Normality of monodromy group in generic convolution group

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Abstract

On an abelian variety A, sheaf convolution gives a Tannakian formalism for perverse sheaves. Let X be an algebraic variety with generic point η. Let K be a family of perverse sheaves on the abelian scheme $p_X : A \times X \to X$. We show that for a very general character sheaf L_χ on A, the monodromy group of the sheaf $R^0 p_{X*}(K \otimes p_A^* L_\chi)$ on X is normal in the Tannakian group $G(K|_{A_\eta})$ of the perverse sheaf $K|_{A_\eta}$ on A_η.

This result is inspired from and could be compared to two other normality results: In the same setting, the Tannakian group $G(K|_{A_{\bar{\eta}}})$ is normal in $G(K|_{A_\eta})$ (due to Lawrence-Sawin). For a polarizable variation of Hodge structure, outside a meager locus, the connected monodromy group is normal in the derived Mumford-Tate group (due to André).

1 Introduction

Given a family of algebraic varieties, a fundamental question is to understand the constraints on the monodromy groups that can arise. A related theme is to construct families with a prescribed, often "large", monodromy group. A useful tool for such constructions is the theory of perverse sheaves. Katz [Kat88, 11.8] exhibits local systems with monodromy group G_2, whose appearance is explained in [CRDRS22] using convolution of perverse sheaves on elliptic curves. To prove Shafarevich type results, [LS25] and [Jav+25] construct local systems with big monodromy, via convolution of perverse sheaves on abelian varieties. A crucial ingredient is a normality result [LS25, Lemma 2.8] for perverse sheaves from geometry. We give a related normality result for general perverse sheaves.

1.1 Comparison of convolution groups and monodromy groups

Setting 1.1.1. Let k be an algebraically closed field of characteristic 0. Let X be an integral algebraic variety over k with generic point η. Let A be an abelian variety over k. Denote by $p_X : A \times X \to X$ and $p_A : A \times X \to A$ the projections.

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Given a family of smooth subvarieties of A, [LS25, Sec. 4] and [Jav+25] study the associated family of perverse sheaves. Hansen and Scholze [HS23] lay out the foundation of *relative perverse sheaves* (Definition 2.3.2). We show how [HS23] provides a conceptual framework for considering more general perverse sheaves, not necessarily associated with subvarieties of A.

When one studies families of sheaves on abelian varieties, twisting by characters typically simplifies cohomology. A recurring phenomenon is that for "generic" twists, the resulting monodromy turns out to be big. Let $D_c^b(X)$ be the triangulated category of bounded constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X. Let $\pi_1(A)$ be the étale fundamental group of A. For a character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, let L_χ be the induced character sheaf on A. By the generic vanishing theorem [KW15b, Thm. 1.1], a quotient of the abelian category $\mathrm{Perv}(A)$ of perverse sheaves on A is Tannakian under sheaf convolution. Moreover, for every $P \in \mathrm{Perv}(A)$, a *general* character χ gives rise to a fiber functor

$$\omega_\chi : \langle P \rangle \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}, \quad Q \mapsto H^0(A, Q \otimes L_\chi)$$

on the generated Tannakian subcategory $\langle P \rangle$. Here, the set of feasible characters χ depends on P. Let $G_\chi(P)$ be the *Tannakian group* of $\langle P \rangle$ at ω_χ. The isomorphism class $G(P)$ of this algebraic group is independent of χ.

Let $\mathrm{Perv}(A \times X/X)$ be the full subcategory of $D_c^b(A \times X)$ of relative perverse sheaves. For every $K \in \mathrm{Perv}(A \times X/X)$ and every character χ, let

$$\rho_{K,\chi} : \Gamma_\eta \to \mathrm{GL}\left(H^0(A_{\bar{\eta}}, K|_{A_\eta} \otimes L_\chi)\right)$$

be the corresponding Galois representation. Let $\mathrm{Mon}(K, \chi)$ be the monodromy group, i.e., the Zariski closure of the image of $\rho_{K,\chi}$. Theorem 1.1.2 is the main result, which is a structural constraint on the monodromy group. It shows that for a very general twist by a character of $\pi_1(A)$, the resulting monodromy group is a normal subgroup of the convolution group.

Theorem 1.1.2 (Theorem 5.2.1)**.** *In Setting 1.1.1, assume* $\dim A > 0$. *If* $K \in \mathrm{Perv}(A \times X/X)$ *is* semisimple[1] *in* $D_c^b(A \times X)$*, then there exist uncountably many characters* $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$*, such that* $\mathrm{Mon}(K, \chi)$ *is a closed* normal *subgroup of* $G_\chi(K|_{A_\eta})$.

Remark 1.1.3. We consider only constructible sheaves with ℓ-adic coefficients and not $\mathbb{C}$-coefficients, because we use the perverse sheaf $K|_{A_\eta}$ on A_η. For a possible extension to $\mathbb{C}$-coefficients when $k = \mathbb{C}$, see Remark 5.2.7.

A dichotomy type result follows from Theorem 1.1.2 directly.

Corollary 1.1.4 (Corollary 5.2.5)**.** *In the notation of Theorem 1.1.2, there exist uncountably many characters* $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$ *such that either* $\mathrm{Mon}(K, \chi)^\circ$ *is a torus, or* $\mathrm{Mon}(K, \chi)$ *contains a simple factor of* $G_\chi(K|_{A_\eta})^*$.

Theorem 1.1.2 is a formal analog for perverse sheaves of André's theorem, which we recall. The category Hdg of pure $\mathbb{Q}$-Hodge structures is Tannakian. For

[1] in the sense of Definition 2.1.3

$V \in \mathrm{Hdg}$, its Tannakian group is denoted $\mathrm{MT}(V)$. Let X be a complex smooth quasi-projective variety. Let $\mathbb{V}$ be a variation of pure $\mathbb{Q}$-Hodge structure on $X(\mathbb{C})$. For every $x \in X(\mathbb{C})$, let H_x be the identity component of the monodromy group of $\rho_{\mathbb{V},x} : \pi_1(X(\mathbb{C}), x) \to \mathrm{GL}(\mathbb{V}_x)$. Theorem 1.1.5 is a special case of [And92, Thm. 1] stated in the mixed case.

Theorem 1.1.5. *If $\mathbb{V}$ is polarizable, then there is a countable union Z of strict Zariski closed subsets of X, such that for every $x \in X(\mathbb{C}) \setminus Z$, H_x is a normal subgroup of $\mathrm{MT}(\mathbb{V}_x)$.*

Remark 1.1.6. (a) By [PS03, Cor. 13], the category of polarizable variations of $\mathbb{Q}$-Hodge structure on $X(\mathbb{C})$ in Theorem 1.1.5 is semisimple. In Theorem 1.1.2, the semisimplicity hypothesis should be compared to the polarizability assumption, and for its proof, we need semisimplicity to apply the decomposition theorem.

(b) In Theorem 1.1.5, the Mumford-Tate group $\mathrm{MT}(\mathbb{V}_x)$ may change with $x \in X(\mathbb{C})$, while H_x is independent of the choice of x up to isomorphism. By contrast, in Theorem 1.1.2, $G_\chi(K|_{A_\eta})$ is independent of the choice of character χ up to isomorphism, while the monodromy group $\mathrm{Mon}(K, \chi)$ depends on χ. Moreover, André shows $H_x \subset [\mathrm{MT}(\mathbb{V}_x), \mathrm{MT}(\mathbb{V}_x)]$ for very general $x \in X(\mathbb{C})$, while $[G_\chi(K|_{A_\eta}), G_\chi(K|_{A_\eta})]$ may not contain $\mathrm{Mon}(K, \chi)$ for very general characters χ (Example 5.1.2).

1.2 Fixed part theorem

The proof of Theorem 1.1.2 is inspired by that of Theorem 1.1.5. As André [And92, p.10] explains, an important ingredient of Theorem 1.1.5 is the fixed part theorem (see, *e.g.*, [Sch73, Thm. 7.22]). For a subgroup H of a group G, if H is normal, then the H-invariant subspace V^H is G-stable for every representation V of G. Under suitable hypotheses, this necessary condition is also sufficient ([And92, Lemma 1]). Restricted to the pure case, the fixed part theorem proves that the monodromy invariant of a variation of Hodge structure naturally forms a sub variation of Hodge structure. Theorem 1.2.1 is an analog in the setting of convolution on abelian varieties essentially due to Lawrence and Sawin. It is an ingredient in the proof of Theorem 1.1.2, and may be of independent interest.

If one takes $k = \mathbb{C}$ and considers perverse sheaves on $A(\mathbb{C})$ with coefficients in $\mathbb{C}$, then the set $\Pi(A)$ of corresponding characters $\pi_1^{\mathrm{top}}(A(\mathbb{C})) \to \mathbb{C}^*$ is naturally a linear algebraic group over $\mathbb{C}$. For general k, Gabber and Loeser [GL96, Sec. 3.2.2] introduce a scheme $\mathcal{C}(A)_\ell$ over $\bar{\mathbb{Q}}_\ell$ to parameterize pro-ℓ characters of $\pi_1(A)$.

Theorem 1.2.1 (Theorem 4.2.1)**.** *In Setting 1.1.1, assume that X is smooth over k. Let $K \in \mathrm{Perv}(A \times X/X)$ be semisimple in $D_c^b(A \times X)$ and p_X-ULA. Then there is a p_X-ULA direct summand $K^0 \subset K$ in $\mathrm{Perv}(A \times X/X)$ such that for a general pro-ℓ character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, one has*

$$H^0(A_{\bar{\eta}}, K^0|_{A_{\bar{\eta}}} \otimes^L L_\chi) = H^0(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi)^{\Gamma_\eta}.$$

Remark 1.2.2. The universal local acyclicity is reviewed in Definition 2.2.1. If K is induced by a family of subvarieties of A as in [LS25, Sec. 4], then K is a ULA relative perverse sheaf and semisimple in $D_c^b(A \times X)$. The ULA hypothesis in Theorem 1.2.1 is mild, because of the generic universal local acyclicity [SGA 4 1/2, Thm. 2.13, p. 242]: For every morphism $f : S \to T$ of irreducible algebraic varieties and every $K \in D_c^b(S)$, there exists a dense open subscheme $U \subset T$ such that $K|_{f^{-1}(U)}$ is ULA relative to $f^{-1}(U) \to U$.

As we explain in Remark 4.2.2, Theorem 1.2.1 is a generalization of [LS25, Lemma 4.4] stated for perverse sheaves $K_\eta \in \mathrm{Perv}(A_\eta)$ *semisimple of geometric origin* in the sense of [BBD, 6.2.4] which admit no simple constituent from A. In their case, one has $K^0 = 0$. The proof of Theorem 1.2.1 uses the projection $p_A : A \times X \to A$, which restricts our results as well as [LS25] and [Jav+25] to constant abelian schemes.

1.3 Existence of very general characters

We sketch the proof of Theorem 1.1.2. Every representation V of $G(K|_{A_\eta})$ comes from a ULA relative perverse sheaf $K_V \in \mathrm{Perv}(A \times X/X)$. Applying Theorem 1.2.1 to K_V, we get a strict closed subset B_V of $\mathcal{C}(A)_\ell$, such that for every $\chi \notin B_V$, $V^{\mathrm{Mon}(K,\chi)}$ is $G(K|_{A_\eta})$-stable. Then normality holds for all $\chi \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus \cup_V B_V$. As $G(K|_{A_\eta})$ is reductive, it has only countably many representations up to isomorphism. Therefore, as in Theorem 1.1.5, this normality holds for χ outside a *countable* union of strict Zariski closed subsets of $\mathcal{C}(A)_\ell$. Contrary to $\Pi(A)$, whenever $\dim A > 0$, $\mathcal{C}(A)_\ell$ is not locally of finite type nor naturally a group scheme over $\bar{\mathbb{Q}}_\ell$. This makes it unclear *a priori* whether the complement of the countable union contains any rational point. We prove that $\pi_1(A)$ has uncountably many *very general* pro-ℓ characters, which in particular guarantees the existence of the desired characters.

Theorem 1.3.1 (Theorem 3.2.11). *Assume* $\dim A > 0$. *Then for every sequence* $(Z_i)_{i \geq 1}$ *of strict closed subsets of* $\mathcal{C}(A)_\ell$*, the set* $\mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus \cup_{i \geq 1} Z_i$ *is uncountable.*

Assume that K is induced by a family of hypersurfaces of A. Then $G(K|_{A_{\bar\eta}})$ is a normal subgroup of $G(K|_{A_\eta})$ ([LS25, Lemma 2.8]). Using this normality and supposing that the derived subgroup $G(K|_{A_{\bar\eta}})^*$ of $G(K|_{A_{\bar\eta}})^\circ$ is *simple*, Lawrence and Sawin show that there is a *finite* list $\{V_i\}_{i=1}^n$ of representations of $G(K|_{A_\eta})$, such that for every reductive subgroup H of $G(K|_{A_\eta})$, if H has no invariants on any V_i, then $H \supset G(K|_{A_{\bar\eta}})^*$. From this, they [LS25, Thm. 4.7] establish $\mathrm{Mon}(K, \chi) \supset G_\chi(K|_{A_{\bar\eta}})^*$ for every character χ outside a single strict closed subset of $\mathcal{C}(A)_\ell$. By contrast, Theorem 1.1.2 removes the simplicity hypothesis on $G(K|_{A_{\bar\eta}})^*$, but needs to exclude a countable union of strict closed subsets of $\mathcal{C}(A)_\ell$. For an application of Corollary 1.1.4 with non-simple $G(K|_{A_{\bar\eta}})^*$, see Example 5.2.6.

Plan of the paper

We recollect universal local acyclicity and relative perverse sheaves in Section 2. In Section 3, we review Gabber and Loeser's construction of a scheme $\mathcal{C}_\ell$, called a cotorus, which parameterizes ℓ-adic characters of $\pi_1(A)$. For this scheme, we prove that the a countable intersection of nonempty open subsets has uncountably many closed points. In Section 4, we recall Krämer and Weissauer's generic vanishing theorem, expressed in terms of cotori. Then we apply it to prove Theorem 1.2.1. In Section 5, we show that for a very general character χ, the monodromy group $\mathrm{Mon}(K,\chi)$ is normal in $G(K|_{A_\eta})$. Together with Theorem 3.2.11, it finishes the proof of Theorem 1.1.2.

Notation and conventions

An object of an abelian category is *semisimple* if it is the direct sum of finitely many simple objects. An abelian category is *semisimple* if every object is semisimple. For a field k, its absolute Galois group is denoted by $\Gamma_k = \Gamma_{\mathrm{Spec}\, k}$. An *algebraic variety* over k means a scheme separated of finite type over k. Given an algebraic group G over an algebraically closed field C, let G° be its identity component, and $G^* := [G^\circ, G^\circ]$ be its derived subgroup. A linear algebraic group is *reductive*, if its identity component is reductive (in the sense of [Mil17, 6.46, p.135]). For a topological group, its $\bar{\mathbb{Q}}_\ell$-characters are assumed to be continuous.

2 Recollections on constructible sheaves

In Section 2, we review universally locally acyclic, relative perverse sheaves. The decomposition theorem [BBD, Thm. 6.2.5] plays a significant role in the proof of Theorem 1.1.2.

