formal KZ-equation:

$$d\,G(z) = \left(\frac{x_0\,dz}{z} + \frac{x_1\,dz}{1-z}\right) G(z)$$

$$G_0(z) \approx z^{x_0} \quad z \to 0 \qquad\qquad G_1(z) \approx (1-z)^{x_1} \quad z \to 1$$

Drinfeld associator:

$$\Phi_{KZ} = G_1^{-1} G_0(1)$$

Meaning: holonomy of the formal connection from 0+ε to 1-ε

Applying ordered exponential we get:

Multiple zeta values (MZVs)

$$\Phi_{KZ}(x_0, x_1) = 1 + \sum_{k_m > 1} (-1)^m \zeta(k_1, \ldots, k_m)\, x_0^{k_m - 1} x_1 \cdots x_0^{k_1 - 1} x_1 +$$

(regularized terms, may be expressed via MZVs by Le–Murakami)

$$\zeta(k_1, \ldots, k_m) = \int_{\{1 > t_1 > \cdots > t_w > 0\}} \omega_1(t_1) \wedge \cdots \wedge \omega_w(t_w)$$

iterated integral

$$w = k_1 + \cdots + k_m$$

$$w_i = 1/z \quad \text{or} \quad 1/(1-z)$$

$$\zeta(k_1,\ldots,k_m) = \sum_{0<n_1<\ldots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

$$\zeta(2)\zeta(2) = \sum_n \frac{1}{n^2}\cdot\sum_m \frac{1}{m^2} = \sum_{n>m}\frac{1}{n^2m^2} + \sum_{n<m}\frac{1}{n^2m^2} + \sum_{n=m}\frac{1}{n^2m^2}$$
$$= 2\zeta(2,2)+\zeta(4)$$

relations between polylogs

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \zeta(\mathbf{k}*\mathbf{l})$$

rDSR

stuffle relations+shuffle relations+regularization=regularized Double Shuffle Realtions

follows from Fubini theorem
for iterated integrals

Ihara-Kaneko-Zagier:
this is all (geometric) relations
between MZVs

???

Racinet: rDSR as coproduct

$$\mathbb{C}\langle y_1, y_2, \dots \rangle$$
$$\|$$
$$1 + \mathbb{C}\langle x_0, x_1\rangle x_1$$

$$\Delta_*(y_n) = \sum_{i=0}^{n} y_i \otimes y_{n-i}$$

$$y_i = -x_0^{i-1} x_1$$

Γ-factor
↓

$$\Phi_{KZ} = 1 + \phi_0 x_0 + \phi_1 x_1 \qquad \Phi_{KZ,y} = 1 + \phi_1 x_1 \qquad \Phi^{mod}_{KZ,y} = \Gamma^{-1}(y_1)\Phi_{KZ,y}$$

$$\Delta_*(\Phi^{mod}_{KZ,y}) = \Phi^{mod}_{KZ,y} \otimes \Phi^{mod}_{KZ,y} \quad \Leftrightarrow \quad \text{rDSR}$$

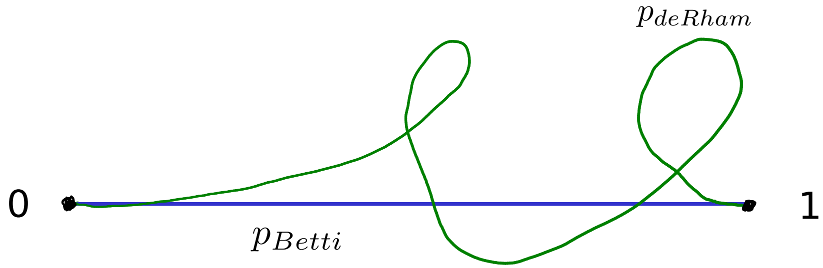
Tannakian formalism:

group-like element in a Hopf algebra <−> automorphism of a fiber functor of a Tannakian category

Deligne-Terasoma:

Category − perverse sheaves
Tensor product − convolution
Fiber functor − vanishing cycles

← how to see it ?
Fourier transform?



