

Quasi-coherent sheaves on complex analytic spaces

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Abstract

We show that in the category of analytic sheaves on a complex analytic space, the full subcategory of quasi-coherent sheaves is an abelian subcategory.

1 Introduction

Let (X, O_X) be a ringed space. The category of O_X -modules is denoted by $\text{Mod}(O_X)$.

Definition 1.1. An O_X -module F is called *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$ of x , two sets I, J and a morphism $O_U^{\oplus J} \rightarrow O_U^{\oplus I}$ with cokernel isomorphic to $F|_U$. The full subcategory of $\text{Mod}(O_X)$ of quasi-coherent modules is denoted by $\text{Qch}(X)$.

According to [Sta24, Tag 01BD], in general $\text{Qch}(X)$ is not an abelian category. By [Sta24, Tag 06YZ], if X is a scheme, then $\text{Qch}(X)$ is a weak Serre subcategory (in the sense of [Sta24, Tag 02MO (2)]) of $\text{Mod}(O_X)$. We show a complex analytic analog of this result, contrary to a guess made in [m]).

Theorem 1.2. *If X is a complex analytic space, then $\text{Qch}(X) \subset \text{Mod}(O_X)$ is a weak Serre subcategory. In particular, it is an abelian subcategory.*

2 Preliminaries

Let (X, O_X) be a ringed space. Every $O_X(X)$ -module naturally induces a quasi-coherent O_X -module.

Example 2.1 ([Sta24, Tag 01BI]). Let $f : (X, O_X) \rightarrow (\{*\}, O_X(X))$ be the morphism of ringed spaces, with $f : X \rightarrow \{*\}$ the unique map and $f_*^{\natural} : O_X(X) \rightarrow O_X(X)$ the identity. For an $O_X(X)$ -module M , its pullback f^*M is called the sheaf associated with M . This O_X -module is quasi-coherent. The functor $f^* : \text{Mod}(O_X(X)) \rightarrow \text{Mod}(O_X)$ is called *localization* and denoted by $\tilde{\cdot}$.

From [EGA I, 4.1.1], on a scheme the direct sum of any family of quasi-coherent modules is quasi-coherent. It fails for complex manifolds, shown by Example 2.2.

Example 2.2. [sta] Let $X \subset \mathbb{C}$ be the unit open disk. For every integer $n \geq 2$, Gabber ([Con06, Eg. 2.1.6]) constructs a locally free (hence quasi-coherent) O_X -module F_n of infinite rank, such that for every open subset $U \subset X$ containing $\{\pm 1/n\}$, one has $\Gamma(U, F_n) = 0$. We prove that $F := \bigoplus_{n \geq 2} F_n$ is not quasi-coherent.

Assume the contrary. Then there is an open neighborhood V of $0 \in X$, a set I and a quotient morphism $q : O_V^{\oplus I} \rightarrow F|_V$. There is an integer $N \geq 2$ with $\{\pm 1/N\} \subset V$. Let $p : F|_V \rightarrow F_N|_V$ be the quotient morphism. Because $\text{Hom}_{\text{Mod}(O_V)}(O_V, F_N|_V) = \Gamma(V, F_N) = 0$, the morphism $pq = 0$. However, it contradicts $F_N|_V \neq 0$.

Let X be a complex analytic space in the sense of [GR04, p.18]. For an inclusion $i : K \rightarrow X$ of a compact subset, let $O_K = i^{-1}O_X$. Then O_K is naturally a sheaf of rings on K .

Definition 2.3. A compact subset K of X is a *Stein compactum*, if K has a fundamental system of open neighborhoods that are Stein subspaces of X . A Stein compactum K is *Noetherian* if $O_K(K)$ is a Noetherian ring.