Let k be a field. Fix an algebraic closure $\bar{k}$ of k. Let ℓ be a prime number invertible in k. For every algebraic variety X over k, denote by $D^b_c(X)$ the triangulated category of complexes of $\bar{\mathbb{Q}}_\ell$-sheaves on X with bounded constructible cohomologies defined in [BBD, p.74]. Let $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$ be the Verdier duality functor. The heart of the standard t-structure on $D^b_c(X)$ is denoted by $\mathrm{Cons}(X)$, which is the category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X. For every integer n, let $\mathcal{H}^n : D^b_c(X) \to \mathrm{Cons}(X)$ be the functor taking the n-th cohomology sheaf. For $F \in \mathrm{Cons}(X)$, set $\mathrm{Supp}\, F := \{x \in X | F_x \neq 0\}$ to be its support. Then $\mathrm{Supp}\, F$ is a quasi-constructible subset of X in the sense of [EGA IV 3, 10.1.1]. Let $\mathrm{Loc}(X) \subset \mathrm{Cons}(X)$ be the full subcategory of $\bar{\mathbb{Q}}_\ell$-lisse sheaves on X.

For every subset $S \subset X$, let $\bar{S}$ be its Zariski closure. Let ${}^pD^{\leq 0}(X) \subset D^b_c(X)$ be the full subcategory of objects K with $\dim \overline{\mathrm{Supp}\, \mathcal{H}^n K} \leq -n$ for all integers n. Let ${}^pD^{\geq 0}(X) \subset D^b_c(X)$ be the full subcategory of objects K with $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. Then $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ defines the (absolute) perverse t-structure on $D^b_c(X)$, whose heart $\mathrm{Perv}(X)$ is the category of perverse sheaves on X. The functor $\mathbb{D}_X$ interchanges ${}^pD^{\leq 0}(X)$ and ${}^pD^{\geq 0}(X)$. For every integer n, let ${}^p\mathcal{H}^n$:

$D^b_c(X) \to \mathrm{Perv}(X)$ be the functor taking the n-th perverse cohomology sheaf. For a morphism $f : X' \to X$ of schemes and $K \in D^b_c(X)$, set $K|_{X'} := f^*K$.

2.1 Constructible sheaves and perverse sheaves

Fact 2.1.1 is a projection formula and can be found in [FK88, Rk. (2), p.100].

Fact 2.1.1. *Let $f : X \to Y$ be a morphism of algebraic varieties over $\bar{k}$. Let $L \in D^b_c(Y)$ be an object with $\mathcal{H}^n L \in \mathrm{Loc}(X)$ for all integers n. Then there is a natural isomorphism $(Rf_*-)\otimes^L L \to Rf_*(-\otimes^L f^*L)$ of functors $D^b_c(X) \to D^b_c(Y)$.*

Fact 2.1.2 can be found in [FK88, Prop. 12.10].

Fact 2.1.2. *Let X be an algebraic variety over k. For every $F \in \mathrm{Cons}(X)$, there is a nonempty Zariski open subset $U \subset X$ with $F|_U \in \mathrm{Loc}(U)$.*

Definition 2.1.3 originates from [BC18, Def. 78].

Definition 2.1.3. Let X be an algebraic variety over k. An object $K \in D^b_c(X)$ is called *semisimple* if it is isomorphic to a finite direct sum of degree shifts of semisimple objects of $\mathrm{Perv}(X)$.

If $K \in D^b_c(X)$ is semisimple, then K is the direct sum of shifts of finitely many simple perverse sheaves, i.e., it is isomorphic to $\oplus_{n\in\mathbb{Z}} {}^p\mathcal{H}^n(K)[-n]$ in $D^b_c(X)$, and each ${}^p\mathcal{H}^n(K)$ is a semisimple object of $\mathrm{Perv}(X)$. A degree shift of a semisimple object of $D^b_c(X)$ is still semisimple. Lemma 2.1.4 is used in the proof of Lemma 5.1.1.

Lemma 2.1.4. *Let X be an algebraic variety over k. Let $U \subset X$ be an open subset of X. Then the functor $(-)|_U : \mathrm{Perv}(X) \to \mathrm{Perv}(U)$ sends every simple object of $\mathrm{Perv}(X)$ to a simple or zero object of $\mathrm{Perv}(U)$. In particular, the functor $(-)|_U : D^b_c(X) \to D^b_c(U)$ preserves semisimplicity.*

Proof. Let K be a simple object of $\mathrm{Perv}(X)$. By [BBD, Thm. 4.3.1 (ii)], there is an irreducible, locally closed and geometrically smooth subvariety $j : V \to X$ and a simple $\bar{\mathbb{Q}}_\ell$-lisse sheaf L on V, such that K is isomorphic to $j_{!*}L[\dim V]$. If V is disjoint from U, then $K|_U = 0$. Now assume that V intersects U. Take a geometric point $\bar{x}$ on $V \cap U$. From [SGA 1, V, Prop. 8.2], as V is normal, the morphism $\pi_1(U \cap V, \bar{x}) \to \pi_1(V, \bar{x})$ is surjective. Thus, the composite representation

$$\pi_1(U \cap V, \bar{x}) \to \pi_1(V, \bar{x}) \to \mathrm{GL}(L_{\bar{x}})$$

is also simple, *i.e.*, the $\bar{\mathbb{Q}}_\ell$-lisse sheaf $L|_{U\cap V}$ is simple. Let $h : U \cap V \to U$ be the immersion. Then $K|_U$ is isomorphic to $h_{!*}L|_{U\cap V}[\dim(U \cap V)]$, hence simple in $\mathrm{Perv}(U)$. $\square$

Theorem 2.1.5 is essentially Kashiwara's conjecture for semisimple perverse sheaves formulated in [Kas98, Sec. 1]; see also [Dri01, Sec. 1.2, 1]. Drinfeld [Dri01] reduces it to de Jong's conjecture, which in turn is proved in [BK06] and

[Gai07]. Theorem 2.1.5 (a) with $k = \mathbb{C}$ follows from Kashiwara's conjecture and the paragraph following [BBD, Thm. 6.2.5]. Part (b) can be deduced from Part (a) as in [BBD, Cor. 6.2.8]. The case of general k follows *via* Lemma 2.1.6; see also [Dri01, Sec. 1.7].

Theorem 2.1.5. *Let k be an algebraically closed field of characteristic 0. Let $f : X \to Y$ be a proper morphism of algebraic varieties over k. Let K be a semisimple object of $D_c^b(X)$.*

(a) (Decomposition theorem) *Then Rf_*K is a semisimple object of $D_c^b(Y)$.*

(b) (Global invariant cycle theorem) *Let i be an integer. By Fact 2.1.2, there is a nonempty connected open subset $V \subset Y$ such that $\mathcal{H}^i Rf_*K|_V$ is a lisse sheaf. Then for every $y \in V(k)$, the canonical map*

$$H^i(X, K) \to H^i(X_y, K|_{X_y})^{\pi_1(V,y)}$$

is surjective.

Lemma 2.1.6. *Let E/F be an extension of algebraically closed fields of characteristic 0. Let X be an algebraic variety over F. Then an object K of $\mathrm{Perv}(X)$ is simple (resp. semisimple) if and only if $K|_{X_E}$ is simple (resp. semisimple) in $\mathrm{Perv}(X_E)$.*

Proof. By [BBD, Thm. 4.3.1 (ii)] and [Esn17, Prop. 5.3], K is simple if and only if $K|_{X_E}$ is simple. Thus, if K is semisimple, then so is $K|_{X_E}$. Conversely, assume that $K|_{X_E}$ is semisimple. For every subobject $P \subset K$ in $\mathrm{Perv}(X)$, there is a morphism $r : K|_{X_E} \to P|_{X_E}$ in $\mathrm{Perv}(X_E)$ with $r|_{(P|_{X_E})} = \mathrm{Id}_{P|_{X_E}}$. By [Jav+25, Lem. A.1], there is a morphism $r' : K \to P$ in $\mathrm{Perv}(X)$ with $r'|_{(K|_{X_E})} = r$ and $r'|_P = \mathrm{Id}_P$. Thus, P admits a direct complement in K. By Lemma 2.1.7 (b), K is semisimple. $\square$

Lemma 2.1.7. *Let $\mathcal{A}$ be an abelian category. Let $X \in \mathcal{A}$ be a Noetherian and Artinian object.*

(a) *Let Y be a simple subquotient of X. Then there is a composition series of X with one graded piece isomorphic to Y. In particular, up to isomorphism X has only finitely many simple subquotients.*

(b) *If every semisimple subobject of X admits a direct complement, then X is semisimple.*

Proof.

(a) There is a subobject $i : X_0 \subset X$ and a quotient $q : X_0 \to Y$ in $\mathcal{A}$. Let $N = \ker(q)$. Both N and X/X_0 are Noetherian and Artinian. They admit composition series. A composition series of X/X_0 is equivalent to a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ by subobjects such that X_i/X_{i-1} is simple for every $1 \le i \le n$. This filtration and every composition series of N glue to a composition series of X with a step $N \subset X_0$, whose factor is isomorphic to Y. By the Jordan-Hölder lemma, up to isomorphism Y has only finitely many choices.

(b) One may assume $X \neq 0$. Let $\mathcal{P}$ be the family of nonzero semisimple subobjects of X. As X is Artinian, it has a simple subobject, so $\mathcal{P}$ is nonempty. Since X is Noetherian, $\mathcal{P}$ has a maximal element $i : X_0 \hookrightarrow X$. As X_0 is semisimple, there is a subobject $F \subset X$ with $X_0 \oplus F = X$. Then $F = 0$. (Otherwise, F has a simple subobject F_0. Then $X_0 \oplus F_0 \in \mathcal{P}$ is strictly larger than X_0, which is a contradiction.) Therefore, $i : X_0 \to X$ is an isomorphism, and X is semisimple.

□

Remark 2.1.8. In a Noetherian and Artinian abelian category, an object may have infinitely many pairwise non isomorphic (non semisimple) subobjects. This should be compared with Lemma 2.1.7 (a).

Lemma 2.1.9. *Let X be an algebraic variety over k. Let L be a $\bar{\mathbb{Q}}_\ell$-lisse sheaf of rank 1 on X. Then $(\cdot) \otimes^L L : D_c^b(X) \to D_c^b(X)$ is an equivalence of categories. It is t-exact for the perverse t-structures.*

Proof. Let L^{-1} be the lisse sheaf dual to L. By associativity of the derived tensor product $\otimes^L$, the pair of functors $(- \otimes^L L, - \otimes^L L^{-1})$ is an equivalence.

(a) Right t-exactness: The functor is t-exact for the standard t-structures. Thus, for every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\mathcal{H}^n(K \otimes^L L) = \mathcal{H}^n(K) \otimes^L L$. Therefore, one has $\operatorname{Supp} \mathcal{H}^n(K \otimes^L L) = \operatorname{Supp} \mathcal{H}^n(K)$. It implies $K \otimes^L L \in {}^pD^{\leq 0}(X)$.

(b) Left t-exactness: By Part (a), for every $K \in {}^pD^{\geq 0}(K)$, one has $L^{-1} \otimes^L \mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. By [KW01, II, Cor. 7.5 f)], one has isomorphisms

$$\mathbb{D}_X(K \otimes^L L) \to \mathrm{R}\mathcal{H}\mathrm{om}(L, \mathbb{D}_X K) \to L^{-1} \otimes^L \mathbb{D}_X K$$

in $D_c^b(X)$. Therefore, one gets $K \otimes^L L \in {}^pD^{\geq 0}(X)$.

□

2.2 Universal local acyclicity

In Section 2.2, all schemes are assumed to be quasi-compact and quasi-separated. For a scheme X and a geometric point $\bar{x}$ on X, denote by $O_{X,\bar{x}}^{\mathrm{sh}}$ the strict henselization of $O_{X,\bar{x}}$. Set $X_{(\bar{x})} := \operatorname{Spec} O_{X,\bar{x}}^{\mathrm{sh}}$.

Let $f : X \to S$ be a separated morphism of finite presentation of schemes over $\mathbb{Z}[1/\ell]$. We recall the definition of universal local acyclicity; see, *e.g.*, [Bar24, Definition 1.2].

Definition 2.2.1. Let K be an object of $D_c^b(X)$.

- If for every geometric point $\bar{x}$ on X and every geometric point $\bar{t}$ on $S_{(\bar{s})}$ with $\bar{s} = f(\bar{x})$, the canonical morphism

$$R\Gamma(X_{(\bar{x})}, K) \to R\Gamma(X_{(\bar{x})} \times_{S_{(\bar{s})}} \bar{t}, K)$$

is an isomorphism, then K is called *f-locally acyclic*.

- If for every morphism $S' \to S$ of schemes, in notation of the cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S, \end{tikzcd} \tag{1}$$

g'^*K is f'-locally acyclic, then K is called *f-universally locally acyclic* (f-ULA). Let $D^{\mathrm{ULA}}(X/S) \subset D_c^b(X)$ be the full subcategory of f-ULA objects.

By [HS23, Thm. 4.4], an object $K \in D_c^b(X)$ is f-ULA if and only if K is universally locally acyclic in the sense of [HS23, Def. 3.2]. Thus, the notation $D^{\mathrm{ULA}}(X/S)$ agrees with that in [HS23]. It is a triangulated subcategory of $D_c^b(X)$.

Fact 2.2.2.

(a) *If $S = \operatorname{Spec} k$, then $D^{\mathrm{ULA}}(X/k) = D_c^b(X)$.*

(b) *If $f : X \to S$ is an isomorphism, then $D^{\mathrm{ULA}}(X/S) \subset D_c^b(X)$ is the full subcategory of objects whose cohomology sheaves are lisse.*

(c) *Let $g : S' \to S$ be a morphism of schemes over $\mathbb{Z}[1/\ell]$. Then in the notation of* (1)*, the functor $g'^* : D_c^b(X) \to D_c^b(X')$ restricts to a functor $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X'/S')$.*

(d) *Let $f' : Y \to S$ be another separated morphism of finite presentation of schemes over $\mathbb{Z}[1/\ell]$. Let $h : X \to Y$ be a morphism of schemes over S. If h is smooth (resp. proper), then the functor $h^* : D_c^b(Y) \to D_c^b(X)$ (resp. $Rh_* : D_c^b(X) \to D_c^b(Y)$) restricts to a functor $D^{\mathrm{ULA}}(Y/S) \to D^{\mathrm{ULA}}(X/S)$ (resp. $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(Y/S)$).*

(e) *Let $g : S \to T$ be a smooth morphism of schemes over $\mathbb{Z}[1/\ell]$. Then $D^{\mathrm{ULA}}(X/S) \subset D^{\mathrm{ULA}}(X/T)$.*

(f) *For $i = 1, 2$, let $f_i : X_i \to S$ be a separated morphism of finite presentation of schemes over $\mathbb{Z}[1/\ell]$, and let $K_i \in D^{\mathrm{ULA}}(X_i/S)$. Then $K_1 \boxtimes_S K_2 \in D^{\mathrm{ULA}}(X_1 \times_S X_2/S)$.*

Results listed in Fact 2.2.2 can be found in [Bar24, Lem. 3.7 (ii)], [Bar24, Lem. 3.7 (i)], [HS23, Prop. 3.4 (i)], [Bar24, Lemma 3.6 (i)] in the smooth case and [Ric14, Lem. 3.15] in th proper case, [HS23, p.643] and [Zhu17, Thm. A.2.5 (4)]. We show that for a ULA complex, if its restriction to the generic fiber vanishes, then it is zero.

Lemma 2.2.3. *Assume that S is Noetherian, irreducible with generic point η. Let $K \in D^{\mathrm{ULA}}(X/S)$. If $K|_{X_{\bar{\eta}}} = 0$ in $D_c^b(X_{\bar{\eta}})$, then $K = 0$.*

Proof. It suffices to prove that for every geometric point $\bar{s}$ on S, one has $K|_{X_{\bar{s}}} = 0$ in $D^b_c(X_{\bar{s}})$. By [EGA II, Prop. 7.1.9], as S is Noetherian, there is a discrete valuation ring R and a separated morphism $g : \mathrm{Spec}(R) = S' \to S$, sending the generic (resp. closed) point ξ (resp. r) of S' to η (resp. s). Let $i : R \to R^h$ be the henselization of R. By [Sta26, Tag 0AP3], R^h is a discrete valuation ring. From [Mil80, I, Exercise 4.9], the local morphism $i : R \to R^h$ is injective. Then $i^* : \mathrm{Spec}(R^h) \to S'$ preserves the generic (resp. closed) point. Replacing R by R^h, one may assume further that R is henselian.