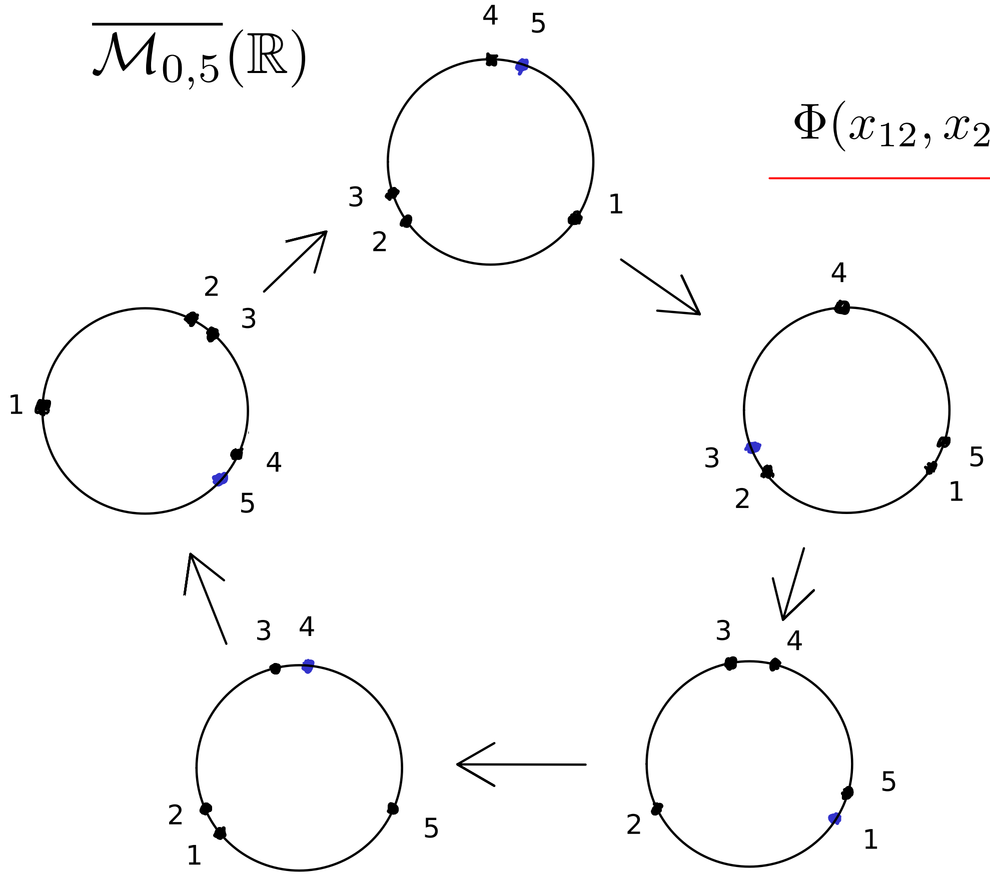
Chen integrals: $\Phi_{KZ} \in \pi_1^{un}(\mathcal{M}_{0,4})$

$\uparrow$

$p_{deRham} \circ p_{Betti}^{-1}$

Esquisse d'un programme: $Aut(\pi_1(\mathcal{M}_{0,n}))$, compatibility

$\overline{\mathcal{M}_{0,5}}(\mathbb{R})$



$$\Phi(x_{12}, x_{23})\Phi(x_{34}, x_{45})\Phi(x_{51}, x_{12})\Phi(x_{23}, x_{34})\Phi(x_{45}, x_{51}) = 1$$

$$x_{ij} = \log(i \text{ around } j) \text{ in } \mathcal{M}_{0,5}$$

pentagon equation

$\Downarrow$

associator relations between MZVs

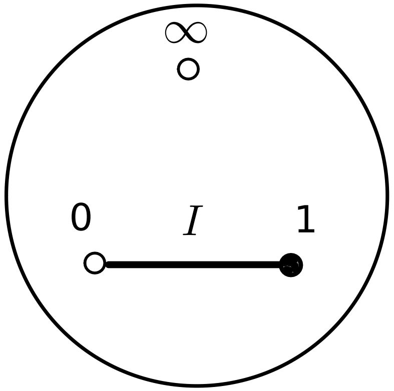
the strongest?