Fact 2.4 ([Fri67, Thm. I, 9, Rem. I, 10]). *Every $x \in X$ admits a neighborhood which is a Noetherian Stein compactum in X .*

Lemma 2.5. *Let F be an O_X -module. Then the following conditions are equivalent:*

1. ([BBP07, Def. 5.1]) *Every $x \in X$ admits a neighborhood K which is a Noetherian Stein compactum, such that $F|_K$ is associated with an $O_K(K)$ -module.*
2. *The O_X -module F is quasi-coherent.*

Proof.

- Assume Condition 1. For every $x \in X$, take such a K and suppose that $F|_K$ is associated with an $O_K(K)$ -module M . There are sets I, J and an exact sequence $O_K(K)^{\oplus I} \rightarrow O_K(K)^{\oplus J} \rightarrow M \rightarrow 0$ in the category of $O_K(K)$ -modules. By [Sta24, Tag 01BH], it induces an exact sequence $O_K^{\oplus I} \rightarrow O_K^{\oplus J} \rightarrow F|_K \rightarrow 0$ in $\text{Mod}(O_K)$. Then the O_K° -module $F|_{K^\circ}$ is quasi-coherent. Thus, Condition 2 is proved.
- Assume Condition 2. By [Sta24, Tag 01BK], because X is locally compact Hausdorff, for every $x \in X$, there is an open neighborhood U of x in X such that $F|_U$ is associated with a $\Gamma(U, O_X)$ -module. From Fact 2.4, there is a neighborhood K of $x \in U$ which is a Noetherian Stein compactum. By [Sta24, Tag 01BJ] applied to the morphism $(K, O_K) \rightarrow (U, O_U)$ of ringed spaces, $F|_K$ is associated with an $O_K(K)$ -module. Thus, Condition 1 is proved.

□

Lemma 2.6. *Let K be a Noetherian Stein compactum in X .*

1. *The natural transformation $\text{Id} \rightarrow \Gamma(K, \tilde{\cdot})$ of functors $\text{Mod}(O_K(K)) \rightarrow \text{Mod}(O_K(K))$ is an isomorphism.*
2. *The localization functor $\tilde{\cdot} : \text{Mod}(O_K(K)) \rightarrow \text{Mod}(O_K)$ is exact, fully faithful.*
3. *For every $O_K(K)$ -module M and every integer $q > 0$, one has $H^q(K, \tilde{M}) = 0$.*

Proof.

1. Let M be an $O_K(K)$ -module. We prove that the morphism $M \rightarrow \Gamma(K, \tilde{M})$ is an isomorphism. Assume first that M is finitely generated. Then the result follows from [Tay02, p.299]. Assume now that M is arbitrary. Let $\{M_i\}_{i \in I}$ be the family of all finitely generated submodules of M . This family is directed in the inclusion relation and

$$M = \sum_{i \in I} M_i. \quad (1)$$

By [Sta24, Tag 01BH (4)], the localization functor preserves colimits. Therefore,

$$\tilde{M} = \text{colim}_{i \in I} \tilde{M}_i. \quad (2)$$

By [God58, Thm. 4.12.1], one has

$$\Gamma(K, \tilde{M}) = \text{colim}_{i \in I} \Gamma(K, \tilde{M}_i) = \text{colim}_{i \in I} M_i = M.$$

2. The exactness is proved in [Tay02, Prop. 11.9.3 (ii)]. For any $M, N \in \text{Mod}(O_K(K))$, we prove that the natural morphism

$$\text{Hom}_{O_K(K)}(M, N) \rightarrow \text{Hom}_{O_K}(\tilde{M}, \tilde{N}) \quad (3)$$

is an isomorphism.

Assume first that M is finitely generated. As the ring $O_K(K)$ is Noetherian, the $O_K(K)$ -module M is of finite presentation. Then by [GW20, Exercise 7.20 (b)], one has $\widetilde{\text{Hom}_{O_K(K)}(M, N)} = \mathcal{H}om_{O_K}(\tilde{M}, \tilde{N})$. By Part 1, the morphism (3) is an isomorphism. Assume now that M is arbitrary. By (1) and (2), the morphism (3) is the inverse limit of the morphisms $\text{Hom}_{O_K(K)}(M_i, N) \rightarrow \text{Hom}_{O_K}(\tilde{M}_i, \tilde{N})$, each of which is an isomorphism.

