HIGHER ROZANSKY–WITTEN INVARIANTS

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E_n is the operad of chains of little disks. For a topological space X, the complex of chains of the multiple loop space $C_*(\Omega^n X)$ is acted by E_n. For an E_∞-algebra A denote by A_{E_n} the E_n-algebra, which is the pullback of A under the forgetful functor $E_n-\mathrm{alg} \to E_\infty-\mathrm{alg}$.

1. **Higher Hochschild cohomology.** [Lur, GTZ14, GG, ...] One says that E_{n+1}-algebra acts on an E_n-algebra if they are acted by the Swiss cheese colored operad. The universal E_{n+1}-algebra acting on a E_n-algebra A is the higher Hochschild cohomology (center) of A denoted by $\mathfrak{Z}(A)$. There is a morphism of L_∞-algebras from the deformation complex of A to the higher Hochschild cohomology $\imath\colon \mathrm{Def}(A) \to \mathfrak{Z}(A)$, where L_∞-structure on $\mathfrak{Z}(A)$ is induced by the map of operads $L_\infty \to E_{n+1}$.

For a manifold M with trivialized tangent bundle of dimension k and an E_n-algebra A, the factorization homology $\int_M \mathfrak{Z}(A)$, which is a E_{n-k+1}-algebra, acts on E_{n-k}-algebra $\int_M A$. By universality, it gives the morphism

$$(\star) \qquad \int_M \mathfrak{Z}(A) \to \mathfrak{Z}(\int_M A)$$

Proposition 1. *For a closed compact M $(\star)$ is an isomorphism.*

2. **Invariants of manifolds.** Given a manifold M with trivialized tangent bundle of dimension k, one gets the pair

(1) Functor $\int_M\colon E_n-\mathrm{alg} \to E_{n-k}-\mathrm{alg}$
(2) Map $(\star)$ for any $A \in E_n-\mathrm{alg}$

One may consider this data as an invariant of M. While the first structure does not seem very powerful, for example, functor $\int_M$ sends $C_*(\Omega^n X)$ to $C_*(\Omega^{n-k}[M, X])$ (nonabelian Poincaré duality, [Lur]), that is, it depends only on the homotopy type of M; the second structure is much subtle, for example, for $A = \Bbbk[x]_{E_n}$ the map $(\star)$ defines Poincaré pairing on the (co)homology of M, which is non-degenerate for closed M, by Proposition 1.

3. **Higher symplectic manifolds.** Call higher n-symplectic structure an E_∞-algebra A with a Maurer–Cartan element $\omega \in MC(\mathrm{Def}(A_{E_n}))$ such that for the corresponding deformation $A_h \in E_n-\mathrm{alg}$ (quantization) the complex $\mathfrak{Z}(A_h \otimes \Bbbk[h^{-1}, h]])$ is trivial. The last condition implies that the symplectic structure is defined by the linear part of the deformation (symplectic form).

This notion of n-symplectic structure is closely related to $(1-n)$-shifted symplectic structure from [CPT$^+$].

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4. **Higher Rozansky–Witten invariants.** Proposition 1 guarantees that for a closed compact manifold M with trivialized tangent bundle $\int_M$ acts on the moduli space of symplectic structures. This action may be presented as follows

$$\text{sympl. form} \xrightarrow{\text{quantization}} E_n\text{−alg} \xrightarrow{\int_M} E_{n-k}\text{−alg} \xrightarrow{\text{dequantization}} \text{sympl. form}$$

In other words, given a symplectic structure A, the symplectic structure on $\int_M A$ is the first-order deformation (symplectic structure) corresponding to the deformation $\int_M A_h$. Its image under $\imath$ gives an element of $\mathfrak{Z}(\int_M A)$. There is another class of the same complex $\int_{[M]} \imath(\omega) \in \int_M \mathfrak{Z}(A) = \mathfrak{Z}(\int_M A)$, where ω is the symplectic form of A and $[M]$ is the fundamental class of M. Difference between them is a particular case of invariants of manifolds as above. The graph complex, which is cohomological complex of the Lie algebra of hamiltonian vector fields, naturally appears in this context ([Mar17]).

Example. *(*[Mar16]*) Let $Ch_h(g)$ be the E_3-algebra (quantum Chevalley algebra) associated with the finite-dimensional Lie algebra g with non-degenerate pairing introduced in* [Mar17, Appendix] *and* [Mar16]*. It is symplectic and may be considered as a deformation of the flat symplectic structure, that is $Ch_h(a)$, where a is abelian. The symplectic structure corresponding to $\int_{S^1} Ch_h(g)$ differs from the "naive" symplectic structure by the Duflo character. This difference corresponds to the one between dg-lagebras $Ch(g, Ug)$ and $Ch(g, S^*g)$. Thus, one should consider the Duflo character as the higher Rozansky–Witten invariant of S^1.*

References

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