Soutenance de l'habilitation à diriger des recherches

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Table of Contents

Atiyah class

X is a smooth complex variety, $\mathcal{E}$ is a vector bundle

The Atiyah class is given by the extension (obstruction to existence of a connection):

$$0 \to \mathcal{E} \otimes \Omega^1 \to j^1(\mathcal{E}) \to \mathcal{E} \to 0 \qquad \mathrm{At}(\mathcal{E}) \in H^1(X, \mathrm{End}(\mathcal{E}) \otimes \Omega^1)$$

Works for quasi-coherent sheaves and complexes as well

Example

$$X = \mathbb{C} = Spec\, \mathbb{C}[x] \qquad \text{sheaf } \mathcal{O}_0 \qquad \text{resolution: } \mathcal{O} \xrightarrow{x} \mathcal{O} \to \mathcal{O}_0$$

$$0 \neq \mathrm{At}(\mathcal{O}_0) \in Ext^1(\mathcal{O}_0, \mathcal{O}_0) = \mathbb{C}$$

HKR isomorphisms

$\mathcal{O}_\Delta$ is the structure sheaf of the diagonal on $X \times X$

$$\mathrm{At}(\mathcal{O}_\Delta)\colon \mathcal{O}_\Delta \to \mathcal{O}_\Delta \otimes \Omega^1[1] \qquad \mathcal{O}_\Delta \to \mathcal{O}_\Delta \otimes \left(\bigoplus_i \Omega^i[i]\right)$$

Hochschild–**K**ostant–**R**osenberg isomorphisms:

$$\mathcal{E}xt^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_i \Lambda^i T[-i]_\Delta \qquad \mathcal{T}or_\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_i \Omega^i[i]_\Delta$$

Hochschild cohomology and homology:

$$\mathrm{Ext}^n(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \bigoplus_i H^i(X, \Lambda^{n-i} T) \quad \mathrm{Tor}_n(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \bigoplus_i H^i(X, \Omega^{n+i})$$

Jacobi identity

The Atiyah class of Ω^1 is symmetric:

$$\begin{tikzcd} \Omega^1 \arrow[r, "\mathrm{At}(\Omega^1)"] \arrow[rr, bend right, dashed] & \Omega^1 \otimes \Omega^1[1] & S^2\Omega^1[1] \arrow[l, hook'] \end{tikzcd}$$

Its symmetrized square vanishes:

$$\Omega^1 \xrightarrow{\mathrm{At}(\Omega^1)} S^2\Omega^1[1] \xrightarrow{\mathrm{At}(\Omega^1)} S^3\Omega^1[2] \qquad = 0$$

This is an analog of the Jacobi identity (Kapranov)

Proof: Consider the filtration on $j_0^3(\mathcal{O})$:

$$\mathrm{Gr}(j_0^3) = \qquad F_3 \qquad F_2/F_3 \qquad F_1/F_2$$

$$S^3\Omega^1 \xleftarrow{\mathrm{At}(\Omega^1)} S^2\Omega^1 \xleftarrow{\mathrm{At}(\Omega^1)} \Omega^1$$

Dictionary

varieties — Lie algebras

Dolbeault complex $(\Omega_X^{0,i}, \overline{\partial})$	Chevalley complex $(\Lambda^* \mathfrak{g}^*, d_{Ch})$
$\mathrm{At}(\Omega^1)$	structure constants of $\mathfrak{g} = d_{Ch}$
$\mathrm{Ext}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$	$U(\mathfrak{g})$
$\mathrm{Tor}_\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$	$F(G)$, formal functions on G
HKR iso for Hoch. cohomology	Poincaré–Birkhoff–Witt iso
HKR iso for Hoch. homology	exponential coordinates on G

Hirzebruch–Riemann–Roch theorem

$$\chi(H^*(\underset{\substack{\uparrow\\ \text{compact}}}{X}, E)) = \int \mathrm{ch}(E)\,\mathrm{td}(X)$$

$$\int : \bigoplus_i H^i(X, \Omega^i) \to H^{\dim X}(X, \Omega^{\dim X}) \to \mathbb{C}$$

$$\bigoplus_i H^i(X, \Omega^i) \ni \mathrm{ch}(E) = \mathrm{Tr}(\exp \mathrm{At}(E))$$

$$\bigoplus_i H^i(X, \Omega^i) \ni \mathrm{td}(E) = \exp(\sum t_i\, \mathrm{ch}(\Omega^1))$$

$$\sum t_i z^i \quad = \quad \log(z/(\underset{\substack{\uparrow\\ \text{Bernoulli numbers}}}{e^z - 1}))$$

Proof (M.)

