Quasi-coherent GAGA

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Abstract

Serre's GAGA asserts that for complex projective varieties, coherent algebraic sheaves are equivalent to coherent analytic sheaves. Hall extends the equivalence to the bounded derived category of coherent sheaves. We extend the equivalence to the derived category of quasi-coherent sheaves.

1 Introduction

Serre's work [Ser56] (known as GAGA) reveals the close relation between algebraic geometry and analytic geometry. An algebraic variety means a finite type, separated scheme over a field. Let X be a complex algebraic variety. Then the set of complex points $X(\mathbb{C})$ underlies a natural complex analytic space (in the sense of [Ser56, Déf. 1]) structure, denoted by X^{an}. When X is a projective variety, Serre [Ser56, Théorèmes 2 et 3] proves that the abelian category of (algebraic) coherent sheaves on X is naturally equivalent to that of (analytic) coherent sheaves on X^{an}. By [Ser56, Thm. 1], the equivalence leaves cohomology groups invariant. Hall [Hal23] extends the equivalence to the bounded derived category of coherent sheaves (Fact 2.1).

A natural question is to find analogous equivalences for the larger category of quasi-coherent sheaves on X. We show that Kashiwara's notion of *good modules* (Definition 2.2) is an analytic counterpart of quasi-coherent sheaves on algebraic varieties.

For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the abelian category of O_X-modules. Let $D(X)$ be the unbounded derived category of $\mathrm{Mod}(O_X)$. For an algebraic variety (resp. a complex analytic space) X, let $\mathrm{Qch}(X) \subset \mathrm{Mod}(O_X)$ (resp. $\mathrm{Good}(X) \subset \mathrm{Mod}(O_X)$) be the full subcategory of quasi-coherent (resp. good) modules. Let $D_{\mathrm{qc}}(X)$ (resp. $D_{\mathrm{gd}}(X)$) be the full subcategory of $D(X)$ comprised of objects with quasi-coherent (resp. good) cohomologies.

Theorem (Proposition 3.2, Theorem 4.2)**.** *Let X be a proper scheme over $\mathbb{C}$. Then the analytification functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(X^{\mathrm{an}})$ is an equivalence of triangulated categories. It restricts to an equivalence of abelian categories $\mathrm{Qch}(X) \to \mathrm{Good}(X^{\mathrm{an}})$.*

2 Review

We recall the work of Serre [Ser56], which gives an equivalence of algebraic coherent sheaves and analytic coherent sheaves on complex projective varieties. The theory is extended to complex, proper algebraic varieties in [SGA 1, Exp. XII].

Let X be a complex algebraic variety. Let An (resp. Set) be the category of complex analytic spaces (resp. sets). Let Ψ_X be the functor $\mathrm{An} \to \mathrm{Set}$ sending a complex analytic space Y to the set $\mathrm{Hom}_{\mathbb{C}}(Y, X)$ of morphisms of spaces with a sheaf of $\mathbb{C}$-algebras. By [SGA 1, Exp. XII, Thm. 1.1], the functor Ψ_X is represented by a complex analytic space X^{an} (called the *analytification* of X) and a *flat* morphism $\psi_X \in \mathrm{Hom}_{\mathbb{C}}(X^{\mathrm{an}}, X)$. From [SGA 1, Exp. XII, Prop. 2.1 (viii)], because X is of finite type over $\mathbb{C}$, the dimension of X^{an} is finite. By [SGA 1, Exp. XII, 1.2], for every morphism $f : X \to Y$ of complex algebraic varieties, there is a commutative square

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "\psi_X"] \arrow[d, "f^{\mathrm{an}}"] & X \arrow[d, "f"] \\ Y^{\mathrm{an}} \arrow[r, "\psi_Y"] & Y \end{tikzcd} \tag{1}$$

in the category of ringed spaces. In other words, the analytification induces a functor $(\cdot)^{\mathrm{an}}$ from the category of complex algebraic varieties to An.

For a ringed space (Y, O_Y), let $\mathrm{Coh}(Y) \subset \mathrm{Mod}(O_Y)$ be the full subcategory comprised of coherent sheaves in the sense of [Sta24, Tag 01BV]. Let $D_c(Y) \subset D(Y)$ and $D^b_c(Y) \subset D^b(Y)$ be the full subcategories of objects with coherent cohomologies. As ψ_X is flat, the pullback functor

$$\psi_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X^{\mathrm{an}}}), \quad F \mapsto F^{\mathrm{an}} \tag{2}$$

is exact, admits a right adjoint and commutes with colimits. It extends to a functor $D(X) \to D(X^{\mathrm{an}})$, which is t-exact relative to the standard t-structures. From [SGA 1, Exp. XII, 1.3], it restricts to a functor $D^b_c(X) \to D^b_c(X^{\mathrm{an}})$ and $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$.