Let $X' = X \times_S S'$. Consider the following cartesian squares

$$\begin{tikzcd}
X'_{\bar{r}} \arrow[r, "\bar{i}"] \arrow[d] \arrow[dr, phantom, "\square"] & X'_{(\bar{r})} \arrow[d] \arrow[dr, phantom, "\square"] & X'_{\bar{\xi}} \arrow[l, "\bar{j}"'] \arrow[d] \\
\bar{r} \arrow[r] & S'_{(\bar{r})} & \bar{\xi}, \arrow[l]
\end{tikzcd}$$

where every vertical morphism is a base change of $f : X \to S$. Let $R\Phi : D^+(X') \to D^+(X'_{\bar{r}})$ be the vanishing cycle functor. Let $R\Psi : D^+(X') \to D^+(X'_{\bar{r}})$ be the nearby cycle functor. In the notation of (1), set $K' := g'^*K \in D^b_c(X')$. By definition, one has $R\Psi(K') = \bar{i}^* R\bar{j}_*(K'|_{X'_{\bar{\xi}}})$. From [Ill06, (1.1.3)], as R is henselian, there is a natural exact triangle

$$K'|_{X'_{\bar{r}}} \to R\Psi(K') \to R\Phi(K') \xrightarrow{+1}$$

in $D^+(X'_{\bar{r}})$. Since $K'|_{X'_{\bar{\xi}}}$ is a pullback of $K|_{X_{\bar{\eta}}} = 0$, one has $K'|_{X'_{\bar{\xi}}} = 0$ and hence $R\Psi(K') = 0$. By [Ill06, Cor. 3.5], the universal local acyclicity of K implies $R\Phi(K') = 0$. Therefore, one gets $K'|_{X'_{\bar{r}}} = 0$.

By [Jav+25, Lem. A.1], since $K'|_{X'_{\bar{r}}}$ is the pullback of $K|_{X_{\bar{s}}}$ under the field extension $k(\bar{r})/k(\bar{s})$, one gets $K|_{X_{\bar{s}}} = 0$. □

2.3 Relative perverse sheaves

Let $f : X \to S$ be a morphism of algebraic varieties over a field k. In particular, f is separated and of finite presentation. As S is "bon" in the sense of [KL85, (1.0)], the relative dualizing complex $K_{X/S} := Rf^!\bar{\mathbb{Q}}_\ell \in D^b_c(X)$ exists. The functor

$$\mathbb{D}_{X/S}(-) = \mathrm{R}\mathcal{H}\mathrm{om}(-, K_{X/S}) : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$$

is called the *relative Verdier dual*. By [KL85, (1.1.5)], as S is "bon", there is a canonical morphism of functors $\mathrm{Id}_{D^b_c(X)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$.

Theorem 2.3.1 is stated for ∞-categories in [HS23], but holds for the associated triangulated categories (described in [HRS23, Lem. 7.9]) by [HS23, Footnote 1].

Theorem 2.3.1.

(a) *There is a unique t-structure* $({}^{p/S}D^{\leq 0}(X/S), {}^{p/S}D^{\geq 0}(X/S))$ *on* $D^b_c(X)$, *called the* relative perverse t-structure, *with the following property: An object* $K \in D^b_c(X)$ *lies in* ${}^{p/S}D^{\leq 0}(X/S)$ *(resp.* ${}^{p/S}D^{\geq 0}(X/S)$*) if and only if for every geometric point* $\bar{s} \to S$, *the restriction* $K|_{X_{\bar{s}}}$ *lies in* ${}^{p}D^{\leq 0}(X_{\bar{s}})$ *(resp.* ${}^{p}D^{\geq 0}(X_{\bar{s}})$*). In particular, for every* $s \in S$, *the functor* $(-)|_{X_s} : D^b_c(X) \to D^b_c(X_s)$ *is t-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse t-structure.*

(b) *The relative perverse t-structure on* $D^b_c(X)$ *restricts to a t-structure*

$$({}^{p/S}D^{\mathrm{ULA}, \leq 0}(X/S), {}^{p/S}D^{\mathrm{ULA}, \geq 0}(X/S))$$

on $D^{\mathrm{ULA}}(X/S)$.

(c) *The functor* $\mathbb{D}_{X/S}$ *preserves* $D^{\mathrm{ULA}}(X/S)$, *and the morphism* $\mathrm{Id}_{D^{\mathrm{ULA}}(X/S)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ *of functors* $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)$ *is an isomorphism. The formation of* $\mathbb{D}_{X/S} : D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)^{\mathrm{op}}$ *commutes with any base change in* S, *so* $\mathbb{D}_{X/S}$ *exchanges* ${}^{p/S}D^{\mathrm{ULA}, \leq 0}(X/S)$ *with* ${}^{p/S}D^{\mathrm{ULA}, \geq 0}(X/S)$.

Results in Theorem 2.3.1 can be found in [HS23, Theorems 1.1 and 1.9, Proposition 3.4].

Definition 2.3.2. Let $\mathrm{Perv}(X/S)$ and $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ be the heart of the relative perverse t-structure on $D^b_c(X)$ and $D^{\mathrm{ULA}}(X/S)$ respectively.

Then $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is an abelian subcategory of $\mathrm{Perv}(X/S)$. By Theorem 2.3.1 (a), an object $K \in D^b_c(X)$ lies in $\mathrm{Perv}(X/S)$ if and only if for every geometric point $\bar{s} \to S$, one has $K|_{X_{\bar{s}}} \in \mathrm{Perv}(X_{\bar{s}})$.

Example 2.3.3 (a) and (c) can be found in [HS23, p.632] and [Bar24, Lem. 3.7 (ii)] respectively.

Example 2.3.3.

(a) If $S = \mathrm{Spec}(k)$, then $\mathrm{Perv}(X/k) = \mathrm{Perv}(X)$.

(b) If $f : X \to S$ is universally injective, then $\mathrm{Perv}(X/S) = \mathrm{Cons}(X)$.

(c) If $f : X \to S$ is smooth of relative dimension r, then the functor $(-)[r] : \mathrm{Loc}(X) \to D^b_c(X)$ factors through $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Example 2.3.4. Let $i : Y \to X$ be a closed immersion of schemes over S. Assume that $Y \to S$ is smooth of relative dimension d and with geometrically connected fibers. If L is a $\bar{\mathbb{Q}}_\ell$-lisse sheaf on Y, then $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Indeed, by Fact 2.2.2 (b), one has $L \in D^{\mathrm{ULA}}(Y/Y)$. From the smoothness of $Y \to S$ and Fact 2.2.2 (e), one has $L \in D^{\mathrm{ULA}}(Y/S)$. Using the properness of $i : Y \to X$ and Fact 2.2.2 (d), one has $i_*L[d] \in D^{\mathrm{ULA}}(X/S)$. For every geometric point $\bar{s} \to S$, let $i_{\bar{s}} : Y_{\bar{s}} \to X_{\bar{s}}$ be the base change of i along the morphism $X_{\bar{s}} \to X$. By the proper base change theorem, one has $(i_*L[d])|_{X_{\bar{s}}} = (i_{\bar{s}})_*(L|_{Y_{\bar{s}}})[d]$ which is in $\mathrm{Perv}(X_{\bar{s}})$. Therefore, $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Theorem 2.3.5 can be found in [HS23, Thm. 1.10 (ii)].

Theorem 2.3.5. *Assume that S is geometrically unibranch and irreducible with generic point η. Then the functor*

$$(-)|_{X_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X_\eta)$$

is exact and fully faithful, and its essential image is stable under subquotients.

For every integer j, let ${}^{p/S}\mathcal{H}^j : D_c^b(X) \to \mathrm{Perv}(X/S)$ be the j-th cohomology functor associated with the relative perverse t-structure.

Lemma 2.3.6. *Suppose that S is smooth over k and irreducible with generic point η. Assume that $K \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is semisimple in $D_c^b(X)$. Then $K|_{X_\eta}$ and $K|_{X_{\bar\eta}}$ are semisimple in $\mathrm{Perv}(X_\eta)$ and $\mathrm{Perv}(X_{\bar\eta})$ respectively.*

Proof. By Theorem 2.3.5, for every subobject $M \subset K|_{X_\eta}$ in $\mathrm{Perv}(X_\eta)$, there is a subobject $K' \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ with $K'|_{X_\eta} \cong M$. By Lemma 2.3.7 and smoothness of S, the morphism $K'[\dim S] \to K[\dim S]$ is a monomorphism in $\mathrm{Perv}(X)$. Because K is semisimple in $D_c^b(X)$, its shift $K[\dim S]$ is semisimple in $\mathrm{Perv}(X)$. Thus, there is a subobject $N \subset K[\dim S]$ in $\mathrm{Perv}(X)$ with $K[\dim S] = (K'[\dim S]) \oplus N$. Then $K = K' \oplus (N[-\dim S])$ in $D_c^b(X)$. For every nonzero integer j, one has

$$0 = {}^{p/S}\mathcal{H}^j(K) = 0 \oplus {}^{p/S}\mathcal{H}^j(N[-\dim S])$$

in $\mathrm{Perv}(X/S)$. Hence ${}^{p/S}\mathcal{H}^j(N[-\dim S]) = 0$ and $N[-\dim S] \in \mathrm{Perv}(X/S)$. Consequently, $K|_{X_\eta} = M \oplus (N|_{X_\eta}[-\dim S])$ in $\mathrm{Perv}(X_\eta)$. By [BBD, Thm. 4.3.1 (i)], the abelian category $\mathrm{Perv}(X_\eta)$ is Noetherian and Artinian. From Lemma 2.1.7 (b), as every subobject of $K|_{X_\eta}$ in $\mathrm{Perv}(X_\eta)$ admits a direct complement, $K|_{X_\eta}$ is semisimple. By [CL25, 2.2.1.1], $K|_{X_{\bar\eta}}$ is also semisimple. $\square$

Lemma 2.3.7 is stated without proof for regular schemes S in [HS23, p.636].

Lemma 2.3.7. *Assume that S is smooth over k of equidimension d. Then the shifted inclusion*

$$\begin{aligned}(-)[d] \colon (D^{\mathrm{ULA}}(X/S), {}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S), {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S))\\ \to (D_c^b(X), {}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))\end{aligned} \tag{2}$$

is t-exact. In particular, it restricts to an exact functor

$$(-)[d] : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X),$$

and a t-exact equivalence

$$\begin{aligned}(-)[d] \colon (D^{\mathrm{ULA}}(X/S), {}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S), {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S))\\ \to (D^{\mathrm{ULA}}(X/S), {}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X)).\end{aligned}$$

Proof.

(a) We prove that the functor

$$(-)[d] : (D_c^b(X), {}^{p/S}D^{\leq 0}(X/S), {}^{p/S}D^{\geq 0}(X/S)) \to (D_c^b(X), {}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$$

is right t-exact. For every geometric point $\bar{s}$ on S, the functor $(-)|_{X_{\bar{s}}}$: $D_c^b(X) \to D_c^b(X_{\bar{s}})$ is t-exact for the standard t-structures. Then for every integer n and every $K \in {}^{p/S}D^{\leq 0}(X/S)$, one has $\mathcal{H}^n(K[d])|_{X_{\bar{s}}} = \mathcal{H}^{n+d}(K|_{X_{\bar{s}}})$ and hence

$$X_{\bar{s}} \cap \operatorname{Supp} \mathcal{H}^n(K[d]) = \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}).$$

As $K|_{X_{\bar{s}}} \in {}^pD^{\leq 0}(X_{\bar{s}})$, one has $\dim \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}) \leq -n-d$. By Lemma 2.3.8 (c), one has

$$\dim \operatorname{Supp} \mathcal{H}^n(K[d]) \leq -n.$$

From Lemma 2.3.8 (a), the Zariski closure of $\operatorname{Supp} \mathcal{H}^n(K[d])$ in X has dimension at most $-n$. Hence, one has $K[d] \in {}^pD^{\leq 0}(X)$.

(b) We prove that (2) is left t-exact. By the proof of [Bar24, Lemma 3.12], as S is smooth, for every $K \in {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S)$, one has

$$\mathbb{D}_X(K[d]) = (\mathbb{D}_{X/S}K)[d](d)$$

in $D_c^b(X)$. From Theorem 2.3.1 (c), one has $\mathbb{D}_{X/S}K \in {}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S)$. By Part (a), one has $(\mathbb{D}_{X/S}K)[d] \in {}^pD^{\leq 0}(X)$ and hence $K[d] \in {}^pD^{\geq 0}(X)$.

□

By convention, the dimension of an empty space is $-\infty$.

Lemma 2.3.8. *Let X be a scheme of finite type over a field F. Let C be a quasi-constructible subset of X.*

(a) *Then* $\dim C = \dim \bar{C}$.

(b) *Let* $\{B_i\}_{i=1}^n$ *be finitely many locally closed subsets of* X. *Set* $B = \cup_{i=1}^n B_i$. *Then* $\dim B = \max_{i=1}^n \dim B_i$.

Let $f : X \to Y$ be a morphism between schemes of finite type over F.

(c) *Let* $n \geq 0$ *be an integer such that* $\dim(C \cap f^{-1}(y)) \leq n$ *for every* $y \in Y$. *Then* $\dim C \leq \dim Y + n$.

(d) *Assume that* Y *is irreducible with generic point* η. *Then* $\dim Y + \dim(C \cap X_\eta) \leq \dim C$.

Proof.

(a) As X is a Noetherian scheme, the topological subspace C is Noetherian. Therefore, C is the union of finitely many irreducible components. Thus, one may assume further that C is nonempty and irreducible. Then the reduced induced closed subscheme $\bar{C}$ of X is integral and of finite type over F. By [Bor91, AG. Prop. 1.3], as C is quasi-constructible, it contains a nonempty open subset of $\bar{C}$. By [Har77, II, Exercise 3.20 (e)], one has $\dim C = \dim \bar{C}$.

(b) For every $1 \leq i \leq n$, since $B_i \subset B$, one has $\dim B_i \leq \dim B$. Then $\max_i \dim B_i \leq \dim B$. By Part (a), as every B_i is quasi-constructible in X, one has $\dim B_i = \dim \overline{B_i}$. As $\{\overline{B_i}\}_{i=1}^n$ is a finite closed cover of $\bar{B}$, one gets $\dim B \leq \dim \bar{B} = \max_i \dim \overline{B_i} = \max_i \dim B_i$.

(c) By Part (b), one may assume that C is locally closed in X. Taking irreducible components, one may assume further that C is irreducible. Let Z be the Zariski closure of $f(C)$ in Y. Then Z is irreducible. With reduced induced subscheme structures, one views C and Z as integral schemes of finite type over F. From [Har77, II, Exercise 3.11 (d)], $f : X \to Y$ induces a dominant morphism $g : C \to Z$ over F. Then for every $y \in f(C) = g(C)$, one has

$$n \geq \dim C \cap f^{-1}(y) = \dim g^{-1}(y) \overset{\text{(a)}}{\geq} \dim C - \dim Z,$$

where (a) uses [Har77, II, Exercise 3.22 (b)]. Hence $\dim C \leq \dim Z + n \leq \dim Y + n$.