On $X \times X$ the Serre duality gives

$$\mathcal{O}_\Delta \xrightarrow{can} \mathcal{O}_X \boxtimes \omega_X[\dim X]$$

Restricting on the diagonal

$$\mathcal{O}_\Delta \overset{L}{\otimes} \mathcal{O}_\Delta \longrightarrow \omega_X[\dim X] \quad \in \quad \mathrm{Ext}^{\dim X}(\bigoplus_i \Omega^i[i], \omega)$$

This is td. To calculate one needs the analog of

$$\mathfrak{g} \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \text{ in terms of PBW} \quad \leftarrow \quad \text{Bernoulli numbers!}$$

Why td reminds the invariant volume form on the Lie group? (Feigin)

Chern–Simons invariants

M is a smooth 3-manifold, $\mathbb{Q}$-homological 3-sphere
G – Lie group (e. g. SU_n), A is a connection on a G-bundle over M

Chern–Simons action for level $k \in \mathbb{Z}$:

$$S(A) = \frac{k}{2\pi} \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

The partition function or the Chern–Simons invariant:

$$Z_k^M = \int e^{iS(A)}$$

where the integral is taken over all connections

What does it mean?

Perturbative Chern–Simons invariants

For $k \gg 1$ let $h = -\frac{2\pi}{ik}$ and apply stationary phase method

Axelrod–Singer expansion:

$$Z(h) = Z^{sc}(h) \cdot \sum_{\text{3-valent graphs } \Gamma} h^{v(\Gamma)/2} \cdot c_{\mathfrak{g}}(\Gamma) \int_{M^{v(\Gamma)}} \bigwedge\nolimits_{e(\Gamma)} p_{s(e),t(e)}$$

where

weight system corresponding to quadratic Lie algebra $\mathfrak{g} = Lie(G)$:

$$c_{\mathfrak{g}}(\Gamma) = \text{contraction of } C^{\otimes e(\Gamma)} \otimes K^{\otimes v(\Gamma)}$$

structure constants $C \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ Killing form $K \in \mathfrak{g} \otimes \mathfrak{g}$

propagator

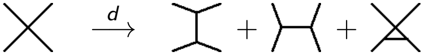
$$p_{ij} = \pi_{ij}^* p \qquad p \in \Omega^2(X \times X) \quad \longleftarrow \quad \text{has singularities}$$

Graph complex

Observation:

$c_{\mathfrak{g}}$ may be replaced with an element of the graph complex (Kontsevich)

Graph complex:
linear functions on graphs, differential is given by inserting edges (co-shrinking).



$c_{\mathfrak{g}}$ is a cocycle due to the Jacobi identity

Lie algebras (co)homology

(V, ω) — (n-)symplectic (graded) vector space

$\mathrm{Ham}(V)$ Lie algebra of Hamiltonian vector fields.

$\mathrm{Ham}(V) = S^* V/\Bbbk \quad v_1, v_2 \in S^1 V: \quad [v_1, v_2] = \omega(v_1, v_2) \quad$ + Leibniz rule

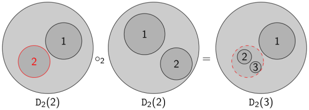
Fuchs: Cohomology of infinite-dimensional Lie algebras

Method:
calculate invariants of the Chevalley complex under the linear group action

$$\text{stabilization of } C^*_{Ch}(\mathrm{Ham}(V))^{sp(V)} = \text{ graph complex}$$

e_n-**algebras**

Operad of little n-discs E_n $\qquad$ $e_n = C_*(E_n)$ is a dg-operad



$H_*(e_n)$ is generated by

$$\begin{array}{ccc} \cdot & \xleftrightarrow{\text{Leibniz rule}} & [\,,\,] \\ \uparrow & & \uparrow \\ \text{commutative, associative, deg } 0 & & \text{Lie bracket, deg } 1-n \end{array}$$

e_1-algebra $= A_\infty$-algebra

Example (May)

Topological space X $\qquad$ n-loop space $\Omega^n X$ $\qquad$ $C_*(\Omega^n X)$ is a e_n-algebra

Factorization homology

A — e_n-algebra, M — (parallelized) n-manifold
$M^{[k]}$ — Fulton-MacPherson compactification of the configuration space of k points in M

$$\int_M A = \underbrace{\left(\bigoplus_i A^{\otimes i}\right)}_{\text{left } e_n\text{-module}} \bigotimes_{e_n} \underbrace{\left(\bigoplus_i C_*(M^{[i]})\right)}_{\text{right } e_n\text{-module}}$$

Example (Hochschild homology)

$n = 1 \qquad M = S^1 \qquad HH_k(A) = \mathrm{Tor}_k^{A\otimes A^{op}}(A, A)$

$$d\left(a_0 \otimes a_1 \otimes \cdots \otimes a_k\right) =$$
$$a_0a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_k + \cdots \pm a_ka_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}$$

Weyl n-algebras (M.)