Fact 2.1 can be retracted from [Hal23, Remark 1.1 and the proof of Theorem A]. Neeman [Nee21, Example A.2] modifies Hall's proof to some extent.

Fact 2.1. Assume that the complex algebraic variety X is proper. Then the functor (2) induces an equivalence $D^b_c(X) \to D^b_c(X^{\mathrm{an}})$ of triangulated categories. In particular, it restricts to an equivalence of abelian categories $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$.

An analytic analog of algebraic quasi-coherent sheaves is introduced by Kashiwara.

Definition 2.2. [Kas03, Def. 4.22] On a complex analytic space X, an O_X-module F called *good* if for every relatively compact open subset $U \subset X$, there exists a directed family $\{G_i\}_{i\in I}$ of coherent O_U-submodules of $F|_U$ such that $F|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ being a directed family means that for any $i, i' \in I$, there is $i'' \in I$ with $G_i + G_{i'} \subset G_{i''}$. (Therefore, $F|_U = \mathrm{colim}_{i\in I} G_i$).

By [Liu24, Lem. A.4.3], $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$, and $D_{\mathrm{gd}}(X)$ is a triangulated subcategory of $D(X)$.

Lemma 2.3. For the complex algebraic variety X, the functor (2) restricts to a functor

$$\mathrm{Qch}(X) \to \mathrm{Good}(X^{\mathrm{an}}) \tag{3}$$

and induces a functor

$$D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(X^{\mathrm{an}}). \tag{4}$$

Proof. By Fact 2.4, every quasi-coherent O_X-module F can be written as the sum of a direct family of coherent O_X-submodules

$$F = \sum_{i\in I} F_i. \tag{5}$$

As ψ_X^* commutes with colimits, one has

$$\psi_X^* F = \mathrm{colim}_{i\in I} \psi_X^* F_i \tag{6}$$

in the category $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Since ψ_X^* is exact, each $\psi_X^* F_i$ is a coherent $O_{X^{\mathrm{an}}}$-*submodule* of $\psi_X^* F$. Therefore, the $O_{X^{\mathrm{an}}}$-module $\psi_X^* F$ is good.

For every $G \in D_{\mathrm{qc}}(X)$ and every integer n, because (2) is an exact functor, the $O_{X^{\mathrm{an}}}$-module $H^n(\psi_X^* G) = \psi_X^*(H^n G)$ is good by last paragraph. Hence $\psi_X^* G \in D_{\mathrm{gd}}(X^{\mathrm{an}})$. $\square$

Fact 2.4 ([EGA I, Cor. 9.4.9], [Sta24, Tag 01PG])**.** On a Noetherian scheme, every quasi-coherent sheaf is the sum of the directed family of all coherent submodules.

3 GAGA for quasi-coherent sheaves

Using Fact 2.4 and that ψ_X^* commutes with colimits, we extend GAGA from coherent sheaves to quasi-coherent sheaves.

When $Y = \operatorname{Spec} \mathbb{C}$, Proposition 3.1 generalizes [Ser56, Thm. 1].

Proposition 3.1. *Let $f : X \to Y$ be a* proper *morphism of complex algebraic varieties. Then the base change natural transformation $(Rf_*\cdot)^{\mathrm{an}} \to Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ (induced by the commutative square* (1)*) induces an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$.*

Proof. By [Lip60, Prop. 3.9.2], for every $F \in D_{\mathrm{qc}}(X)$, one has $Rf_*F \in D_{\mathrm{qc}}(Y)$. By Lemma 2.3, one has $F^{\mathrm{an}} \in D_{\mathrm{gd}}(X^{\mathrm{an}})$ and $(Rf_*F)^{\mathrm{an}} \in D_{\mathrm{gd}}(Y^{\mathrm{an}})$. From [SGA 1, Exp. XII, Prop. 3.2 (v)], since f is proper, the morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper. By [Liu24, Thm. 3.1.6], as X^{an} has finite dimension, one has $Rf_*^{\mathrm{an}} F^{\mathrm{an}} \in D_{\mathrm{gd}}(Y^{\mathrm{an}})$. Therefore, both functors $(Rf_*\cdot)^{\mathrm{an}}$ and $Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ restrict to functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$.

