \in S_1^L$ with $f_1(y) = x$. Let C be an irreducible component of S_1^L containing y. By Assumption 2.1 1, $\dim C$ is positive. As $f_1 : S_1 \to S$ is finite, $f_1(C)$ is irreducible and positive dimensional. From $x \in f_1(C)$, every irreducible component of $S^{L\infty}$ containing x is positive dimensional.

□

In Lemma 4.4 2, for another étale cover of Shimura varieties $S_2 \to S_1$, the composition $S_2^L \to S_1^L \xrightarrow{f_1} S^{L\infty}$ is still surjective.

Lemma 4.5. *Let $f : T \to S$ be an étale cover of Shimura varieties over $\bar{\mathbb{Q}}$. Then $f^{-1}(S^{L\infty}) = T^{L\infty}$. In particular, $T^{L\infty} = T$ is equivalent to $S^{L\infty} = S$, and $S^{L\infty} = S^L$ implies $T^{L\infty} = T^L$.*

Proof. There is an open subgroup K_1 of K, such that T is an irreducible component of $\mathrm{Sh}_{K_1}(G,X)$, and $p_{K_1,K} : \mathrm{Sh}_{K_1}(G,X) \to \mathrm{Sh}_K(G,X)$ restricts to $f : T \to S$.

- We show $T^{L\infty} \subset f^{-1}(S^{L\infty})$.

Fix $t \in T^{L\infty}$, and set $s = f(t)$. For every open normal subgroup K' of K, there is $t' \in \mathrm{Sh}_{K' \cap K_1}(G,X)^L$ with $p_{K' \cap K_1, K_1}(t') = t$. By Assumption 2.1 3, one has $s' := p_{K' \cap K_1, K'}(t') \in \mathrm{Sh}_{K'}(G,X)^L$. Then $s = p_{K',K}(s') \in p_{K',K}(\mathrm{Sh}_{K'}(G,X)^L)$. By Lemma 4.4 1, one has $s \in \mathrm{Sh}_K(G,X)^{L\infty}$ and hence $s \in S^{L\infty}$.

- We show $T^{L\infty} \supset f^{-1}(S^{L\infty})$.

Take $t \in f^{-1}(S^{L\infty})$. Then $s := f(t) \in S^{L\infty}$. For every étale cover of Shimura varieties $u : Z \to T$ associated with an open subgroup K_2 of K_1, there is a *normal* open subgroup K_3 of K with $K_3 \subset K_2$. Let N be an irreducible component of $p_{K_3,K_2}^{-1}(Z)$. Then the restriction $v : N \to Z$ of $p_{K_3,K_2} : \mathrm{Sh}_{K_3}(G,X) \to \mathrm{Sh}_{K_2}(G,X)$ is an étale cover of Shimura varieties, and the composition $N \xrightarrow{v} Z \xrightarrow{u} T \xrightarrow{f} S$ is a Galois cover. One has

$$\begin{aligned}
&(u \circ v)^{-1}(t) \subset (f \circ u \circ v)^{-1}(s) \subset (f \circ u \circ v)^{-1}(S^{L\infty}) \\
\subset&(f \circ u \circ v)^{-1}((f \circ u \circ v)(N^L)) \overset{\text{(a)}}{=} N^L,
\end{aligned}$$

where (a) uses Lemma 2.5 3. Thus, one has $u^{-1}(t) \subset v(N^L) \subset Z^L$ and $t \in u(Z^L)$. Hence, one has $t \in T^{L\infty}$.

- The equality $T^{L_\infty} = T$ is equivalent to $S^{L_\infty} = S$.

If $T^{L_\infty} = T$, then $S^{L_\infty} = f(f^{-1}(S^{L_\infty})) = f(T^{L_\infty}) = f(T) = S$. Conversely, if $S^{L_\infty} = S$, then $T^{L_\infty} = f^{-1}(S^{L_\infty}) = f^{-1}(S) = T$.

- The equality $S^{L_\infty} = S^L$ implies $T^{L_\infty} = f^{-1}(S^{L_\infty}) = f^{-1}(S^L) \overset{(b)}{\supset} T^L$, where (b) uses Assumption 2.1 3. Hence, one obtains $T^{L_\infty} = T^L$.

$\square$

Assume $S = [X^+, 1]_K$. Let Z be a special subvariety of S over $\bar{\mathbb{Q}}$. Let $(H, X_H) \subset (G, X)$ be an irreducible Shimura subdatum, X_H^+ and $\pi : \tilde{Z} \to Z$ be as in Fact 3.7.