(d) The statement is topological, so one may assume that Y is reduced. As in the proof of Part (c), one may assume that C is an irreducible, locally closed subset of X and view C as an integral scheme of finite type over F. One may assume that $C \cap X_\eta$ is nonempty. As C_η is homeomorphic to $C \cap X_\eta$, the morphism $C \to Y$ induced by f is dominant. By [Har77, II, Exercise 3.22 (c)], as Y is integral, one has $\dim C \cap X_\eta = \dim C_\eta = \dim C - \dim Y$.

□

Lemma 2.3.9. *Assume that S is irreducible with generic point η and geometrically reduced. Let $d := \dim S$. Then the functor $(-)|_{X_\eta}[-d] : D_c^b(X) \to D_c^b(X_\eta)$ is t-exact for the absolute perverse t-structures. In particular, it restricts to an exact functor*

$$(-)|_{X_\eta}[-d] : \mathrm{Perv}(X) \to \mathrm{Perv}(X_\eta). \tag{3}$$

Proof.

(a) Right t-exactness: For every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) = \operatorname{Supp} \mathcal{H}^{n-d}(K|_{X_\eta}) = X_\eta \cap \operatorname{Supp} \mathcal{H}^{n-d}(K)$. By Lemma 2.3.8 (d), one has

$$\dim \operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) \leq \dim \operatorname{Supp}(\mathcal{H}^{n-d}(K)) - d \leq -n.$$

From Lemma 2.3.8 (a), one has $K|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$.

(b) To prove left t-exactness, one may shrink S to a nonempty open. As S is geometrically reduced, shrinking S one may assume that S is smooth. From [SGA 4 1/2, Thm. 2.13, p.242], shrinking S one may assume $K \in D^{\mathrm{ULA}}(X/S)$. Then for every integer n, we prove

$$\operatorname{Supp}\mathcal{H}^n\left(\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])\right) = \operatorname{Supp}\mathcal{H}^n\left((\mathbb{D}_X K)|_{X_\eta}[-d]\right). \tag{4}$$

Indeed, by the proof of [Bar24, Lem. 3.12], as S is smooth, one has $\mathbb{D}_X K = (\mathbb{D}_{X/S}K)(d)[2d]$. From Theorem 2.3.1 (c), as K is ULA, $(\mathbb{D}_X K)|_{X_\eta}[-d]$ is a Tate twist of $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])$, which proves (4).

Now assume $K \in {}^pD^{\geq 0}(X)$. Then $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. From Part (a), one has $(\mathbb{D}_X K)|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$. By (4), one has $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d]) \in {}^pD^{\leq 0}(X_\eta)$, or equivalently, $K|_{X_\eta}[-d] \in {}^pD^{\geq 0}(X_\eta)$.

□

A nonempty full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if for every exact sequence $X \to Y \to Z$ in $\mathcal{A}$ with $X, Z \in \mathcal{B}$, one has $Y \in \mathcal{B}$. This definition originates from [Ser53, Sec. 1, (I)]. Assume that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is an abelian subcategory of $\mathcal{A}$, and $\mathcal{A}/\mathcal{B}$ has a natural structure of abelian category. For every $X \in \mathcal{B}$, X is simple (resp. semisimple) in $\mathcal{B}$ if and only if X is simple (resp. semisimple) in $\mathcal{A}$.

Lemma 2.3.10. *Assume that S is smooth over k, integral with generic point η and $\dim S = d$. Then:*

(a) *Let $A \in \operatorname{Perv}^{\mathrm{ULA}}(X/S)$, and let $B[d]$ be a subquotient of $A[d]$ in $\operatorname{Perv}(X)$. If the image $B|_{X_\eta} \in \operatorname{Perv}(X_\eta)$ of $B[d]$ under the functor (3) is zero, then $B[d] = 0$ in $\operatorname{Perv}(X)$.*

(b) *The functor $(-)[d] : \operatorname{Perv}^{\mathrm{ULA}}(X/S) \to \operatorname{Perv}(X)$ exhibits a Serre subcategory.*

Proof.

(a) By regularity of S and [HS23, Cor. 1.12], one has $B \in D^{\mathrm{ULA}}(X/S)$. Then by Lemma 2.2.3, since $B|_{X_\eta} = 0$, one has $B = 0$.

(b) The functor is fully faithful. Its essential image is closed under extensions in $\operatorname{Perv}(X)$, because $\operatorname{Perv}^{\mathrm{ULA}}(X/S)$ is closed under extensions in the triangulated subcategory $D^{\mathrm{ULA}}(X/S)$ of $D^b_c(X)$.

We claim that the essential image is closed under taking subobjects. Take $K \in \operatorname{Perv}^{\mathrm{ULA}}(X/S)$ and a subobject $L[d]$ of $K[d]$ in $\operatorname{Perv}(X)$. By Lemma 2.3.9, as S is integral and smooth, $L|_{X_\eta}$ is a subobject of $K|_{X_\eta}$ in $\operatorname{Perv}(X_\eta)$. By smoothness of S and Theorem 2.3.5, there is a subobject $L' \subset K$ in $\operatorname{Perv}^{\mathrm{ULA}}(X/S)$ with $L'|_{X_\eta} = L|_{X_\eta}$. Set $M = K/L' \in \operatorname{Perv}^{\mathrm{ULA}}(X/S)$. Let $N[d]$ be the image of $L[d]$ under the morphism $K[d] \to M[d]$ in $\operatorname{Perv}(X)$. From Lemma 2.3.9, as the sequence

$$0 \to L'[d] \cap L[d] \to L[d] \to N[d] \to 0$$

is exact in $\mathrm{Perv}(X)$, the sequence

$$0 \to L'|_{X_\eta} \cap L|_{X_\eta} \to L|_{X_\eta} \to N|_{X_\eta} \to 0$$

is exact in $\mathrm{Perv}(X_\eta)$. It implies $N|_{X_\eta} = 0$. By Part (a), since $N[d]$ is a subobject of $M[d]$ in $\mathrm{Perv}(X)$, one has $N[d] = 0$. Then $L[d]$ is a subobject of $L'[d]$ in $\mathrm{Perv}(X)$. Since $L'[d]/(L[d])$ is a quotient of $L'[d]$ in $\mathrm{Perv}(X)$ and $L'|_{X_\eta}/(L|_{X_\eta}) = 0$ in $\mathrm{Perv}(X_\eta)$, one gets $L'[d]/(L[d]) = 0$ in $\mathrm{Perv}(X)$. Therefore, $L[d] = L'[d]$. The claim is proved.

Similarly, the essential image is closed under taking quotients. By a slight generalization of [Ser53, Proposition 1], the essential image is a Serre subcategory of $\mathrm{Perv}(X)$.

□

3 Cotori

We review Gabber and Loeser's construction of a scheme structure on the set of pro-ℓ characters. Then we show that there exist uncountably many closed points outside a countable union of strict closed subsets of this scheme.

3.1 Scheme parameterizing characters

By [Rob00, p.127], there is a canonical absolute value on $\bar{\mathbb{Q}}_\ell$ extending the discrete absolute value $|\cdot|_\ell$ on $\mathbb{Q}_\ell$. It induces a topology on $\bar{\mathbb{Q}}_\ell$ which is totally disconnected. For a profinite group G, let $\mathcal{C}(G)$ be the group of ℓ-adic characters, *i.e.*, continuous morphisms $\chi : G \to \bar{\mathbb{Q}}_\ell^\times$. Then $\chi(G)$ are compact subgroup of $\bar{\mathbb{Q}}_\ell^\times$. Let $\mathcal{C}(G)_{\ell'}$ (resp. $\mathcal{C}(G)_\ell$) be the subgroup of $\mathcal{C}(G)$ consisting of characters of finite order prime to ℓ (resp. that are pro-ℓ). Then there is a canonical isomorphism $\mathcal{C}(G)_\ell \xrightarrow{\sim} \mathcal{C}\big((G^{(\ell)})^{\mathrm{ab}}\big)$. By Lemma 3.1.1, one has $\mathcal{C}(G) = \mathcal{C}(G)_{\ell'} \times \mathcal{C}(G)_\ell$.

Lemma 3.1.1. *Let G be a compact subgroup of $\bar{\mathbb{Q}}_\ell^\times$. Let $G^{(\ell)}$ (resp. $G^{(\ell')}$) be the ℓ-Sylow subgroup (resp. maximal prime-to-ℓ quotient) of G. Then $G^{(\ell')}$ is finite, and $G \cong G^{(\ell)} \times G^{(\ell')}$ as topological groups.*

Proof. By [KS99, Remark 9.0.7], there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ containing G. By [Ser92, Thm. 1 2, p.122], one has $G \subset O_E^\times$. Then from [Ser92, Cor., p.155], G is an ℓ-adic Lie group. From Lazard's theorem (see, *e.g.*, [GSK09, p.711]), there is a pro-ℓ open subgroup U of G. By [RZ10, Cor. 2.3.6 (b)], there is an ℓ-Sylow subgroup $H \leq G$ containing U. Since G is compact, $[G : U]$ is finite. Thus, the group G/H is finite of order prime to ℓ. By [RZ10, Prop. 2.3.8], G is isomorphic to $(G/H) \times H$. By [RZ10, Cor. 2.3.6 (c)], since G is commutative, it has exactly one ℓ-Sylow subgroup. □

We review the contents of [GL96, Sec. 3.2]. Fix an integer $n \geq 0$. Let A_n be a free $\hat{\mathbb{Z}}$-module of rank n. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a $\mathbb{Z}_\ell$-basis of $A_n^{(\ell)}$. Let

$\mathcal{R} = \{O_E : E/\mathbb{Q}_\ell \text{ is a finite subextension of } \bar{\mathbb{Q}}_\ell\}$, which is a directed set under inclusion. For every $R \in \mathcal{R}$, let m_R be the maximal ideal of R. Let $R[[A_n^{(\ell)}]] := \varprojlim_{i,j\geq 1}(R/m_R^i)[A_n^{(\ell)}/\ell^j]$ be the completed group ring. There is a canonical injective morphism $A_n^{(\ell)} \to R[[A_n^{(\ell)}]]^\times$ of groups.

Fact 3.1.2 can be found in [GL96, p.509].

Fact 3.1.2. *The ring $R[[A_n^{(\ell)}]]$ is a Noetherian, regular, complete, local domain of Krull dimension $1+n$. There is an isomorphism of topological rings*

$$R[[A_n^{(\ell)}]] \xrightarrow{\sim} R[[X_1, \ldots, X_n]], \quad \gamma_i \mapsto 1 + X_i. \tag{5}$$

Gabber and Loeser introduce a scheme of ℓ-adic characters.

Definition 3.1.3. Write $R_n = \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[A_n^{(\ell)}]]$. Define the "cotorus" associated with A_n to be $\mathcal{C}_\ell := \operatorname{Spec} R_n$.

By [GL96, Prop. A.2.2.3 (ii)], the scheme $\mathcal{C}_\ell$ is integral, Noetherian and regular. Its set of closed points coincides with $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$, and it is Zariski dense in $\mathcal{C}_\ell$. If $n > 0$, then $\mathcal{C}_\ell$ is **not** locally of finite type over $\bar{\mathbb{Q}}_\ell$.

Lemma 3.1.4. *Every character $\chi : A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ extends canonically to a surjective morphism $R_n \to \bar{\mathbb{Q}}_\ell$ of $\bar{\mathbb{Q}}_\ell$-algebras.*

Proof. There is a finite subextension $E/\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$ containing all the $\chi(\gamma_i)$. Then by completeness of E, for every $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{Z}_\ell[[X_1, \ldots, X_n]]$, the series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha \prod_{i=1}^n (\chi(\gamma_i) - 1)^{\alpha_i}$ converges in E. Denote its limit by $f(\chi(\gamma_1) - 1, \ldots, \chi(\gamma_n) - 1)$. The composition $\mathbb{Z}_\ell[[A_n^{(\ell)}]] \to E$ of (5) followed by

$$\mathbb{Z}_\ell[[X_1, \ldots, X_n]] \to E, \quad f \mapsto f(\chi(\gamma_1) - 1, \ldots, \chi(\gamma_n) - 1)$$

extends χ. It induces the stated surjection. The construction is independent of the choice of the $\mathbb{Z}_\ell$-basis $\{\gamma_1, \ldots, \gamma_n\}$ of $A_n^{(\ell)}$. $\square$

By Lemma 3.1.4, for every $\chi \in \mathcal{C}(A_n)_\ell$, the corresponding character $A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ induces a surjection $R_n \to \bar{\mathbb{Q}}_\ell$. Let $\Psi(\chi)$ be the corresponding element of $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$. Thus, there is a map

$$\Psi : \mathcal{C}(A_n)_\ell \to \mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell). \tag{6}$$

Fact 3.1.5 can be found in [GL96, p.519]. It shows that the scheme $\mathcal{C}_\ell$ parameterizes the pro-ℓ characters of A_n.

Fact 3.1.5. *The map (6) is bijective.*

3.2 Cotori are Baire

Fix an uncountable, algebraically closed field k. The objective of Section 3.2 is Theorem 3.2.11, used in the proof of Theorem 5.2.1. We show that over k, a reasonable scheme has uncountably many rational points outside a countable union of strict closed subsets.

Baire schemes

Definition 3.2.1. A scheme X over k is called *k-Baire*, if $\dim X$ is finite and $X(k) \setminus \cup_{i\geq 1} Z_i(k)$ is uncountable for every countable sequence $\{Z_i\}_{i\geq 1}$ of closed subschemes of X with $\dim Z_i < \dim X$ for all $i \geq 1$. A k-algebra R is called k-Baire if $\operatorname{Spec}(R)$ is k-Baire.

Example 3.2.2. Assume $k = \mathbb{C}$. Let X be a complex algebraic variety with $\dim X > 0$. The analytification X^{an} of X is locally compact Hausdorff. Then by the Baire category theorem (see, *e.g.*, [Wil70, Cor. 25.4 a)]), X is $\mathbb{C}$-Baire.

Remark 3.2.3. As k is uncountable, an algebraic curve over k is k-Baire. In Definition 3.2.1, the underlying reduced induced closed subscheme $X_{\mathrm{red}} \to X$ induces a bijection $X_{\mathrm{red}}(k) \to X(k)$, so X is k-Baire if and only X_{red} is k-Baire. If X is irreducible and k-Baire, then $X \setminus \cup_{i\geq 1} Z_i$ is Zariski dense in X.

Lemma 3.2.4. *Let $f : X \to Y$ be a finite surjective morphism of schemes over k. If Y is k-Baire, then so is X.*

Proof. Let $\{Z_i\}_i$ be a sequence of closed subschemes of X with $\dim Z_i < \dim X$. Since f is a closed morphism, every $Y_i := f(Z_i)$ is closed in Y. Endow each Y_i with the reduced induced structure. Let $Z'_i := f^{-1}(Y_i) = Y_i \times_Y X$. Then there is a canonical closed immersion $Z_i \to Z'_i$. The restriction $Z_i \to Y_i$ of f is a finite surjective morphism. By [Sta26, Tag 0ECG], one has $\dim X = \dim Y$ and $\dim Y_i = \dim Z_i$. In particular, $\dim X$ is finite and $\dim Y_i < \dim Y$.

As k is algebraically closed, the induced map $X(k) \to Y(k)$ is surjective. Then the induced map

$$X(k) \setminus (\cup_{i\geq 1} Z'_i(k)) \to Y(k) \setminus (\cup_i Y_i(k))$$

is surjective. Because Y is k-Baire, the target is uncountable. Then $X(k) \setminus (\cup_{i\geq 1} Z_i(k))$ is also uncountable, as it contains the source. □

Lemma 3.2.5. *Let X be a Noetherian scheme over k.*

(a) *Then X is k-Baire if and only if X has an irreducible component C with $\dim C = \dim X$, such that the reduced induced closed subscheme C is k-Baire.*

(b) *Assume that $n := \dim X - 1$ is finite. If X has uncountably many (pairwise set-theoretically distinct) irreducible, k-Baire, closed subschemes of dimension n, then X is k-Baire.*

Proof.