We prove that the morphism $(Rf_*F)^{\mathrm{an}} \to Rf_*^{\mathrm{an}} F^{\mathrm{an}}$ is an isomorphism. By [Liu24, Lem. 3.1.10] (resp. [Lip60, Prop. 3.9.2]), the functor $Rf_*^{\mathrm{an}} : D(X^{\mathrm{an}}) \to$

$D(Y^{\mathrm{an}})$ (resp. $Rf_* : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$) is bounded. From [Sta24, Tag 06YZ], the inclusion functor $\mathrm{Qch}(X) \to \mathrm{Mod}(O_X)$ exhibits $\mathrm{Qch}(X)$ as a weak Serre subcategory of $\mathrm{Mod}(O_X)$ in the sense of [Sta24, Tag 02MO]. Then by the way-out argument [Har66, I, Prop. 7.1 (iii)], one may assume $F \in \mathrm{Qch}(X)$. By [KS06, Prop. 13.1.5 (ii), p.320], it suffices to check that for every integer $n \geq 0$, the natural morphism $(R^n f_* F)^{\mathrm{an}} \to R^n f_*^{\mathrm{an}}(F^{\mathrm{an}})$ in $\mathrm{Mod}(O_{Y^{\mathrm{an}}})$ is an isomorphism.

One can write F as in (5). By [Sta24, Tag 07TB], one has

$$\mathrm{colim}_{i\in I} R^n f_* F_i \xrightarrow{\sim} R^n f_* F.$$

The analytification commutes with colimits, so

$$\mathrm{colim}_{i\in I} (R^n f_* F_i)^{\mathrm{an}} \xrightarrow{\sim} (R^n f_* F)^{\mathrm{an}}.$$

By [SGA 1, XII, Thm. 4.2], the natural morphisms $(R^n f_* F_i)^{\mathrm{an}} \to R^n f_*^{\mathrm{an}}(F_i^{\mathrm{an}})$ are isomorphisms. By [Liu24, Lem. 3.1.8], the natural morphism

$$\mathrm{colim}_{i\in I} R^n f_*^{\mathrm{an}}(F_i^{\mathrm{an}}) \to R^n f_*^{\mathrm{an}}(F^{\mathrm{an}})$$

is an isomorphism. □

Proposition 3.2 shows that goodness on complex analytic spaces is an analytic counterpart of quasi-coherence on complex algebraic varieties.

Proposition 3.2. *Suppose that the complex algebraic variety X is proper. Then (3) is an equivalence of abelian categories.*

Proof.

- The functor (3) is essentially surjective: Indeed, by properness of X and [SGA 1, Exp. XII, Prop. 3.2 (v)], X^{an} is compact. Then for every good $O_{X^{\mathrm{an}}}$-module G, one can write $G = \sum_{i\in I} G_i$ as the sum of a directed family of coherent $O_{X^{\mathrm{an}}}$-submodules. From the equivalence $\psi_X^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ in [SGA 1, XII, Thm. 4.4], there is a filtered inductive system $\{H_i\}_{i\in I}$ in $\mathrm{Coh}(X)$ whose analytification is the filtered inductive system $\{G_i\}_{i\in I}$. By [Sta24, Tag 01LA (4)], the colimit H of $\{H_i\}_{i\in I}$ in $\mathrm{Mod}(O_X)$ exists and lies in $\mathrm{Qch}(X)$. Because ψ_X^* commutes with colimits, one has $H^{\mathrm{an}} = \mathrm{colim}_{i\in I} G_i$. In particular, H^{an} is isomorphic to G in $\mathrm{Good}(X^{\mathrm{an}})$.

- The functor (3) is fully faithful: For any quasi-coherent O_X-modules F and G, we have to show that the canonical morphism

$$\mathrm{Hom}_{O_X}(F, G) \to \mathrm{Hom}_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}) \tag{7}$$

  is an isomorphism. Assume first that F is coherent.

  - From [GW20, Exercise 7.20 (b)], one has

$$[\mathcal{H}om_{O_X}(F, G)]^{\mathrm{an}} = \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}).$$

– As F is of finite presentation, the O_X-module $\mathcal{H}om_{O_X}(F, G)$ is quasi-coherent.

Therefore, by Proposition 3.1, the canonical morphism

$$H^0(X, \mathcal{H}om_{O_X}(F, G)) \to H^0(X^{\mathrm{an}}, \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}))$$

is an isomorphism, which is exactly (7).

By (5) and (6), the general case follows.

□

4 Derived category of quasi-coherent sheaves

Theorem 4.2 extends GAGA further to the derived category of quasi-coherent sheaves.

By [Sta24, Tag 0BKN], for every ringed space Y, the derived category $D(Y)$ has products and derived limits. This together with Definition 4.1 plays an essential role in the step 4 of the proof of Theorem 4.2.

Definition 4.1.