Lemma 4.6. *There is a open subgroup K' of K, such that*

1. *the induced étale cover of Shimura varieties $f : S'(:= [X^+, 1]_{K'}) \to S$ is Galois and satisfies $S'^L = S'^{L_\infty}$;*
2. *for every irreducible component Z' of $f^{-1}(Z)$, the restriction $f|_{Z'} : Z' \to Z$ factors through an étale cover $h : Z' \to \tilde{Z}$ with $h(Z'^L) = \tilde{Z}^{L_\infty}$.*

Proof. By Lemma 4.4 2, there is an open subgroup $K_{0,H} \leq K \cap H(\mathbb{A}_f)$, such that the induced étale cover of Shimura varieties $g_0 : \tilde{Z}_0(:= [X_H^+, 1]_{K_{0,H}}) \to \tilde{Z}$ satisfies $g_0(\tilde{Z}_0^L) = \tilde{Z}^{L_\infty}$. Similarly, there is an open subgroup $K_1 \leq K$, such that $K_1 \cap H(\mathbb{A}_f) \subset K_{0,H}$ and the étale cover $f_1 : S_1(:= [X^+, 1]_{K_1}) \to S$ satisfies $f_1(S_1^L) = S^{L_\infty}$. By Lemma 4.5, one has $S_1^L = S_1^{L_\infty}$. From Lemma 3.3, there is a normal open subgroup K' of K contained in K_1, such that the induced morphism $\mathrm{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K'}(G, X)$ is a closed immersion. Set $S' := [X^+, 1]_{K'}$. Let $f : S' \to S$ be the restriction of $p_{K',K} : \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$.

1. Since K' is normal in K, the étale cover $f : S' \to S$ is Galois. By Lemma 4.5 and $S_1^L = S_1^{L_\infty}$, one has $S'^L = S'^{L_\infty}$.
2. Let $\tilde{Z}_1 := [X_H^+, 1]_{K' \cap H(\mathbb{A}_f)}$. The morphism of Shimura varieties $\tilde{Z}_1 \to S'$ is a closed immersion, so is the induced morphism $\pi' : \tilde{Z}_1 \to f^{-1}(Z)$. Let $h : \tilde{Z}_1 \to \tilde{Z}$ be the restriction of the morphism

$$p_{K' \cap H(\mathbb{A}_f), K \cap H(\mathbb{A}_f)} : \mathrm{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H).$$

Then h is an étale cover of Shimura varieties. The situation is depicted as a commutative diagram

$$\begin{tikzcd}
\tilde{Z}_1 \arrow[r, hook, "\pi'"] \arrow[d] \arrow[dd, bend right, "h"'] & f^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \arrow[dd, bend left, "f"] \\
\tilde{Z}_0 \arrow[r] \arrow[d, "g_0"] & f_1^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S_1 \arrow[d, "f_1"'] \\
\tilde{Z} \arrow[r, two heads, "\pi"] & Z \arrow[r, hook] & S.
\end{tikzcd}$$

Since h factors through $\tilde{Z}_0$, one has $h(\tilde{Z}_1^L) = \tilde{Z}^{L_\infty}$.

Claim 4.7. The closed immersion $\pi' : \tilde{Z}_1 \to f^{-1}(Z)$ identifies $\tilde{Z}_1$ with an irreducible component of $f^{-1}(Z)$.

The Galois group of the Galois cover $f : S' \to S$ permutes the irreducible components of $f^{-1}(Z)$, so Z' has properties similar to that of $\tilde{Z}_1$.

□

Proof of Claim 4.7. Since $\tilde{Z}_1$ is irreducible, it is contained in an irreducible component C of $f^{-1}(Z)$. By [Sta25, Tag 0BAC], as $\pi : \tilde{Z} \to Z$ is birational, there is a nonempty open subset $\tilde{U} \subset \tilde{Z}$, such that $U := \pi(\tilde{U})$ is open in Z, and $\pi|_{\tilde{U}} : \tilde{U} \to U$ is an isomorphism. Consider the commutative square of algebraic varieties

$$\begin{tikzcd} h^{-1}(\tilde{U}) \arrow[r, hook, "\pi'|_{h^{-1}(\tilde{U})}"] \arrow[d, "h|_{h^{-1}(\tilde{U})}"'] & f^{-1}(U) \arrow[d, "f"] \\ \tilde{U} \arrow[r, "\pi|_{\tilde{U}}"] & U. \end{tikzcd}$$

The morphism $h^{-1}(\tilde{U}) \to \tilde{U}$ (resp. $f^{-1}(U) \to U$) is a base change of the étale morphism $h : \tilde{Z}_1 \to \tilde{Z}$ (resp. $f : S' \to S$), so it is étale. By [Sta25, Tag 03PC (10)], the morphism $\pi'|_{h^{-1}(\tilde{U})} : h^{-1}(\tilde{U}) \to f^{-1}(U)$ is étale. From [Sta25, Tag 03PC (9)], $h^{-1}(\tilde{U})$ is an open subset of $f^{-1}(U)$, hence a nonempty open subset of C. Since C is irreducible, $h^{-1}(\tilde{U})$ is dense in C. Therefore, $C = \tilde{Z}_1$. □

Let $S = S_g$ be a Shimura variety associated with (G, X, K). Lemma 4.8 is used in the proof of Theorem 5.1.