(a) Assume that there is such a component C. Consider a sequence of closed subschemes $\{Z_i\}_{i\geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Then for every $i \geq 1$, one has $\dim C \cap Z_i \leq \dim Z_i < \dim X = \dim C$. Since C is k-Baire, the set $C(k) \setminus \cup_i (C \cap Z_i)(k)$ is uncountable. Therefore, $X(k) \setminus \cup_i Z_i(k)$ is also uncountable.

Conversely, assume that every component of X of maximum dimension is not k-Baire. As X is Noetherian, one can write $X = \cup_{j=1}^{n} C_j$ as a finite union of the irreducible components. For every j with $\dim C_j = \dim X$, the scheme C_j is not k-Baire. Therefore, there is a sequence $\{Z_i^j\}_{i \geq 1}$ of closed subschemes of C_j such that $\dim Z_i^j < \dim C_j$ for all i and that $C_j(k) \setminus \cup_i Z_i^j(k)$ is countable. The finite family of components C_k with $\dim C_k < \dim X$, joint with the sequences $\{Z_i^j\}_i$ for all j with $\dim C_j = \dim X$, gives a countable family $\{Z_s\}_s$ of closed subschemes of X with $\dim Z_s < \dim X$ for all s. Then $X(k) \setminus (\cup_s Z_s(k))$ is countable, so X is not k-Baire.

(b) Consider a sequence of closed subschemes $\{Z_i\}_{i \geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Every Z_i is a Noetherian scheme, so it has only finitely many irreducible components. The set of irreducible components of the family $\{Z_i\}_i$ is countable. Thus, one may assume that every Z_i is irreducible. By assumption, X has an n-dimensional, irreducible, k-Baire closed subscheme X' which is set-theoretically distinct from any Z_i. For every $i \geq 1$, because $\dim X' = n \geq \dim Z_i$ and Z_i is irreducible, one has $X' \not\subset Z_i$ and $X' \cap Z_i \neq X'$. Since X' is irreducible, one has $\dim(X' \cap Z_i) < \dim X'$. As X' is k-Baire, the set $X'(k) \setminus \cup_{i \geq 1}(X' \times_X Z_i)(k)$ is uncountable. It is a subset of $X(k) \setminus \cup_{i \geq 1} Z_i(k)$. Therefore, X is k-Baire.

□

Lemma 3.2.6 is well known.

Lemma 3.2.6. *If X is a finite type scheme over k with $\dim X > 0$, then X is k-Baire.*

Proof. Since X is of finite type over k, its dimension m is finite and X has only finitely many irreducible components. Replacing X with an irreducible component of dimension m, one may assume that X is irreducible. Then by [Har77, Exercise 3.20 (e), p.94], every nonempty open subset of X has dimension m. Replacing X by an affine open, one may assume that X is affine. By Noether's normalization lemma, there is a finite surjective morphism $p : X \to \mathbf{A}_k^m$ over k. By Lemma 3.2.4, one may assume $X = \mathbf{A}_k^m$.

By induction on $m > 0$, we prove that $\mathbf{A}_k^m$ is k-Baire. When $m = 1$, one has $\dim \mathbf{A}_k^1 = 1$, and $\mathbf{A}_k^1(k)$ is uncountable. By Remark 3.2.3, $\mathbf{A}_k^1$ is k-Baire. Assume the statement for $m-1$ with $m \geq 2$. The set of hyperplanes in $\mathbf{A}_k^m$ is uncountable. By the inductive hypothesis, every hyperplane is k-Baire. From Lemma 3.2.5 (b), so is $\mathbf{A}_k^m$. The induction is completed. □

Baireness of cotori

We show that every positive dimensional cotorus is $\bar{\mathbb{Q}}_\ell$-Baire. Definition 3.2.7 is a slight variant of [BGR84, Def. 1, p.205].

Definition 3.2.7. Let A be a k-algebra, and let $A[X] \to B$ be an injective ring map. We say that B is *k-Rückert* over A if there is a nonempty family W of *monic* polynomials in $A[X]$ such that the following axioms are fulfilled.

Axiom (1) If $f, g \in A[X]$ are monic polynomials with $fg \in W$, then $f, g \in W$.

Axiom (2) For every $w \in W$, the A-algebra B/w is isomorphic to $A[X]/w$.

Axiom (3) For every $b \in B \setminus \{0\}$, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ satisfying $u\sigma(b) \in W$.

Remark 3.2.8. From Axiom (1), one gets $1 \in W$. If $W = \{1\}$, then by Axiom (3), for every $b \in B \setminus \{0\}$, one has $b \in B^\times$, *i.e.*, B is a field. Conversely, if B is a field, then B is k-Rückert over A with $W = \{1\}$.

Assume $W \neq \{1\}$. We prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Indeed, take $w(\neq 1) \in W$. By Axiom (2), there is an A-isomorphism $B/w \to A[X]/w$, hence an isomorphism $\mathrm{Spec}(A[X]/w) \to \mathrm{Spec}(B/w)$ of schemes over A. Because w is a monic polynomial different from 1, the ring map $A \to A[X]/w$ is injective and finite. The induced morphism $\mathrm{Spec}(A[X]/w) \to \mathrm{Spec}(A)$ is surjective, so $\mathrm{Spec}(B/w) \to \mathrm{Spec}(A)$ is surjective.

For a commutative ring R and an ideal $I \subset R$, let $V_R(I) = \mathrm{Spec}\, R/I (\subset \mathrm{Spec}\, R)$. For $r \in R$, let $V_R(r) = V_R(rR)$. Lemma 3.2.9 is used in the induction step of the proof of Theorem 3.2.11.

Lemma 3.2.9. *Let A be Noetherian k-algebra of dimension n. Let B be a domain, but not a field, containing $A[X]$. Assume that B is k-Rückert over A.*

(a) *The ring B is Noetherian of dimension $n + 1$.*

(b) *Suppose that A is k-Baire. Let S be an uncountable subset of A such that for every $s \in S$, one has $\dim V_A(s) = n - 1$. Suppose that the family $\{V_A(s)\}_{s \in S}$ is pairwise disjoint. Then B is k-Baire.*

Proof. By Axiom (3), for every $b \in B \setminus (B^\times \cup \{0\})$, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ such that $w := u\sigma(b)$ is in W. Since b is not a unit, one has $w \neq 1$. By Axiom (2), the A-algebra B/w is isomorphic to $A[X]/w$. Since $w(\neq 1)$ is a monic polynomial over A, the ring map $A \to A[X]/w$ is injective finite.

(a) One has

$$\dim B/b = \dim B/w = \dim A[X]/w \overset{\text{(a)}}{=} \dim A = n, \tag{7}$$

where (a) uses [Sta26, Tag 00OK]. The domain B is not a field, so $\dim B = n + 1$. By [BGR84, Prop. 2, p.206], the ring B is Noetherian.

(b) The morphism $\mathrm{Spec}\, A[X]/w \to \mathrm{Spec}\, A$ is finite surjective. Then by Lemma 3.2.4, the algebra $A[X]/w$ is k-Baire. As σ is over k, the k-algebra B/b is isomorphic to B/w. Then B/b is k-Baire.

For every $s \in S$, one has $\dim V_A(s) < \dim A$, so $s \neq 0$. From Remark 3.2.8, as B is not a field, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. The preimage of $V_A(s)$ under the surjection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is $V_B(s)$, so $V_B(s)$ is nonempty. In particular, $s \notin B^\times$ and B/s is k-Baire. Moreover, the family $\{V_B(s)\}_{s\in S}$ is pairwise disjoint. By (7), one gets $\dim V_B(s) = n$.

By Part (a), B is Noetherian. Then by Lemma 3.2.5 (a), for every $s \in S$, there is a k-Baire irreducible component $C_s \subset \mathrm{Spec}(B/s)$ of dimension n. The family $\{C_s\}_{s\in S}$ is pairwise disjoint. From Lemma 3.2.5 (b), B is k-Baire.

□

Set $S_n := \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[X_1, \ldots, X_n]]$. By [GL96, Prop. 3.2.2 (1)], the natural morphism $S_n \to \bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]$ is injective. Then the isomorphism (5) identifies R_n with the $\bar{\mathbb{Q}}_\ell$-subalgebra $S_n \subset \bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]$.

Fact 3.2.10. *For every integer $n \geq 0$,*

(a) ([GL96, Thm. A.2.1, Prop, A.2.2.1]) *the ring S_n is a Noetherian, regular, Jacobson domain of Krull dimension n;*

(b) ([GL96, Prop A.2.2.2, proof of A.2.2.3 (ii)]) *S_{n+1} is $\bar{\mathbb{Q}}_\ell$-Rückert over S_n.*

Theorem 3.2.11. *For every integer $n \geq 1$, the algebra S_n is $\bar{\mathbb{Q}}_\ell$-Baire.*

Proof. Since $\bar{\mathbb{Q}}_\ell$ is a flat $\mathbb{Z}_\ell$-module, the injection $\mathbb{Z}_\ell[X_1, \ldots, X_n] \to \mathbb{Z}_\ell[[X_1, \ldots, X_n]]$ induces an injection $\bar{\mathbb{Q}}_\ell[X_1, \ldots, X_n] \to S_n$. The natural morphism

$$\mathrm{Spec}(\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]]) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell} \tag{8}$$

of schemes over $\bar{\mathbb{Q}}_\ell$ factors through a morphism $p_n : \mathrm{Spec}(S_n) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell}$.

Let $\mathcal{M} = \cup_E m_E$, where E runs through all finite subextensions of $\mathbb{Q}_\ell \subset \bar{\mathbb{Q}}_\ell$, and m_E is the maximal ideal of the ring of integers of E. Then $\mathcal{M}$ is the maximal ideal of the integral closure $\overline{\mathbb{Z}_\ell}$ of $\mathbb{Z}_\ell$ inside $\bar{\mathbb{Q}}_\ell$. By [Rob00, Prop., p.128], the residue field $\overline{\mathbb{Z}_\ell}/\mathcal{M}$ is an algebraic closure of the finite field F_ℓ, so it is countable. As $\mathbb{Z}_\ell$ is uncountable, so is the set $\mathcal{M}$.

For every $(a_1, \ldots, a_n) \in \mathcal{M}^n$, there is a surjective morphism of $\bar{\mathbb{Q}}_\ell$-algebras:

$$\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]] \to \bar{\mathbb{Q}}_\ell, \quad f \mapsto f(a_1, \ldots, a_n).$$

Its kernel is a $\bar{\mathbb{Q}}_\ell$-point of $\mathrm{Spec}(\bar{\mathbb{Q}}_\ell[[X_1, \ldots, X_n]])$, whose image under (8) is $(a_1, \ldots, a_n) \in A^n_{\bar{\mathbb{Q}}_\ell}(\bar{\mathbb{Q}}_\ell)$. Hence $\mathcal{M}^n \subset p_n(\mathrm{Spec}(S_n)(\bar{\mathbb{Q}}_\ell))$. In particular, $\mathrm{Spec}(S_n)(\bar{\mathbb{Q}}_\ell)$ is uncountable.

By induction on $n > 0$, we prove that S_n is $\bar{\mathbb{Q}}_\ell$-Baire, and $\{V_{S_n}(X_1 - a)\}_{a\in\mathcal{M}}$ is a pairwise disjoint family of $(n-1)$-dimensional subsets. When $n = 1$, by Fact 3.2.10 (a), S_1 is $\bar{\mathbb{Q}}_\ell$-Baire. Moreover, $\{V_{S_1}(X_1 - a)\}_{a\in\mathcal{M}}$ is a pairwise distinct family of closed point of $\mathrm{Spec}(S_1)$. The statement is proved for $n = 1$. Assume the statement for $n - 1$ with $n \geq 2$. By Fact 3.2.10, (7), and Lemma 3.2.9 (b), the statement holds for n. The induction is completed. □

4 Perverse sheaves on abelian varieties

In Section 4, we review the generic vanishing theorem of Krämer and Weissauer for perverse sheaves on abelian varieties. As an application, we prove Theorem 1.2.1. Then we recall the construction of the convolution and the Tannakian formalism.

Let k be a field of characteristic zero. Let Vec_k be the category of finite dimensional k-vector spaces. Choose an algebraic closure $\bar{k}$ of k. Let $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ be the category of continuous, finite dimensional $\bar{\mathbb{Q}}_\ell$-representations of Γ_k. Let A be an abelian variety over k of dimension g. Recall that $\pi_1(A_{\bar{k}})$ is a free $\hat{\mathbb{Z}}$-module of rank $2g$. With the notation of Section 3, set

- $\mathcal{C}(A) = \mathcal{C}(\pi_1(A_{\bar{k}}))$: the group of characters $\pi_1(A_{\bar{k}}) \to \bar{\mathbb{Q}}_\ell^\times$;
- $\mathcal{C}(A)_{\ell'} = \mathcal{C}(\pi_1(A_{\bar{k}}))_{\ell'}$: the group of characters of finite order prime to ℓ;
- $\mathcal{C}(A)_\ell$: the cotorus assigned to $\pi_1(A_{\bar{k}})$.

4.1 Generic vanishing theorem

For an object $K \in \mathrm{Perv}(A)$, set

$$\mathcal{S}(K) := \{\chi \in \mathcal{C}(A) | H^i(A_{\bar{k}}, K \otimes^L L_\chi) \neq 0 \text{ for some integer } i \neq 0\}.$$

Theorem 4.1.1 is a combination of [Wei16, Vanishing Theorem and Thm. 2]. It is first proved in [KW15b, Thm. 1.1] for complex abelian varieties, and perverse sheaves with coefficient $\mathbb{C}$ instead of $\bar{\mathbb{Q}}_\ell$. For a nonzero abelian subvariety $B \subset A_{\bar{k}}$, let $K(B) \subset \mathcal{C}(A)$ be the subset of characters $\chi : \pi_1(A_{\bar{k}}) \to \bar{\mathbb{Q}}_\ell^\times$ whose restriction to $\pi_1(B)$ is trivial. A subset of $\mathcal{C}(A)$ is called *thin*, if it is in the form of $\cup_{i=1}^n \chi_i \cdot K(A_i)$ for some integer $n \geq 0$, certain $\chi_i \in \mathcal{C}(A)$ and certain nonzero abelian subvarieties $A_i \subset A_{\bar{k}}$.

Theorem 4.1.1. *For every perverse sheaf $K \in \mathrm{Perv}(A)$ and every character $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$,*

$$\{\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) | \chi_{\ell'}\chi_\ell \in \mathcal{S}(K)\} \tag{9}$$

is the set of $\bar{\mathbb{Q}}_\ell$-points of a strict Zariski closed subset of the scheme $\mathcal{C}(A)_\ell$. It is contained in a thin subset of $\mathcal{C}(A)$.

Remark 4.1.2. In Theorem 4.1.1, if further $K|_{A_{\bar{k}}}$ is *semisimple of geometric origin* in $\mathrm{Perv}(A_{\bar{k}})$, then by [KW15b, Lemma 11.2 (c)], (9) is a thin subset $\cup_{i=1}^n \chi_i \cdot K(A_i)$ with the χ_i torsion.