- ([Sta24, Tag 07LS]) Let $\mathcal{A}$ be an additive category with arbitrary direct sums. An object $K \in \mathcal{A}$ is called *compact*, if $\mathrm{Hom}_{\mathcal{A}}(K, \cdot) : \mathcal{A} \to \mathrm{Ab}$ preserves direct sums.
- ([Sta24, Tag 090Z]) Let $\mathcal{D}$ be a triangulated category. Let $(K_n, f_n)_{n>0}$ be a system of objects of $\mathcal{D}$. An object K is a *homotopy colimit* of the system if the direct sum $\oplus_{n>0} K_n$ exists and there is an exact triangle $\oplus_{n>0} K_n \overset{(1-f_n)_n}{\to} \oplus_{n>0} K_n \to K \overset{+1}{\to} \oplus_{n>0} K_n[1]$. In this case, we write $K = \mathrm{hocolim}_n K_n$.

A triangulated category $\mathcal{D}$ with arbitrary direct sums is *compactly generated*, if there is a family of compact objects $\{E_i\}_{i \in I}$ such that $\oplus_{i \in I} E_i$ generates $\mathcal{D}$. Let S be an algebraic variety. By [Nee96, Prop. 2.5], $D_{\mathrm{qc}}(S)$ is compactly generated. By [Sta24, Tag 09M1], the compact objects of $D_{\mathrm{qc}}(S)$ are exactly the perfect complexes in $D(S)$. By [Orl06, p.1827], when S is smooth, the full subcategory of perfect complexes coincides with $D^b_c(S)$.

Theorem 4.2. *If the complex algebraic variety X is proper, then the functor (4) is an equivalence of triangulated categories.*

Proof. By compactness of X^{an} and Lemma 4.6, the perfect complex $O_{X^{\mathrm{an}}}$ is a compact object of $D(X^{\mathrm{an}})$. Then from the proof of [Hal23, Lem. 4.3], the functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ admits a right adjoint functor $R\psi_{\mathrm{qc},*} : D(X^{\mathrm{an}}) \to D_{\mathrm{qc}}(X)$ which preserves small coproducts.

1. The functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ is fully faithful.

The unit of the adjunction $\eta : \mathrm{Id} \to R\psi_{\mathrm{qc},*}\psi_X^*$ is a natural transformation of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$. From Fact 2.1, it restricts to an isomorphism of functors $D_c^b(X) \to D_c^b(X)$. By [BB03, Thm. 3.1.1 1], the compact objects of $D_{\mathrm{qc}}(X)$ are precisely the perfect complexes. From [Nee96, Prop. 2.5], $D_{\mathrm{qc}}(X)$ is generated by a family of perfect complexes $\{E_i\}_{i\in I}$. By [Sta24, Tag 0FXU (1)], every perfect complex in $D(X)$ belongs to $D_c^b(X)$, so the η_{E_i} are isomorphisms. From Lemma 4.5, the unit η is an isomorphism. Thus, 1 is proved.

2. The functor (4) restricts to an equivalence $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$.

We prove that every object $F \in D^b_{\mathrm{gd}}(X^{\mathrm{an}})$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. Induction on the cohomological length of F. By Proposition 3.2, it holds when F has length zero. Suppose that it is true for objects of length $\leq n$ and F has length $n+1$. There is an integer i such that the truncations $\tau^{\leq i}F$ and $\tau^{>i}F$ have length $\leq n$. There is a canonical exact triangle

$$\tau^{\leq i}F \to F \to \tau^{>i}F \xrightarrow{+1} \tau^{\leq i}F[1]$$

in $D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. By 1 and the inductive hypothesis, the morphism $+1 : \tau^{>i}F \to \tau^{\leq i}F[1]$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. Then so is F. The essential surjectivity together with 1 proves 2.

3. The functor $\psi_X^* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{gd}}(X^{\mathrm{an}})$ is an equivalence.

By Lemma 4.3, for every $F \in D^+_{\mathrm{gd}}(X^{\mathrm{an}})$, one has $\mathrm{hocolim}_{n>0}\, \tau^{\leq n}F \xrightarrow{\sim} F$. Every $\tau^{\leq n}F$ is in $D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. From 2, there is a system $(K_n)_{n>0}$ of objects of $D^b_{\mathrm{qc}}(X)$, whose image under ψ_X^* is isomorphic to the system $(\tau^{\leq n}F)_{n>0}$. Since (4) respects coproducts, it respects homotopy colimits. Since $\mathrm{Qch}(X)$ is closed under filtered colimits in $\mathrm{Mod}(O_X)$, the subcategory $D_{\mathrm{qc}}(X)$ is closed under homotopy colimits in $D(X)$. Then F is isomorphic to the image of $K := \mathrm{hocolim}_{n>0}\, K_n \in D_{\mathrm{qc}}(X)$ under ψ_X^*. There is an integer q, such that $H^i(F) = 0$ for every integer $i < q$. Then $\psi_X^* H^i(K) = H^i(\psi_X^* K) = 0$. By Proposition 3.2, one has $H^i(K) = 0$. Hence $K \in D^+_{\mathrm{qc}}(X)$. Thus, the functor $\psi_X^* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{gd}}(X^{\mathrm{an}})$ is essentially surjective. By 1, it is an equivalence.