V is a $\mathbb{Z}$-graded vector space
with a non-degenerate skew-symmetric pairing of degree $1-n$

$$\omega\colon V \otimes V \to \Bbbk$$

The Weyl e_n-algebra $\mathcal{W}^n_h(V)$ is an e_n-algebra over $\Bbbk[[h]]$ generated by V such that for $v_{1,2} \in V$, $[v_1, v_2] = h\omega(v_1, v_2)$

It is a formal deformation of the polynomial algebra $\Bbbk[V]$

Example (Weyl algebra)

$$W_{2n} \qquad < x_1, \ldots, x_n, \partial_1, \ldots, \partial_n > \qquad h \neq 0$$

$$[x_i, x_j] = [\partial_i, \partial_j] = 0 \qquad [x_i, \partial_j] = h\delta_{ij}$$

Factorization homology of Weyl n-algebras

The main statement (M.): For n-manifold M

$$\text{total dimension of } H^*\left(\int_M \mathcal{W}^n_h(V)\right) \otimes \mathbb{k}[h^{-1}, h]] = 1$$

Proof: Deformation.

Example (Hochschild homology of a Weyl algebra)

$$HH_i(W_{2n}) = \begin{cases} \mathbb{k}, & \text{if } i = 2n \\ 0, & \text{otherwise} \end{cases}$$

From Lie algebra homology to factorization homology

The map of operads $L_\infty[1-n] \to e_n$ gives the functor

$$L\colon e_n\text{-algebras} \longrightarrow L_\infty\text{-algebras}$$

For associative algebras it gives the commutator Lie algebra

For e_n-algebra A and n-manifold M there is the natural map

$$C^{Ch}_\bullet(L(A)) \to \int_M A$$

Example (Hochschild homology)

$$a_1 \wedge \cdots \wedge a_k \mapsto \sum_\sigma (-1)^{sgn\,\sigma} 1 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$$

Chern–Simons invariant via factorization homology

For n-manifold M and $(1-n)$-symplectic vector space V

$$
\begin{array}{ccc}
H_*(L(\mathcal{W}_h^n(V))) & \longrightarrow & H_*\left(\int_M \mathcal{W}_h^n(V)\right) \\
\| & & \| \\
H_*(\mathrm{Ham}(V)) & & \Bbbk[h^{-1}, h]] \\
\| & & \\
\text{graph complex} & &
\end{array}
$$

Theorem (M.)

A variant of the above map gives perturbative Chern–Simons invariants.

This explains the appearance of the graph complex

Knots

$K\colon S^1 \to M^3$ is a knot, $q \in \mathcal{W}_h^3(V)$ is a MC element: $[q,q]=0$.
K induces

$$HH_*(A) = H_*\left(\int_{S^1} A\right) \to H_*\left(\int_M \mathcal{W}_h^3(V)\right) = \Bbbk[h^{-1},h]]$$

where A is $\mathcal{W}_h^3(V)$ as an e_1-dg-algebra with differential $[q,\cdot]$.

Take $V=\mathfrak{g}[-1]$ for a simple Lie algebra $\mathfrak{g}$, symplectic form is given by the Killing form, $q\in\Lambda^3\mathfrak{g}$ is the structure constants. Then $A = H^*_{Ch}(\mathfrak{g})$ after localizing by h.

$$HH_*(A)\otimes \Bbbk[h^{-1},h]] = H^*_{Ch}(\mathfrak{g}, F(G))\otimes \Bbbk[h^{-1},h]]$$

Knot invariant is in the dual space.

This is the **Kontsevich integral** $I(K)$ of the knot.