We review [KW15a, p.725]. Because of $\mathrm{char}(k) = 0$, for every $K \in \mathrm{Perv}(A)$, its Euler characteristic $\chi(A, K) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\bar{\mathbb{Q}}_\ell} H^i(A_{\bar{k}}, K)$ satisfies

$$\chi(A, K) \geq 0. \tag{10}$$

Let $N(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $\chi(A, K) = 0$. From the additivity of the function $\chi(A, -) : \mathrm{Ob}(\mathrm{Perv}(A)) \to \mathbb{N}$ and (10), $N(A)$ is a Serre subcategory of $\mathrm{Perv}(A)$. Let

$$\bar{P}(A) := \mathrm{Perv}(A)/N(A)$$

be the quotient abelian category. For every $\chi \in \mathcal{C}(A)$, set

$$\mathcal{E}^{\chi}(A_{\bar{k}}) = \{K \in \mathrm{Perv}(A_{\bar{k}}) | \chi \notin \mathcal{S}(K)\}.$$

Then $\mathcal{E}^{\chi}(A_{\bar{k}})$ is closed under extensions in $\mathrm{Perv}(A_{\bar{k}})$. Let $P^{\chi}(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $Q \in \mathcal{E}^{\chi}(A_{\bar{k}})$ for every simple subquotient Q of $K|_{A_{\bar{k}}}$ in $\mathrm{Perv}(A_{\bar{k}})$.

By [BBD, Thm. 4.3.1 (i)], every object $K \in \mathrm{Perv}(A)$ is Noetherian and Artinian. Then by Theorem 4.1.1 and Lemma 2.1.7 (a), for every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, the set $\{\chi_{\ell} \in \mathcal{C}(A)_{\ell}(\bar{\mathbb{Q}}_{\ell}) | K \in P^{\chi_{\ell'}\chi_{\ell}}(A)\}$ is the set of $\bar{\mathbb{Q}}_{\ell}$-points of a strict Zariski closed subset of $\mathcal{C}(A)_{\ell}$.

Lemma 4.1.3. *Let $\mathcal{A}$ be a Noetherian and Artinian abelian category. Let $\mathcal{E}$ be a class of objects of $\mathcal{A}$ closed under isomorphisms. Let $\mathcal{S} \subset \mathcal{A}$ be the full subcategory of objects whose all simple subquotients are in $\mathcal{E}$.*

(a) *Then $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$.*

(b) *If further $\mathcal{E}$ is closed under extensions, then $\mathcal{S} \subset \mathcal{E}$.*

Proof.

(a) (i) We prove that $\mathcal{S}$ is closed under subquotients. Let X be an object of $\mathcal{S}$ with a subquotient Y in $\mathcal{A}$. Every simple subquotient of Y is that of X, hence in $\mathcal{E}$. Thus, $Y \in \mathcal{S}$.

Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathcal{A}$ with $L, N \in \mathcal{S}$. Let Q be a simple subquotient of M. We prove $Q \in \mathcal{E}$.

(ii) First, assume that Q is a quotient of M. The natural morphism $L \to Q$ is either an epimorphism or zero, in which case Q is a simple quotient of L or N respectively. Hence $Q \in \mathcal{E}$.

(iii) Now assume that Q is general. There is a subobject $M_0 \subset M$ and an epimorphism $M_0 \to Q$. Then

$$0 \to f^{-1}(M_0) \to M_0 \to g(M_0) \to 0$$

is a short exact sequence in $\mathcal{A}$ with $f^{-1}(M_0)$ (resp. $g(M_0)$) a subobject of L (resp. N). From Part (i), both $f^{-1}(M_0)$ and $g(M_0)$ are in $\mathcal{S}$. From Part (ii), one has $Q \in \mathcal{E}$.

From Part (iii), one has $M \in \mathcal{S}$, and $\mathcal{S}$ is closed under extensions. The result follows from a slight generalization of [Ser53, Proposition 1].

(b) By the Jordan-Hölder lemma, every object $X \in \mathcal{S}$ admits a filtration in $\mathcal{A}$

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

by subobjects such that each X_i/X_{i-1} is a simple subquotient of X. Then $X_i/X_{i-1} \in \mathcal{E}$. As $\mathcal{E}$ is closed under extensions, one has $X \in \mathcal{E}$.

□

By Lemma 4.1.3 (a), for every $\chi \in \mathcal{C}(A)$, $P^{\chi}(A)$ is a Serre subcategory of $\mathrm{Perv}(A)$. From Lemma 4.1.3 (b), for every $K \in P^{\chi}(A)$ and every integer $i \neq 0$, one has

$$H^i(A_{\bar{k}}, K \otimes^L L_\chi) = 0. \tag{11}$$

Denote the functor $H^0(A_{\bar{k}}, \cdot \otimes^L L_\chi) : \mathrm{Perv}(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ by ω_χ. From the proof of [LS25, Lemma 2.5 (3)], the restriction

$$\omega_\chi : P^{\chi}(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \tag{12}$$

is exact. Let $N^{\chi}(A)$ be the full subcategory of $P^{\chi}(A)$ of objects in $N(A)$. By [KW15b, Cor. 4.2], for every $K \in N^{\chi}(A)$, one has $\chi(A, K \otimes^L L_\chi) = 0$. From (11), one has $H^0(A_{\bar{k}}, K \otimes^L L_\chi) = 0$. Then by [Sta26, Tag 02MS], the functor ω_χ factors uniquely through an exact functor (still denoted by ω_χ)

$$\omega_\chi \colon P^{\chi}(A)/N^{\chi}(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}. \tag{13}$$

4.2 A fixed part theorem for relative perverse sheaves

Theorem 1.2.1 is an analog of the fixed part theorem for relative perverse sheaves on constant abelian schemes. It follows from Theorems 4.1.1 (with $\chi_{\ell'} = 1$) and 4.2.1, because in Theorem 4.2.1, there are only finitely many integers j with ${}^p\mathcal{H}^j(Rp_{A*}K) \neq 0$.

Theorem 4.2.1. *In Setting 1.1.1, assume that X is smooth. Let $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ be semisimple $D_c^b(A \times X)$. Then there exists a perverse sheaf F on A, such that $K^0 := p_A^* F$ is a direct summand of K in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$, and that for every*

$$\chi \in \mathcal{C}(A) \setminus \cup_{j \in \mathbb{Z}} \mathcal{S}({}^p\mathcal{H}^j(Rp_{A*}K)),$$

one has $\omega_{\chi_\eta}(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$.

Proof. As K is ULA, $\mathcal{H}^0 Rp_{X*}(K \otimes p_A^* L_\chi)$ is a lisse sheaf on X, and since X is smooth, the canonical morphism $\Gamma_{k(\eta)} \to \pi_1(X, \bar{\eta})$ is surjective (see also Remark 4.4.5). Thus, from Theorem 2.1.5 (b), as k is algebraically closed of characteristic 0 and K is semisimple in $D_c^b(A \times X)$, the natural map

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) \to \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}} \tag{14}$$

is surjective.

By Fact 2.1.1, one has

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) = H^0(A, (Rp_{A*}K) \otimes^L L_\chi). \tag{15}$$

By Lemma 2.1.9, the spectral sequence in [Max19, Rk. 8.1.14 (6)] becomes

$$E_2^{i,j} = H^i(A, {}^p\mathcal{H}^j(Rp_{A*}K) \otimes^L L_\chi) \Rightarrow H^{i+j}(A, (Rp_{A*}K) \otimes^L L_\chi).$$

For every integer j, from $\chi \notin \mathcal{S}({}^p\mathcal{H}^j(Rp_{A*}K))$, one has $E_2^{i,j} = 0$ for all integers $i \neq 0$. Set $F := {}^p\mathcal{H}^0(Rp_{A*}K) \in \mathrm{Perv}(A)$. The spectral sequence degenerates at page E_2, hence an isomorphism

$$H^0(A, (Rp_{A*}K) \otimes^L L_\chi) = H^0(A, F \otimes^L L_\chi). \tag{16}$$

Let $d := \dim X$. From Lemma 2.3.7, as X is smooth and K is ULA, one has $K[d] \in \mathrm{Perv}(A \times X)$. As K is semisimple in $D_c^b(A \times X)$, $K[d]$ is semisimple in $\mathrm{Perv}(A \times X)$. Set $K^0 := p_A^* F \in D_c^b(A \times X)$. From Fact 2.2.2, one has $K^0 \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By [KW01, III, Theorem 11.3], as the projection $p_A : A \times X \to A$ is smooth with geometrically connected fibers of pure dimension d, there is a natural injection $K^0[d] \hookrightarrow K[d]$ in $\mathrm{Perv}(A \times X)$. By Lemma 2.3.10 (b), K is semisimple in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$, and there is a natural injection $K^0 \hookrightarrow K$. Thus, K^0 is a direct summand of K in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$.

By Theorem 2.3.1 (a), the functor $(-)|_{A_\eta} : \mathrm{Perv}(A \times X/X) \to \mathrm{Perv}(A_\eta)$ is exact. Then $K^0|_{A_\eta} = (p_A|_{A_\eta})^* F$ is a direct summand of $K|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$. As $\omega_{\chi_\eta} : \mathrm{Perv}(A_\eta) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ is an additive functor, the natural morphism $\omega_{\chi_\eta}(K^0|_{A_\eta}) \to \omega_{\chi_\eta}(K|_{A_\eta})$ is injective. By [Mil80, VI, Corollary 4.3], one has

$$\omega_{\chi_\eta}(K^0|_{A_\eta}) = H^0(A, F \otimes^L L_\chi). \tag{17}$$

Combining (14), (15), (16) with (17), one gets $\omega_{\chi_\eta}(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$.
$\square$

Remark 4.2.2. We explain how to recover [LS25, Lemma 4.4] from Theorem 4.2.1. Let $K_\eta \in \mathrm{Perv}(A_\eta)$ be semisimple of geometric origin such that no simple constituent of K_η is a pullback from A. Lawrence and Sawin show that for every $\chi \in \mathcal{C}(A)$ outside a thin subset $\cup_{i=1}^n \chi_i \cdot K(A_i)$ with the χ_i torsion, one has $\omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}} = 0$.

In fact, as K_η is semisimple of geometric origin, shrinking X one may find $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ such that K is semisimple of geometric origin in $D_c^b(A \times X)$ with $K|_{A_\eta} = K_\eta$. For every integer j, every simple constituent K' of ${}^p\mathcal{H}^j(Rp_{A*}K)$ in $\mathrm{Perv}(A)$ is of geometric origin. So by Remark 4.1.2, $\mathcal{S}(K')$ is a thin set with torsion translates. Thus, $\mathcal{S}({}^p\mathcal{H}^j(Rp_{A*}K))$ is contained in a thin subset with torsion translates. By assumption, $K^0|_{A_\eta}$ from Theorem 4.2.1 is zero. Therefore, for $\chi \in \mathcal{C}(A)$ outside a thin subset with torsion translates, one has $\omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}} = \omega_{\chi_\eta}(K^0|_{A_\eta}) = 0$.

4.3 Tannakian groups

Let $(\mathcal{C}, \otimes)$ a neutral Tannakian category (in the sense of [DM22, Def. 2.19]) over an algebraically closed field Q of characteristic 0, with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_Q$. Let $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ be the corresponding affine group scheme over Q. By [Del90, Sec. 9.2, p.187], up to isomorphism of group schemes, $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ is independent of the choice of ω. See [Wib22, Thm. 1.2] for an elementary proof.

For an object $K \in \mathcal{C}$, let $\iota : \langle K \rangle \hookrightarrow \mathcal{C}$ be the full subcategory whose objects are the subquotients of $\{(K \oplus K^{\vee})^{\otimes n}\}_{n \geq 1}$. Then $(\langle K \rangle, \otimes)$ is a neutral Tannakian subcategory of $\mathcal{C}$ (in the sense of [Mil07, 1.7]), for which $\omega \circ \iota : \langle K \rangle \to \mathrm{Vec}_Q$ is a fiber functor. The group scheme $\mathrm{Aut}^{\otimes}(\langle K \rangle, \omega \circ \iota)$ is the image of the natural morphism $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega) \to \mathrm{GL}(\omega(K))$. The algebraic group $\mathrm{Aut}^{\otimes}(\langle K \rangle, \omega \circ \iota)$ is called the *Tannakian group* of K at ω and is denoted by $G_{\omega}(K)$. By [Sim92, p.69], $G_{\omega}(K)$ is reductive if and only if K is semisimple in $\mathcal{C}$.

Example 4.3.1. With tensor product, $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ is a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. The forgetful functor $\omega : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ is a fiber functor. The Tannakian group of an object $\rho : \Gamma_k \to \mathrm{GL}(V)$ at ω is the Zariski closure of $\rho(\Gamma_k)$ in $\mathrm{GL}(V)$.

4.4 Convolution of perverse sheaves

Let $m : A \times_k A \to A$ be the group law on A. Let $p_i : A \times_k A \to A$ be the projection to i-th factor ($i = 1, 2$). The bifunctor

$$* : D_c^b(A) \times D_c^b(A) \to D_c^b(A), \quad K * K' := Rm_*(p_1^* K \otimes^L p_2^* K')$$

is called the *convolution* on A.

Example 4.4.1. For every closed reduced subvariety $i : X \to A$, let $\delta_X := i_* \bar{\mathbb{Q}}_{\ell, X} \in D_c^b(A)$. Then for every closed point $x \in A$, one has $\delta_x * \delta_X = \delta_{x+X}$.

By [Wei11] and [Jav+25, Sec. 3.1], the pair $(D_c^b(A), *)$ is a rigid, symmetric monoidal category, with unit δ_0. For every $K \in D_c^b(A)$, its adjoint dual is $K^{\vee} := [-1]_A^* \mathbb{D}_A K$. Theorem 4.4.2 can be found in [Jav+25, Prop. 3.1]. It is essentially from the proof of [KW15b, Thm. 13.2] (for k algebraically closed) and [LS25, Lemma 2.5 (4)] (for general k).

Theorem 4.4.2. *The convolution on A induces a bifunctor*

$$\bar{P}(A) \times \bar{P}(A) \to \bar{P}(A), \quad (K, K') \mapsto {}^p\mathcal{H}^0(K * K')$$

fitting into a commutative square

$$\begin{tikzcd} \mathrm{Perv}(A) \times \mathrm{Perv}(A) \arrow[r, "*"] \arrow[d] & D_c^b(A) \arrow[d, "{}^p\mathcal{H}^0"] \\ \bar{P}(A) \times \bar{P}(A) \arrow[r, dashed] & \bar{P}(A). \end{tikzcd}$$

It makes $\bar{P}(A)$ a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. For every $\chi \in \mathcal{C}(A)$, the subcategory $P^\chi(A)/N^\chi(A) \subset \bar{P}(A)$ is a Tannakian subcategory, on which (13) is a fiber functor.

Remark 4.4.3. In Setting 1.1.1, for every character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$ and every $K \in P^\chi(A_\eta)$, $G_\chi(K|_{A_\eta})$ is related to its geometric variant $G_\chi^{\mathrm{geo}}(K) := G_\chi(K|_{A_{\bar\eta}})$ as follows. Let $e : \eta \to A_\eta$ be the zero element. Then $e_* : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_\eta) \hookrightarrow \bar{P}(A_\eta)$ is a fully faithful exact tensor functor. Let $\mathrm{Gal}_\chi(K)$ be the Tannakian group of $\mathrm{Rep}(\Gamma_\eta) \cap \langle K|_{A_\eta}\rangle$ at the fiber functor ω_χ. By [Jav+25, Thm. 4.3], the sequence

$$\mathrm{Rep}(\Gamma_\eta) \cap \langle K|_{A_\eta}\rangle \hookrightarrow \langle K|_{A_\eta}\rangle \xrightarrow{(-)|_{A_{\bar\eta}}} \langle K|_{A_{\bar\eta}}\rangle$$

induces a short exact sequence of algebraic groups

$$1 \to G_\chi^{\mathrm{geo}}(K) \to G_\chi(K|_{A_\eta}) \to \mathrm{Gal}_\chi(K) \to 1. \tag{18}$$

The composition $\mathrm{Rep}(\Gamma_\eta) \cap \langle K|_{A_\eta}\rangle \hookrightarrow \langle K|_{A_\eta}\rangle \xrightarrow{\omega_\chi} \langle \omega_\chi(K|_{A_\eta})\rangle$ is the inclusion of a Tannakian subcategory, so $\mathrm{Mon}(K,\chi) \hookrightarrow G_\chi(K|_{A_\eta}) \to \mathrm{Gal}_\chi(K)$ is faithfully flat.