3. When M is finitely generated, it follows from [Tay02, Prop. 11.9.2] and [Car57, Thm. 1 (B)]. Assume now that M is arbitrary. By (2) and [God58, Thm. 4.12.1], one has $H^q(K, \tilde{M}) = \text{colim}_i H^q(K, \tilde{M}_i) = 0$.

□

3 Proof of Theorem 1.2

1. For every morphism $f : F \rightarrow G$ in $\text{Qch}(X)$, we prove that $\ker(f), \text{coker}(f)$ in $\text{Mod}(O_X)$ lie in $\text{Qch}(X)$.

By Lemma 2.5, for every $x \in X$, there is a neighborhood A (resp. B) of $x \in X$ which is a Noetherian Stein compactum and an $O_A(A)$ -module M (resp. $O_B(B)$ -module N), such that $F|_A$ (resp. $G|_B$) is associated with M (resp. N). By Fact 2.4, there is a neighborhood C of x in $A^\circ \cap B^\circ$ which is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F|_C$ (resp. $G|_C$) is associated with $M \otimes_{O_A(A)} O_C(C)$ (resp. $N \otimes_{O_B(B)} O_C(C)$). By Lemma 2.6 2, there is a morphism

$$\phi : M \otimes_{O_A(A)} O_C(C) \rightarrow N \otimes_{O_B(B)} O_C(C)$$

in $\text{Mod}(O_C(C))$ whose localization is $f|_C : F|_C \rightarrow G|_C$. The restriction functor $\text{Mod}(O_X) \rightarrow \text{Mod}(O_{C^\circ})$ is exact, so $\ker(f)|_{C^\circ}$ (resp. $\text{coker}(f)|_{C^\circ}$) is the localization of $\ker(\phi \otimes_{O_C(C)} \text{Id}_{O_X(C^\circ)})$ (resp. $\text{coker}(\phi \otimes_{O_C(C)} \text{Id}_{O_X(C^\circ)})$) in $\text{Mod}(O_X(C^\circ))$. Therefore, the O_X -modules $\ker(f)$ and $\text{coker}(f)$ are quasi-coherent.

2. Let

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \quad (4)$$

be a short exact sequence in $\text{Mod}(O_X)$, with F' and F'' quasi-coherent. We prove that F is quasi-coherent.

By Lemma 2.5, for every $x \in X$, there is a neighborhood K' (resp. K'') of x which is a Noetherian Stein compactum, and an $O_{K'}(K')$ -module M' (resp. $O_{K''}(K'')$ -module M'') whose localization is $F'|_{K'}$ (resp. $F''|_{K''}$). By Fact 2.4, there is a neighborhood K of $x \in K'^\circ \cap K''^\circ$ that is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F'|_K$ (resp. $F''|_K$) is associated with the $O_K(K)$ -module $M' \otimes_{O_{K'}(K')} O_K(K)$ (resp. $M'' \otimes_{O_{K''}(K'')} O_K(K)$).

Let $P = \Gamma(K, F)$. By Lemma 2.6 1 and 3, the sequence (4) induces a short exact sequence in $\text{Mod}(O_K(K))$:

$$0 \rightarrow M' \otimes_{O_{K'}(K')} O_K(K) \rightarrow P \rightarrow M'' \otimes_{O_{K''}(K'')} O_K(K) \rightarrow 0.$$

From Lemma 2.6 2, by localization it induces a shot exact sequence in $\text{Mod}(O_K)$:

$$0 \rightarrow M' \otimes_{O_{K'}(K')} \widetilde{O_K(K)} \rightarrow \tilde{P} \rightarrow M'' \otimes_{O_{K''}(K'')} \widetilde{O_K(K)} \rightarrow 0.$$