4. Every $Z \in D_{\mathrm{gd}}(X^{\mathrm{an}})$ is in the essential image of (4).

By Lemma 4.4, the canonical morphism $Z \to \mathrm{Rlim}_{n>0}\, \tau^{\geq -n}Z$ is an isomorphism in $D(X^{\mathrm{an}})$. By 3, there is an inverse system $(Y^{-n})_{n>0}$ of objects of $D^+_{\mathrm{qc}}(X)$, whose image is isomorphic to the inverse system $(\tau^{\geq -n}Z)_{n>0}$. Let Y be $\mathrm{Rlim}_{n>0}\, Y^{-n}$ in $D(X)$. For any integers $n \geq 1$ and q, the functor ψ_X^* transforms the morphism $H^q(Y^{-n-1}) \to H^q(Y^{-n})$ in $\mathrm{Qch}(X)$ to $H^q(\tau^{\geq -n-1}Z) \to H^q(\tau^{\geq -n}Z)$ in $\mathrm{Good}(X^{\mathrm{an}})$.

The morphism $H^q(\tau^{\geq -n-1}Z) \to H^q(\tau^{\geq -n}Z)$ is a surjection, and when $n \geq -q$, it is an isomorphism. By Proposition 3.2, so is $H^q(Y^{-n-1}) \to H^q(Y^{-n})$. Then by [Sta24, Tag 0A0J (1)], the canonical morphism $H^q(Y) \to$

$H^q(Y^{\min(q,-1)})$ is an isomorphism. In particular, the O_X-module $H^q(Y)$ is quasi-coherent. Hence, Y is in $D_{\mathrm{qc}}(X)$.

For every integer $m > 0$, the functor ψ_X^* transforms the projection $\prod_{n>0} Y^{-n} \to Y^{-m}$ to $\psi_X^*(\prod_{n>0} Y^{-n}) \to \tau^{\geq -m} Z$. Thus, one has a morphism $\psi_X^*(\prod_{n>0} Y^{-n}) \to \prod_{n>0} \tau^{\geq -n} Z$ in $D(X^{\mathrm{an}})$. It fits to a commutative diagram

$$\begin{tikzcd}
\psi_X^*(\prod_{n>0} Y^{-n})[-1] \arrow[r] \arrow[d] & \psi_X^*(\prod_{n>0} Y^{-n})[-1] \arrow[r] \arrow[d] & \psi_X^* Y \arrow[r] \arrow[d, dashed] & \psi_X^*(\prod_{n>0} Y^{-n}) \arrow[d] \\
\prod_{n>0} \tau^{\geq -n} Z[-1] \arrow[r] & \prod_{n>0} \tau^{\geq -n} Z[-1] \arrow[r] & Z \arrow[r] & \prod_{n>0} \tau^{\geq -n} Z
\end{tikzcd}$$

in $D(X^{\mathrm{an}})$, where the rows are exact triangles. By TR3, it induces a morphism of triangles. Hence, one has a commutative square

$$\begin{tikzcd}
H^q(\psi_X^* Y) \arrow[r, "\sim"] \arrow[d, dashed] & H^q(\psi_X^* Y^{\min(q,-1)}) \arrow[d, "\sim"] \\
H^q(Z) \arrow[r, "\sim"] & H^q(\tau^{\geq \min(q,-1)} Z)
\end{tikzcd}$$

in $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Therefore, for every integer q, the induced morphism $H^q(\psi_X^* Y) \to H^q(Z)$ is an isomorphism. Consequently, the morphism $\psi_X^* Y \to Z$ is an isomorphism in $D_{\mathrm{gd}}(X^{\mathrm{an}})$. Thus, 4 is proved.

By 4 and 1, the functor (4) is an equivalence. □

Lemma 4.3. Let $\mathcal{A}$ be an abelian category, where colimits over $\mathbb{Z}_{>0}$ exist and are exact. Then the natural transformation $\mathrm{hocolim}_{n>0} \tau^{\leq n} \cdot \to \mathrm{Id}$ is an isomorphism of functors $\mathcal{A} \to \mathcal{A}$.