known $\Downarrow \Uparrow$???

rDSR

$\mathrm{Perv}(\mathbb{G}m, 1)$

category of perverse sheaves on $\mathbb{G}m = \mathbb{P}^1 \setminus \{0, \infty\}$ smooth outside 1



vanishing cycles at 1

$\Phi_1(\mathcal{F}) = H^0_{(0,1]}(\mathbb{P}^1, \mathcal{F})$

nearby cycles at 1

$\Psi_1(\mathcal{F}) = H^0_{(0,1)}(\mathbb{P}^1, \mathcal{F})$

Galligo−Granger−Maisonobe
Kapranov−Schechtman

Stratification $((0,1],\, 1)$ gives $\quad i_1 : 1 \to \mathbb{P}^1$

$$0 \longrightarrow H^0(i_1^! \mathcal{F}) \longrightarrow \Phi_1(\mathcal{F}) \xrightarrow{var} \Psi_1(\mathcal{F}) \longrightarrow H^1(i_1^! \mathcal{F}) \longrightarrow 0$$

We consider $\mathcal{F}$ such that $H^0(i_1^! \mathcal{F}) = 0 \quad \Rightarrow \quad \Phi_1(\mathcal{F}) \overset{var}{\hookrightarrow} \Psi(\mathcal{F})$

Basic example: $\mathcal{F} = j_{1*}\mathcal{L}[1]$

monodromy

$$\Phi \underset{can}{\overset{var}{\rightleftarrows}} \Psi \qquad var \circ can = 1 - T \qquad can \circ var = 1 - T$$

For $\mathcal{F} \in \mathrm{Perv}(\mathbb{G}m, 1)$ $\pi_1(\mathcal{M}_{0,4})$ acts on $\Psi_1(\mathcal{F})$ (and other fibers)

What does act on $\Phi_1(\mathcal{F})$? $a, b \in \pi_1(\mathcal{M}_{0,4})$

action of a action of b

$$\Phi_1(\mathcal{F}) \xrightarrow{var} \Psi_1(\mathcal{F}) \xrightarrow{a} \Psi_1(\mathcal{F}) \xrightarrow{can} \Phi_1(\mathcal{F}) \xrightarrow{var} \Psi_1(\mathcal{F}) \xrightarrow{b} \Psi_1(\mathcal{F}) \xrightarrow{can} \Phi_1(\mathcal{F})$$

$1 - X_1$

Transport algebra W is ($1 - X_1$)-homotope of the group algebra of $\pi_1(\mathcal{M}_{0,4})$

Jacobson: α -homotope of an algebra is an algebra with the new product $x\ {}_\alpha y = x\alpha y$

$$W = 1 + \mathbb{C}\langle X_0, X_1\rangle(1 - X_1)$$ right ideal

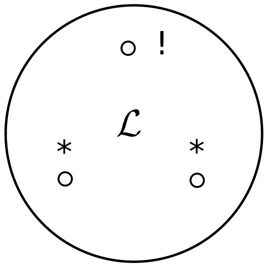
Variant: replace var with Var $X_1 - 1$ with $x_1 = \log X_1$

$$W = 1 + \mathbb{C}\langle x_0, x_1\rangle x_1 = \mathbb{C}\langle y_1, y_2, \ldots\rangle$$ algebra from the Racinet's formalism

$\mathcal{F} \in \mathrm{Perv}(\mathbb{G}m, 1)$

$\varphi_I : \Phi_1(\mathcal{F}) \xrightarrow{\sim} H^0(\mathbb{P}^1, j_{\infty!} j_{0*} \mathcal{F})$

A good way to define Drinfeld associator?