Lemma 4.8. *If $S^{L_\infty} \neq \emptyset$ is a finite union of special subvarieties of S, then $S^L = S$.*

Proof. The Hecke isomorphism $T(g) : \mathrm{Sh}_{gKg^{-1}}(G, X) \to \mathrm{Sh}_K(G, X)$ sends $[X^+, 1]_{gKg^{-1}}$ to S_g. It preserves the special subvarieties. Thus, one may assume $g = 1$ (by replacing K with gKg^{-1}). Write $S^{L_\infty} = \cup_{i=1}^n Z_i$ for the irreducible decomposition. By assumption, for every $1 \leq i \leq n$, the subvariety $Z_i \subset S$ is special. Let $\pi_i : \tilde{Z}_i \to Z_i$ be a finite birational morphism given by Fact 3.7.

Claim 4.9. One has $\tilde{Z}_1^{L_\infty} = \tilde{Z}_1$.

For every $q \in G(\mathbb{Q})_+$, we prove

$$T_q Z_1 \subset S^L. \tag{1}$$

Let $(H, X_H) \subset (G, X)$ be an irreducible Shimura subdatum, $\tilde{Z}_1 = [X_H^+, 1]_{K \cap H(\mathbb{A}_f)}$, and $\pi_1 : \tilde{Z}_1 \to Z_1$ be as in Fact 3.7. For every irreducible component $Z_q \subset \alpha_q^{-1}(Z_1)$, there is an irreducible component $\tilde{Z}_q$ of $\mathrm{Sh}_{K \cap q^{-1}Kq \cap H(\mathbb{A}_f)}(H, X_H)$ with the following properties:

- The morphism $\mathrm{Sh}_{K \cap q^{-1}Kq \cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$ restricts to an étale cover of Shimura varieties $\alpha'_q : \tilde{Z}_q \to \tilde{Z}_1$.

- The image of $\tilde{Z}_q$ under the morphism

$$\mathrm{Sh}_{K\cap q^{-1}Kq\cap H(\mathbb{A}_f)}(H,X_H)\to \mathrm{Sh}_{K\cap q^{-1}Kq}(G,X)$$

  is Z_q.

Conjugating by q gives another irreducible Shimura subdatum $(qHq^{-1}, q\cdot X_H)\subset (G,X)$, and a morphism of Shimura data $(H,X_H)\to(qHq^{-1},q\cdot X_H)$. Let $\tilde{Z}'_q$ be the image of $\tilde{Z}_q$ under the induced morphism

$$\mathrm{Sh}_{K\cap q^{-1}Kq\cap H(\mathbb{A}_f)}(H,X_H)\to \mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1},q\cdot X_H).$$

Then $\tilde{Z}'_q$ is an irreducible component of $\mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1},q\cdot X_H)$, and the restriction $\beta'_q:\tilde{Z}_q\to\tilde{Z}'_q$ is an étale cover of Shimura varieties. By Fact 3.7, the morphism $\mathrm{Sh}_{K\cap qH(\mathbb{A}_f)q^{-1}}(qHq^{-1},q\cdot X_H)\to \mathrm{Sh}_K(G,X)$ restricts to a finite birational morphism $\pi'_q:\tilde{Z}'_q\to\beta_q(Z_q)$. Consider the commutative diagram

$$\begin{tikzcd}
\tilde{Z}_1 \arrow[d, two heads, "\pi_1"'] & \tilde{Z}_q \arrow[l, "\alpha'_q"'] \arrow[r, "\beta'_q"] \arrow[d, two heads] & \tilde{Z}'_q \arrow[d, "\pi'_q"] \\
Z_1 & Z_q \arrow[l, "\alpha_q"] \arrow[r, "\beta_q"'] & \beta_q(Z_q).
\end{tikzcd}$$

From Claim 4.9 and Lemma 4.5, one has $\tilde{Z}'_q=\tilde{Z}'^{L_\infty}_q=\tilde{Z}'^{L}_q$. Then

$$\beta_q(Z_q)=\pi'_q(\tilde{Z}'_q)=\pi'_q(\tilde{Z}'^{L}_q)\overset{\text{(a)}}{\subset}\beta_q(Z_q)^L,$$

where (a) uses Assumption 2.1 5. By Assumption 2.1 2, one has $\beta_q(Z_q)\subset S^L$, which proves (1).

From [KUY18, Lem. 2.5], the special subvariety Z_1 of S contains a special point z. By [LZ19, Rk. 2.7], $\{T_qz\}_{q\in G(\mathbb{Q})_+}$ is dense in the complex manifold $S(\mathbb{C})$. By (1), the algebraic Zariski closed subset $S^L\subset S$ contains $\{T_qz\}_{q\in G(\mathbb{Q})_+}$, which implies $S^L=S$. □

Proof of Claim 4.9. First, we prove

$$S^{L_\infty}\subset\cup_{i=1}^n\pi_i(\tilde{Z}_i^{L_\infty}). \tag{2}$$

Let K_i be a normal open subgroup of K given by Lemma 4.6 for $\pi_i:\tilde{Z}_i\to Z_i$. There is a normal open subgroup K' of K with $K'\subset\cap_{i=1}^nK_i$. Set $S_i:=[X^+,1]_{K_i}$ and $S':=[X^+,1]_{K'}$. Let $g_i:S'\to S_i$, $f_i:S_i\to S$, and $f:S'\to S$ be the induced Galois cover of Shimura varieties. Then for every $1\le i\le n$, one has $f_ig_i=f$. By construction of $f:S'\to S$, one has $S'^L=S'^{L_\infty}$ and $f(S'^L)=S^{L_\infty}$.