Kontsevich integral of unknot

Duflo character

$$I(\bigcirc) = \exp\left(\sum b_{2n} w_{2n}\right) \qquad w_n = \mathrm{Tr}\,\mathrm{Ad}_{\mathfrak{g}}^n \in S^n\mathfrak{g}^*$$

$$\sum b_{2n} z^{2n} = \frac{1}{2} \ln\left(\frac{e^{z/2} - e^{-z/2}}{z/2}\right) \longleftarrow \text{Bernoulli numbers!}$$

Other possible applications

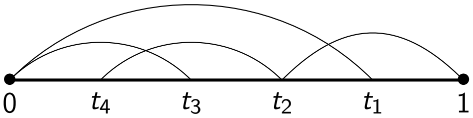
- Kontsevich formality theorem
- Chan–Galatius–Payne homology classes of M_g
- Grothendieck–Teichmüller group and graph complex after Willwacher, Fresse and others

Cell zeta values

Operad of little discs e_2 has Hodge structure

Example of its period (period of a Tate motive over $\mathbb{Z}$):

$$\int_{0<t_4<t_3<t_2<t_1<1} d\,log(1-t_2) \wedge d\,log\,t_1 \wedge d\,log(t_2-t_4) \wedge d\,log\,t_3$$

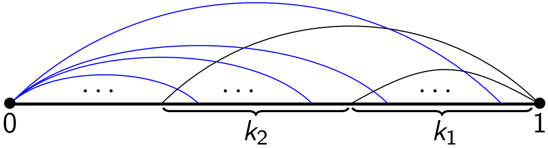


Brown: cell zeta values = multiple zeta values

Multiple zeta values

Multiple zeta values are cell zeta values without $d\log(x_i - x_j)$ i. e.

Iterated integrals:



$$\| $$

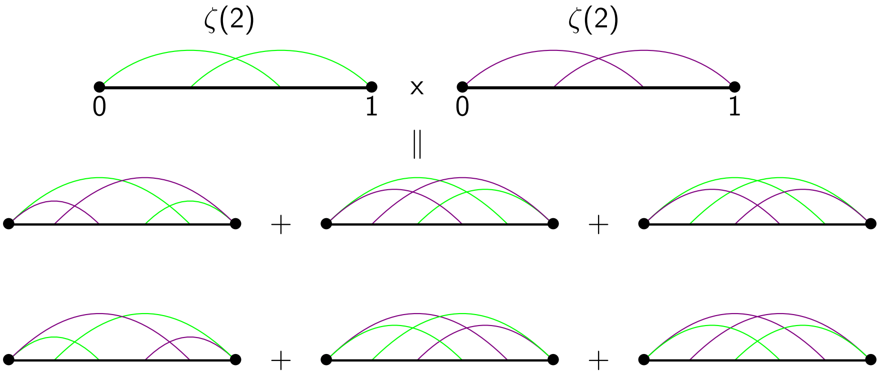
$$\zeta(k_1, k_2, \dots)$$

$$\|$$

$$\sum_{n_1 > n_2 > \dots} \frac{1}{n_1^{k_1} n_2^{k_2} \dots} \qquad n_i \in \mathbb{N}$$

Multiple zeta values

Shuffle relations



$$\zeta(2)\,\zeta(2) = 4\,\zeta(3,1) + 2\,\zeta(2,2)$$

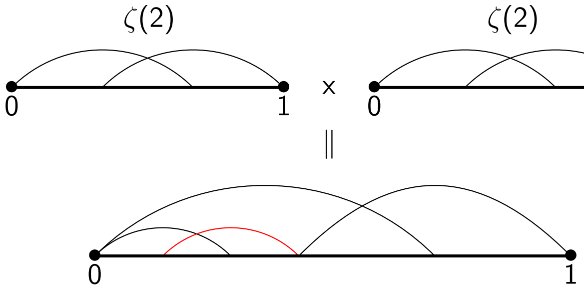
Stuffle relations

$$\zeta(2)\,\zeta(2) = \sum_{n} \frac{1}{n^2} \cdot \sum_{m} \frac{1}{m^2}$$

$$= \sum_{n>m} \frac{1}{n^2 m^2} + \sum_{n<m} \frac{1}{n^2 m^2} + \sum_{n=m} \frac{1}{n^2 m^2}$$

$$= 2\,\zeta(2,2) + \zeta(4)$$

Stuffle relations via integrals



This is not an iterated integral! $\uparrow$

But may be presented as a sum of iterated by means of birational transforms, cubical coordinates (Cartier):

$$x_i = t_i / t_{i-1}$$

Generalized shuffle and stuffle relations

Question

How to describe all (geometric=motivic) relation between multiple zeta values?

Two set of relations between **cell** zeta values (M.):

Generalized shuffle relations Fubini theorem

Generalized stuffle relations Fubini theorem + reflection invariance

Theorem (M.)

These relations imply double shuffle relations between MZVs.

Merci de votre attention !

Thank you for your attention!