Proof. It follows from [Sta24, Tag 0949] and the construction of canonical truncations. □

Lemma 4.4. Let X be a complex analytic space. Then the natural transformation $\mathrm{Id} \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} \cdot$ is an isomorphism of functors $D(X) \to D(X)$.

Proof. For every $x \in X$, there is an integer $d_x \geq 0$, and a fundamental system $\mathcal{U}_x$ of open neighborhoods of x, such that every $U \in \mathcal{U}_x$ is a closed complex subspace of a domain in $\mathbb{C}^{d_x}$. By [Liu24, Fact 3.1.9], for every $E \in D(X)$, any integers $p > 2d_x$ and q, one has $H^p(U, H^q(E)) = 0$. Then by [Sta24, Tag 0D63], the canonical morphism $E \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} E$ is an isomorphism in $D(X)$. □

Lemma 4.5. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories. Assume that $\mathcal{C}$ has direct sums. Let $\{E_i\}_{i \in I}$ be a family of compact objects of $\mathcal{C}$ such that $\oplus_{i \in I} E_i$ generates $\mathcal{C}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be triangulated functors preserving direct sums. Let $\eta : F \to G$ be a natural transformation. If for every $i \in I$, the morphism $\eta_{E_i} : F(E_i) \to G(E_i)$ is an isomorphism in $\mathcal{D}$, then η is an isomorphism.

Proof. From [Sta24, Tag 09SN], every object $X \in \mathcal{C}$ can be written as $X = \mathrm{hocolim}_{n>0} X_n$, where

- X_1 is a direct sum of shifts of the E_i,
- each transition morphism $X_n \to X_{n+1}$ fits into an exact triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$,
- and Y_n is a direct sum of shifts of the E_i.

Since F, G preserve direct sums, and the η_{E_i} are isomorphisms, so are the $\{\eta_{Y_n}\}_{n>0}$ and η_{X_1}. By [Sta24, Tag 014A] and induction on $n > 0$, one proves that the η_{X_n} are isomorphisms. By [BN93, Lem. 4.1], $F, G : \mathcal{C} \to \mathcal{D}$ preserve homotopy colimits. Therefore, η_X is an isomorphism. $\square$

Lemma 4.6. Let X be a compact complex analytic space. Then every perfect object of $D(X)$ belongs to $D^b_c(X)$. It is a compact object of $D(X)$ and of $D_{\mathrm{gd}}(X)$.

Proof. Let $E \in D(X)$ be a perfect object. By definition, there is an open covering $X = \cup_{i \in I} U_i$, such that for every $i \in I$, there is a morphism of complexes $E_i^\bullet \to E|_U$ which is a quasi-isomorphism, with $E_i^j = 0$ for all but finite many integers j, and every E_i^j is a direct summand of a finite free O_X-module. Since X is compact, one has $E \in D^b(X)$. By [Sta24, Tag 01BY (1)], every E_i^j is coherent. Therefore, every $H^j(E)|_{U_i}$ is coherent over O_{U_i}. Thus, $H^j(E)$ is coherent over O_X for all j. Hence $E \in D^b_c(X)$. In particular, E is in $D_{\mathrm{gd}}(X)$.

Let $E^\vee := R\mathcal{H}om(E, O_X) \in D(X)$. From [Sta24, Tag 08DQ], there is a natural isomorphism $\mathrm{Hom}_{D(X)}(E, \cdot) \to H^0(X, E^\vee \otimes^L_{O_X} \cdot)$ of functors $D(X) \to \mathrm{Ab}$. The functor $E^\vee \otimes^L_{O_X} \cdot : D(X) \to D(X)$ commutes with direct sums. Since X is compact, $\dim X$ is finite. Then by Lemma 4.7, the functor $H^0(X, \cdot) : D(X) \to \mathrm{Ab}$ also commutes with direct sums. Therefore, E is a compact object of $D(X)$. By [Liu24, Lem. A.4.3 2], $D_{\mathrm{gd}}(X)$ is closed under direct sums in $D(X)$. Then E is also a compact object of $D_{\mathrm{gd}}(X)$. $\square$

Lemma 4.7. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. If $\dim X$ is finite, then the functor $Rf_* : D(X) \to D(Y)$ commutes with direct sums.