Now we introduce monodromy groups. Let $\psi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$ be a character.

Definition 4.4.4. For every $K \in \mathrm{Perv}(A)$, let $\mathrm{Mon}(K,\psi)$ be the Tannakian group of $\omega_\psi(K)$ in $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ at the forgetful fiber functor $\omega : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$.

Set $\bar\psi := \psi|_{\pi_1(A_{\bar k})}$. For every $K \in P^{\bar\psi}(A)$, $\mathrm{Mon}(K,\psi)$ is naturally a closed subgroup of $G_{\omega_{\bar\psi}}(K)$ in the following sense. The functor

$$\omega_\psi : \mathrm{Perv}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \quad K \mapsto H^0(A_{\bar k}, K \otimes^L L_\psi)$$

fits into a commutative square

$$\begin{tikzcd}
\mathrm{Perv}(A) \arrow[r, "\omega_\psi"] & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \arrow[d, "\omega"] \\
P^{\bar\psi}(A) \arrow[u, hook] \arrow[r, "(12)"] & \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}
\end{tikzcd}$$

The quotient functor $P^{\bar\psi}(A)/N^{\bar\psi}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ of $\omega_{\bar\psi} = \omega_\psi|_{P^{\bar\psi}(A)}$ induces a morphism of affine groups schemes

$$\omega_\psi^* : \mathrm{Aut}^\otimes(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \omega) \to \mathrm{Aut}^*(P^{\bar\psi}(A)/N^{\bar\psi}(A), \omega_{\bar\psi}). \tag{19}$$

Since $K \in P^{\bar\psi}(A)$, the functor $\omega_\psi|_{\langle K\rangle} : \langle K\rangle \to \langle \omega_\psi(K)\rangle$ induces a closed immersion of linear algebraic groups $\omega_\psi^* : \mathrm{Mon}(K,\psi) \hookrightarrow G_{\omega_{\bar\psi}}(K)$, which is the projection of (19) in $\mathrm{GL}(\omega_{\bar\psi}(K))$.

Remark 4.4.5. We explain how Definition 4.4.4 is related to the monodromy group of lisse sheaves. In the notation of Setting 1.1.1, let $K \in D^{\mathrm{ULA}}(A \times X/X)$. For every character $\chi \in \mathcal{C}(A)$, since L_χ is a lisse sheaf of rank 1, one has $K \otimes p_A^* L_\chi \in D^{\mathrm{ULA}}(A \times X/X)$. By Fact 2.2.2 (d), as $p_X : A \times X \to X$ is proper, one has $Rp_{X*}(K \otimes p_A^* L_\chi) \in D^{\mathrm{ULA}}(X/X)$. Then from Fact 2.2.2 (b), for every integer n, $L^n(K, \chi) := R^n p_{X*}(K \otimes p_A^* L_\chi)$ is a lisse sheaf on X. By the proper base change theorem, one has

$$L^n(K, \chi)_{\bar{\eta}} = H^n(A_{\bar{\eta}}, K|_{A_\eta} \otimes^L L_\chi).$$

Suppose that X is normal. Then the natural morphism $\eta_* : \Gamma_{k(\eta)} \to \pi_1(X, \bar{\eta})$ is surjective. Its composition with the monodromy representation

$$\pi_1(X, \bar{\eta}) \to \mathrm{GL}(L^n(K, \chi)_{\bar{\eta}})$$

is the natural Galois representation

$$\Gamma_{k(\eta)} \to \mathrm{GL}\left(H^n(A_{\bar{\eta}}, K|_{A_\eta} \otimes^L L_\chi)\right).$$

Therefore, the monodromy group of $H^n(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi)$ in $\mathrm{Rep}(\Gamma_{k(\eta)})$ coincides with that of $L^n(K, \chi)$ in $\mathrm{Loc}(X)$.

5 Normality of monodromy group

In Section 5, we prove Theorem 1.1.2. More precisely, we show that for a very general character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, the monodromy group $\mathrm{Mon}(K, \chi)$ is normal in $G_\chi(K|_{A_\eta})$. For this, we first show that $\mathrm{Mon}(K, \chi)$ is reductive in Section 5.1. Combining it with Theorem 4.2.1, we finish the proof in Section 5.2.

An algebraic subgroup H of a linear algebraic group G is *observable* in the sense of [BBHM63, p.134], if every finite dimension rational representation of H is an H-submodule of a finite dimensional rational representation of G. Lemma 5.0.1 is a well-known normality criterion.

Lemma 5.0.1. *Let G be a linear algebraic group over an algebraically closed field C. Let H be a closed, reductive subgroup of G. Then H is normal in G if and only if for every $V \in \mathrm{Rep}_C(G)$, the subspace V^H is G-stable.*

Proof. We only prove the "if" part. By [Gro06, Cor. 2.4], as H is reductive, it is observable in G. From [And21, Prop. C.3], H is normal in G. □

Consider Setting 1.1.1. For every character $\chi \in \mathcal{C}(A)$, denote the pullback of χ along $(p_A|_{A_\eta})_* : \pi_1(A_\eta) \to \pi_1(A)$ by $\chi_\eta : \pi_1(A_\eta) \to \bar{\mathbb{Q}}_\ell^\times$. Then the restriction $\chi_\eta|_{\pi_1(A_{\bar{\eta}})}$ is identified with χ *via* the isomorphism $(p_A|_{A_{\bar{\eta}}})_* : \pi_1(A_{\bar{\eta}}) \to \pi_1(A)$. Let $K \in \mathrm{Perv}(A \times X/X)$. Write $\mathrm{Mon}(K, \chi)$ for $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$. By Remark 4.4.5, when K is ULA and X is normal, $\mathrm{Mon}(K, \chi)$ is the monodromy group of the lisse sheaf $L^0(K, \chi)$.

By Lemma 5.0.1, to prove Theorem 1.1.2, it suffices to show that $\mathrm{Mon}(K, \chi)$ is reductive, and to consider the monodromy invariants of *all* the representations of $G(K|_{A_\eta})$. Such representations are from perverse sheaves.

5.1 Monodromy group is reductive

One can prove that $G(K|_{A_\eta})$ is reductive (Theorem 5.2.1 (b)). A normal subgroup of a reductive group is also reductive. To prove that $\mathrm{Mon}(K,\chi)$ is normal in $G(K|_{A_\eta})$, we first prove that $\mathrm{Mon}(K,\chi)$ is reductive.

Lemma 5.1.1. *In the notation of Setting 1.1.1, let $K \in \mathrm{Perv}(A \times X/X)$ be semisimple $D_c^b(A\times X)$. Then for every $\chi \in \mathcal{C}(A) \setminus \mathcal{S}(K|_{A_\eta})$, the monodromy group $\mathrm{Mon}(K,\chi)$ is reductive.*

Proof. By Lemma 2.1.4, when X is replaced by a nonempty open subset, the semisimplicity of K in $D_c^b(A\times X)$ is preserved. Moreover, the $\Gamma_{k(\eta)}$-representation $\omega_{\chi_\eta}(K|_{A_\eta})$ and hence the group $\mathrm{Mon}(K,\chi)$ remain unchanged. Thus, one may assume that X is smooth. From Lemma 2.1.9, as K is semisimple in $D_c^b(A\times X)$, so is $K \otimes^L p_A^* L_\chi$. By Theorem 2.1.5 (a), as k is algebraically closed, $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is semisimple in $D_c^b(X)$.

Since $\chi \notin \mathcal{S}(K|_{A_\eta})$, for every integer $n \neq 0$, one has $H^n(A_{\bar\eta}, K|_{A_{\bar\eta}} \otimes^L L_\chi) = 0$. By Fact 2.1.2, there is a nonempty open subset U_0 (resp. U_n for every integer $n \neq 0$) of X such that $\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)|_{U_0}$ is a lisse sheaf (resp. $\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi)|_{U_n} = 0$). The set

$$J := \{n \in \mathbb{Z} : \mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) \neq 0\}$$

is finite and X is irreducible, so $U := U_0 \cap \cap_{n\in J} U_n$ is a dense open subset of X. Shrinking X to U, one may assume further that $\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) = 0$ for every integer $n \neq 0$, and that $\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is a lisse sheaf on X.

Thus, the semisimple object $Rp_{X*}(K \otimes^L p_A^* L_\chi)[\dim X]$ of $D_c^b(X)$ lies in $\mathrm{Perv}(X)$, so it is semisimple in $\mathrm{Perv}(X)$. By [Ach21, Prop. 3.4.1], the object $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ of $\mathrm{Loc}(X)$ is also semisimple. Because X is smooth, the algebraic group $\mathrm{Mon}(K,\chi)$ is reductive. ☐

Example 5.1.2. In the setting of variation of Hodge structures, André shows that the connected monodromy group is contained in the derived subgroup of the Mumford-Tate group. By contrast, in Lemma 5.1.1, $\mathrm{Mon}(K,\chi)^\circ$ may be neither semisimple nor contained in the derived subgroup of $G_\chi(K|_{A_\eta})$. Let X be a smooth, projective, integral algebraic curve over k of genus 1. Then $\pi_1(X,\bar\eta) \cong \hat{\mathbb{Z}}^2$. There exists a character $\rho : \pi_1(X,\bar\eta) \to \bar{\mathbb{Q}}_\ell^\times$ of *infinite* order. Let L_ρ be the character sheaf on X induced by ρ. Let $e : X \to A \times X$ be the zero section. Then $K := e_* L_\rho \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By Fact 2.1.1, for every character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, one has $K \otimes^L_{A\times X} p_A^* L_\chi = e_*(L_\rho \otimes^L_X e^* p_A^* L_\chi) = K$. It implies $R^0 p_{X*}(K \otimes^L_{A\times X} p_A^* L_\chi) = L_\rho$ and hence $\mathrm{Mon}(K,\chi) = \mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$. Since $K|_{A_\eta}$ is the pushforward of the Galois representation $\rho|_{\Gamma_\eta} : \Gamma_\eta \to \bar{\mathbb{Q}}_\ell^\times$, one has $G_\chi(K|_{A_\eta}) = \mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$ and $G_\chi(K|_{A_{\bar\eta}}) = 1$.

5.2 Monodromy group is normal in convolution group

By [Jav+25, Thm. 4.3], for every character $\chi \in \mathcal{C}(A)$, the geometric generic convolution group $G_\chi(K|_{A_{\bar\eta}})$ is a normal closed subgroup of the generic convolution group $G_\chi(K|_{A_\eta})$. Theorem 5.2.1 shows that for uncountably many characters, the corresponding monodromy group is also a normal closed subgroup of $G_\chi(K|_{A_\eta})$.

For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$ and every $\chi_\ell \in \mathcal{C}(A)_\ell$, set $\chi = \chi_{\ell'}\chi_\ell$.

Theorem 5.2.1. *In the notation of Setting 1.1.1, let $K \in \mathrm{Perv}(A \times X/X)$ be semisimple $D_c^b(A \times X)$. Fix $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$. Then there is a countable union $B = \cup_{i\geq 1} B_i$ of strict closed subsets of $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus B$,*

(a) *one has $K|_{A_\eta} \in P^\chi(A_\eta)$;*

(b) *$G_\chi(K|_{A_\eta})$ and $G_\chi(K|_{A_{\bar\eta}})$ are reductive groups;*

(c) *and $\mathrm{Mon}(K,\chi)$ is a* normal *closed subgroup of $G_\chi(K|_{A_\eta})$.*

By Theorem 3.2.11, when $\dim A > 0$, the set $\mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus B$ is uncountable. Thus, Theorem 1.1.2 follows from Theorem 5.2.1.

We outline the strategy of proving Theorem 5.2.1 first. By Theorem 1.2.1, for every representation V of the Tannakian group $G(K|_{A_\eta})$ and every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, there is a strict Zariski closed subset B_V of the cotorus $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_V)(\bar{\mathbb{Q}}_\ell)$, the monodromy invariant $V^{\mathrm{Mon}(K,\chi)}$ is a $G(K|_{A_\eta})$-subrepresentation. Choose $B = \cup_V B_V(\bar{\mathbb{Q}}_\ell)$. From Lemma 5.0.1, normality holds when $\chi_\ell \notin B$.

Proof. Both $\mathrm{Mon}(K,\chi)$ and $G_\chi(K|_{A_\eta})$ depend only on the generic fiber of $p_X : A \times X \to X$. Therefore, shrinking X to a nonempty open subset does not change them. Thus, one may assume that X is smooth. By [SGA 4 1/2, Thm. 2.13, p.242], one may assume further $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$.

We prove Part (b). By smoothness of X and Lemma 2.3.6, the object $K|_{A_\eta}$ of $\mathrm{Perv}(A_\eta)$ is semisimple. From Lemma 5.2.4 (a), $K|_{A_\eta}$ is also semisimple in $\bar{P}(A_\eta)$. Therefore, a (hence every) Tannakian group of the neutral Tannakian category $\langle K|_{A_\eta}\rangle (\subset \bar{P}(A_\eta))$ is a *reductive*, algebraic group over $\bar{\mathbb{Q}}_\ell$. For every character $\chi \in \mathcal{C}(A)$ with $K|_{A_\eta} \in P^\chi(A_\eta)$, $G_\chi(K|_{A_\eta})$ is well-defined and reductive. Similarly, $G_\chi(K|_{A_{\bar\eta}})$ is reductive.

Then by Lemma 5.2.3, there is a sequence of objects $\{\bar{K}_i\}_{i\geq 1}$ in $\langle K|_{A_\eta}\rangle$, such that every object of $\langle K|_{A_\eta}\rangle$ is isomorphic to some $\bar{K}_i$. To apply Theorem 1.2.1, we need semisimple objects of $D_c^b(A \times X)$.

Claim 5.2.2. For every object $N \in \langle K|_{A_\eta}\rangle$, there is $L \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ that is semisimple in $D_c^b(A \times X)$, such that $L|_{A_\eta}$ is isomorphic to N in $\bar{P}(A_\eta)$.

From Claim 5.2.2, for every integer $i \geq 1$, there is $K_i \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ that is semisimple in $D_c^b(A \times X)$ with $K_i|_{A_\eta} \cong \bar{K}_i$ in $\bar{P}(A_\eta)$. From smoothness of X and Theorem 1.2.1, there is a subobject $K_i^0 \subset K_i$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ and a strict Zariski closed subset

$$B_i := \cup_{j\in\mathbb{Z}} \mathcal{S}\left({}^p\mathcal{H}^j(Rp_{A*}K_i)\right)$$

of $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_i)(\bar{\mathbb{Q}}_\ell)$, one has $K_i|_{A_\eta} \in P^\chi(A_\eta)$ and

$$\omega_{\chi_\eta}(K_i|_{A_\eta})^{\Gamma_{k(\eta)}} = \omega_{\chi_\eta}(K_i^0|_{A_\eta}). \tag{20}$$

Set $B := \cup_{i\geq 1} B_i$. For every $\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus B$, one has $K|_{A_\eta} \in P^\chi(A_\eta)$, which proves Part (a). It remains to prove Part (c). For every $i \geq 1$, by $\chi_\ell \notin B_i(\bar{\mathbb{Q}}_\ell)$ and (20), the subspace $\omega_{\chi_\eta}(K_i|_{A_\eta})^{\mathrm{Mon}(K,\chi)}$ is $G_\chi(K|_{A_\eta})$-stable. By Lemmas 5.0.1 and 5.1.1, the subgroup $\mathrm{Mon}(K,\chi)$ of $G_\chi(K|_{A_\eta})$ is *normal*. $\square$

Proof of Claim 5.2.2. From Lemma 5.2.3, N is semisimple in $\bar{P}(A_\eta)$. There is an integer $n \geq 0$ such that N is a subquotient of $(K|_{A_\eta} \oplus K|_{A_\eta}^\vee)^{*n}$ in $\bar{P}(A_\eta)$.