For every irreducible component $C\subset S'^L$, the subset $f(C)$ of S^{L_∞} is irreducible. Then there is $1\le i\le n$ with $f(C)\subset Z_i$. Thus, $g_i(C)$ is an irreducible subset of $f_i^{-1}(Z_i)$. There is an irreducible component Z' of $f_i^{-1}(Z_i)$ containing $g_i(C)$.

By Lemma 4.6 2, the morphism $f_i|_{Z'} : Z' \to Z_i$ factors through an étale cover $h : Z' \to \tilde{Z}_i$. Therefore, Z' and $g_i^{-1}(Z')$ are smooth. One has

$$g_i^{-1}(Z') \subset f^{-1}(Z_i) \subset f^{-1}(S^{L_\infty}) \overset{\text{(a)}}{=} S'^L,$$

where (a) uses Lemma 4.5. Then C is an irreducible component of $g_i^{-1}(Z')$. As $g_i^{-1}(Z')$ is smooth, $g_i|_C : C \to Z'$ is an étale cover. The setting is summarized in the diagram

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[dr] & g_i^{-1}(Z') \arrow[rr, hook] \arrow[d] \arrow[drr, phantom, "\square"] & & S' \arrow[d, "g_i"'] \arrow[dd, bend left, "f"] \\
& Z' \arrow[r, hook] \arrow[d, dashed, "h"] & f_i^{-1}(Z_i) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S_i \arrow[d, "f_i"'] \\
& \tilde{Z}_i \arrow[r, two heads, "\pi_i"] & Z_i \arrow[r, hook] & S.
\end{tikzcd}$$

One has

$$f(C) \overset{\text{(a)}}{=} f(C^L) = f_i g_i(C^L) \overset{\text{(b)}}{\subset} f_i(Z'^L) \overset{\text{(c)}}{\subset} \pi_i(\tilde{Z}_i^{L_\infty}),$$

where (a), (b) and (c) use Lemma 2.5 2, Assumption 2.1 3 and Lemma 4.6 2 respectively. This proves (2).

From (2), one has $Z_1 \subset \cup_{i=1}^n \pi_i(\tilde{Z}_i^{L_\infty})$. Since Z_1 is irreducible, there is $1 \leq j \leq n$ with $Z_1 \subset \pi_j(\tilde{Z}_j^{L_\infty}) \subset Z_j$. As Z_1 is an irreducible component of S^{L_∞}, one has $j = 1$. Then $\dim \tilde{Z}_1^{L_\infty} \geq \dim Z_1 = \dim \tilde{Z}_1$. The irreducibility of $\tilde{Z}_1$ proves $\tilde{Z}_1^{L_\infty} = \tilde{Z}_1$. $\square$

5 Ullmo-Yafaev alternative principle

Let (G, X) be a Shimura datum. Let $(\cdot)^L$ be a locus formation satisfying Assumption 2.1. Let $(\cdot)^{L_\infty}$ be the corresponding locus formation at infinite level for Shimura varieties. Theorem 5.1 is an alternative principle due to Ullmo and Yafaev.

Theorem 5.1. *Let S be a Shimura variety associated with (G, X, K). Then either $S^{L_\infty} = \emptyset$ or $S^{L_\infty} = S$.*

Proof. By Hecke isomorphisms, one may assume $S = [X^+, 1]_K$. By Lemma 4.5, one may replace S by an étale cover induced by an open subgroup of K. One may thereby assume $S^L = S^{L_\infty} \neq \emptyset$. By Assumption 2.1 1 (resp. Lemma 2.5 2), for every irreducible component Z of S^L, one has $\dim Z > 0$ (resp. $Z^L = Z$).

Claim 5.2. The subvariety $Z \subset S$ is special.

By Claim 5.2, the locus S^L is a finite union of special subvarieties. From Lemma 4.8, one has $S^{L_\infty} = S$. $\square$

Proof of Claim 5.2. Let $S_M \subset S$ be the smallest special subvariety containing Z. From Fact 3.7, there is a Shimura subdatum $(H, X_H) \subset (G, X)$, such that the restriction $\pi : \tilde{S}_M (:= [X_H^+, 1]_{K\cap H(\mathbb{A}_f)}) \to S_M$ of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H) \to \mathrm{Sh}_K(G, X)$ is finite birational.