Proof. First, we prove that for every integer q, there is a natural isomorphism

$$R^q f_* \xrightarrow{\sim} R^q f_* \tau_{\geq q-2\dim X} : D(X) \to \mathrm{Mod}(O_Y). \tag{8}$$

Indeed, by [Sta24, Tag 08J5], for every object $E \in D(X)$, there is an exact triangle $\tau_{\leq q-2\dim X-1} E \to E \to \tau_{\geq q-2\dim X} E \to (\tau_{\leq q-2\dim X-1} E)[1]$. It induces an exact sequence

$$R^q f_* \tau_{\leq q-2\dim X-1} E \to R^q f_* E \to R^q f_* \tau_{\geq q-2\dim X} E \to R^{q+1} f_* \tau_{\leq q-2\dim X-1} E$$

in $\mathrm{Mod}(O_Y)$. From [Liu24, Lem. 3.1.10], one has

$$R^q f_* \tau_{\leq q-2\dim X-1} E = R^{q+1} f_* \tau_{\leq q-2\dim X-1} E = 0.$$

Hence, one has an isomorphism $R^q f_* E \to R^q f_* \tau_{\geq -q-2\dim X} E$ which is functorial in E.

Let $\{E_i\}_{i\in I}$ be a family of objects of $D(X)$. Set $E = \oplus_{i\in I} E_i$. To prove that the canonical morphism $\oplus_{i\in I} Rf_* E_i \to Rf_* E$ in $D(Y)$ is an isomorphism, it suffices to show that for every integer q, the induced morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Since $\tau_{\geq q-2\dim X} E = \oplus_{i\in I} \tau_{\geq q-2\dim X} E_i$, by (8), one may assume that E and all the E_i are in $D^{\geq q-2\dim X}(X)$. Then from [Sta24, Tag 015J], one has canonical spectral sequences

$$R^s f_* H^t(E) \Rightarrow R^{s+t} f_* E, \quad R^s f_* H^t(E_i) \Rightarrow R^{s+t} f_* E_i.$$

By [Liu24, Lem. 3.1.8], for any integers s and t, the canonical morphism $\oplus_{i\in I} R^s f_* H^t(E_i) \to R^s f_* H^t(E)$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Consequently, the canonical morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ is an isomorphism. □

Corollary 4.8. If the complex algebraic variety X is proper, then the functors

1. $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ and
2. $D(\mathrm{Good}(X^{\mathrm{an}})) \to D_{\mathrm{gd}}(X^{\mathrm{an}})$

are equivalences of triangulated categories.

Proof. 1. For every $F \in D_c(X)$ and every integer i, the $O_{X^{\mathrm{an}}}$-module $H^i(\psi_X^* F) = \psi_X^* H^i(F)$ is coherent. Thus, the functors $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is well-defined. By Theorem 4.2, the functor $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is fully faithful. By Theorem 4.2, for every $F \in D_c(X^{\mathrm{an}})$, there is $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. Then the $O_{X^{\mathrm{an}}}$-module $\psi_X^* H^i(G) = H^i(\psi_X^* G) \xrightarrow{\sim} H^i(F)$ is coherent. By Fact 2.1 and Proposition 3.2, the O_X-module $H^i(G)$ is coherent, so $G \in D_c(X)$. Therefore, $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is essential surjective and hence an equivalence.

2. By Proposition 3.2 and Theorem 4.2, the functor is identified with the natural functor $D(\mathrm{Qch}(X)) \to D_{\mathrm{qc}}(X)$, which is an equivalence of triangulated categories by [Sta24, Tag 08DB].

□

5 Compact objects and objects of finite presentation

We determine the compact objects of the derived category of an algebraic compact complex manifold.

Corollary 5.1. Let X be a proper complex algebraic variety. Then the compact objects of $D_{\mathrm{gd}}(X^{\mathrm{an}})$ are precisely the perfect complexes in $D(X^{\mathrm{an}})$.

Proof. By compactness of X^{an} and Lemma 4.6, prefect complexes are compact objects of $D_{\mathrm{gd}}(X^{\mathrm{an}})$. Conversely, let F be a compact object of $D_{\mathrm{gd}}(X^{\mathrm{an}})$. By Theorem 4.2, there is a compact object $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. By [Sta24, Tag 09M1], G is a perfect complex in $D(X)$. By definition, F is a perfect complex in $D(X^{\mathrm{an}})$. □

Let X be a compact complex manifold.

Question 5.2. Does the full subcategory of $D_{\mathrm{gd}}(X)$ of compact objects coincide with $D^b_c(X)$?

Question 5.3. Is the category $D_{\mathrm{gd}}(X)$ compactly generated?

When X is the analytification of a smooth proper complex algebraic variety, Corollary 5.1 (resp. Theorem 4.2) answers Questions 5.2 (resp. 5.3) affirmatively.