We “globalize" the fiberwise convolution functors as follows. Let p_{ij} be the projections on $A\times A\times X$. Define a bifunctor

$$\begin{aligned} *_X : &D_c^b(A\times X)\times D_c^b(A\times X) \to D_c^b(A\times X),\\ &(-,+)\mapsto R(m\times \mathrm{Id}_X)_*(p_{13}^* - \otimes^L p_{23}^* +). \end{aligned} \tag{21}$$

By the proper base change theorem, for every $x\in X(k)$, there exist isomorphisms of bifunctors

$$\begin{aligned} (- *_X +)|_{A_x} &\xrightarrow{\sim} (-|_{A_x}) * (+|_{A_x})\colon D_c^b(A\times X)\times D_c^b(A\times X)\to D_c^b(A_x),\\ (- *_X +)|_{A_\eta} &\xrightarrow{\sim} (-|_{A_\eta}) * (+|_{A_\eta})\colon D_c^b(A\times X)\times D_c^b(A\times X)\to D_c^b(A_\eta). \end{aligned}$$

We prove that the bifunctor (21) restricts to a bifunctor $D^{\mathrm{ULA}}(A\times X/X)\times D^{\mathrm{ULA}}(A\times X/X)\to D^{\mathrm{ULA}}(A\times X/X)$. By Fact 2.2.2 (f), as the square

$$\begin{tikzcd} A\times A\times X \arrow[r, "p_{23}"] \arrow[d, "p_{13}"] \arrow[dr, phantom, "\square"] & A\times X \arrow[d, "p_X"] \\ A\times X \arrow[r, "p_X"] & X \end{tikzcd}$$

is cartesian, for any $K', K''\in D^{\mathrm{ULA}}(A\times X/X)$, one has

$$p_{13}^*K'\otimes^L p_{23}^*K''\in D^{\mathrm{ULA}}(A\times A\times X/X).$$

By Fact 2.2.2 (d), one gets $K' *_X K''\in D^{\mathrm{ULA}}(A\times X/X)$.

Set $K^\vee := ([-1]_A\times \mathrm{Id}_X)^*\mathbb{D}_{A\times X/X}K$. By Theorem 2.3.1 (c), one has $K^\vee\in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ and $(K^\vee)|_{A_\eta} = (K|_{A_\eta})^\vee$. Then

$$(K\oplus K^\vee)^{*_X n})\in D^{\mathrm{ULA}}(A\times X/X).$$

Set $M := {}^{p/X}\mathcal{H}^0((K\oplus K^\vee)^{*_X n})\in \mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$. Then

$$M|_{A_\eta} = {}^p\mathcal{H}^0(\big(K|_{A_\eta}\oplus (K|_{A_\eta})^\vee\big)^{*n})$$

in $\mathrm{Perv}(A_\eta)$. By Lemma 5.2.4 (c), there is a semisimple subquotient L' of $M|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$, whose image in $\bar{P}(A_\eta)$ is N. By smoothness of X and Theorem 2.3.5, there is a semisimple subquotient L of M in $\mathrm{Perv}^{\mathrm{ULA}}(A\times X/X)$ with $L|_{A_\eta} = L'$. By smoothness of X and Lemma 2.3.10 (b), $L[\dim X]$ is a semisimple object of $\mathrm{Perv}(A\times X)$. Then L is semisimple in $D_c^b(A\times X)$. $\square$

Lemmas 5.2.3 and 5.2.4 are well known, but we include a proof for lack of references. For a category $\mathcal{C}$, let $\mathcal{C}/\sim$ be the class of isomorphism classes of objects in $\mathcal{C}$.

Lemma 5.2.3. *Let $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over an algebraically closed field k of characteristic 0 with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_k$. Assume that $\mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$ is a reductive, algebraic group over k. Then the underlying abelian category $\mathcal{C}$ is semisimple, and $\mathcal{C}/\sim$ is countable.*

Proof. Set $G = \mathrm{Aut}^{\otimes}(\mathcal{C}, \omega)$. Let $\mathrm{Rep}(G)$ be the category of k-rational representations of G. Then $\mathcal{C}$ is equivalent to $\mathrm{Rep}(G)$. By [Mil17, Cor. 22.43], because k has characteristic 0, the abelian category $\mathrm{Rep}(G)$ is semisimple. By [AHR20, Thm. 2.16], as k is algebraically closed, there is an at most countable set X^+ and for every $\lambda \in X^+$, a unital k-algebra $\mathscr{A}^\lambda$ with the following property: The set $\mathrm{Irr}(G)$ of isomorphism classes of simple objects of $\mathrm{Rep}(G)$ is in bijection with the set of pairs (λ, E), where $\lambda \in X^+$ and E is an isomorphism class of simple left $\mathscr{A}^\lambda$-modules. From [AHR20, Lem. 2.19], for every $\lambda \in X^+$, the algebra $\mathscr{A}^\lambda$ is semisimple. Then by [Lan02, XVII, Thm. 4.3, Cor. 4.5], the set of isomorphism classes of simple left $\mathscr{A}^\lambda$-modules is finite. Therefore, $\mathrm{Irr}(G)$ is at most countable. Consequently, $\mathrm{Rep}(G)/\sim$ is countable. $\square$

Lemma 5.2.4. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Consider the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$.*

(a) *Let $X \in \mathcal{A}$. Let $i : Y \to F(X)$ be a monomorphism in $\mathcal{A}/\mathcal{B}$. Then there is a monomorphism $j : Z \to X$ in $\mathcal{A}$ and an isomorphism $u : Y \to F(Z)$ in $\mathcal{A}/\mathcal{B}$ fitting into a commutative diagram in $\mathcal{A}/\mathcal{B}$*

$$\begin{tikzcd} & F(Z) \arrow[dr, "F(j)"] & \\ Y \arrow[ur, "u"] \arrow[rr, hook, "i"] & & F(X). \end{tikzcd}$$

Dually, up to isomorphism every quotient in $\mathcal{A}/\mathcal{B}$ lifts to a quotient in $\mathcal{A}$. In particular, if $X \in \mathcal{A}$ is a simple object, then $F(X)$ is either simple or zero in $\mathcal{A}/\mathcal{B}$.

(b) *Let $V \in \mathcal{A}$ be a Noetherian and Artinian object. If $F(V)$ is simple in $\mathcal{A}/\mathcal{B}$, then there is a simple subquotient W of V in $\mathcal{A}$ such that $F(W)$ is isomorphic to $F(V)$ in $\mathcal{A}/\mathcal{B}$.*

(c) *Assume that $\mathcal{A}$ is Noetherian and Artinian. Let $X \in \mathcal{A}$. Let Y be a simple subquotient of $F(X)$ in $\mathcal{A}/\mathcal{B}$. Then there is a simple subquotient W of X, with $F(W)$ isomorphic to Y in $\mathcal{A}/\mathcal{B}$.*

Proof.

(a) By the right calculus of fractions, there is a diagram $Y \xleftarrow{f} M \xrightarrow{g} X$ in $\mathcal{A}$, such that $F(f)$ is an isomorphism and $F(g) = i \circ F(f)$ in $\mathcal{A}/\mathcal{B}$. Therefore, $F(g)$ is a monomorphism. Since F is exact, one has $F(\ker(g)) = \ker(F(g)) = 0$, so $\ker(g) \in \mathcal{B}$. Let $q : M \to M/\ker(g)$ be the epimorphism in $\mathcal{A}$, and let $j : M/\ker(g) \to X$ be the monomorphism in $\mathcal{A}$ induced by g. Then $F(q)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Set $u : Y \to F(M/\ker(g))$ to be the morphism $F(q) \circ F(f)^{-1}$ in $\mathcal{A}/\mathcal{B}$. Then u is an isomorphism with the stated property.

(b) Let $\mathcal{P}$ be the family of subobjects V' of V in $\mathcal{A}$ with $V/V' \in \mathcal{B}$. Then $\mathcal{P}$ is nonempty since $V \in P$. As V is Artinian in $\mathcal{A}$, there is a minimal object $U \in \mathcal{P}$. Moreover, the morphism $F(U) \to F(V)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Let $\mathcal{Q}$ be the family of subobjects of U in $\mathcal{A}$ that lie in $\mathcal{B}$. Then $\mathcal{Q}$ is nonempty since $0 \in \mathcal{Q}$. As V is Noetherian in $\mathcal{A}$, so is U. Thus, $\mathcal{Q}$ has a maximal object U_0. Then $W := U/U_0$ is a subquotient of V in $\mathcal{A}$, and the morphism $F(U) \to F(W)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. In particular, $W \neq 0$ in $\mathcal{A}$.

It remains to prove that W is simple in $\mathcal{A}$. For this, let $U' \to W$ be a subobject in $\mathcal{A}$. Then there is a subobject U'' of U in $\mathcal{A}$ containing U_0 with $U''/U_0 = U'$. As $F(U'')$ is a subobject of a simple object $F(U)$ in $\mathcal{A}/\mathcal{B}$, either the morphism $F(U'') \to F(U)$ is an isomorphism or $F(U'') = 0$. If $F(U'') = 0$, then $U'' \in \mathcal{B}$ and $U'' \in \mathcal{Q}$. Since U_0 is maximal in $\mathcal{Q}$, one has $U_0 = U''$, so $U' = 0$. Now assume that $F(U'') \to F(U)$ is an isomorphism. Then $U/U'' \in \mathcal{B}$. Since the sequence

$$0 \to U/U'' \to V/U'' \to V/U \to 0$$

is exact in $\mathcal{A}$, and $\mathcal{B}$ is closed under extensions, one gets $V/U'' \in \mathcal{B}$ and $U'' \in \mathcal{P}$. Since U is minimal in $\mathcal{P}$, one has $U'' = U$. The morphism $U' \to W$ is thus an isomorphism in $\mathcal{A}$.

(c) By Part (a), there is a subquotient Z of X in $\mathcal{A}$ with $F(Z)$ isomorphic to Y. Then $F(Z)$ is simple in $\mathcal{A}/\mathcal{B}$. By assumption, Z is Noetherian and Artinian in $\mathcal{A}$. Thus from Part (b), there is a simple subquotient W of Z in $\mathcal{A}$ with $F(W)$ isomorphic to $F(Z)$ and to Y in $\mathcal{A}/\mathcal{B}$. □

We deduce a dichotomy result from Theorem 5.2.1. For a reductive group G over an algebraically closed field C, G^* is a connected semisimple algebraic group. If G is connected semisimple, then $G = G_1 \dots G_r$ is the almost-direct product of finitely many almost-simple normal subgroups. We call such a G_i a *simple factor* of G.

Corollary 5.2.5. *In the notation of Theorem 5.2.1, for a very general character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, either $\mathrm{Mon}(K,\chi)^\circ$ is a torus, or $\mathrm{Mon}(K,\chi)$ contains a simple factor of $G_\chi(K|_{A_\eta})^*$.*

Proof. By Theorem 5.2.1 (b) and (c), $\mathrm{Mon}(K,\chi)^*$ is a connected normal subgroup of the semisimple group $G_\chi(K|_{A_\eta})^*$. By [Mil17, Thm. 21.51], either

$\mathrm{Mon}(K,\chi)^*$ is trivial, or it contains a simple factor of $G_\chi(K|_{A_\eta})^*$. By Lemma 5.1.1, if $\mathrm{Mon}(K,\chi)^*$ is trivial, then $\mathrm{Mon}(K,\chi)^\circ$ is a torus. $\square$

In the setting of Theorem 5.2.1, assume further that K arises from a closed subvariety Z of $A \times X$ not constant up to translation in $A(X)$. By [Jav+25, Thm. 4.10], if $G(K|_{A_{\bar\eta}})^*$ is *simple*, then for a *general* character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, one has $G(K|_{A_{\bar\eta}})^* \triangleleft \mathrm{Mon}(K,\chi)$. However, the simplicity hypothesis on $G(K|_{A_{\bar\eta}})^*$ may fail, for instance when $Z_{\bar\eta}$ decomposes into a product of strict subvarieties ([Jav+25, Prop. 6.5]).

Example 5.2.6. In Setting 1.1.1, let Y, Z be closed subvarieties of $A \times X$, smooth over X of relative dimensions d and d' respectively. Then $P := \bar{\mathbb{Q}}_{\ell,Y}[d]$ and $Q := \bar{\mathbb{Q}}_{\ell,Z}[d']$ are in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Set $K := P \oplus Q$. In general, $G(K|_{A_{\bar\eta}})^*$ may not be simple even if both $G(P|_{A_{\bar\eta}})^*$ and $G(Q|_{A_{\bar\eta}})^*$ are simple. For every character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, as a Γ_η-representation $\omega_\chi(K|_{A_\eta})$ decomposes into $\omega_\chi(P|_{A_\eta}) \oplus \omega_\chi(Q|_{A_\eta})$. Therefore, $\mathrm{Mon}(P,\chi) \times \mathrm{Mon}(Q,\chi) \subset \mathrm{GL}(\omega_\chi(K|_{A_\eta}))$ contains $\mathrm{Mon}(K,\chi)$, and both projections

$$\mathrm{Mon}(K,\chi) \to \mathrm{Mon}(P,\chi), \quad \mathrm{Mon}(K,\chi) \to \mathrm{Mon}(Q,\chi)$$

are surjective.

Assume that for a very general χ, $\mathrm{Mon}(P,\chi)^\circ$ is not a torus. (Specific examples of subvarieties Y with "big" monodromy group $\mathrm{Mon}(P,\chi)^\circ$ can be found in [LS25, Thm. 4.7] and [Jav+25, p.1049], but we only need the weaker hypothesis that $\mathrm{Mon}(P,\chi)^\circ$ is not a torus.) Then $\mathrm{Mon}(K,\chi)^\circ$ is not a torus, neither. By Corollary 5.2.5, $\mathrm{Mon}(K,\chi)$ contains a simple factor of $G_\chi(K|_{A_\eta})^*$.

Remark 5.2.7. In view of [CL25, Cor. 1.5 (2)], the following analog of Theorem 5.2.1 seems plausible. In Setting 1.1.1, assume that X is smooth over $k = \mathbb{C}$ of dimension d, and consider constructible sheaves with $\mathbb{C}$-coefficients. Let F be a semisimple perverse sheaf on $A \times X$. For every character $\chi : \pi_1^{\mathrm{top}}(A(\mathbb{C})) \to \mathbb{C}^*$, the constructible sheaf $R^{-d}p_{X*}(F \otimes p_A^* L_\chi)$ on X restricts to a $\mathbb{C}$-local system on a dense Zariski open subset U of X. For $x \in U(\mathbb{C})$, let $\mathrm{Mon}_x(F[-d],\chi)$ be the monodromy group of

$$\rho_{F[-d],\chi} : \pi_1^{\mathrm{top}}(U(\mathbb{C}),x) \to \mathrm{GL}\left(H^{-d}(A_x, F|_{A_x} \otimes L_\chi)\right).$$

Then for a very general χ and a very general $x \in X(\mathbb{C})$, $\mathrm{Mon}_x(F[-d],\chi) \cap G_\chi(F|_{A_x}[-d])$ is a closed normal subgroup of $G_\chi(F|_{A_x}[-d])$.

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Conflict of interest

There is no conflict of interest.

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There is no associated data.

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