Take a Galois cover $f : S' \to S$ given by Lemma 4.6 for the special subvariety $S_M \subset S$. Since f is finite surjective, there is an irreducible component $T \subset f^{-1}(Z)$ with $f(T) = Z$. Since $Z \subset S^L$ is an irreducible component, T is an irreducible component of

$$f^{-1}(S^L) = f^{-1}(S^{L_\infty}) \overset{\text{(a)}}{=} S'^{L_\infty} \overset{\text{(b)}}{=} S'^L.$$

Here (a) and (b) use Lemma 4.5. There is an irreducible component S'_M of $f^{-1}(S_M)$ containing T.

By Lemma 4.6, $f : S'_M \to S_M$ factors through an étale cover $h : S'_M \to \tilde{S}_M$. Consider the commutative diagram

$$\begin{tikzcd} T \arrow[r, hook] & S'_M \arrow[r, hook] \arrow[d, dashed, "h"] & f^{-1}(S_M) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d, "f"] \\ & \tilde{S}_M \arrow[r, "\pi"] & S_M \arrow[r, hook] & S. \end{tikzcd}$$

One has

$$h(T) \overset{\text{(a)}}{=} h(T^L) \subset h(S'^L_M) \overset{\text{(b)}}{=} \tilde{S}_M^{L_\infty}, \tag{3}$$

where (a) and (b) use Lemma 2.5 2 and Lemma 4.6 2 respectively.

We show that the nonempty, irreducible, closed subset $h(T)$ is Hodge generic in $\tilde{S}_M$. Since $\pi : \tilde{S}_M \to S_M$ is finite surjective, there is an irreducible component $\tilde{Z}$ of $\pi^{-1}(Z)$ with $\pi(\tilde{Z}) = Z$. By [KY14, p.879], for every special subvariety V of $\tilde{S}_M$ containing $h(T)$, the image $\pi(V)$ is a special subvariety of S containing $\pi h(T) = f(T) = Z$. Hence, one has $\pi(V) = S_M$. Therefore, $\dim V \geq \dim S_M = \dim \tilde{S}_M$. Since $\tilde{S}_M$ is irreducible, one has $V = \tilde{S}_M$.

Then by (3) and Lemma 5.3, one has $\tilde{S}_M = \tilde{S}_M^{L_\infty} = \tilde{S}_M^L$. As a result, $S_M = \pi(\tilde{S}_M) = \pi(\tilde{S}_M^L) \overset{\text{(c)}}{\subset} S_M^L \subset S^L$, where (c) uses Assumption 2.1 5. Since Z is an irreducible component of S^L and S_M is irreducible, one has $Z = S_M$. □

Lemma 5.3. *Let $S = [X^+, 1]_K \subset \mathrm{Sh}_K(G, X)$. If S^{L_∞} contains a nonempty irreducible closed subset that is Hodge generic in S, then $S^{L_\infty} = S$.*

Proof. By Lemma 4.5, for every $q \in G(\mathbb{Q})^+$, one has $T_q S^{L_\infty} = \beta_q(\alpha_q^{-1}(S^{L_\infty})) = \beta_q(S_q^{L_\infty}) = S^{L_\infty}$. Write $S^{L_\infty} = U_1 \cup U_2$, where U_1 is the union of irreducible components of S^{L_∞} that are Hodge generic in S, and U_2 is the union of the remaining irreducible components. By Lemma 4.4 3 and assumption, one has $\dim U_1 > 0$.

Let C be an irreducible component of $T_q U_2$. There is an irreducible subvariety $C_q \subset S_q$ with $\beta_q(C_q) = C$ and $\alpha_q(C_q) \subset U_2$. Then $\alpha_q(C_q)$ is not Hodge generic in S. Thus, there is a strict, special subvariety V of S containing

$\alpha_q(C_q)$. Then $C \subset T_q(\alpha_q(C_q)) \subset T_qV$. There is an irreducible component W of T_qV containing C. By [LZ19, Remark 2.7], W is a special subvariety of S. Since $\dim W \leq \dim V < \dim S$, the subvariety C of S is not Hodge generic. As every irreducible component of T_qU_2 is not Hodge generic in S, by $U_1 \subset T_qS^{L_\infty} = T_qU_1 \cup T_qU_2$, one has $U_1 \subset T_qU_1$. By $\dim U_1 > 0$ and [UY10, Thm. 1.2], one has $U_1 = S$ and hence $S^{L_\infty} = S$. $\square$

Corollary 5.4. *Let (G, X) be a Shimura datum. Then there is a compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup K of K_0 and every Shimura variety S associated with (G, X, K), one has either $S^L = \emptyset$ or $S^L = S$.*

Proof. By Lemmas 4.4 2 and 4.5, there is a neat compact open subgroup K_0 of $G(\mathbb{A}_f)$, such that for every open subgroup $K \leq K_0$, one has $S^L = S^{L_\infty}$. The result follows from Theorem 5.1. $\square$

6 "All or nothing" principle for integral points

We define a locus concerning integral points, analogous to the Lang locus $(\cdot)^{\mathrm{UY}}$ concerning rational points. For this locus, we verify Assumption 2.1. Then an alternative principle follows.