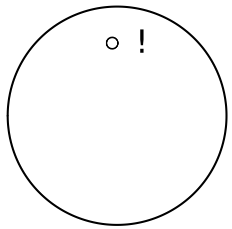


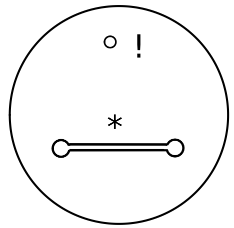
$$H^0(\mathbb{P}^1, j_{\infty!} j_{0*} j_{1*} \mathcal{L}[1])$$

$$\Phi_0 = \Psi_0 \longleftrightarrow \Psi_1 = \Phi_1$$

holonomy along I

Proof: For $\mathcal{F} \in \mathrm{Perv}(\mathbb{A}^1)$ with singularities at 0 and 1, $H^*(\mathbb{P}^1, j_{\infty!}\mathcal{F}) = H^*(I, i_I^! \mathcal{F})$





$$i_{I*} i_I^! \longrightarrow j_{\infty!} \longrightarrow j_{\setminus I*} j_{\infty!}$$

cohomology vanishes

Graphical representation: vanishing cycles may be thought as cohomology with support on 1-submanifolds = "homological cycles with coefficients"

$\mathbb{G}m = \mathbb{P}^1 \setminus \{0, \infty\}$ $\quad m \colon \mathbb{G}m \times \mathbb{G}m \to \mathbb{G}m$ $\quad \mathcal{E}, \mathcal{F} \in \mathrm{Perv}(\mathbb{G}m, 1)$

convolution: $\mathcal{E} *^! \mathcal{F} = \tau^{\leq 0} \mathbf{R}\, m_!(\mathcal{E} \boxtimes \mathcal{F})$

Künneth formula $\Rightarrow$ $H^0(\mathbb{P}^1, \underline{j_{\infty !} j_{0*}(\mathcal{E} *^! \mathcal{F})}) = H^0(\mathbb{P}^1, j_{\infty !} j_{0*}\mathcal{E}) \otimes H^0(\mathbb{P}^1, j_{\infty !} j_{0*}\mathcal{F})$