By the Gabriel-Rosenberg reconstruction theorem (see, e.g., [Bra18, Thm. 1.1]), a quasi-separated scheme can be recovered from its category of quasi-coherent sheaves. Theorem 4.2 shows that good sheaves are complex analytic analog of quasi-coherent sheaves, which leads naturally to Question 5.4.

Question 5.4. Can one recover a compact complex manifold from its category of good sheaves?

We answer Question 5.4 negatively in Corollary 5.9, which relies on Verbitsky's theorem. For its proof, we use a notion related to compact objects.

Definition 5.5 ([KS06, Def. 6.3.3])**.** In a category $\mathcal{C}$ with small filtered colimits, an object A is *of finite presentation*, if the functor $\mathrm{Hom}_{\mathcal{C}}(A,\cdot) : \mathcal{C} \to \mathrm{Set}$ commutes with small filtered colimits.

Remark 5.6. In an additive category with arbitrary direct sums, an object of finite presentation is necessarily compact, but the converse is false. For example, there is a finite module M over a commutative ring R that is not a module of finite presentation. By [Ren69, no. 2], M is a compact object of the abelian category $\mathrm{Mod}(R)$. From [Sta24, Tag 0G8P], M is not an object of finite presentation.

Lemma 5.7. Let (X, O_X) be a ringed space. If the topology of X is Hausdorff compact, then every O_X-module of finite presentation is an object of finite presentation of $\mathrm{Mod}(O_X)$.

Proof. Let $G = \mathrm{colim}_{i\in I} G_i$ be a filtered colimit in $\mathrm{Mod}(O_X)$. Let F be an O_X-module of finite presentation. By [Sta24, Tag 0GMV], the canonical morphism $\mathrm{colim}_{i\in I}\mathcal{H}om_{O_X}(F, G_i) \to \mathcal{H}om_{O_X}(F, G)$ is an isomorphism. By compactness of X and [God58, Thm. 4.12.1], the canonical map $\mathrm{colim}_{i\in I} H^0(X, \mathcal{H}om_{O_X}(F, G_i)) \to H^0(X, \mathrm{colim}_{i\in I}\mathcal{H}om_{O_X}(F, G_i))$ is bijective. Then the canonical map

$$\mathrm{colim}_{i\in I}\mathrm{Hom}_{\mathrm{Mod}(O_X)}(F, G_i) \to \mathrm{Hom}_{\mathrm{Mod}(O_X)}(F, G)$$

is bijective. □

For a category $\mathcal{C}$, let $\mathrm{Ind}(\mathcal{C})$ be its indization in the sense of [KS06, Def. 6.1.1 (ii)].

Lemma 5.8. Let X be a compact complex analytic space. Then the full subcategory of $\mathrm{Good}(X)$ of objects of finite presentation is $\mathrm{Coh}(X)$. In particular, the canonical functor $\mathrm{Ind}(\mathrm{Coh}(X)) \to \mathrm{Good}(X)$ is an equivalence.

Proof. Let $F \in \mathrm{Good}(X)$ be an object of finite presentation. By compactness of X, there is a directed family of coherent submodules $\{F_i\}_{i \in I}$ of F with $F = \sum_{i \in I} F_i$. Then the canonical morphism $\mathrm{colim}_{i \in I} \mathrm{Hom}(F, F_i) \to \mathrm{Hom}(F, F)$ of abelian groups is an isomorphism. Thus, there is $i_0 \in I$ and an element $\mathrm{Hom}(F, F_{i_0})$ lying over Id_F. As Id_F factors through F_{i_0}, one has $F = F_{i_0} \in \mathrm{Coh}(X)$.

Conversely, by [Sta24, Tag 01BW], every object of $\mathrm{Coh}(X)$ is an O_X-module of finite presentation. From Lemma 5.7 and compactness of X, it is an object of finite presentation in $\mathrm{Mod}(O_X)$. By [Liu24, Lem. A.4.3 3], it is also an object of finite presentation in $\mathrm{Good}(X)$. Then by [KS06, Cor. 6.3.5], the functor $\mathrm{Ind}(\mathrm{Coh}(X)) \to \mathrm{Good}(X)$ is an equivalence. $\square$

Corollary 5.9. Let X and X' be either

1. complex analytic K3 surfaces having no non-trivial integer $(1, 1)$-cycles, or

2. complex tori of same dimension > 2 that are generic in the sense of [Ver04, Def. 1.1].

Then the abelian categories $\mathrm{Good}(X)$ and $\mathrm{Good}(X')$ are equivalent.

Proof. By either [Ver08, Thm. 1.1] or [Ver04, Thm. 1.2], the abelian categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X')$ are equivalent. Consequently, the result follows from Lemma 5.8. $\square$

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