Let X be an integral algebraic variety over $\bar{\mathbb{Q}}$. As in Example 2.2, there is a number field $F \subset \bar{\mathbb{Q}}$, an integral algebraic variety X_F over F and an isomorphism $X_F \otimes_F \bar{\mathbb{Q}} \to X$ over $\bar{\mathbb{Q}}$. For every finite set Σ of places of F including all archimedean ones, let $O_{F,\Sigma}$ be the ring of Σ-integers. When Σ is sufficiently large, there exists an integral scheme $\mathcal{X}$ and a dominant, separated morphism $\mathcal{X} \to \operatorname{Spec} O_{F,\Sigma}$ of finite type, whose generic fiber is X_F. (From [Har77, III, Prop. 9.7], $\mathcal{X}$ is flat over $O_{F,\Sigma}$.) We call $\mathcal{X}$ an *integral model* for X relative to (F, Σ). By a finite extension $(M, \Omega)/(F, \Sigma)$, we mean a finite extension M/F inside $\bar{\mathbb{Q}}$ together with a finite set Ω of places of M containing all the places above Σ. For every $(M, \Omega)/(F, \Sigma)$, let $X(\mathcal{X}, M, \Omega)$ be the image of the injection

$$\mathcal{X}(O_{M,\Omega}) \to X(\bar{\mathbb{Q}}), \quad x \mapsto x|_{\operatorname{Spec} \bar{\mathbb{Q}}}.$$

Definition 6.1. Let $\mathcal{X}^I$ be the Zariski closure of

$$\cup_{(M,\Omega)/(F,\Sigma)} \overline{X(\mathcal{X}, M, \Omega)}^{>0}$$

inside X, where (M, Ω) runs though all finite extensions of (F, Σ). We call $\mathcal{X}^I$ the *integral Lang locus* of X relative to the integral model $(\mathcal{X}, F, \Sigma)$.

The integral Lang locus $\mathcal{X}^I$ is a subvariety of the Lang locus X^{UY} of X.

Lemma 6.2. *Given two integral models $\mathcal{X}_i$ over O_{F_i,Σ_i} $(i = 1, 2)$ for X, one has $\mathcal{X}_1^I = \mathcal{X}_2^I$.*

Proof. For $i = 1, 2$, let $\phi_i : X_{F_i} \otimes_{F_i} \bar{\mathbb{Q}} \to X$ be the isomorphism in the corresponding model. By [EGA IV 3, Cor. 8.8.2.5], there is a common finite extension (F_3, Σ_3) of (F_1, Σ_1) and (F_2, Σ_2), such that there is an O_{F_3,Σ_3}-isomorphism

$$\mathcal{X}_1 \otimes_{O_{F_1,\Sigma_1}} O_{F_3,\Sigma_3} \to \mathcal{X}_2 \otimes_{O_{F_2,\Sigma_2}} O_{F_3,\Sigma_3}$$

extending the isomorphism $\phi_2^{-1}\phi_1 : X_{F_1} \otimes_{F_1} \bar{\mathbb{Q}} \to X_{F_2} \otimes_{F_2} \bar{\mathbb{Q}}$. For every finite extension $(M_1, \Omega_1)/(F_1, \Sigma_1)$, there is a common finite extension (M_2, Ω_2) of (F_3, Σ_3) and (M_1, Ω_1). Then

$$\mathcal{X}_1(O_{M_1,\Omega_1}) \subset \mathcal{X}_1(O_{M_2,\Omega_2}) = \mathcal{X}_2(O_{M_2,\Omega_2}),$$

so $X(\mathcal{X}_1, M_1, \Omega_1) \subset X(\mathcal{X}_2, M_2, \Omega_2)$. Therefore,

$$\overline{X(\mathcal{X}_1, M_1, \Omega_1)}^{>0} \subset \overline{X(\mathcal{X}_2, M_2, \Omega_2)}^{>0} \subset \mathcal{X}_2^I.$$

Hence $\mathcal{X}_1^I \subset \mathcal{X}_2^I$. The other inclusion follows by symmetry. □

By Lemma 6.2, one may use the notation X^I for $\mathcal{X}^I$ and call it the *integral Lang locus* of X. We extend the definition to reducible algebraic varieties as in Section 2.

Remark 6.3. Let X be a proper integral algebraic variety over $\bar{\mathbb{Q}}$. Then there is an integral model $(\mathcal{X}, F, \Sigma)$ for X, such that $\mathcal{X}$ is proper over $O_{F,\Sigma}$. Then by [Poo17, Thm. 3.2.13 (ii)], X^I coincides with the Lang locus X^{UY}.

Definition 6.4. [Ull04, Déf. 2.3] An integral algebraic variety X over $\bar{\mathbb{Q}}$ is *arithmetically hyperbolic* if $X^I = \emptyset$.