$||$

$m_!\,(j_{\infty !} j_{0*}\mathcal{E} \boxtimes j_{\infty !} j_{0*}\mathcal{F})$

$\Downarrow$

$H^0(\mathbb{P}^1, j_{\infty !} j_{0*} -)$

$\varphi_I \;\wr\!\uparrow$

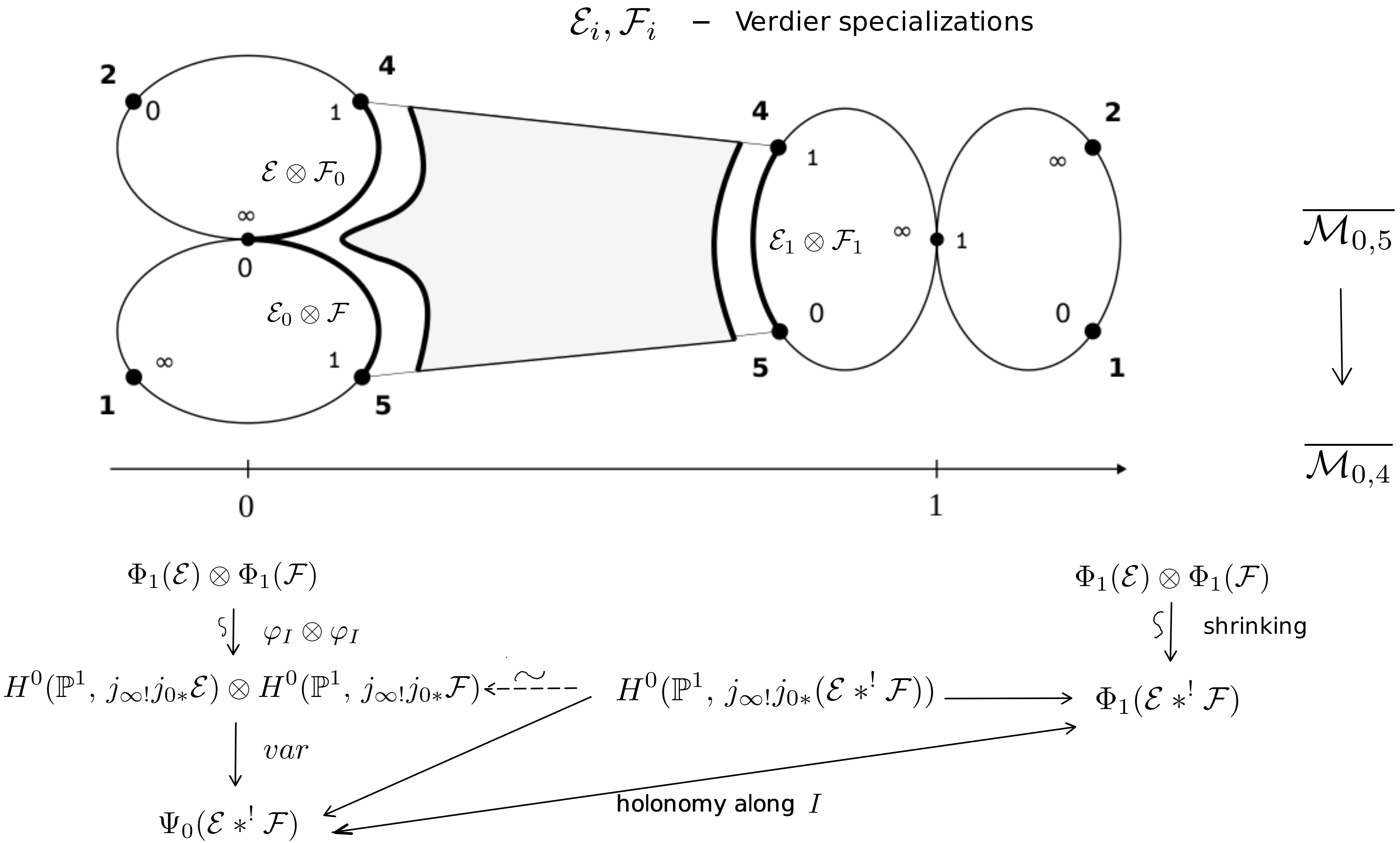
$\Phi_1(-)$

tensor functors

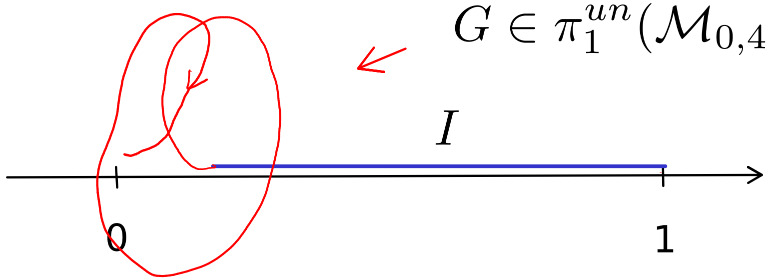
Katz 2012:
Convolution and equidistribution

Fiber functors for the Serre quotient $\mathrm{Perv}(\mathbb{G}m, 1)/(\text{shvs smooth at } 1)$ (=W-mod)

$\mathcal{M}_{0,5} \to \mathcal{M}_{0,4} \quad \subset \quad \mathbb{G}m \times \mathbb{G}m \to \mathbb{G}m$ $\qquad R\, p_{3*}(p_5^*(-) \otimes p_4^*(-))$