By restricting to K° and [Sta24, Tag 01BJ], one has a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' \otimes_{O_{K'}(K')} \widetilde{O_X(K^\circ)} & \longrightarrow & P \otimes_{O_K(K)} \widetilde{O_X(K^\circ)} & \longrightarrow & M'' \otimes_{O_{K''}(K'')} \widetilde{O_X(K^\circ)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F'|_{K^\circ} & \longrightarrow & F|_{K^\circ} & \longrightarrow & F''|_{K^\circ} \longrightarrow 0 \end{array}$$

in $\text{Mod}(O_{K^\circ})$. The vertical morphisms are given by the canonical morphism $P \otimes_{O_K(K)} O_X(K^\circ) \rightarrow \Gamma(K^\circ, F)$ in $\text{Mod}(O_X(K^\circ))$, and the adjunction between functors $\tilde{\cdot} : \text{Mod}(O_X(K^\circ)) \rightarrow \text{Mod}(O_{K^\circ})$ and $\Gamma(K^\circ, \cdot) : \text{Mod}(O_{K^\circ}) \rightarrow \text{Mod}(O_X(K^\circ))$. The rows are exact, and the two outside vertical arrows are isomorphisms. By the five lemma, the middle vertical morphism is an isomorphism. By Example 2.1, the O_{K° -module $F|_{K^\circ}$ is quasi-coherent. Consequently, F is quasi-coherent.

By 1, 2 and [Sta24, Tag 0754], $\text{Qch}(X)$ is a weak Serre subcategory of $\text{Mod}(O_X)$.

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References

- [BBP07] Oren Ben-Bassat, Jonathan Block, and Tony Pantev. “Non-commutative tori and Fourier-Mukai duality”. In: *Compositio Mathematica* 143.2 (2007), pp. 423–475. DOI: 10.1112/S0010437X06002636.
- [Car57] Henri Cartan. “Variétés analytiques réelles et variétés analytiques complexes”. In: *Bulletin de la Société Mathématique de France* 85 (1957), pp. 77–99.
- [Con06] Brian Conrad. “Relative ampleness in rigid geometry”. In: *Annales de l’institut Fourier* 56.4 (2006), pp. 1049–1126.
- [EGA I] Alexander Grothendieck. “Éléments de géométrie algébrique : I. Le langage des schémas”. In: *Publications Mathématiques de l’IHÉS* 4 (1960), pp. 5–228.
- [Fri67] Jacques Frisch. “Points de platitude d’un morphisme d’espaces analytiques complexes.” In: *Inventiones mathematicae* 4 (1967), pp. 118–138.
- [God58] Roger Godement. “Topologie algébrique et théorie des faisceaux”. In: *Publications de l’institut de Mathématique de L’Université de Strasbourg XIII* 1 (1958).
- [GR04] Hans Grauert and Reinhold Remmert. *Theory of Stein spaces*. Springer Science & Business Media, 2004. DOI: 10.1007/978-3-642-18921-0.
- [GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. 2nd. Springer, 2020.
- [m)] Z. M (<https://mathoverflow.net/users/176381/z-m>). *Does quasi-coherent modules over $(X^{an}, \mathcal{O}_{X^{an}})$ have enough injectives?* <https://mathoverflow.net/q/421073> (version: 2022-04-25).
- [sta] Jason Starr (<https://mathoverflow.net/users/13265/jason-starr>). *Is an infinite direct sum of quasi-coherent $\mathcal{O}_X$ -modules quasi-coherent on a complex manifold?* <https://mathoverflow.net/q/453047> (version: 2023-08-18).

- [Sta24] The Stacks project authors. *The Stacks project*. <https://stacks.math.columbia.edu>. 2024.
- [Tay02] Joseph L. Taylor. *Several complex variables with connections to algebraic geometry and Lie groups*. Vol. 46. American Mathematical Society Providence, 2002.