An integral algebraic variety X over $\bar{\mathbb{Q}}$ is arithmetically hyperbolic if and only if for one (hence for every by Lemma 6.2) integral model $(\mathcal{X}, F, \Sigma)$, the set of integral points $\mathcal{X}(O_{M,\Omega})$ is finite for every finite extension $(M, \Omega)/(F, \Sigma)$. Thus, [Ull04, Lem. 2.4] follows from Lemma 6.2.

Example 6.5. Let $X = \mathbf{P}^1 \setminus \{0, 1, \infty\} = Y(2)$ be a modular curve over $\bar{\mathbb{Q}}$. Its Baily-Borel compactification is $X^* = \mathbf{P}^1$, and $X^{\mathrm{UY}} = X$. By the Siegel-Mahler theorem (see, *e.g.*, [HS00, Thm. D.8.1]), X is arithmetically hyperbolic.

A complex analytic space is called *Kobayashi hyperbolic*, if its Kobayashi pseudo-distance (in the sense of [Kob98, p.50]) is a metric. Every Kobayashi hyperbolic, complex analytic space is Brody hyperbolic. Conversely, Brody [Bro78, p.213] proves that every compact, Brody hyperbolic complex analytic space is Kobayashi hyperbolic. In view of Remark 6.3, Conjecture 6.6 implies Conjecture 1.1.

Conjecture 6.6 ([Lan91, IX, Conjecture 5.1], [Ull04, Conjecture 2.5])**.** *Let X be a quasi-projective, integral algebraic variety over $\bar{\mathbb{Q}}$. If the complex analytic space $X(\mathbb{C})$ is Kobayashi hyperbolic, then X is arithmetically hyperbolic.*

Fact 6.7 is an evidence of Conjecture 6.6. It relies on Faltings's solution [Fal83, Satz 6] to Shafarevich's conjecture.

Fact 6.7 ([Ull04, Thm. 3.2 (a)])**.** *Let (G, X) be an adjoint Shimura datum of abelian type* (in the sense of [Ull04, p.4118]). *Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Then every irreducible component of $\mathrm{Sh}_K(G, X)_{\bar{\mathbb{Q}}}$ is arithmetically hyperbolic.*

Remark 6.8. By [Moo98b, 2.17], the model over $\bar{\mathbb{Q}}$ of $\mathrm{Sh}_K(G,X)$ defined by Faltings [Fal82] (used in [Ull04, Thm. 3.2 (a)]) is the scalar extension of the canonical model along the field embedding $E(G,X) \to \bar{\mathbb{Q}}$.

We prove that an alternative principle holds for integral points on Shimura varieties, by checking Assumption 2.1. Since an irreducible component of X^I of dimension 0 would be an isolated point, Assumption 2.1 1 holds. Lemma 6.9 verifies Assumptions 2.1 2, 3 and 5.

Lemma 6.9. *Let $f : Z_1 \to Z_2$ be a morphism of integral algebraic varieties over $\bar{\mathbb{Q}}$. If f has finite geometric fibers, then $f(Z_1^I) \subset Z_2^I$.*

Proof. One may choose a number field F, a finite set Σ of places of F containing all the archimedean ones, an integral model $\mathcal{Z}_i$ over $O_{F,\Sigma}$ for Z_i ($i = 1,2$) and a morphism $f' : \mathcal{Z}_1 \to \mathcal{Z}_2$ over $O_{F,\Sigma}$ extending $f : X \to Y$. For every finite extension $(M,\Omega)/(F,\Sigma)$, one has $f'(\mathcal{Z}_1(O_{M,\Omega})) \subset \mathcal{Z}_2(O_{M,\Omega})$. Hence, one obtains

$$f(Z_1(\mathcal{Z}_1,M,\Omega)) \subset Z_2(\mathcal{Z}_2,M,\Omega), \quad f(\overline{Z_1(\mathcal{Z}_1,M,\Omega)}) \subset \overline{Z_2(\mathcal{Z}_2,M,\Omega)}.$$

Let $C \subset \overline{Z_1(\mathcal{Z}_1,M,\Omega)}$ be an irreducible component of positive dimension. Then $f(C)$ is irreducible but not a singleton. (For otherwise, C is a finite set by assumption, which is a contradiction). Hence

$$f(C) \subset \overline{Z_2(\mathcal{Z}_2,M,\Omega)}^{>0} \subset Z_2^I.$$

Therefore, $f(\overline{Z_1(\mathcal{Z}_1,M,\Omega)}^{>0}) \subset Z_2^I$ and $f(Z_1^I) \subset Z_2^I$. □

Corollary 6.10 ([Ull04, Prop. 2.6])**.** *A locally closed subvariety of an arithmetically hyperbolic variety is also arithmetically hyperbolic.*

Proof. It follows from Lemma 6.9. □

Lemma 6.11 verifies Assumption 2.1 4 for integral Lang locus.

