MZV RELATIONS

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The general idea I would like to test is the following: pentagon relations are filtered by an integer non-negative parameter s. The deepest part of filtration given by $s = 0$ is rDSR. I am trying to imagine other relations. The meaning of the parameter s is discussed at the end of the first section.

The first section presents a conjectural form of relations corresponding to $s = 1$. The second section presents conjectural relations for all s, but of depth 2 only.

1. Double semi-shuffle relations

Let $\mathbf{k} = (k_1, k_2, \dots)$ and $\mathbf{l} = (l_1, l_2, \dots)$ be multi-indices. Consider the integral

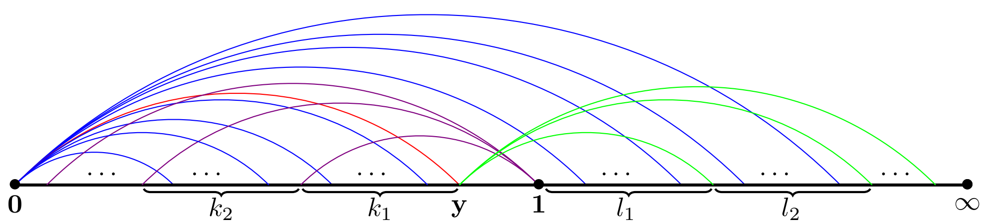

Figure 1

Here an arc designates the 1-form $d\log(x_i - x_j)$, where x_i and x_j are ends of the arc. Integration is taken by all positions in the presented order of free ends of arcs, that is, which are not 0 or 1. The interval $(y, 1)$ contains no ends of arcs.

Expressing the integrand in cubical coordinates, one sees that this integral equals to the sum

$$\sum_{\substack{n_1 > n_2 > \dots \\ m_1 > m_2 > \dots}} \frac{1}{(n_1 + m_1) \cdot n_1^{k_1} n_2^{k_2} \cdots m_1^{l_1} m_2^{l_2} \cdots} \tag{1}$$

Applying the identity $\frac{1}{nm} = \frac{1}{n(n+m)} + \frac{1}{m(n+m)}$ and regrouping summands in the same way as for the stuffle product, one may express this integral in terms of MZVs.

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Example 1.

$$\sum_{n,m} \frac{1}{(n+m)n^k m^l} = \sum_{n,m} \frac{1}{n^{k+1} m^l} - \sum_{n,m} \frac{1}{(n+m)n^{k+1} m^{l-1}} =$$

$$\sum_{n,m} \frac{1}{n^{k+1} m^l} - \sum_{n,m} \frac{1}{n^{k+2} m^{l-1}} + \sum_{n,m} \frac{1}{(n+m)n^{k+2} m^{l-2}} = \cdots =$$

$$\zeta(k+1,l)+\zeta(l,k+1)-\zeta(k+2,l-1)-\zeta(l-1,k+2)+\cdots\pm\zeta(k+l,1)+\epsilon\,\zeta(k+l+1),$$

where ϵ is 0 or 1 depending of parity of l.

There is another way to rewrite the sum, given by switching roles of k and l. The difference between them vanishes if $(k+l+1)$ is odd, and for even $(k+l+1)$ it is equal to

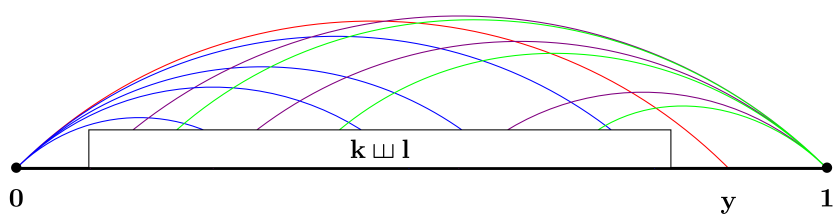
$$\pm(2\sum_{i=2}^{k+l}(-1)^i\,\zeta(i,k+l+1-i)-\zeta(k+l+1))$$

This expression vanishes due to [GKZ06]*, see also formula (5) from* [Wan13].

On the other hand, this integral equals to

$$\int_0^1 p_{\mathbf{k}}(y) p_{\mathbf{l}}(y)\, dy/y,$$

where $p_{\mathbf{k}}$ and $p_{\mathbf{l}}$ are multiple polylogarithms. This integral may be presented in a way analogous to the shuffle product, namely as the sum of integrals given by all possible shuffles of ends of non-red arcs:



One may see that this integral is given by the half shuffle operation $\prec$ from [Cha22], it is equal to

$$\zeta((\mathbf{k}+1) \prec \mathbf{l}) = \zeta((\mathbf{l}+1) \prec \mathbf{k}),$$

where $\mathbf{n}+1 = (n_1+1, n_2, \dots)$. We will call it semi-shuffle operation.