$\mathcal{E}_i, \mathcal{F}_i$ – Verdier specializations
2
4
0
1
$\mathcal{E} \otimes \mathcal{F}_0$
∞
0
$\mathcal{E}_0 \otimes \mathcal{F}$
∞
1
1
5
4
1
$\mathcal{E}_1 \otimes \mathcal{F}_1$
∞
1
0
5
2
∞
0
1
$\overline{\mathcal{M}_{0,5}}$
$\overline{\mathcal{M}_{0,4}}$
0
1
$\Phi_1(\mathcal{E}) \otimes \Phi_1(\mathcal{F})$
$\varphi_I \otimes \varphi_I$
$H^0(\mathbb{P}^1, j_{\infty!}j_{0*}\mathcal{E}) \otimes H^0(\mathbb{P}^1, j_{\infty!}j_{0*}\mathcal{F})$
$\sim$
$H^0(\mathbb{P}^1, j_{\infty!}j_{0*}(\mathcal{E} *^! \mathcal{F}))$
$\Phi_1(\mathcal{E} *^! \mathcal{F})$
var
$\Psi_0(\mathcal{E} *^! \mathcal{F})$
holonomy along I
$\Phi_1(\mathcal{E}) \otimes \Phi_1(\mathcal{F})$
shrinking

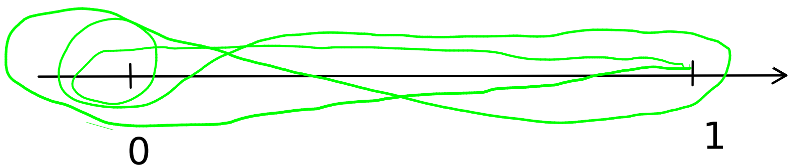


$G \in \pi_1^{un}(\mathcal{M}_{0,4})$

$$\varphi_G : \Phi_1(\mathcal{F}) \xrightarrow{\sim} H^0(\mathbb{P}^1,\, j_{\infty !} j_{0*}\mathcal{F})$$

$$\uparrow i_{G*}$$

$$i_G : I \to \mathbb{P}^1 \qquad H^0(i_G{}^! \mathcal{F})$$



Fox derivative: $\dfrac{\partial}{\partial x_i} x_{i_1} \cdots x_{i_n} = \sum x_{i_1} \cdots x_{i_{s-1}} \delta^i_{i_s}$

$$\downarrow$$

$$var(\varphi_G) = (1 + \frac{\partial G}{\partial X_1}(X_1 - 1))\, var(\varphi_I)$$