Lemma 6.11. *Let X be an integral algebraic variety over $\bar{\mathbb{Q}}$. Then $X^I \subset (X^I)^I$.*

Proof. Take an integral model $(\mathcal{X},F,\Sigma)$ for X. Write $X^I = \cup_{i=1}^n Y_i$ for the irreducible decomposition. For every $1 \leq i \leq n$, endow Y_i with the reduced induced closed subscheme structure. Let $\mathcal{Y}_i$ be the scheme-theoretic image of the composition $Y_i \to X \to \mathcal{X}$. As Y_i is integral, so is $\mathcal{Y}_i$. Let η be the generic point of $\operatorname{Spec} O_{F,\Sigma}$. By [Sta25, Tag 054Z], $\mathcal{Y}_{i,\eta}$ is a reduced scheme. The morphism $Y_i \to \mathcal{Y}_{i,\eta}$ is a closed immersion. By [Sta25, Tag 054V], it is surjective, hence an isomorphism. The morphism $\mathcal{Y}_i \to \operatorname{Spec} O_{F,\Sigma}$ is dominant as its generic fiber Y_i is nonempty. Therefore, $\mathcal{Y}_i$ is an integral model of Y_i relative to (F,Σ).

For every finite extension $(M,\Omega)/(F,\Sigma)$, the Zariski closed subset $\overline{X(\mathcal{X},M,\Omega)}$ of X is the disjoint union of $\overline{X(\mathcal{X},M,\Omega)}^{>0}$ with a finite subset $\{p_1,\ldots,p_t\}$ of

$X(\bar{\mathbb{Q}})$. Consider $x \in \mathcal{X}(O_{M,\Omega})$, *i.e.*, a section $x : \mathrm{Spec}(O_{M,\Omega}) \to \mathcal{X}$ to the structure morphism $\mathcal{X} \to \mathrm{Spec}(O_{M,\Omega})$. If $x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \notin \{p_1, \ldots, p_t\}$, then

$$x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \in \overline{X(\mathcal{X}, M, \Omega)}^{>0} \subset X^I.$$

Thus, there exists an index $1 \leq i \leq n$ with $x|_{\mathrm{Spec}\,\bar{\mathbb{Q}}} \in Y_i$. Since $\mathcal{Y}_i$ is Zariski closed in $\mathcal{X}$, the section x factors through $\mathcal{Y}_i$, *i.e.*, $x \in \mathcal{Y}_i(O_{M,\Omega})$. Therefore,

$$X(\mathcal{X}, M, \Omega) \subset \cup_{i=1}^n Y_i(\mathcal{Y}_i, M, \Omega) \cup \{p_1, \ldots, p_t\}.$$

Then

$$\overline{X(\mathcal{X}, M, \Omega)}^{>0} \subset \cup_{i=1}^n \overline{Y_i(\mathcal{Y}_i, M, \Omega)}^{>0} \subset \cup_{i=1}^n Y_i^I = (X^I)^I,$$

so $X^I \subset (X^I)^I$. □

Lemma 6.12 implies [Ull04, Prop. 2.8].

Lemma 6.12 (Chevalley-Weil). *Let $f : X \to Y$ be an étale cover over $\bar{\mathbb{Q}}$. Then $f(X^I) = Y^I$. In particular, $X^I = X$ (resp. $X^I = \emptyset$) is equivalent to $Y^I = Y$ (resp. $Y^I = \emptyset$).*

Proof. By Lemma 6.9, one has $f(X^I) \subset Y^I$. There is a number field F, a finite set Σ of places of F containing all the archimedean ones, and a finite étale morphism $f' : \mathcal{X} \to \mathcal{Y}$ between integral models over $O_{F,\Sigma}$ extending f.

From the Chevalley-Weil theorem (see, *e.g.*, [Ser97, p.50]), for every finite extension $(M, \Omega)/(F, \Sigma)$, there is a finite extension $(M', \Omega')/(M, \Omega)$ with

$$Y(\mathcal{Y}, M, \Omega) \subset f(X(\mathcal{X}, M', \Omega')).$$

Since zero dimensional schemes are discrete, one has

$$\overline{Y(\mathcal{Y}, M, \Omega)}^{>0} \subset f(\overline{X(\mathcal{X}, M', \Omega')}^{>0}) \subset f(X^I).$$

Hence $Y^I \subset f(X^I)$.

If $X^I = X$, then $Y = f(X) = f(X^I) = Y^I$. Conversely, if $Y^I = Y$, then $\dim X = \dim Y = \dim Y^I \leq \dim X^I$. Since X is irreducible, one has $X^I = X$.

If $X^I = \emptyset$, then $Y^I = f(X^I) = \emptyset$. Conversely, if $Y^I = \emptyset$, then by Assumption 2.1 3, one has $X^I = \emptyset$. □

Theorem 6.13. *The integral Lang locus S^I of a Shimura variety S is either empty or S itself.*

Proof. Take the locus formation $(\cdot)^L := (\cdot)^I$ to be that of integral Lang locus. By Lemmas 6.9 and 6.11, it satisfies Assumption 2.1. By Lemma 6.12, one has $S^I = S^{L_\infty}$. The result follows from Theorem 5.1. □

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