It is natural to call operation (1) semi-stuffle. Denote it by $\lhd$. Thus,

$$\zeta(\mathbf{k} \lhd \mathbf{l}) = \sum_{\substack{n_1>n_2>\dots \\ m_1>m_2>\dots}} \frac{1}{(n_1+m_1)\cdot n_1^{k_1-1} n_2^{k_2}\cdots m_1^{l_1} m_2^{l_2}\cdots}$$

where in the left hand side mean the linear combination of MZVs given by the series. Obviously, $\zeta(\mathbf{k} \lhd \mathbf{l}) + \zeta(\mathbf{l} \lhd \mathbf{k}) = \zeta(\mathbf{k} * \mathbf{l})$.

The relations above by by formulated as equality of semi-shuffle and semi-stuffle products. Call it double semi-shuffle relations (DHSR). There is a regularized version of them rDHSR. One may show that these relations are implied by the associator relations.

Conjecture 1. *rDHSR are not implied by rDSR. rDHSR are equivalent to the following condition. Let* $f \in \pi_1^{un}(\mathcal{M}_{0,4})$ *be the formal associator given by formal MZVs if* $f \in \pi_1^{un}(\mathcal{M}_{0,4})$ *is symmetric, that is* $f(x,y) = f^{-1}(y,x)$*, then* $\pentagon(f) \in \ker p_i$ *for any* i*, consequently* $\pentagon(f) \in K = \cap_{i=1,2,3} \ker p_i$*, where* $\pentagon$ *is the pentagon (5-term product). The condition states that* $\pentagon(f) \equiv 1 \mod [K,[K,K]]$.

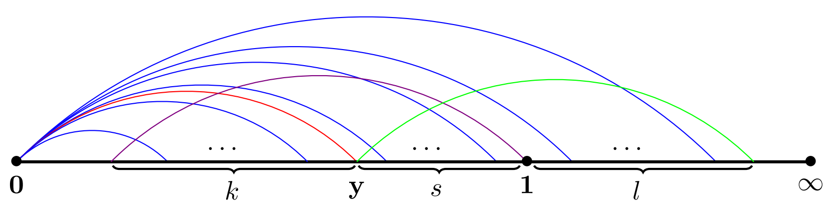
It is instructive to compare the conjecture with the following unpublished result.

Theorem 1 (Enriquez–Furusho, M)**.** *Let* $f \in \pi_1^{un}(\mathcal{M}_{0,4})$ *be the symmetric formal associator given by formal MZVs as above. Then coefficients of* f *obey rDSR iff* $\pentagon(f) \equiv 1 \mod [K,K]$*, where* K *and* $\pentagon$ *are as above.*

It should be possible to formulate a generalization of the above conjecture and theorem for the s-th term of the lower central series of K. The corresponding integral is somehow analogous to the one given by the Figure 1, but the overlapping region of green and violet structures contains $(s-1)$ nodes. Correspondingly, one may expect that the algebra of formal zeta is equipped with a hierarchy of operations, each one is presented in two versions: shuffle-like and stuffle-like, and relations are given by their equality. All these conjectural relations in total should be equivalent to the pentagon relations.

2. Depth 2

As above, consider the integral



Applying the same arguments as in the previous section, one identifies the value of this integral with a corresponding shuffle-like sum of iterated integrals and get the following relation for $k, l > 1$

$$\sum_{i=2}^{k+l-1} \left(\binom{i-1}{k-1} + \binom{i-1}{l-1} \right) \zeta(i+s, k+l-i) = \sum_{n,m} \frac{1}{(n+m)^s n^k m^l} \tag{2}$$

generalizing double shuffle relation for depth 2, to which it specializes at $s = 0$. Here the right-hand side means the linear combination of MZVs, which one can get inductively decreasing s like in Example 1. Interestingly enough, for $k = 1$, for example, both sides of the equality are well defined, but are not equal. It must be a regularized version of this equality for this case.

It seems that one may show that pentagon relations imply (2).

Conjecture 2. *Relations (2) are equivalent to double shuffle relations plus the Euler relation* $\sum_{i=2}^{w-1} \zeta(i, w-i) = \zeta(w)$.

I checked this conjecture up to weight 25. Indeed, the Euler relation appears already at $s = 1$, at least, this is true up to weight 45.

References

[Cha22] Frédéric Chapoton. Zinbiel algebras and multiple zeta values. *Documenta Mathematica*, 27:519–533, 2022. 11 pages, 1 figure.

[GKZ06] H. Gangl, M. Kaneko, and D. Zagier. Double zeta values and modular forms. In *Automorphic forms and zeta functions*, pages 71–106. World Scientific Publishing, 2006.

[Wan13] James Wan. Some notes on weighted sum formulae for double zeta values. In Jonathan M. Borwein, Igor Shparlinski, and Wadim Zudilin, editors, *Number Theory and Related Fields*, pages 361–379, New York, NY, 2013. Springer New York.

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