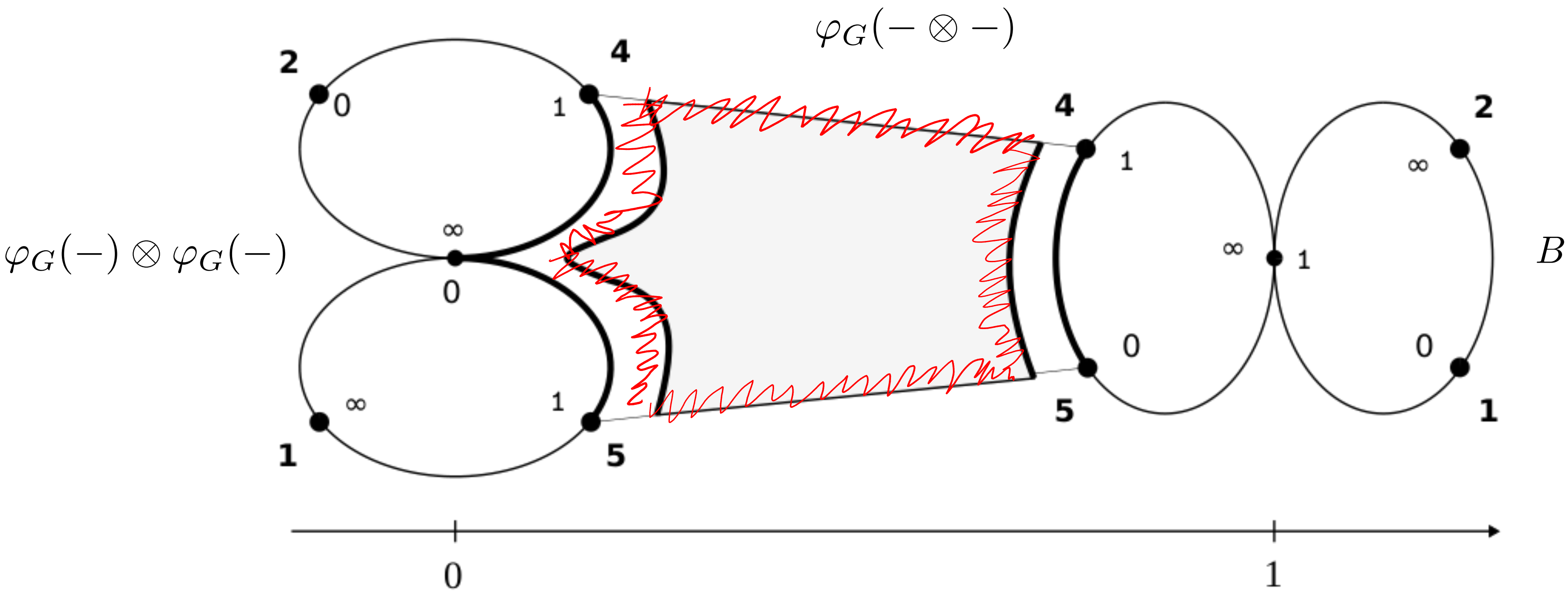
$$\varphi_G = (1 + \frac{\partial G}{\partial X_1}) \cdot \varphi_I$$

Alien derivative?

$$\uparrow$$

$\alpha(G) \in W,$

non-multiplicative wrt G



associator $f(x,y)$ symmetric: $f(y,x) = f(x,y)^{-1}$ + pentagon equation

$\Downarrow$

$$\Phi_1(\mathcal{F}) \underset{\varphi_G}{\overset{\varphi_I}{\rightrightarrows}} H^0(\mathbb{P}^1,\, j_{\infty !} j_{0*}\mathcal{F}) \circlearrowleft \alpha(G)$$

"automorphism of the tensor functor"

Details: Γ-factors, ...

$$\Phi_{KZ} = 1 + \phi_0 x_0 + \phi_1 x_1$$

$$G = 1 + \frac{\partial G}{\partial X_0}(X_0 - 1) + \frac{\partial G}{\partial X_1}(X_1 - 1) \longleftarrow$$

fundamental formula
of Fox differential calculus

Deligne−Terasoma: Betti and de Rham fiber functors, fake Hodge structure

Homological pentagon equation: (for symmetric $f(x,y)$)

the pentagonal cycle in $\mathcal{M}_{0,5}$ with coefficients in $p_4^*\mathcal{E} \otimes p_5^*\mathcal{F}$ is homologically trivial

HPE for the universal local system

$\pi_1^{un}(\mathcal{M}_{0,5}) \to \pi_1^{un}(\mathcal{M}_{0,4}) \times \pi_1^{un}(\mathcal{M}_{0,4})$ ⟵ Reminds Hain's construction of the Turaev cobracket

$\Updownarrow$

$\pentagon(f)^{ab} \in K^{ab}$ vanishes

$\pentagon(f) \in K \qquad p_i : \pi_1^{un}(\mathcal{M}_{0,5}) \to \pi_1^{un}(\mathcal{M}_{0,4})$

$K = \cap_{i=3,4,5} \ker p_i$

$\Updownarrow$ Enriquez−Furusho, unpublished

rDSR

$\Uparrow$

intermediate relations?

$\Uparrow$

Associator relations